A Note on Fuzzy Set–Valued Brownian Motion

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Abstract

In this paper, we prove that a fuzzy set–valued Brownian motion B_t, as defined in Li and Guan (2007), can be handle by an $\mathbb{R}^d$–valued Wiener process b_t, in the sense that $B_t = \mathbb{I}_{b_t}$; i.e. it actually is the indicator function of a Wiener process.

Keywords: Fuzzy random sets; Fuzzy Brownian motion; Gaussian fuzzy process; defuzzification of randomness;

Revised Personal Version of the text of the final journal article:

E. G. Bongiorno. *A Note on Fuzzy Set–Valued Brownian Motion*. Statistics and Probability Letters 82 (2012), pp. 827-832. (doi: 10.1016/j.spl.2012.01.011).
`http://dx.doi.org/10.1016/j.spl.2012.01.011`

It reflects changes made in the peer review process.

1. Introduction

Stochastic (fuzzy) set–valued evolution is a relevant topic that was studied largely by different authors and in different frameworks (e.g. Kloeden and Lorenz (2011); Li and Guan (2007); Li et al. (2002); Mitoma et al. (2010); Molchanov (2005) and references therein). The following question was stated in (Molchanov, 2005, Open Problem 1.24, p.316):

> Define a set–valued analogue of the Wiener process and the corresponding stochastic integral.

In Li and Guan (2007), the authors tackle the proposed problem defining a fuzzy set–valued Brownian motion in $\mathbb{F}_{kc}$, the family of convex fuzzy subsets of $\mathbb{R}^d$ with compact support. In the sequel we prove that such a process is equivalent to consider simply a Wiener process in $\mathbb{R}^d$. This is based upon the fact that the Brownian motion is a zero–mean Gaussian (fuzzy set–valued) process. In fact, it is widely known (cf. Li et al., 2002, Theorem 6.1.7) that a Gaussian random fuzzy set decomposes according to

$$X = \mathbb{E}X \oplus \mathbb{I}_\xi, \tag{1}$$

where $\mathbb{E}X$ is in the Aumann sense, ξ is a Gaussian random element in $\mathbb{R}^d$ with $\mathbb{E}\xi = 0$ and $\mathbb{I}_A : \mathbb{R}^d \to \{0, 1\}$ denotes the indicator function of any $A \subseteq \mathbb{R}^d$

$$\mathbb{I}_A(x) = \begin{cases} 1, & \text{if } x \in A, \\ 0, & \text{otherwise.} \end{cases}$$

We write $\mathbb{I}_a$ instead of $\mathbb{I}_{\{a\}}$. Equation (1) means that X is just its expected value $\mathbb{E}X$ up to a random Gaussian translation ξ. In other words, $\mathbb{E}X$ represents the "deterministic" part of X whilst ξ represents its random part; this is a situation in which the randomness of X is defuzzificated although X is still a fuzzy set with $\mathbb{E}X$. Moreover, it is also known (cf. Molchanov, 2005, Proposition 1.30, p.161) that a zero–mean random set is actually a random element in $\mathbb{R}^d$ with zero–mean. Such a result can be easily extended to the fuzzy case (see Corollary 8) and, jointly to decomposition (1), implies

$$X = \mathbb{I}_0 \oplus \mathbb{I}_\xi = \mathbb{I}_\xi.$$

Roughly speaking, the definition of Brownian motion in Li and Guan (2007) for random fuzzy sets drives down the complexity of the chosen (fuzzy) framework. In fact, a Gaussian fuzzy random set with zero–mean is reduced to be a random Gaussian element in $\mathbb{R}^d$. In other words, in the case of a fuzzy Brownian motion, both the randomness of X and the expectation $\mathbb{E}X$ are defuzzificated.

The paper is organized as follow. Section 2 is devoted to preliminaries such as random fuzzy sets, embedding theorems and Brownian motion for fuzzy sets (according to Li and Guan, 2007; Li et al., 2002). In Section 3 we prove the main result of the paper, whilst in Appendix A we provide, using selections, an alternative proof to (Molchanov, 2005, Proposition 1.30, p.161).

2. Preliminaries

We refer to Li et al. (2002) for what concerns classical results on fuzzy set–valued random variable, and to Li and Guan (2007) for Gaussian and Brownian fuzzy set-valued processes. Denote by $\mathbb{K}_{kc}$ the class of non–empty compact convex subsets of $\mathbb{R}^d$, endowed with the Hausdorff metric

$$\delta_H(A, B) = \max\{\sup_{a \in A} \inf_{b \in B} \|a - b\|, \sup_{b \in B} \inf_{a \in A} \|a - b\|\},$$

and the operations

$$A + B = \{a + b : a \in A,\ b \in B\}, \qquad \lambda \cdot A = \lambda A = \{\lambda a : a \in A\}.$$

A *fuzzy set* is a map $\nu : \mathbb{R}^d \to [0, 1]$. Let $\mathbb{F}_{kc}$ denote the family of all fuzzy sets, which satisfy the following conditions.

1. Each ν is an upper semicontinuous function, i.e. for each $\alpha \in (0, 1]$, the cut set $\nu_\alpha = \{x \in \mathbb{R}^d : \nu(x) \geq \alpha\}$ is a closed subset of $\mathbb{R}^d$.
2. The cut set $\nu_1 = \{x \in \mathbb{R}^d : \nu(x) = 1\} \neq \emptyset$.
3. The support set $\nu_{0+} = \overline{\{x \in \mathbb{R}^d : \nu(x) > 0\}}$ of ν is compact; hence every ν_α is compact for $\alpha \in (0, 1]$.
4. For any $\alpha \in [0, 1]$, ν_α is a convex subset of $\mathbb{R}^d$.

Let us endow $\mathbb{F}_{kc}$ with the metric

$$\delta_H^\infty(\nu^1,\nu^2) = \sup\{\alpha \in [0,1] : \delta_H(\nu^1_\alpha,\nu^2_\alpha)\}.$$

and the operations

$$(\nu^1 \oplus \nu^2)_\alpha = \nu^1_\alpha + \nu^2_\alpha, \qquad (\lambda \odot \nu^1)_\alpha = \lambda \cdot \nu^1_\alpha.$$

Let $(\Omega, \mathfrak{F}, \mathbb{P})$ be a complete probability space. A *fuzzy set–valued random variable* (FRV) is a measurable function $X : \Omega \to \mathbb{F}_{kc}$ with respect to the δ_H^∞–Borel σ–algebra on $\mathbb{F}_{kc}$. It is known (cf. Colubi et al., 2002) that this measurability notion implies the levelwise measurability; i.e. $X_\alpha : \omega \mapsto X(\omega)_\alpha$ are random compact convex sets for every $\alpha \in (0,1]$, that is X_α is a $\mathbb{K}_{kc}$–valued function measurable with respect to the δ_H–Borel σ–algebra. This implication is used in the proof of Corollary 8.

A FRV X is *integrably bounded* and we write $X \in L^1[\Omega, \mathfrak{F}, \mu; \mathbb{F}_{kc}] = L^1[\Omega; \mathbb{F}_{kc}]$, if $\|X_{0+}\|_H := \delta_H(X_{0+}, \{0\}) \in L^1[\Omega; \mathbb{R}]$. The *expected value* of $X \in L^1[\Omega; \mathbb{F}_{kc}]$, denoted by $\mathbb{E}[X]$, is a fuzzy set such that, for every $\alpha \in (0,1]$,

$$(\mathbb{E}[X])_\alpha = \int_\Omega X_\alpha \mathrm{d}\mu = \{\mathbb{E}(f) : f \in L^1[\Omega;\mathbb{R}^d], f \in X_\alpha \ \mu - \text{a.e.}\}.$$

Embedding Theorem, Support function of a FRV and some useful properties. Let S^{d-1} be the unit sphere in $\mathbb{R}^d$. Let $C(S^{d-1})$ denote the Banach space of all continuous functions v on S^{d-1} with respect to the norm $\|v\|_C = \sup_{x\in S^{d-1}} |v(x)|$. Let $\overline{\mathbf{C}} := \overline{C}([0,1], C(S^{d-1}))$ be the set of all functions $f : [0,1] \to C(S^{d-1})$ such that f is bounded, left continuous with respect to $\alpha \in (0,1]$, right continuous at 0, and f has right limit for any $\alpha \in (0,1)$. Then we have that $\overline{\mathbf{C}}$ is a Banach space with the norm $\|f\|_{\overline{C}} = \sup_{\alpha\in[0,1]} \|f(\alpha)\|_C$, and the following embedding theorem holds.

Proposition 1. *(Li et al., 2002, Theorem 6.1.2)* There exists a function $j : \mathbb{F}_{kc} \to \overline{\mathbf{C}}$ such that:

1. j is an isometric mapping, i.e.

$$\delta_H^\infty(\nu^1,\nu^2) = \|j(\nu^1) - j(\nu^2)\|_{\overline{C}}, \quad \nu^1,\nu^2 \in \mathbb{F}_{kc},$$

2. $j(r\nu^1 \oplus t\nu^2) = rj(\nu^1) + tj(\nu^2)$, $\nu^1,\nu^2 \in \mathbb{F}_{kc}$ and $r,t \geq 0$.
3. $j(\mathbb{F}_{kc})$ is a closed subset in $\overline{\mathbf{C}}$.

As a matter of fact, we can define an injection $j : \mathbb{F}_{kc} \to \overline{\mathbf{C}}$ that satisfies above proposition by $j(\nu) = h_\nu$, i.e. $j(\nu)(x,\alpha) = h_\nu(x,\alpha)$ for every $(x,\alpha) \in S^{d-1} \times [0,1]$, where $h_\nu : S^{d-1} \times [0,1] \to \mathbb{R}$ is the *support function* associated to $\nu \in \mathbb{F}_{kc}$ defined by

$$h_\nu(x,\alpha) = \begin{cases} h_{\nu_\alpha}(x) & \text{if } \alpha > 0, \\ h_{\nu_{0+}}(x) & \text{if } \alpha = 0, \end{cases} \qquad \text{for} \qquad (x,\alpha) \in S^{d-1} \times [0,1],$$

and where $h_K(x) = \sup\{\langle x, a\rangle : a \in K\}$, for any $x \in S^{d-1}$ and $K \in \mathbb{K}_{kc}$.
From Proposition 1 it follows that every FRV X can be regarded as a random element of $\overline{\mathbf{C}}$

by considering $j(X) = h_X : (\Omega, \mathfrak{F}) \to (\overline{\mathbf{C}}, \mathcal{B}_{\overline{\mathbf{C}}})$ with $\omega \mapsto j(X(\omega)) = (h_X)(\omega) = h_{X(\omega)}$, where $j(X(\cdot))$ is a measurable map with respect to $\mathcal{B}_{\overline{\mathbf{C}}}$ (the Borel σ–algebra on $\overline{\mathbf{C}}$ generated by $\|\cdot\|_{\overline{C}}$) since it is the composition of the $\mathcal{B}_{\delta_H^\infty}$–measurable map $X(\cdot)$ with the continuous one $j(\cdot)$. Moreover, it is known (cf. Li et al., 2002, Lemma 6.1.6) that whenever $X \in L^1[\Omega; \mathbb{F}_{kc}]$, then $h_{X(\cdot)}(x,\alpha) \in L^1[\Omega;\mathbb{R}]$ and $\mathbb{E}[h_X(x,\alpha)] = h_{\mathbb{E}[X]}(x,\alpha)$, for any $(x,\alpha) \in \mathbb{R}^d \times [0,1]$.
We stress out that such a representation of a FRV X, by means of its support function $j(X)$, is a standard technic used widely in the (fuzzy) set–valued random variable framework (e.g. Li et al., 2002) and it is extensively used in this paper as well.

Fuzzy set–valued Brownian motion. A FRV $X : \Omega \to \mathbb{F}_{kc}$ is *Gaussian* if h_X is a Gaussian random element of $\overline{\mathbf{C}}$ (cf. Li et al., 2002, Definition 6.1.5). A random element h_X taking values in $\overline{\mathbf{C}}$ is Gaussian if and only if, for any $n \in \mathbb{N}$ and $f_1, f_2, \ldots, f_n \in \overline{\mathbf{C}}^*$, the real vector–valued random variable $(f_1(h_X), f_2(h_X), \ldots, f_n(h_X))$ is Gaussian, where $\overline{\mathbf{C}}^*$ is the topological dual of $\overline{\mathbf{C}}$ (i.e. the set of all continuous linear functionals on $\overline{\mathbf{C}}$).
It follows from the properties of h_X and elements in $\overline{\mathbf{C}}^*$ that $X \oplus Y$ is Gaussian if X and Y are Gaussian FRV. Also λX is Gaussian whenever X is Gaussian and $\lambda \in \mathbb{R}$.

Proposition 2. *(Li et al., 2002, Theorem 6.1.7)* A FRV X is Gaussian if and only if X is representable in the form

$$X = \mathbb{E}[X] \oplus \mathbb{I}_\xi,$$

where ξ is a Gaussian random element of $\mathbb{R}^d$ with zero mean.

Assume that $\{\mathfrak{F}_t : t \geq 0\}$ is a σ–filtration satisfying the usual condition (complete and right continuous). $\{X_t : t \geq 0\}$ is called an *adaptive fuzzy set–valued stochastic process* if for any $t \in \mathbb{R}_+$, X_t is an $\mathfrak{F}_t$–measurable FRV. An adaptive fuzzy set–valued stochastic process $\{X_t : t \geq 0\}$ is called *Gaussian* if, for any $t \in \mathbb{R}_+$, X_t is Gaussian. Thus, an adaptive fuzzy set–valued stochastic process $X = \{X_t : t \geq 0\}$ is Gaussian if and only if $\{(f_1(h_{X_t}), \ldots, f_n(h_{X_t})) : t \geq 0\}$ is a real vector–valued Gaussian process, for any $n \in \mathbb{N}$ and $f_1, f_2, \ldots, f_n \in \overline{\mathbf{C}}^*$.

Definition 3. An adaptive fuzzy set–valued stochastic process $\{B_t : t \in \mathbb{R}_+\}$ is called a fuzzy set–valued Brownian motion if and only if $\{h_{B_t} : t \in \mathbb{R}_+\}$ is a Brownian motion in $\overline{\mathbf{C}}$.

Proposition 4. Assume that a fuzzy set–valued stochastic process $\{B_t : t \geq 0\}$ satisfies $B_0 = \mathbb{I}_0$. Then $\{B_t : t \geq 0\}$ is a fuzzy set–valued Brownian motion if and only if it is a Gaussian process and, for any $s, t \geq 0$ and $f_1, \ldots, f_n \in \overline{\mathbf{C}}^*$ and $i, j \in \{1, \ldots, n\}$ with $i \neq j$,

1. $\mathbb{E}[f_i(h_{B_t})] = 0$,
2. $\mathbb{E}[f_i(h_{B_t}) f_i(h_{B_s})] = t \wedge s$,
3. $\mathbb{E}[f_i(h_{B_t}) f_j(h_{B_s})] = 0$.

The following results provide properties of a fuzzy Brownian motion similar to those of the real case.

Proposition 5. *(Li and Guan, 2007, Theorem 4.3 and Theorem 4.4)* Let $\{B_t\}_{t\geq 0}$ be a fuzzy set–valued Brownian motion. The following hold.

1. $\{B_{t+t_0}\}_{t\geq 0}$ is a fuzzy set–valued Brownian motion for any $t_0 \geq 0$.
2. $\{\nu \oplus B_t\}_{t\geq 0}$ is a fuzzy set-valued Brownian motion for any fuzzy set $\nu \in \mathbb{F}_k$.
3. $\{\frac{1}{\sqrt{\lambda}} B_{\lambda t}\}_{t\geq 0}$ is a fuzzy set-valued Brownian motion for any $\lambda > 0$.
4. $\{tB_{\frac{1}{\sqrt{t}}}\}_{t\geq 0}$ is a fuzzy set-valued Brownian motion.
5. If $\mathfrak{F}_t = \sigma\{B_s : s \leq t\}$, then $\{B_t, \mathfrak{F}_t\}_{t\geq 0}$ is a fuzzy set–valued martingale.

3. A FRV Brownian motion is a Wiener process in $\mathbb{R}^d$

This section is devoted to prove Theorem 6: the main result of this paper.

Theorem 6. A fuzzy set–valued process $\{B_t : t \geq 0\}$ is a Brownian motion, if and only if, for each $t \geq 0$,

$$B_t = \mathbb{I}_{b_t}, \qquad \mu\text{–a.e.}$$

where $\{b_t : t \geq 0\}$ is a Wiener process in $\mathbb{R}^d$.

According to Definition 3 a fuzzy set–valued Brownian motion B_t is a process taking values in $\mathbb{F}$ (that is a functional space over $\mathbb{R}^d$). On the other hand, the previous result provides a way to handle a fuzzy set–valued Brownian motion simply using a random vector of $\mathbb{R}^d$. In other words, we observe a "complexity reduction"(from $\mathbb{F}$ to $\mathbb{R}^d$) or a "defuzzification of randomness".
In view of Theorem 6, Property 2 in Proposition 5 is not completely true. More precisely, it claims that, for any $\nu \in \mathbb{F}_{kc}$, $\{\nu \oplus B_t\}_{t\geq 0}$ is a fuzzy set-valued Brownian motion as well as $\{B_t\}_{t\geq 0}$. On the other hand, by Theorem 6, let $\{b_t\}_{t\geq 0}$ and $\{b'_t\}_{t\geq 0}$ be the Wiener processes associated to $\{B_t\}_{t\geq 0}$ and $\{\nu\oplus B_t\}_{t\geq 0}$ respectively. For any $t \geq 0$, it holds $\mathbb{I}_{b'_t} = \nu\oplus B_t = \nu\oplus\mathbb{I}_{b_t}$ μ–a.e.. Passing to the support function and computing expectation, we get, for $t \geq 0$,

$$h_{\mathbb{I}_{b'_t - b_t}} = h_\nu \qquad \mu\text{–a.e.,} \qquad \text{and} \qquad h_{\mathbb{I}_{\mathbb{E}[b'_t - b_t]}} = h_{\mathbb{I}_0} = h_\nu,$$

that is $\nu = \mathbb{I}_0$. As a consequence, Property 2 in Proposition 5 must be rewrite as: $\{\nu\oplus B_t\}_{t\geq 0}$ *is a fuzzy set-valued Brownian motion if and only if* $\nu = \mathbb{I}_0$.
Actually the "complexity reduction" stated in Theorem 6 is strictly related to the characterization of Gaussian FRV (cf. Proposition 2), to Property 1 of Proposition 4, and to Corollary 8.

Proposition 7. Let X be a random compact convex set with $\|X\|_H \in L^1[\Omega;\mathbb{R}]$ and let $a \in \mathbb{R}^d$. $\int_\Omega X d\mu = \{a\}$ if and only if there exists a $x \in L^1[\Omega;\mathbb{R}^d]$ such that $X = \{x\}$ μ–a.e. and $\int_\Omega x d\mu = a$.

Proof. See (Molchanov, 2005, Proposition 1.30, p.161) or Appendix A. ■

Corollary 8. Let X be in $L^1[\Omega;\mathbb{F}_{kc}]$ and let $a \in \mathbb{R}^d$. $\mathbb{E}X = \mathbb{I}_a$ if and only if there exists a $x \in L^1[\Omega;\mathbb{R}^d]$ such that $X = \mathbb{I}_x$ μ–a.e. and $\int_\Omega x d\mu = a$.

Proof. By definition of expectation, $\mathbb{E}X = \mathbb{I}_a$ is equivalent to $\mathbb{E}[X_\alpha] = (\mathbb{E}[X])_\alpha = \{a\}$, for each $\alpha \in (0,1]$, that, by Proposition 7, is equivalent to say that, for each $\alpha \in (0,1]$, X_α is μ–a.e. a random singleton (in general depending on α) with mean value a; i.e. $X_\alpha = \{x_\alpha\}$ μ–a.e. with x_α being a random element of $\mathbb{R}^d$ and with $\mathbb{E}[x_\alpha] = a$. By definition of α–level sets for fuzzy set, $\{x_\alpha\} = X_\alpha \supseteq X_\beta = \{x_\beta\}$ for any $0 \leq \alpha \leq \beta \leq 1$; i.e. $x_\alpha = x_\beta$ for any $\alpha, \beta \in (0,1]$ and hence, setting $x = x_1$, $X = \mathbb{I}_x$ μ–a.e.. ■

Lemma 9. For each $(x,\alpha) \in \mathbb{R}^d \times [0,1]$, the following map belongs to $\overline{\mathbf{C}}^*$

$$\begin{array}{rccl} \varphi_{x,\alpha}: & \overline{\mathbf{C}} & \to & \mathbb{R} \\ & s & \mapsto & \varphi_{x,\alpha}(s) = s(x,\alpha). \end{array}$$

Proof. It is easy to check that $\varphi_{x,\alpha}$ is linear. By linearity, it is sufficient to prove the continuity at $0 \in \overline{\mathbf{C}}$: for each $\varepsilon > 0$ and $h \in \overline{\mathbf{C}}$ such that $\|h\|_{\overline{C}} < \varepsilon$,

$$|\varphi_{x,\alpha}(h) - \varphi_{x,\alpha}(0)| = |\varphi_{x,\alpha}(h)| = |h(\alpha,x)| \leq \|h\|_{\overline{C}} < \varepsilon.$$

■

Proof of Theorem 6. Consider the "if"part. Let $\{b_t\}_{t\geq 0}$ be a Wiener process in $\mathbb{R}^d$ and $\{B_t\}_{t\geq 0}$ be defined by $B_t := \mathbb{I}_{b_t}$ for any $t \geq 0$. Since $\mathbb{I}_{B_t} = \mathbb{I}_0 \oplus \mathbb{I}_{b_t}$ for $t \geq 0$, by Proposition 2, B_t is a Gaussian FRV for any $t \geq 0$ and hence $\{B_t\}_{t\geq 0}$ is a Gaussian process on $\mathbb{F}_{kc}$ as well as $\{h_{B_t}\}_{t\geq 0}$ is a Gaussian process on $\overline{\mathbf{C}}$. By Definition 3, it remains to prove that $\{h_{B_t}\}_{t\geq 0}$ has stationary independent zero mean increments. In particular, let $t_1 \geq s_1 \geq t_2 \geq s_2 \geq 0$, then $h_{B_{t_1}} - h_{B_{s_1}} = h_{\mathbb{I}_{b_{t_1}}} - h_{\mathbb{I}_{b_{s_1}}} = h_{\mathbb{I}_{b_{t_1-s_1}}}$ is clearly independent of $h_{B_{t_2}} - h_{B_{s_2}} = h_{\mathbb{I}_{b_{t_2-s_2}}}$, and if $h \geq 0$, then $h_{B_{t_1+h}} - h_{B_{t_1}}$ has the same probability law of $h_{B_{t_2+h}} - h_{B_{t_2}}$. Finally, let us consider

$$\mathbb{E}[h_{B_{t_1}} - h_{B_{s_1}}] = h_{\mathbb{I}_{\mathbb{E}(b_{t_1-s_1})}} = h_{\mathbb{I}_0} \equiv 0$$

that completes the proof of the "if"part.

In order to prove the "only if"part let us consider the fuzzy set–valued Brownian motion $\{B_t : t \geq 0\}$. According to Proposition 4 and Proposition 2, for any $t \geq 0$ and $f \in \overline{\mathbf{C}}^*$, it satisfies

$$0 = \mathbb{E}[f(h_{B_t})] = \mathbb{E}[f(h_{\mathbb{E}[B_t]\oplus\mathbb{I}_{\xi_t}})].$$

where ξ_t is an Gaussian random element of $\mathbb{R}^d$ with $\mathbb{E}\xi_t = 0$. By the fact that, for any $\nu^1, \nu^2 \in \mathbb{F}_{kc}$, $h_{\nu^1\oplus\nu^2} = h_{\nu^1} + h_{\nu^2}$ (cf. Proposition 1), using the linearity of the expectation and of f, we get

$$\begin{aligned} 0 &= \mathbb{E}[f(h_{\mathbb{E}[B_t]})] + \mathbb{E}[f(h_{\mathbb{I}_{\xi_t}})] = f(h_{\mathbb{E}[B_t]}) + f(\mathbb{E}[h_{\mathbb{I}_{\xi_t}}]) \\ &= f(h_{\mathbb{E}[B_t]}) + f(h_{\mathbb{I}_{\mathbb{E}[\xi_t]}}) = f(h_{\mathbb{E}[B_t]}), \end{aligned} \tag{2}$$

for any $t \geq 0$ and $f \in \overline{\mathbf{C}}^*$, where for the last two equalities we use $h_{\mathbb{E}X} = \mathbb{E}h_X$ and the fact that ξ_t is zero mean. By the arbitrariness of $f \in \overline{\mathbf{C}}^*$ in (2), $h_{\mathbb{E}[B_t]} \equiv 0$. More precisely, if this was not the case, there would exist $(\alpha, x) \in [0,1] \times \mathbb{R}^d$ such that $h_{\mathbb{E}[B_t]}(\alpha, x) \neq 0$ and, one could consider the map $\varphi_{x,\alpha}$, defined by $\varphi_{x,\alpha}(s) = s(x, \alpha)$; it is an element of $\overline{\mathbf{C}}^*$ (cf. Lemma 9) with $\varphi_{x,\alpha}(h_{\mathbb{E}[B_t]}) \neq 0$ that would contradict Equation (2). As a consequence, $\mathbb{E}[B_t] = \mathbb{I}_0$ for each $t \geq 0$ and, by Proposition 2 or by Corollary 8, $B_t = \mathbb{I}_0 \oplus \mathbb{I}_{\xi_t} = \mathbb{I}_{\xi_t}$ μ–a.e. for each $t \geq 0$ with ξ_t being a Gaussian zero mean random element of $\mathbb{R}^d$. Moreover, $\{\xi_t\}_{t\geq 0}$ is a process with stationary independent zero mean increments, because $\{B_t\}_{t\geq 0}$ and $\{h_{B_t}\}_{t\geq 0}$ are so, and hence $\{\xi_t\}_{t\geq 0}$ is a Brownian motion in $\mathbb{R}^d$. The thesis is complete considering the process $\{b_t\}_{t\geq 0}$ defined by $b_t = \xi_t$ for all $t \geq 0$. ■

Note that Proof of Theorem 6 only uses the fact that $\{B_t\}_{t\geq 0}$ is a Gaussian process for which any finite distribution, at any time t, has null expectation.

We want to point out that, although one can associate a fuzzy set–valued Brownian motion at any Brownian motion in $\overline{\mathbf{C}}$ (using embedding j in Proposition 1), in general, the contrary is not possible. In particular, $(\mathbb{F}_{kc}, \oplus, \cdot)$ can be viewed, by means of j, as a proper subset and subcone of the vector space $(\overline{\mathbf{C}}, +, \cdot)$. As a consequence, while a Gaussian element in $\overline{\mathbf{C}}$ can also take "negative" values (with respect to $+$), this could not happen in $\mathbb{F}_{kc}$ because the fuzzy sets that admit inverse elements with respect to $\oplus$ have the form $\mathbb{I}_a$. Roughly speaking, embedding j could not carry back all the possible "fluctuations" of gaussian element, and the only Gaussian elements that could pull back by j are actually defuzzificated.

Appendix A. Proof of Proposition 7

In (Molchanov, 2005, Proposition 1.30, p.161) the author proposed a proof of Proposition 7, in the case of random closed sets, that involves the support function. Here we propose a different approach, via random sets selections, which simply relies on Hahn–Banach Theorem, and that leads to the same result (with compactness and convexity hypothesis that could be dropped).

For the sake of generality, here we consider $\mathfrak{X}$ to be a separable Banach space with $\mathcal{B}_{\mathfrak{X}}$ its borel σ–algebra, and $(\Omega, \mathfrak{F})$ to be a measurable space endowed with a positive finite measure μ. Till now $\mathfrak{X}$ was $\mathbb{R}^d$ and μ was a probability measure.
We need the following two lemmas. Roughly speaking, the former says that any non–null vector in $\mathfrak{X}$ can be separated from zero using a suitable countable family of elements of $\mathfrak{X}^*$ (it is a classical expression of the hyperplane separation theorem implied by the geometric Hahn–Banach Theorem (cf. Lax, 2002, Theorem 5, p.23)). The second lemma says that, for any couple of different (on some set of positive measure) integrable random elements in $\mathfrak{X}$, there exists an element of $\mathfrak{X}^*$ that separates (on a set of positive measure) these two random elements of $\mathfrak{X}$.

Lemma 10. There exists $\{\phi_n\}_{n\in\mathbb{N}} \subset \mathfrak{X}^*$ such that whenever $x \in \mathfrak{X} \setminus \{0\}$ there exists $n \in \mathbb{N}$ for which $\phi_n(x) \neq 0$.

Lemma 11. Let $x_1, x_2 \in L^1[\Omega; \mathfrak{X}]$ and $A = \{\omega \in \Omega : x_1(\omega) \neq x_2(\omega)\}$ with $\mu(A) > 0$. Then there exists $\varphi \in \mathfrak{X}^*$ such that

$$A_\varphi = \{\omega \in \Omega : \varphi[x_1(\omega)] > \varphi[x_2(\omega)]\}$$

has positive measure (i.e. $\mu(A_\varphi) > 0$).

Proof. Let $x = (x_1 - x_2)$ then $A = \{\omega \in \Omega : x(\omega) \neq 0\}$ and let $\{\phi_n\}_{n\in\mathbb{N}} \subset \mathfrak{X}^*$ as in Lemma 10. We claim that there exists $n \in \mathbb{N}$ such that $\mu(A_{\phi_n}) + \mu(A_{-\phi_n}) > 0$. Otherwise, if $A_n = A_{\phi_n} \cup A_{-\phi_n}$, we have

$$\mu(A_n) \leq \mu(A_{\phi_n}) + \mu(A_{-\phi_n}) = 0, \qquad \forall n \in \mathbb{N}.$$

Now we prove that $A \subseteq \bigcup_{n\in\mathbb{N}} A_n$: let $\omega \in A$ then $x(\omega) \neq 0$ and, by Lemma 10, there exists $n \in \mathbb{N}$ such that $\phi_n(x(\omega)) \neq 0$. Hence $\phi_n(x(\omega)) > 0$ or $\phi_n(x(\omega)) < 0$ i.e. $\omega \in A_n$ and thus $A \subseteq \bigcup_{n\in\mathbb{N}} A_n$.
This means that $\mu(A) \leq \mu(\bigcup_{n\in\mathbb{N}} A_n) = 0$ that contradicts hypothesis ($\mu(A) > 0$) and concludes the proof. ■

Proof of Proposition 7. The "if" part is trivial. Conversely, let S_X be the family of integrable selections of X, that is $S_X = \{x \in L^1[\Omega; \mathfrak{X}] : x \in X \ \mu - \text{a.e.}\}$, and let us suppose that $\int_\Omega x d\mu = a$ holds for all $x \in S_X$, where integral is in the Bochner sense. Let us recall that a Bochner integrable map is also Pettis integrable and by definition (e.g. Talagrand, 1984) we have

$$\int_\Omega \phi(x) d\mu = \phi(a), \qquad \forall \phi \in \mathfrak{X}^*, \ \forall x \in S_X. \tag{A.1}$$

Now, let us suppose that x_1, x_2 are distinct elements of S_X i.e. $A = \{\omega \in \Omega : x_1(\omega) \neq x_2(\omega)\}$ has positive measure. Then, by Lemma 11, there exists $\varphi \in \mathfrak{X}^*$ such that $A_\varphi = \{\omega \in \Omega : \varphi[x_1(\omega)] > \varphi[x_2(\omega)]\}$ has positive measure. Let us consider $x_\varphi = \mathbb{I}_{A_\varphi} x_1 + \mathbb{I}_{A_\varphi^C} x_2$. Clearly x_φ is a selection of X (i.e. $x_\varphi \in S_X$), and

$$\begin{aligned}\int_\Omega \varphi(x_\varphi) d\mu &= \int_{A_\varphi} \varphi(x_1) d\mu + \int_{A_\varphi^C} \varphi(x_2) d\mu \\ &> \int_{A_\varphi} \varphi(x_2) d\mu + \int_{A_\varphi^C} \varphi(x_2) d\mu = \varphi(a)\end{aligned}$$

which contradicts Pettis integrability (A.1). ■

Acknowledgements

The author would like to thank Prof. G. Aletti and Prof. V. Capasso for the helpful discussions and encouragements during the preparation of this paper. The author is also very grateful to the referee for the valuable comments that helped to improve the paper.

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