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WEAKLY Z-SYMMETRIC MANIFOLDS

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Abstract. We introduce a new kind of Riemannian manifold that includes weakly-, pseudo- and pseudo projective- Ricci symmetric manifolds. The manifold is defined through a generalization of the so called Z tensor; it is named *weakly Z-symmetric* and denoted by $(WZS)_n$. If the Z tensor is singular we give conditions for the existence of a proper concircular vector. For non singular Z tensor, we study the closedness property of the associated covectors and give sufficient conditions for the existence of a proper concircular vector in the conformally harmonic case, and the general form of the Ricci tensor. For conformally flat $(WZS)_n$ manifolds, we derive the local form of the metric tensor.
Date: May 2011.

1. Introduction

In 1993 Tamassy and Binh [31] introduced and studied a Riemannian manifold whose Ricci tensor satisfies the equation:

$$\nabla_k R_{jl} = A_k R_{jl} + B_j R_{kl} + D_l R_{kj}. \tag{1}$$

The manifold is called *weakly Ricci symmetric* and denoted by $(WRS)_n$. The Ricci tensor and the scalar curvature are $R_{kl} = -R_{mkl}{}^m$ and $R = g^{ij} R_{ij}$. ∇_k is the covariant derivative with reference to the metric g_{kl}. We also put $\|\eta\| = \sqrt{\eta^k \eta_k}$. The covectors A_k, B_k and D_k are the *associated 1-forms*. The same manifold with the 1-form A_k replaced by $2A_k$ was studied by Chaki and Koley [6], and called *generalized pseudo Ricci symmetric*. The two structures extend *pseudo Ricci symmetric* manifolds, $(PRS)_n$, introduced by Chaki [4], where $\nabla_k R_{jl} = 2A_k R_{jl} + A_j R_{kl} + A_l R_{kj}$ (this definition differs from that of R. Deszcz [17]).
Later on, other authors studied the manifolds [10, 20, 12]; in [12] some global properties of $(WRS)_n$ were obtained, and the form of the Ricci tensor was found. In [10] generalized pseudo Ricci symmetric manifolds were considered, where the conformal curvature tensor

$$\begin{aligned} C_{jkl}{}^m = R_{jkl}{}^m &+ \frac{1}{n-2}(\delta_j{}^m R_{kl} - \delta_k{}^m R_{jl} + R_j{}^m g_{kl} - R_k{}^m g_{jl}) \\ &- \frac{R}{(n-1)(n-2)}(\delta_j{}^m g_{kl} - \delta_k{}^m g_{jl}) \end{aligned} \tag{2}$$

vanishes (for $n = 3$: $C_{jkl}{}^m = 0$ holds identically, [27]) and the existence of a proper concircular vector was proven. In [20] a quasi conformally flat $(WRS)_n$ was studied, and again the existence of a proper concircular vector was proven.

2010 *Mathematics Subject Classification.* 53B20, 53B21.
Key words and phrases. Weakly-Ricci Symmetric manifolds, Pseudo- Projective Ricci symmetric, conformal curvature tensor, quasi conformal curvature tensor, conformally symmetric, conformally recurrent, Riemannian manifolds. Weakly Z symmetric Manifolds.

In [2] $(PRS)_n$ with harmonic curvature tensor (i.e. $\nabla_m R_{jkl}{}^m = 0$) or with harmonic conformal curvature tensor (i.e. $\nabla_m C_{jkl}{}^m = 0$) were considered.
Chaki and Saha considered the projective Ricci tensor P_{kl}, obtained by a contraction of the projective curvature tensor $P_{jkl}{}^m$ [18]:

$$P_{kl} = \frac{n}{n-1}\left(R_{kl} - \frac{R}{n} g_{kl}\right), \tag{3}$$

and generalized $(PRS)_n$ to manifolds such that

$$\nabla_k P_{jl} = 2A_k P_{jl} + A_j P_{kl} + A_l P_{kj}. \tag{4}$$

The manifold is called *pseudo projective Ricci symmetric* and denoted by $(PWRS)_n$ [8]. Recently another generalization of a $(PRS)_n$ was considered in [5] and [11], whose Ricci tensor satisfies the condition

$$\nabla_k R_{jl} = (A_k + B_k)R_{jl} + A_j R_{kl} + A_l R_{kj}, \tag{5}$$

The manifold is called *almost pseudo Ricci symmetric* and denoted by $A(PRS)_n$. In ref.[11] the properties of conformally flat $A(PRS)_n$ were studied, pointing out their importance in the theory of General Relativity.

It seems worthwhile to introduce and study a new manifold structure that includes $(WRS)_n$, $(PRS)_n$ and $(PWRS)_n$ as special cases.

Definition 1.1. A (0,2) symmetric tensor is a *generalized Z tensor* if

$$Z_{kl} = R_{kl} + \phi\, g_{kl}, \tag{6}$$

where ϕ is an arbitrary scalar function. The Z scalar is $Z = g^{kl} Z_{kl} = R + n\phi$.

The classical Z tensor is obtained with the choice $\phi = -\frac{1}{n}R$. Hereafter we refer to the generalized Z tensor simply as the Z tensor.
The Z tensor allows us to reinterpret several well known structures on Riemannian manifolds.
1) If $Z_{kl} = 0$ the (Z-flat) manifold is an Einstein space, $R_{ij} = (R/n)g_{ij}$ [3].
2) If $\nabla_i Z_{kl} = \lambda_i Z_{kl}$, the ($Z$-recurrent) manifold is a generalized Ricci recurrent manifold [9, 26]: the condition is equivalent to $\nabla_i R_{kl} = \lambda_i R_{kl} + (n-1)\,\mu_i\, g_{kl}$ where $(n-1)\mu_i \equiv (\lambda_i - \nabla_i)\phi$. If moreover $0 = (\lambda_i - \nabla_i)\phi$, a Ricci Recurrent manifold is recovered.
3) If $\nabla_k Z_{jl} = \nabla_j Z_{kl}$ (i.e. Z is a Codazzi tensor, [16]) then $\nabla_k R_{jl} - \nabla_j R_{kl} = (g_{kl}\nabla_j - g_{jl}\nabla_k)\phi$. By transvecting with g^{jl} we get $\nabla_k[R + 2(n-1)\phi] = 0$ and, finally,

$$\nabla_k R_{jl} - \nabla_j R_{kl} = \frac{1}{2(n-1)}(g_{jl}\nabla_k - g_{kl}\nabla_j)R.$$

This condition defines a *nearly conformally symmetric* manifold, $(NCS)_n$. The condition was introduced and studied by Roter [29]. Conversely a $(NCS)_n$ has a Codazzi Z tensor if $\nabla_k[R + 2(n-1)\phi] = 0$.
4) Einstein's equations [14] with cosmological constant Λ and energy-stress tensor T_{kl} may be written as $Z_{kl} = kT_{kl}$, where $\phi = -\frac{1}{2}R + \Lambda$, and k is the gravitational constant. The Z tensor may be thought of as a generalized Einstein gravitational tensor with arbitrary scalar function ϕ.
Conditions on the energy-momentum tensor determine constraints on the Z tensor: the vacuum solution $Z = 0$ determines an Einstein space with $\Lambda = \frac{n-2}{2n}R$; conservation of total energy-momentum ($\nabla^l T_{kl} = 0$) gives $\nabla^l Z_{kl} = 0$ and $\nabla_k(\frac{1}{2}R + \phi) = 0$;

the condition $\nabla_i Z_{kl} = 0$ describes a space-time with conserved energy-momentum density.

Several cases accomodate in a new kind of Riemannian manifold:

Definition 1.2. A manifold is *Weakly Z-symmetric*, and denoted by $(WZS)_n$, if the generalized Z tensor satisfies the condition:

$$\nabla_k Z_{jl} = A_k Z_{jl} + B_j Z_{kl} + D_l Z_{kj}. \tag{7}$$

If $\phi = 0$ we recover a $(WRS)_n$ and its particular case $(PRS)_n$. If $\phi = -R/n$ (classical Z tensor) and if A_k is replaced by $2A_k$, $B_k = D_k = A_k$, then $Z_{jl} = \frac{n-1}{n} P_{jl}$ and the space reduces to a $(PWRS)_n$.

Different properties follow from the Z tensor being singular or not. Z is singular if the matrix equation $Z_{ij} u^j = 0$ admits (locally) nontrivial solutions, i.e. Z cannot be inverted.

In sect.2 we obtain general properties of $(WZS)_n$ that descend directly from the definition and strongly depend on Z_{ij} being singular or not. The two cases are examined in sections 3 and 4. In sect.3 we study $(WZS)_n$ that are conformally or pseudo conformally harmonic with $B - D \neq 0$; we show that $B - D$, after normalization, is a proper concircular vector. Sect.4 is devoted to $(\mathrm{WZS})_n$ with non-singular Z tensor, and gives conditions for the closedness of the 1-form $A - B$ that involve various generalized curvature tensors. In sect.5 we study conformally harmonic $(\mathrm{WZS})_n$ and obtain the explicit form of the Ricci tensor. In the conformally flat case we also give the local form of the metric.

2. General properties

From the definition of a $(WZS)_n$ and its symmetries we obtain

$$0 = \eta_j Z_{kl} - \eta_l Z_{kj}, \tag{8}$$

$$\nabla_k Z_{jl} - \nabla_j Z_{kl} = \omega_k Z_{jl} - \omega_j Z_{kl}, \tag{9}$$

with covectors

$$\eta = B - D, \qquad \omega = A - B \tag{10}$$

that will be used throughout.
Let's consider eq.(8) first, it implies the following statements:

Proposition 2.1. *In a $(WZS)_n$, if the Z tensor is non-singular then $\eta_k = 0$.*

Proof. If the Z tensor is non singular, there exists a (2,0) tensor Z^{-1} such that $(Z^{-1})^{kh} Z_{kl} = \delta^h{}_l$. By transvecting eq.(8) with $(Z^{-1})^{kh}$ we obtain $\eta_j \delta_l{}^h = \eta_l \delta_j{}^h$; put $h = l$ and sum to obtain $(n-1)\eta_j = 0$. □

Proposition 2.2. *If $\eta_k \neq 0$ and the scalar $Z \neq 0$, then the Z tensor has rank one:*

$$Z_{ij} = Z \, \frac{\eta_i \eta_j}{\eta^k \eta_k} \tag{11}$$

Proof. Multiply eq.(8) by η^j and sum: $\eta^j \eta_j Z_{kl} = \eta_l \eta^j Z_{kj}$. Multiply eq.(8) by g^{jk} and sum: $\eta^k Z_{kl} = Z \eta_l$. The two results imply the assertion. □

The result translates to the Ricci tensor, whose expression is characteristic of *quasi Einstein* Riemannian manifolds [7], and generalizes the results of [12]:

Proposition 2.3. *A $(WZS)_n$ with $\eta_k \neq 0$, is a quasi Einstein manifold:*

$$R_{ij} = -\phi\, g_{ij} + Z\, T_i T_j, \qquad T_i = \frac{\eta_i}{\|\eta\|}, \tag{12}$$

Next we consider eq.(9). If Z_{ij} is a Codazzi tensor, then the l.h.s. of the equation vanishes by definition, and the above discussion of eq.(8) can be repeated. We merely state the result:

Proposition 2.4. *In a $(WZS)_n$ with a Codazzi Z tensor, if Z is singular then $\omega_k \neq 0$. Conversely, if rank $[Z_{kl}] > 1$ then $\omega_k = 0$.*

3. Harmonic conformal or quasi conformal $(\mathrm{WZS})_n$ with $\eta \neq 0$

In this section we consider manifolds $(WZS)_n$ $(n > 3)$ with $\eta_k \neq 0$, and the property $\nabla_m C_{jkl}{}^m = 0$ (i.e. harmonic conformal curvature tensor [3]) or $\nabla_m W_{jkl}{}^m = 0$ (i.e. harmonic quasi conformal curvature tensor [34]). We provide sufficient conditions for $\eta/\|\eta\|$ to be a proper concircular vector [28, 32].

We begin with the case of harmonic conformal tensor. From the expression for the divergence of the conformal tensor,

$$\nabla_m C_{jkl}{}^m = \frac{n-3}{n-2}\left[\nabla_k R_{jl} - \nabla_j R_{kl} + \frac{1}{2(n-1)}(g_{kl}\nabla_j - g_{jl}\nabla_k)R\right] \tag{13}$$

we read the condition $\nabla_m C_{jkl}{}^m = 0$:

$$\nabla_k R_{jl} - \nabla_j R_{kl} = \frac{1}{2(n-1)}(g_{jl}\nabla_k - g_{kl}\nabla_j)R. \tag{14}$$

We need the following theorem, whose proof given here is different from that in [13] (see also [10]):

Theorem 3.1. *Let M be a $n > 3$ dimensional manifold, with harmonic conformal curvature tensor, and Ricci tensor $R_{kl} = \alpha g_{kl} + \beta T_k T_l$, where α, β are scalars, and $T^k T_k = 1$. If*

$$(T_j \nabla_k - T_k \nabla_j)\beta = 0, \tag{15}$$

then T_k is a proper concircular vector.

Proof. Since M is conformally harmonic, eq.(14) gives:

$$\beta[\nabla_k(T_j T_l) - \nabla_j(T_k T_l)] = \frac{1}{2(n-1)}(g_{jl}\nabla_k - g_{kl}\nabla_j)S, \tag{16}$$

where $S = -(n-2)\alpha + \beta$, and condition (15) was used. The proof is in four steps. 1) We show that $T^l\nabla_l T_k = 0$: multiply eq.(16) by g^{jl} to obtain: a) $-\beta\nabla^l(T_k T_l) = \frac{1}{2}\nabla_k S$. The result a) is multiplied by T^k to give: b) $-\beta\nabla_l T^l = \frac{1}{2}T^l\nabla_l S$. a) and b) combine to give: c) $-\beta T^l\nabla_l T_k = \frac{1}{2}[\nabla_k - T_k T^l\nabla_l]S$. Finally multiply eq.(16) by $T^k T^l$ and use the property $T^l\nabla_k T_l = 0$ to obtain:

$$\beta T^k\nabla_k T_j = \frac{1}{2(n-1)}(T_j T^k\nabla_k - \nabla_j)S$$

which, compared to c) shows that d) $T^l\nabla_l T_k = 0$ and $(T_jT^k\nabla_k - \nabla_j)S = 0$.
2) We show that T is a closed 1-form: multiply eq.(16) by T^l

$$\beta[\nabla_k T_j - \nabla_j T_k] = \frac{1}{2(n-1)}(T_j\nabla_k - T_k\nabla_j)S.$$

T is a closed form if the r.h.s. is null. This is proven by using identity a) to write: $(T_j\nabla_k - T_k\nabla_j)S = -2\beta[T_j\nabla^l(T_kT_l) - T_k\nabla^l(T_jT_l)] = 0$ by property d).
3) With condition d) in mind, transvect eq.(16) with T^k and obtain

$$-\beta\nabla_j T_l = \frac{1}{2(n-1)}(g_{jl}T^k\nabla_k - T_l\nabla_j)S$$

Use d) to replace $T_l\nabla_j S$ with $T_lT_jT^k\nabla_k S$. Then:

$$\nabla_j T_l = f\,(T_jT_l - g_{jl}), \quad f \equiv \frac{T^k\nabla_k S}{2\beta(n-1)} \tag{17}$$

which means that T_k is a concircular vector.
4) We prove that T_k is a proper concircular vector, i.e. fT_k is a closed 1-form: from d) by a covariant derivative we obtain $\nabla_j\nabla_k S = (\nabla_j T_k)(T^l\nabla_l S) + T_k\nabla_j(T^l\nabla_l S)$; subtract same equation with indices k and j exchanged. Since T_k is a closed 1-form we obtain: $T_k\nabla_j(T^l\nabla_l S) = T_j\nabla_k(T^l\nabla_l S)$. Multiply by T^k:

$$(T_jT^k\nabla_k - \nabla_j)(T^l\nabla_l S) = 0$$

From the relation (15), one obtains: $(T_kT^l\nabla_l - \nabla_k)\beta = 0$. It follows that the scalar function f has the property $\nabla_j f = \mu T_j$ where μ is a scalar function. Then the 1-form fT_k is closed. □

With the identifications $\alpha = -\phi$ and $\beta = Z$, $T_i = \eta_i/\|\eta\|$ (see Prop. 2.3) the condition (15) is $(\eta_j\nabla_k - \eta_k\nabla_j)Z = 0$. Since $Z = S-(n-2)\phi$ and $(\eta_j\nabla_k - \eta_k\nabla_j)S = 0$, the condition can be rewritten as $(\eta_j\nabla_k - \eta_k\nabla_j)\phi = 0$. Thus we can state the following:

Theorem 3.2. *In a $(WZS)_n$ manifold with $\eta_k \neq 0$ and harmonic conformal curvature tensor, if*

$$(\eta_j\nabla_k - \eta_k\nabla_j)\phi = 0 \tag{18}$$

then $\eta_i/\|\eta\|$ is a proper concircular vector.

Remark 1. *If $\phi = 0$ or $\nabla_k\phi = 0$, the condition (18) is fulfilled automatically. In the case $\phi = 0$ we recover a $(WRS)_n$ manifold (and the results of refs* [10, 12]*).*

Now we consider the case of a $(WZS)_n$ manifold with harmonic quasi conformal curvature tensor. In 1968 Yano and Sawaki [34] defined and studied a tensor $W_{jkl}{}^m$ on a Riemannian manifold of dimension $n > 3$, which includes as particular cases the conformal curvature tensor $C_{jkl}{}^m$, eq.(2), and the concircular curvature tensor

$$\tilde{C}_{jkl}{}^m = R_{jkl}{}^m + \frac{R}{n(n-1)}(\delta_j{}^m g_{kl} - \delta_k{}^m g_{jl}). \tag{19}$$

The tensor is known as the *quasi conformal* curvature tensor:

$$W_{jkl}{}^m = -(n-2)\,b\,C_{jkl}{}^m + [a + (n-2)b]\tilde{C}_{jkl}{}^m; \tag{20}$$

a and b are nonzero constants. From the expressions (13) and (33) we evaluate

$$\nabla_m W_{jkl}{}^m = (a+b)\nabla_m R_{jkl}{}^m + \frac{2a - b(n-1)(n-4)}{2n(n-1)}(g_{kl}\nabla_j - g_{jl}\nabla_k)R. \tag{21}$$

A manifold is *quasi conformally harmonic* if $\nabla_m W_{jkl}{}^m = 0$. By transvecting the condition with g^{jk} we get:

$$(1-2/n)[a+b(n-2)]\,\nabla_j R = 0, \tag{22}$$

which means that either $a+b(n-2)=0$ or $\nabla_j R = 0$. The first condition implies $W=C$, and gives back the harmonic conformal case. If $\nabla_j R=0$ it is $\nabla_m R_{jkl}{}^m = 0$ by (21), and the equations in the proof of theorem 3.1 simplify and we can state the following (analogous to theorem 3.2):

Theorem 3.3. *Let $(WZS)_n$ be a quasi conformally harmonic manifold, with $\eta_k \neq 0$. If $(\eta_j \nabla_k - \eta_k \nabla_j)\phi = 0$, then $\eta/\|\eta\|$ is a proper concircular vector.*

4. $(\mathrm{WZS})_n$ with non-singular Z tensor: conditions for closed ω

In this section we investigate in a $(WZS)_n$ $(n>3)$ the conditions the 1-form ω_k to be closed: $\nabla_i\omega_j - \nabla_j\omega_i = 0$. We need:

Lemma 4.1 (Lovelock's differential identity, [23, 24])**.** *In a Riemannian manifold the following identity is true:*

$$\begin{aligned}&\nabla_i\nabla_m R_{jkl}{}^m + \nabla_j\nabla_m R_{kil}{}^m + \nabla_k\nabla_m R_{ijl}{}^m\\ &\quad = -R_{im}R_{jkl}{}^m - R_{jm}R_{kil}{}^m - R_{km}R_{ijl}{}^m\end{aligned} \tag{23}$$

and also the contracted second Bianchi identity in the form

$$\nabla_m R_{jkl}{}^m = \nabla_k Z_{jl} - \nabla_j Z_{kl} + (g_{kl}\,\nabla_j - g_{jl}\,\nabla_k)\phi. \tag{24}$$

Now we prove the relevant theorem (see also [24]):

Theorem 4.2. *In a $(WZS)_n$ $(n>3)$ with non singular Z tensor, ω_k is a closed 1-form if and only if:*

$$R_{im}R_{jkl}{}^m + R_{jm}R_{kil}{}^m + R_{km}R_{ijl}{}^m = 0. \tag{25}$$

Proof. The covariant derivative of eq.(24) and eq.(9) give: $\nabla_i\nabla_m R_{jkl}{}^m = (\nabla_i\omega_k)Z_{jl} + \omega_k(\nabla_i Z_{jl}) - (\nabla_i\omega_j)Z_{kl} - \omega_j(\nabla_i Z_{kl}) + (g_{kl}\nabla_i\nabla_j\phi - g_{jl}\nabla_i\nabla_k\phi)$. Cyclic permutations of the indices i,j,k are made, and the resulting three equations are added:

$$\begin{aligned}&\nabla_i\nabla_m R_{jkl}{}^m + \nabla_j\nabla_m R_{kil}{}^m + \nabla_k\nabla_m R_{ijl}{}^m\\ &= (\nabla_i\omega_k - \nabla_k\omega_i)Z_{jl} + (\nabla_j\omega_i - \nabla_i\omega_j)Z_{kl} + (\nabla_k\omega_j - \nabla_j\omega_k)Z_{il}\\ &\quad +\omega_j(\nabla_k Z_{il} - \nabla_i Z_{kl}) + \omega_k(\nabla_i Z_{jl} - \nabla_j Z_{il}) + \omega_i(\nabla_j Z_{kl} - \nabla_k Z_{jl}).\end{aligned}$$

Cancellations occur by eq.(9). By lemma 4.1, one obtains:

$$\begin{aligned}&-R_{im}R_{jkl}{}^m - R_{jm}R_{kil}{}^m - R_{km}R_{ijl}{}^m\\ &= (\nabla_i\omega_k - \nabla_k\omega_i)Z_{jl} + (\nabla_j\omega_i - \nabla_i\omega_j)Z_{kl} + (\nabla_k\omega_j - \nabla_j\omega_k)Z_{il}.\end{aligned}$$

If ω_k is a closed 1-form then eq.(25) is fulfilled. Conversely, suppose that eq.(25) holds: if the Z tensor is non singular, there is a $(2,0)$ tensor such that $Z_{kl}(Z^{-1})^{km} = \delta_l{}^m$. Multiply the last equation by $(Z^{-1})^{hl}$: $(\nabla_i\omega_k - \nabla_k\omega_i)\delta_j{}^h + (\nabla_j\omega_i - \nabla_i\omega_j)\delta_k{}^h + (\nabla_k\omega_j - \nabla_j\omega_k)\delta_i{}^h = 0$. Set $h=j$ and sum: $(n-2)(\nabla_i\omega_k - \nabla_k\omega_i) = 0$. Since $n>2$, ω_k is a closed 1-form. □

Remark 2. *By Lovelock's identity, the condition (25) is obviously true if* $\nabla_m R_{ijk}{}^m = 0$*, i.e. the* $(WZS)_n$ *is a harmonic manifold. However, we have shown in ref.*[24] *that there is a broad class of generalized curvature tensors for which the case* $\nabla_m K_{ijk}{}^m = 0$ *implies the same condition. This class includes several well known curvature tensors, and is the main subject of this section.*

Definition 4.3. A tensor $K_{jkl}{}^m$ is a *generalized curvature tensor*[1] if:
1) $K_{jkl}{}^m = -K_{kjl}{}^m$,
2) $K_{jkl}{}^m + K_{klj}{}^m + K_{ljk}{}^m = 0$.

The second Bianchi identity does not hold in general, and is modified by a tensor source $B_{ijkl}{}^m$ that depends on the specific form of the curvature tensor:

$$\nabla_i K_{jkl}{}^m + \nabla_j K_{kil}{}^m + \nabla_k K_{ijl}{}^m = B_{ijkl}{}^m \tag{26}$$

Proposition 4.4 ([24])**.** *If* $K_{jkl}{}^m$ *is a generalized curvature tensor such that*

$$\nabla_m K_{jkl}{}^m = A\nabla_m R_{jkl}{}^m + B(a_{lk}\nabla_j - a_{lj}\nabla_k)\psi, \tag{27}$$

where $A \neq 0$*,* B *are constants,* ψ *is a scalar field, and* a_{ij} *is a symmetric* $(0,2)$ *Codazzi tensor (i.e.* $\nabla_i a_{kl} = \nabla_k a_{il}$*), then the following relation holds:*

$$\begin{aligned} &\nabla_i \nabla_m K_{jkl}{}^m + \nabla_j \nabla_m K_{kil}{}^m + \nabla_k \nabla_m K_{ijl}{}^m \\ &\quad = -A(R_{im} R_{jkl}{}^m + R_{jm} R_{kil}{}^m + R_{km} R_{ijl}{}^m). \end{aligned} \tag{28}$$

Remark 3. *In* [16] *it is proven that any smooth manifold carries a metric such that* (M, g) *admits a non trivial Codazzi tensor (i.e. proportional to the metric tensor) and the deep consequences on the structure of the curvature operator are presented (see also* [25]*).*
Given a Codazzi tensor it is possible to exhibit a K *tensor that satisfies the condition (27):*

$$K_{jkl}{}^m = A\, R_{jkl}{}^m + B\, \psi\, (\delta_j{}^m a_{kl} - \delta_k{}^m a_{jl}). \tag{29}$$

Its trace is: $K_{kl} = -K_{mkl}{}^m = A\, R_{kl} - B(n-1)\psi\, a_{kl}$*. Note that for* $a_{kl} = g_{kl}$ *the tensor* K_{kl} *is up to a factor a* Z *tensor. Thus* Z *tensors arise naturally from the invariance of Lovelock's identity.*

Remark 4. *In the literature one meets generalized curvature tensors whose divergence has the form (27), with trivial Codazzi tensor:*

$$\nabla_m K_{jkl}{}^m = A\, \nabla_m R_{jkl}{}^m + B(g_{kl}\nabla_j - g_{jl}\nabla_k)R. \tag{30}$$

They are the projective curvature tensor $P_{jkl}{}^m$ [18]*, the conformal curvature tensor* $C_{jkl}{}^m$ [27]*, the concircular tensor* $\tilde{C}_{jkl}{}^m$ [28, 32]*, the conharmonic tensor* $N_{jkl}{}^m$ [26, 30] *and the quasi conformal tensor* $W_{jkl}{}^m$ [34].

Definition 4.5. A manifold is K-harmonic if $\nabla_m K_{jkl}{}^m = 0$.

Proposition 4.6. *In a* K*-harmonic manifold, if* K *is of type (30) and* $A \neq 2(n-1)B$*, then* $\nabla_j R = 0$*.*

Proof. By transvecting eq.(30) with g^{kl} and by the second contracted Bianchi identity, we obtain $\frac{1}{2}[A - 2(n-1)B]\nabla_j R = 0$. □

[1] The notion was introduced by Kobayashi and Nomizu [22], but with the further antisymmetry in the last pair of indices.

Hereafter, we specialize to $(WZS)_n$ manifolds with non singular Z tensor, and with a generalized curvature tensor of the type (30). From eqs. (24) and (9) we obtain:

$$\nabla_m K_{jkl}{}^m = A(\omega_k Z_{jl} - \omega_j Z_{kl}) + (g_{kl}\nabla_j - g_{jl}\nabla_k)(A\phi + B\,R). \tag{31}$$

Then, the manifold is K-harmonic if:

$$A(\omega_k Z_{jl} - \omega_j Z_{kl}) = (g_{jl}\nabla_k - g_{kl}\nabla_j)\,(A\phi + B\,R). \tag{32}$$

Lemma 4.7. *In a K-harmonic $(WZS)_n$ with non singular Z tensor:*
1) $\omega_k = 0$ if and only if $\nabla_k(A\phi + BR) = 0$;
2) If $A \neq 2(n-1)B$, then $\omega_k = 0$ if and only if $\nabla_k\phi = 0$.

Proof. If $\nabla_k(A\phi + BR) = 0$ then $\omega_k Z_{jl} = \omega_j Z_{kl}$: if the Z tensor is non singular, by transvecting with $(Z^{-1})^{lh}$ we obtain $\omega_j \delta^h{}_k = \omega_k \delta^h{}_j$. Now put $h = j$ and sum to obtain $(n-1)\omega_k = 0$. On the other hand if $\omega_k = 0$ eq.(32) gives $[g_{jl}\nabla_k - g_{kl}\nabla_j](A\phi + BR) = 0$ and transvecting with g^{kl} we get the result.
If $A \neq 2B(n-1)$ then $\nabla_k R = 0$ and part 1) applies. □

Theorem 4.8. *In a K-harmonic $(WZS)_n$ with non-singular Z tensor and K of type (30), if $\omega \neq 0$ then ω is a closed 1-form.*

This theorem extends theorem 4.2 (where $K = R$), and has interesting corollaries according to the various choices $K = C$, W, P, $\tilde{C}$, N.

Corollary 4.9. *Let $(WZS)_n$ have non singular Z tensor and $\omega \neq 0$. If $\nabla_m K_{jkl}{}^m = 0$, and $K = P$, $\tilde{C}$, N, then ω is a closed 1-form.*

Proof. 1) Harmonic conformal curvature: $\nabla_m C_{jkl}{}^m = 0$. Note that in this case $A = 2B(n-1)$; theorem 4.8 applies.
2) Harmonic quasi conformal curvature: $\nabla_m W_{jkl}{}^m = 0$: Eq.(22) gives either $\nabla_j R = 0$ or $a + b(n-2) = 0$. If $\nabla_j R = 0$ then $\nabla_m R_{jkl}{}^m = 0$ and theorem 4.2. If $a + b(n-2) = 0$ it is $\nabla_m C_{jkl}{}^m = 0$ and case 1) applies.
3) Harmonic projective curvature: $\nabla_m P_{jkl}{}^m = 0$. The components of the projective curvature tensor are [18, 30]:

$$P_{jkl}{}^m = R_{jkl}{}^m + \frac{1}{n-1}(\delta_j{}^m R_{kl} - \delta_k{}^m R_{jl}).$$

One evaluates $\nabla_m P_{jkl}{}^m = \frac{n-2}{n-1}\nabla_m R_{jkl}{}^m$, and theorem 4.2 applies.
4) Harmonic concircular curvature: $\nabla_m \tilde{C}_{jkl}{}^m = 0$. The concircular curvature tensor is given in eq.(19), [28, 32]. Its divergence is

$$\nabla_m \tilde{C}_{jkl}{}^m = \nabla_m R^m_{jkl} + \frac{1}{n(n-1)}(g_{kl}\nabla_j - g_{jl}\nabla_k)R \tag{33}$$

Theorem 4.8 applies.
5) Harmonic conharmonic curvature: $\nabla_m N_{jkl}{}^m = 0$. The conharmonic curvature tensor [26, 30] is:

$$N_{jkl}{}^m = R_{jkl}{}^m + \frac{1}{n-2}(\delta_j{}^m R_{kl} - \delta_k{}^m R_{jl} + R^m_j g_{kl} - R^m_k g_{jl}).$$

A covariant derivative and the second contracted Bianchi identity give:

$$\nabla_m N_{jkl}{}^m = \frac{n-3}{n-2}\nabla_m R_{jkl}{}^m + \frac{1}{2(n-2)}(g_{kl}\nabla_j - g_{jl}\nabla_k)R.$$

Theorems 4.8 applies. □

There are other cases where the 1-form ω_k is closed for a $(WZS)_n$ manifold.

Definition 4.10 ([24, 21])**.** A n-dimensional Riemannian manifold is *K-recurrent*, $(KR)_n$, if the generalized curvature tensor is recurrent, $\nabla_i K_{jkl}{}^m = \lambda_i K_{jkl}{}^m$, for some non zero covector λ_i.

Theorem 4.11 ([24])**.** *In a $(KR)_n$, if λ_i is closed then:*

$$R_{im}R_{jkl}{}^m + R_{jm}R_{kil}{}^m + R_{km}R_{ijl}{}^m = -\frac{1}{A}\nabla_m B_{ijkl}{}^m. \tag{34}$$

where B is the source tensor in eq.(26). In particular, for $K = C$, P, $\tilde{C}$, N, W the tensor $\nabla_m B_{ijkl}{}^m$ either vanishes or is proportional to the l.h.s. of eq.(34).

Corollary 4.12. *Let $(WZS)_n$ have non singular Z tensor, and be K recurrent with closed λ_i. If $K = C$, P, $\tilde{C}$, N, W, then ω is a closed 1-form.*

Definition 4.13. A Riemannian manifold is *pseudosymmetric in the sense of R. Deszcz* [17] if the following condition holds:

$$\begin{aligned}(\nabla_s\nabla_i - \nabla_i\nabla_s)R_{jklm} = L_R\,(&g_{js}R_{iklm} - g_{ji}R_{sklm} + g_{ks}R_{jilm} - g_{ki}R_{jslm}\\ &+g_{ls}R_{jkim} - g_{li}R_{jksm} + g_{ms}R_{jkli} - g_{mi}R_{jkls}),\end{aligned} \tag{35}$$

where L_R is a non null scalar function.

In ref.[24] the following theorem is proven:

Theorem 4.14. *In a Riemannian manifold which is pseudosymmetric in the sense of R. Deszcz, it is $R_{im}R_{jkl}{}^m + R_{jm}R_{kil}{}^m + R_{km}R_{ijl}{}^m = 0$.*

Then we can state the following:

Proposition 4.15. *In a $(WZS)_n$ which is pseudosymmetric in the sense of R. Deszcz, if the Z tensor is non-singular then ω_k is a closed 1-form.*

Definition 4.16. A Riemannian manifold is *generalized Ricci pseudosymmetric in the sense of R. Deszcz*, [15], if the following condition holds:

$$\begin{aligned}(\nabla_s\nabla_i - \nabla_i\nabla_s)R_{jklm} = L_S(&R_{js}R_{iklm} - R_{ji}R_{sklm} + R_{ks}R_{jilm} - R_{ki}R_{jslm} +\\ &+R_{ls}R_{jkim} - R_{li}R_{jksm} + R_{ms}R_{jkli} - R_{mi}R_{jkls}),\end{aligned} \tag{36}$$

where L_S is a non null scalar function.

Theorem 4.17. *In a generalized Ricci pseudosymmetric manifold in the sense of R. Deszcz, it is either $L_S = -\frac{1}{3}$, or $R_{im}R_{jkl}{}^m + R_{jm}R_{kil}{}^m + R_{km}R_{ijl}{}^m = 0$.*

Proof. Equation (36) is transvected with g^{mj} to obtain

$$(\nabla_s\nabla_i - \nabla_i\nabla_s)R_{kl} = L_S[R_{im}(R_{skl}{}^m + R_{slk}{}^m) - R_{sm}(R_{ikl}{}^m + R_{ilk}{}^m)].$$

Then:

$$\begin{aligned}&(\nabla_i\nabla_k - \nabla_k\nabla_i)R_{jl} + (\nabla_j\nabla_i - \nabla_i\nabla_j)R_{kl} + (\nabla_k\nabla_j - \nabla_j\nabla_k)R_{il}\\ &= 3L_S(R_{im}R_{jkl}{}^m + R_{jm}R_{kil}{}^m + R_{km}R_{ijl}{}^m)\end{aligned}$$

By Lovelock's identity (4.1), the l.h.s. of the previous equation is:

$$\begin{aligned}&\nabla_i\nabla_m R_{jkl}{}^m + \nabla_j\nabla_m R_{kil}{}^m + \nabla_k\nabla_m R_{ijl}{}^m\\ &= (\nabla_i\nabla_k - \nabla_k\nabla_i)R_{jl} + (\nabla_j\nabla_i - \nabla_i\nabla_j)R_{kl} + (\nabla_k\nabla_j - \nabla_j\nabla_k)R_{il}\\ &= -R_{im}R_{jkl}{}^m - R_{jm}R_{kil}{}^m - R_{km}R_{ijl}{}^m.\end{aligned}$$

Compare the two results and conclude that either $L_S = -\frac{1}{3}$, or $R_{im}R_{jkl}{}^m + R_{jm}R_{kil}{}^m + R_{km}R_{ijl}{}^m = 0$. □

Finally we state:

Proposition 4.18. *In a $(WZS)_n$ which is also a generalized Ricci pseudosymmetric manifold in the sense of R.Deszcz, if the Z tensor is non-singular and $L_S \neq -\frac{1}{3}$, then ω_k is a closed 1-form.*

5. Conformally harmonic $(\mathrm{WZS})_n$: form of the Ricci tensor

In this section we study conformally harmonic $(WZS)_n$ in depth. We show the existence of a proper concircular vector in such manifolds, and obtain the form of the Ricci tensor. The proof only requires the Z tensor to be non singular. For the conformally flat case, in particular, we give the explicit local form of the metric tensor.

The condition $\nabla_m C_{jkl}{}^m = 0$ is eq.(14) which, by using $R_{ij} = Z_{ij} - g_{ij}\,\phi$ and the property eq.(9), becomes:

$$\omega_k Z_{jl} - \omega_j Z_{kl} = \frac{1}{2(n-1)}(g_{jl}\nabla_k - g_{kl}\nabla_j)[R + 2(n-1)\phi]. \tag{37}$$

This is the starting point for the proofs. By prop 4.7, since Z is non singular, $\omega_k \neq 0$ if and only if $\nabla_k[R + 2(n-1)\phi] \neq 0$.

Remark 5. *1) The condition $\nabla_m C_{jkl}{}^m = 0$ implies that the manifold is a $(NCS)_n$.*
2) If $\nabla_k[R + 2(n-1)\phi] = 0$ the Z tensor is a Codazzi tensor.

The following theorem generalizes a result in [11] for $A(PRS)_n$:

Theorem 5.1. *In a conformally harmonic $(WZS)_n$ the 1-form ω is an eigenvector of the Z tensor.*

Proof. By transvecting eq.(37) with g^{kl} we obtain

$$\omega_j Z - \omega^m Z_{jm} = \frac{1}{2}\nabla_j[R + 2(n-1)\phi]; \tag{38}$$

the result is inserted back in eq.(37),

$$\omega_k Z_{jl} - \omega_j Z_{kl} = \frac{1}{(n-1)}[(\omega_k Z - \omega^m Z_{km})g_{jl} - (\omega_j Z - \omega^m Z_{jm})g_{kl}],$$

and transvected with $\omega^j\omega^l$ to obtain $\omega_k(\omega^j\omega^l Z_{jl}) = (\omega_j\omega^j)\omega^l Z_{kl}$. The last equation can be rewritten as: $Z_{kl}\omega^l = \zeta\omega_k$ □

Now eq.(38) simplifies: $\omega_j(\zeta - Z) = -\frac{1}{2}\nabla_j[R + 2(n-1)\phi]$. The result is a natural generalization of a similar one given in ref.[11] for $A(PRS)_n$.

Theorem 5.2. *Let M be a conformally harmonic $(WZS)_n$. Then:*
1) M is a quasi Einstein manifold;
2) if the Z tensor is non singular and if $(\omega_j\nabla_k - \omega_k\nabla_j)\phi = 0$, then:

$$(\omega_j\nabla_k - \omega_k\nabla_j)\left[\frac{n\zeta - Z}{n-1}\right] = 0, \tag{39}$$

and M admits a proper concircular vector.

Proof. Eq.(37) is transvected with ω^j and theorem 5.1 is used to show that

$$R_{kl} = \left[\frac{Z-\zeta}{n-1} - \phi\right] g_{kl} + \left[\frac{n\zeta - Z}{n-1}\right] \frac{\omega_k\omega_l}{\omega_j\omega^j},$$

i.e. R_{kl} has the structure $\alpha g_{kl} + \beta T_k T_l$ and the manifold is quasi Einstein [7]. By transvecting eq.(24) with g^{jl} we obtain

$$\frac{1}{2}\nabla_k Z + \frac{n-2}{2}\nabla_k\phi = \omega_k Z - \omega^l Z_{kl}.$$

This and theorem (5.1) imply:

$$\frac{1}{2}\nabla_k Z + \frac{n-2}{2}\nabla_k\phi = \omega_k(Z-\zeta). \tag{40}$$

A covariant derivative gives $\frac{1}{2}\nabla_j\nabla_k Z + \frac{n-2}{2}\nabla_j\nabla_k\phi = \nabla_j[\omega_k(Z-\zeta)]$. Subtract the equation with indices k and j exchanged:

$$(Z-\zeta)(\nabla_j\omega_k - \nabla_k\omega_j) + (\omega_k\nabla_j - \omega_j\nabla_k)(Z-\zeta) = 0.$$

According to corollary 4.9, in a conformally harmonic $(WZS)_n$ with non singular Z the 1-form ω_k is closed. Then

$$(\omega_k\nabla_j - \omega_j\nabla_k)(Z-\zeta) = 0 \tag{41}$$

Multiply eq.(40) by ω_j and subtract from it the equation with indices k and j exchanged: $(\omega_j\nabla_k - \omega_k\nabla_j)Z + (n-2)(\omega_j\nabla_k - \omega_k\nabla_j)\phi = 0$. Suppose that ω_k, besides being a closed 1-form, has the property $(\omega_j\nabla_k - \omega_k\nabla_j)\phi = 0$, then one obtains the further equation:

$$(\omega_k\nabla_j - \omega_j\nabla_k)Z = 0. \tag{42}$$

Eqs.(41,42) imply the assertion eq.(39). The existence of a proper concircular vector follows from Theorem 3.1. □

Let us specialize to the case $C_{ijk}{}^m = 0$ (conformally flat $(WZS)_n$).
It is well known [1] that if a conformally flat space admits a proper concircular vector, then the space is subprojective in the sense of Kagan.
From theorem 5.2 we state the following:

Theorem 5.3. *Let $(WZS)_n$ $(n > 3)$ be conformally flat with nonsingular Z tensor and $(\omega_j\nabla_k - \omega_k\nabla_j)\phi = 0$, then the manifold is a subprojective space.*

In [33] K. Yano proved that a necessary and sufficient condition for a Riemannian manifold to admit a concircular vector, is that there is a coordinate system in which the first fundamental form may be written as:

$$ds^2 = (dx^1)^2 + e^{q(x^1)} g^*_{\alpha\beta}(x^2, \dots, x^n) dx^\alpha dx^\beta, \tag{43}$$

where $\alpha, \beta = 2, \dots, n$. Since a conformally flat $(WZS)_n$ with non singular Z tensor admits a proper concircular vector field, this space is the warped product $1 \times e^q M^*$, where (M^*, g^*) is a $(n-1)-$dimensional Riemannian manifold. Gebarosky [19] proved that the warped product $1 \times e^q M^*$ has the metric structure (43) if and only if M^* is Einstein. Thus the following theorem holds:

Theorem 5.4. *Let M be a n dimensional conformally flat $(WZS)_n$ $(n > 3)$. If Z_{kl} is non singular and $(\omega_j\nabla_k - \omega_k\nabla_j)\phi = 0$, then M is the warped product $1 \times e^q M^*$, where M^* is Einstein.*

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