Università degli Studi di Milano

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Scuola di Dottorato in Scienze Matematiche
Corso di Dottorato di Ricerca in Matematica
XXIV Ciclo

Tesi di Dottorato di Ricerca

Rigidity results for Lichnerowicz Bakry–Emery Ricci tensors

MAT/03

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Anno Accademico 2010/2011

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Introduction

The study of rigidity phenomena in Riemannian geometry deals with geometric conditions on a space such as curvature inequalities, volume growth restrictions, spectral assumptions, existence of isometric imbeddings, or, often, a combination of them, which can be satisfied only in the presence of a special geometric structure of the space. This can appear at the level of the metric (e.g. isometric splitting) or of the topology (e.g. structure of the fundamental group). If the space was already equipped with an extra structure (e.g. the choice of a differential form) rigidity can be realized as triviality of this structure.

In this thesis we shall investigate rigidity properties in the more general class of Riemannian manifolds with density. By a weighted manifold (or manifold with density) we mean a triple $(M, \langle\,,\,\rangle, e^{-f}d\mathrm{vol})$, where $(M, \langle\,,\,\rangle)$ is a Riemannian manifold with Riemannian measure $d\mathrm{vol}$, $f : M \to \mathbb{R}$ is a smooth function and $e^{-f}d\mathrm{vol}$ is the weighted measure (absolute continuous with respect to the Riemannian one). The geometry of weighted manifolds is visible in the weighted metric structure, i.e., in the weighted measures of (intrinsic) metric objects (e.g. weighted length of curves and weighted volume of metric balls). Associated to a weighted manifold $(M, \langle\,,\,\rangle, e^{-f}d\mathrm{vol})$ there is also a natural divergence form second order diffusion operator: the f–Laplacian. This is defined on $u \in C^2(M)$ by $\Delta_f u = e^f \mathrm{div}(e^{-f}\nabla u) = \Delta u - \langle \nabla u, \nabla f\rangle$. A natural question that arise in the setting of weighted manifolds is what is the right concept of curvature on these spaces. Actually there is not a canonical choice. Good choices are those that reveal interplays with metric and topological properties of the space, see e.g. [**64**]. We are going to focus our attention on Lichnerowicz Bakry–Emery's Ricci tensors $\mathrm{Ric}_f^k = \mathrm{Ric} + \mathrm{Hess}(f) - \frac{1}{k} df \otimes df$, $k \in (0, +\infty]$, which were first introduced in [**54**], [**3**].

Recently it has been found that these curvature tensors are strictly related with geometric objects whose importance is outstanding in mathematics. Imposing the constancy of Ric_f^k one introduces on the manifold an additional structure which goes under the name of (gradient) Ricci soliton structure ($k = \infty$) or k–quasi–Einstein structure ($k < \infty$). More precisely, given a Riemannian manifold $(M, \langle\,,\,\rangle)$, a Ricci soliton structure on M is the choice of a smooth vector field X (if any) satisfying the soliton equation $\mathrm{Ric} + \frac{1}{2}\mathcal{L}_X \langle\,,\,\rangle = \lambda \langle\,,\,\rangle$, for some $\lambda \in \mathbb{R}$. The Ricci soliton $(M, \langle\,,\,\rangle, X)$ is said to be shrinking, steady or expanding according to whether $\lambda > 0$, $\lambda = 0$ or $\lambda < 0$. In the special case where $X = \nabla f$ for some smooth function

$f : M \to \mathbb{R}$, we have that

$$\mathrm{Ric}_f^\infty = \mathrm{Ric} + \mathrm{Hess}(f) = \lambda \langle\, , \,\rangle\,, \tag{0.1}$$

and we say that $(M, \langle\, , \,\rangle, \nabla f)$ is a gradient Ricci soliton with potential f. On the other hand we say that the Riemannian manifold $(M, \langle\, , \,\rangle)$ supports a k–quasi–Einstein structure, $k \in \mathbb{N}$, if there is some smooth function $f : M \to \mathbb{R}$ such that

$$\mathrm{Ric}_f^k = \mathrm{Ric} + \mathrm{Hess}(f) - \frac{1}{k} df \otimes df = \lambda \langle\, , \,\rangle\,, \tag{0.2}$$

for some $\lambda \in \mathbb{R}$. Clearly, both equations (0.1) and (0.2), can be considered as perturbations of the Einstein equation. When f is constant we will call these structures trivial.

The importance of gradient Ricci solitons is due to Perelman's solution of Poincaré conjecture. They correspond to "self–similar" solutions to Hamilton's Ricci flow and often arise as limits of dilations of singularities developed along the Ricci flow. On the other hand the importance of k–quasi–Einstein manifolds comes from a problem (proposed by A. Besse in [**5**]) on the existence of Einstein manifolds realized as warped products. Indeed in [**17**], following the results in [**51**], it is proved the following characterization.

Theorem 0.1. *(Theorem 2.2 in* [**17**]*) Let $M^m \times_u F^k$ be an Einstein warped product with Einstein constant λ, warping function $u = e^{-\frac{f}{k}}$ and Einstein fibre F^k. Then the weighted manifold $(M^m, g_M, e^{-f}\mathrm{dvol})$ satisfies the quasi–Einstein equation* (0.2)*. Furthermore the Einstein constant μ of the fibre satisfies*

$$\Delta f - |\nabla f|^2 = k\lambda - k\mu e^{\frac{2}{k}f}. \tag{0.3}$$

Conversely if the weighted manifold $(M^m, g_M, e^{-f}\mathrm{dvol})$ satisfies (0.2)*, then f satisfies* (0.3) *for some constant $\mu \in \mathbb{R}$. Consider the warped product $N^{m+k} = M^m \times_u F^k$, with $u = e^{-\frac{f}{k}}$ and Einstein fibre F with ${}^F Ric = \mu g_F$. Then N is Einstein with ${}^N Ric = \lambda g_N$.*

Actually in this thesis we also introduce an extension of the concept of gradient Ricci soliton, the Ricci almost soliton, allowing λ in the soliton equation (0.1) to be a generic smooth function on the weighted manifold $(M, \langle\, , \,\rangle, e^{-f}d\mathrm{vol})$. In view of the fact that the soliton function λ is not necessarily constant, one expects that a certain flexibility on the almost soliton structure is allowed and, consequently, the existence of almost solitons is easier to prove than in the classical situation. This feeling is confirmed by a number of different examples of almost solitons. On the other hand we prove a rigidity result which indicates that almost solitons should reveal a reasonably broad generalization of the fruitful concept of classical soliton.

Our results will follow from considering elliptic equations and inequalities for various geometric quantities and rely on analytic techniques. This is the same philosophy used for instance in [**33**]. More specifically, we will see that the differential (in)equalities at hand naturally involve the f–Laplace operator. We are thus led to introduce a number of weighted manifold tools whose range of application goes beyond the investigation of the geometric objects under study. This point of view is in the spirit of [**94**]. Important istances of these tools are maximum principles at infinity for Δ_f, a–priori estimates for solutions of certain classes of differential inequalities, f–parabolicity and weighted L^p–Liouville type results. These are obtained under both weighted Ricci lower bounds and weighted volume growth conditions.

The results contained in the thesis exemplify different aspects of rigidity of weighted manifolds. For what concerns metric rigidity, we obtain scalar curvature estimates, classification results and gap theorems for some geometric quantities. For instance, the next result shows that under a pointwise control on the soliton function λ, the scalar curvature of an almost soliton is bounded from below. Furthermore the lower bound of the scalar curvature can be estimated both from the above and from below and, applying some abstract structure theorems for domains of nontrivial solutions of the resulting differential equations, also some rigidity at the endpoints occurs. Note that the case $\mu = 0$ contains, of course, the soliton case. Contrary to previous investigations ([**74**]) we do not assume that the scalar curvature is neither constant nor bounded.

Theorem 0.2 (Theorem 3 in [**84**], Theorem 0.4 in [**76**]). *Let $(M, \langle\, ,\, \rangle, \nabla f)$ be a complete gradient Ricci almost soliton with scalar curvature S and soliton function λ such that $\Delta\lambda \leq 0$ on M. Set*

$$S_* = \inf_M S, \quad \lambda_* = \inf_M \lambda, \quad \lambda^* = \sup_M \lambda.$$

(i) *If the almost soliton satisfies $-\infty < \lambda_* \leq \lambda \leq 0$, $\lambda \not\equiv 0$ (and in particular if the almost soliton is expanding), then $m\lambda_* \leq S_* \leq 0$. Moreover, if $m \geq 3$ and there exists x_o such that $S(x_o) = S_* = m\lambda_*$, then the soliton is trivial and M is Einstein; while if $S(x_0) = S_* = 0$ for some $x_0 \in M$, then M is Ricci flat and isometric to $\mathbb{R}^m$.*

(ii) *If the almost soliton is a steady soliton then $S_* = 0$. Morever, if $m \geq 3$ and there exists x_0 such that $S(x_0) = 0$, then M is a cylinder over a totally geodesic hypersurface.*

(iii) *If the almost soliton satisfies $0 \leq \lambda$, $\lambda \not\equiv 0$ (and in particular if the almost soliton is shrinking), then $0 \leq S_* \leq m\lambda^*$. Moreover if $m \geq 3$ and there exists x_o such that $S(x_o) = S_* = 0$ then M is isometric to $\mathbb{R}^m$. Finally if $S_* = m\lambda^*$ and $(M, \langle\, ,\, \rangle, e^{-f}\mathrm{dvol})$ is f-parabolic, then the almost soliton is trivial and $(M, \langle\, ,\, \rangle)$ is compact Einstein. This latter case occurs in particular if*

$$A^2 \left(1 + r\left(x\right)\right)^{-\mu} \leq \lambda\left(x\right)$$

on M for some $A > 0$, $0 \leq \mu \leq 1$.

In the same spirit, we are able to generalize scalar curvature estimates obtained in [**17**] to k–quasi–Einstein manifolds with non–constant scalar curvature, again discussing possible rigidity at the endpoints.

Theorem 0.3. *(Theorem 3 in [**88**]) Let $(M^m, g_M, e^{-f}\mathrm{dvol})$ be a geodesically complete k–quasi–Einstein manifold, $1 < k < +\infty$, with scalar curvature S and let $S_* = \inf_M S$.*

(a) *If* $\lambda > 0$, *then* M *is compact and*

$$\frac{m(m-1)}{m+k-1}\lambda < S_* \leq m\lambda.$$

Moreover $S_* \neq m\lambda$ *unless* M *is Einstein.*

(b) *If* $\lambda = 0$ *and* $\inf_M f = f_* > -\infty$ *then* $S_* = 0$. *Moreover, either* $S > 0$ *or* $S(x) \equiv 0$. *In this latter case, either* f *is constant (and* M *is trivial) or* M *is isometric to the Riemannian product* $\mathbb{R} \times \Sigma$ *where* Σ *is a Ricci–flat, totally geodesic hypersurface.*

(c) *If* $\lambda < 0$ *and* $\inf_M f = f_* > -\infty$, *then*

$$m\lambda \leq S_* \leq \frac{m(m-1)}{m+k-1}\lambda$$

and $S(x) > m\lambda$ *unless* M *is Einstein.*

It is well known from Perelman's original work that compact expanding Ricci solitons are necessarily trivial. In the following theorem we generalize this result to the complete non–compact setting. This is an important instance of how rigidity can appear as triviality of an additional structure supported by the weighted manifold.

Theorem 0.4. *(Theorem 2 in* [**84**]*) A complete, expanding, gradient Ricci soliton* $(M, \langle\, ,\, \rangle, \nabla f)$ *is trivial provided* $|\nabla f| \in L^p\left(M, e^{-f}d\mathrm{vol}\right)$, *for some* $1 \leq p \leq +\infty$.

As a matter of fact, the above statement encloses three different results according to the assumption that $p = +\infty$, $1 < p < +\infty$ and $p = 1$. The L^∞ situation is dealt with using the weak Omori–Yau maximum principle for Δ_f. On the other hand, the $L^{1<p<\infty}$ and the L^1 results will rely on Liouville properties of diffusion operators. Using a refined and generalized version in the weighted setting of the weak Omori–Yau maximum principle and a weighted volume comparison result in presence of a nonnecessarily constant lower bound on Ric_f^∞, we are able also to prove the following triviality theorem for gradient Ricci almost solitons. It permits, in some sense, to extend the case $p = \infty$ of Theorem 0.4.

Theorem 0.5 (Theorem 0.2 in [**76**]). *Let* $(M, \langle\, ,\, \rangle, \nabla f)$ *be a complete, expanding gradient Ricci almost soliton with soliton function* λ. *Let* α, σ, $\mu \in \mathbb{R}$ *be such that*

$$\alpha > -2\,;\ 0 \leq \sigma \leq 2/3$$

$$\min\{0, -\alpha\} \leq \mu \leq \begin{cases} 1 - 3\sigma/2 & \text{if } \ \sigma \geq \alpha \\ 1 - \sigma - \alpha/2 & \text{if } \ \sigma < \alpha \end{cases}$$

Assume

$$\limsup_{r(x)\to+\infty} \frac{|\nabla f|^2}{r(x)^\sigma} \begin{cases} = 0 & \text{if } \ 0 < \sigma \leq 2/3 \\ < +\infty & \text{if } \ \sigma = 0 \end{cases}$$

$$-(m-1)B^2\left(1 + r(x)^2\right)^{\frac{\alpha}{2}} \leq \lambda(x) \leq -(m-1)A^2\left(1 + r(x)^2\right)^{-\frac{\mu}{2}}$$

on M *for some constants* $B \geq A > 0$.
Suppose either $m = 2$ *or*

$$\langle \nabla\lambda, \nabla f\rangle \leq 0 \text{ on } M.$$

Then, the almost soliton is trivial.

Making use again of weighted L^p–Liouville–type theorems, we can also prove triviality results for k–quasi–Einstein manifolds under L^p conditions. For instance, using the scalar curvature estimates of Theorem 0.3, we get the next

Theorem 0.6 (Theorem 5 in [**88**]). *Let* $(M^m, g_M, e^{-f}d\mathrm{vol})$ *be a geodesically complete non–compact* k*–quasi–Einstein manifold,* $1 \leq k < +\infty$*. If the quasi–Einstein constant* λ *is non–positive and* f *satisfies, for some* $1 < p < +\infty$,

$$f \in L^p(M, e^{-\frac{f}{k}}d\mathrm{vol}),$$

and $\inf_M f = f_* > -\infty$*, then either* $f \equiv const \leq 0$ *and* M *is Einstein or* $f > 0$.

We now exemplify how triviality results for quasi–Einstein metrics affect conclusions on the existence of Einstein warped products. By combining Theorem 0.1 with Theorem 0.6, we are able to extend to the case of non–compact bases a recent theorem by D.–S. Kim and Y.–H. Kim, [**51**].

Theorem 0.7 (Theorem 1 in [**88**]). *Let* $N^{m+k} = M^m \times_u F^k$, $k > 1$, *be a complete Einstein warped product with non–positive scalar curvature* ${}^N S \leq 0$, *warping function* $u(x) = e^{-\frac{f(x)}{k}}$ *satisfying* $\inf_M f = f_* > -\infty$ *and complete Einstein fibre* F*. Then* N *is simply a Riemannian product if the base manifold* M *is complete and non–compact, the warping function satisfies* $\int_M |f|^p e^{-\frac{f}{k}}d\mathrm{vol} < +\infty$*, for some* $1 < p < +\infty$*, and* $f(x_0) \leq 0$ *for some point* $x_0 \in M$.

Finally we manage with topological rigidity for weighted manifolds under curvature restrictions. The situation for Ric_f^∞ and for Ric_f^k is quite different. An extension of Myers' theorem to weighted manifolds with a positive lower bound on Ric_f^k, $k < \infty$, has been obtained by Z. Qian in [**86**]. As in the classical (non–weighted) case this result can be generalized à la Galloway ([**39**]), proving compactness also in the case the positive constant lower bound for Ric_f^k is perturbed by the derivative of some bounded function. Following the results obtained in the classical case in [**61**] (which grow around an idea of E. Calabi, [**10**]), we extend Qian's theorem by allowing some negativity for Ric_f^k.

Theorem 0.8. *Let* $\mathrm{Ric}_f^k \geq -(m+k-1)B^2$*, for some constant* $B \geq 0$, $k < +\infty$*. Suppose there is a point* $q \in M$ *such that along each geodesic* $\gamma : [0, +\infty) \to M$ *parameterized by arc–length, with* $\gamma(0) = q$*, it holds either*

$$\int_a^b t\frac{\mathrm{Ric}_f^k(\dot\gamma,\dot\gamma)}{m+k-1}dt > B\left\{b + a\frac{e^{2Ba}+1}{e^{2Ba}-1}\right\} + \frac{1}{4}\log\left(\frac{b}{a}\right).$$

or

$$\int_a^b t^\alpha \frac{\mathrm{Ric}_f^k(\dot\gamma,\dot\gamma)}{m+k-1}dt > B\left\{b^\alpha + a^\alpha \frac{e^{2Ba}+1}{e^{2Ba}-1}\right\} + \frac{\alpha^2}{4(1-\alpha)}\left\{a^{\alpha-1} - b^{\alpha-1}\right\}$$

for some $0 < a < b$ and $\alpha \neq 1$. Then M is compact.

The lack of an appropriate Bochner formula for Ric_f^∞ prevents Myers–type compactness conclusions. Nevertheless, as initially investigated in works of M. Fernández–López and E. García–Río in the compact case, [**35**], and later in the complete non–compact case by W. Wylie, [**96**], there is a close relationship between Ric_f^∞ and the fundamental group of a weighted manifold. Namely, Myers–type results in this context establish the finiteness of the fundamental group if $Ric_f^\infty \geq K > 0$. The next result extends in the direction of the classical Ambrose theorem ([**1**]) topological results obtained in [**96**], [**67**], [**35**], [**33**].

Theorem 0.9. *(Theorem 8.1 in [**76**]) Let $\left(M, \langle\,,\,\rangle\,, e^{-f}\mathrm{dvol}\right)$ be a geodesically complete weighted manifold, and assume that there exists a point $o \in M$ and functions $\mu \geq 0$ and g bounded such that for every unit speed geodesic γ issuing from $\gamma(0) = o$ we have*

$$\mathrm{Ric}_f^\infty(\dot\gamma,\dot\gamma) \geq \mu\circ\gamma + \langle \nabla g\circ\gamma, \dot\gamma\rangle$$

and

$$\int_0^{+\infty} \mu\circ\gamma(t)dt = +\infty.$$

Then, the following hold:

(a) *If the above conditions hold then $|\pi_1(M)| < \infty$.*
(b) *If in addition $\mathrm{Ric} \leq c < +\infty$ and $\mu = \mu_o(r(x))$ is radial, where $r(x) = \mathrm{dist}(x,o)$, then M is diffeomorphic to the interior of a compact manifold N with $\partial N \neq \emptyset$.*
(c) *If $\mu(x) \geq \mu_0 > 0$ and $\sup_M(|\nabla f| + |g|) \leq F < +\infty$, then M is compact and $diam(M) \leq \frac{1}{\mu_0}\left[2F + \sqrt{4F^2 + \pi^2(m-1)c}\right]$.*

CHAPTER 1

Differential equations and domain rigidity

The general strategy for metric rigidity in Riemannian geometry is to encode the geometric data in a family of functions which obey differential equations and then try to apply abstract structure theorems for domains of nontrivial solutions of these equations. The aim of this introductory chapter is to present some of these structure theorems. In particular we will focus on Obata–type second order differential systems, extending classical works on characterization of space–forms ([**69**], [**92**], [**50**]) to general model manifolds. Along the way, we will also discover new characterizations of space–forms.

Having fixed a smooth, even function $G:\mathbb{R}\to\mathbb{R}$, we let M^m_{-G} denote the m–dimensional (not necessarily complete) model manifold with radial sectional curvature $-G(r)$. More precisely, we set

$$M^m_{-G}=\left([0,r_{-G})\times\mathbb{S}^{m-1},dr^2+g(r)^2\,d\theta^2\right),$$

where $g:\mathbb{R}\to\mathbb{R}$ is the unique solution of the problem

$$\begin{cases} g''=Gg\\ g(0)=0\\ g'(0)=1,\end{cases}$$

and $r_{-G}\in(0,+\infty]$ is the first zero of $g(r)$ on $(0,+\infty)$. Obviously, in case $g(r)>0$ for every $r>0$, we are using the convention $r_{-G}=+\infty$. In this case, the model is geodesically complete.

Examples of models come from the standard space–forms.

(i) Let $G(r)\equiv -k<0$. Then $g(r)=k^{-1/2}\sin\left(k^{1/2}r\right)$, $r_k=\pi/k^{1/2}$ and M^m_k is isometric to the standard sphere of constant curvature k punctured at one point. Equivalently, M^m_k is isometric to the geodesic ball $B_{\pi/\sqrt{k}}(o)$ in the standard sphere of constant curvature k.

(ii) Let $G(r)\equiv k>0$. Then $g(r)=k^{-1/2}\sinh\left(k^{1/2}r\right)$ and M^m_{-k} is isometric to the standard hyperbolic space of constant curvature $-k$.

(iii) Let $G(r)\equiv 0$. Then $g(r)=r$ and M^m_0 is isometric to the standard Euclidean space.

Characterizations of space–forms as complete manifolds supporting solutions of second order differential systems of the form

$$\operatorname{Hess}(u)(x)=(au(x)+b)\left\langle\,,\,\right\rangle_x,$$

have been classically investigated by M. Obata, [**69**], Y. Tashiro, [**92**], and M. Kanai, [**50**]. The following theorem encloses in a single statement their results.

Theorem 1.1. *Let $(M^m, \langle\, ,\rangle)$ be a complete, connected m–dimensional Riemannian manifold. Then:*

(i) *A necessary and sufficient condition for M to be isometric to the sphere of constant curvature $k > 0$ is that M supports a smooth, non trivial solution $u : M \to \mathbb{R}$ of the differential system*

$$\operatorname{Hess}(u)(x) = -ku(x)\langle\, ,\rangle . \tag{1.1}$$

(ii) *A necessary and sufficient condition for M to be isometric to the hyperbolic space of constant curvature $-k < 0$ is that M supports a smooth, non trivial solution $u : M \to \mathbb{R}$ of the differential system*

$$\operatorname{Hess}(u)(x) = ku(x)\langle\, ,\rangle , \tag{1.2}$$

with precisely one critical point.

(iii) *A necessary and sufficient condition for M to be isometric to the Euclidean space is that M supports a smooth, non trivial solution $u : M \to \mathbb{R}$ of the differential system*

$$\operatorname{Hess}(u)(x) = h\langle\, ,\rangle , \tag{1.3}$$

for some constant $h \neq 0$.

Since space–forms are very special cases of model manifolds, a natural question is whether a general model manifold M^m_{-G} can be characterized in the same perspective of Theorem 1.1. This chapter aims to answer the question in the affirmative.

Quite naturally, one expects that a characterization of the model M^m_{-G}, in the spirit of Theorem 1.1 above, must involve more general differential systems of the form

$$\operatorname{Hess}(u)(x) = H(r(x))\,u(x)\langle\, ,\rangle , \tag{1.4}$$

where $r(x)$ denotes the geodesic distance from a fixed origin o. First of all, we need to find the right form of the radial coefficient H. Let $u(x) = \alpha(r(x))$ be a radial solution of (1.4). We suppose to have normalized u in such a way that $u(0) = 1$ and we require u to have a critical point at 0. Then, recalling that

$$\operatorname{Hess}(r) = \frac{g'}{g}\{\langle\, ,\rangle - dr \otimes dr\} = gg'd\theta^2, \tag{1.5}$$

we have

$$\operatorname{Hess}(u) = \alpha'' dr \otimes dr + \alpha' gg' d\theta^2$$

On the other hand

$$\operatorname{Hess}(u) = H\alpha\langle\, ,\rangle = H\alpha dr \otimes dr + H\alpha g^2 d\theta^2.$$

Comparing these two equations gives the ordinary differential system

$$\begin{cases} \alpha'' = H\alpha \\ \alpha' gg' = H\alpha g^2, \end{cases}$$

that is,

$$\begin{cases} \alpha'' = \alpha' g'/g \\ H = \alpha' g'/\alpha g, \end{cases}$$

where, we recall, $\alpha(0) = 1$, $\alpha'(0) = 0$. Integrating the first equation gives

$$\alpha(r) = A \int_0^r g(s)\, ds + 1, \tag{1.6}$$

with $A \neq 0$ any constant. Inserting this expression into the second equation we finally deduce

$$H(r) = \frac{A g'(r)}{A \int_0^r g(s)\, ds + 1}.$$

In order that H is defined on all of $[0, r_{-G})$ we need to impose that

$$\inf \left\{ t > 0 : A \int_0^t g(s)\, ds + 1 \leq 0 \right\} \geq r_{-G}.$$

We have thus obtained the following

Lemma 1.2 (Lemma 2.1 in [**83**])**.** *A necessary and sufficient condition for equation (1.4) on M^m_{-G} to possess a radial solution u is that*

$$H(r) = \frac{A g'(r)}{A \int_0^r g(s)\, ds + 1}.$$

for any constant $A \neq 0$ such that

$$\inf \left\{ t > 0 : A \int_0^t g(s)\, ds + 1 \leq 0 \right\} \geq r_{-G}.$$

Note that, in particular,

- On the punctured standard sphere $M^m_1 = \mathbb{S}^m \backslash \{\text{point}\} = B_\pi(0)$, for every $A \in \mathbb{R} \backslash \{0\}$ such that either $A > -1/2$ or $A = -1$, there is a smooth function u_A with exactly one critical point at 0 and satisfying the equation

$$\operatorname{Hess}(u_A)(x) = \frac{A \cos r(x)}{-A \cos r(x) + 1 + A} u_A(x) \langle\, , \rangle. \tag{1.7}$$

  As a matter of fact, the function $u(x) = -A \cos r(x) + 1 + A$ is well defined and solves the equation on all of $\mathbb{S}^m$. Note finally that, in the special case $A = -1$, (1.7) reduces to (1.1).
- On the standard hyperbolic model $M^m_{-1} = \mathbb{H}^m_{-1}$, for every $A > 0$, there exists a smooth function u_A with exactly one critical point at 0 and satisfying the equation

$$\operatorname{Hess}(u_A)(x) = \frac{A \cosh r(x)}{A \cosh r(x) + 1 - A} u_A(x) \langle\, , \rangle. \tag{1.8}$$

  In the special case $A = 1$, (1.8) reduces to (1.2).

- On the standard Euclidean space $M_0^m = \mathbb{R}^m$, for every $A > 0$, there exists a function u_A with exactly one critical point at 0 and satisfying the equation

$$\operatorname{Hess}(u_A)(x) = \frac{2A}{Ar(x)^2 + 2} u_A(x) \langle\, , \,\rangle . \tag{1.9}$$

We shall prove the following result. Recall that a twisted sphere of dimension n is a differentiable manifold N, homeomorphic to the standard sphere $\mathbb{S}^n$, which is obtained by gluing two n–dimensional closed, unit disks $D^n \subset \mathbb{R}^n$ via a boundary diffeomorphism.

Theorem 1.3 (Theorem 2.2 in [**83**]). *Let $(M^m, \langle\, , \,\rangle)$ be a complete m–dimensional Riemannian manifold, and let $o \in M$ be a reference origin. Then, a necessary and sufficient condition for the existence of an isometric imbedding $\Phi : M_{-G}^m \to M$ is that there exists a smooth solution $u : B_{r_{-G}}(o) \to \mathbb{R}$ of the problem*

$$\begin{cases} \operatorname{Hess}(u)(x) = H(r(x))\, u(x) \langle\, , \,\rangle \\ u(o) = 1 \\ |\nabla u|(o) = 0, \end{cases} \tag{1.10}$$

where $r(x) = \operatorname{dist}_{(M, \langle\, , \,\rangle)}(x, o)$, $H : [0, R^*] \to \mathbb{R}$ is the smooth function

$$H(t) = \frac{Ag'(t)}{A \int_0^t g(s)\, ds + 1}, \tag{1.11}$$

for some real number $A \neq 0$, and

$$R^* = \sup \{T > 0 : H(t) \text{ well defined on } [0, T]\} > r_{-G}.$$

Furthermore, if u is a solution of (1.10) on all of M, then the following holds:

(i) *In case $r_{-G} = +\infty$, then M is isometric to the model M_{-G}^m.*
(ii) *In case $r_{-G} < +\infty$ and $H(r_{-G}) \neq 0$, then $\operatorname{cut}(o) = \{O\}$ for some $O \in M$, and $\Phi(M_{-G}^m) = M \setminus \{O\}$. Furthermore, M is diffeomorphically a twisted sphere.*

As a direct consequence of Theorem 1.3 we point out the following result that generalizes, in some directions, Theorem 1.1 above.

Corollary 1.4 (Corollary 2.3 in [**83**]). *Let $(M^m, \langle\, , \,\rangle)$ be a complete Riemannian manifold, $o \in M$ a reference origin and $r(x) = \operatorname{dist}_{(M, \langle\, , \,\rangle)}(x, o)$. Then:*

(i) *M is isometric to the standard sphere $\mathbb{S}^m$ if and only if M supports a real valued function $u \not\equiv 0$ with a critical point at o and satisfying the differential system (1.7), for some $A \neq 0$ such that either $A > -1/2$ or $A = -1$.*
(ii) *M is isometric to the standard hyperbolic space if and only if M supports a real valued function $u \not\equiv 0$ with a critical point at o and satisfying the differential system (1.8) for some $A > 0$.*

(iii) *M is isometric to the standard Euclidean space if and only if M supports a real valued function $u \not\equiv 0$ with a critical point at o and satisfying the differential system (1.9) for some $A > 0$.*

Before proving Theorem 1.3 we make some observations on case (a) of the previous Corollary.

(i) First of all, to deduce that M is a standard sphere one simply observes that, as established in (b) of Theorem 1.3, M is simply connected and $M \backslash \{O\}$ is isometric to a standard punctured sphere. Therefore, by continuity, M itself has positive constant curvature and we can apply the Hopf classification theorem. Alternatively, we can recall that a necessary and sufficient condition for the model metric $dr \otimes dr + g(r)^2 d\theta^2$ of M^m_{-G} to smoothly extend on all of $[0, r_{-G}] \times \mathbb{S}^{m-1}$ is that $g^{(2k)}(r_{-G}) = 0$ and $g'(r_{-G}) = -1$; see [**73**]. In the present situation we have $g(r) = \sin(r)$ and therefore we deduce that the isometry Φ extends to cover the removed point O.

(ii) Comparing with case (i) of Theorem 1.1 we see that, on the one hand, we enlarge the class of differential systems characterizing the sphere but, on the other hand, we make the additional assumption that u has a critical point at o. As first noted by Obata, the existence of a critical point is automatically guaranteed if $H(r) \equiv -k < 0$. To see this, one can argue as follows. By contradiction, suppose u has no critical point at all. Then, the vector field $X = \nabla u / |\nabla u|$ is defined on all of M. Using the differential system $\operatorname{Hess}(u) = -ku \langle \, , \rangle$ it is readily seen that the integral curves $\gamma(t) : \mathbb{R} \to M$ of X are unit speed, but not necessarily minimizing, geodesics. Indeed

$$\begin{aligned} D_{\dot\gamma}\dot\gamma = D_{\dot\gamma} X_\gamma &= |\nabla u|^{-1} Hess(u)\left(\dot\gamma, \cdot\right)^{\#} - |\nabla u|^{-1} Hess(u)\left(\dot\gamma, X\right) X \\ &= -ku\,|\nabla u|^{-1} X + ku\,|\nabla u|^{-1} X \\ &= 0. \end{aligned}$$

Note that the same argument works if u solves the more general equation $\operatorname{Hess}(u) = f \langle \, , \rangle$, for any real–valued function f. Now consider $y(t) = u \circ \gamma(t)$. Then, y satisfies the oscillatory o.d.e.

$$y'' = -ky.$$

Let $t_0 > 0$ be a critical point of y. Since

$$\begin{aligned} 0 = \frac{dy}{dt}(t_0) &= \left\langle \nabla u(\gamma(t_0)), \dot\gamma(t_0) \right\rangle = \left\langle \nabla u(\gamma(t_0)), \frac{\nabla u}{|\nabla u|}(\gamma(t_0)) \right\rangle \\ &= |\nabla u|(\gamma(t_0)), \end{aligned}$$

we have that $\gamma(t_0)$ is a critical point of u. Contradiction. Thus, u has a critical point p and we can always take $p = o$ as the reference origin in our Theorem 1.3.

In case the coefficient H in the differential equation depends on the distance function $r(x)$, if we try to adapt the previous argument to the present situation, we encounter two obvious difficulties.

(a) As observed above, an integral curve $\gamma(t) : \mathbb{R} \to M$ of the vector field X is a geodesic but it can be non–minimizing. Therefore, for large values of $|t|$, $H(r(\gamma(t))) \neq H(t)$. It follows that the reduction procedure of the P.D.E. to an o.d.e., via composition with γ, cannot be carried over for large values of $|t|$.

(b) Even if we were able to prove that u has a critical point at some $p \in M$, since the coefficient H depends on the distance from the reference origin o, we could not take $p = o$.

The rest of the section is entirely devoted to a proof of Theorem 1.3. The necessity part has been already discussed above. Therefore we may concentrate on the sufficiency part.

The following density result due to R. Bishop, [**7**], will play a key role in our argument. For a nice and simplified proof, see F. Wolter, [**95**]. Following Bishop, recall that, given a complete manifold $(M, \langle\,,\,\rangle)$ and a reference point $o \in M$, then $p \in \mathrm{cut}(o)$ is an *ordinary cut point* if there are at least two distinct minimizing geodesics from o to p. Using the infinitesimal Euclidean law of cosines, it is not difficult to show that at an ordinary cut point p the distance function $r(x) = \mathrm{dist}_{(M,\langle\,,\,\rangle)}(x, o)$ is not differentiable, [**95**].

Theorem 1.5. *Let $(M, \langle\,,\,\rangle)$ be a complete Riemannian manifold and let $o \in M$ be a reference point. Then the ordinary cut–points of o are dense in* $\mathrm{cut}(o)$. *In particular, if the distance function $r(x)$ from o is differentiable on the (punctured) open ball $B_R(o) \setminus \{o\}$ then $B_R(o) \cap \mathrm{cut}(o) = \emptyset$.*

Proof (of Theorem 1.3). To simplify the exposition we will proceed by steps.

Step 1. First of all, we note that the function $u : B_{r_{-G}}(o) \to \mathbb{R}$ must be radial and, more precisely, $u(x) = \alpha(r(x))$, where

$$\alpha(t) = A \int_0^t g(s)\, ds + 1.$$

Indeed, fix x and choose a unit speed, minimizing geodesic $\gamma : [0, r(x)] \to B_{r_{-G}}(o)$ from o to x. Then, composing with γ, we deduce that $y(t) = u \circ \gamma(t)$ is the solution of the Cauchy problem

$$\begin{cases} y''(t) = \frac{A g'(t)}{A \int_0^t g(s) ds + 1} y(t) \\ y(0) = 1 \\ y'(0) = \left\langle \nabla u(o), \dot{\gamma}(0) \right\rangle = 0. \end{cases}$$

It follows that

$$y(t) = A \int_0^t g(s)\, ds + 1,$$

and, taking $t = r(x)$, we get

$$u(x) = y(r(x)) = A \int_0^{r(x)} g(s)\, ds + 1.$$

Step 2. The open ball $B_{r_{-G}}(o)$ is inside the cut–locus of o. Indeed, recall that $u(x) = \alpha(r(x))$ and note that α is a diffeomorphism on $(0, r_{-G})$

because $\alpha'(t) = Ag(t) \neq 0$ on that interval. Therefore, $r(x) = \alpha^{-1} \circ u(x)$ is smooth on $B_{r_{-G}}(o) \setminus \{o\}$ as a composition of smooth functions. By Theorem 1.5, it follows that $B_{r_{-G}}(o) \cap \operatorname{cut}(o) = \emptyset$.

Step 3. According to Step 2, we can introduce geodesic polar coordinates on $B_{r_{-G}}(o)$. We claim that the corresponding map

$$\Phi(r,\theta) = \exp_o(r\theta) : M^m_{-G} \approx \mathbf{B}^m_{r_{-G}}(0) \subseteq T_oM \to B_{r_{-G}}(o) \subseteq M$$

is a Riemannian isometry. To see this, let v be the function

$$v(x) = \frac{u(x)-1}{A} = \int_0^{r(x)} g(s)\,ds$$

on $B_{r_{-G}}(o)$ and note that

$$\begin{cases} \operatorname{Hess}(v) = A^{-1}Hu\langle\,,\,\rangle \\ v(o) = 0 \\ |\nabla v|(o) = 0. \end{cases} \tag{1.12}$$

Furthermore,

$$\nabla r = \frac{\nabla v}{|\nabla v|}. \tag{1.13}$$

Using geodesic polar coordinates $(r,\theta) \in (0, r_{-G}) \times \mathbf{S}^{m-1} \approx \mathbf{B}^m_{r_{-G}}(0) \setminus \{0\} \subseteq T_oM$, keeping a local orthonormal frame $\{\theta^\alpha\}$ on $\mathbf{S}^{m-1} \subset T_oM$, and recalling Gauss lemma, we now express $\exp_o^*\langle\,,\,\rangle = dr \otimes dr + \sigma_{\alpha\beta}(r,\theta)\,\theta^\alpha \otimes \theta^\beta$, where $d\theta^2 = \sum \theta^\alpha \otimes \theta^\alpha$ denotes the standard metric on $\mathbf{S}^{m-1}$ and the coefficient matrix $(\sigma_{\alpha\beta})$ satisfies the asymptotic condition

$$\sigma_{\alpha\beta}(r,\theta) = r^2\delta_{\alpha\beta} + o\left(r^2\right), \text{ as } r \to 0. \tag{1.14}$$

By the fundamental equations of Riemannian geometry, we know that, within the cut locus of o,

$$\mathcal{L}_{\nabla r}\langle\,,\,\rangle = 2\operatorname{Hess}(r),$$

where, furthermore, $\nabla r = \partial_r$ the radial vector field. Therefore, on $B_{r_{-G}}(o)$, we have

$$\partial_r\sigma_{\alpha\beta}(r,\theta) = 2\operatorname{Hess}(r)_{\alpha\beta}. \tag{1.15}$$

But, according to (1.12) and (1.13), we have, for every $X, Y \in (\nabla r)^\perp$,

$$\begin{aligned} \operatorname{Hess}(r)(X,Y) &= \left\langle D_X \frac{\nabla v}{|\nabla v|}, Y \right\rangle = \frac{1}{|\nabla v|}\operatorname{Hess}(v)(X,Y) \\ &= \frac{1}{|\nabla v|}A^{-1}Hu\langle X,Y\rangle = \frac{g'}{g}\langle X,Y\rangle. \end{aligned}$$

Using this information into (1.15) and recalling (1.14) we deduce that

$$\begin{cases} \partial_r\sigma_{\alpha\beta}(r,\theta) = 2\dfrac{g'}{g}(r)\,\sigma_{\alpha\beta}(r,\theta) \\ \\ \sigma_{\alpha\beta}(r,\theta) = r^2\delta_{\alpha\beta} + o\left(r^2\right), \text{ as } r \to 0, \end{cases} \tag{1.16}$$

which integrated gives

$$\sigma_{\alpha\beta}(r,\theta) = g(r)^2\,\delta_{\alpha\beta}.$$

We have thus shown that

$$\exp_o^* \langle \, , \, \rangle = dr \otimes dr + g\,(r)^2\, d\theta^2,$$

proving that $\exp_o : M^m_{-G} \backslash \{0\} \to B_R\,(o) \backslash \{o\}$ is a Riemannian isometry. To conclude, note that, by the assumptions on g, this isometry smoothly extends even to the origin 0.

Step 4. We now assume that u is a solution of (1.10) on all of M. In case $r_{-G} = +\infty$, then it follows directly from Step 3 that $\Phi : M^m_{-G} \to M$ is a Riemannian isometry. Accordingly, in what follows, we assume $r_{-G} < +\infty$.

Step 5. We show that $\partial B_{r_{-G}}\,(o)$ is discrete, hence a finite set. Indeed, for every $x \in \partial B_{r_{-G}}\,(o)$, let γ be a unit speed, minimizing geodesic from o to x. Then $|\nabla u| \circ \gamma\,(t) = Ag\,(t) \to 0$ as $t \to r_{-G}$. Therefore, $\partial B_{r_{-G}}\,(o)$ is made up by critical points of u. Since u satisfies the differential equation $\mathrm{Hess}\,(u)\,(x) = H\,(r\,(x))\,u\,(x) \langle \, , \, \rangle$ and, by assumption, $H\,(r_{-G}) \neq 0$ and $u \neq 0$ on $\partial B_{r_{-G}}\,(o)$, we deduce that such critical points are non–degenerate (i.e., the quadratic form $\mathrm{Hess}\,(u)$ has no zero eigenvalues) hence, by Morse Lemma, they are isolated. Accordingly, $\partial B_{r_{-G}}(o) = \{p_1, ..., p_k\}$, as claimed.

Step 6. We prove that $\mathrm{cut}\,(o) = \{O\} = \partial B_{r_{-G}}\,(o)$, for some $O \in M$. Indeed, by Step 2, the standard m–dimensional ball $\mathbf{B}^m_{r_{-G}}\,(0) \subset T_oM$ of radius r_{-G} lies in the domain $D_o \subset T_oM$ of the normal coordinates at o. Therefore, it suffices to show that

$$\exp_o\left(\partial \mathbf{B}^m_{r_{-G}}\,(0)\right) = \partial B_{r_{-G}}\,(o) = \{O\}\,. \tag{1.17}$$

If this occurs then $\partial \mathbf{B}^m_{r_{-G}}\,(0)$ is precisely the tangential cut–locus of o and, hence, $\mathrm{cut}\,(o) = \{O\}$. Note that, in particular, all the geodesics issuing from o will meet at O (and cannot minimizes distances past r_{-G}).

Now for the proof of (1.17). Let us observe that $\exp_o(\partial \mathbf{B}^m_{r_{-G}}\,(0)) \subseteq \overline{B_{r_{-G}}\,(o)}$ and $\exp_o(\partial \mathbf{B}^m_{r_{-G}}\,(0) \cap D_o) = \partial B_{r_{-G}}\,(o) \cap (M \backslash \mathrm{cut}\,(o))$. Since $B_{r_{-G}}\,(o)$ does not contain any cut–point of o, it follows that also the tangential cut points in $\partial \mathbf{B}^m_{r_{-G}}\,(0)$ are mapped on $\partial B_{r_{-G}}\,(o)$ by $\exp_o$. Thus, $\exp_o(\partial \mathbf{B}^m_{r_{-G}}\,(0))$ $= \partial B_{r_{-G}}\,(o)$. Now, recall from Step 5 that $\partial B_{r_{-G}}\,(o)$ is a finite set. Since $\partial \mathbf{B}^m_{r_{-G}}\,(0)$ is connected and $\exp_o$ is a continuous map, we conclude the validity of (1.17).

Step 7. We note that $\Phi\left(M^m_{-G}\right) = M \backslash \{O\} = B_{r_{-G}}\,(o)$. Indeed, this follows directly from Step 3 and Step 6.

Step 8. We finally deduce that M is, diffeomorphically, a twisted sphere. To this end, recall that, by Step 6, M is compact. Moreover, u is a smooth function on M with precisely two critical points, o and O. According to (1.10) and Step 5, these critical points are non–degenerate. Therefore, to conclude, we can apply the (differentiable version of) the classical result by G. Reeb.

This completes the proof of the Theorem. □

Note that in [**83**] we were able to extend the metric rigidity established in Theorem 1.3 also to complete manifolds supporting solutions of third order differential systems of the form

$$\nabla Hess(u) = H(r(x)) \langle \, , \, \rangle \otimes du.$$

The first result in this direction with $H \equiv const.$ and du replaced by a general 1–form ω was obtained by E. García–Río, D. Kupeli and B. Unal in [**40**].

Theorem 1.6 (Theorem 3.2 in [**83**]). *Let $(M^m, \langle \, , \, \rangle)$ be an m–dimensional, complete Riemannian manifold, let $o \in M$ be a reference origin and set $r(x) = \operatorname{dist}_{(M,\langle \, , \, \rangle)}(x, o)$. A necessary and sufficient condition for the existence of an isometric imbedding $\Phi : M^m_{-G} \to M$ is that there exists a non–trivial, smooth solution $u : B_{r_{-G}}(o) \to \mathbb{R}$ of the problem*

$$\left\{ \begin{array}{l} (\nabla \mathrm{Hess}\,(u))\,(X; Y, W) = G\,(r\,(x))\,\langle \nabla u, X \rangle \,\langle Y, W \rangle \\ \mathrm{Hess}\,(u)\,(o) = A\,\langle \, , \, \rangle \\ |\nabla u|\,(o) = 0, \end{array} \right. \tag{1.18}$$

for some $A \neq 0$. Furthermore, if u is a solution of (1.18) on all of M, then the following holds:

(a) *If $r_{-G} = +\infty$, then M is isometric to the model M^m_{-G}.*
(b) *In case $r_{-G} < +\infty$, and $g'(r_{-G}) \neq 0$, then $\mathrm{cut}\,(o) = \{O\}$ for some $O \in M$, and $\Phi\left(M^m_{-G}\right) = M \backslash \{O\}$. Moreover, M is diffeomorphically a twisted sphere.*

CHAPTER 2

Weighted manifolds and comparison geometry

Many problems in geometric analysis lead to consider Riemannian manifolds endowed with a measure that has a smooth positive density with respect to the Riemannian measure. This turns out to be compatible with the metric structure of the manifold and the resulting spaces take the name of weighted manifolds, also known in literature as manifolds with density. The geometry of weighted manifolds is visible in the weighted metric structure, i.e., in the weighted measures of (intrinsic) metric objects, and it is controlled by suitable concepts of curvature adapted to the density of the measure. Weighted manifolds first arose in the study of diffusion processes on manifolds in works of Bakry and Emery. Moreover in [**3**] (see also [**54**]), they introduced a generalization of Ricci curvature to such spaces, known as the Bakry–Emery Ricci tensor. Recently, mainly in reaction to Perelman's work on Ricci flow which have carried to the solution of the Poincaré conjecture, the study of weighted manifolds and of comparison geometry for the Bakry–Emery Ricci tensor has become the subject of a rapidly increasing investigation.

2.1. Weighted manifolds and Lichnerowicz–Bakry–Emery Ricci tensors

A weighted manifold is a triple $(M^m, \langle\, ,\, \rangle, e^{-f} d\mathrm{vol})$, where $(M^m, \langle\, ,\, \rangle)$ is a complete m–dimensional Riemannian manifold, $f \in C^\infty(M)$ and $d$vol denotes the canonical Riemannian volume form on M. If $B_r(p)$ and $\partial B_r(p)$ denote respectively the metric ball and the metric sphere of $(M, \langle\, ,\, \rangle)$ of radius $r > 0$ and centered at $p \in M$, we define

$$\mathrm{vol}_f(B_r(p)) = \int_{B_r(p)} e^{-f} d\mathrm{vol}, \qquad \mathrm{vol}_f(\partial B_r(p)) = \int_{\partial B_r(p)} e^{-f} d\mathrm{vol}_{m-1},$$

where $d\mathrm{vol}_{m-1}$ stands for the $(m-1)$–Hausdorff measure. Observe that weighted areas and volumes are all computed with respect to the same weight, thus no conformal change of the metric is involved.

Associated to a weighted manifold $(M, \langle\, ,\, \rangle, e^{-f} d\mathrm{vol})$ there is a natural divergence form second order diffusion operator: the f–Laplacian defined on u by

$$\Delta_f u = e^f \operatorname{div}\left(e^{-f} \nabla u\right) = \Delta u - \langle \nabla u, \nabla f \rangle,$$

which is clearly symmetric on $L^2\left(M, e^{-f} d\mathrm{vol}\right)$.

A basic principle in Riemannian geometry is that a lower bound on the Ricci curvature implies that the Riemannian measure is bounded above by the measure in the corresponding model space. To develop a similar theory in weighted geometry one has to generalize the Ricci tensor in this more general framework. As we pointed out in the Introduction there is not a canonical choice. Many generalization have been considered, depending on the geometrical problem under study. For example, following F. Morgan, [**64**], one can consider the tensor

$$\mathcal{R}ic_f = \mathrm{Ric} + \mathrm{Hess}(f) - \Delta f \langle \, , \, \rangle .$$

Then $\mathcal{R}ic_f \geq k > 0$ implies a weighted diameter estimate á la Myers, and $\mathcal{R}ic_f$ is related e.g. to weighted Bishop–Gromov inequality and to weigthed isoperimetry.

We will concentrate on the Lichnerowicz Bakry–Emery Ricci tensors, which were first introduced in [**54**] and [**3**]. The k–Bakry–Emery Ricci tensor on the weighted manifold $(M, \langle \, , \, \rangle , e^{-f} d\mathrm{vol})$ is defined by

$$\mathrm{Ric}_f^k = \mathrm{Ric} + \mathrm{Hess}(f) - \frac{1}{k} df \otimes df, \qquad \text{for} \qquad 0 < k \leq \infty.$$

When f is constant, this is the usual Ricci tensor and when $k = \infty$ this is the Bakry–Emery Ricci tensor, usually denoted by Ric_f. Note that, if $k_1 \geq k_2$ then $\mathrm{Ric}_f^{k_1} \geq \mathrm{Ric}_f^{k_2}$ so that, for example, a lower bound on Ric_f^k for some $k < \infty$, implies a lower bound on Ric_f.

In the next section we will see that the classical scheme of comparison geometry works perfectly for Ric_f^k, $k < \infty$. The reason is that, in this case, we have the validity of a good Bochner formula for Ric_f^k. We briefly remind his derivation. Recall that on a complete Riemannian manifold $(M^m, \langle \, , \, \rangle)$, given $u \in C^\infty(M)$ we have that

$$\frac{1}{2}\Delta|\nabla u|^2 = |\,\mathrm{Hess}(u)|^2 + \langle \nabla u, \nabla \Delta u \rangle + \mathrm{Ric}(\nabla u, \nabla u). \tag{2.1}$$

Observe that, applying the Schwarz inequality,

$$|Hess(u)|^2 \geq \frac{(\Delta u)^2}{m},$$

we obtain the following inequality

$$\frac{1}{2}\Delta|\nabla u|^2 \geq \frac{(\Delta u)^2}{m} + \langle \nabla u, \nabla \Delta u \rangle + \mathrm{Ric}(\nabla u, \nabla u). \tag{2.2}$$

Consider now the operator Δ_f on the complete weighted manifold $(M, \langle \, , \, \rangle ,$ $e^{-f} d\mathrm{vol})$ and observe that the following identities hold

$$\begin{aligned} \Delta_f |\nabla u|^2 &= \Delta |\nabla u|^2 - 2\,\mathrm{Hess}(u)(\nabla u, \nabla f), \\ \langle \nabla u, \nabla \Delta_f u \rangle &= \langle \nabla u, \nabla \Delta u \rangle - \mathrm{Hess}(u)(\nabla u, \nabla f) - \mathrm{Hess}(f)(\nabla u, \nabla u). \end{aligned}$$

Plugging these into (2.1) we get the Bochner formula for Ric_f^k

$$\frac{1}{2}\Delta_f|\nabla u|^2 = |\,\mathrm{Hess}(u)|^2 + \langle \nabla u, \nabla \Delta_f u \rangle + \mathrm{Ric}_f^k(\nabla u, \nabla u) + \frac{1}{k}|\,\langle \nabla f, \nabla u \rangle\,|^2. \tag{2.3}$$

When $k = \infty$, we get

$$\frac{1}{2}\Delta_f|\nabla u|^2 = |\,\mathrm{Hess}(u)|^2 + \langle \nabla u, \nabla (\Delta_f u) \rangle + \mathrm{Ric}_f(\nabla u, \nabla u). \tag{2.4}$$

As observed in [**93**], this formula is quite similar to (2.1) except for the important fact that $tr(\mathrm{Hess}(u)) = \Delta u$ not $\Delta_f u$. Nevertheless, in case $k < \infty$, we can overcome this difficulty. Using the inequality

$$\frac{(\Delta u)^2}{m} + \frac{1}{k} | \langle \nabla f, \nabla u \rangle |^2 \geq \frac{(\Delta_f u)^2}{m+k},$$

we deduce

$$\frac{1}{2}\Delta_f |\nabla u|^2 \geq \frac{(\Delta_f u)^2}{m+k} + \langle \nabla u, \nabla \Delta_f u \rangle + \mathrm{Ric}_f^k(\nabla u, \nabla u). \tag{2.5}$$

Hence, comparing with (2.2) we can say that a Bochner formula for Ric_f^k, $k < \infty$ holds and it looks like the Bochner formula for Ric of an $(m+k)$–dimensional manifold. For this reason when we impose a lower bound on Ric_f^k we can say that, in some sense, we are introducing a “virtual” dimension k.

As we will see in the next section, the case $k = \infty$ is more critical and the difficulties that arise are absolutely non–technical.

2.2. Comparison results in weighted geometry

Classically in comparison geometry, combining the lower bound on the Ricci tensor with the Bochner formula, one proves the Laplacian comparison for the distance function. This, in turn, permits to derive Myers–type theorems, diameter estimates and area and volume comparison theorems. Hence the common root of all these results is the validity of a Bochner formula for smooth, real valued functions.

Our first aim is to present the generalization of such a scheme to weighted manifolds with a lower bound on the k–Bakry–Emery Ricci tensor, $k < \infty$. For nice accounts on this topic see e.g. [**56**], [**86**] and [**93**]. As a first application of the Bochner formula (2.5) for Ric_f^k, $k < \infty$, we want to derive the following generalization to Ric_f^k of a well–known integral estimate of the Ricci tensor along minimizing geodesics. This, in the classical case, is usually obtained via the second variation formula for the arc–length and Jacobi fields. However, it can be derived directly from the Bochner formula applied to the distance function; the proof is modelled on [**86**].

Lemma 2.1. *Let $(M^m, \langle\, , \,\rangle, e^{-f}d\mathrm{vol})$ be a complete weighted manifold, and consider the k–Bakry–Emery Ricci tensor Ric_f^k for k finite. Fix $o \in M$ and let $r(x) = dist(x, o)$. For any point $q \in M$, let $\gamma_q : [0, r(q)] \to M$ be a minimizing geodesic from o to q such that $|\dot\gamma_q| = 1$. If $h \in Lip_{loc}(\mathbb{R})$ is such that $h(0) = h(r(q)) = 0$, then for every $q \in M$, it holds*

$$0 \leq \int_0^{r(q)} (m+k-1)\left(h'\right)^2 ds - \int_0^{r(q)} h^2\, \mathrm{Ric}_f^k(\dot\gamma_q, \dot\gamma_q) ds. \tag{2.6}$$

PROOF. Fix a point $q \notin \mathrm{cut}(o)$. By the Cauchy–Schwarz inequality, on noting also that $Hess(r)(\nabla r, \cdot) = 0$ we have that

$$\frac{(\Delta_f r)^2}{m+k-1} \leq \frac{(\Delta r)^2}{m-1} + \frac{|\langle \nabla f, \nabla r\rangle|^2}{k}, \tag{2.7}$$

$$|Hess(r)|^2 \geq \frac{(\Delta r)^2}{m-1}. \tag{2.8}$$

Using (2.7) and (2.8), from the Bochner formula (2.5) applied to the distance function $r(x)$ we obtain that

$$0 \geq \frac{(\Delta_f r)^2}{m+k-1} + \langle \nabla r, \nabla \Delta_f r\rangle + \mathrm{Ric}_f^k(\nabla r, \nabla r).$$

Evaluating this along a minimizing geodesic γ_q such that $|\dot\gamma_q| = 1$, we get

$$0 \geq \frac{(\Delta_f r \circ \gamma_q)^2}{m+k-1} + \frac{d}{ds}(\Delta_f (r \circ \gamma_q)) + \mathrm{Ric}_f^k(\dot\gamma_q, \dot\gamma_q). \tag{2.9}$$

If $h \in Lip_{loc}(\mathbb{R})$, $h \geq 0$, $h(0) = 0$, multiplying (2.9) by h^2 and integrating on $[0, t]$, we obtain

$$0 \geq \int_0^t h^2 \frac{(\Delta_f r \circ \gamma_q)^2}{m+k-1} ds + \int_0^t \frac{d}{ds}(\Delta_f r \circ \gamma_q) h^2 + \int_0^t h^2 \,\mathrm{Ric}_f^k(\dot\gamma_q, \dot\gamma_q).$$

Since $(\Delta_f r \circ \gamma_q) h^2 \to 0$ as $r \to 0$, integrating by parts we have that

$$\begin{aligned} 0 \geq & \int_0^t h^2 \frac{(\Delta_f r \circ \gamma_q)^2}{m+k-1} ds + h^2(t)(\Delta_f r \circ \gamma_q)(t) \\ & - 2\int_0^t h h' (\Delta_f r \circ \gamma_q) ds + \int_0^t h^2 \,\mathrm{Ric}_f^k(\dot\gamma_q, \dot\gamma_q) ds. \end{aligned}$$

Since

$$-2hh'(\Delta_f r \circ \gamma_q) \geq \frac{-h^2(\Delta_f r \circ \gamma_q)^2}{m+k-1} - (m+k-1)(h')^2,$$

we deduce that

$$0 \geq h^2(t)(\Delta_f r \circ \gamma_q) - \int_0^t (m+k-1)(h')^2 ds + \int_0^t \mathrm{Ric}_f^k(\dot\gamma_q, \dot\gamma_q) h^2 ds$$

Thus, taking $t = r(q)$ and choosing h such that $h^2(r(q)) = 0$, we get (2.6) for $q \notin \mathrm{cut}(o)$. To treat the general case one can use the Calabi trick. Namely suppose that $q \in \mathrm{cut}\,(o)$. Translating the origin o to $o_\epsilon = \gamma_q(\epsilon)$ so that $q \notin \mathrm{cut}\,(o_\epsilon)$, using the triangle inequality and, finally, taking the limit as $\epsilon \to 0$, one checks that (2.6) holds also in this case. □

Direct application of Lemma 2.1 yields Myers–type conclusions. Here we state only the following result which is due to Z. Qian, [**86**]. Further generalizations will be presented in Chapter 6. The proof is a particular case of that of Theorem 6.1.

Theorem 2.2 (Theorem 5 in [**86**])**.** *Let $(M^m, \langle\, , \rangle, e^{-f}\mathrm{dvol})$ be a complete weighted manifold. Given two different points $p, q \in M$, let $\gamma_{p,q}$ be a minimizing geodesic from p to q parameterized by arc–length. Suppose that there exists a constant c such that for each pair of points p, q it holds*

$$Ric_f^k(\dot\gamma_{p,q}, \dot\gamma_{p,q})|_{\gamma_{p,q}(t)} \geq (m+k-1)c^2,$$

for $k < +\infty$. Then M is compact and

$$\mathrm{diam}(M) \leq \frac{\pi}{c}. \tag{2.10}$$

Another consequence of the Bochner formula (2.5) applied to the distance function $r(x)$ is the weighted Laplacian comparison theorem. Version of this result have been obtained by A. G. Setti, [**89**], for the case $k = 1$ and later by Z. Qian, [**86**], in the general case $k \in \mathbb{N}$. Here we state a more general version proved in [**56**], where the lower bound for Ric_f^k is allowed to be a function of the distance from a reference point.

Theorem 2.3 (Proposition 2.3 in [**56**])**.** *Let $(M^m, \langle\, , \rangle, e^{-f}\mathrm{dvol})$ be a complete weighted manifold. Denote with $\mathrm{cut}(o)$ the cut locus of the reference point $o \in M$. Assume that*

$$\mathrm{Ric}_f^k(\nabla r, \nabla r) \geq -(m+k-1)G(r)$$

for some $G \in C^0([0, +\infty])$, let $h \in C^2([0, +\infty])$ be a solution of the problem

$$\begin{cases} h'' - Gh \geq 0 \\ h(0) = 0, \qquad h'(0) = 1, \end{cases}$$

and let $(0, R)$, $R \leq +\infty$, be the maximal interval where $h > 0$. Then for every $x \in M$ we have $r(x) \leq R$, and the inequality

$$\Delta_f r(x) \leq (m+k-1)\frac{h'(r(x))}{h(r(x))}$$

holds pointwise in $M \setminus (\mathrm{cut}(o) \cup \{o\})$ and weakly on M.

As in the classical case, the weighted Laplacian comparison allows to obtain weighted volume comparison estimates. Again, in the case of constant lower bounds on Ric_f^k, this was originally proven in [**86**], but here we state the general version which can be found in [**56**].

Theorem 2.4 (Theorem 2.4 in [**56**])**.** *Let $(M, \langle\, , \rangle, e^{-f}\mathrm{dvol})$ be as in the above theorem, and assume the same lower bound as above on Ric_f^k. Then the functions*

$$r \longmapsto \frac{\mathrm{vol}_f \partial B_r(o)}{h(r)^{m+k-1}}$$

and

$$r \longmapsto \frac{\mathrm{vol}_f B_r(o)}{\int_0^r h(t)^{m+k-1} dt}$$

are respectively non–increasing a.e. and non–increasing in $(0, R)$. In particular, for every $0 < r_0 < R$ there exists a constant C depending on the *geometry of* M *in* $B_{r_0}(o)$ *such that*

$$\mathrm{vol}_f(B_r(o)) \le C \begin{cases} r^m & 0 \le r \le r_0, \\ \int_0^r h(t)^{m+k-1} dt & r \ge r_0. \end{cases}$$

In particular in case the lower bound for Ric_f^k is a negative constant, we recover (up to consider the virtual dimension k) the usual hyperbolic growth for weighted volumes originally proved in [**86**].

Corollary 2.5 (Corollary 2 in [**86**])**.** *Let* $(M, \langle\, ,\, \rangle, e^{-f}\mathrm{dvol})$ *be a complete weighted manifold and suppose that*

$$\mathrm{Ric}_f^k \ge -(m+k-1)B^2,$$

for some constant $B > 0$. *Then for every* $r > 0$ *there exists a constant* C *such that*

$$\mathrm{vol}_f(B_r(o)) \le C B^{-(m+k-1)} \int_0^r (\sinh(Bt))^{(m+k-1)} dt.$$

When we impose a lower bound on Ric_f the situation is quite different. This clearly comes out from the next two examples.

Example 2.6. *The Gaussian space* $(\mathbb{R}^m, \langle\, ,\, \rangle_{can}, e^{-f}\mathrm{dvol})$ *with*

$$f(x) = \frac{1}{2}A^2|x|^2,$$

for arbitrary $A \in \mathbb{R}$, *satisfies* $\mathrm{Ric}_f = A^2 > 0$ *but it is non–compact.*

Hence, in general, when we assume that

$$\mathrm{Ric}_f \ge c^2 \langle\, ,\, \rangle, \tag{2.11}$$

for some constant c, compactness is not guaranteed. In order to recover compactness we have to impose further conditions on the growth of f or on its gradient. However, the validity of (2.11) implies that the fundamental group must be finite. This type of results have been recently investigated in [**35**], [**96**], [**67**] and [**33**]. In Chapter 6 we are going to extend some of these results no longer assuming the lower bound in (2.11) to be constant.

Example 2.7. *The anti–Gaussian space* $(\mathbb{R}^m, \langle\, ,\, \rangle_{can}, e^{-f}\mathrm{dvol})$ *with*

$$f(x) = -\frac{1}{2}A^2|x|^2,$$

for arbitrary $A \in \mathbb{R}$ *satisfies* $\mathrm{Ric}_f = -A^2$ *but, integrating in polar coordinates and indicating with* C_{m-1} *the volume of the* $(m-1)$*–dimensional ball in* $\mathbb{R}^{m-1}$,

$$\begin{aligned} \mathrm{vol}_f(B_r(0)) &= \int_0^r \left(\int_{\partial B_t(0)} e^{\frac{A^2}{2}t^2} \mathrm{dvol}_{m-1} \right) dt \\ &= C_{m-1} \int_0^r e^{\frac{A^2}{2}t^2} t^{m-1} dt \\ &\asymp e^{\frac{A^2}{2}t^2} t^{m-2}. \end{aligned}$$

Thus, the volume growth is more than hyperbolic.

Nevertheless, there are again mutual relations between Ric_f–bounds and vol_f–growth properties. The comparison geometry in this case is more subtle and most of it has been developed in [**85**] and [**94**]. First, we recall a weighted-volume comparison established in [**85**], [**94**].

Theorem 2.8 (Theorem 1.4 in [**85**] and Theorem 4.1 in [**94**]). *Let* $(M, \langle\,,\,\rangle, e^{-f}\mathrm{dvol})$ *be a geodesically complete weighted manifold. Suppose that*

$$\mathrm{Ric}_f \geq \lambda,$$

for some constant $\lambda \in \mathbb{R}$*. Then, having fixed* $R_0 > 0$*, there are constants* $A, B, C > 0$ *such that, for every* $r \geq R_0$*,*

$$\mathrm{vol}_f\left(B_r(o)\right) \leq A + B \int_{R_0}^{r} e^{-\lambda t^2 + Ct} dt.$$

Remark 2.9. In the case where $\lambda > 0$ Theorem 2.8 gives that $vol_f(M)$ is finite. This was first observed in [**63**].

We present now two results which can be useful in order to deal with the case in which the lower bound for Ric_f is not necessarily a constant. The first is an improvement of Theorem 1.2 (a) of Wei and Wylie [**94**], where they assume θ and G below to be constant.

Theorem 2.10 (Theorem 3.1 in [**76**]). *Let* $\left(M, \langle\,,\,\rangle, e^{-f}\mathrm{dvol}\right)$ *be a complete weighted manifold such that*

$$\langle \nabla r, \nabla f \rangle \geq -\theta\left(r\right), \tag{2.12}$$

for some non–decreasing function $\theta \in C^0\left(\mathbb{R}_0^+\right)$*. Assume*

$$\mathrm{Ric}_f \geq -\left(m-1\right) G\left(r\right) \tag{2.13}$$

for a smooth positive function G *on* $\mathbb{R}_0^+$*, even at the origin. Let* g *be a solution on* $\mathbb{R}_0^+$ *of*

$$\begin{cases} g'' - Gg \geq 0 \\ g\left(0\right) = 0, \;\; g'\left(0\right) \geq 1. \end{cases} \tag{2.14}$$

Then there exists a constant $D > 0$ *such that* $\forall r \geq 0$

$$vol_f\left(B_r(o)\right) \leq D \int_0^r g\left(t\right)^{m-1} e^{\int_0^t \theta(s)ds} dt. \tag{2.15}$$

Proof. Let h be the solution on $\mathbb{R}_0^+$ of the Cauchy problem

$$\begin{cases} h'' - Gh = 0 \\ h\left(0\right) = 0, \;\; h'\left(0\right) = 1. \end{cases} \tag{2.16}$$

Note that $h > 0$ on $\mathbb{R}^+$ since $G \geq 0$. Fix $x \in M \setminus \left(\mathrm{cut}\left(o\right) \cup \{o\}\right)$ and let $\gamma : [0, l] \to M$, $l = length\left(\gamma\right)$, be a minimizing geodesic with $\gamma\left(0\right) = o$, $\gamma\left(l\right) = x$. Note that $G\left(r \circ \gamma\right)\left(t\right) = G\left(t\right)$. From Bochner formula applied to the distance function r we have

$$0 = \left|Hess\left(r\right)\right|^2 + \langle \nabla r, \nabla \Delta r \rangle + \mathrm{Ric}\left(\nabla r, \nabla r\right) \tag{2.17}$$

so that, using the Schwarz inequality, it follows that the function $\varphi(t) = (\Delta r) \circ \gamma(t)$, $t \in (0, l]$, satisfies the Riccati inequality

$$\varphi' + \frac{1}{m-1}\varphi^2 \leq -\operatorname{Ric}(\nabla r \circ \gamma, \nabla r \circ \gamma) \tag{2.18}$$

on $(0, l]$. With h as in (2.16) and using the definition of Ric_f, (2.13) and (2.18) we compute

$$\begin{aligned}
\left(h^2\varphi\right)' =& 2hh'\varphi + h^2\varphi' \\
\leq& 2hh'\varphi - \frac{h^2\varphi^2}{m-1} + (m-1)\,G(t)\,h^2 + Hess(f)(\nabla r \circ \gamma, \nabla r \circ \gamma)\,h^2 \\
=& -\left(\frac{h\varphi}{\sqrt{m-1}} - \sqrt{m-1}h'\right)^2 + (m-1)\left(h'\right)^2 + (m-1)\,G(t)\,h^2 \\
&+ h^2 (f \circ \gamma)''.
\end{aligned}$$

We let

$$\varphi_G(t) = (m-1)\frac{h'}{h}(t)$$

so that, using (2.16)

$$\left(h^2\varphi_G\right)' = (m-1)\left(h'\right)^2 + (m-1)\,G(t)\,h^2.$$

Inserting into the above inequality we obtain

$$\left(h^2\varphi\right)' \leq \left(h^2\varphi_G\right)' + h^2 (f \circ \gamma)'' \tag{2.19}$$

Integrating (2.19) on $[0, r]$ and using (2.16) yields

$$h^2(r)\,\varphi(r) \leq h^2(r)\,\varphi_G(r) + \int_0^r h^2 (f \circ \gamma)''\,dt. \tag{2.20}$$

Next we recall that

$$\varphi_f = (\Delta_f r) \circ \gamma = (\Delta r) \circ \gamma - \langle \nabla f, \nabla r\rangle \circ \gamma = \varphi - (f \circ \gamma)' \tag{2.21}$$

Thus, using (2.20), (2.16) and integrating by parts we compute

$$\begin{aligned}
h^2\varphi_f \leq& h^2\varphi_G - h^2 (f \circ \gamma)' + \int_0^r h^2 (f \circ \gamma)''\,dt \\
=& h^2\varphi_G - h^2 (f \circ \gamma)' + \left(h^2 (f \circ \gamma)'\right)\Big|_0^r - \int_0^r \left(h^2\right)' (f \circ \gamma)'\,dt \\
=& h^2\varphi_G - \int_0^r \left(h^2\right)' (f \circ \gamma)'\,dt,
\end{aligned}$$

that is,

$$h^2\varphi_f \leq h^2\varphi_G - \int_0^r \left(h^2\right)' (f \circ \gamma)'\,dt \tag{2.22}$$

on $(0, l]$. We observe that, because of (2.16) and $G \geq 0$, $\left(h^2\right)' = 2hh' \geq 0$ so that, using (2.12), (2.16) and the monotonicity of θ, (2.22) yields

$$h^2\varphi_f \leq h^2\varphi_G + \theta(r)\,h^2$$

on $(0, l]$, and

$$\varphi_f \leq \varphi_G + \theta(r)$$

on $(0,l]$. In particular

$$\Delta_f r(x) \leq (m-1)\frac{h'(r(x))}{h(r(x))} + \theta(r(x)) \tag{2.23}$$

on $M\setminus(\{o\}\cup \mathrm{cut}(o))$. Proceeding as in Theorem 2.4 of [**82**] one shows that (2.23) holds weakly on all of M and reasoning as in Theorem 2.14 of [**82**] one shows that

$$vol_f(\partial B_r(o)) \leq Dh(r)^{m-1} e^{\int_0^r \theta(t)dt} \tag{2.24}$$

for some constant $D > 0$. Integrating over $[0,r]$ and using the co-area formula we get

$$vol_f(B_r(o)) \leq D\int_0^r h(t)^{m-1} e^{\int_0^t \theta(s)ds} dt. \tag{2.25}$$

Since g in (2.14) is a subsolution of (2.16) it follows, by Lemma 2.1 in [**82**], that $h \leq g$ on $\mathbb{R}_0^+$ so that (2.25) immediately implies (2.15) $\square$

A second estimate on φ_f can also be derived, replacing assumption (2.12) with

$$\xi(r) \leq f \leq \omega(r), \tag{2.26}$$

for some functions $\omega, \xi \in C^1(\mathbb{R}_0^+)$ with ω non decreasing and such that $\xi'(r) \leq \omega'(r)$.
Towards this aim we integrate (2.22) again by parts to obtain

$$h^2\varphi_f \leq h^2\varphi_G - \left[\left(h^2\right)'(f\circ\gamma)\right]\Big|_0^r + \int_0^r \left(h^2\right)''(f\circ\gamma)\,dt.$$

Now, using (2.16),

$$\left(h^2\right)'' = 2\left(h'\right)^2 + 2Gh^2 \geq 0,$$

because of the sign of G. Thus using (2.26), (2.16) and the fact that ω is non-decreasing, from the above we obtain

$$\begin{aligned} h^2\varphi_f \leq & h^2\varphi_G - \left(h^2\right)'(f\circ\gamma)|_0^r + \omega(r)\left(h^2\right)'\Big|_0^r \\ \leq & h^2\varphi_G - \left(h^2\right)'(r)(f\circ\gamma)(r) + \left(h^2\right)'(r)\,\omega(r) \\ \leq & h^2\varphi_G + \left(h^2\right)'(r)\left[\omega(r) - (f\circ\gamma)(r)\right]. \end{aligned}$$

Now

$$\left(h^2\right)' = 2hh' = \frac{2}{m-1}h^2(m-1)\frac{h'}{h} = \frac{2}{m-1}h^2\varphi_G,\ r>0$$

so that the above inequality may be rewritten as

$$h^2\varphi_f \leq h^2\left(1+\frac{2}{m-1}\left(\omega(r) - (f\circ\gamma)(r)\right)\right)\varphi_G,\ r>0$$

and using (2.26)

$$\varphi_f \leq \left(1+\frac{2}{m-1}\left(\omega(r) - \xi(r)\right)\right)\varphi_G,\ r>0.$$

Let $\widetilde{\omega}(r) \geq \omega(r) - \xi(r) \geq 0$. Similarly to what we did in Theorem 2.10 we arrive at (2.24), where $\theta(t)$ is now substituted by $\frac{2}{m-1}\widetilde{\omega}(t)\varphi_G(t)$. Thus we need to estimate $e^{\int_{r_0}^{r}\frac{2}{m-1}\widetilde{\omega}(t)\frac{h'}{h}}$.

$$\begin{aligned}&\int_{r_0}^{r}\frac{2}{m-1}\widetilde{\omega}(t)\frac{h'}{h}\\ =&\frac{2}{m-1}\widetilde{\omega}(r)\log h^{m-1}(r) - \frac{2}{m-1}\widetilde{\omega}(r_0)\log h^{m-1}(r_0)\\ &-\int_{r_0}^{r}\frac{2}{m-1}\widetilde{\omega}'(t)\log h^{m-1}(t)\,dt.\end{aligned}$$

Now, by (2.16), $h(t) \nearrow +\infty$ as $t \to +\infty$. Choose r_0 sufficiently large that $h(r_0) \geq 1$. Since $\widetilde{\omega}' \geq 0$

$$\int_{r_0}^{r}\frac{2}{m-1}\widetilde{\omega}(t)\left(\log h^{m-1}\right)' dt \leq \log\left(h(r)\right)^{2\widetilde{\omega}(r)} - A,$$

and

$$e^{\int_{r_0}^{r}\frac{2}{m-1}\widetilde{\omega}(t)\varphi_G} \leq h(r)^{2\widetilde{\omega}(r)}e^{-A}.$$

Hence, from (2.24),

$$vol_f\left(\partial B_r(o)\right) \leq Dh(r)^{m-1+2\widetilde{\omega}(r)}$$

Since $h \leq g$ we have thus proven the following result, which improves on Theorem 1.2 (b) of Wei and Wylie [**94**].

Theorem 2.11 (Theorem 3.2 in [**76**]). *Let* $\left(M, \langle\,,\,\rangle, e^{-f}d\mathrm{vol}\right)$ *be a complete weighted manifold such that*

$$\xi(r) \leq f \leq \omega(r)$$

for some functions $\omega, \xi \in C^1\left(\mathbb{R}_0^+\right)$ *with* ω *non decreasing and such that* $\xi'(r) \leq \omega'(r)$. *Assume*

$$\mathrm{Ric}_f \geq -(m-1)G(r)$$

for a smooth positive function G *on* $\mathbb{R}_0^+$, *even at the origin.*
Let $\widetilde{\omega}(r) = \omega(r) - \xi(r)$ *and* g *be a solution on* $\mathbb{R}_0^+$ *of*

$$\begin{cases} g'' - Gg \geq 0 \\ g(0) = 0,\ g'(0) \geq 1\end{cases}$$

Then there exist constants C,$B > 0$ *such that,* $\forall r \geq r_0 > 0$,

$$vol_f\left(B_r(o)\right) \leq C + B\int_{r_0}^{r} g(t)^{(m-1)+2\widetilde{\omega}(t)}\,dt.$$

We end this discussion with the following simple proposition which marginally extends a previous result by Wei and Wylie, [**94**].

Proposition 2.12 (Proposition 3.3 in [**76**]). *Let* $(M, \langle\,,\,\rangle, e^{-f}d\mathrm{vol})$ *be a weighted manifold and assume that*

$$\mathrm{Ric}_f \geq D(1+r)^{-\mu}.$$

(i) *If $D > 0$ and $0 \le \mu \le 1$, then there exist constants $C_j > 0$ such that for every $r > 2$,*

$$\mathrm{vol}_f(\partial B_r(o)) \le \begin{cases} C_1 e^{-C_2 r \log(1+r)} & \text{if } \mu = 1 \\ C_1 e^{-C_2 r^{2-\mu}} & \text{if } 0 \le \mu < 1 \end{cases} \quad \text{and} \quad \mathrm{vol}_f(B_r(o)) \le C_3$$

(ii) *If $D = 0$ then there exist constants $C_j > 0$ such that for every $r > 2$,*

$$\mathrm{vol}_f(\partial B_r(o)) \le C_1 e^r \quad \text{and} \quad \mathrm{vol}_f(B_r(o)) \le C_2 e^r.$$

(iii) *If $D < 0$ then there exist constants $C_j > 0$ such that for every $r > 2$,*

$$\mathrm{vol}_f(\partial B_r(o)) \le \begin{cases} C_1 e^{C_2 r} & \text{if } \mu > 1 \\ C_1 e^{C_2 r \log r} & \text{if } \mu = 1 \\ C_1 e^{C_2 r^{2-\mu}} & \text{if } 0 \le \mu < 1 \end{cases}$$

and

$$\mathrm{vol}_f(B_r(o)) \le \begin{cases} C_3 e^{C_2 r} & \text{if } \mu > 1 \\ C_3 (\log r)^{-1} e^{C_2 r \log r} & \text{if } \mu = 1 \\ C_3 r^{\mu-1} e^{C_2 r^{2-\mu}} & \text{if } 0 \le \mu < 1. \end{cases}$$

Proof. Maintaining the notation introduced above, it follows from (2.18), (2.21) and the definition of Ric_f that if $\varphi_f = \Delta_f r \circ \gamma$ then

$$\varphi_f' = \varphi' - (f \circ \gamma)'' \le -\frac{\varphi^2}{m-1} - \mathrm{Ric}(\dot\gamma, \dot\gamma) - \mathrm{Hess} f(\dot\gamma, \dot\gamma) \le -\mathrm{Ric}_f(\dot\gamma, \dot\gamma).$$

Thus, if we assume that $\mathrm{Ric}_f \ge \theta(r(x))$ and that the ball $B_\varepsilon(o)$ is contained in the domain of the normal coordinates at o, setting $C = \max_{\partial B_\varepsilon(o)} \Delta_f r$ and integrating between ε and $r(x)$ we obtain

$$\Delta_f r(x) \le C - \int_\varepsilon^{r(x)} \theta(t) dt$$

pointwise in the $M \setminus (B_\varepsilon(o) \cup \mathrm{cut}(o))$ and weakly on $M \setminus B_\varepsilon(o)$. From this, arguing as in [**82**] Theorem 2.14 we deduce that

$$\mathrm{vol}_f(\partial B_r(o)) \le e^{C(r-r_0) - \int_{r_0}^r (\int_\varepsilon^t \theta(s) ds) dt} \mathrm{vol}_f(\partial B_{r_0}(o)).$$

The conclusion now follows estimating the integral on the right hand side for $\theta(s) = D(1+s)^{-\mu}$. □

2.3. Analysis of the f–Laplacian

In Section 2.1, we have introduced the natural diffusion operator Δ_f associated to a weighted manifold $(M, \langle\, , \,\rangle, e^{-f} d\mathrm{vol})$. The general idea at the basis of this section is that analytic theorems for the Laplace–Beltrami operator on M, which are proved under metric–measure assumptions, using for example the divergence theorem, comparison arguments or heat semigroup methods, can be translitterated into theorems, under weighted metric–measure assumptions, for the f–Laplacian on $(M, \langle\, , \,\rangle, e^{-f} d\mathrm{vol})$.

Since it will be applied repeatedly in the next chapters, we start recalling the classical maximum principle for the f–Laplacian.

Theorem 2.13. *Let* $\Omega \subseteq (M, \langle\, , \,\rangle, e^{-f}d\mathrm{vol})$ *be a connected domain. Then the following hold:*

(1) *If* $\Delta_f u \geq 0$ *in* Ω *and* $u(x_0) = \sup_\Omega u$ *then* $u \equiv u(x_0)$ *in* Ω.
(2) *If* $\Delta_f u \leq cu$ *in* Ω, *for a generic constant* $c \in \mathbb{R}$, $u \geq 0$ *in* Ω *and* $u(x_0) = 0$ *then* $u \equiv 0$ *in* Ω.

The classical maximum principle is an invaluable tool in the study of the qualitative behavior of solutions of PDE's on (domains of) $\mathbb{R}^m$. Due to its local nature, it can be succesfully applied on general Riemannian manifolds to investigate equations of great geometrical interest. However, precisely because of its local nature, the maximum principle is not sensitive to the specific geometric properties of the manifold M. In [**71**], H. Omori established a global version of the maximum principle for the Laplace–Beltrami operator on a Riemannian manifold with sectional curvature bounded from below. This was later refined by S. T. Yau , [**97**], and S.Y. Cheng and S. T. Yau, [**27**], relaxing the curvature assumption to Ricci bounded below, and permitted to find elegant solutions to a number of outstanding geometric problems. For these reasons this principle is also known as the Omori–Yau maximum principle. Asking the validity of a relaxed form of this maximum principle, the weak Omori–Yau maximum principle, one should be able to relax also the geometrical assumptions on the manifold. Moreover, in the case of the Laplace–Beltrami operator, the validity of this new form of the maximum principle is equivalent to the stochastic completeness of the manifold, [**80**], and thus is strictly related to volume growth properties of geodesic balls, [**41**].

In [**80**] these concepts were also extended to other differential operators both of linear and non–linear nature. For instance, in the linear setting, one can replace the Laplace–Beltrami operator with the operator Δ_f on the weighted manifold $(M, \langle\, , \,\rangle, e^{-f}d\mathrm{vol})$.

Definition 2.14. *Let* $\left(M, \langle\, , \,\rangle, e^{-f}d\mathrm{vol}\right)$ *be a (not necessarily complete) weighted manifold. We say that the (full) Omori–Yau maximum principle for* Δ_f *holds if for any* C^2 *function* $u : M \to \mathbb{R}$ *satisfying* $\sup_M u = u^* < +\infty$, *there exists a sequence* $\{x_n\} \subset M$ *along which*

$$(2.27)\quad (i)\ u\left(x_n\right) \geq u^* - \frac{1}{n}, \quad (ii)\ \left|\nabla u\left(x_n\right)\right| \leq \frac{1}{n}, \ \text{and } (iii)\ \Delta_f u\left(x_n\right) \leq \frac{1}{n}.$$

Analogously to the case of the Laplace–Beltrami operator we have also the following

Definition 2.15. *Let* $\left(M, \langle\, , \,\rangle, e^{-f}d\mathrm{vol}\right)$ *be a (not necessarily complete) weighted manifold. We say that the weak Omori–Yau maximum principle for* Δ_f *holds if for any* C^2 *function* $u : M \to \mathbb{R}$ *satisfying* $\sup_M u = u^* < +\infty$, *there exists a sequence* $\{x_n\} \subset M$ *along which*

$$(2.28)\qquad (i)\ u\left(x_n\right) \geq u^* - \frac{1}{n}, \quad \text{and} \quad (ii)\ \Delta_f u\left(x_n\right) \leq \frac{1}{n}.$$

It happens that the f–Laplacian is related to a suitable stochastic process X_t, called a symmetric diffusion. Following J. Dodziuk construction, [**32**], the diffusion operator Δ_f has a minimal, positive heat kernel $p_f(t,x,y)$ and its total mass $\int_M p_f(t,x,y)\, e^{-f} d\mathrm{vol}(y)$ turns out to be related to the intrinsic explosion time of the associated diffusion process X_t. Let us introduce the following

Definition 2.16. *A weighted manifold* $(M, \langle\, ,\, \rangle, e^{-f} d\mathrm{vol})$ *is said to be* f*–stochastically complete if*

$$\int_M p_f(t,x,y)\, e^{-f} d\mathrm{vol}(y) = 1 \tag{2.29}$$

holds, for every $t>0$ *and for every* $x \in M$.

Note that, from a probabilistic viewpoint, condition (2.29) states that the diffusion process with transition probabilities $p_f(t,x,y)$ is Markovian, hence stochastically complete. According to this definition in [**80**] it is proven that the weak Omori–Yau maximum principle for the operator Δ_f holds if and only if the underlying manifold is f–stochastically complete.

The next result states the validity of the weak Omori–Yau maximum principle for Δ_f, under weighted volume growth conditions. It can be deduced from [**80**] Theorem 3.11, making minor modifications in the proofs of Lemma 3.13, Lemma 3.14, Theorem 3.15 and Corollary 3.16.

Theorem 2.17. *Let* $\left(M, \langle\, ,\, \rangle, e^{-f} d\mathrm{vol}\right)$ *be a geodesically complete weighted manifold satisfying the volume growth condition*

$$\frac{r}{\log \mathrm{vol}_f(B_r)} \notin L^1(+\infty). \tag{2.30}$$

Then, the weak Omori–Yau maximum principle for Δ_f *holds on* M.

Remark 2.18. Combining Theorem 2.17, Theorem 2.4 and Theorem 2.8 immediately gives that the weak Omori–Yau maximum principle for Δ_f holds on a weighted manifold $(M, \langle\, ,\, \rangle, e^{-f} d\mathrm{vol})$ provided one of the following curvature assumption is satisfied for some $\lambda \in \mathbb{R}$

(a) $Ric_f^k \geq \lambda, \quad k < \infty$;
(b) $Ric_f \geq \lambda$.

Theorem 2.17 can be seen also as a consequence of the following version of Theorem 5.1 of [**56**], which represents a refined and generalized version in the weighted setting of the weak Omori–Yau maximum principle; [**77**], [**80**]. Indeed, taking $\sigma = \mu = 0$ in the next theorem, we deduce that, for a smooth function u on M satisfying $\sup_M u = u^* < +\infty$, there exists a sequence $\{x_n\}$ along which (2.28) holds.

Theorem 2.19 (Theorem 5.1 in [**56**])**.** *Let* $\left(M, \langle\, ,\, \rangle, e^{-f} d\mathrm{vol}\right)$ *be a complete weighted manifold. Given* $\sigma, \mu \in \mathbb{R}$*, let* $\nu = \mu + 2(\sigma - 1)$ *and assume that* $\sigma \geq 0$*,* $\sigma - \nu > 0$*. Let* $u \in C^1(M)$ *be a function such that*

$$\widehat{u} = \limsup_{r(x)\to+\infty} \frac{u(x)}{r(x)^{\sigma}} < +\infty$$

and suppose that

$$\liminf_{r\to+\infty}\frac{\log vol_f(B_r)}{r^{\sigma-\nu}}=d_0<+\infty. \tag{2.31}$$

Then given $\gamma\in\mathbb{R}$ *such that*

$$\Omega_\gamma=\{x\in M\,:\,u(x)>\gamma\}\neq\emptyset$$

we have

$$\inf_{\Omega_\gamma}(1+r(x))^\mu\,\Delta_f u\leq C\max\{\widehat{u},0\}$$

with

$$C=\begin{cases}0 & \text{if }\ \sigma=0\\ d_0(\sigma-\nu)^2 & \text{if }\ 0<\nu<\sigma\\ d_0\sigma(\sigma-\nu) & \text{if }\ \sigma>0,\ \nu\geq\sigma.\end{cases}$$

In general, under volume growth conditions, nothing can be said about the behavior of the gradient of u and hence on the validity of the full Omori–Yau maximum principle for Δ_f. The following result, which is a generalization of Theorem 1.9 in [**80**], gives function-theoretic sufficient conditions for a weighted Riemannian manifold $\left(M,\langle\,,\,\rangle\,,e^{-f}d\mathrm{vol}\right)$ to satisfy the Omori–Yau maximum principle.

Theorem 2.20 (Theorem 4.1 in [**76**] and Corollary 11 in [**60**])**.** *Let* $(M,\langle\,,\,\rangle\,,$ $e^{-f}d\mathrm{vol})$ *be a weighted Riemannian manifold and assume that there exists a non–negative* C^2 *function* γ *satisfying the following conditions*

$$\gamma(x)\to+\infty\ \text{ as }\ x\to\infty \tag{2.32}$$

$$\exists A>0\ \text{ such that }\ |\nabla\gamma|\leq A\gamma^{\frac{1}{2}}\ \text{ off a compact set} \tag{2.33}$$

$$\exists B>0\ \text{ such that }\ \Delta_f\gamma\leq B\gamma^{\frac{1}{2}}G\left(\gamma^{\frac{1}{2}}\right)^{\frac{1}{2}}\ \text{ off a compact set} \tag{2.34}$$

where G *is a smooth function on* $[0,+\infty)$ *satisfying*

$$\begin{array}{ll}(i)\ G(0)>0 & (ii)\ G'(t)\geq0\ \text{ on }\ [0,+\infty)\\ (iii)\ G(t)^{-\frac{1}{2}}\notin L^1(+\infty) & (iv)\ \limsup_{t\to+\infty}\frac{tG\left(t^{\frac{1}{2}}\right)}{G(t)}<+\infty.\end{array} \tag{2.35}$$

Then, given any function $u\in C^2(M)$ *with* $u^*=\sup_M u<+\infty$*, there exists a sequence* $\{x_n\}_n\subset M$ *such that*

$$(i)\ u(x_k)>u^*-\tfrac{1}{k};\quad (ii)\ |\nabla u(x_k)|<\tfrac{1}{k};\quad (iii)\ \Delta_f u(x_k)<\tfrac{1}{k}; \tag{2.36}$$

for each $k\in\mathbb{N}$*, i.e. the Omori–Yau maximum principle for* Δ_f *holds on* $\left(M,\langle\,,\,\rangle\,,e^{-f}d\mathrm{vol}\right)$.

The proof of this theorem is similar to that of Theorem 1.9 in [**80**] and we refer to this one for more details.

Proof. We define the function

$$\varphi(t)=e^{\int_0^t G(s)^{-\frac{1}{2}}ds}.$$

Proceeding as in [**80**] and using assumption (2.35) (iv), we have that

$$0\leq\frac{\varphi'(t)}{\varphi(t)}<c\left(tG\left(t^{\frac{1}{2}}\right)\right)^{-\frac{1}{2}} \tag{2.37}$$

for some constant $c > 0$. Next, we fix a point $p \in M$ and, $\forall k \in \mathbb{N}$, we define

$$F_k(x) = \frac{u(x) - u(p) + 1}{\varphi(\gamma(x))^{\frac{1}{k}}}.$$

Then $F_k(p) = 1/\varphi(\gamma(p))^{1/k} > 0$. Moreover, since $u^* < +\infty$ and $\varphi(\gamma(x)) \to +\infty$ as $x \to +\infty$, we have $\limsup_{x\to\infty} F_k(x) \leq 0$. Thus, F_k attains a positive absolute maximum at $x_k \in M$. Iterating this procedure, we produce a sequence $\{x_k\}$. It is shown in [**80**] that

$$\limsup_{k\to+\infty} u(x_k) = u^*,$$

and by passing to a subsequence if necessary, we may assume that

$$\lim_{k\to+\infty} u(x_k) = u^*.$$

If $\{x_k\}$ remains in a compact set, then $x_k \to \bar{x}$ as $k \to +\infty$ and the sequence $z_k = \bar{x}$, for each k, clearly satisfies (2.36). We only need to consider the case when $x_k \to \infty$ so that, according to (2.32), $\gamma(x_k) \to +\infty$. Since F_k attains a positive maximum at x_k we have

(2.38)
$$(i)\ (\nabla \log F_k)(x_k) = 0; \quad (ii)\ \Delta_f(\log F_k)(x_k) = \Delta(\log F_k)(x_k) \leq 0.$$

Note that from (2.38)(i) we have that

$$\nabla u(x_k) = \frac{1}{k}(u(x_k) - u(p) + 1)\frac{\varphi'(\gamma(x_k))}{\varphi(\gamma(x_k))}\nabla\gamma(x_k).$$

Furthermore, reasoning as in [**80**] we have

$$\Delta u(x_k) \leq \frac{u(x_k) - u(p) + 1}{k}\left\{\frac{\varphi'(\gamma(x_k))}{\varphi(\gamma(x_k))}\Delta(\gamma)(x_k) + \frac{1}{k}\left(\frac{\varphi'(\gamma(x_k))}{\varphi(\gamma(x_k))}\right)^2 |\nabla\gamma(x_k)|^2\right\}.$$

Assume now that (2.33) and (2.34) hold so that they hold at x_k for sufficiently large k. A computation shows that

$$|\nabla u(x_k)| \leq \frac{a}{k}\cdot\frac{u(x_k) - u(p) + 1}{G\left(\gamma(x_k)^{1/2}\right)^{1/2}}$$

for some costant $a > 0$. Therefore, using (2.34) and (2.37), we obtain

$$\begin{aligned}\Delta_f u(x_k) &= \Delta u(x_k) - \langle \nabla u, \nabla f\rangle (x_k)\\ &\le \frac{u(x_k)-u(p)+1}{k}\left\{\frac{\varphi'(\gamma(x_k))}{\varphi(\gamma(x_k))}\Delta\gamma(x_k) + \frac{1}{k}\left(\frac{\varphi'(\gamma(x_k))}{\varphi(\gamma(x_k))}\right)^2 |\nabla\gamma|^2(x_k)\right.\\ &\quad\left. - \frac{\varphi'(\gamma(x_k))}{\varphi(\gamma(x_k))}\langle\nabla\gamma,\nabla f\rangle(x_k)\right\}\\ &= \frac{u(x_k)-u(p)+1}{k}\left\{\frac{\varphi'(\gamma(x_k))}{\varphi(\gamma(x_k))}\Delta_f\gamma(x_k) + \frac{1}{k}\left(\frac{\varphi'(\gamma(x_k))}{\varphi(\gamma(x_k))}\right)^2 |\nabla\gamma|^2(x_k)\right\}\\ &\le \frac{u(x_k)-u(p)+1}{k}\left\{\frac{c}{\gamma^{1/2}G\left(\gamma^{1/2}\right)^{1/2}}B\gamma^{1/2}G\left(\gamma^{1/2}\right)^{1/2} + \frac{1}{k}\cdot\frac{c^2}{\gamma G\left(\gamma^{1/2}\right)}A^2\gamma\right\}\end{aligned}$$

and the RHS tends to zero as $k \to +\infty$. □

Important situations where Theorem 2.20 applies concerns with general weighted manifolds whose k–Bakry–Emery Ricci tensor, $0 < k \le \infty$ is suitably controlled.

Corollary 2.21. *Let $\left(M^m, \langle\,,\,\rangle, e^{-f}d\mathrm{vol}\right)$ be a complete weighted manifold such that*

$$Ric_f \ge -(m-1)\,G(r)\,\langle\,,\,\rangle \tag{2.39}$$

for a smooth positive function G satisfying (2.35), even at the origin. Assume also that

$$|\nabla f| \le CG(r)^{1/2} \tag{2.40}$$

Then, the full Omori–Yau maximum principle for the f–Laplacian holds on M.

Proof. Let h be as in the proof of Theorem 2.10. Then out of $\mathrm{cut}(o)$ we have that $r \in C^2$ and satisfies

$$\begin{aligned}\Delta_f r \le&(m-1)\frac{h'}{h} + |\nabla r|\,|\nabla f|\\ &\le (m-1)\frac{h'}{h} + CG(r)^{1/2} \le DG(r)^{1/2},\end{aligned}$$

and thus

$$\Delta_f r^2 = 2 + 2r\Delta_f r \le 2 + 2rG(r)^{1/2} \le CrG(r)^{1/2},$$

We want now to apply Theorem 2.20 with $\gamma = r^2$. An ispection of the proof shows that we need γ to be C^2 only in a neighborhood of the points of the sequence $\{x_k\}$. If only a finite number of points x_k of the sequence stays in $\mathrm{cut}(o)$, we can overcome the problem by passing to a subsequence. Otherwise we need to make some further considerations. This will be done with an adaptation of the Calabi trick.

Let $x_k \in \mathrm{cut}(o)$ and let σ be a minimizing geodesic between o and x_k. Up to translate the origin o to $o_k = \sigma(\varepsilon_k)$ we have that $x_k \notin \mathrm{cut}(o_k)$. Defining

$r_k(x) = d(o_k, x)$, since $r(x) \leq \varepsilon_k + r_k(x)$ for every $x \in M$ and since by (2.35)(ii) G is non–decreasing on $[0, +\infty)$, we have that

$$-G(r(x)) \geq -G_k(r_k) := -G(r_k(x) + \varepsilon_k).$$

Hence, by (2.39) and (2.40), we have that

$$\mathrm{Ric}_f \geq -G_k(r_k), \tag{2.41}$$

and

$$|\nabla f| \leq C G_k(r_k)^{\frac{1}{2}}. \tag{2.42}$$

If we define h_k to be the solution on $\mathbb{R}_0^+$ of the Cauchy problem

$$\begin{cases} h_k'' - G_k h_k = 0 \\ h_k(0) = 0, \ h_k'(0) = 1. \end{cases} \tag{2.43}$$

we have that

$$\frac{h_k'}{h_k} \leq D_k G_k(r_k)^{\frac{1}{2}}, \tag{2.44}$$

for some constant $D_k > 0$ sufficiently large. Using (2.41), (2.42) and (2.44) we then obtain

$$\Delta_f r_k \leq D G_k(r_k)^{\frac{1}{2}},$$

for some constant $D > 0$, and thus

$$\Delta_f r_k^2 \leq C r_k G_k(r_k)^{\frac{1}{2}}.$$

Applying the proof of Theorem 2.20 with $\gamma = r_k^2$ and taking the limit as $\varepsilon_k \to 0$ we obtain hence the validity of (2.36) along x_k. □

Corollary 2.22 (Corollary 11 in [**60**])**.** *Let $(M^m, \langle\, ,\, \rangle, e^{-f}\mathrm{dvol})$ be a complete weighted manifold such that*

$$Ric_f^k(\nabla r, \nabla r) \geq -(m + k - 1) G(r) \tag{2.45}$$

for a smooth positive function G satisfying (2.35)*, even at the origin. Then the full Omori-Yau maximum principle for the f–Laplacian holds on M.*

Proof. Let h be as in the proof of Theorem 2.10. Then, by Theorem 2.3, the inequality

$$\Delta_f r \leq (m + k - 1)\frac{h'}{h} \leq C G(r)^{\frac{1}{2}},$$

holds pointwise in $M \setminus (\mathrm{cut}(o) \cup \{o\})$ for some constant C. Thus, arguing as in the proof of Corollary 2.21 to deal with the cut points and applying Theorem 2.20, we obtain the thesis. □

As in the classical case, a way to use the Omori–Yau maximum principle to obtain analytic results is that of proving an “a–priori” estimate for a class of semilinear PDEs. In Chapter 5, we shall apply this estimate to obtain a triviality result for quasi–Einstein manifolds under L^p conditions. The following theorem is a weighted version of Theorem 1.31 in [**80**], which can be proved by minor changes to the proof of this latter.

Theorem 2.23 (Theorem 10 in [**60**])**.** *Assume on the complete weighted manifold* $(M, \langle\, ,\, \rangle, e^{-f}\mathrm{dvol})$ *the validity of the full Omori–Yau maximum principle for the* f*–Laplacian. Let* $v \in C^2(M)$ *be a solution of the differential inequality*

$$\Delta_f v \geq \Phi(v, |\nabla v|),$$

with $\Phi(t, y)$ *continuous in* t*,* C^2 *in* y *and such that*

$$\frac{\partial \Phi}{\partial y}(t, y) \geq 0.$$

Set $\varphi(t) = \Phi(t, 0)$*. Then a sufficient condition to guarantee that*

$$v^* = \sup_M v < +\infty$$

is the existence of a continuous function F *positive on* $[a, +\infty)$ *for some* $a \in \mathbb{R}$ *satisfying*

$$\left\{ \int_a^t F(s)ds \right\}^{-\frac{1}{2}} \in L^1(+\infty), \tag{2.46}$$

$$\limsup_{t \to +\infty} \frac{\int_a^t F(s)ds}{tF(t)} < +\infty, \tag{2.47}$$

$$\liminf_{t \to +\infty} \frac{\varphi(t)}{F(t)} > 0 \tag{2.48}$$

and

$$\liminf_{t \to +\infty} \frac{\left\{ \int_a^t F(s)ds \right\}^{\frac{1}{2}}}{F(t)} \left.\frac{\partial \Phi}{\partial y}\right|_{(t,0)} > -\infty. \tag{2.49}$$

Furthermore in this case

$$\varphi(v^*) \leq 0.$$

For the scalar curvature estimates we will present in Section 4.1 we need also the following "a–priori" estimate for weak solutions of semi–linear elliptic inequalities under volume assumptions. It is an adaptation of Theorem B in [**78**].

Theorem 2.24 (Theorem B in [**78**] and Theorem 12 in [**84**])**.** *Let* $\left(M, \langle\, ,\, \rangle, e^{-f}\mathrm{dvol}\right)$ *be a complete, weighted manifold. Let* $a(x)$, $b(x) \in C^0(M)$*, set* $a_-(x) = \max\{-a(x), 0\}$ *and assume that*

$$\sup_M a_-(x) < +\infty$$

and

$$b(x) \geq \frac{1}{Q(r(x))} \text{ on } M,$$

for some positive, non–decreasing function $Q(t)$ *such that* $Q(t) = o\left(t^2\right)$*, as* $t \to +\infty$*. Assume furthermore that, for some* $H > 0$*,*

$$\frac{a_-(x)}{b(x)} \leq H, \text{ on } M.$$

Let $u \in Lip_{loc}(M)$ *be a non–negative solution of*

$$\Delta_f u \geq a(x)\, u + b(x)\, u^{\sigma},$$

weakly on $\left(M, e^{-f}d\mathrm{vol}\right)$*, with* $\sigma > 1$*. If*

$$\liminf_{r\to+\infty} \frac{Q(r)\log \mathrm{vol}_f(B_r)}{r^2} < +\infty,$$

then

$$u(x) \leq H^{\frac{1}{\sigma-1}}, \text{ on } M.$$

Proof. We have only to verify that the integral inequality stated in Lemma 1.5 on page 1309 of [**78**] holds with respect to the weighted measure $e^{-f}d$vol. This in turn can be deduced exactly as in [**78**] provided (the weighted version of) inequality (1.21) on page 1310 is satisfied. Now, by assumption, for every compactly supported $\rho \in W^{1,2}_{loc}\left(M, e^{-f}d\mathrm{vol}\right)$, $\rho \geq 0$, we have

$$-\int \langle \nabla u, \nabla \rho\rangle\, e^{-f}d\mathrm{vol} \geq \int (au\rho + bu^{\sigma}\rho)\, e^{-f}d\mathrm{vol}.$$

Therefore, the desired inequality (1.21) follows by taking

$$\rho = \lambda(u)\, \psi^{2(\alpha+\sigma-1)} u^{\alpha-1}$$

with $\alpha \geq 2$. □

It is well known that a non–negative, L^p subharmonic function, $1 < p < +\infty$, on a complete Riemannian manifold must be constant, [**98**]. This classical Liouville–type theorem has been extended in various directions to both linear and non–linear operators. Here we recall the following version for the f–Laplacian established in [**81**], Theorem 1.1. See also [**82**]. Recently, somewhat less general forms of this result have been independently rediscovered in [**67**], [**74**], [**75**].

Theorem 2.25 (Theorem 1.1 in [**82**])**.** *Let* $\left(M, \langle\, ,\, \rangle, e^{-f}d\mathrm{vol}\right)$ *be a geodesically complete weighted manifold. Assume that* $u \in Lip_{loc}(M)$ *satisfy*

$$u\Delta_f u \geq 0, \text{ weakly on } M. \tag{2.50}$$

If, for some $p > 1$,

$$\frac{1}{\int_{\partial B_r} |u|^p\, e^{-f}d\mathrm{vol}_{m-1}} \notin L^1(+\infty), \tag{2.51}$$

then u *is constant.*

Remark 2.26. Observe that if $u \in L^p\left(M, e^{-f}d\mathrm{vol}\right)$ then condition (2.51) is satisfied. Indeed, by the Cauchy–Schwarz inequality, we have that $\forall R > 0$ and $r > R$,

$$(r-R)^2 \leq \left(\int_R^r \int_{\partial B_s} |u|^p e^{-f}d\mathrm{vol}_{m-1}\, ds\right)\left(\int_R^r \frac{1}{\int_{\partial B_s} |u|^p e^{-f}d\mathrm{vol}_{m-1}}\, ds\right).$$

Hence, by the co–area formula,

$$(r-R)^2 \le \left(\int_{B_r\setminus B_R} |u|^p e^{-f} d\mathrm{vol}\right)\left(\int_R^r \frac{1}{\int_{\partial B_s} |u|^p e^{-f} d\mathrm{vol}_{m-1}}\, ds\right).$$

Taking now the limit as $r \to \infty$ and using the fact that $u \in L^p(M, e^{-f}d\mathrm{vol})$ we obtain (2.51).

Note also that no sign condition is required on u. Moreover, if the locally Lipschitz function u satisfies both $\Delta_f u \ge 0$ and the non-integrability condition (2.51) then, applying Theorem 2.25 to $u_+ = \max\{u, 0\}$, gives that either u is constant or $u \le 0$.

As in the non–weighted setting, a L^1–Liouville–type theorem for f–subharmonic functions is in general false if we do not require some extra assumptions. Indeed in the following adaptation of an example of P. Li and R. Schoen, [**53**] (see also [**87**]) we construct an example of a non–costant, L^1, f–subharmonic function on a complete manifold.

Example 2.27. Let $\sigma \in C^\infty([0,+\infty))$ be a positive function such that $\sigma(t) = t$ for $t \in [0,1]$ and define

$$\langle\,,\,\rangle = dr^2 + \sigma^2(r)d\vartheta^2,$$

where (r,ϑ) are the polar coordinates on $\mathbb{R}^m \setminus \{0\} = (0,+\infty) \times S^{m-1}$, and $d\vartheta^2$ denotes the standard metric on S^{m-1}. Clearly, $\langle\,,\,\rangle$ extends to a smooth complete metric on $\mathbb{R}^m$. We now consider the weighted manifold $(\mathbb{R}^m, \langle\,,\,\rangle, e^{-f}d\mathrm{vol})$, for some smooth radial function $f = f(r)$ on $\mathbb{R}^m$. Next, let $a \in C^0([0,+\infty))$ be a non–negative continuous function such that $a(t) \equiv 1$ for $t \in [0,1]$. We define the non-negative function

$$u(x) = \int_0^{r(x)} e^{f(t)}\sigma(t)^{-(m-1)}\left(\int_0^t a(s)\sigma(s)^{m-1}ds\right)dt,$$

where $r(x)$ denotes the distance function from 0. Since u is radial for ease of notation we will write $u(r)$. It is easily verified that $u \in C^2$ and satisfies

$$\Delta_f(u(r)) = e^{f(r)}a(r) \ge 0$$

on $(\mathbb{R}^m, \langle\,,\,\rangle)$. Thus u is a noncostant f–subharmonic function.

To construct the required example, we fix $T_0 > 1$, and choose the function $a(t)$ and $\sigma(t)$ so as to satisfy the further conditions

$$a(t) = 0 \quad \sigma(t) = (t\log^\varepsilon t)^{-\frac{1}{m-1}} e^{-\frac{t^2\log^\varepsilon t}{m-1}}$$

on $[T_0,+\infty)$ for some $\varepsilon > 0$. Moreover assume that $f'(r) = o(r\log^\varepsilon r)$ as $r \to +\infty$. Inserting these in the definition of u, we deduce, using l'Hopital's rule, that there exist constants C_1, C_2 such that

$$u(r) \sim C_1 e^{f(r)} e^{r^2\log^\varepsilon r}$$

and

$$\int_{\partial B_r} u e^{-f(t)} d\mathrm{vol}_{m-1} \sim \frac{C_2}{r\log^\varepsilon r}$$

as $r \to +\infty$. Thus, if $\varepsilon > 1$, $0 \le u \in L^1(\mathbb{R}^m, e^{-f}d\mathrm{vol})$ is a non–constant, f–subharmonic function on M.

This explain the role of assumptions (2.52) in the following result, which follows from the proof of Theorem 4.3 in [**82**].

Theorem 2.28. *Let* $\left(M, \langle\, ,\, \rangle, e^{-f}\mathrm{dvol}\right)$ *be a geodesically complete weighted manifold. Let* $0 \le u \in Lip_{loc}(M)$ *be a weak solution of* $\Delta_f u \ge 0$ *satisfying*

$$\text{(2.52)} \qquad \textit{(i) } u \in L^1(M, e^{-f}\mathrm{dvol}), \qquad \textit{(ii) } u(x) = O\left(e^{\alpha r(x)^{2-\epsilon}}\right),$$

as $r(x) \to +\infty$*, for some constants* $\alpha, \epsilon > 0$*. Then* u *is constant.*

Proof. Arguing as in [**82**] Theorem 4.3 shows that if u is non-constant then (4.20) therein holds, and therefore, for any $\beta > 0$,

$$\frac{1}{\int_{\partial B_r} u(1+\log(1+u))(1+\log^{1+\beta}(1+\log(1+u)))e^{-f}\mathrm{dvol_{m-1}}} \in L^1(+\infty).$$

Using the pointwise bounds on u shows that

$$\frac{1}{\int_{\partial B_r} r^{2-\epsilon}(1+\log^{1+\beta} r)ue^{-f}\mathrm{dvol_{m-1}}} \in L^1(+\infty),$$

and therefore, by Proposition 1.3 in [**87**],

$$\frac{r}{\int_{B_r} r^{2-\epsilon}(1+\log^{1+\beta} r)ue^{-f}\mathrm{dvol}} \in L^1(+\infty).$$

Since $u \in L^1(M, e^{-f}\mathrm{dvol})$ this yields a contradiction. □

In particular, applying the theorem to the positive part $u_+ = \max\{u, 0\}$ of the given solution u yields the following

Corollary 2.29. *Let* $\left(M, \langle\, ,\, \rangle, e^{-f}\mathrm{dvol}\right)$ *be a geodesically complete weighted manifold. If* $u \in Lip_{loc}(M) \cap L^1\left(M, e^{-f}\mathrm{dvol}\right)$ *is a solution of* $\Delta_f u \ge 0$ *satisfying* $u(x) \le Ce^{\alpha r(x)^{2-\epsilon}}$*, for some constants* $C, \alpha, \epsilon > 0$*, then either* u *is constant or* $u \le 0$*.*

We end this section making some further observations about L^1–Liouville–type theorems which will be useful in the next chapters.

Following classical terminology in linear potential theory we have the following

Definition 2.30. *A weighted manifold* $\left(M, \langle\, ,\, \rangle, e^{-f}\mathrm{dvol}\right)$ *is said to be* f*–parabolic if every solution of* $\Delta_f u \ge 0$ *satisfying* $u^* = \sup_M u < +\infty$ *must be identically constant.*

Equivalently, $\left(M, \langle\, ,\, \rangle, e^{-f}\mathrm{dvol}\right)$ is f–non–parabolic if and only if Δ_f possesses a positive, minimal Green kernel $G_f(x, y)$. It can be shown that a sufficient condition for $\left(M, \langle\, ,\, \rangle, e^{-f}\mathrm{dvol}\right)$ to be f–parabolic is that M is geodesically complete and

$$\text{(2.53)} \qquad \mathrm{vol}_f(\partial B_r)^{-1} \notin L^1(+\infty).$$

All these facts can be easily established adapting to the diffusion operator Δ_f standard proofs for the Laplace–Beltrami operator; [**43**], [**87**].

Remark 2.31. By Theorem 2.8 and Proposition 2.12 we hence obtain the f–parabolicity of weighted manifolds $(M, \langle\, ,\, \rangle, e^{-f}d\mathrm{vol})$ satisfying one of the following curvature assumptions

(a) $Ric_f \geq \lambda > 0, \quad \lambda$ constant;
(b) $Ric_f \geq D(1+r)^{-\mu}$ with $D > 0$ and $0 \leq \mu \leq 1$.

Observe that it can be shown that f–parabolicity implies the validity of the weak Omori–Yau maximum principle for the operator Δ_f. This follows in a way similar to the case $f = 0$, noting that the weak maximum principle is equivalent to the property that if u is a non–negative bounded function satisfying $\Delta_f u \geq \mu u$ for some $\mu > 0$ then $u \equiv 0$ (see [**80**], Theorem 3.11).

Moreover, in case $f \equiv 0$, it is known that stochastic completeness with respect to the Brownian motion on $(M, \langle\, ,\, \rangle)$ is related to L^1–Liouville type properties for super–harmonic functions, [**42**]. Rephrasing these properties for the operator Δ_f, we have the following

Definition 2.32. *The L^1–Liouville property for Δ_f–superharmonic functions holds if every Lip_{loc} solution of $\Delta_f u \leq 0$ satisfying $0 \leq u \in L^1\left(M, \langle\, ,\, \rangle, e^{-f}d\mathrm{vol}\right)$ must be constant.*

Using exactly the same proof as in the case $f \equiv 0$, [**42**], one shows that this is equivalent to the fact that for some, hence for all, $x \in M$,

$$\int_M G_f(x,y)\, e^{-f}d\mathrm{vol}(y) = +\infty. \tag{2.54}$$

Recalling that the Green kernel G_f is related to the heat kernel p_f by the formula

$$G_f(x,y) = \int_0^{+\infty} p_f(t,x,y)\, dt, \tag{2.55}$$

from the above circle of ideas one obtains

Theorem 2.33 (Theorem 24 in [**84**])**.** *If the weak Omori–Yau maximum principle holds for Δ_f then the L^1–Liouville property for Δ_f–superharmonic functions holds.*

CHAPTER 3

Geometric structures on weighted manifolds

A celebrated question in Riemannian geometry, which goes back to the book by A. Besse, [**5**], asks if there are best Riemannian structures on a given Riemannian manifold. It might be natural to consider as "best" metrics those of constant curvature. On the other hand, on a simply connected manifold there is one and only one complete Riemannian metric structure of constant sectional curvature +1, 0 or −1. The corresponding manifolds are isometric to the standard Euclidean space, sphere and hyperbolic space. Moreover, on any compact manifold of any dimension there exist Riemannian metrics of constant scalar curvature. Thus, constancy of sectional curvature is too strong. On the other hand examples show that constancy of scalar curvature is too weak and we are left with constancy of Ricci curvature: a good candidate for a privileged metric on a given manifold is an Einstein metric.

In Chapter 2 we have introduced some other concepts of curvature that generalize Ricci curvature on a weighted manifold, namely the Lichnerowicz–Bakry–Emery's Ricci tensors

$$Ric_f^k = Ric + Hess(f) - \frac{1}{k} df \otimes df,$$

$0 < k \leq \infty$. Recently it has been found that these curvature tensors are strictly related with geometric objects whose importance is outstanding in mathemathics. Imposing the constancy of the Bakry–Emery Ricci tensor one introduce on the manifold an additional structure which goes under the name of (gradient) Ricci soliton structure ($k = \infty$) or k–quasi–Einstein structure ($k < \infty$). The importance of Ricci solitons is due to Perelman's solution of Poincaré conjecture. They correspond to "self–similar" solution to Hamilton's Ricci flow and often arise as limits of dilations of singularities which arise along the Ricci flow. On the other hand the importance of k–quasi–Einstein manifolds comes from a problem (proposed by A. Besse in [**5**]) on the existence of Einstein manifolds realized as warped products.

3.1. Ricci solitons

Definition 3.1. *Let $(M, \langle\, ,\, \rangle)$ be a Riemannian manifold. A Ricci soliton structure on M is the choice of a smooth vector field X (if any) satisfying the soliton equation*

$$\mathrm{Ric} + \frac{1}{2}\mathcal{L}_X \langle\, ,\, \rangle = \lambda \langle\, ,\, \rangle \tag{3.1}$$

for some constant $\lambda \in \mathbb{R}$.

Here, Ric denotes the Ricci tensor of M and $\mathcal{L}_X$ stands for the Lie derivative in the direction of X. The Ricci soliton $(M, \langle\,,\,\rangle, X)$ is said to be shrinking. steady or expanding according to whether the coefficient λ appearing in equation (3.1) satisfies $\lambda > 0$, $\lambda = 0$ or $\lambda < 0$.

In the special case where $X = \nabla f$ for some smooth function $f : M \to \mathbb{R}$, we say that $(M, \langle\,,\,\rangle, \nabla f)$ is a gradient Ricci soliton with potential f. In this case the soliton equation (3.1) reads

$$\operatorname{Ric} + \operatorname{Hess}(f) = \lambda \langle\,,\,\rangle\,. \tag{3.2}$$

Clearly, equations (3.1) and (3.2) can be considered as perturbations of the Einstein equation

$$\operatorname{Ric} = \lambda \langle\,,\,\rangle$$

and reduce to this latter in case X or ∇f are Killing vector fields. When $X = 0$ or f is constant we call the underlying Einstein manifold a trivial Ricci soliton. It is easy to show that Einstein, gradient Ricci solitons are either trivial or Ricci flat; see e.g. Theorem 4.7.

Although we are interested in the elliptic point of view, it is important to stress that Ricci solitons are closely related to Hamilton's Ricci flow, [**45**],

$$\frac{\partial}{\partial t} \langle\,,\,\rangle_t = -2 \operatorname{Ric}(\langle\,,\,\rangle_t). \tag{3.3}$$

Firstly, they arise as blow–up limits of the Ricci flow when singularities develop and this clearly justify the importance of understandig geometrical and topological properties of Ricci solitons and their classification, (for more background see e.g. [**11**]).

Secondly, as we are going to explain, there is a strict relationship between complete Ricci solitons and self–similar solutions of the Ricci flow. Recall that, if $(M, \langle\,,\,\rangle_t)$ is a smooth Riemannian manifold with a solution $\langle\,,\,\rangle_t$ of the Ricci flow on a time interval (a, b) containing 0, then $(M, \langle\,,\,\rangle_t)$ is called a self–similar solution, with initial metric $\langle\,,\,\rangle_0$, if there exist scalars $\sigma(t)$ such that $\langle\,,\,\rangle_t = \sigma(t)\varphi_t^*(\langle\,,\,\rangle_0)$, where φ_t is a one–parameter group of diffeomorphisms, which is generated by some vector field on M. In other words, these solutions move by diffeomorphisms and also shrink or expand by a time–dependent factor at the same time.

Let now $(M, \langle\,,\,\rangle_0, X)$ be a Ricci soliton satisfying (3.1). Consider the vector field $Y \in \mathfrak{X}(M \times J_1)$, given by

$$Y(x,t) = \frac{X(x)}{1 - 2\lambda t} + \frac{\partial}{\partial t},$$

where $J_1 \subseteq \mathbb{R}$ is defined by

$$J_1 = \begin{cases} \left(\frac{1}{2\lambda}, +\infty\right) & \text{if } \lambda < 0 \\ \mathbb{R} & \text{if } \lambda = 0 \\ \left(-\infty, \frac{1}{2\lambda}\right) & \text{if } \lambda > 0. \end{cases}$$

We observe that the requirement

$$|X| \le c(1 + r(x)), \tag{3.4}$$

for some constant $c > 0$, allows us to conclude that for every $t_0 < \frac{1}{2\lambda}$, $t_1 > \frac{1}{2\lambda}$ fixed, and for every $t \in \mathbb{R}$ there exist diffeomorphisms $\psi_t : M \times J_2 \to M \times J_2$ such that

$$\begin{cases} \frac{d}{dt}\psi_t = Y \circ \psi_t & \text{on } M \times J_2 \\ \psi_0 = id_{M \times J_2} & , \end{cases} \tag{3.5}$$

where

$$J_2 = \begin{cases} (t_1, +\infty) & \text{if } \lambda < 0 \\ \mathbb{R} & \text{if } \lambda = 0 \\ (-\infty, t_0) & \text{if } \lambda > 0. \end{cases}$$

Towards this aim we have to show that for any fixed $(y, \bar{t}) \in M \times J_2$ the maximal interval $J((y, \bar{t})) = (a((y, \bar{t})), b((y, \bar{t})))$ where the integral curve of Y issuing from $(y, \bar{t})$ is defined, coincides with J_1. Let us suppose by contradiction e.g. that, in case $\lambda \leq 0$, $b((y, \bar{t})) < +\infty$. By a well–known "escape" lemma (see e.g. Lemma 12.11 in [**52**]) we then know that the integral curve $\Phi_{(y,\bar{t})} : J((y, \bar{t})) \to M \times \mathbb{R}$ is a divergent curve. Now, let $\varepsilon = \inf \left\{ s \in J((y, \bar{t})) : \Phi_{(y,\bar{t})}(s) \in ({}^M B_1(y))^c \times J_2 \right\} > 0$ and for every $t < b((y, \bar{t}))$ consider the restriction

$$\gamma = \Phi_{(y,\bar{t})}\Big|_{[\varepsilon,t]} : [\varepsilon, t] \to M \times \mathbb{R}.$$

Then

$$l(\gamma) = \int_\varepsilon^t |\dot{\gamma}(s)| ds.$$

By (3.4) we have that outside ${}^M B_1(y)$

$$|X(x)| \leq 2cr(x).$$

Let $\widetilde{r}((x, t)) = d_{M \times \mathbb{R}}\left((x, t), (y, \bar{t})\right)$. Hence we have that in $({}^M B_1(y))^c \times J_2$

$$|Y((x, t))| \leq B\widetilde{r}\,((x, t))\,,$$

for some constant $B > 0$ which depends either on t_0 or on t_1 according to the sign of λ. Thus

$$\begin{aligned} \widetilde{r}(\gamma(t)) &= d_{M \times \mathbb{R}}(\gamma(t), (y, \bar{t})) \\ &\leq \widetilde{r}\,(\gamma(\varepsilon)) + d_{M \times \mathbb{R}}\,(\gamma(\varepsilon), \gamma(t)) \\ &\leq \widetilde{r}\,(\gamma(\varepsilon)) + l(\gamma) \\ &= \widetilde{r}\,(\gamma(\varepsilon)) + \int_\varepsilon^t |\dot{\gamma}(s)| ds \\ &= \widetilde{r}\,(\gamma(\varepsilon)) + \int_\varepsilon^t |Y(\gamma(s))| ds \\ &\leq \widetilde{r}\,(\gamma(\varepsilon)) + B \int_\varepsilon^t \widetilde{r}(\gamma(s)) ds. \end{aligned}$$

Writing this in terms of $I(t) = \int_\varepsilon^t \widetilde{r}(\gamma(s))ds$ and integrating the resulting differential inequality one obtains

$$\int_\varepsilon^t \widetilde{r}\,(\gamma(s))\, ds \leq B' e^{Bt},$$

for some constant $B' > 0$. Recalling that γ has to be divergent we thus get that $b((y,\bar{t})) = +\infty$, getting the desired contradiction.

By the definition of Y we have that for every $(x,t) \in M \times J_2$ the diffeomorphisms ψ_t, $t \in J_2$, can be written in the form

$$\psi_t(x,t) = (\varphi_t(x), t)\,,$$

for some diffeomorphisms $\varphi_t : M \to M$. Moreover we have that $\varphi_0 = id_M$ and $\frac{d}{dt}\varphi_t(x) = \frac{1}{1-2\lambda t}X(x)$. Let now $\langle\, , \rangle(t)$ be defined by

$$\langle\, , \rangle(t) = (1-2\lambda t)\varphi_t^* \langle\, , \rangle\,,$$

We then have that

$$\begin{aligned}
\frac{d}{dt}\langle\, , \rangle(t) &= \frac{d}{dt}\left((1-2\lambda t)\varphi_t^*\langle\, , \rangle\right) \\
&= -2\lambda\varphi_t^*\langle\, , \rangle + (1-2\lambda t)\varphi_t^*(\mathcal{L}_{\frac{X}{1-2\lambda t}}\langle\, , \rangle) \\
&= -2\varphi_t^*(\lambda\langle\, , \rangle - \frac{(1-2\lambda t)}{2}\mathcal{L}_{\frac{X}{1-2\lambda t}}\langle\, , \rangle) \\
&= -2\varphi_t^*(\lambda\langle\, , \rangle - \frac{1}{2}\mathcal{L}_X\langle\, , \rangle) \\
&= -2\varphi_t^*(\mathrm{Ric}(\langle\, , \rangle)) = -2(\mathrm{Ric}(\langle\, , \rangle(t))).
\end{aligned}$$

Thus, for every $t_0 < \frac{1}{2\lambda}$ and $t_1 > \frac{1}{2\lambda}$, we can define a self–similar solution of the Ricci flow $(M, \langle\, , \rangle(t))$, defined respectively on $(-\infty, t_0)$ if $\lambda > 0$, on $\mathbb{R}$ if $\lambda = 0$, and on $(t_1, +\infty)$ if $\lambda < 0$. In particular, a complete Ricci soliton $(M, \langle\, , \rangle, X)$ for which (3.4) holds always corresponds to the "self–similar" solution of the Ricci flow it generates.

Recently Z.–H. Zhang, [**101**], has observed that for any complete gradient Ricci soliton with potential function f, $X = \nabla f$ satisfies (3.4) (see Lemma 3.9 below), and hence we can carry out the construction above. In particular a complete gradient Ricci soliton always corresponds to the self–similar solution of the Ricci flow it generates. On the other hand, in general, for complete (non–compact) Ricci solitons which are not necessarily gradient Ricci solitons, we have no control on the growth rate of the norm of the soliton field X and the diffeomorphisms ψ_t may not exist.

Remark 3.2. Note that condition (3.4) turns out to be crucial for the study of nonnecessarily gradient Ricci solitons, see [**59**] for more details on this topic.

In this thesis we will focus our attention on geodesically complete, gradient Ricci solitons. Here are some typical examples, [**74**].

Example 3.3. The standard Euclidean space $(\mathbb{R}^m, \langle\, , \rangle, \nabla f)$ with

$$f(x) = \frac{1}{2}A\,|x|^2 + \langle x, B\rangle + C,$$

for arbitrary $A \in \mathbb{R}$, $B \in \mathbb{R}^m$ and $C \in \mathbb{R}$. Note that f is the essentially unique solution of the equation $\mathrm{Hess}(f) = A\langle\, , \rangle$ on $\mathbb{R}^m$. This follows by integrating on $[0,|x|]$ the equation

$$\frac{d^2}{ds^2}\left(f\left(vs\right)\right)=A,$$

with $v\in\mathbb{R}^m$ such that $|v|=1$. In fact, as we have seen in case (iii) of Theorem 1.1, a kind of converse also holds; see also [**67**], [**74**].

Example 3.4. The Riemannian product

$$\left(\mathbb{R}^m\times N^k,\langle\,,\,\rangle_{\mathbb{R}^m}+\langle\,,\,\rangle_{N^k},\nabla f\right)$$

where $\left(N^k,\langle\,,\,\rangle_{N^k}\right)$ is any k–dimensional Einstein manifold with Ricci curvature $\lambda\neq 0$, and $f\left(t,x\right):\mathbb{R}^m\times N^k\to\mathbb{R}$ is defined by

$$f\left(x,p\right)=\frac{\lambda}{2}\left|x\right|^2_{\mathbb{R}^m}+\langle x,B\rangle_{\mathbb{R}^m}+C, \tag{3.6}$$

with $C\in\mathbb{R}$ and $B\in\mathbb{R}^m$. Note that, according to the terminology of P. Petersen and W. Wylie, [**74**], often in literature a gradient Ricci soliton is said to be rigid if it is isometric to a quotient of one of these Riemannian products and f is as in (3.6).

As generalizations of Einstein manifolds, Ricci solitons enjoy some rigidity properties, which can take the form of classification, curvature estimates (metric rigidity), or alternatively, triviality of the soliton structure (soliton rigidity).

For instance, it has been known for some time that compact expanding Ricci soliton are necessarily trivial, [**33**]. In Chapter 5 we will present generalizations of this result to the complete, non–compact setting obtained recently in [**84**] and [**76**].

On the other hand, since the appearance of the seminal works by R. Hamilton, [**44**], and G. Perelman, [**73**], the classification of shrinking gradient Ricci solitons has become the subject of a rapidly increasing investigation. In this direction, we limit ourselves to quote the far–reaching [**12**] by H.–D. Cao, B.–L. Chen and X.–P. Zhu where a complete classification in the three–dimensional case is given, [**100**] by Z.–H. Zhang for the extension in the conformally flat, higher dimensional case, [**66**] by O. Monteanu and N. Sesum where, on the base of the rigidity works by P. Petersen and W. Wylie, [**75**], [**74**], and M. Fernández–López and E. García–Río, [**36**], the authors extend Zhang classification result to complete shrinkers with harmonic Weyl tensor, and the very recent [**13**] where a even weaker condition is considered (namely the Bach flatness).

For classification results in the steady case we refer to the papers by H.–D. Cao and Q. Chen, [**14**], and G. Catino and C. Mantegazza, [**22**], in which they indipendently give the classification of locally conformally flat steady gradient Ricci solitons. More recently, also in this case, weaker conditions were considered, see [**9**], [**26**], [**15**].

The classification of expanding Ricci solitons appears to be more difficult and relatively few results are known. For instance, the reader may consult

[**75**] for the case of constant scalar curvature expanders, and [**22**] for the case of nonnegative Ricci tensor.

As an instance of curvature estimates, we quote the recent papers by B.-L. Chen, [**24**], and by Z.-H. Zhang, [**101**], where it is shown that the scalar curvature of any gradient Ricci soliton is bounded below. In another direction, upper and lower estimates for the infimum of the scalar curvature of a gradient Ricci soliton obtained in [**84**] will be presented in Section 4.1.

3.1.1. Basic equations. The geometric quantities related to gradient Ricci solitons satisfy a number of differential identities that have been explored in several papers. We are interested in the elliptic point of view, therefore we limit ourselves to quoting the interesting papers [**33**] and [**74**], [**75**], which are particularly relevant to our investigation.

In the sequel we will use the following well known Bochner–type identities. For a proof (in a more general case) see Lemma 3.22 below.

Lemma 3.5. *Let $(M, \langle\, , \rangle, \nabla f)$ be a gradient Ricci soliton. Then*

$$\frac{1}{2}\Delta\left|\nabla f\right|^2 = \left|\mathrm{Hess}\,(f)\right|^2 - \mathrm{Ric}\,(\nabla f, \nabla f)$$

and

$$\frac{1}{2}\Delta_f\left|\nabla f\right|^2 = \left|\mathrm{Hess}\,(f)\right|^2 - \lambda\left|\nabla f\right|^2, \tag{3.7}$$

where λ is defined in (3.2).

In particular, combining Lemma 3.5 with Kato's inequality

$$\left|\mathrm{Hess}\,(f)\right|^2 \geq \left|\nabla\left|\nabla f\right|\right|^2,$$

we deduce the next

Corollary 3.6. *Let $(M, \langle\, , \rangle, \nabla f)$ be a gradient Ricci soliton. Then, $|\nabla f| \in Lip_{loc}(M)$ satisfies*

$$\left|\nabla f\right|\Delta\left|\nabla f\right| \geq -\,\mathrm{Ric}\,(\nabla f, \nabla f)$$

weakly on M and

$$\left|\nabla f\right|\Delta_f\left|\nabla f\right| \geq -\lambda\left|\nabla f\right|^2, \tag{3.8}$$

weakly on $\left(M, e^{-f}d\mathrm{vol}\right)$.

Thus, not surprisingly, from the Bochner equation viewpoint, the vector field $X = \nabla f$ behaves like a Killing field. Therefore, the standard Bochner technique implies that if $(M, \langle\, , \rangle, \nabla f)$ is a compact gradient Ricci soliton with $\mathrm{Ric} \leq 0$ then f must be constant and, hence, M is Einstein. Similar conclusions can be obtained in the non–compact setting and, in fact, a little amount of positive Ricci curvature is also allowed as explained in [**82**].

We shall also use the next computations concerning the scalar curvature of a gradient Ricci soliton; [**33**], [**74**]. For a proof (in a more general case) and for some related results we refer to Lemma 3.24.

Theorem 3.7. *Let $(M, \langle\, ,\,\rangle, \nabla f)$ be a gradient Ricci soliton with scalar curvature S and Ricci curvature Ric. Then*

$$\frac{1}{2}\Delta_f S = \lambda S - |\mathrm{Ric}|^2 . \tag{3.9}$$

3.1.2. Some function theoretic properties of gradient Ricci solitons. In the last part of this section we state explicitly, with the terminology introduced above, some analytical consequences of the results obtained in the general setting of weighted manifolds in Chapter 2. These results will represent essential tools to obtain the geometric results we are going to present in the next chapters.

For instance, as a consequence of Remark 2.18 we obtain the validity of the following

Corollary 3.8 (Corollary 10 in [**84**]). *Let $(M, \langle\, ,\,\rangle, \nabla f)$ be a geodesically complete Ricci soliton which is either shrinking, steady or expanding. Then, the weak Omori–Yau maximum principle for Δ_f holds on M.*

Before stating the next result, which is a further application of Theorem 2.20, we recall explicitly some known estimates for the potential and the gradient of the potential of a gradient Ricci soliton. For general gradient Ricci solitons Z.-H. Zhang, [**101**], (see also [**12**]), has proved the following

Lemma 3.9. *Let $(M, \langle\, ,\,\rangle, \nabla f)$ be a complete gradient Ricci soliton. Then there exist positive constant a and b depending only on the soliton such that*

$$|\nabla f| \leq b + |\lambda| r(x); \quad |f(x)| \leq \tfrac{|\lambda|}{2}(r(x))^2 + br(x) + a, \tag{3.10}$$

where $r(x)$ is the distance from some fixed point $o \in M$.

For shrinking gradient Ricci solitons a precise estimate is obtained by H.–D. Cao and Q. Chen in [**16**]: for these solitons the potential function has to grow quadratically in $r(x)$.

Lemma 3.10 (Theorem 1.1 in [**16**]). *Let $(M, \langle\, ,\,\rangle, \nabla f)$ be a complete gradient shrinking Ricci soliton. Then there exist positive constants c_1 and c_2 such that the potential function f satisfies the estimates*

$$\frac{\lambda}{2}(r(x) - c_1)^2 \leq f(x) \leq \frac{\lambda}{2}(r(x) + c_2)^2.$$

Using these estimates M. Fernàndez–Lòpez and E. Garcìa–Rìo in [**34**] have proved the following result.

Corollary 3.11 (Theorem 2.2. in [**34**]). *Let $(M, \langle\, ,\,\rangle, \nabla f)$ be a gradient shrinking Ricci soliton. Then, the full Omori-Yau maximum principle for Δ_f holds.*

Proof. Set $G(t) = t^2 + 1$ and $\gamma = f$. By Lemma 3.10 we get that

$$\gamma \to +\infty \text{ as } x \to \infty.$$

Now, recall that the scalar curvature of a gradient shrinking Ricci soliton is nonnegative, [**24**], and that on a gradient Ricci solitons holds that(see e.g. [**33**])

$$S + |\nabla f|^2 - 2\lambda f = c,$$

for some constant c. Using these facts we obtain that there exists $A > 0$ such that off a compact set

$$|\nabla\gamma| = |\nabla f| \leq A\sqrt{f} = A\sqrt{\gamma}.$$

Finally, using again Lemma 3.10, the fact that $R \geq 0$ and the trace of the soliton equation (3.2), we get that there exists $B > 0$ such that, off a compact set,

$$\begin{aligned} \Delta_f \gamma &= \Delta f - |\nabla f|^2 \leq \Delta f \\ &\leq m\lambda \leq \frac{\lambda}{2}(r - c_1)^2 \leq f \\ &\leq B\sqrt{f}\sqrt{f+1} = B\gamma^{1/2} G\left(\gamma^{1/2}\right)^{1/2}. \end{aligned}$$

We have thus proved that conditions (2.32), (2.33), (2.34) of Theorem 2.20 are satisfied. □

Remark 3.12. Very recently G. P. Bessa, S. Pigola and A. G. Setti in [**4**], using a refined version of Theorem 2.20 and Qian's estimates for $\Delta_f r$, [**85**], have proved that the full Omori–Yau maximum principle for Δ_f actually holds on every gradient Ricci soliton.

The "a–priori" estimate given in Theorem 2.24, in virtue of Theorem 2.8 specializes to the following

Corollary 3.13. *Let $(M, \langle\, ,\,\rangle, \nabla f)$ be a complete Ricci soliton and let $u \in Lip_{loc}(M)$ be a non-negative weak solution of*

$$\Delta_f u \geq au + bu^\sigma,$$

for some constants $a \in \mathbb{R}$, $b > 0$ and $\sigma > 1$. Then

$$u(x)^{\sigma-1} \leq \frac{\max\{-a, 0\}}{b}.$$

According to Remark 2.31 we have

Corollary 3.14 (Theorem 22 in [**84**]). *A complete, gradient shrinking Ricci soliton $(M, \langle\, ,\,\rangle, \nabla f)$ is f–parabolic.*

Moreover, combining Theorem 2.33 with Corollary 3.8 we conclude the validity of the next Liouville–type property of gradient Ricci solitons.

Corollary 3.15 (Theorem 25 in [**84**]). *Let $(M, \langle\, ,\,\rangle, \nabla f)$ be a complete, gradient Ricci soliton. Then the L^1–Liouville property for Δ_f–superharmonic functions holds.*

Remark 3.16. Since, by Corollary 3.14, shrinking solitons are f–parabolic, in this situation the same conclusion holds without any integrability assumption.

3.2. Ricci almost solitons

In this section we present an extension of the concept of Ricci soliton, introduced in [**76**], that, as we are going to explain, appears to be natural and meaningful. First of all we set the following

Definition 3.17. *We say that $(M, \langle\,,\,\rangle\,, \nabla f)$ is a gradient Ricci almost soliton (almost soliton for short) with potential f and soliton function λ if (3.2) holds on M with λ a smooth function on M.*

Clearly, the above definition generalizes the notion of gradient Ricci solitons. One could consider almost Ricci solitons which are not necessarily gradient, replacing the Hessian of f with the Lie derivative $\frac{1}{2}\mathcal{L}_X\langle\,,\,\rangle$ of the metric along a vector field, and study the properties of this new object. For instance, it is an interesting problem to find under which conditions an almost Ricci soliton is necessarily gradient. Here we are going to deal with gradient almost Ricci solitons.

We also note that generalizations in different directions have been recently considered. For instance G. Maschler in [**57**], replaced (1) by what the author calls the "Ricci–Hessian equation", namely,

$$\alpha \mathrm{Hess}\, f + \mathrm{Ric} = \gamma\langle\,,\,\rangle,$$

where α and γ are functions. Note that since the author is interested in conformal changes of Kähler–Ricci solitons which give rise to new Kähler metrics, the presence of the function α is vital in his investigation. Further generalizations will be considered in Section 3.3.

Extending to our new setting the soliton terminology, we say that the gradient Ricci almost soliton $(M, \langle\,,\,\rangle\,, \nabla f)$ is shrinking, steady or expanding if respectively λ is positive, null or negative on M. If λ has no definitive sign the gradient Ricci almost soliton will be called indefinite. In case f is constant the almost soliton is called trivial and if $\dim M \geq 3$ the underlying manifold $(M, \langle\,,\,\rangle)$ is Einstein by Schur's Theorem. This also suggests that for an almost soliton an appropriate terminology could be that of an almost Einstein manifold.

In view of the fact that the soliton function λ is not necessarily constant, one expects that a certain flexibility on the almost soliton structure is allowed and, consequently, the existence of almost solitons is easier to prove than in the classical situation. This feeling is confirmed in Subsection 3.2.1 below where we shall give a number of different examples of almost solitons, showing in particular that all the previous possibilities (shrinking, steady and expanding) with a non–constant soliton function λ can indeed occur. On the other hand, the rigidity result contained in Theorem 4.7 below indicates that almost solitons should reveal a reasonably broad generalization of the fruitful concept of classical soliton. In particular, one obtains that not every complete manifold supports an almost soliton structure; see Example 4.8.

3.2.1. Examples of Ricci almost solitons. Let $M = I \times_g \Sigma$ denote the g–warped product of the real interval $I \subseteq \mathbb{R}$ with $0 \in I$, and the Riemannian manifold $(\Sigma, (\ ,\ )_\Sigma)$ of dimension $\dim \Sigma = m$. Namely, the $(m+1)$–dimensional, smooth product manifold $I \times \Sigma$ is endowed with the metric

$$\langle\, ,\, \rangle = dt \otimes dt + g\,(t)^2\,(\ ,\ )_\Sigma\,,$$

where t is a global parameter of I and $g : I \to \mathbb{R}_0^+$ is a smooth function. Using the moving–frame formalism, the geometry of M can be described as follows.

Fix the index convention $1 \le i, j, k, l, t ... \le m$ and $1 \le \alpha, \beta, \gamma, ... \le m+1$. Let $\{e_j\}$ be a local orthonormal frame of Σ with dual frame $\{\theta^j\}$ so that $(,)_\Sigma = \sum \theta^j \otimes \theta^j$. We denote the corresponding connection 1–forms by $\theta^j_k = -\theta^k_j$ and the curvature 2–forms by $\Theta^i_j = -\Theta^j_i$. Accordingly, the structural equations of Σ are

$$\begin{aligned} d\theta^j &= -\theta^j_k \wedge \theta^k, \\ d\theta^j_i &= -\theta^j_k \wedge \theta^k_i + \Theta^j_i. \end{aligned}$$

Furthermore, the curvature forms are related to the (components of) the Riemann tensor by

$$\Theta^j_i = \frac{1}{2}\, {}^{\Sigma}R^j_{ikl}\theta^k \wedge \theta^l.$$

Let us introduce the local orthonormal coframe $\{\varphi^\alpha\}$ on M such that

$$\varphi^j = g\,(t)\,\theta^j,\ \varphi^{m+1} = dt.$$

The corresponding connection and curvature forms are denoted, respectively, by $\varphi^\alpha_\beta = -\varphi^\beta_\alpha$ and $\Phi^\alpha_\beta = -\Phi^\beta_\alpha = \frac{1}{2}\, {}^{M}R^\alpha_{\beta\delta\gamma}\varphi^\delta \wedge \varphi^\gamma$. A repeated use of exterior differentiations of φ^α and φ^α_β and of the structure equations of M and Σ, together with the well known characterization of the Levi Civita connection forms, yield

$$\begin{aligned} \varphi^k_j &= \theta^k_j \\ \varphi^k_{m+1} &= \frac{g'}{g}\varphi^k = -\varphi^{m+1}_k, \end{aligned} \tag{3.11}$$

and consequently,

$$\begin{aligned} \Phi^k_j &= -\left(\frac{g'}{g}\right)^2 \varphi^k \wedge \varphi^j + \Theta^k_j \\ \Phi^{m+1}_k &= \left\{\left(\frac{g'}{g}\right)^2 + \left(\frac{g'}{g}\right)'\right\} \varphi^k \wedge \varphi^{m+1} = \frac{g''}{g}\varphi^k \wedge \varphi^{m+1} = -\Phi^k_{m+1}. \end{aligned}$$

Let $\{E_\alpha\}$ denote the dual frame of $\{\varphi^\alpha\}$ so that $E_j = g\,(t)^{-1}\,e_j$. Then,

$${}^{M}\operatorname{Ric}_{\alpha\beta} = \Phi^\gamma_\alpha\,(E_\gamma, E_\beta)\,, \text{ and } {}^{\Sigma}\operatorname{Ric}_{kt} = g^2\Theta^j_k\,(E_j, E_t)\,.$$

It follows from (3.11) that

$$
\begin{aligned}
{}^{M}\mathrm{Ric}_{kt} &= \left\{ -(m-1)\left(\frac{g'}{g}\right)^2 - \frac{g''}{g} \right\} \delta_{kt} + \frac{1}{g^2}\ {}^{\Sigma}\mathrm{Ric}_{kt} \qquad (3.12)\\
{}^{M}\mathrm{Ric}_{m+1t} &= 0\\
{}^{M}\mathrm{Ric}_{m+1\ m+1} &= -m\frac{g''}{g}.
\end{aligned}
$$

In light of these relations, we have that M is Einstein with ${}^{M}\mathrm{Ric} = -mc\langle , \rangle$ if and only if

$$
{}^{\Sigma}\mathrm{Ric}_{kt} = \left\{ (m-1)\left(\frac{g'}{g}\right)^2 + \frac{g''}{g} - mc \right\} g^2\delta_{kt},
$$

and

$$
g'' = cg. \tag{3.13}
$$

Therefore

$$
{}^{\Sigma}\mathrm{Ric}_{kt} = -(m-1)\left(-g'^2 + cg^2\right)\delta_{kt}. \tag{3.14}
$$

We explicitly note that the general solution of (3.13) is given by

$$
g(t) = g'(0)\,\mathrm{sn}_{-c}(t) + g(0)\,\mathrm{cn}_{-c}(t), \tag{3.15}
$$

where

$$
\mathrm{sn}_k(t) = \begin{cases} \frac{1}{\sqrt{-k}}\sinh\left(\sqrt{-k}t\right) & \text{if} \quad k<0 \\ t & \text{if} \quad k=0 \\ \frac{1}{\sqrt{k}}\sin\left(\sqrt{k}t\right) & \text{if} \quad k>0. \end{cases}
$$

and

$$
\mathrm{cn}_k(t) = \mathrm{sn}'_k(t).
$$

Inserting (3.15) into (3.14) we obtain the following

Lemma 3.18. *Let $(\Sigma, (\ ,\ )_\Sigma)$ be a Riemannian manifold of dimension $m \geq 3$. Consider the warped product $M = I \times_g \Sigma$ where $0 \in I \subseteq \mathbb{R}$ and $g : I \to \mathbb{R}^+$ is a smooth function. Then, M is Einstein with*

$$
{}^{M}\mathrm{Ric} = -mc\langle\ ,\ \rangle, \quad c \in \mathbb{R},
$$

if and only

$$
g(t) = g'(0)\,\mathrm{sn}_{-c}(t) + g(0)\,\mathrm{cn}_{-c}(t) \tag{3.16}
$$

and Σ is Einstein with

$$
{}^{\Sigma}\mathrm{Ric} = -(m-1)\left\{-g'(0)^2 + cg(0)^2\right\}(\ ,\ )_\Sigma. \tag{3.17}
$$

Now, consider a smooth function $f : M \to \mathbb{R}$ of the form $f(t,x) = f(t)$. Its Hessian expresses as

$$
\mathrm{Hess}(f) = f'\frac{g'}{g}\sum \varphi^k \otimes \varphi^k + f''\varphi^{m+1} \otimes \varphi^{m+1}. \tag{3.18}
$$

Thus, in case M is an Einstein manifold with ${}^{M}\mathrm{Ric} = -mc\langle\, ,\rangle$ (hence Σ is so), the almost Ricci soliton equation on M with respect to the potential $f(x,t) = f(t)$ reads

$$\begin{cases} f'\frac{g'}{g} - mc = \lambda \\ f'' - mc = \lambda. \end{cases}$$

Integrating this latter we deduce

$$\begin{cases} f(t) = a\int_0^t g(s)\,ds + b \\ \lambda(t) = ag'(t) - mc, \end{cases} \tag{3.19}$$

for some constants $a, b \in \mathbb{R}$. Summarizing, we have obtained the following examples of Einstein almost Ricci solitons.

Proposition 3.19. *Let $g(t) : I \to \mathbb{R}^{+}$ be the smooth function defined in (3.16), $0 \in I \subseteq \mathbb{R}$. Let $(\Sigma, (\, ,)_{\Sigma})$ be an m–dimensional Einstein manifold satisfying (3.17). Then, the warped product $M = I \times_g \Sigma$ is Einstein with ${}^{M}\mathrm{Ric} = -mc\langle\, ,\rangle$ and it is an almost Ricci soliton with potential $f(t)$ and soliton function $\lambda(t)$ defined in (3.19).*

Now, suppose that we are given a warped product $M = I \times_g \Sigma$ where $(\Sigma, (\, ,))$ is an m–dimensional Einstein manifold and $0 \in I$. If $m \geq 3$, then, for some constant a,

$${}^{\Sigma}\mathrm{Ric} = -(m-1)\,a\,(\, ,)_{\Sigma}\,.$$

According to Lemma 3.18 in order that M be Einstein with ${}^{M}\mathrm{Ric} = -mc\langle, \rangle$ for some $c \in \mathbb{R}$, g must be given by (3.16) and then $cg(0)^2 - g'(0)^2 = a$. Therefore if (3.16) is not satisfied, then M is not Einstein. We consider a function $f(x,t) = f(t)$, so that, using (3.12) and (3.18) we see that to give $M = I \times_g \Sigma$ the structure of an almost soliton we need to solve the system

$$\begin{cases} f'\frac{g'}{g} = \lambda + (m-1)\left(\frac{g'}{g}\right)^2 + \frac{g''}{g} + \frac{(m-1)a}{g^2} \\ f'' = \lambda + m\frac{g''}{g} \end{cases} \tag{3.20}$$

on I. Subtracting the first equation from the second we obtain

$$\left(\frac{f'}{g}\right)' = (m-1)\,\frac{gg'' - (g')^2 - a}{g^3} = (m-1)\,h(t)$$

on I, and integrating

$$f(t) = B + \int_0^t g(s)\left[A + (m-1)\int_0^s \frac{g''g - (g')^2 - a}{g^3}dx\right]ds \tag{3.21}$$

for some constants $A, B \in \mathbb{R}$. Going back to (3.20) we then deduce

$$\lambda(t) = -(m-1)\,\frac{(g')^2 + a}{g^2} - \frac{g''}{g} + g'\left[A + (m-1)\int_0^t \frac{g''g - (g')^2 - a}{g^3}dx\right]. \tag{3.22}$$

Summarizing we have obtained the following new set of examples.

Example 3.20. Let $M = I \times_g \Sigma^m$ where $(\Sigma^m, (\ ,\ )_\Sigma)$ is an Einstein manifold satisfying ${}^\Sigma \operatorname{Ric} = -(m-1)\, a\, (\ ,\ )_\Sigma$. Then, M supports an almost soliton structure $f' \frac{\partial}{\partial t}$ whith soliton function $\lambda(t)$ where $f(t)$ and $\lambda(t)$ are defined respectively in (3.21) and (3.22).

Remark 3.21. As observed above , if g does not satisfy (3.16), these almost solitons are not Einstein hence necessarily different from those produced in Proposition 3.19 above. We also note that if Σ is the standard $(m-1)$–dimensional sphere, and g is defined on $I = [-1, +\infty)$ satisfies $g^{(2k)}(-1) = 0$, $g'(-1) = 1$ then we obtain a model manifold in the sense of Greene and Wu (with radial variable $r = t + 1$), and the almost soliton structure, which is in general defined only on $(-1, +\infty)$ extends to $[-1, +\infty)$ provided the functions f and λ can be smoothly extended in $t = -1$. We note that expanding the function h as $t \to -1^+$ we obtain that

$$h(t) \sim \frac{-a - 1 + o\left((t-1)^3\right)}{(t-1)^3}.$$

Thus h is integrable in a neighborhood of $t = -1$ and f and λ can be extended to $t = -1$ if and only if $a = -1$.

3.2.2. Some basic formulas. We want now to prove some basic formulas for gradient Ricci almost solitons. Some of them are well known for solitons and have been recalled in Section 3.1, but we have chosen to reproduce computations here since in this more general setting λ is a function and significant extra terms appear along the way. Throughout this section computations are performed with the method of the moving frame in a local orthonormal coframe for the metric $\langle\ ,\ \rangle$.

Lemma 3.22. *Let $(M, \langle\ ,\ \rangle, \nabla f)$ be a gradient Ricci almost soliton. Then*

$$\frac{1}{2}\Delta_f |\nabla f|^2 = |Hess\,(f)|^2 - \lambda |\nabla f|^2 - (m-2)\langle \nabla\lambda, \nabla f\rangle . \tag{3.23}$$

Proof. We recall the defining equations

$$R_{ij} = \lambda\delta_{ij} - f_{ij}. \tag{3.24}$$

Taking covariant derivatives

$$R_{ij,k} = \lambda_k\delta_{ij} - f_{ijk}. \tag{3.25}$$

Tracing with respect to j and k

$$R_{ik,k} = \lambda_i - f_{ikk}. \tag{3.26}$$

Next tracing the second Bianchi identities

$$R_{ijkl,s} + R_{ijls,k} + R_{ijsk,l} = 0$$

with respect to i and s we have

$$R_{ijkl,i} = R_{jl,k} - R_{jk,l}$$

and tracing again with respect to j and l

$$2R_{ik,i} = S_k\ , \tag{3.27}$$

where S denotes the scalar curvature. Using the commutation relations

$$R_{ij,k} = R_{ji,k}$$

we then deduce

$$R_{ki,i} = \frac{1}{2} S_k \tag{3.28}$$

Using the commutation rule

$$f_{ijk} - f_{ikj} = R_{lijk} f_l$$

and (3.28) into (3.26) we finally obtain

$$\frac{1}{2} S_i = \lambda_i - f_{kki} - f_t R_{ti}. \tag{3.29}$$

Now, tracing (3.25) with respect to i and j yields

$$S_i = m\lambda_i - f_{kki} \tag{3.30}$$

so that, substituting into (3.29) gives

$$S_i = 2\,(m-1)\,\lambda_i + 2 f_k R_{ki}. \tag{3.31}$$

In particular, from (3.31) we obtain

$$\langle \nabla S, \nabla f \rangle = 2\,(m-1)\,\langle \nabla\lambda, \nabla f \rangle + 2\,\mathrm{Ric}\,(\nabla f, \nabla f)\,. \tag{3.32}$$

Next we recall Bochner formula

$$\frac{1}{2} \Delta \left|\nabla f\right|^2 = \left|Hess\,(f)\right|^2 + \mathrm{Ric}\,(\nabla f, \nabla f) + \langle \nabla\Delta f, \nabla f \rangle\,. \tag{3.33}$$

Tracing (3.24)

$$S = m\lambda - \Delta f$$

so that

$$\nabla\Delta f = m\nabla\lambda - \nabla S. \tag{3.34}$$

Inserting (3.34) into (3.33) and using (3.32)

$$\begin{aligned} \frac{1}{2} \Delta \left|\nabla f\right|^2 &= \left|Hess\,(f)\right|^2 + \mathrm{Ric}\,(\nabla f, \nabla f) + m\,\langle \nabla\lambda, \nabla f \rangle - \langle \nabla S, \nabla f \rangle \\ &= \left|Hess\,(f)\right|^2 - \mathrm{Ric}\,(\nabla f, \nabla f) - (m-2)\,\langle \nabla\lambda, \nabla f \rangle\,. \end{aligned}$$

On the other hand, using

$$\frac{1}{2} \left\langle \nabla \left|\nabla u\right|^2, X \right\rangle = Hess\,(u)\,(\nabla u, X)$$

and (3.24) from the above we obtain

$$\begin{aligned} \frac{1}{2} \Delta_f \left|\nabla f\right|^2 &= \frac{1}{2} \Delta \left|\nabla f\right|^2 - \frac{1}{2} \left\langle \nabla f, \nabla \left|\nabla f\right|^2 \right\rangle \\ &= \left|Hess\,(f)\right|^2 - (m-2)\,\langle \nabla\lambda, \nabla f \rangle \\ &\qquad - \mathrm{Ric}\,(\nabla f, \nabla f) - Hess\,(f)\,(\nabla f, \nabla f) \\ &= \left|Hess\,(f)\right|^2 - \lambda \left|\nabla f\right|^2 - (m-2)\,\langle \nabla\lambda, \nabla f \rangle\,, \end{aligned}$$

that is, (3.23). □

Corollary 3.23.

$$|\nabla f|\,\Delta_f\,|\nabla f| \geq -\lambda\,|\nabla f|^2 - (m-2)\,\langle\nabla\lambda,\nabla f\rangle \tag{3.35}$$

PROOF. From Kato's inequality

$$|Hess\,(f)|^2 \geq |\nabla\,|\nabla f||^2$$

Inserting into (3.23) we obtain (3.35). □

We let S denote the scalar curvature and W the Weyl tensor of $(M,\langle\,,\,\rangle)$.

Lemma 3.24. *Let $(M,\langle\,,\,\rangle,\nabla f)$ be a gradient Ricci almost soliton of dimension $m\geq 3$. Then*

$$\Delta_f R_{ik} = \Delta\lambda\delta_{ik} + (m-2)\,\lambda_{ik} + 2\lambda R_{ik} - \frac{2}{m-2}\left(|\mathrm{Ric}|^2 - \frac{S^2}{m-1}\right)\delta_{ik}$$
$$-\frac{2m}{(m-1)\,(m-2)}SR_{ik} + \frac{4}{m-2}R_{is}R_{sk} - 2W_{ijks}R_{sj}. \tag{3.36}$$

Therefore, tracing with respect to i and k

$$\frac{1}{2}\Delta_f S = \lambda S - |\mathrm{Ric}|^2 + (m-1)\,\Delta\lambda. \tag{3.37}$$

Remark 3.25. Note that for (3.37) we do not need the restriction $m\geq 3$. Indeed (3.37) can also be obtained by tracing (3.40) below for which it is not required $m\geq 3$.

PROOF (OF LEMMA 3.24). It follows from (3.25) and the commutations relations $f_{ijk} - f_{ikj} = R_{lijk}f_l$ that

$$R_{ik,j} - R_{jk,i} = f_s R_{ijks} + \lambda_j\delta_{ki} - \lambda_i\delta_{kj}, \tag{3.38}$$

and taking covariant derivatives we obtain the commutation relations

$$R_{ik,jt} - R_{jk,it} = f_{st}R_{ijks} + f_s R_{ijks,t} + \lambda_{jt}\delta_{ki} - \lambda_{it}\delta_{kj}. \tag{3.39}$$

Also, from the commutation relations for the second covariant derivative of R_{ik} we have

$$R_{ij,kl} - R_{ij,lk} = R_{it}R_{tjkl} + R_{jt}R_{tikl},$$

whence, contracting we obtain

$$R_{jk,ij} = R_{jk,ji} + R_{si}R_{sk} + R_{jiks}R_{sj}.$$

We now use (3.39) to obtain

$$\Delta R_{ik} = R_{ik,jj} = R_{jk,ij} + f_s R_{ijks,j} + f_{sj}R_{ijks} + \Delta\lambda\delta_{ki} - \lambda_{ik}.$$

On the other hand, from the second Bianchi identities we have

$$f_s R_{ijks,j} = R_{ik,s}f_s - R_{is,k}f_s$$

and inserting this into the above identity yields

$$\Delta R_{ik} = f_{sj}R_{ijks} - f_s R_{is,k} + f_s R_{ik,s}$$
$$+ R_{jk,ji} + R_{sk}R_{is} + R_{sj}R_{jiks} + \Delta\lambda\delta_{ik} - \lambda_{ik}.$$

Hence, from (3.27) and (3.24)

$$\Delta R_{ik} = \frac{1}{2}S_{ki} + \lambda R_{ik} + R_{sk}R_{is} + \Delta\lambda\delta_{ik} - \lambda_{ik} - 2R_{ijks}R_{sj} - R_{is,k}f_s + R_{ik,s}f_s. \tag{3.40}$$

We shall now deal with the sum

$$Z = \frac{1}{2}S_{ki} + R_{sk}R_{is} - R_{is,k}f_s. \tag{3.41}$$

Towards this aim we first observe that taking covariant derivative of (3.31) we have

$$\frac{1}{2}S_{ik} = f_{jk}R_{ij} + R_{ij,k}f_j + (m-1)\,\lambda_{ik}.$$

Substituting this into (3.41) and using the almost soliton equation (3.24) we obtain

$$Z = (m-1)\lambda_{ik} + \lambda R_{ik}.$$

Substituing into (3.40) we therefore obtain

(3.42)

$$\begin{aligned}\Delta_f R_{ik} =& \Delta R_{ik} - f_s R_{ik,s}\\ =&(m-1)\lambda_{ik} + 2\lambda R_{ik} + \Delta\lambda g_{ik} - \lambda_{ik} - 2R_{ijks}R_{js} + R_{ik,s}f_s - R_{ik,s}f_s\\ =&(m-2)\lambda_{ik} + 2\lambda R_{ik} + \Delta\lambda g_{ik} - 2R_{ijks}R_{js}.\end{aligned}$$

The conclusion now follows recalling the decomposition of the curvature tensor into its irreducible components.

$$\begin{aligned}R_{ijks} =& W_{ijks} + \frac{1}{m-2}\left(R_{ik}\delta_{js} - R_{is}\delta_{jk} + R_{js}\delta_{ik} - R_{jk}\delta_{is}\right)\\ &- \frac{S}{(m-1)(m-2)}\left(\delta_{ik}\delta_{js} - \delta_{is}\delta_{jk}\right).\end{aligned}$$

Substituting into (3.42) we obtain (3.36). □

Corollary 3.26. *Let $(M, \langle\, ,\, \rangle, \nabla f)$ be a conformally flat gradient Ricci almost soliton of dimension $m \geq 3$. Then*

$$\begin{aligned}\Delta_f R_{ik} =& \Delta\lambda\delta_{ik} + (m-2)\,\lambda_{ik} + 2\lambda R_{ik} - \frac{2}{m-2}\left(|\mathrm{Ric}|^2 - \frac{S^2}{m-1}\right)\delta_{ik}\\ &- \frac{2m}{(m-1)(m-2)}SR_{ik} + \frac{4}{m-2}R_{is}R_{sk}.\end{aligned}$$

Corollary 3.27. *Let $(M, \langle\, ,\, \rangle, \nabla f)$ be a gradient Ricci almost soliton of dimension $m \geq 3$ and let $T = \mathrm{Ric} - \frac{S}{m}\langle\, ,\, \rangle$ be the traceless Ricci tensor. Then*

$$\begin{aligned}\frac{1}{2}\Delta_f |T|^2 =& |\nabla T|^2 + 2\left(\lambda - \frac{m-2}{m(m-1)}S\right)|T|^2\\ &+ (m-2)\left\langle Hess\,(\lambda)\,, T\right\rangle + \frac{4}{m-2}tr\left(T^3\right) - 2W_{ijks}T_{js}T_{ik},\end{aligned} \tag{3.43}$$

with $T^3 = T \circ T \circ T$. *In particular, using Okumura's Lemma*

$$\frac{1}{2}\Delta_f |T|^2 \geq 2\left(\lambda - \frac{m-2}{m(m-1)}S\right)|T|^2 - \frac{4}{\sqrt{m(m-1)}}|T|^3 + (m-2)\langle Hess(\lambda), T\rangle - 2W_{ijks}T_{js}T_{ik}. \tag{3.44}$$

Proof. We compute

$$\Delta_f |T|^2 = 2|\nabla T|^2 + 2\langle T, \Delta T\rangle - \left\langle \nabla f, \nabla |T|^2 \right\rangle = 2T_{ik,l}T_{ik,l} + 2T_{ik}\Delta_f T_{ik}.$$

Using equations (3.36), (3.37) and the definition of T, we have

$$\begin{aligned}
\Delta_f T_{ik} &= \Delta_f R_{ik} - \frac{1}{m}\delta_{ik}\Delta_f S \\
&= \Delta\lambda\delta_{ik} + (m-2)\lambda_{ik} + 2\lambda R_{ik} - \frac{2}{m-2}|\mathrm{Ric}|^2\delta_{ik} \\
&+ \frac{2}{(m-2)(m-1)}S^2\delta_{ik} - \frac{2m}{(m-1)(m-2)}SR_{ik} \\
&+ \frac{4}{m-2}R_{is}R_{sk} - 2W_{ijks}R_{sj} \\
&- \frac{2}{m}\lambda S\delta_{ik} + \frac{2}{m}|\mathrm{Ric}|^2\delta_{ik} - \frac{2}{m}(m-1)\Delta\lambda\delta_{ik} \\
&= -\frac{m-2}{m}\delta_{ik}\Delta\lambda + (m-2)\lambda_{ik} + 2\lambda T_{ik} \\
&- \frac{2mS}{(m-1)(m-2)}T_{ik} - \frac{4}{m(m-2)}\delta_{ik}|\mathrm{Ric}|^2 \\
&+ \frac{4}{m-2}\big(T_{is}T_{sk} + \frac{S^2}{m^2}\delta_{ik} + \frac{2S}{m}T_{ik}\big) - 2W_{ijks}R_{js}.
\end{aligned}$$

Thus, recalling that T is trace free and that all the traces of the Weyl tensor vanish, we obtain

$$\begin{aligned}
T_{ik}\Delta_f T_{ik} &= 2\lambda|T|^2 + (m-2)\lambda_{ik}T_{ik} \\
&\quad - \frac{2(m-2)S}{m(m-1)}|T|^2 + \frac{4}{m-2}\mathrm{tr}(T^3) - 2W_{ijks}T_{js}T_{ik},
\end{aligned}$$

and (3.43) follows. Inequality (3.44) follows immediately, since by Okumura's Lemma, [**70**],

$$tr\left(T^3\right) \geq -\frac{m-2}{\sqrt{m(m-1)}}|T|^3.$$

□

3.2.3. Some function theoretic properties of gradient Ricci almost solitons. We can now state explicitly some consequences for Ricci almost solitons of the results we have obtained in Chapter 2. These clearly estend to this more general setting the properties of gradient Ricci solitons collected in section 3.1.2.

As a consequence of Remark 2.18 and Theorem 2.33 we obtain the validity of the following

Corollary 3.28. *Let $(M, \langle\,,\,\rangle, \nabla f)$ be a complete gradient Ricci almost soliton with soliton function λ. Then, the weak Omori–Yau maximum principle for Δ_f and the L^1–Liouville property for Δ_f–superharmonic functions holds on M provided that λ satisfies $\lambda \geq \lambda_* = \inf_M \lambda > -\infty$.*

Rephrasing Corollary 2.21 we also get this other result.

Corollary 3.29. *Let $(M, \langle\,,\,\rangle, \nabla f)$ be a gradient Ricci almost soliton with soliton function λ such that*

$$\lambda \geq -(m-1)G(r)\langle\,,\,\rangle$$

for a smooth positive function G satisfying (2.35), even at the origin. Assume also that

$$|\nabla f| \leq CG(r)^{1/2}.$$

Then the full Omori–Yau maximum principle for Δ_f holds on M.

Finally, according to Remark 2.31 we have the following result concerning the f–parabolicity of gradient Ricci almost solitons.

Corollary 3.30. *Let $(M, \langle\,,\,\rangle, \nabla f)$ be a complete gradient Ricci almost soliton with soliton function λ. Then M is f–parabolic if λ satisfies one of the following conditions:*

(a) $\lambda \geq \lambda_* = \inf_M \lambda > 0$;
(b) $\lambda \geq D(1+r)^{-\mu}$ *with* $D > 0$ *and* $0 \leq \mu \leq 1$

3.3. Quasi–Einstein manifolds

Definition 3.31. *A complete weighted manifold $\left(M^m, g_M, e^{-f}d\mathrm{vol}\right)$ is said to be quasi–Einstein if*

$$Ric + Hess(f) - \nu\, df \otimes df = \lambda g_M\,. \tag{3.45}$$

When $\nu = 0$, quasi–Einstein manifolds correspond to gradient Ricci solitons and when f is constant (3.45) gives the Einstein equation and we call the quasi–Einstein metric trivial. We notice that, for $\nu = \frac{1}{2-m}$, $m \geq 3$ the metric $\widetilde{g} = e^{-\frac{2}{m-2}f}g$ is Einstein. Indeed, from the expression of the Ricci tensor of a conformal metric, we get

$$\begin{aligned} Ric_{\widetilde{g}} &= Ric_g + Hess(f) + \frac{1}{m-2}df \otimes df + \frac{1}{m-2}(\Delta f - |\nabla f|^2)g \\ &= \frac{1}{m-2}(\Delta f - |\nabla f|^2 + (m-2)\lambda)e^{\frac{2}{m-2f}}\widetilde{g}, \end{aligned}$$

and Schur lemma applies. In particular, if g is also locally conformally flat, then $\widetilde{g}$ has constant curvature. Observe also that in case $\frac{1}{2-m} \leq \nu < 0$. the definition of quasi–Einstein metrics was used by D. Chen in [**25**] in the context of finding conformally Einstein product metrics on $M^m \times F^k$ for $\nu = \frac{1}{2-k-m}$, $k \in \mathbb{N} \cup \{0\}$.

Quasi–Einstein manifolds have been recently introduced by J. Case, Y.-S. Shu and G. Wei in [**17**]. In that work the authors focus mainly on the case $\nu \geq 0$. In case $\nu = \frac{1}{k}$ for some $k \in \mathbb{N}$, on the LHS of equation (3.45) we recover the k–Bakry–Emery Ricci tensor. In particular we have the following

Definition 3.32. *The metric g_M is said to be k–quasi–Einstein if the k–Bakry–Emery Ricci tensor satisfies the equation*

$$Ric_f^k = \lambda g_M, \tag{3.46}$$

for some $\lambda \in \mathbb{R}$.

This last situation is particularly relevant due to the link with Einstein warped products. Indeed in [**17**], following the results in [**51**], it is proved the following characterization of k–quasi–Einstein metrics as base manifolds of Einstein warped product metrics.

Theorem 3.33. *Let $M^m \times_u F^k$ be an Einstein warped product with Einstein constant λ, warping function $u = e^{-\frac{f}{k}}$ and Einstein fibre F^k. Then the weighted manifold $(M^m, g_M, e^{-f}\mathrm{dvol})$ satisfies the quasi–Einstein equation* (3.46). *Furthermore the Einstein constant μ of the fibre satisfies*

$$\Delta f - |\nabla f|^2 = k\lambda - k\mu e^{\frac{2}{k}f}. \tag{3.47}$$

Conversely if the weighted manifold $(M^m, g_M, e^{-f}\mathrm{dvol})$ satisfies (3.46)*, then f satisfies* (3.47) *for some constant $\mu \in \mathbb{R}$. Consider the warped product $N^{m+k} = M^m \times_u F^k$, with $u = e^{-\frac{f}{k}}$ and Einstein fibre F with ${}^F Ric = \mu g_F$. Then N is Einstein with ${}^N Ric = \lambda g_N$.*

The importance of this characterization relies on the fact that on the one hand it enables to translate results from one setting to the other and on the other hand it permits to furnish several examples of k–quasi–Einstein manifolds, $k < \infty$.

Example 3.34. In [**5**, Theorem 9.119] it is obtained the complete classification of Einstein warped products with one and two dimensional bases. This hence translates into a complete classification of m–dimensional k–quasi–Einstein manifolds when $m = 1, 2$, $k < \infty$ (see also [**46**]). This gives examples with $\lambda < 0$ and μ of arbitrary sign and $\lambda = 0$ and $\mu \geq 0$. Moreover in this latter case, all non–trivial examples have $\mu > 0$, while the trivial quasi–Einstein metrics with $\lambda = 0$ necessarily satisfy $\mu = 0$.

Example 3.35. Other examples are constructed by H. Lü, Don N. Page, and C. N. Pope in [**55**]. For $2 \leq k < \infty$ they construct non–trivial cohomogeneity one examples of k–quasi–Einstein metrics on some $\mathbb{S}^2$ and $\mathbb{R}^2$ bundles over Kähler Einstein metrics. These examples have $\mu > 0$ and the $\mathbb{S}^2$–bundles have $\lambda > 0$ while the $\mathbb{R}^2$ bundles have $\lambda = 0$. Observe that, since if $k < \infty$ and $\lambda > 0$ by Theorem 2.2 (see also Theorem 3.41 below) M is necessarily compact. Thus the maximum principle applied to (3.47) yields that μ has to be positive in this situation.

Example 3.36. It is well known that a compact locally conformally flat gradient shrinking Ricci soliton has constant curvature (see [**33**]). Such a conclusion cannot be extended to quasi–Einstein metrics. Indeed, C. Böhm, [**8**], has found Einstein metrics on $\mathbb{S}^{l+1} \times \mathbb{S}^k$ for $l, k \geq 2$ and $l + k \leq 8$ and these induce a k–quasi–Einstein metric on $\mathbb{S}^{l+1}$ and with the metric on $\mathbb{S}^{l+1}$ being conformally flat (see also [**46**]).

Example 3.37. In Proposition 4.2 of [**17**] it is given the following classification of non–trivial k–quasi–Einstein metrics which are Einstein at the same time. A complete k–quasi–Einstein metric $(M^m, g_M, e^{-f}d\mathrm{vol})$ is Einstein if and only if either M is isometric to $\mathbb{R}^m$ with the warped product structure $\mathbb{R} \times_{a^{-1}e^{ar}} N^{m-1}$, where N^{m-1} is Ricci flat and a is a constant, or to $(\mathbb{H}^m, dr^2 + a^{-2}\sinh^2(ar)g_{\mathbb{S}^{m-1}})$. The constant μ vanishes in the former case, while $\mu < 0$ in the latter case.

We can make a scheme of the examples we have presented that will be useful also to visualize the results we are going to present in the next chapters.

Table 3.1. Examples of quasi–Einstein manifolds

	$\lambda < 0$	$\lambda = 0$	$\lambda > 0$
$\mu < 0$	Example 3.34		Trivial
$\mu = 0$	Example 3.34		Trivial
$\mu > 0$	Example 3.34	Example 3.34	Example 3.35

Some further remarks on the case $k = 1$ are in order. Since for 1–dimensional manifolds $Ric = 0$, the characterization of Theorem 3.33 can be applied only if (3.47) holds with $\mu = 0$. Note that any 1–quasi–Einstein metric which has $\mu = 0$ (and so corresponds to a warped product Einstein metric) necessarily has constant scalar curvature $S \equiv (m-1)\lambda$. This follows simply by taking the trace of the quasi–Einstein equation (3.46) and using equation (3.47). Warped product Einstein metrics which correspond to these latters are more commonly known as *static metrics* and have been studied extensively due to their connections to scalar curvature, the positive mass theorem, and general relativity, (see e.g. [**2**], [**30**]).

Being another generalization of Einstein manifolds, quasi–Einstein manifolds exhibit a certain rigidity, also in the case $\nu > 0$. This is expressed, once again, by triviality and classification results and curvature estimates.

For instance, analogously to the case $\nu = 0$, D.–S. Kim and Y.–H. Kim in [**51**] prove that if $\lambda \leq 0$, compact k–quasi–Einstein manifolds are trivial. A generalization to the complete non–compact setting of this result, obtained in [**88**], will be presented in Section 5.2 where also some other triviality results obtained in [**20**], [**88**] and [**60**] are discussed.

For classification results we refer to the paper by G. Catino, C. Mantegazza, L. Mazzieri and M. Rimoldi, [**23**], where it is proved that any complete locally conformally flat quasi–Einstein manifold of dimension $m \geq 3$ is locally a warped product with $(m-1)$–dimensional fibers of constant curvature, and to [**46**] where C. He, P. Petersen, and W. Wylie reach the same conclusion, in the case when $0 < \nu < 1$, assuming a slightly weaker condition than locally conformally flatness. In another direction in [**46**] and [**47**] classification and rigidity results obtained in [**17**] are generalized allowing the k–quasi–Einstein manifold to have non–empty boundary.

For what concern curvature estimates we refer to Section 4.1 where estimates for the infimum of the scalar curvature for k–quasi–Einstein manifolds in the case of non–constant scalar curvature from [**88**] are presented (extending the previous work in [**17**]).

Finally it is worth to mention that, in the two recent preprints [**18**] and [**19**], J. Case has delineated a different perspective to look to quasi–Einstein metrics introducing the concept of conformally warped manifolds. This seems to offer a unifying and inspiring way to look to these geometrical structures.

3.3.1. Some formulas. We remind some basic formulas for k–quasi–Einstein manifolds which will be useful in the sequel. Recall that the defining equation of a k–quasi–Einstein manifold $(M^m, \langle\,,\,\rangle, e^{-f}d\mathrm{vol})$, $k<\infty$, is

$$Ric + Hess(f) - \frac{1}{k} df \otimes df = \lambda \langle\,,\,\rangle \,.$$

Taking the trace of this we get

$$S + \Delta f - \frac{1}{k}|\nabla f|^2 = m\lambda. \tag{3.48}$$

Analogously to the the structures we have presented in the preceeding sections we can obtain the following Bochner–type identities.

Lemma 3.38. *Let $(M^m, g_M, e^{-f}d\mathrm{vol})$ be a geodesically complete weighted manifold such that $\mathrm{Ric}^k_f = \lambda g_M$ for some $\lambda \in \mathbb{R}$ and $k < \infty$. Then*

$$\frac{1}{2}\Delta|\nabla f|^2 = |Hess(f)|^2 - \mathrm{Ric}(\nabla f, \nabla f) + \frac{2}{k}|\nabla f|\Delta f \tag{3.49}$$

and

$$\frac{1}{2}\Delta_f|\nabla f|^2 = |Hess(f)|^2 + \frac{2m-k}{k}\,\lambda|\nabla f|^2 - \frac{2}{k}\,S|\nabla f|^2 + \frac{2-k}{k^2}\,|\nabla f|^4. \tag{3.50}$$

Proof. Equation (3.49) is proven in Lemma 3.2 in [**17**].
Substituing (3.46) and (3.48) in (3.49) we compute

$$\begin{aligned}\frac{1}{2}\Delta|\nabla f|^2 \;&=\; |Hess(f)|^2 + Hess(f)(\nabla f, \nabla f) - \frac{1}{k}|\nabla f|^2 - \lambda|\nabla f|^2 \\ &\quad + \frac{2}{k}|\nabla f|^2\left(-S + \frac{1}{k}|\nabla f|^2 + m\lambda\right),\end{aligned}$$

and (3.50) follows easily. □

In particular, combining Lemma 3.38 with Kato's inequality

$$|\mathrm{Hess}\,(f)|^2 \geq |\nabla\,|\nabla f||^2\,,$$

we deduce

Corollary 3.39.

$$|\nabla f|\Delta_f|\nabla f| \geq \frac{2m-k}{k}\,\lambda|\nabla f|^2 - \frac{2}{k}\,S|\nabla f|^2 + \frac{2-k}{k^2}\,|\nabla f|^4. \tag{3.51}$$

Moreover, in the proof of scalar curvature estimates for k–quasi–Einstein manifolds, we will require the following formula obtained in [**17**], which generalizes to the case $k < \infty$ formula (3.37) for Ricci solitons, obtained previously by P. Petersen and W. Wylie, [**74**].

Lemma 3.40. *Let $Ric_f^k = \lambda g_M$, for some $\lambda \in \mathbb{R}$ and $k < \infty$. Set $\widetilde{f} = \frac{k+2}{k} f$. Then*

$$\frac{1}{2}\Delta_{\widetilde{f}} S = -\frac{k-1}{k}|Ric - \frac{1}{m} S g_M|^2 - \frac{m+k-1}{mk}(S - m\lambda)(S - \frac{m(m-1)}{m+k-1}\lambda). \tag{3.52}$$

The majority of the other result we are going to present relies in an essential way on Theorem 3.33, and hence on the study of equation (3.47).

3.3.2. Some geometric and functional theoretic properties of k–quasi–Einstein manifolds. We now apply the results presented in Chapter 2 to deduce some properties of k–quasi–Einstein manifolds, $k < \infty$. For instance, by Theorem 2.2 (see also Remark 6.9), we immediately get the following compactness result.

Corollary 3.41. *Let $(M, \langle\, ,\, \rangle, e^{-f}\mathrm{dvol})$ be a complete k–quasi–Einstein manifold with quasi–Einstein constant $\lambda > 0$. Then M is compact, $|\pi_1(M)| < +\infty$, and*

$$\mathrm{diam}(M) \leq \frac{\pi}{\sqrt{\frac{\lambda}{m+k-1}}}.$$

Next, as a consequence of Remark 2.18 and Theorem 2.33, we obtain the validity of this other

Corollary 3.42. *Let $(M, \langle\, ,\, \rangle, e^{-f}\mathrm{dvol})$ be a complete k–quasi–Einstein manifold, $k < \infty$. Then the weak Omori–Yau maximum principle for Δ_f and the L^1–Liouville property for Δ_f–superharmonic functions holds on M.*

Finally, choosing $G(t) = t^2 + \frac{|\lambda|+\varepsilon}{m+k-1}$, for some $\varepsilon > 0$, in Corollary 2.22, we conclude the validity of the following

Corollary 3.43. *Let $(M^m, \langle\, ,\, \rangle, e^{-f}\mathrm{dvol})$ be a k–quasi–Einstein manifold, $k < \infty$. Then the full Omori–Yau maximum principle for Δ_f holds on M*

CHAPTER 4

Metric rigidity

When asking a weighted manifold to support a gradient Ricci soliton or a k–quasi–Einstein structure, we are indirectly restricting its geometry. In this chapter we analyze rigidity phenomena that involve the underlying Riemannian structure and appear in various way, such as curvature estimates, classification results and gap theorems for some curvature quantities.

4.1. Scalar curvature estimates

Estimates for the scalar curvature of shrinking and expanding Ricci solitons, both from above and from below, were first obtained by P. Petersen and W. Wylie in [**74**] using equation (3.9). Nevertheless in their result only the case of constant scalar curvature is treated. B.–L. Chen, [**24**], and Z.–H. Zhang, [**101**], have shown with different techniques that the scalar curvature of any gradient Ricci soliton is bounded from below. In Corollary 4.2 below the results in [**74**] and [**24**], [**101**] are improved by no longer assuming the constancy of the scalar curvature and by showing an explicit bound. Namely we obtain upper and lower estimates for the infimum of the scalar curvature of a gradient Ricci soliton. Note that the lower bound for expanders has been independently obtained, with different methods, in [**99**].

Very recently some works appeared on arXiv in which lower bounds for the scalar curvature of gradient Ricci solitons are given under additional assumptions. The lower bounds are explicitly expressed in terms of the dimension of the manifold and the potential function or the distance function from a fixed point. In this direction we limit ourselves to quote [**29**] for the shrinking case and [**28**], [**37**] for the steady one.

In the next Theorem 4.1 we present scalar curvature estimates in the more general contest of Ricci almost solitons. Namely, we show that under a pointwise control on the soliton function, the scalar curvature of an almost soliton is bounded from below. Furthermore, the lower bound of the scalar curvature can be estimated both from above and from below and, applying some abstract structure theorems for domains of nontrivial solutions of the resulting differential equations, some strong rigidity at the endpoints occurs. As a consequence we recover scalar curvature estimates in the classical soliton case.

Theorem 4.1 (Theorem 0.4 in [**76**]). *Let $(M, \langle\, , \,\rangle, \nabla f)$ be a complete gradient Ricci almost soliton with scalar curvature S and soliton function λ such that $\Delta\lambda \le 0$ on M. Set*

$$S_* = \inf_M S, \quad \lambda_* = \inf_M \lambda, \quad \lambda^* = \sup_M \lambda.$$

(i) *If the almost soliton satisfies* $-\infty < \lambda_* \leq \lambda \leq 0$, $\lambda \not\equiv 0$ *(and in particular if the almost soliton is expanding), then* $m\lambda_* \leq S_* \leq 0$*. Moreover, if* $m \geq 3$ *and there exists* x_o *such that* $S(x_o) = S_* = m\lambda_*$*, then the soliton is trivial and* M *is Einstein; while if* $S(x_0) = S_* = 0$ *for some* $x_0 \in M$*, then* M *is Ricci flat and isometric to* $\mathbb{R}^m$*.*

(ii) *If the almost soliton is a steady soliton then* $S_* = 0$*. Morever, if* $m \geq 3$ *and there exists* x_0 *such that* $S(x_0) = 0$*, then* M *is a cylinder over a totally geodesic hypersurface.*

(iii) *If the almost soliton satisfies* $0 \leq \lambda$, $\lambda \not\equiv 0$ *(and in particular if the almost soliton is shrinking), then* $0 \leq S_* \leq m\lambda^*$*. Moreover if* $m \geq 3$ *and there exists* x_o *such that* $S(x_o) = S_* = 0$ *then* M *is isometric to* $\mathbb{R}^m$*. Finally if* $S_* = m\lambda^*$ *and* $(M, \langle\, ,\, \rangle, e^{-f}\mathrm{dvol})$ *is* f*–parabolic, then the almost soliton is trivial and* $(M, \langle\, ,\, \rangle)$ *is compact Einstein. This latter case occurs in particular if*

$$A^2 (1 + r(x))^{-\mu} \leq \lambda(x)$$

on M *for some* $A > 0$, $0 \leq \mu \leq 1$*.*

Note that the case $\mu = 0$ contains, of course, the soliton case. In particular, as a consequence, we deduce the validity of the following

Corollary 4.2 (Theorem 3 in [**84**], Theorem 3.4 in [**34**])**.** *Let* $(M, \langle\, ,\, \rangle, \nabla f)$ *be a geodesically complete gradient Ricci soliton with scalar curvature* S *and let* $S_* = \inf_M S$*.*

(i) *If* M *is an expanding Ricci soliton then* $m\lambda \leq S_* \leq 0$ *and* $S(x) > m\lambda$ *unless* M *is Einstein and the soliton is trivial; while if* $S(x_0) = S_* = 0$ *for some* $x_0 \in M$*, then* M *is Ricci flat and isometric to* $\mathbb{R}^m$*.*

(ii) *If* M *is a steady Ricci soliton then* $S_* = 0$*. Moreover, if* $m \geq 3$ *and there exists* x_0 *such that* $S(x_0) = 0$*, then* M *is a cylinder over a totally geodesic hypersurface.*

(iii) *If* M *is a shrinking Ricci soliton then* $0 \leq S_* \leq m\lambda$*. Moreover,* $S_* < m\lambda$ *unless* M *is compact Einstein and the soliton is trivial, and* $S(x) > 0$ *on* M *unless* $S(x) \equiv 0$ *on* M*, and* M *is isometric to* $\mathbb{R}^m$*.*

Proof (of Theorem 4.1). Since $|\mathrm{Ric}|^2 \geq S^2/m$ by the Cauchy–Schwarz inequality, and $\Delta\lambda \leq 0$ by assumption, (3.37) in Lemma 3.24 yields

$$\frac{1}{2}\Delta_f S = \lambda S - |Ric|^2 + (m-1)\,\Delta\lambda \leq \lambda S - \frac{S^2}{m}. \tag{4.1}$$

Note next that since $\mathrm{Ric}_f = \lambda \geq \lambda_* > -\infty$, by Proposition 2.12 we have

$$\mathrm{vol}_f(B_r) \leq C_1 e^{C_2 r^2},$$

for some positive constants C_1, C_2, and, in particular,

$$\liminf_{r\to\infty} \frac{\log \mathrm{vol}_f(B_r)}{r^2} \leq C_2 < +\infty. \tag{4.2}$$

Applying Theorem 2.24 to the function $S_- = \max\{-S, 0\}$, which is a weak solution of

$$\Delta_f S_- \geq 2\lambda S_- - \frac{2}{m}S_-^2,$$

with $a(x) = 2\lambda(x)$, $b(x) = \frac{2}{m}$, $\sigma = 2$, and deduce that

$$S_{-}(x) \leq \sup_{M} \frac{\lambda_{-}(x)}{1/m},$$

from which we conclude that

$$S(x) \geq \min\{m\lambda_{*}, 0\}.$$

In particular, $S_{*} \geq 0$ if $\lambda \geq 0$, and $S_{*} \geq m\lambda_{*}$ if $\lambda_{*} \leq \lambda \leq 0$.

Next, by Corollary 3.28, the weak Omori–Yau minimum principle for Δ_f holds. Therefore we may find a sequence $\{x_n\}$ such that $\Delta_f S(x_n) \geq -1/n$ and $S(x_n) \to S_{*}$. Computing the liminf of (4.1) along this sequence and setting $\overline{\lambda} = \liminf \lambda(x_n)$ we deduce that

$$0 \leq \overline{\lambda} S_{*} - S_{*}^2/m.$$

Thus, if $\overline{\lambda} = 0$, then $S_{*} = 0$, while if $\overline{\lambda} \neq 0$, then solving the inequality yields $m\overline{\lambda} \leq S_{*} \leq 0$ if $\overline{\lambda} < 0$ and $0 \leq S_{*} \leq m\overline{\lambda}$ if $\overline{\lambda} > 0$. Since obviously $\lambda_{*} \leq \overline{\lambda} \leq \lambda^{*}$, this gives the scalar curvature estimates in (i), (ii) and (iii).

We now suppose that the scalar curvature achieves one of its bounds and, according to the classification that we will prove in Theorem 4.7, we prove rigidity.

In case (i) using (4.1), we see that $S(x) \geq S_{*} \geq m\lambda_{*}$ satisfies

$$\frac{1}{2}\Delta_f S \leq -\frac{S}{m}(S - m\lambda) \leq -\frac{S}{m}(S - m\lambda_{*}) \tag{4.3}$$

on the open set $\Omega = \{x \in M : S(x) < 0\}$. Therefore, if $S(x_0) = S_{*} = m\lambda_{*} < 0$ for some x_0, we deduce that the function $u = S - m\lambda_{*} \geq 0$ achieves its minimum value $u(x_0) = 0$ and satisfies the differential inequality

$$\frac{1}{2}\Delta_f u + \lambda_{*} u \leq 0$$

on Ω. By the minimum principle, $u(x) = 0$ on the connected component Ω_0 of Ω containing x_0. It follows that the open set Ω_0 is also closed, thus $\Omega_0 = M$ and $u(x) = 0$ on M. This means that $S(x) = m\lambda_{*}$ is constant. Using this information into (4.1) we get that λ is constant. Going back to (4.1), by the equality case in the Cauchy–Schwarz inequality, we obtain

$$Ric = \lambda \langle\, ,\, \rangle\,,$$

showing that M is Einstein and the soliton is trivial. On the other hand, if $S(x_0) = S_{*} = 0$ for some x_0, we deduce again from (4.1) that $S(x) \geq S_{*} = 0$ satisfies

$$\frac{1}{2}\Delta_f S - \lambda S \leq 0.$$

and so, by the minimum principle $S(x) = 0$ is constant and all the inequalities in (4.1) become equalities. In particular by the equality case in the Cauchy–Schwarz inequality we obtain that M is Ricci flat. Using now case (a.2) of Theorem 4.7 M has to be isometric to the standard Euclidean plane.

Assume next that the soliton is steady, so that $\lambda \equiv 0$. Then $S(x) \geq S_{*} = 0$ solves

$$\frac{1}{2}\Delta_f S \leq 0.$$

Therefore, if $S(x_0) = 0$, arguing as above we conclude that M must be Ricci-flat and, by case (a.1) of Theorem 4.7, M is a cylinder over a Ricci-flat, totally geodesic hypersurfaces Σ.

Finally we consider case (iii). Then, $S(x) \geq S_* = 0$ satisfies

$$\frac{1}{2}\Delta_f S \leq \lambda S.$$

If $S(x_0) = 0$ for some $x_0 \in M$, by the minimum principle (see Theorem 2.13 or e.g. page 35 in [**31**]) we deduce that $S(x) = 0$ is constant and all the inequalities used in (4.1) become equalities. In particular, $|Ric|^2 = S^2/m = 0$ proving that M is Ricci–flat. By case (a.2) in Theorem 4.7, λ is a positive constant and M must be isometric to the standard Euclidean space.

It remains to prove the last statement. Suppose then that $S_* = m\lambda^* > 0$. Since $S \geq S_* = m\lambda^* \geq m\lambda > 0$, it follows that $m\lambda S - S^2 \leq 0$ on M. Thus from (4.1),

$$\Delta_f S \leq 0$$

and $S > 0$ is a nonnegative Δ_f–superharmonic function. It follows that, if $\left(M, \langle\, ,\, \rangle, e^{-f}d\mathrm{vol}\right)$ is f–parabolic, then $S = m\lambda^*$ is constant. Using (4.1) we immediately deduce $S\left(\lambda - \frac{S}{m}\right) = 0$ so that $\lambda = \frac{S}{m}$ is constant. From (4.1) we have that $|Ric|^2 = \frac{S^2}{m}$, and, again by the equality case in the Cauchy-Schwarz inequality

$$Ric = \lambda \langle\, ,\, \rangle,$$

with $\lambda > 0$. Thus M is Einstein and compact by Myers' Theorem. Now from (3.2) and the above considerations it follows that $Hess\,(f) = 0$ on M and compactness implies that f is constant. Finally, if $\lambda \geq A^2(1+r)^{-\mu}$ with $0 \leq \mu \leq 1$, then, by Corollary 3.30, we know that $\left(M, \langle\, ,\, \rangle, e^{-f}d\mathrm{vol}\right)$ is f–parabolic. $\square$

As an application of Corollary 3.15 and Corollary 3.28 one can deduce interesting rigidity results for Ricci solitons and Ricci almost solitons with integrable scalar curvature that should be compared with [**74**], [**75**]. Note that, combining Lemma 2.3 in [**16**] with Remark 2.9 it follows that the scalar curvature of a shrinking Ricci soliton is p–integrable, for every $p > 0$. We are grateful to M. Fernández–López for pointing out this to us. In the expanding case we shall prove the next result. It shows that some further rigidity occurs for expanders at the end–point case $S_* = 0$ in Theorem 4.1 and Corollary 4.2.

Theorem 4.3. *Let $(M, \langle\, ,\, \rangle, \nabla f)$ be a complete, expanding, gradient Ricci almost soliton with soliton function λ satisfying $\Delta\lambda \leq 0$ and $0 > \lambda \geq \lambda_* = \inf_M \lambda > -\infty$. Let S be the scalar curvature of M. If $S \geq 0$ and $S \in L^1\left(M, e^{-f}d\mathrm{vol}\right)$ then M is isometric to the standard Euclidean space.*

As an immediate consequence we obtain,

Corollary 4.4 (Theorem 4 in [**84**])**.** *Let $(M, \langle\, ,\, \rangle, \nabla f)$ be a complete, expanding, gradient Ricci soliton. Let S be the scalar curvature of M. If $S \geq 0$ and $S \in L^1\left(M, e^{-f}d\mathrm{vol}\right)$ then M is isometric to the standard Euclidean space.*

Proof (of Theorem 4.3). Recall that, by formula (3.37) of Lemma 3.24, it holds

$$\frac{1}{2}\Delta_f S = \lambda S - |Ric|^2 + (m-1)\Delta\lambda. \tag{4.4}$$

Since $S \geq 0$, $\lambda < 0$ and $\Delta\lambda \leq 0$, from the above we deduce

$$\Delta_f S \leq 0.$$

Since by assumption $\lambda \geq \lambda_* > -\infty$, applying Corollary 3.28, we obtain that S is constant. Using this information into (4.4) implies that $Ric \equiv 0$, and the required conclusion follows by case (a.2) of Theorem 4.7. ☐

We now switch to quasi–Einstein manifolds. In the same spirit of Theorem 4.1, we generalize the scalar curvature estimates in Proposition 3.6 of [**17**] to k–quasi–Einstein manifolds with non–constant scalar curvature. Again, possible rigidity at the endpoints is discussed.

Theorem 4.5. *(Theorem 3 in [**88**]) Let $(M^m, g_M, e^{-f}\mathrm{dvol})$ be a geodesically complete k–quasi–Einstein manifold, $1 < k < +\infty$, with scalar curvature S and let $S_* = \inf_M S$.*

(a) *If $\lambda > 0$, then M is compact and*

$$\frac{m(m-1)}{m+k-1}\lambda < S_* \leq m\lambda. \tag{4.5}$$

Moreover $S_* \neq m\lambda$ unless M is Einstein.

(b) *If $\lambda = 0$ and $\inf_M f = f_* > -\infty$ then $S_* = 0$. Moreover, either $S > 0$ or $S(x) \equiv 0$. In this latter case, either f is constant (and M is trivial) or M is isometric to the Riemannian product $\mathbb{R} \times \Sigma$ where Σ is a Ricci–flat, totally geodesic hypersurface.*

(c) *If $\lambda < 0$ and $\inf_M f = f_* > -\infty$, then*

$$m\lambda \leq S_* \leq \frac{m(m-1)}{m+k-1}\lambda \tag{4.6}$$

and $S(x) > m\lambda$ unless M is Einstein.

Proof. First of all, we show that $\inf_M S > -\infty$. According to Corollary 3.41 this is obvious if $\lambda > 0$ because M is compact. In the general case $\lambda \in \mathbb{R}$ we proceed as follows. Since

$$-\left|Ric - \frac{1}{m}Sg_M\right|^2 = -|Ric|^2 + \frac{S^2}{m},$$

from (3.52) we obtain

$$\begin{aligned} \frac{1}{2}\Delta_{\tilde{f}} S &= -\frac{k-1}{k}|Ric|^2 - \frac{1}{k}S^2 + \frac{k+2m-2}{k}\lambda S - \frac{m(m-1)}{k}\lambda^2. \\ &\leq -\frac{1}{k}S^2 + \frac{k+2m-2}{k}\lambda S. \end{aligned} \tag{4.7}$$

Let $S_-(x) = \max\{-S(x), 0\}$. Then

$$\Delta_{\tilde{f}} S_- \geq \frac{2}{k}S_-^2 + \frac{2(k+2m-2)}{k}\lambda S_-. \tag{4.8}$$

Observe now that from Qian's estimates of weighted volumes, recalled in Corollary 2.5, since $vol_{\widetilde{f}}(B_r) \leq e^{-\frac{2}{k}f_*} vol_f(B_r)$, we can apply the "a-priori" estimate in Theorem 2.24 to inequality (4.8) on the complete weighted manifold $(M, g_M, e^{-\widetilde{f}} d\text{vol})$ and we obtain that S_- is bounded from above, or equivalently, $S_* = \inf_M S > -\infty$. Again from Qian's estimates and by Theorem 2.17 applied to $(M, g_M, e^{-\widetilde{f}} d\text{vol})$, the weak Omori–Yau maximum principle for the $\widetilde{f}$–Laplacian holds on M. This produces a sequence $\{x_n\}$ such that $\Delta_{\widetilde{f}} S(x_n) \geq -\frac{1}{n}$ and $S(x_n) \to S_*$. Taking the lim inf in (3.52) along $\{x_n\}$ shows that, for $k > 1$,

$$0 \leq -\frac{m+k-1}{mn}(S_* - m\lambda)(S_* - \frac{m(m-1)}{m+k-1}\lambda). \tag{4.9}$$

We now distinguish three cases.

(a) Assume $\lambda > 0$, so that M is compact. Equation (4.9) yields $\frac{m(m-1)}{m+k-1}\lambda \leq S_* \leq m\lambda$. Assume now that $S_* = m\lambda > 0$. Then $S \geq m\lambda \geq \frac{m(m-1)}{m+k-1}\lambda$ and from (3.52) we get

$$\frac{1}{2}\Delta_{\widetilde{f}} S \leq -\frac{m+k-1}{mk}(S - m\lambda)(S - \frac{m(m-1)}{m+k-1}\lambda) \leq 0.$$

Since M is compact, S must be constant. Hence $S = S_* = m\lambda$. Substituting in (3.52) we obtain that $Ric = \frac{1}{m} S g_M$ and thus that M is Einstein.

Now we show that $S_* > \frac{m(m-1)}{m+k-1}\lambda$. Indeed, suppose that S attains its minimum $\frac{m(m-1)}{m+k-1}\lambda$. Since the non-negative function $v(x) = S(x) - \frac{m(m-1)}{m+k-1}\lambda$ satisfies

$$\frac{1}{2}\Delta_{\widetilde{f}} v \leq -\frac{m+k-1}{mk} v^2 + \lambda v \leq +\lambda v,$$

and v attains its minimum $v(x_0) = 0$, it follows from the minimum principle, (see Theorem 2.13 or e.g. p. 35 in [**31**]), that v vanishes identically. Hence $S \equiv \frac{m(m-1)}{m+k-1}\lambda$ is constant and, substituting in (3.52), we get that M is Einstein with

$$Ric = \frac{m-1}{m+k-1}\lambda g_M.$$

Using this information into (3.46) we obtain that

$$Hess(f) = \frac{1}{k}|\nabla f|^2 + \frac{k}{m+k-1}\lambda g_M > 0.$$

But this is clearly impossible because M is compact.

(b) Assume $\lambda = 0$. From (4.9) we conclude that $S_* = 0$. Note that, according to (3.52), $\Delta_{\widetilde{f}} S \leq 0$. Therefore, by the minimum principle, either $S(x) > 0$ on M or $S(x) \equiv 0$. In this latter case, substituting in (3.52), we obtain that M is Ricci flat and the k–quasi–Einstein equation reads $Hess(f) - \frac{1}{k} df \otimes df = 0$. Therefore, either f is constant and M is Einstein, or the non constant function $u = e^{-\frac{f}{k}}$ satisfies $\text{Hess}(u) = 0$. A Cheeger–Gromoll type argument now shows that M is isometric to the Riemannian product $\mathbb{R} \times \Sigma$ along a Ricci–flat, totally geodesic hypersurface Σ of M, which is any level hypersurface of u.

(c) Assume $\lambda < 0$. From (4.9) we deduce that $m\lambda \le S_* \le \frac{m(m-1)}{m+k-1}\lambda$. Suppose that $S(x_0) = m\lambda < 0$ for some $x_0 \in M$. Since the non-negative function $w(x) = S(x) - m\lambda$ satisfies

$$\frac{1}{2}\Delta_{\widetilde{f}} w \le -\frac{m+k-1}{mk}\, w^2 - \lambda w \le -\lambda w,$$

and w attains its minimum $w(x_0) = 0$, it follows from the minimum principle that w vanishes identically. Hence $S \equiv m\lambda$ is constant and substituting in (3.52) we get that M is Einstein. □

Remark 4.6. Note that in Theorem 4.5 we have assumed that $k > 1$. The reason is that when $k = 1$ the first term on the LHS in (3.52) vanishes. This prevents to use some of the arguments employed in case $k > 1$. The right version of Theorem 4.5 in case $k = 1$ which can be obtained with minor variations to the proof of Theorem 4.5 is the following.

Let $(M^m, g_M, e^{-f}\mathrm{dvol})$ be a geodesically complete 1–quasi–Einstein manifold, with scalar curvature S and let $S_* = \inf_M S$.

(a) *If $\lambda > 0$ then M is compact and*

$$(m-1)\lambda < S_* \le m\lambda.$$

Moreover $S_* \neq m\lambda$ unless $S \equiv m\lambda$ and M is Einstein.

(b) *If $\lambda = 0$ and $\inf_M f = f_* > -\infty$ then $S_* = 0$. Moreover, either $S > 0$ or $S(x) \equiv 0$.*

(c) *If $\lambda < 0$ and $\inf_M f = f_* > -\infty$, then*

$$m\lambda \le S_* \le (m-1)\lambda$$

and $S(x) > m\lambda$ unless $S \equiv m\lambda$.

4.2. Rigidity of Einstein Ricci almost solitons

The next rigidity theorem, in the complete case, shows that basically there are no further examples of Einstein Ricci almost solitons than that constructed in Proposition 3.19.

Theorem 4.7. *(Theorem 1.3 in* [**76**]*) Let $(M, \langle\, ,\rangle)$ be a complete, connected, Einstein manifold of dimension $m \ge 4$ and*

$$^M\mathrm{Ric} = -(m-1)\, c\, \langle\, ,\rangle\,, \quad c \in \mathbb{R}.$$

Assume that M is an almost Ricci soliton, namely, for some $\lambda \in C^\infty(M)$, there is a solution $f \in C^\infty(M)$ of the equation

$$^M\mathrm{Ric} + \mathrm{Hess}\,(f) = \lambda\,(x)\, \langle\, ,\rangle\,.$$

(a) *If $c = 0$, then λ must be constant and the following possibilities occur:*

(a.1): *If $\lambda = 0$ then M is isometric to a cylinder $\mathbb{R} \times \Sigma$ over a totally geodesic, Ricci flat hypersurface $\Sigma \subset M$. Furthermore, $f(t,x) = at + b$, for some constants $a, b \in \mathbb{R}$.*

(a.2): *If* $\lambda = \text{const.} \neq 0$ *then* M *is isometric to* $\mathbb{R}^m$ *and*

$$f(x) = \frac{\lambda}{2}|x|^2 + \langle b, x\rangle + c, \tag{4.10}$$

for some $b \in \mathbb{R}^m$ *and* $c \in \mathbb{R}$.

(b) *If* $c \neq 0$, *then either* λ *is constant and the soliton is trivial, or one of the following cases occurs:*

(b.1): $c \in \mathbb{R}\setminus\{0\}$ *and* M *is a space–form of constant curvature* $-c$. *Furthermore*

$$\begin{cases} \lambda(x) = a\mathrm{cn}_{-c}(r(x)) - (m-1)c \\ f(x) = c^{-1}a\mathrm{cn}_{-c}(r(x)) + b, \end{cases} \tag{4.11}$$

for some constants $a, b \in \mathbb{R}$. *Here,* $r(x)$ *denotes the distance from a fixed origin.*

(b.2): $c > 0$ *and* M *is isometric to the warped product* $\mathbb{R} \times_g \Sigma$ *where*

$$g(t) = \frac{g'(0)}{\sqrt{c}}\sinh\left(\sqrt{c}t\right) + g(0)\cosh\left(\sqrt{c}t\right) > 0,$$

and $\Sigma \subset M$ *is an Einstein hypersurface with*

$${}^{\Sigma}\mathrm{Ric} = -(m-2)(-g'(0)^2 + cg(o)^2).$$

Furthermore

$$\begin{cases} \lambda(t,x) = ag'(t) - mc \\ f(t,x) = a\int_0^t g(s)\,ds + b, \end{cases} \tag{4.12}$$

for some constants $a, b \in \mathbb{R}$. *Here,* t *is a global coordinate on* $\mathbb{R}$.

Proof. By assumption, with respect to a local orthonormal coframe, we have

$$f_{ij} = ((m-1)c + \lambda)\delta_{ij}. \tag{4.13}$$

Differentiating both sides and using the commutation rule

$$f_{ijk} - f_{ikj} = R_{lijk}f_l, \tag{4.14}$$

we deduce

$$R_{lijk}f_l = \lambda_k\delta_{ij} - \lambda_j\delta_{ik}.$$

Tracing this latter with respect to i and k, recalling that $R_{ij} = -(m-1)c\delta_{ij}$, and simplifying we conclude that

$$cf_j = \lambda_j. \tag{4.15}$$

We now distinguish several cases.

(a) Suppose $c = 0$, i.e., M is Ricci flat. Then $\lambda_j = 0$ proving that λ is constant. The soliton equation reads

$$\mathrm{Hess}(f) = \lambda\langle\,,\,\rangle.$$

(a.1) In case $\lambda = 0$, then f is affine. In particular $|\nabla f|$ is constant proving that either f is constant, and the soliton is trivial, or f has no critical point at all. Suppose this latter case occurs. Up to rescaling f we can assume that $|\nabla f| = 1$, i.e., f is a function of distance type. Then, a Cheeger–Gromoll type argument (see (b.2.ii$_1$) below for details) shows that the flow ϕ of the vector field $X = \nabla f$ establishes a Riemannian isometry $\phi : \mathbb{R} \times \Sigma \to M$, where Σ is any of the (totally geodesic) level sets of f and f is a linear function of t. Finally, since M is Ricci flat then also Σ must be Ricci flat. This proves the first part of statement (a) of the Theorem.

(a.2) Assume $\lambda \neq 0$. Then, from case (iii) of Theorem 1.1, we known that M is isometric to $\mathbb{R}^m$ and $f(x)$ takes the form given in (4.10). See [**92**] and the Appendix in [**84**] for a straightforward proof. The proof of case (a) is completed.

(b) Suppose $c \neq 0$. By (4.15) we have

$$f = c^{-1}\lambda + d, \tag{4.16}$$

for some constant $d \in \mathbb{R}$. Inserting into (4.13) gives

$$\mathrm{Hess}(\lambda) = c\{(m-1)c + \lambda\}\langle\,,\rangle .$$

If $\lambda(x) = $ const., i.e. M is a classical Ricci soliton, then, in view of (4.16), f must be constant and the soliton is trivial.

Assume then that $\lambda(x)$ is nonconstant. Note that the function

$$v(x) = (m-1)c + \lambda(x) \tag{4.17}$$

is a nontrivial solution of

$$\mathrm{Hess}(v) = cv\langle\,,\,\rangle . \tag{4.18}$$

(b.1.i) If $c < 0$, then by the classical Obata theorem, [**69**], which we recalled in case (i) of Theorem 1.1, M is isometric to a spaceform of constant curvature $-c > 0$ and

$$v(x) = a\cos\left(\sqrt{-c}r(x)\right),$$

for some constant $a \neq 0$. Here, $r(x)$ denotes the distance function from a fixed origin. It follows that the functions $\lambda(x)$ and $f(x)$ take the form given in **(b.1)**, (4.11), for $c < 0$.

It remains to consider the case $c > 0$. Two possibilities can occur:

(b.1.ii) The function v, which is a nontrivial solution of (4.18), has at least one critical point $o \in M$ and, therefore, it is a nontrivial solution of the problem

$$\begin{cases} \mathrm{Hess}(v) = cv\langle,\rangle \\ |\nabla v|(o) = 0, \end{cases}$$

with $c > 0$. Thus, for every unit speed geodesic γ issuing from o, the function $y = v \circ \gamma$ satisfies the initial value problem

$$\begin{cases} y'' = cy \\ y(0) = v(o),\ y'(0) = \langle \nabla v(o), \dot\gamma(0)\rangle, \end{cases}$$

and since v is nonconstant, we must have $v(o) \neq 0$. Using Kanai's version of Obata theorem, [**50**], recalled in case (ii) of Theorem 1.1, we conclude that M is isometric to hyperbolic space of constant curvature $-c < 0$ and $v(x) = v(o)\cosh(\sqrt{c}r(x))$ where $r(x)$ is the distance function from o. Inserting this expression into (4.17) and (4.16) completes the proof of case **(b.1)**.

(b.2) The function v has no critical points. A classification of M under this assumption, and the corresponding form of $v, f, \lambda,$ can be deduced from some works by Ishihara and Tashiro, [**49**], and Tashiro, [**92**]. However we provide a concise and complete proof for the sake of completeness. Let $\Sigma = \{v(x) = s\}$ be a non–empty, smooth, level hypersurface.

Note that, up to multiplying v by a non–zero constant, we can always assume that either $s = 0$ or $s = 1$. A computation that uses (4.18) shows that the integral curves of the complete vector field $X = \nabla v / |\nabla v|$ are unit speed geodesics orthogonal to Σ. Moreover, the flow of X gives rise to a smooth map $\phi : \mathbb{R} \times \Sigma \to M$ which coincides with the normal exponential map $\exp^{\perp}$ of Σ. In particular, ϕ is surjective. Evaluating (4.18) along the integral curve $\phi(t,x)$ issuing from $x \in \Sigma$ we deduce that $y(t) = v(\phi(t,x))$ satisfies

$$\begin{cases} y'' = cy \\ y(0) = s \in \{0,1\} \\ y'(0) = |\nabla v|(x) \end{cases}$$

and therefore

$$v(\phi(t,x)) = |\nabla v|(x)\,\mathrm{sn}_{-c}(t) + s\,\mathrm{cn}_{-c}(t)\,. \tag{4.19}$$

Since

$$\frac{dv(\phi(t,x))}{dt} = |\nabla v| \circ \phi(t,x) > 0 \tag{4.20}$$

it follows from (4.19) that, necessarily, $c > 0$. Moreover, if $s = 1$ we have the further restriction $|\nabla v|(x) \geq \sqrt{c}$. The function v is strictly increasing along the geodesic curves $\phi_x(t)$ issuing from $x \in \Sigma$. Whence, it is easy to conclude that ϕ is also injective, hence a diffeomorphism. Since $M \approx \mathbb{R} \times \Sigma$ is connected, also Σ must be connected. As a consequence, $|\nabla v|$ is constant on Σ. Indeed, for any smooth curve $\gamma \subset \Sigma$, we have

$$\begin{aligned} \frac{d}{dt}(|\nabla v| \circ \gamma) &= \mathrm{Hess}(v)\left(\frac{\nabla v}{|\nabla v|} \circ \gamma, \dot{\gamma}(t)\right) \\ &= cv(\gamma)\langle X_\gamma, \dot{\gamma}(t)\rangle \\ &= 0, \end{aligned}$$

because $\dot{\gamma}(t) \in T\Sigma$ and X_γ is orthogonal to Σ. Therefore $|\nabla v|(x) = a \geq \sqrt{c}$, for every $x \in \Sigma$. Using this information into (4.19) with $c > 0$ gives

$$v(\phi(t,x)) = \alpha(t)$$

where we have set

$$\alpha(t) = \frac{a}{\sqrt{c}}\sinh\left(\sqrt{c}t\right) + s\cosh\left(\sqrt{c}t\right).$$

In particular, ϕ moves Σ onto every other level set of v. To conclude, we show that

$$\phi^* \langle\, , \,\rangle = dt^2 + \left(\alpha'\right)^2(t)\,\langle\, , \,\rangle_{\Sigma_0}\,, \tag{4.21}$$

where $\langle\, , \rangle_\Sigma = (\phi_0)^* \langle\, , \rangle$ denotes the metric induced by M on the smooth hypersurface Σ. Indeed, by the above reasonings (or applying Gauss Lemma) we have

$$\phi^* \langle\, , \,\rangle = dt^2 + (\phi_t)^* \langle\, , \,\rangle\,.$$

Furthermore, using (4.18), (4.20) and the definition of the Lie derivative, we see that, on $T\Sigma_{\phi_t} = X^{\perp}_{\phi_t}$,

$$\frac{d}{dt}(\phi_t)^* \langle\, ,\, \rangle = \frac{2\alpha''}{\alpha'}\phi_t^* \langle\, ,\, \rangle\,.$$

Whence, integrating on $[0,t]$ we conclude the validity of (4.21). Summarizing, we have obtained that, if v has no critical point, then $(M, \langle\, ,\, \rangle)$ is isometric to the warped product manifold

$$\left(\mathbb{R}\times\Sigma, dt^2 + \alpha'(t)^2 \langle\, ,\, \rangle_\Sigma\right),$$

with Σ a smooth hypersurface of M. By assumption, M is Einstein with constant Ricci curvature $-(m-1)c$, therefore Σ is Einstein and the expression of its Ricci curvature follows from Lemma 3.18.

(b.2.ii$_2$) To conclude, assume that v possesses at least one critical point $o \in M$ and, therefore, it is a nontrivial solution of the problem

$$\begin{cases} \mathrm{Hess}(v) = cv\langle\, ,\, \rangle \\ |\nabla v|(o) = 0, \end{cases}$$

with $c > 0$. Since v is nonconstant, we have $v(o) \neq 0$. Using case (ii) of Theorem 1.1 we conclude that M is isometric to the hyperbolic space of constant curvature $-c < 0$ and $v(x) = v(o)\cosh(\sqrt{c}r(x))$ where $r(x)$ is the distance function from o. Inserting this expression into (4.17) and (4.16) completes the proof of case (b) and, hence, of the theorem. □

Example 4.8. The rigidity expressed in Theorem 4.7 enables us to produce examples of manifolds that do not admit any non–trivial almost soliton structure. Let M be any (possibly trivial) quotient of the Riemannian product of standard spheres $\mathbb{S}^2\times\mathbb{S}^2$ or a non trivial quotient of $\mathbb{S}^m$. Then, M is Einstein, and according to Theorem 4.7 (**b.1**), M has no nontrivial almost Ricci soliton structure.

A similar conclusion holds for possibly trivial quotients of the Riemannian product of standard hyperbolic spaces $\mathbb{H}^2\times\mathbb{H}^2$. Clearly it suffices to consider $\mathbb{H}^2\times\mathbb{H}^2$ itself. Since $\mathbb{H}^2\times\mathbb{H}^2$ is Einstein with $\mathrm{Ric} = -\langle\,,\rangle$, if it had the structure of a nontrivial almost soliton structure, by Theorem 4.7 it would be isometric to the warped product $\mathbb{R}\times_g\Sigma$ where Σ is a 3 dimensional Einstein hypersurface and g has the form given in the statement of the Theorem. It follows that Σ has constant negative curvature, and, from the expression of the Riemann tensor of a warped product (see e.g., [**72**]), $\mathbb{R}\times_g\Sigma \approx \mathbb{H}^2\times\mathbb{H}^2$ would have strictly negative sectional curvature. Notice that the above reasoning shows that in case (**b.2**) if $m = 4$ and M is simply connected then Σ is a hyperbolic space.

4.3. A gap theorem for the traceless Ricci tensor

Denoting by $T = Ric - \frac{1}{m}S\langle\, ,\, \rangle$ the traceless Ricci tensor of $(M, \langle\, ,\, \rangle)$ the next result is a gap theorem for the values of

$$|T|^* = \sup_M |T|\,.$$

Theorem 4.9. *(Theorem 0.9 in* [**76**]*, Theorem 4 in* [**59**]*) Let* $(M, \langle\,,\,\rangle, \nabla f)$ *be a complete Ricci almost soliton with scalar curvature* S*, traceless Ricci tensor* T*, Weyl tensor* W *and soliton function* λ *such that*

$$\langle Hess\,(\lambda)\,, T\rangle \geq 0 \tag{4.22}$$

on M*. Assume* $m = \dim M \geq 3$*,*

$$S^* = \sup_M S < +\infty, \tag{4.23}$$

$$|W|^* = \sup_M |W| < +\infty, \tag{4.24}$$

$$\lambda_* = \inf_M \lambda > -\infty. \tag{4.25}$$

Then either $(M, \langle\,,\,\rangle)$ *is Einstein and the classification of Theorem 4.7 applies or*

$$|T|^* \geq \frac{1}{2}\left(\sqrt{m\,(m-1)}\lambda_* - \frac{m-2}{\sqrt{m\,(m-1)}}S^* - \sqrt{\frac{m(m-2)}{2}}|W|^*\right).$$

Proof. By Remark 2.18, since Ric_f is bounded below by (4.25), the weak maximum principle for Δ_f holds on $\big(M, \langle\,,\,\rangle\,, e^{-f}d\mathrm{vol}\big)$. Next, by Corollary 3.27, (4.22), (4.23), and (4.25), we deduce that

$$\frac{1}{2}\Delta_f\,|T|^2 \geq 2\left(\lambda_* - \frac{m-2}{m\,(m-1)}S^*\right)|T|^2 - \frac{4}{\sqrt{m\,(m-1)}}\,|T|^3 - 2W_{ijks}T_{js}T_{ik}.$$

Now we recall the next estimate due to G. Huisken, see [**48**] Lemma 3.4:

$$|W_{ijks}T_{js}T_{ik}| \leq \frac{\sqrt{2}}{2}\sqrt{\frac{m-2}{m-1}}|W||T|^2. \tag{4.26}$$

Using (4.26) and (4.24) we thus obtain

$$\begin{aligned}\frac{1}{2}\Delta_f|T|^2 \;&\geq\; 2\left(\lambda_* - \frac{m-2}{m(m-1)}S^* - \sqrt{\frac{m-2}{2(m-1)}}|W|^*\right)|T|^2\\ &\quad - \frac{4}{\sqrt{m(m-1)}}|T|^3.\end{aligned}$$

Assuming that $|T|^* = \sup_M |T| < +\infty$ (for otherwise there is nothing to prove) we may apply the weak maximum principle for Δ_f and deduce that either $|T|^* = 0$ or $|T|^* \geq \frac{1}{2}\left(\sqrt{m\,(m-1)}\lambda_* - \frac{m-2}{\sqrt{m(m-1)}}S^* - \sqrt{\frac{m(m-2)}{2}}|W|^*\right)$. In the former case, $T = 0$ that is $\mathrm{Ric} = S/m\langle\,,\rangle$ and since $m \geq 3$, S is constant by Schur's lemma and M is Einstein, as required to conclude the proof. □

Remark 4.10. Following the arguments in [**21**], we can observe that if the complete weighted manifold $(M^m, \langle\,,\,\rangle\,, e^{-f}d\mathrm{vol})$ satisfies

(i) $\mathrm{Ric} = -(m-1)c\,\langle\,,\,\rangle$, $c \neq 0$;
(ii) $\mathrm{Ric}_f = \lambda\,\langle\,,\,\rangle$, for some $0 < \lambda \in C^\infty(M)$, such that $\Delta\lambda \leq 0$ and $A^2(1 + r(x))^{-\mu} \leq \lambda^* < +\infty$ on M for some $A > 0$, $0 \leq \mu \leq 1$;

(iii) $S_* = m\lambda^*$;

then, by Theorem 4.1, S is a positive constant and thus $(M, \langle\,,\,\rangle)$ is compact by Myers' Theorem. In this latter case, if $|W|$ is sufficiently small, precisely if $|W|^2 \le \frac{4}{(m+1)m(m-1)(m-2)} S^2$, then $(M, \langle\,,\,\rangle)$ has positive curvature operator ([**48**], Corollary 2.5). Thus, from Tachibana, [**91**], $(M, \langle\,,\,\rangle)$ has positive constant sectional curvature and therefore is a finite quotient of $\mathbb{S}^m$. Note that, in the gradient Ricci soliton case we don't need assumption (iii) to conclude since by assumptions (i) and (ii) the soliton is trivial (see e.g. the proof of Theorem 4.7) and thus the stronger condition $\lambda = -(m-1)c = \frac{S}{m}$ holds.

4.4. A complement to scalar curvature estimates for 1–quasi–Einstein manifolds

In this section we use a Liouville result for k–quasi–Einstein manifolds, in case $k = 1$, to obtain a further result concerning the scalar curvature of 1–quasi–Einstein manifolds. This, jointly with (4.32) and (4.33), essentially states that, under suitable geometric assumption, when the scalar curvature is confined in a particular interval it has to be constant and identically equal to one of the extremes of the interval.

As we have said in Section 3.3, the efforts in the study of 1–quasi–Einstein manifolds have been put mainly on the (phisically relevant) case $\mu = 0$. Nevertheless, as observed in [**18**], also the study of quasi–Einstein metrics with $k = 1$ and $\mu \neq 0$ is interesting. Since we cannot apply Theorem 3.33 to construct the related Einstein warped products, their existence proves that, even restricting to integer hidden dimension k, quasi–Einstein manifolds form a strictly larger class of manifolds that those which are the base of an Einstein warped product manifold. For some examples of these manifolds, constructed in the more general setting of conformally warped manifolds, see the last section of [**18**].

Our Liouville result, which is relevant exactly in the $\mu \neq 0$ case, will follow from an adaptation to the f–Laplacian under weighted volume growth conditions of Theorem A in [**79**]. This can be deduced from the proof of the latter, making minor modifications in the proofs of Theorem A, Lemma 1.2, Theorem A′ in [**78**] and Theorem 2.5 in [**79**].

Theorem 4.11. *Let ϕ be a continuous function on $[0, +\infty)$ satisfying the conditions*

(4.27)
$$(i)\ \phi(0) = \phi(a) = 0, \quad (ii)\ \phi(s) > 0 \text{ in } (0, a), \quad (iii)\ \phi(s) < 0 \text{ in } (a, +\infty),$$

for some $a > 0$, and

$$\liminf_{s \to +\infty} \frac{-\phi(s)}{s^\sigma} > 0, \tag{4.28}$$

for some $\sigma > 1$; let also $b(x) \in C^0(M)$ and suppose that

$$b(x) \ge \frac{C}{(1 + r(x))^\mu} \text{ on } M,$$

for some $C > 0$ *and* $0 \le \mu < 2$. *Let* u *be a non-negative solution of*

$$\Delta_f u = -b(x)\phi(u) \text{ on } M. \tag{4.29}$$

Assume that

$$\liminf_{r \to +\infty} \frac{\log vol_f(B_r)}{r^{2-\mu}} < +\infty \tag{4.30}$$

and, if

$$(vol_f(\partial B_r))^{-1} \in L^1(+\infty)$$

assume furthermore that

$$\phi(t) \ge ct^{\xi} \qquad 0 < t \ll 1$$

for some $\xi > 0$ *and* $c > 0$. *Finally, if* $\xi \ge 1$ *suppose also that*

$$u(x) \ge Dr(x)^{-\theta}, \qquad r(x) \gg 1$$

for some $\theta \ge 0$, $D > 0$ *and that*

$$\liminf_{r \to +\infty} \frac{\log vol_f(B_r)}{r^{2-\theta(\xi-1+\varepsilon)-\mu}} < +\infty$$

for some $\varepsilon > 0$. *Then* u *is constant and identically equal to* 0 *or* a.

Now, consider a k–quasi–Einstein manifold $(M^m, g_M, e^{-f}\mathrm{dvol})$, $k < +\infty$. We recall that, according to Lemma 3.40, setting $\widetilde{f} = \frac{k+2}{k} f$, the scalar curvature S of a quasi–Einstein manifold satisfies the following relation,

$$\frac{1}{2}\Delta_{\widetilde{f}} S = -\frac{k-1}{k}\left|Ric - \frac{1}{m} S g_M\right|^2 - \frac{k+m-1}{km}(S - m\lambda)\left(S - \frac{m(m-1)}{k+m-1}\lambda\right). \tag{4.31}$$

We have seen in Theorem 4.5 and Remark 4.6 that exploiting (4.31), one can obtain estimates for the infimum of the scalar curvature $S_* = \inf_M S$. In particular we have that for $\lambda > 0$

$$\frac{m(m-1)}{m+k-1} < S_* \le m\lambda, \tag{4.32}$$

and for $\lambda < 0$ and $\inf_M f > -\infty$

$$m\lambda \le S_* \le \frac{m(m-1)}{m+k-1}\lambda. \tag{4.33}$$

In the special case $k = 1$ equation (4.31) becomes

$$\Delta_{\widetilde{f}} S = -2(S - m\lambda)(S - (m-1)\lambda). \tag{4.34}$$

Making an essential use of (4.34), we now prove the announced result. Note that some extra rigidity in case $\lambda > 0$ is also discussed.

Theorem 4.12. *(Theorem 16 in* [**60**]*) Let* $(M^m, g_M, e^{-f}\mathrm{dvol})$ *be a geodesically complete* 1*–quasi–Einstein manifold with quasi–Einstein constant* λ *and scalar curvature* S. *Set* $\widetilde{f} = 3f$ *and suppose that* $f_* = \inf_M f > -\infty$.
If

$$(vol_{\widetilde{f}}(\partial B_r))^{-1} \in L^1(+\infty),$$

letting

$$u(x) = \begin{cases} -S(x) + (m-1)\lambda & \lambda < 0 \\ -S(x) + m\lambda & \lambda > 0, \end{cases}$$

assume furthermore that

$$u(x) \geq Dr(x)^{-\theta}, \qquad r(x) \gg 1$$

for some $\theta \geq 0$*,* $D > 0$*, and that*

$$\liminf_{r \to +\infty} \frac{\log vol_{\widetilde{f}}(B_r)}{r^{2-\theta\varepsilon}} < +\infty$$

for some $\varepsilon > 0$*.*

(a) *If* $\lambda < 0$ *and* $S \leq (m-1)\lambda$ *we obtain that* S *is constant and identically equal to either* $(m-1)\lambda$ *or* $m\lambda$*.*
(b) *If* $\lambda > 0$ *and* $S \leq m\lambda$ *then* S *is constant, identically equal to* $m\lambda$ *and* M *is Einstein.*

Proof. (a) Assume $\lambda < 0$. Considering $u = -S + (m-1)\lambda$, which is non-negative for $S \leq (m-1)\lambda$, from (4.34) we obtain that

$$\Delta_{\widetilde{f}} u = 2u(u + \lambda). \tag{4.35}$$

We want now to apply Theorem 4.11 to equation (4.35) on the weighted manifold $(M, g_M, e^{-\widetilde{f}}d\mathrm{vol})$. If we choose $\phi(t) = -2t(t+\lambda)$, it clearly satisfies assumptions (4.27) and (4.28) with $a = -\lambda$ and the equation (4.35) can be written in the form

$$\Delta_{\widetilde{f}} u = -\phi(u),$$

as in the statement of Theorem 4.11.

Furthermore, according to Qian weighted volume estimates, that we recalled in Corollary 2.5, since by assumption $f_* > -\infty$, we have the validity of the condition on the $\widetilde{f}$–volume growth of the form (4.30). Hence by Theorem 4.11 we are able to conclude that S is constant and identically equal to either $m\lambda$ or $(m-1)\lambda$.

(b) Assume $\lambda > 0$. Consider $u = -S + m\lambda$ which is non-negative for $S \leq m\lambda$, and choose $\phi(t) = -2t(t - \lambda)$; applying Theorem 4.11 with $a = \lambda$, we conclude that S is constant and identically equal to either $(m-1)\lambda$ or $m\lambda$.

Now we show that the first case cannot happen. Indeed, suppose that $S \equiv (m-1)\lambda$. Substituing in the trace of the quasi–Einstein equation we get that $\Delta f \geq 0$ and since M is compact we obtain that f is constant and M is Einstein with $Ric = \lambda g_M$. But this is clearly impossible, since tracing this latter equation we get a contradiction. Hence $S \equiv m\lambda$. Substituing again in the trace of the quasi–Einstein equation we obtain, reasoning as above, that f is constant, and thus that M is Einstein. ∎

CHAPTER 5

Rigidity as triviality of an additional structure

If a weighted manifold is endowed with an additional structure often rigidity can appear as triviality of this structure. In this chapter we present some istances of this type of results when the weighted manifold supports a gradient Ricci (almost) soliton or a k–quasi–Einstein structure. Namely we obtain triviality results for gradient Ricci almost solitons with $L^{1\leq p\leq\infty}$ soliton structures and for k–quasi–Einstein manifolds with some kind of boundedness or integral control on the potential (or on some related function).

5.1. Triviality for Ricci solitons and Ricci almost solitons

5.1.1. Triviality under $L^{1\leq p\leq\infty}$ conditions. The main result we are going to prove in this section is the following.

Theorem 5.1. *(Theorem 2 in* [**84**]*) A complete, expanding, gradient Ricci soliton* $(M, \langle\,,\,\rangle, \nabla f)$ *is trivial provided* $|\nabla f| \in L^p\left(M, e^{-f}d\text{vol}\right)$, *for some* $1 \leq p \leq +\infty$.

As a matter of fact, the above statement encloses three different results according to the assumption that $p = +\infty$, $1 < p < +\infty$ and $p = 1$. These will be obtained using different arguments. The L^∞ situation will be dealt with using the form of the weak Omori–Yau maximum principle for diffusion operators we have presented in Section 2.3, [**80**].

On the other hand, the $L^{1<p<\infty}$ and the L^1 results will rely on Liouville properties of the diffusion operators, [**81**], [**82**], [**80**].

Concerning triviality under L^∞ conditions, it is known, [**35**], that a complete, shrinking Ricci soliton $(M, \langle\,,\,\rangle, X)$ satisfying $|X| \in L^\infty$ must be compact. We now show that, in case the soliton is gradient and expanding, the L^∞ condition implies triviality.

Theorem 5.2. *Let* $(M, \langle\,,\,\rangle, \nabla f)$ *be a geodesically complete, expanding Ricci soliton with* $\sup_M |\nabla f| < +\infty$. *Then the Ricci soliton is trivial.*

Proof. According to (3.7) the smooth function $|\nabla f|^2$ satisfies

$$\frac{1}{2}\Delta_f |\nabla f|^2 \geq -\lambda |\nabla f|^2 \geq 0. \tag{5.1}$$

Applying Corollary 3.8 we deduce that there exists a sequence $\{x_n\} \subset M$ such that,

$$|\nabla f|^2 (x_n) \geq \sup_M |\nabla f|^2 - \frac{1}{n},$$

and

$$\Delta_f |\nabla f|^2 (x_n) \leq \frac{1}{n}.$$

Evaluating (5.1) along $\{x_n\}$ and taking the limit as $n \to +\infty$ we conclude

$$-\lambda \sup_M |\nabla f|^2 = 0,$$

proving that f is constant. □

Remark 5.3. Note that using (3.23) instead of (3.7), and applying Corollary 3.28, one can readily deduce the triviality of a geodesically complete, gradient, expanding Ricci almost soliton $(M, \langle\, ,\, \rangle, \nabla f)$ with $sup_M|\nabla f| < +\infty$ such that either $m = 2$ or $\langle \nabla\lambda, \nabla f\rangle \leq 0$ and satisfying $-\infty < \lambda_* = \inf_M \lambda \leq \lambda < 0$. This result however is contained in the more general Theorem 5.8 we are presenting below.

On the other hand, to deal with triviality of expanders under $L^{1<p<\infty}$ conditions we apply Theorem 2.25. Recall that, as observed in Remark 2.26, if $|\nabla f| \in L^p(M, e^{-f}d\mathrm{vol})$, then condition (5.2) below is satisfied.

Theorem 5.4. *Let $(M, \langle\, ,\, \rangle, \nabla f)$ be a geodesically complete, expanding Ricci soliton. If*

$$\frac{1}{\int_{\partial B_r} |\nabla f|^p \, e^{-f} d\mathrm{vol}_{m-1}} \notin L^1(+\infty), \tag{5.2}$$

for some $p > 1$ then the soliton is trivial.

Proof. Recall from equation (3.8) that

$$|\nabla f|\, \Delta_f\, |\nabla f| \geq -\lambda\, |\nabla f|^2 \geq 0, \text{ weakly on } (M, e^{-f}d\mathrm{vol}).$$

An application of Theorem 2.25 gives that $|\nabla f|$ is constant. Using this information into (3.7) we conclude that $|\nabla f| = 0$ and f is a constant function. □

Remark 5.5. Note again that, using (3.35), we can deduce triviality also for a geodesically complete gradient expanding Ricci almost soliton $(M, \langle\, ,\, \rangle, \nabla f)$ satisfying (5.2) for some $p > 1$ and such that either $m = 2$ or $\langle \nabla f, \nabla\lambda\rangle \leq 0$, with λ the soliton function.

Finally, in order to apply Theorem 2.28 and conclude triviality of expanders under solely L^1 conditions, we also need Zhang's estimates given in Lemma 3.9. In particular, if $(M, \langle\, ,\, \rangle, \nabla f)$ is a complete, expanding Ricci soliton we know that fixed a reference origin $o \in M$, there exists a constant $c > 0$ such that

(1) $|f(x)| \leq c(1 + r(x)^2)$,
(2) $|\nabla f| \leq c(1 + r(x))$.

Remark 5.6. Observe that, according to the scalar curvature estimates of Theorem 4.2, the above constant $c > 0$ can be expressed in terms of the soliton constant $\lambda < 0$ and the dimension of M.

As an immediate consequence of Theorem 2.28 we hence obtain the case $p = 1$ of Theorem 5.1.

Theorem 5.7. *Let $(M, \langle\, ,\, \rangle, \nabla f)$ be a geodesically complete, expanding Ricci soliton. If $|\nabla f| \in L^1(M, e^{-f}d\mathrm{vol})$ then the soliton is trivial.*

The following triviality theorem for gradient Ricci almost solitons extends in some direction Remark 5.3 and permit also to extend Theorem 5.2.

Theorem 5.8 (Theorem 0.2 in [**76**]). *Let* $(M, \langle\, ,\, \rangle, \nabla f)$ *be a complete, expanding gradient Ricci almost soliton with soliton function* λ. *Let* α, σ, $\mu \in \mathbb{R}$ *be such that*

$$\alpha > -2\,;\ 0 \leq \sigma \leq 2/3$$

$$\min\{0, -\alpha\} \leq \mu \leq \begin{cases} 1 - 3\sigma/2 & \text{if } \sigma \geq \alpha \\ 1 - \sigma - \alpha/2 & \text{if } \sigma < \alpha \end{cases} \tag{5.3}$$

Assume

$$\limsup_{r(x)\to+\infty} \frac{|\nabla f|^2}{r(x)^\sigma} \begin{cases} = 0 & \text{if } 0 < \sigma \leq 2/3 \\ < +\infty & \text{if } \sigma = 0 \end{cases} \tag{5.4}$$

$$-(m-1)\,B^2\left(1 + r(x)^2\right)^{\frac{\alpha}{2}} \leq \lambda(x) \leq -(m-1)\,A^2\left(1 + r(x)^2\right)^{-\frac{\mu}{2}} \tag{5.5}$$

on M *for some constants* $B \geq A > 0$.
Suppose either $m = 2$ *or*

$$\langle \nabla\lambda, \nabla f\rangle \leq 0 \text{ on } M. \tag{5.6}$$

Then, the almost soliton is trivial.

Note that (5.3) implies that (5.5) is meaningful.

Corollary 5.9. *Let* $(M, \langle\, ,\, \rangle, \nabla f)$ *be a complete, expanding gradient Ricci soliton such that*

$$\limsup_{r(x)\to+\infty} \frac{|\nabla f|^2}{r(x)^\sigma} \begin{cases} = 0 & 0 < \sigma \leq \frac{2}{3} \\ < +\infty & \sigma = 0. \end{cases} \tag{5.7}$$

Then the soliton is trivial.

The case $\sigma = 0$ of Corollary 5.9 recovers Theorem 5.2.

To prove Theorem 5.8 we will use the comparison result we have proven in Theorem 2.10 and the version of Theorem 5.1 in [**56**] stated in Theorem 2.19.

Proof (of Theorem 5.8). First of all from Lemma 3.22 and assumption (5.6) we know that $|\nabla f|^2$ satisfies the differential inequality

$$\Delta_f |\nabla f|^2 \geq -2\lambda\, |\nabla f|^2 \tag{5.8}$$

on M. Furthermore, from (5.4) we deduce

$$\langle \nabla r, \nabla f\rangle \geq -a\,(1 + r)^{\frac{\sigma}{2}}\,,$$

for some constant $a > 0$. Using (5.5) we apply Theorem 2.10 with the choice $\theta(r) = a\,(1 + r)^{\frac{\sigma}{2}}$ to obtain

$$vol_f(B_r) \leq D \int_0^r h(t)^{m-1}\, e^{\frac{2a}{(\sigma+2)(m-1)}(1+t)^{\frac{\sigma+2}{2}}}\, dt$$

for some constant $D > 0$ and where h solves (2.16) with $G(r) = B^2 \left(1 + r^2\right)^{\frac{\alpha}{2}}$. By Proposition 2.11 of [**82**] it follows that, for $r \gg 1$,

$$h(r) \leq \begin{cases} C_1 \exp(C_2 r^{\frac{\alpha+2}{2}}) & \text{if } \alpha \geq 0 \\ C_1 r^{-\frac{\alpha}{4}} \exp(C_2 (1+r)^{\frac{\alpha+2}{2}}) & \text{if } -2 < \alpha \leq 0, \end{cases}$$

for some constant C_1, C_2. Thus, a simple computation shows that

$$\frac{\log \mathrm{vol}_f (B_r)}{r^{2-\mu-\sigma}} \leq C \left(r^{\mu+\sigma-1+\frac{\alpha}{2}} + r^{\mu-1+\frac{3}{2}\sigma} \right)$$

for $r >> 1$ and some constant $C > 0$. Using (5.3) we see that assumption (2.31) of Theorem 2.19 is satisfied with $\nu = \mu + 2(\sigma - 1)$ so that $\sigma - \nu = 2 - \mu - \sigma$. On the other hand, from (5.8) and (5.5) we have

$$(1 + r(x))^{\mu} \Delta_f |\nabla f|^2 \geq H |\nabla f|^2 \tag{5.9}$$

for some appropriate constant $H > 0$. Assume that $|\nabla f|$ is different from 0 and choose $\gamma > 0$ so that

$$\Omega_\gamma = \left\{ x \in M : |\nabla f|^2 > \gamma \right\} \neq \emptyset.$$

From (5.4), (5.9) and Theorem 2.19 we immediately obtain a contradiction. □

5.1.2. Triviality in the presence of weighted Poincaré–Sobolev inequalities. In the next triviality result we shall assume the validity of a weighted Poincaré–Sobolev inequality on M. In a sense it can be considered as an isolation result for the soliton function of almost solitons with L^p soliton structure.

Theorem 5.10. *(Theorem 0.6 in [**76**]) Let $(M, \langle\, ,\, \rangle, \nabla f)$ be a complete gradient Ricci almost soliton with soliton function λ. For some $0 \leq \alpha < 1$ assume on M the validity of*

$$\int_M |\nabla \varphi|^2 e^{-f} d\mathrm{vol} \geq S(\alpha)^{-1} \left\{ \int_M |\varphi|^{\frac{2}{1-\alpha}} e^{-f} d\mathrm{vol} \right\}^{1-\alpha} \tag{5.10}$$

for all $\varphi \in C_c^\infty(M)$ and some constant $S(\alpha) > 0$. Suppose that

$$\int_{B_r} |\nabla f|^p e^{-f} d\mathrm{vol} = o\left(r^2\right) \tag{5.11}$$

as $r \to +\infty$, for some $p > 1$, and that

$$\|\lambda_+(x)\|_{L^{\frac{1}{\alpha}}\left(M, e^{-f} d\mathrm{vol}\right)} < \frac{4}{S(\alpha)} \frac{p-1}{p^2},$$

with $\lambda_+(x) = \max\{0, \lambda(x)\}$. Suppose finally that either $m = 2$ or (5.6) is satisfied. Then the almost soliton is trivial and, when $m \geq 3$, M is Einstein with non–positive Ricci curvature.

As an immediate consequence we obtain,

Corollary 5.11. *Let $(M, \langle\, ,\, \rangle \nabla f)$ be a complete, expanding, gradient Ricci soliton and assume that (5.10) and (5.11) hold for some $0 \leq \alpha < 1$ and $p > 1$. Then the soliton is trivial.*

Remark 5.12. (a) Note that, according to the variational characterization of the bottom of the spectrum of the f–Laplacian, assumption (5.10) with $\alpha = 0$ means

$$\lambda_1^{-\Delta_f}(M, e^{-f}d\mathrm{vol}) > 0.$$

Thus, in particular, inequality (5.10) with $\alpha = 0$ holds if the almost soliton $(M, \langle\, , \,\rangle, \nabla f)$ is expanding and satisfies:

$$Sec_{rad} \leq -K \leq 0 \text{ and } \frac{\partial f}{\partial r} \leq 0.$$

This follows from Theorem 3.4 in [**89**].

(b) Condition (5.10) implies that $\mathrm{vol}_f(M) = +\infty$. Indeed, let $\varphi \in C_c^\infty$ be such that $\varphi = 1$ on B_R, $\varphi = 0$ off B_{2R}, $|\nabla\varphi| \leq \frac{C}{R}$. Then, from (5.10)

$$\begin{aligned}\frac{C^2}{R^2}\int_{B_{2R}\setminus B_R} e^{-f}d\mathrm{vol} \geq &\int_M |\nabla\varphi|^2 e^{-f}d\mathrm{vol}\\ \geq &S(\alpha)^{-1}\left(\int_M |\varphi|^{\frac{2}{1-\alpha}} e^{-f}d\mathrm{vol}\right)^{1-\alpha}\\ \geq &S(\alpha)^{-1}\left(\int_{B_R} e^{-f}d\mathrm{vol}\right)^{1-\alpha},\end{aligned}$$

i.e

$$\frac{C^2}{R^2}\left\{vol_f(B_{2R}) - vol_f(B_R)\right\} \geq S(\alpha)^{-1} vol_f(B_R)^{1-\alpha}.$$

To prove Theorem 5.10 we need again a preliminary result. The next proposition can be deduced by simple modifications to the proof of Theorem 9.12 in [**82**].

Proposition 5.13. *Let $\left(M, \langle\, , \,\rangle, e^{-f}d\mathrm{vol}\right)$ be a complete weighted manifold and assume that, for some $0 \leq \alpha < 1$, the Poincaré–Sobolev inequality*

$$\int_M |\nabla\varphi|^2 e^{-f}d\mathrm{vol} \geq S(\alpha)^{-1}\left\{\int_M |\varphi|^{\frac{2}{1-\alpha}} e^{-f}d\mathrm{vol}\right\}^{1-\alpha}, \tag{5.12}$$

holds for all $\varphi \in C_0^\infty(M)$ and some constant $S(\alpha) > 0$. Let $B \in \mathbb{R}$, $q(x) \in C^0(M)$ and let $\psi \in Lip_{loc}(M)$ be a non–negative weak solution of

$$\psi\Delta_f\psi + q(x)\psi^2 \geq -B|\nabla\psi|^2$$

on M. Assume that, for some

$$\sigma > \max\{1, B+1\}$$

we have

$$\int_{B_r} \psi^\sigma e^{-f}d\mathrm{vol} = o\left(r^2\right)$$

as $r \to +\infty$. Then either $\psi \equiv 0$ or otherwise

$$\|q_+(x)\|_{L^{\frac{1}{\alpha}}\left(M, e^{-f}d\mathrm{vol}\right)} \geq \frac{4}{S(\alpha)}\frac{p-1}{p^2}.$$

An immediate consequence of Proposition 5.13 is the following

Theorem 5.14. *Let $(M, \langle\, ,\, \rangle, \nabla f)$ be a complete, gradient Ricci almost soliton with soliton function λ. Suppose either $m = 2$ or otherwise*

$$\langle \nabla f, \nabla \lambda \rangle \leq 0. \tag{5.13}$$

Assume the validity of the Poincaré–Sobolev inequality (5.12) for some $0 \leq \alpha < 1$ and suppose that for some $p > 1$

$$\int_{B_r} |\nabla f|^p \, e^{-f} d\mathrm{vol} = o\left(r^2\right)$$

as $r \to +\infty$. Then either the almost soliton is trivial or

$$\|\lambda_+(x)\|_{L^{\frac{1}{\alpha}}\left(M, e^{-f} d\mathrm{vol}\right)} \geq \frac{4}{S(\alpha)} \frac{\sigma - 1 - B}{\sigma^2}. \tag{5.14}$$

Proof. From Corollary 3.23 and (5.13) we deduce

$$|\nabla f| \, \Delta_f |\nabla f| + \lambda |\nabla f|^2 \geq 0.$$

Thus we can apply Proposition 5.13 with $p = \sigma$, $B = 0$, $q(x) = \lambda(x)$ to deduce that either the almost soliton is trivial or (5.14) holds. □

Theorem 5.10 now follows immediately from Theorem 5.14. Indeed, since $m \geq 3$, a trivial almost soliton is necessarily Einstein and the soliton function λ must be constant. On the other hand, the Poincaré–Sobolev inequality implies that M has infinite volume, therefore $\lambda_+ = 0$.

5.2. Triviality results for k–quasi–Einstein manifolds and Einstein warped products

5.2.1. Triviality under $L^{1<p<\infty}$ conditions. It is well known that steady or expanding compact Ricci solitons are necessarily trivial. The same result is proven in [**51**] for quasi–Einstein metrics on compact manifolds with finite k. For Ricci solitons a generalization to the complete non–compact setting was obtained in Theorem 5.1. In this section using the scalar curvature estimates of Theorem 4.5, we get triviality for (non–necessarily compact) k–quasi–Einstein metrics with $k < +\infty$, $\lambda \leq 0$.

Theorem 5.15 (Theorem 5 in [**88**])**.** *Let $(M^m, g_M, e^{-f} d\mathrm{vol})$ be a geodesically complete non–compact k–quasi–Einstein manifold, $1 \leq k < +\infty$. If the quasi–Einstein constant λ is non-positive and f satisfies, for some $1 < p < +\infty$,*

$$f \in L^p(M, e^{-\frac{f}{k}} d\mathrm{vol}), \tag{5.15}$$

and $\inf_M f = f_* > -\infty$, then either $f \equiv const \leq 0$ and M is Einstein or $f > 0$.

Proof. (of Theorem 5.15) Tracing (3.46) and letting $\widehat{f} = \frac{1}{k} f$ we have that

$$\Delta_{\widehat{f}} f = m\lambda - S. \tag{5.16}$$

Since $\lambda \leq 0$ and $f_* > -\infty$, from (4.6) of Theorem 4.5 we obtain that $\Delta_{\widehat{f}} f \leq 0$. Applying Theorem 2.25 to $f_- = \max\{-f, 0\} \in L^p(M, e^{-\widehat{f}} d\mathrm{vol})$, gives that f_- is constant. Hence, if there exists a point $x_0 \in M$ such that $f(x_0) \leq 0$ then $f \equiv f(x_0) \leq 0$. □

Remark 5.16. From the proof it follows that if either M is compact or f attains its absolute minimum then $f \equiv const$. Actually, it was pointed out to us by Dezhong Chen that the same conclusion holds if we merely assume that f attains a local minimum at some point $x_0 \in M$. Indeed the following proposition holds.

Proposition 5.17. *Let $(M, g_M, e^{-f}\mathrm{dvol})$ be a geodesically complete non-compact k–quasi–Einstein manifold, $1 < k < +\infty$. If the quasi–Einstein constant λ is non positive and f satisfies $f_* > -\infty$, then any local minimum of f is actually an absolute minimum.*

Proof. Assume that f attains a local minimum $x_0 \in M$. Evaluating (5.16) at x_0, we get

$$S(x_0) \leq m\lambda.$$

Hence, since $\lambda \leq 0$, by Theorem 4.5, M is Einstein and S is identically $m\lambda$. Thus the quasi–Einstein equation (3.46) reads

$$Hess(f) = \frac{1}{k} df \otimes df. \tag{5.17}$$

In particular $Hess(f)$ is positive semi-definite on M and this implies the thesis. □

Now, exploiting the link between Einstein warped product metrics and k–quasi–Einstein metrics recalled in Theorem 3.33 we are able to prove the following triviality result for Einstein warped products which extends, to the case of non–compact bases, a recent theorem by D.–S. Kim and Y.–H. Kim, [**51**].

Theorem 5.18 (Theorem 1 in [**88**])**.** *Let $N^{m+k} = M^m \times_u F^k$, $k > 1$, be a complete Einstein warped product with non–positive scalar curvature ${}^N S \leq 0$, warping function $u(x) = e^{-\frac{f(x)}{k}}$ satisfying $\inf_M f = f_* > -\infty$ and complete Einstein fibre F. Then N is simply a Riemannian product if either one of the following further conditions is satisfied:*

(a) *f has a local minimum.*
(b) *the base manifold M is complete and non-compact, the warping function satisfies $\int_M |f|^p e^{-\frac{f}{k}}\mathrm{dvol} < +\infty$, for some $1 < p < +\infty$, and $f(x_0) \leq 0$ for some point $x_0 \in M$.*

Note that, in case M is compact, from the point (a) we recover the main result in [**51**]. In a sense, also (b) can be considered as a natural extension of this result because, if M is compact, we can always take $\widetilde{f} = f - c$ so that $\widetilde{f} \leq 0$ and $\widetilde{f} \in L^p(M, e^{-\frac{\widetilde{f}}{k}}\mathrm{dvol})$ and the triviality follows.

Proof. According to Theorem 3.33, M is k–quasi–Einstein. Statement (a) follows immediately from Remark 5.16 and Proposition 5.17. In case (b), since $(m + k)\lambda = {}^N S \leq 0$, we get by Theorem 5.15 that f, and so u, is a constant function. □

Our goal now is to deduce a triviality result for complete Einstein warped products, which is a corollary of Theorem 5.18, replacing the integrability assumption with weight $e^{-\frac{f}{k}}$ in the aforementioned theorem with a more natural condition. This will be done making an essential use of the Motomiya–type theorem we stated in Theorem 2.23.

Indeed, consider the equation which relates the Einstein constants of the product and of the fibre

$$\Delta_f f = k\lambda - k\mu e^{\frac{2}{k}f}. \tag{5.18}$$

and let $\mu < 0$. If we choose $\varphi(t) = \Phi(t, y) = m\lambda - m\mu e^{\frac{2}{k}t}$ and $F(t) = (t-a)^\sigma$, with $\sigma > 1$, then F satisfies the assumptions of Theorem 2.23. Moreover by Corollary 3.43 we know that the full Omori–Yau maximum principle for Δ_f holds on a generic k–quasi–Einstein manifold, $k < \infty$.

Hence, using Theorem 2.23, we can deduce the following result.

Corollary 5.19. *Let $N^{m+k} = M^m \times_u F^k$ be a complete Einstein warped product with non–positive scalar curvature ${}^N S \leq 0$, warping function $u(x) = e^{-\frac{f(x)}{k}}$ satisfying $\inf_M f = f_* > -\infty$ and complete Einstein fibre F. Suppose also that ${}^F S < 0$. Then $f^* < +\infty$. In particular Riemannian volumes are equivalent to f–weighted volumes.*

Proof. Recall that $\lambda = \frac{{}^N S}{m+k} < 0$. Applying Theorem 2.23 to equation (5.18) we obtain that $f^* < +\infty$. Since, by assumption, we know also that $f_* > -\infty$ the thesis follows. □

From Corollary 5.19 and Theorem 5.18 we thus immediately get the desired corollary of Theorem 1 in [**88**].

Corollary 5.20 (Corollary 13 in [**60**], see also [**58**])**.** *Let $N^{m+k} = M^m \times_u F^k$, $k > 1$, be a complete Einstein warped product with non-positive scalar curvature $(m+k)\lambda = {}^N S \leq 0$, warping function $u(x) = e^{-\frac{f(x)}{k}}$ satisfying $\inf_M f = f_* > -\infty$ and complete Einstein fibre F. Suppose also that ${}^F S < 0$. Then N is simply a Riemannian product if the base manifold M is complete and non-compact, the warping function satisfies $f \in L^p(M, e^{-f}\mathrm{dvol})$, for some $1 < p < +\infty$, and $f(x_0) \leq 0$ for some point $x_0 \in M$.*

From Theorem 2.23 we can deduce also the following

Theorem 5.21 (Theorem 14 in [**60**])**.** *Let $N^{m+k} = M^m \times_u F^k$ be a complete Einstein warped product with non–positive scalar curvature $(m+k)\lambda = {}^N S \leq 0$, warping function $u(x) = e^{-\frac{f(x)}{k}}$ satisfying $\inf_M f = f_* > -\infty$ and complete Einstein fibre F with ${}^F S < 0$. Then ${}^M S_* = m\lambda$.*

Proof. As above, by Theorem 2.23, we have that $f^* < +\infty$ and so

$$vol_{\widehat{f}}(M) \leq vol_f(M) e^{\frac{k-1}{k} f^*}$$

From the weighted volume estimates by Qian of Theorem 2.5 and Theorem 2.17 we get that the weak Omori–Yau maximum principle for the $\widehat{f}$-Laplacian holds on M. Hence we can construct a sequence $\{x_n\}$ such that $f(x_n) \to f_*$

and $\Delta_{\widehat{f}} f(x_n) \geq -\frac{1}{n}$. Thus, since tracing (3.46) we have that $\Delta_{\widehat{f}} f = m\lambda - {}^M S$, we obtain that

$$-\frac{1}{n} \leq m\lambda - {}^M S(x_n) \leq m\lambda - {}^M S_* \leq 0,$$

where in the last inequality we have used the estimates of Theorem 4.5. The conclusion now follows taking the limit for $n \to +\infty$. □

Remark 5.22. Let us mention that applying Corollary 3.39 one can try to obtain triviality results under L^p conditions on $|\nabla f|$. However this can be carried out only if $k = 1, 2$ and imposing additional assumptions on f and on ${}^M S$. Thus the resulting conclusions are rather unsatisfactory and we are not going to present them here.

5.2.2. Other Triviality results. Another triviality result for Einstein warped products has been obtained by J. Case in [**20**].

Theorem 5.23 (Corollary 1.3 in [**20**]). *Let $N^{m+k} = M^m \times_u F^k$ be a complete Einstein warped product with warping function $u(x) = e^{-\frac{f(x)}{k}}$, scalar curvature ${}^N S \geq 0$ and complete Einstein fibre F. Then N is simply a Riemannian product provided the base manifold M is complete and the scalar curvature of F satisfies ${}^F S \leq 0$.*

In the following theorem we obtain the same conclusion in case the fibers have non–negative scalar curvature, up to assume an integrability condition on the warping function u. Note that, as observed in Example 3.34, non–trivial examples with ${}^N S \leq 0$ and ${}^F S \geq 0$ are constructed in [**5**, Theorem 9.119]. Thus the integrabilty assumption is necessary.

Theorem 5.24. *Let $N^{m+k} = M^m \times_u F^k$ be a complete Einstein warped product with warping function $u(x) = e^{-\frac{f(x)}{k}}$, scalar curvature ${}^N S \leq 0$, and complete Einstein fibre F. Then N is simply a Riemannian product provided the base manifold M is complete, the warping function satisfies $\int_M e^{-(\frac{p+k}{k})f} d\mathrm{vol} < +\infty$ for some $1 < p < +\infty$, and the scalar curvature of F satisfies ${}^F S \geq 0$. In this case M and F are Ricci flat and M is compact.*

Thus combining Theorem 5.23 and Theorem 5.24 immediately gives the following

Corollary 5.25. *Let N be a complete Ricci flat warped product with complete Einstein fibre F and warping function $u(x) = e^{-\frac{f(x)}{k}}$ satisfying $u \in L^p(M, e^{-f} d\mathrm{vol})$, for some $1 < p < +\infty$. Then N is simply a Riemannian product.*

Proof (of Theorem 5.24). Just observe that computing the f–Laplacian of u and using (3.47) one obtains the following equation

$$\Delta_f u = \mu u^{-1} - \lambda u + \frac{u}{k^2} |\nabla f|^2, \tag{5.19}$$

where $\lambda = \frac{{}^N S}{m+k} \leq 0$ and $\mu = \frac{{}^F S}{k} \geq 0$. Thus, in our assumptions, we obtain that $\Delta_f u \geq 0$. Since $0 < u \in L^p(M, e^{-f} d\mathrm{vol})$, by Theorem 2.25, we obtain the constancy of u. Up to a rescaling of the metric of F we can suppose $u = 1$.

Now, since the Riemannian product $M \times F$ is Einstein, both M and F are Einstein manifolds with the same Einstein constant. In particular, ${}^{M}S$ and ${}^{F}S$ have the same sign. By our assumption on the signs of ${}^{N}S$ and ${}^{F}S$ we thus obtain that both M and F are Ricci flat. Finally, since u (and thus f) is constant, from the integrability condition we obtain that $\mathrm{vol}(M) < +\infty$. Since by a result of Calabi and Yau, [Y], a complete, non–compact manifold with $\mathrm{Ric} \geq 0$ has at least linear volume growth, we obtain that M must be compact. □

To prove the next result it suffices to observe that by Corollary 3.42 the L^1–Liouville property for Δ_f–superharmonic functions holds on any k–quasi-Einstein manifold $(M, g_M, e^{-f}d\mathrm{vol})$, $k < \infty$, and apply it to equation (5.18).

Theorem 5.26. *Let $N^{m+k} = M^m \times_u F^k$ be a complete Einstein warped product with warping function $u(x) = e^{-\frac{f(x)}{k}}$, scalar curvature ${}^{N}S \leq 0$, and complete Einstein fibre F. Then N is simply a Riemannian product provided the base manifold M is complete, f satisfies $0 \leq f \in L^1(M, e^{-f}d\mathrm{vol})$, and the scalar curvature of F satisfies ${}^{F}S \geq 0$.*

In [**20**] J. Case deals with the triviality of quasi–Einstein metrics, and hence of Einstein warped products, by considering only equation (5.18). The proof of Theorem 5.23 is a consequence of a gradient estimate for solutions of weighted Poisson equation. However in that work only the case $\lambda \geq 0$ is studied. Obtaining a similar estimate in case $\lambda < 0$, in [**60**] (see also [**58**]) it is proved the following Theorem. For a proof we refer to [**60**], [**58**].

Theorem 5.27 (Theorem 7 in [**60**])**.** *Let $N = M^m \times_u F^k$ be a complete Einstein warped product with warping function $u = e^{-\frac{f}{k}}$, scalar curvature ${}^{N}S = (m+k)\lambda < 0$ and complete Einstein fibre F^k with scalar curvature ${}^{F}S = k\mu < 0$. Suppose that*

$$f \geq \frac{k}{2} \log\left(\frac{\lambda}{2\mu}\frac{m+2k}{m+k}\right) \quad \text{for all } x \in M.$$

Then N is simply a Riemannian product.

The last result we present is a non–existence result. Recall that by Corollary 3.42 the weak maximum principle for Δ_f holds on any k–quasi–Einstein manifold $(M, g_M, e^{-f}d\mathrm{vol})$, $k < +\infty$.

Theorem 5.28. *There is no complete Einstein warped product $N = M^m \times_u F^k$ with warping function $u \in L^\infty(M)$, scalar curvature ${}^{N}S < 0$ and Einstein fibre F with ${}^{F}S \geq 0$.*

Proof. Set, as usual, $u = e^{-\frac{f}{k}}$. Since $k\mu =^F S \geq 0$ and $(m+k)\lambda =^N S$, from (5.19), we have that

$$\Delta_f u \geq -u\lambda. \tag{5.20}$$

Since, by assumption, u satisfies $\sup_M u = u^* < +\infty$, by the weak Omori–Yau maximum principle for Δ_f, there exists a sequence $\{x_n\} \subset M$ along which $u(x_n) \geq u^* - \frac{1}{n}$ and $\Delta_f u(x_n) \leq \frac{1}{n}$. Thus evaluating (5.20) along $\{x_n\}$ and taking the limit as $n \to +\infty$ we obtain that $\lambda u^* \geq 0$ and since $u^* > 0$ we cannot have $\lambda < 0$. □

In Section 3.3 we have represented in a table most of the known examples of k–quasi–Einstein manifolds. To better visualize under what sign assumptions on λ and μ the various conditions on the potential function f or on the warping function u imply triviality, we now fill Table 3.1 with the results we have discussed in this section.

TABLE 5.1. Triviality results for quasi–Einstein manifolds

	$\lambda < 0$	$\lambda = 0$	$\lambda > 0$
$\mu < 0$	Ex. 3.34 $\begin{cases} f_* > +\infty \\ f \in L^{1<p<\infty}(M, e^{-f}\,\mathrm{dvol}) \\ \exists\, x_0 \in M \quad \text{s.t.} \quad f(x_0) \leq 0 \end{cases}$ ⇓ Trivial (Cor. 5.20) $f \geq \frac{k}{2} \log\left(\frac{\lambda}{2\mu}\frac{m+2k}{m+k}\right)$ ⇓ Trivial (Th.5.27)	Trivial (Th. 5.23)	Trivial
$\mu = 0$	Ex. 3.34 $u = e^{-\frac{f}{k}} \notin L^\infty(M)$ (Th. 5.28)	$\begin{cases} f_* > +\infty \\ \exists \text{ loc. min. of } f \end{cases}$ or $\begin{cases} f_* > +\infty \\ f \in L^{1<p<\infty}(M, e^{-\frac{f}{k}}\,\mathrm{dvol}) \\ \exists\, x_0 \in M \quad \text{s.t.} \quad f(x_0) \leq 0 \end{cases}$ ⇓ Trivial (Th.5.18) Trivial (Th. 5.23)	Trivial
$\mu > 0$	Ex. 3.34 $u \in L^{1<p<\infty}(M, e^{-f}\,\mathrm{dvol})$ ⇓ Trivial (Cor. 5.25) $u = e^{-\frac{f}{k}} \notin L^\infty(M)$ (Th. 5.28)	Ex. 3.34	Ex. 3.35

CHAPTER 6

Topological rigidity

Rigidity of weighted manifolds can appear also at the level of the topology and is visible, for instance, in the structure of their fundamental group under weighted Ricci curvature restrictions. In this direction the situation for Ric_f and for Ric_f^k is quite different. The reason can be traced, once again, in the different nature of Bochner formulas involving these two tensors. We will see that, up to consider the virtual (co)dimension, Myers–type compactness conclusions can be proven also for weighted manifold with a lower bound on Ric_f^k. Simple examples show that analogous results cannot be obtained for Ric_f. Nevertheless, as initially investigated in works of M. Fernández–López and E. García–Río in the compact case, [**35**], and later in the complete non–compact case by W. Wylie, [**96**], a close relationship beetween Ric_f and the fundamental group of a weighted manifold still survive. Namely, Myers–type results in this contest establish the finiteness of the fundamental group.

6.1. Myers–type results for Ric_f^k

An extension of the Myers' theorem to weighted manifolds with a positive lower bound on the k–Bakry–Emery Ricci tensor ($k < \infty$) is obtained by Z. Qian in [**86**]. For generalizations of Myers' theorem in a different direction see [**57**].

In this section we extend Qian's theorem by allowing some negativity of the k–Bakry–Emery Ricci tensor. The viewpoint is the same which we adopted, in the non–weighted case, in [**61**].

The starting point of our considerations is the integral estimate of Ric_f^k along minimizing geodesics presented in Lemma 2.1. From this some Myers–type results can be proven. For example we state the following which generalizes a theorem of G. J. Galloway, [**38**], where the constant lower bound for the curvature is perturbed by the derivative in radial direction of some bounded function.

Theorem 6.1. *Let $(M^m, \langle\,,\,\rangle, e^{-f}d\mathrm{vol})$ be a complete weighted manifold. Given two different points $p, q \in M$, let $\gamma_{p,q}$ be a minimizing geodesic from p to q parameterized by arc–length. Suppose that there exist constants $c > 0$ and $G \geq 0$ such that for each pair of points p, q it holds*

$$\mathrm{Ric}_f^k(\dot\gamma_{p,q}, \dot\gamma_{p,q})|_{\gamma_{p,q}(t)} \geq (m + k - 1)\left[c^2 + \frac{d}{dt}(g \circ \gamma_{p,q})\right],$$

for some $C^1(M)$ function g satisfing $\sup_M |g| \leq G$, $k < +\infty$. Then M is compact and

$$\operatorname{diam}(M) \leq \frac{1}{c}\left[\frac{2G}{c} + \sqrt{\frac{4G^2}{c^2} + \pi^2}\right]. \tag{6.1}$$

Proof. Define L to be the length of $\gamma_{p,q}$ between p and q and set $h(t) := \sin(\frac{\pi}{L}t)$. Compute

$$\int_0^L h^2(t)dt = \int_0^L \sin^2(\frac{\pi}{L}t)dt = \frac{L}{2}; \quad \int_0^L h'^2(t)dt = \frac{\pi^2}{L^2}\int_0^L \cos^2(\frac{\pi}{L}t)dt = \frac{\pi^2}{2L}.$$

Then, applying Lemma 2.1, we have

$$\begin{aligned}\frac{\pi^2(m+k-1)}{2L} &= \int_0^L (m+k-1)\, h'^2 \geq \int_0^L h^2 \operatorname{Ric}_f^k(\dot\gamma_{p,q}, \dot\gamma_{p,q})|_{\gamma_{p,q}} ds \\ &\geq c^2(m+k-1)\int_0^L h^2 + (m+k-1)\int_0^L h^2 \frac{d}{dt}(g\circ\gamma_{p,q}) \\ &= \frac{c^2(m+k-1)L}{2} + (m+k-1)h^2 g(\gamma_{p,q})\Big|_0^L \\ &\quad - (m+k-1)\left[\int_0^{\frac{L}{2}} (\frac{d}{dt}h^2)(g\circ\gamma_{p,q}) + \int_{\frac{L}{2}}^L (\frac{d}{dt}h^2)(g\circ\gamma_{p,q})\right] \\ &\geq \frac{c^2(m+k-1)L}{2} - (m+k-1)G\left[\int_0^{\frac{L}{2}} (\frac{d}{dt}h^2) + \int_{\frac{L}{2}}^L \left|\frac{d}{dt}h^2\right|\right] \\ &\geq \frac{c^2(m+k-1)L}{2} - 2(m+k-1)G\end{aligned} \tag{6.2}$$

Finally, this latter can be written as

$$c^2L^2 - 4GL - \pi^2 \leq 0,$$

which in turn implies (6.1), because p and q are arbitrary. □

Reasoning as in the classical case, ([**39**], [**10**], [**61**]) the validity of (2.6) and an integration by parts shows that the compactness of M depends on the behavior, and on the position of the zeros, of the solutions of the differential equation along minimizing geodesics

$$-h''(t) - K_\gamma^{f,k}h(t) = 0,$$

where, from now on, we denote

$$K_\gamma^{f,k} = \frac{\operatorname{Ric}_f^k(\dot\gamma,\dot\gamma)}{m+k-1}.$$

More precisely we have the validity of the following lemma, due to G. J. Galloway, [**39**].

Lemma 6.2 (Lemma 1 in [**39**]). *Let $(M^m, \langle\, ,\, \rangle, e^{-f}d\mathrm{vol})$ be a complete weighted manifold. Suppose there exists a point $q \in M$ such that for all* *geodesic* $\gamma : [0, +\infty) \to M$, *parameterized by arc–length, with* $\gamma(0) = q$, *the differential equation*

$$J\,h(t) = -h''(t) - K_\gamma^{f,k} h(t) = 0 \tag{6.3}$$

has a nontrivial weak solution $\widetilde{h}$ *with* $\widetilde{h}(t_1) = \widetilde{h}(t_2) = 0$ *for some* $0 \leq t_1 < t_2$ *depending on* γ, $k < +\infty$. *Then* M *is compact and*

$$\operatorname{diam} M \leq 2 \max_{\gamma:\gamma(0)=q} t_2. \tag{6.4}$$

Remark 6.3. Observe that the existence of a locally Lipschitz solution, globally defined in $[0, +\infty)$, of the Cauchy problem (6.3) with initial condition $h(0) = 0$ is guaranteed by minor changes to Proposition A.1 in [**6**].

For the sake of completeness we provide a somewhat direct proof of the above lemma.

Proof (of Lemma 6.2). First, we fix γ and show that γ stops minimizing beyond t_2. Without loss of generality we can suppose γ minimizes distances on $[0, t_2]$. Moreover we can assume t_2 is the first zero of $\widetilde{h}$ greater than t_1. This is well defined since $\widetilde{h}(t) > 0$ on $[t_1, t_1 + \eta]$ for some η small enough. Indeed $\widetilde{h}$ is an eigenfunction of J on $[t_1, t_2]$ corresponding to the eigenvalue 0. If, by contradiction $\widetilde{h}(t) = 0$ on a sequence $\{t_1 + \eta_n\}_1^\infty$ for some $\eta_n \searrow 0$, it would be $\widetilde{h} \equiv 0$ on $[t_1, t_1 + \eta]$ by the unique continuation principle of eigenfuctions. Hence, up to changing sign, we take $\widetilde{h} > 0$ on (t_1, t_2). Denote the bottom of the spectrum of the operator J restricted to the interval $[t_1, t_2]$ by

$$\lambda_1^{-J}([t_1, t_2]) = \inf_{\substack{h \in H^2([t_1, t_2]) \\ h(t_1) = h(t_2) = 0}} \frac{\int_{t_1}^{t_2} hJh}{\int_{t_1}^{t_2} h^2}.$$

From the above considerations, we have $\lambda_1^{-J}([t_1, t_2]) \leq 0$. On the other hand, by Lemma 2.1 and integrating by parts, we have that

$$\int_{t_1}^{t_2} h(t) Jh(t) dt = -\int_{t_1}^{t_2} h^2(t) K_\gamma^{f,k} dt + \int_{t_1}^{t_2} {h'}^2(t) dt \geq 0 \tag{6.5}$$

for all $0 \leq h \in Lip_{loc}(\mathbb{R})$ such that $h(t_1) = h(t_2) = 0$. In particular, replacing $\widetilde{h}$ to h in (6.5) gives that $\lambda_1^{-J}([t_1, t_2]) \geq 0$. Thus $\lambda_1^{-J}([t_1, t_2]) = 0$. Now, fix $\varepsilon > 0$ and define a new function $\widetilde{h}_\varepsilon$ on $[t_1, t_2 + \varepsilon]$ as

$$\widetilde{h}_\varepsilon(t) := \begin{cases} \widetilde{h}(t) & t \in [t_1, t_2] \\ 0 & t \in [t_2, t_2 + \varepsilon]. \end{cases}$$

We have that $\widetilde{h}_\varepsilon \in H^2([t_1, t_2 + \varepsilon])$ since it is H^2 on both $[t_1, t_2]$ and $[t_2, t_2 + \varepsilon]$ and it is $Lip_{loc}([t_1, t_2 + \varepsilon])$. This gives

$$\lambda_1^{-J}([t_1, t_2 + \varepsilon]) = \inf_{\substack{h \in H^2([t_1, t_2+\varepsilon]) \\ h(t_1) = h(t_2+\varepsilon) = 0}} \frac{\int_{t_1}^{t_2+\varepsilon} hJh}{\int_{t_1}^{t_2+\varepsilon} h^2} \leq \frac{\int_{t_1}^{t_2+\varepsilon} \widetilde{h}_\varepsilon J \widetilde{h}_\varepsilon}{\int_{t_1}^{t_2+\varepsilon} \widetilde{h}_\varepsilon^2} = 0. \tag{6.6}$$

We show that the inequality is strict. By contradiction, let $\lambda_1^{-J}([t_1,t_2+\varepsilon]) = 0$. Since $\widetilde{h}_\varepsilon$ realizes the minimum in (6.6), it would be an eigenfunction. Then it would be $\widetilde{h}_\varepsilon \equiv 0$ by unique continuation. Contradiction.
Thus there exists an eigenfunction v on $[t_1,t_2+\varepsilon]$ such that $v(t_1) = v(t_2+\varepsilon) = 0$, $v \geq 0$ and $Jv = \lambda_1^{-J}([t_1,t_2+\varepsilon])v$ is nonpositive and not identically 0. Applying Lemma 2.1, we obtain that γ can not minimize distances on $[t_1,t_2+\varepsilon]$, hence it stops minimizing at t_2 as claimed.
Now, fix a point $q \in M$ and let Γ be the set of geodesics γ_q parameterized by arc length such that $\gamma_q(0) = q$, define

$$\mathrm{conj}(q,\gamma_q) := \inf_{\gamma_q\in\Gamma} \{t : \gamma_q \text{ does not minimize on } [0,t]\}.$$

Set $\mathrm{conj}(q) = \cup_{\gamma:\gamma(0)=q}\mathrm{conj}(q,\gamma)$. Since M is complete, M is compact provided $\mathrm{conj}(q)$ is bounded (see [**1**]). This is trivial since the function $\mathrm{conj}(q,\gamma)$ is continuous with respect to the outgoing geodesic vector $\dot\gamma(0) \in \mathbb{S}^m$ by a result of M. Morse (Lemma 13.1 in [**65**]).
Finally let $p_1, p_2 \in M$ and consider the geodesics γ_1 and γ_2 joining respectively p_1 and p_2 to q. Both γ_1 and γ_2 are shorter than $\max_{\gamma:\gamma(0)=q} t_2$. Hence (6.4) is proved because of the arbitrarity of p_1 and p_2. □

Thus we are reduced to find sufficient condition on Ric_f^k for which solutions of the differential equation (6.3) have a first zero at finite time.

At this point, usually one applies oscillation theory to get geometric assumptions to guarantee that M is compact; we refer to [**90**] and [**39**] for a more detailed discussion on oscillation theory and compactness. In particular, applying Theorem 2 in [**62**] by R. Moore, we obtain as in [**39**] the following Ambrose–type theorem (see also [**1**]). Recall that we have defined

$$K_\gamma^{f,k} = \frac{\mathrm{Ric}_f^k(\dot\gamma,\dot\gamma)}{m+k-1}.$$

Theorem 6.4. *Let $(M, \langle\, ,\, \rangle, e^{-f}d\mathrm{vol})$ be a complete weighted manifold and suppose that there is a point $q \in M$ such that along each geodesic $\gamma : [0,+\infty) \to M$ parameterized by arc–length with $\gamma(0) = q$ the condition*

$$\int_0^\infty t^\alpha K_\gamma^{f,k}(t)dt = +\infty \tag{6.7}$$

holds for some $0 \leq \alpha < 1$. Then M is compact.

Under the further assumption $\mathrm{Ric}_f^k \geq 0$, condition (6.7) can be improved. The following result applies an oscillation theorem of Nehari's, see Theorem III in [**68**].

Theorem 6.5. *Let $\mathrm{Ric}_f^k \geq 0$. Suppose that there is a point $q \in M$ such that along each geodesic $\gamma : [0,+\infty) \to M$ parameterized by arc–length with $\gamma(0) = q$ the condition*

$$\int_{t_0}^\infty t^\alpha K_\gamma^{f,k}(t)dt > \frac{(2-\alpha)^2}{4(1-\alpha)}\frac{1}{t_0^{1-\alpha}}$$

holds for some $t_0 > 0$ and $0 \leq \alpha < 1$. Then M is compact.

As a matter of fact, as we observed above, to conclude that M is compact oscillation theory is not strictly necessary and one could improve Theorem 6.4 and Theorem 6.5 by focusing attention upon the more general problem of the existence of a zero for solutions of (6.3). To the best of our knowledge, few steps have been done in this direction, also in the classical case. We point out the following adaptation to the weighted setting of a result in [**10**] by E. Calabi, where compactness is reached under assumptions which seem to be neither weaker nor stronger than those of Nehari's result.

Theorem 6.6. *Let $(M, \langle\, ,\, \rangle, e^{-f}\mathrm{dvol})$ be a complete weighted manifold with $\mathrm{Ric}_f^k \geq 0$. Suppose there is a point $q \in M$ such that along each geodesic $\gamma : [0, +\infty) \to M$ parameterized by arc–length with $\gamma(0) = q$ it holds*

$$\limsup_{a\to+\infty} \left\{ \int_0^a \sqrt{K_\gamma^{f,k}(t)}dt - \frac{1}{2\sqrt{m-1}} \log a \right\} = +\infty.$$

Then M is compact.

In [**61**], adapting the techniques introduced by Calabi, we were able to extend Theorem 6.4 and Theorem 6.5 to the case where the Ricci tensor is bounded from below by a negative constant. Minor changes to the proof of Theorem 5 in [**61**] lead to a similar compactness result in the weighted setting thus obtaining a Myers–type conclusion assuming a nonpositive lower bound on Ric_f^k.

Theorem 6.7. *Let $\mathrm{Ric}_f^k \geq -(m+k-1)B^2$, for some constant $B \geq 0$, $k < +\infty$. Suppose there is a point $q \in M$ such that along each geodesic $\gamma : [0, +\infty) \to M$ parameterized by arc–length, with $\gamma(0) = q$, it holds either*

$$\int_a^b tK_\gamma^{f,k}(t)dt > B\left\{b + a\frac{e^{2Ba}+1}{e^{2Ba}-1}\right\} + \frac{1}{4}\log\left(\frac{b}{a}\right). \tag{6.8}$$

or

$$\int_a^b t^\alpha K_\gamma^{f,k}(t)dt > B\left\{b^\alpha + a^\alpha\frac{e^{2Ba}+1}{e^{2Ba}-1}\right\} + \frac{\alpha^2}{4(1-\alpha)}\left\{a^{\alpha-1} - b^{\alpha-1}\right\} \tag{6.9}$$

for some $0 < a < b$ and $\alpha \neq 1$. Then M is compact.

Remark 6.8. In case $B = 0$ the expressions in Theorem 6.7 have to be intended in a limit sense. Namely (6.8) and (6.9) have to be replaced respectively by

$$\int_a^b tK_\gamma^{f,k}(t)dt > 1 + \frac{1}{4}\log\left(\frac{b}{a}\right) \tag{6.8'}$$

and

$$\int_a^b t^\alpha K_\gamma^{f,k}(t)dt > \frac{(2-\alpha)^2}{4\,(1-\alpha)\,a^{1-\alpha}} - \frac{\alpha^2}{4\,(1-\alpha)\,b^{1-\alpha}} \tag{6.9'}$$

Moreover we note that for $B > 0$ and $\alpha = 0$ assumption (6.9) has the more compact expression

$$(1 - e^{-2Ba})\int_a^b K_\gamma^{f,k}(t)dt > 2B. \tag{6.9''}$$

Remark 6.9. Consider a complete weighted manifold $(M, \langle\, ,\, \rangle, e^{-f}d\mathrm{vol})$ and its universal covering $\widetilde{M}$. Since the projection $\pi_M : \widetilde{M} \to M$ is a local isometry we note that geodesics of M (not necessarily minimizing) lift to geodesics of $\widetilde{M}$ and Ricci curvature is preserved. Define the function $\widetilde{f} = f \circ \pi_M$. Supposing we are in the assumptions of one of the theorems above, we have that also the weighted manifold $(\widetilde{M}, \widetilde{\langle\, ,\, \rangle}, e^{-\widetilde{f}}\widetilde{d\mathrm{vol}})$ satisfies the same set of assumptions and so it is compact. Hence we can also conclude that the fundamental group $\pi_1(M)$ is finite.

To prove Theorem 6.7 we will use a comparison result for Riccati equations, which is a generalization of Corollary 2.2 in [**82**].

Lemma 6.10 (Lemma 18 in [**61**])**.** *Let G and $0 < v$ be $C^0([0, +\infty))$ functions and let $q_i \in AC((\bar{t}, T_i))$, $i = 1, 2$, be solutions of the Riccati differential inequalities*

$$q_1'(t) - \frac{q_1^2(t)}{v(t)} - G(t) \geq 0, \qquad q_2'(t) - \frac{q_2^2(t)}{v(t)} - G(t) \leq 0, \tag{6.10}$$

a.e. in $(\bar{t}, T_i)$ satisfying $q_1(\bar{t}) = q_2(\bar{t})$ for some $\bar{t} > 0$. Then $T_1 \leq T_2$ and $q_1(t) \geq q_2(t)$ in $[\bar{t}, T_1)$.
Conversely, if $q_i \in AC((T_i, \bar{t}))$, $i = 1, 2$, are solutions of (6.10) a.e. in $(T_i, \bar{t})$ satisfying $q_1(\bar{t}) = q_2(\bar{t})$, then $T_1 \geq T_2$ and $q_1(t) \leq q_2(t)$ in $(T_1, \bar{t}]$.

This lemma is proven with minor changes to the proof of Corollary 2.2 in [**82**] and we refer to this for more details.

Proof. Let $q_i \in AC((\bar{t}, T_i))$, $i = 1, 2$, be solutions of (6.10) a.e. in $(\bar{t}, T_i)$, with $q_1(\bar{t}) = q_2(\bar{t})$. Setting $y_i = -q_i$ we obtain that

$$y_1'(t) + \frac{y_1^2(t)}{v(t)} + G(t) \leq 0, \qquad y_2'(t) + \frac{y_2^2(t)}{v(t)} + G(t) \geq 0. \tag{6.11}$$

Let $\phi_i \in C^1([\bar{t}, T_i))$ be the positive function on $[\bar{t}, T_i)$ defined by

$$\phi_i = exp\left\{\int_{\bar{t}}^{t} \left(\frac{y_i(s)}{v(s)}\right) ds\right\}. \tag{6.12}$$

Then $\phi_i(\bar{t}) = 1$, $\phi_i > 0$ on $(\bar{t}, T_i)$, $\phi_i' \in AC(\bar{t}, T_i)$ and a straightforward computation shows that

$$\phi_i'(t) = \frac{y_i}{v}\phi_i(t),$$
$$\phi_1'(\bar{t}) = \frac{y_1(\bar{t})}{v(\bar{t})}\phi_1(\bar{t}) = \frac{y_2(\bar{t})}{v(\bar{t})}\phi_2(\bar{t}) = \phi_2'(\bar{t})$$

and

(6.13)
$$\left(v\phi_1'\right)' + G\phi_1 \leq 0 \text{ a.e. in } (\bar{t}, T_1), \qquad \left(v\phi_2'\right)' + G\phi_2 \geq 0 \text{ a.e. in } (\bar{t}, T_2).$$

Adapting the Sturm comparison result of Theorem 2.1 in [**82**] to the differential inequalities (6.13) we have that if $\phi_i \in C^1([\bar{t}, T_i))$ are solutions of

(6.13) with the properties obtained above then

$$\frac{\phi_1'}{\phi_1} \le \frac{\phi_2'}{\phi_2}, \qquad T_1 \le T_2 \quad \text{and} \qquad \phi_1 \le \phi_2 \text{ on } [\bar{t}, T_1)\,.$$

This shows that $-q_1 = y_1 = \frac{\phi_1'}{\phi_1} v \le \frac{\phi_2'}{\phi_2} v = y_2 = -q_2$ on $(\bar{t}, T_1)$, as required. The second part of the lemma can be proven similarly making a change of variable from t to $-t$. ☐

We are now in the position to prove Theorem 6.7.

PROOF (OF THEOREM 6.7). First consider the case $B > 0$. Suppose M is noncompact. By Lemma 6.2 for each $q \in M$ there exists a geodesic γ parameterized by arc–length with $\gamma(0) = q$ such that each nontrivial Lip_{loc} solution h of the problem

$$\begin{cases} h''(t) + K_\gamma^{f,k}(t)h(t) = 0 \\ h(0) = 0, \end{cases}$$

which exists by the considerations at the beginning of this section, should satisfy $h(t) \ne 0$ for all $t > 0$. Hence the function $g(t) := -\frac{h'(t)}{h(t)}$ satisfies the differential equation

$$g'(t) = g^2(t) + K_\gamma^{f,k}(t). \tag{6.14}$$

We want to prove that

$$-\frac{e^{2Bt}+1}{e^{2Bt}-1} \le \frac{g(t)}{B} \le 1, \tag{6.15}$$

for all $t > 0$. To this purpose consider the functions

$$\widetilde{g}_C(t) = B\frac{C + e^{2Bt}}{C - e^{2Bt}}, \qquad C \ge 1,$$

which are solutions of the equation

$$\widetilde{g}'(t) = \widetilde{g}^2(t) - B^2$$

and note that for all $t > 0$ the lower bound on Ricci yields $g'(t) \ge \widetilde{g}_C'(t)$ at points where $g(t) = \widetilde{g}_C(t)$. Moreover $g'(t), \widetilde{g}_C'(t) \ge 0$ where $|g(t)| \ge B$ and

$$\begin{aligned} \widetilde{g}_C(t) &\to +\infty, \qquad \text{as } t \to (\log C/(2B))^-\,,\ C > 1, \\ \widetilde{g}_C(t) &\to -\infty, \qquad \text{as } t \to (\log C/(2B))^+\,,\ C \ge 1. \end{aligned}$$

By contradiction, suppose there is a value t_1 for which $g(t_1) = H_1 > B$. Then we have that

$$\widetilde{g}_{C_1}(t_1) = G_1 = g(t_1), \qquad \text{for } C_1 = \frac{G_1 + B}{G_1 - B} e^{2Bt_1} > 1.$$

Applying the first part of Lemma 6.10 with $q_1 = g$, $q_2 = \widetilde{g}_{C_1}$, $G \equiv -B^2$, $v \equiv 1$ and $\bar{t} = t_1$, we can conclude that $g(t) \to +\infty$ as $t \to t_0$ for some $0 < t_0 < \frac{\log C_1}{2B}$. Thus h is not globally defined. Contradiction. Similarly, suppose there is a value t_2 for which

$$g(t_2) = G_2 < -B\frac{e^{2Bt_2}+1}{e^{2Bt_2}-1}.$$

Then we have that

$$\widetilde{g}_{C_2}(t_2) = G_2 = g(t_2), \qquad \text{for } C_2 = \frac{G_2 + B}{G_2 - B} e^{2Bt_2} > 1.$$

As above, we achieve a contradiction by applying the second part of Lemma 6.10 with $q_1 = g$, $q_2 = \widetilde{g}_{C_2}$ and $\bar{t} = t_2$.
Now we want to use (6.14) and (6.15) to contradict (6.9). Then, for $\alpha \neq 1$,

$$\begin{aligned}
\int_a^b t^\alpha K_\gamma^{f,k}(t)dt &= \int_a^b (t^\alpha g'(t) - t^\alpha g^2(t))dt \\
&= \int_a^b \left[(t^\alpha g(t))' - t^\alpha \left(g(t) + \frac{\alpha}{2t} \right)^2 + \frac{\alpha^2}{4} t^{\alpha-2} \right] dt \\
&\leq b^\alpha g(b) - a^\alpha g(a) + \frac{\alpha^2}{4(\alpha-1)} \left[b^{\alpha-1} - a^{\alpha-1} \right] \\
&\leq B \left\{ b^\alpha + a^\alpha \frac{e^{2Ba}+1}{e^{2Ba}-1} \right\} + \frac{\alpha^2}{4(1-\alpha)} \left\{ a^{\alpha-1} - b^{\alpha-1} \right\}
\end{aligned} \tag{6.16}$$

for all $b > a > 0$. The case $\alpha = 1$ can be treated similarly. Finally observe that the computations above work even if we intend all the expressions in a limit sense as $B \to 0$. This concludes the proof. ☐

Remark 6.11. Reasoning as in the proof of Theorem 6.7, we can even find diameter estimates as follows. Suppose $\operatorname{diam} M > D$. Hence by Lemma 6.2 there exists a geodesic ray $\bar{\gamma}$, with $\bar{\gamma}(0) = q$, such that $\bar{\gamma}$ is minimizing at least on $(0, D/2)$. With notation as above, we have that g has to be defined and continuous at least on $(0, D/2)$. In analogy with (6.15), this fact and Riccati comparison force g to satisfy

$$-B \frac{e^{2Bt}+1}{e^{2Bt}-1} \leq g(t) \leq B \frac{e^{2B\left(\frac{D}{2}-t\right)}+1}{e^{2B\left(\frac{D}{2}-t\right)}-1}. \tag{6.17}$$

This estimate, together with the fact that $K_\gamma^{f,k} = g' - g^2$, leads to obtain integral conditions on $K_\gamma^{f,k}$, in the spirit of (6.16). For instance one can prove that $\operatorname{diam} M \leq D$ provided that

$$2 \int_0^{D/4} t^2 K_\gamma^{f,k}(t)dt > D.$$

6.2. Topological results for Ric_f

We have already pointed out in Section 2.2 that the full conclusion of the classical Myers–Bonnet theorem cannot be extended to Ric_f. Indeed the Gaussian space $(\mathbb{R}^m, \langle\, ,\, \rangle_{can}, e^{-|x|^2/2} d\mathrm{vol})$ is a non–compact, complete weighted manifold with $\mathrm{Ric}_f = 1 > 0$. In order to recover compactness we have to impose, besides the positive constant lower bound on Ric_f, further conditions on the growth of f or on its gradient. However, Myers–type results in this contest establish the finiteness of the fundamental group if $\mathrm{Ric}_f \geq c^2 > 0$. The next result extends in the direction of the classical Ambrose theorem ([**1**]) topological results obtained in [**96**], [**67**], [**35**] and [**33**].

Theorem 6.12. *Let $\left(M, \langle\, , \rangle\, , e^{-f}\mathrm{dvol}\right)$ be a geodesically complete weighted manifold, and assume that there exists a point $o \in M$ and functions $\mu \geq 0$ and g bounded such that for every unit speed geodesic γ issuing from $\gamma(0) = o$ we have*

$$\mathrm{Ric}_f(\dot{\gamma}, \dot{\gamma}) \geq \mu \circ \gamma + \langle \nabla g \circ \gamma, \dot{\gamma} \rangle$$

and

$$\int_0^{+\infty} \mu \circ \gamma(t) dt = +\infty.$$

Then, the following hold:

(a) *If the above conditions hold then $|\pi_1(M)| < \infty$.*
(b) *If in addition* $\mathrm{Ric} \leq c < +\infty$ *and $\mu = \mu_o(r(x))$ is radial, where $r(x) = \mathrm{dist}(x, o)$, then M is diffeomorphic to the interior of a compact manifold N with $\partial N \neq \emptyset$.*
(c) *If $\mu(x) \geq \mu_0 > 0$ and $\sup_M(|\nabla f| + |g|) \leq F < +\infty$, then M is compact and $diam(M) \leq \frac{1}{\mu_0}\left[2F + \sqrt{4F^2 + \pi^2(m-1)c}\right]$*

Clearly the theorem applies to almost Ricci solitons for which the soliton function λ satisfies the conditions listed in the statement.

Remark 6.13. Nevertheless hypothesis (6.12) brings to mind an Ambrose–type condition, it is not completely analogous to this latter. Indeed in Theorem 6.12 we are also assuming $\mathrm{Ric}_f \geq 0$ while in the genuine Ambrose theorem no sign assumption on the curvature is imposed. Actually, it would be interesting to find if this sign assumption can be removed. This would be an appetizer for the following more general problem.

Problem 6.14. Is it true, in some sense, that generically, the weighted counterpart of Myers–type results is the finiteness of the fundamental group?

The three conclusions of Theorem 6.12 can be deduced from the following lemmas which estimates the integral of Ric along geodesics.

Lemma 6.15. *Let $(M, \langle\, , \rangle)$ be a Riemannian manifold. Fix $o \in M$ and let $r(x) = dist(x, o)$. For any point $q \in M$, let $\gamma_q : [0, r(q)] \to M$ be a minimizing geodesic from o to q such that $|\dot{\gamma}_q| = 1$.*

(A) *If $h \in Lip_{loc}(\mathbb{R})$ is such that $h \geq 0$ and $h(0) = 0$, then, for every $q \notin \mathrm{cut}(o)$,*

$$h^2(r(q))\, \Delta r(q) \leq (m-1) \int_0^{r(q)} \left(h'\right)^2 ds - \int_0^{r(q)} h^2\, \mathrm{Ric}(\dot{\gamma_q}, \dot{\gamma_q})\, ds.$$

If in addition $h(r(q)) = 0$, then for every $q \in M$,

$$0 \leq (m-1) \int_0^{r(q)} \left(h'\right)^2 ds - \int_0^{r(q)} h^2\, \mathrm{Ric}(\dot{\gamma_q}, \dot{\gamma_q})\, ds.$$

(B) *For all $q \in M$ such that $r(q) > 2$, we have*

$$\int_0^{r(q)} \mathrm{Ric}(\dot{\gamma_q}, \dot{\gamma_q}) \leq 2(m-1) + H_o + H_q,$$

where, as in [**96**], *we have set*

$$H_p = \max\left\{0, \sup_{B_1(p)} \mathrm{Ric}\right\}, \forall p \in M$$

PROOF. Part (A) is well known. For a proof which avoids the use of the second variation formula for arc–length it suffices to consider f constant in the proof of Lemma 2.1.

To prove part (B) we note that if $h \in Lip_{loc}(\mathbb{R})$ is such that $h \geq 0$ and $h(0) = h(r(q)) = 0$, then we may rewrite (A) in the form

$$\int_0^{r(q)} \mathrm{Ric}(\dot{\gamma_q}, \dot{\gamma_q})\, ds \leq (m-1)\int_0^{r(q)} (h')^2\, ds + \int_0^{r(q)} \left(1-h^2\right) \mathrm{Ric}(\dot{\gamma_q}, \dot{\gamma_q})\, ds.$$

Choosing

$$h(s) = \begin{cases} s & 0 \leq s \leq 1 \\ 1 & 1 \leq s \leq r(q)-1 \\ r(q)-s & r(q)-1 \leq s \leq r(q), \end{cases}$$

where $r(q) > 2$, we obtain

$$\begin{aligned} \int_0^{r(q)} \mathrm{Ric}(\dot{\gamma_q}, \dot{\gamma_q})\, ds \leq & 2(m-1) + \int_0^1 \left(1-s^2\right) \mathrm{Ric}(\dot{\gamma_q}, \dot{\gamma_q})\, ds \\ & + \int_{r(q)-1}^{r(q)} \left(1-(r(q)-s)^2\right) \mathrm{Ric}(\dot{\gamma_q}, \dot{\gamma_q})\, ds \\ \leq & 2(m-1) + H_o + H_q. \end{aligned}$$

□

Lemma 6.16. *Let* $\left(M, \langle\,,\,\rangle, e^{-f}\mathrm{dvol}\right)$ *be a complete weighted Riemannian manifold. Fix* $o \in M$ *and let* $r(x) = dist(x, o)$ *and assume that there exist functions* μ *and* g *bounded such that for every unit speed geodesic* γ *issuing from* o

$$\mathrm{Ric}_f(\dot{\gamma}, \dot{\gamma}) \geq \mu(\gamma(t)) + \langle \nabla g, \dot{\gamma}\rangle.$$

Then for every such geodesic

$$\begin{aligned} \int_0^t \mathrm{Ric}(\dot{\gamma}, \dot{\gamma}) &= \langle \nabla f, \dot{\gamma}(0)\rangle - \langle \nabla f, \dot{\gamma}(t)\rangle + \int_0^t \mu(\gamma(s))\, ds + g(\gamma(t)) - g(o) \\ &\geq -\left|\nabla f_{\gamma(0)}\right| - \left|\nabla f_{\gamma(t)}\right| - 2\sup|g| + \int_0^t \mu(\gamma(s))\, ds. \end{aligned}$$

PROOF. By assumption

$$\mathrm{Ric}(\dot{\gamma}, \dot{\gamma}) + Hess(f)(\dot{\gamma}, \dot{\gamma}) \geq \mu \circ \gamma + \langle \nabla g, \dot{\gamma}\rangle, \tag{6.18}$$

which can be written in the form

$$\mathrm{Ric}(\dot{\gamma}, \dot{\gamma}) + \frac{d}{dt}\langle \nabla f(\gamma), \dot{\gamma}\rangle \geq \mu \circ \gamma + \frac{d}{dt}(g \circ \gamma).$$

Now integrating on $[0, t]$,

$$\int_0^t \mathrm{Ric}(\dot{\gamma}, \dot{\gamma}) + \langle \nabla f, \dot{\gamma}(t)\rangle - \langle \nabla f, \dot{\gamma}(0)\rangle \geq \int_0^t \mu(\gamma(s))\, ds + g(\gamma(t)) - g(o).$$

□

We are now in the position to give the

PROOF (OF THEOREM 6.12). Following [**96**], let us consider the Riemannian universal covering $P : M' \to M$ of M. Since P is a local isometry then M' is a weighted complete Riemannian manifold with weight $f' = f \circ P$. Moreover, since every unit speed geodesic γ' projects to a unit speed geodesic $\gamma = P \circ \gamma'$ we see that

$$\mathrm{Ric}'_{f'}(\dot\gamma', \dot\gamma') = \mathrm{Ric}_f(\dot\gamma, \dot\gamma) \geq \mu \circ \gamma + \frac{d}{dt}(g \circ \gamma) = \mu' \circ \gamma' + \frac{d}{dt}(g' \circ \gamma'),$$

where the function $g' = g \circ P$ is bounded and
$\mu' = \mu \circ P \geq 0$. satisfies

$$\int_0^{+\infty} \mu' \circ \gamma'(t) dt = \int_0^{+\infty} \mu \circ \gamma(t) dt = +\infty. \tag{6.19}$$

We identify

$$\pi_1 (M, o) = Deck(M'),$$

the covering transformation group, and recall that there is a bijective correspondence $\pi_1 (M, o) \longleftrightarrow P^{-1} (o)$. Therefore it suffices to show that $P^{-1} (o) \subset B'_R (o')$ for some $R >> 1$. Since $\pi_1 (M, o) = Deck(M')$ acts transitively on the fibre $P^{-1} (o)$, we have

$$P^{-1} (o) = \left\{ h \left(o' \right) : h \in Deck(M') \right\},$$

and we are reduced to showing that

$$r' \left(h \left(o' \right) \right) \leq R < \infty, \quad \forall h \in Deck(M'),$$

where we have set $r' (x') = dist_{M'} (o', x')$. Fix $h \in Deck(M')$ and a unit speed, minimizing geodesic $\gamma'_{h(o')} : [0, r' (h (o'))] \to M'$, issuing from $\gamma'_{h(o')} (0) = o'$. Recalling that $\mathrm{Ric}'(\dot\gamma', \dot\gamma') = \mathrm{Ric}'_{f'}(\dot\gamma', \dot\gamma') - \frac{d}{dt}\langle \nabla' f' \circ \gamma', \dot\gamma' \rangle$ and using Lemma 6.15 (B) and Lemma 6.16 we get

$$\begin{aligned}\int_0^{r'(h(o'))} \mu' \circ \ \gamma'_{h(o')} (s) \, ds \leq{}& 2 (m-1) + H_{o'} + H_{h(o')} \\ &+ \ \left| \nabla' f' \right| \left(o' \right) + \left| \nabla' f' \right| \left(h \left(o' \right) \right) + 2 \sup_{M'} |g'|.\end{aligned}$$

Since $P : M' \to M$ is a local isometry and $o', h (o') \in P^{-1} (o)$ we deduce

$$\left| \nabla' f' \right| \left(o' \right) = |\nabla f| (o) = \left| \nabla' f' \right| \left(h \left(o' \right) \right).$$

On the other hand $Deck(M') \subset Iso(M')$, so $h(B'_1(o'))$ is isometric to $B'_1 (h (o'))$ and we have

$$|H_{o'}| = \left| H_{h(o')} \right|.$$

Summarizing, we have obtained that, for every $h \in Deck (M')$,

$$\int_0^{r'(h(o'))} \mu' \circ \ \gamma'_{h(o')} (s) \, ds \leq 2 \left\{ (m-1) + H_{o'} + |\nabla f| (o) \right\} + 2 \sup_M |g|. \tag{6.20}$$

With this preparation, we now argue by contradiction and suppose that there exists a sequence of transformations $\{h_n\} \subset Deck (M')$ such that

$$r' \left(h_n \left(o' \right) \right) \to +\infty, \text{ as } n \to +\infty. \tag{6.21}$$

Let $\gamma'_{h_n(o')}(s) = \exp_{o'}(s\xi'_n)$, where $\{\xi'_n\} \subset \mathbb{S}^{m-1}_{o'} \subset T_{o'}M'$. Then, there exists a subsequence $\left\{\xi'_{n_k}\right\} \to \xi' \in \mathbb{S}^{m-1}_{o'}$ as $k \to +\infty$ and, by the Ascoli–Arzelà's Theorem, the sequence of minimizing geodesics $\left\{\gamma'_{h_{n_k}(o')}\right\}$ converges uniformly on compact subintervals of $[0,+\infty)$ to the unit speed geodesic $\gamma'(s) = \exp_{o'}(s\xi')$. Since, by (6.19)

$$\int_0^{+\infty} \mu' \circ \gamma'(s)\, ds = +\infty,$$

we can choose $T >> 1$ such that

$$\int_0^{T} \mu' \circ \gamma'(s)\, ds > 2\left\{(m-1) + H_{o'} + |\nabla f|(o)\right\}. \tag{6.22}$$

On the other hand, according to (6.21) we find $k_0 > 0$ such that, for every $k \geq k_0$, $r'(h_{n_k}(o')) > T$. It follows from this, from inequality (6.20) and the definition of $\mu'(x') = \mu \circ P(x') \geq 0$ that

$$\begin{aligned}\int_0^{T} \mu' \circ\ \gamma'_{h_{n_k}(o')}(s)\, ds &\leq \int_0^{r'\left(h_{n_k}(o')\right)} \mu' \circ\ \gamma'_{h_{n_k}(o')}(s)\, ds \\ &\leq 2\left\{(m-1) + H_{o'} + |\nabla f|(o)\right\}.\end{aligned}$$

Whence, letting $k \to +\infty$ we deduce

$$\int_0^{T} \mu' \circ \gamma'(s)\, ds \leq 2\left\{(m-1) + H_{o'} + |\nabla f|(o)\right\}$$

which contradicts (6.22).

Now for the proof of (b), suppose $\mathrm{Ric} \leq c$. Fix $q \in M$ such that $r(q) = \mathrm{dist}(o,q) > 2$, and let γ_q be a minimizing geodesic joining o to q. As above, combining (B) of Lemma 6.15 and Lemma 6.16, and recalling that $\mu(x) = \mu_o(r(x))$ is radial we obtain

$$\begin{aligned}-|\nabla f(o)| - |\nabla f(\gamma(q))| - 2\sup_M |g| + \int_0^{r(q)} \mu_o(s) ds &\leq 2(m-1) + H_q + H_o \\ &\leq 2(m-1) + 2c,\end{aligned}$$

which implies

$$|\nabla f(q)| \geq \int_0^{r(q)} \mu_o(s) ds + \{-|\nabla f(o)| - 2(m-1) - 2c\} - 2\sup_M |g|.$$

Since $0 < \mu_o \notin L^1(+\infty)$ if $r(q) >> 1$, say $r(q) \geq R_0$, we have $|\nabla f(q)| > 0$. Thus f has no critical point in $M \setminus B_{R_0}(o)$. Again from Lemmas 6.15 and 6.16, for every $0 \leq t \leq r(q)$,

$$\int_0^t \mu_o(s) ds - \langle \nabla f \circ \gamma_q, \dot\gamma_q\rangle + \langle \nabla f \circ \gamma_q, \dot\gamma_q\rangle_{|s=0} + g(q) - g(o) \leq 2(m-1) + 2c,$$

so that

$$\frac{d}{ds} f \circ \gamma_q\,|_{s=t} \geq \int_0^t \mu_o(s) ds - \{|\nabla f(o)| + 2\sup_M |g| + 2(m-1) + 2c\}.$$

Thus, integrating on $[2, r(q)]$,

$$
\begin{aligned}
f(q) \geq & \int_2^{r(q)} \int_0^t \mu_o(\gamma(s))ds - |f(\gamma_q(2))| \\
& \qquad - \big\{|\nabla f(o)| + 2\sup_M |g| + 2(m-1)2c\big\}\big(r(q)-2\big) \\
\geq & \int_2^{r(q)} \int_0^{r(q)} \mu_o(s)ds - \max_{\partial B_2(o)} |f| \\
& \qquad - \big\{|\nabla f(o)| + 2\sup_M |g| + 2(m-1) + 2c\big\}\big(r(q)-2\big) \to +\infty,
\end{aligned}
$$

for $r(q) \to +\infty$. Therefore f is a smooth exhaustion function whose critical points are confined in a compact set. By standard Morse theory, there exists a compact manifold N with boundary such that M is diffeomorphic to the interior of N.

Finally, we prove (c). Suppose that $\sup_M(|\nabla f| + |g|) \leq F < +\infty$. Then, by (6.18) in Lemma 6.16, for every unit speed geodesic γ issuing from o we have

$$\mathrm{Ric}(\dot\gamma, \dot\gamma) \geq \mu_0 + \frac{d}{dt} G \circ \gamma,$$

where $G \circ \gamma = -\langle \nabla f \circ \gamma, \dot\gamma \rangle + g \circ \gamma$ satisfies $|G \circ \gamma| \leq \sup_M(|\nabla f| + |g|)$. Using Theorem 1.2 in [**38**] we obtain the desired diameter estimate. $\square$

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