Chapter 1

Phase transitions.

(third draft)

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Abstract

A review is presented to orientate the reader in the vast and complex literature concerning phase transitions in matrix models that are invariant under a symmetry group. These phase transitions often have relevant applications in physics. We also mention on phase transitions that occur in some matrix ensembles with preferred basis, like the Anderson transition.

1.1 Introduction

Phase transitions were discovered almost thirty years ago in some random matrix ensembles which had a definite interpretation in theoretical physics. Perhaps the most relevant ones are the Gross - Witten phase transition in two dimensional Yang - Mills theory and the models of matter coupled to two dimensional gravity. The following decades witnessed a vast increase of interest in the study of random matrices in fields of pure and applied mathematics and in theoretical physics, together with a development of accurate mathematical techniques. Phase transitions were found in several models, with universal properties which are often the main subject of interest.

The simplest setting is the ensemble of $n \times n$ Hermitian matrices with probability density that is invariant under the action of the unitary group. The partition function $Z_n = \int dH\, e^{-n\,\mathrm{tr}V(H)}$ can be expressed in terms of the eigenvalues $\{\lambda_i\}$ of the random matrix H and defines the equilibrium statistical mechanics of a Dyson gas of n particles with positions λ_i in the line, in the potential $V(\lambda)$. Neglecting irrelevant constants,

$$Z_n = \int \prod_{j>k} (\lambda_j - \lambda_k)^2 \prod_{j=1}^{n} e^{-nV(\lambda_j)} d\lambda_j \;=\; \int e^{-E_n(\vec{\lambda})} d\lambda_1 \cdots d\lambda_n \qquad (1.1.1)$$

The particles interact by the repulsive electrostatic potential of 2D world and are bounded by $V(\lambda)$. The energy of a configuration is $E_n(\vec{\lambda}) = -\sum_{j>k} \log(\lambda_j - \lambda_k)^2 + n \sum_j V(\lambda_j)$. For large n, the partition function can be reformulated as a functional integral on normalized particle densities ρ_n, $Z_n = \int \mathcal{D}\rho_n e^{-n^2 E_n}$, with Boltzmann weight

$$E_n[\rho_n] = \int \rho_n(\lambda) V(\lambda)\, d\lambda + \frac{1}{n} \int \rho_n(\lambda)\, \log \rho_n(\lambda)\, d\lambda$$
$$- \int\int d\lambda\, d\mu\, \rho_n(\lambda)\, \rho_n(\mu)\, \log|\lambda - \mu| - 2\gamma \left(\int \rho_n(\lambda)\, d\lambda - 1 \right)$$

Two new terms appear: an entropic one resulting from the Jacobian (negligible in the large n limit) and the Lagrange multiplier γ enforcing normalization. The large n limit is both the thermodynamic and the zero temperature limit of the model, and allows for a saddle point evaluation of the partition function. Under certain conditions on V, there is a unique limit spectral density $\rho(\lambda)$. It is the solution of the limit saddle-point equation

$$\frac{1}{2} V(\lambda) - \int_\sigma d\mu\, \log|\lambda - \mu|\, \rho(\mu) = \gamma, \quad \lambda \in \sigma, \qquad (1.1.2)$$

that also minimizes the limit free energy:

$$\mathcal{F} = \lim_{n\to\infty} \frac{1}{n^2} \log Z_n = \int \rho(\lambda) V(\lambda) d\lambda - \int\int d\lambda\, d\mu\, \rho(\lambda)\rho(\mu)\, \log|\lambda - \mu| \qquad (1.1.3)$$

Generically $\mathcal{F}$ is an analytic function of the parameters of the potential, except for possible critical points or lines. When they occur, they divide the parameter space into different phases of the model[1].

[1] It may be useful and important to consider also complex potentials. Early papers on the subject are [Dav91][Moo90] [Fok91]. In the paper by Eynard [Eyn07] many important results are summarized.
We have chosen to limit ourselves to real potentials to stress the analogy with statistical mechanics and Boltzmann weight.

In sect.2 we summarize the results for the simplest model with a non trivial set of phases, the one-matrix Hermitian model with polynomial potential. We refer the reader to the beautiful lecture notes by P. Bleher [Ble08] for proofs and references to the mathematical literature.

We present a view of the several solutions of the saddle point equation, which simplifies the current analysis of the phases of the model. Generically, the limit eigenvalue density ρ has support on different numbers of intervals, in different phases of the model. Its behaviour near an edge of the support is typically a square root. Parameters can be adjusted to soften this edge singularity and, in the one-cut phase, the continuum limit of the model is approached. The universal distributions that describe various scalings of the density in the bulk or close to an edge are affected by phase transitions; we refer the reader to Ch.6 of this book. In the orthogonal polynomial approach, a phase transition manifests in the doubling phenomenon of recurrence equations.

For several matrix models that are invariant under a continuous group it is possible to obtain a partition function for the eigenvalues only. We call them Eigenvalue Matrix models. Quite often they display phase transitions. In sect.3 we review *circular* models and their Cayley transform to Hermitian models, and *fixed trace* models. Models with *normal, chiral, Wishart and rectangular* matrices are discussed briefly, because they are subjects of chapters in this book. We do not include results about matrix models in nonzero spacetime.

Though non-Hermitian random matrices are discussed in Ch.18, it is appropriate to present here, in sect.4, the curious single-ring theorem, restricting the phase transitions of complex spectra with rotational symmetry.

Multi-matrix models are recalled in sect. 5, with their spectacular success in describing phase transitions of classical statistical models on fluctuating two-dimensional surfaces. We refer the reader to chapters 15 and 16.

A large and important number of matrix models have a preferred basis; their analysis is usually performed with specific tools. In sect.6 the delocalization transition is summarized for the Anderson, Hatano-Nelson, and Euclidean RM models.

1.2 One-matrix models with polynomial potential

In our view the occurrence of phase transitions is best understood in the steepest descent solution of matrix models. As it happens with many discoveries, the two-cut and other one-cut asymmetric solutions of the saddle point equation were discovered at least twice, first by Shimamune in $D = 0, 1$, next by Cicuta et al.[Shi82][Cic86]. Soon after, two puzzling features emerged: 1) Multi-cut

solutions of the (derivative of the) saddle point equation

$$\frac{1}{2}V'(\lambda) = P\int_\sigma d\mu\,\frac{\rho(\mu)}{\lambda-\mu}, \qquad \lambda\in\sigma \tag{1.2.4}$$

seemed to have an insufficient number of constraints to fully determine them, 2) the recurrence relations which determine the large n behaviour of orthogonal polynomials were found to need two interpolating functions in correspondence of the two-cut phase [Mol88]. The intriguing feature was that the recurrence relations seemed to exhibit unstable behaviour [Jur91] [Sen92].
The following decades witnessed a rigorous derivation of the limit spectral density as the solution of a variational problem [Joh98], and the development of Riemann-Hilbert and resolvent-based approaches [Dei],[Pas06]. The latter seem crucial for the present understanding of the quasi-periodic asymptotics of recurrence coefficients. The rigorous derivations confirmed much of the previous heuristic work of theoretical physicists, including the set of missing equations (1.2.10), which were predicted [Jur90],[Lec91].

We outline the multi-cut solutions of the saddle point equation for polynomial potentials. The interest in this model was revived by recent work on the "birth of a cut" [Ble03],[Eyn06],[Cla08],[Mo07] [2]. Consider the Hermitian one-matrix model with a polynomial potential of even degree p, and positive leading coefficient. The limit density $\rho(\lambda)$ with its support σ can be evaluated from the Green function

$$F(z) = \int_\sigma d\lambda\frac{\rho(\lambda)}{z-\lambda}, \qquad \rho(\lambda) = \frac{1}{\pi}\mathrm{Im}\,F(\lambda-i\epsilon) \tag{1.2.5}$$

The saddle point approximation [Bre78] or the loop equations [Wad81] provide

$$F(z) = \frac{1}{2}V'(z) - \frac{1}{2}\sqrt{V'(z)^2-4Q(z)}, \tag{1.2.6}$$

$$Q(z) = \int_\sigma d\lambda\,\rho(\lambda)\frac{V'(z)-V'(\lambda)}{z-\lambda}. \tag{1.2.7}$$

$Q(z)$ is a polynomial of degree $p-2$, that contains unknown parameters $\langle x\rangle$, $\ldots$, $\langle x^{p-2}\rangle$, which are moments of the density. Since $F(z)\approx Q(z)/V'(z) = 1/z$ for $|z|\to\infty$, normalization of ρ is ensured. The relevant question arises about the polynomial $V'(z)^2-4Q(z)$: *how many are the pairs of simple real zeros?* The pairs of zeros are the endpoints of cuts of the function $F(z)$ that become intervals in the support of ρ: $\sigma = \bigcup_{j=1}^q [a_j,b_j]$. The remaining $2(p-q-1)$ zeros

[2] The merging of the extrema of a cut, leading to its disappearance, and the reverse phenomenon, had been known to some experts for a long time. See for instance sect.6.2 in [Ake97].

must be such to factor a squared polynomial $M(z)^2$:

$$V'(z)^2 - 4Q(z) = M(z)^2 \prod_{j=1}^{q} (z - a_j)(z - b_j) \tag{1.2.8}$$

The density is then evaluated,

$$\rho(\lambda) = \frac{1}{2\pi} |M(\lambda)| \sqrt{\prod_{j=1}^{q} |(\lambda - a_j)(b_j - \lambda)|}, \quad \lambda \in \sigma. \tag{1.2.9}$$

The polynomial identity (1.2.8) provides $2p-1$ equations for $2q+(p-q) = p+q$ unknowns in the r.h.s. (the endpoints and the coefficients of M). However, the l.h.s. is fully determined only in its monomials $z^{2p-2}, \ldots, z^{p-2}$. Therefore, only $p+1$ equations are useful to fix the $p+q$ parameters of the density. The remaining $p-2$ are equations for the moments of the density (they are not self-consistency relations, and add nothing to the knowledge of the density).

The 1-cut density ($q = 1$) is then completely (but not uniquely!) determined, while the q-cut density ($q > 1$) still depends on $q-1$ unknown parameters. It is a continuum set of normalized solutions of the *derivative* of the saddle point equation (1.1.2). This equation, (1.2.4), ensures that the chemical potential γ is constant for arbitrary λ in a single interval. But the saddle point equation requires γ to be the same for λ in the whole support σ: this gives extra $q-1$ conditions. They can be chosen as the vanishing of the integrals of the density on the gaps between the cuts [Jur90][Lec91]:

$$\int_{b_j}^{a_{j+1}} d\lambda \, \rho(\lambda) = 0, \quad j = 1, \ldots, q-1. \tag{1.2.10}$$

Necessarily $M(z)$ must have at least one real zero in each gap. Hint: the equations can be obtained by integrating on a gap eq.(1.2.6), with the input of eq.(1.2.8).

As the parameters of the potential V are changed, the zeros of ρ move continuously, and eventually two of them collide on the real axis; collisions provide mechanisms for phase transitions:

- a pair of complex zeros coalesce into a double real zero *inside* an interval. The density ceases to be positive, and an extra cut must be considered in (1.2.6). The reverse is the closure of a gap, with the ends turning into a complex conjugate pair.

- a pair of complex zeros coalesce into a double real zero *outside* an interval and an extra interval is born. The reverse is the closure of an interval, with the ends turning into a complex conjugate pair.

- a zero other that an endpoint coalesces with an endpoint, thus changing the edge's singular behaviour.

The first two are *multicut transitions*, the third is an *edge singularity transition.*

With the conditions (1.2.10) the number of equations equals the number of parameters that determine the density. However:
1) the decoupling of the algebraic equations produces equations for the single unknowns which may have several solutions with same q;
2) the requirement that ρ must be non-negative on its support eliminates solutions.
In general one remains with more than one q-cut solution, and selects *the* saddle point solution by comparing their free energies.

We have recently redone the analysis of the phase diagram for the quartic model.[3] We summarize the results of the new analysis, which suggest that multiple solutions with different numbers of cuts are trivially related.

The quartic potential.
The general quartic potential can be rescaled and shifted to $V(\lambda) = hz + \frac{1}{2}\alpha\lambda^2 + \frac{1}{4}\lambda^4$, which depends on two parameters h and α. Let us first summarize the simple case $h = 0$: for $\alpha \geq -2$ there is the BIPZ 1-cut solution [Bre78]. At $\alpha = -2$ the density has a zero in the middle of its support. For $\alpha < -2$ one must consider a two cut solution $\rho(\lambda) \approx |\lambda|\sqrt{(b^2 - \lambda^2)(\lambda^2 - a^2)}$.
In the general case, the plane (h, α) is partitioned in three phases (I,II,III). In I, which includes the half-plane $\alpha \geq 0$, the solution is one-cut; in II only the two-cut solution exists. In III, three solutions coexist: the two-cut solution and *two* one-cut solutions. As we show below, the two-cut solution has lower free energy. III has the line $h = 0$ in its interior ($\alpha < -\sqrt{15}$).
In III, before the condition (1.2.10) is imposed, one has a one-parameter family of two-cut solutions of (1.2.4):

$$\rho(\lambda) = \frac{1}{2\pi}|\lambda - R|\sqrt{\prod_{j=1,2} |(\lambda - a_j)(\lambda - b_j)|}, \qquad \lambda \in [a_1, b_1] \cup [a_2, b_2] \quad (1.2.11)$$

If R is chosen as free parameter, it is fixed by eq.(1.2.10)[4]. Another choice is x, that gives the filling fractions of the intervals: $x = \int_{a_1}^{b_1} \rho(\lambda)\, d\lambda$ and $1 - x = \int_{a_2}^{b_2} \rho(\lambda)\, d\lambda$ [Bon00]. One checks that the two one-cut solutions which also exist in III correspond to the limit values $x = 0$ and $x = 1$ where an interval degenerates to a point. The corresponding free energies are then higher. Since

[3] We studied this same problem long ago [Cic87]. At the time we did not identify the correct missing equation (1.2.10), which led us to conclusions quite different from those presented here. A detailed presentation together with a less inadequate bibliography is now being prepared.

[4] The determination of ρ in the gap (b_1, a_2) has the factor $\lambda - R$ without modulus.

a two-cut solution is analytic in h for small h, there is no spontaneous breaking of the Z_2 symmetry.

The three different phases are separated by lines of third-order phase transitions. Their occurrence finds application in the study of Homolumo gap in one-fermion spectrum [And07], structural changes in glasses [Deo02], gluons in baryons (with addition of log-term in potential) [Kri06].

1.2.1 Orthogonal Polynomials

Multicut phase transitions arise when the matrix potential is endowed by two or more minima, and the large n limit suppresses tunnelling among the wells. In the saddle point solution this is manifest in the multicut support of the eigenvalue density. In the approach with orthogonal polynomials, see Ch.4 (and Ch.5), the phenomenon appears in a different guise.

The partition function, the spectral density and correlators can be evaluated formally for all n, by expanding the Vandermonde determinant in monic polynomials $P_r(x)$ constrained by orthogonality: $\int dx e^{-nV} P_r P_s = h_r \delta_{rs}$. Starting with $P_0(x) = 1$, the relation allows in principle to evaluate all coefficients and the partition function $Z_n = n! h_0 \cdots h_{n-1}$. In practice, one looks for asymptotics in n as follows. Because of orthogonality the polynomials are linked by a three term relation $xP_r = P_{r+1} + A_r P_r + R_r P_{r-1}$. The coefficients A_r and R_r solve themselves finite order recurrency equations, resulting from the identities

$$\frac{k}{n} = \frac{1}{h_{k-1}} \int dx e^{-nV} V' P_{k-1} P_k, \quad 0 = \int dx e^{-nV} V'(x) P_k^2. \tag{1.2.12}$$

The initial conditions $R_0 = 0$, A_0, ... must be computed explicitly. Since $R_r = h_{r+1}/h_r > 0$, the free energy is evaluated:

$$\mathcal{F}_n = -\frac{1}{n^2} \log Z_n = -\frac{1}{n^2} \log(n!) - \frac{1}{n} \log h_0 - \frac{1}{n} \sum_{r=1}^{n-1} (1 - \frac{r}{n}) \log R_r \tag{1.2.13}$$

For large n, the *string equations* (1.2.12) become equations for interpolating functions $R(x)$ and $A(x)$, with $R(k/n) = R_k$ and $A(k/n) = A_k$: $x = w_1(R(x), A(x); R'(x), \ldots)$ and $0 = w_2(R(x), A(x); R'(x), \ldots)$. The leading order in n is purely algebraic in R and A, and does not involve derivatives. The solutions must comply with the initial conditions $R(0) = 0 = R_1 = \ldots$ and $A(0) = A_0 = A_1 = \ldots$. However, the finite sets of initial conditions $\{R_k\}$ and $\{A_k\}$ may not collapse in the large n limit to unique values $R(0)$ and $A(0)$. This is the signal of a *multicut phase*: coefficients have to be interpolated by as many functions $R_\ell(x)$ and $A_\ell(x)$ as the initial conditions.

The quartic potential.
For the symmetric quartic potential $V(x) = \frac{1}{2}\alpha x^2 + \frac{1}{4}x^4$ the orthogonal polynomials have definite parity ($A_k = 0$) and only the first string equation is used:

$$\frac{k}{n} = \alpha R_k + R_k(R_{k+1} + R_k + R_{k-1}) \tag{1.2.14}$$

with initial conditions $R_0 = 0$ and $R_1 = \langle x^2 \rangle$. If $\alpha > 0$ it is $R_1 \to 0$ and a unique function $R(x)$ is needed, with $R(0) = 0$. The recurrence equation becomes the quadratic equation $x = \alpha R(x) + 3R(x)^2$ that leads to the one-cut solution [Bre78]. If $\alpha < 0$ it is $R_1 \to |\alpha|$ and two interpolating functions $R_1(x)$ and $R_2(x)$ are needed: $R_1(2k/n) = R_{2k}$ with $R_1(0) = 0$ and $R_2(2k/n + 1/n) = R_{2k+1}$ with $R_2(0) = |\alpha|$. They solve: $x = \alpha R_1 + R_1^2 + 2R_1R_2$ and $x = \alpha R_2 + R_2^2 + 2R_1R_2$. One recovers the solution $R_1 = R_2$ but also the new ones $R_2(x) = -R_1(x) - \alpha$ and $x = gR_1(x)R_2(x)$. In the latter case the equation for $R_1(x)$ yields a real positive solution only for $\alpha < -2$. This is the two-cut phase.
The asymmetric quartic potential is more complicated because of $A_k \neq 0$. One reobtains regions I,II,III of the previous discussion.

1.2.2 The edge singularity limit

A special status is owned by the edge singularity limit in the 1-cut phase. The critical line marks the boundary of analyticity of the perturbative phase where the partition function can be expanded in all couplings above quadratic. The perturbative expansion provides numbers that enumerate Feynman diagrams with fixed numbers of vertices. It is a useful tool in the theory of graphs and in statistical mechanics, where each graph may correspond to a configuration in the partition function of some model on random graphs [Amb][Gro92] (see chapters 30 and 31).

In the *topological expansion* ('t Hooft) the perturbative series is rearranged

$$\mathcal{F}_n = \sum_{h=0}^{\infty} \frac{1}{n^{2h-2}} \mathcal{F}_h(g), \qquad \mathcal{F}_h(g) = \sum_V g^V \mathcal{F}_{h,V} \tag{1.2.15}$$

to enumerate the vacuum graphs of given genus h and number V of vertices. The leading term $\mathcal{F}_0$ is the *planar* free energy. A remarkable feature of the expansion is that $\mathcal{F}_h(g)$ has a finite radius of convergence g_c. This singularity is where two zeros of the eigenvalue density collide, and in the single-well phase it corresponds to an edge-singularity limit.
The *area* of a graph is defined as the number of vertices (faces of the dual graph). The average area at fixed genus diverges near the critical point, thus providing the continuum limit:

$$\langle area \rangle = \frac{1}{\mathcal{F}_h} \sum_V V g^V \mathcal{F}_{h,V} = \frac{1}{\mathcal{F}_h} g \frac{\partial}{\partial g} \mathcal{F}_h \approx const. \frac{g_c}{g_c - g}. \tag{1.2.16}$$

The double scaling is a prescription to reach the critical point, in order to enhance subleading orders of the topological expansion.

1.3 Eigenvalue Matrix models

Several matrix models with continuous symmetry, after integration of angular degrees of freedom, produce a partition function that generalizes eq.(1.1.1) and describes only the eigenvalues or the singular values. Many models, but not all, derive from classification schemes of the symmetries (Ch.3). Depending on the eigenvalue measure, phase transitions may appear. The analytic methods are almost the same as in the previous section.

1.3.1 Unitary circular ensembles

Perhaps the first and most influential phase transition was found by Gross and Witten in the study of one plaquette in QCD [Gro80], see Ch. 17,

$$Z_n = \int dU\, e^{\frac{1}{g^2}\mathrm{tr}(U+U^\dagger)} \quad , \quad U \in U(n) \tag{1.3.17}$$

Let $\{e^{i\alpha_k}\}$ be the n eigenvalues of the random unitary matrix U, then

$$Z_n = \int_{-\pi}^{\pi} d\alpha_1 \cdots d\alpha_n\, e^{\sum_{i\neq j}\log|\sin\frac{\alpha_i-\alpha_j}{2}|+\frac{2}{g^2}\sum_j\cos\alpha_j} \tag{1.3.18}$$

In the large-n limit the spectral density $\rho(\alpha)$ which makes the energy stationary is solution of the integral equation

$$\frac{1}{g^2 n}\cos\alpha + \int \rho(\beta)\, \log|\sin\frac{\alpha-\beta}{2}|\, d\beta = 0 \tag{1.3.19}$$

As it is well known, there are a strong and a weak coupling solution:

$$\rho(\alpha) = \frac{1}{2\pi} \quad (g^2 n \geq 2), \qquad \rho(\alpha) = \frac{1}{2\pi}\left[1 + \frac{2}{g^2 n}\cos\alpha\right] \quad (g^2 n \leq 2)$$

with support on the whole circle or an arc. The Cayley map provides a one-to-one correspondence between unitary and Hermitian matrices:

$$U = \frac{i-H}{i+H}, \quad dU = \frac{dH}{\det(1+H^2)}$$

Accordingly, the model (1.3.18) can be mapped into a Hermitian model. The corresponding partition function is written in terms of the eigenvalues λ_j of the Hermitian matrix

$$Z_n = \int_{-\infty}^{\infty} \prod_j d\lambda_j\, e^{-n\sum_j\log(1+\lambda_j^2)+\sum_{i\neq j}\log|\lambda_i-\lambda_j|+\frac{2}{g^2}\sum_j\frac{1-\lambda_j^2}{1+\lambda_j^2}}$$

Instead of eq.(1.3.19) one has the integral equation

$$\log(1+\lambda^2) - \frac{2}{g^2 n}\frac{1-\lambda^2}{1+\lambda^2} = 2\int d\mu\, \rho(\mu)\, \log|\lambda-\mu| \tag{1.3.20}$$

One finds a strong coupling solution with support on the whole real axis,

$$\rho(\lambda) = \frac{1}{\pi}\left[\frac{1}{1+\lambda^2} + \frac{2}{g^2 N}\frac{1-\lambda^2}{(1+\lambda^2)^2}\right] \quad , \quad 2 \leq g^2 n \tag{1.3.21}$$

Mizoguchi [Miz05] recently studied the weak coupling solution. It has support on a finite interval

$$\rho(\lambda) = \frac{2}{\pi}\frac{\sqrt{1+b^2}}{b^2}\frac{\sqrt{b^2-\lambda^2}}{(1+\lambda^2)^2} \quad , \quad b^2 = \frac{g^2 n}{2-g^2 n} \tag{1.3.22}$$

Therefore, the Cayley map takes the Gross - Witten phase transition to a phase transition in a Hermitian matrix model, where it separates a compact support phase (typical of confining potentials) and an infinite support phase (typical of logarithmic external potentials). The Gross - Witten model was solved for polynomial potentials [Man90] [Dem91], $V(U) = \sum_k c_k U^k + c_k^* U^{\dagger k}$, and phases with support on several arcs of the unit circle were found. The Cayley map takes them to multi-cut solutions of Hermitian models.

1.3.2 Restricted trace ensembles

Restricted trace ensembles of Hermitian matrices have joint probability density

$$p(\lambda_1, \cdots, \lambda_n) = const. \quad \Phi\left(r^2 - \frac{1}{n}\sum_{j=1}^{n} V(\lambda_j)\right) \prod_{1\leq i<j\leq n} |\lambda_i - \lambda_j|^2$$

with $\Phi(x) = \delta(x)$ for the fixed trace ensembles or $\Phi(x) = \theta(x)$ for the bounded trace ensembles. The constraint replaces the standard Boltzmann factor of matrix models. These invariant matrix ensembles bear the same relation to unrestricted ensembles as microcanonical ensembles do with canonical ones in statistical physics. They were studied at the early stages of random matrix theory [Ros63], with quadratic potential. It was interesting to generalize them to polynomial potential, as phase transitions in these models provide good examples for the limited equivalence of ensembles in the thermodynamic limit. [Ake99]. Further recent works include [Del00],[Got08], later extended to the fixed trace beta-ensembles [LeC07],[Liu09].

1.3.3 External field

Ensembles of unitary or Hermitian matrices coupled to a fixed matrix source, called external field, had been investigated for a long time [Bre80][Gro91]. The

limiting eigenvalue density of the ensemble depends on the eigenvalue density of the fixed matrix. Several important tools of one-matrix models may be generalized to the external field problem to prove that the short distance behaviour of correlation functions are not affected by the source [Zin98], [Ble08]. We refer the reader to Ch.16.
If the eigenvalue density of the external source has a gap which may be tuned to vanish, a new class of universality appears [BrH98]. This phase transition has a relevant role in several problems, including the spectral statistics of low-lying eigenvalues of QCD Dirac operator, see Ch.32.

1.3.4 Other models

Several models are described in chapters of this book, and display various critical behaviours. We list some.
1) Normal matrices have complex spectrum. With the potential $XX^\dagger + V(X) + V(X)^\dagger$ the boundary of the support of eigenvalues describes the growth of a 2D fluid droplet, as the parameters change. At singular points the boundary develops cusp-like singularities. [Teo06] (Ch.38).
2) Chiral matrix ensembles were introduced by Verbaarschot to study spectral properties of the QCD Dirac operator (Ch.32) linked to the formation of a condensate, with beautiful accordance with lattice calculations. A natural application of chiral RME is the study of single particle excitations in bulk type-II superconductors [Bah96].
3) Wishart matrices have the form $W = R^\dagger R$, where R is a rectangular matrix. They occur in multivariate statistics (Ch.28 and 39), in transport (as suggested by Buttiker's Landauer formula for conductance) and in the study of rare events (Ch.36).
4) Transfer matrix ensembles were introduced to reproduce the statistics of conductance (Ch.35).
5) Models of rectangular matrices $n \times nL$ with $O(L)$ symmetry can be viewed as $L-$matrix models with square matrices of size n, and the large n limit is the planar limit of a vector model [Cic87b][And91][Fei97b]. The singular values undergo a phase transition in the double well potential, where a gap opens near the origin. Multicritical behaviour and underlying random surfaces are studied. An ample introduction and refs. to rectangular matrices is in a paper on colored graphs [DiF03].
6) Non-polynomial potentials may originate different universality classes for correlation functions. Some of these models may be seen as arising from integration over additional bosonic or fermionic fields. Early relevant examples include [Kaz90][Kon92]. Recent examples include [Ake02][Jan02].

1.4 Complex matrix ensembles

The powerful spectral methods available for Hermitian matrices, based on the resolvent, do not apply to complex matrices. With the exclusion of normal matrices, it is very difficult to extract a measure for the eigenvalues. By regarding the eigenvalues as point charges in the complex plane, the search of the density can be formulated as an electrostatic problem [Cri88]. The ensemble-averaged logarithmic potential

$$U(z,z^*) = \langle \frac{1}{n} \log[\det(zI_n - X)\det(z^*I_n - X^\dagger)]\rangle \tag{1.4.23}$$

and the average eigenvalue density $\rho(x,y) = \langle \frac{1}{n}\sum_a \delta(x - \mathrm{Re}z_a)\delta(y - \mathrm{Im}z_a)\rangle$ are linked by the Poisson equation $\partial\partial^* U = \pi\rho$, where $z = x+iy$ and $\partial = \frac{1}{2}(\partial_x - i\partial_y)$. The problem is simplified by introducing the Green function,

$$G(z,z^*) = \langle \frac{1}{n}\mathrm{tr}\frac{1}{z-M}\rangle = \int d^2w \frac{\rho(w,w^*)}{z-w}, \tag{1.4.24}$$

then: $\rho(z,z^*) = \frac{1}{\pi}\partial^* G(z,z^*)$. By noting that, up to a constant, the potential U contains the determinant of the $2n \times 2n$ Hermitian matrix

$$H(z,z^*) = \begin{bmatrix} 0 & X-z \\ X^\dagger - z^* & 0 \end{bmatrix} \tag{1.4.25}$$

Feinberg and Zee proposed the *Method of Hermitian reduction* [Fei97]. It allows to evaluate the Green function $G(z,z^*)$ from the resolvent matrix of $H(z,z^*)$: $F(\eta)_{ab} = \langle(\eta - H)^{-1}_{ab}\rangle$. For the Ginibre ensemble [Gin65] the resolvent is $G(z,z^*) = 1/z$ for $|z| > 1$ and z^* for $|z| < 1$. Then the eigenvalue distribution is uniform in the disk $|z| < 1$. Other tools are the extension to non-Hermitian matrices of *Blue functions* [Jar06] (the functional inverse of the Green function $G(B(z)) = B(G(z)) = z$), or the fermionic replica trick [Nis02].

1.4.1 The single ring theorem

The method of Hermitian reduction can be worked out for the probability distribution

$$p(X) = Z^{-1}e^{-n\mathrm{tr}V(XX^\dagger)} \tag{1.4.26}$$

where V is a polynomial. The eigenvalue density $\rho(x,y)$ for X only depends on $r = \sqrt{x^2+y^2}$. By resummation of the planar diagrams in the perturbative expansion of the resolvent F of the hermitized model, Feinberg, Zee and Scalettar showed that the fraction $\gamma(r)$ of eigenvalues with modulus less than r solves in the large n limit the algebraic equation

$$r^2\frac{\gamma(r)}{1-\gamma(r)}F\left(r^2\frac{\gamma(r)}{1-\gamma(r)}\right) = 1 \tag{1.4.27}$$

It admits various solutions, but only two are $r-$independent, namely $\gamma = 0$ and $\gamma = 1$. The actual solution is obtained by a smooth matching of solutions such that $\gamma(r)$ is non decreasing, with b.c. $\gamma(0) = 0$ and $\gamma(\infty) = 1$. Surprisingly, the support of the density ρ can only be a disk or a single annulus. This is the single ring theorem [Fei97][Jan97]. The proof is simple: two or more annuli would imply a gap, hence a solution $\gamma(r)$ which is constant on the gap interval of r, and different from the only two allowed values $0, 1$.

Although the eigenvalues of the positive matrix $XX^\dagger$ may distribute with a multicut density, the complex eigenvalues of X are only allowed to coalesce in a disk or in an annulus. As the parameters of the potential V are changed, one observes phase transitions between the two configurations. For the double well potential $V = \mathrm{tr}(2\alpha XX^\dagger + g(XX^\dagger)^2)$, the phase transition takes place at $\alpha = -\sqrt{2g}$. An interesting insight was provided in [Fyo07], where eq.(1.4.27) was recovered with the hypothesis that, for large n, the log can be taken out of the ensemble average (1.4.23), thus simplifying the evaluation.

A disk–annulus transition was observed in other ensembles. Complex tridiagonal random matrices have exponentially localized eigenvectors, with inverse localization length given by the Thouless formula

$$\gamma(z) = \int d^2w \log|z - w|\rho(w, w^*) + \text{const.} \tag{1.4.28}$$

If boundary conditions (b.c.) $u_{n+1} = e^{n\xi}u_1$ and $u_0 = e^{-n\xi}u_n$ are used, then for $\xi > \xi_c$ a hole opens in the support of the spectrum [Mol09b]. The eigenvalues that are removed from the hole belong to eigenvectors that are delocalized by the b.c., i.e. $\gamma(z) < \xi$. For large ξ the complex spectrum becomes circular.

The transition was also observed in the ensemble $A + H_0$, where H_0 is a matrix with a highly unstable zero eigenvalue, and A is random and asymmetric [Kho96].

1.5 Multi-matrix models.

The free energy of a group-invariant matrix model has a topological large-n expansion (1.2.15), where the terms associated to inverse powers of n^2 (unitary ensembles) or n (orthogonal and symplectic ensembles) are the generating functions of Feynman graphs embeddable on orientable (or non-orientable, in the second case) surfaces of different genus.

With two or more matrix variables, each graph has both the meaning of a random triangulation and fixes a "configuration" of two or more variables associated to its vertices or links. The matrix integrations perform simultaneously the summation on configurations of variables on a graph (a random triangulation) and the summation on inequivalent triangulations. Therefore, multi-matrix models may be considered as a definition for classical statistical

mechanics on random lattices. By tuning the couplings of the potential to critical values it is possible to interpret the triangulations provided by the Feynman graphs, or their duals, as becoming dense and reaching a continuum limit.

In chain models, the potential of two (or more) random matrices has the form $V(A,B) = V_1(A) + V_2(B) + AB$. The analysis of such chain-linked ensembles was possible after two major discoveries:
1) integration of angular variables by Itzykson and Zuber [Itz80] through the Harish-Chandra-Izkyson-Zuber formula, which reduces the partition function to integrals on the eigenvalues of the two (or more) random matrices,
2) bi-orthogonal polynomials by Mehta [Meh81], which allow the formal exact evaluation of the free energy and asymptotic behaviour.

It is well known that near the critical singularities of statistical mechanical models on fixed, regular 2D lattices, the correlation lengths become much bigger than the lattice spacing. The critical behaviour of thermodynamic functions are then described by classes of universality of 2D field theories in the continuum. Analogous universality classes occur for phase transitions of statistical models on random surfaces.

If the critical singularities of a model on a fixed lattice are described by a conformal field theory with central charge c, the Knizhnik-Polyakov-Zamolodchikov relation predicts the conformal dimensions of operators of the corresponding model on inequivalent random triangulations (i.e. dressed by gravity). This relation was checked in several multi-matrix models: Ising, Potts, $O(n)$, 8-vertex, edge-colouring, ... These impressive accomplishments are described in classical reviews [Amb][DiF95]. More recent ones by Di Francesco [DiF02] focus on the structure of random lattices. Recent results on the Riemann-Hilbert problem for the two matrix model are found in [Dui08],[Mo08].

1.6 Matrix ensembles with preferred basis.

1.6.1 Lattice Anderson models

Anderson models describe the dynamics of a particle in a lattice, in a random potential; the hopping amplitudes may also be random. They arise as tight-binding descriptions of a particle in a crystal with impurities, disordered materials, random alloys.[5]

The Hamiltonian is $H = T + V$ with kinetic term $(Tu)_k = \sum_\mu u_{k+\mu}$ and potential $(Vu)_k = v_k u_k$, where k labels the lattice sites. The sum runs on all sites linked to k, the numbers v_k are i.i.d. random variables with uniform

[5]For physics we address the reader to the reviews [Kra93],[Vol],[Jan98]. For the RMT approach see Ch.35 and refs therein, [Bee97],[Eve08]. The mathematically oriented reader can consult the review [Bel04], the books [Pas92],[Car] or (as a start) [Fro]. Material can be found at the web sites of the Newton and Poincaré Institutes [NI][IHP].

distribution in the interval $[-w/2, w/2]$ (w is the disorder parameter) or else. For the Cauchy distribution, $p(v) = (w/\pi)(v^2+w^2)^{-1}$, the density of states can be computed analytically. The lattice is usually taken as Z^D, but also other structures have been studied. In direct space the Hamiltonian has a matrix representation where T is fixed (a Laplacian matrix, up to a diagonal shift) and V is diagonal and random. The matrix size is equal to the number of sites. The matrix H is block-tridiagonal, with the number of blocks being equal to the number of sites in one direction ($n = n_z$), and the size of blocks being equal to the number of sites in a section ($m = n_x n_y$). The eigenvalue equation in terms of the diagonal blocks h_k and off diagonal ones (unit matrices if hopping amplitude is one) is

$$h_k u_k + u_{k+1} + u_{k-1} = \lambda u_k, \quad u_k \in \mathrm{C}^m \tag{1.6.29}$$

The components of u_k are the occupation amplitudes of the m sites having longitudinal coordinate k, λ is the energy of the particle. A fundamental analytic tool is the transfer matrix, of size $2m \times 2m$, that links the wave-vector components at the ends of the sample of length n:

$$\begin{bmatrix} u_{n+1} \\ u_n \end{bmatrix} = T(\lambda) \begin{bmatrix} u_1 \\ u_0 \end{bmatrix}, \quad T(\lambda) = \prod_k \begin{bmatrix} \lambda - h_k & -I_m \\ I_m & 0 \end{bmatrix}. \tag{1.6.30}$$

With Dirichlet b.c. it is $u_{n+1} = u_0 = 0$. The transfer matrix is the product of n random matrices; its eigenvalues describe the long range behaviour of eigenstates of the Hamiltonian. Oseledec's theorem [Pas92][Cri93] states that the matrix $TT^\dagger$ in the limit $n \to \infty$ converges to a nonrandom matrix $e^{-n\Gamma(\lambda)}$ and the eigenvalues of Γ come in pairs $\pm\gamma(\lambda)$ (Lyapunov spectrum). The smallest positive one is the inverse localization length, that controls transport properties. So far no analytic expression is known for the Lyapunov spectrum (in $D = 1$ it is Thouless formula). However, for any n and m, the eigenvalues of the transfer matrix can be linked to the spectrum of the Hamiltonian matrix, but with non-Hermitian boundary conditions, via an algebraic spectral duality [Mol09a]:

$$z^{nm} \det[\lambda - H(z^n)] = (-1)^m \det[T(\lambda) - z^n] \tag{1.6.31}$$

The main feature of Anderson models is a phase transition that occurs in 3D in the infinite-size limit: for $w < w_c$ the spectrum of H is a.c. while for $w > w_c$ the spectrum is p.p. The eigenvectors are, respectively, extended or exponentially localized. Anderson's transition (also named *metal insulator transition*, MIT) and Mott's transition (due to interaction effects on the band filling) are cornerstones of the theory of electronic transport (both physicists earned the Nobel prize in 1977 with van Vleck). The transition is observed in various experimental situations [Kra93]. Numerically, it can be detected in various

ways: scaling in transverse dimensions of the smallest Lyapunov exponent of the transfer matrix [Mac81], spacing distribution of the energy levels [Shk93], response of energy levels to changes of boundary conditions [Zyc94] (Thouless' approach to conductance). In the metallic phase the Thouless conductance is well described by the distribution of level curvatures of the RM ensemble $H(\varphi) = A\cos\varphi + B\sin\varphi$, A and B in GOE or GUE, which was conjectured in [Zak93] and analytically proved in [vOp94].

The two phases are characterized by order parameters: the localization length (localized phase) and the adimensional conductance g (delocalized phase), that diverge approaching the transition. A finite size one-parameter scaling theory was established by Abrahams et al. which implies the phase transition in $D = 3$, but no transition in $D = 1, 2$. In $D = 1, 2$ such as wires or electron layers in heterostructures, the eigenstates are localized. This is crucial for explaining the occurrence of plateaux in the integer Quantum Hall effect. Anderson localization is caused by destructive interference on random scatterers, and has been observed in diverse wave phenomena as electrons, sound, light, Bose-Einstein condensates.

Mathematical proofs of localization in 1D were available in the late seventies; Molchanov proved localization in 1D continuous case and the Poisson law for energy levels. Theorems for D>1 appeared in the early eighties (Frohlich, Spencer, Martinelli and Scoppola [Mar86]) and established localization at large disorder. Minami proved the Poisson law for energy levels of lattice models. Multilevel correlators are studied by various groups (see [Aiz08]). The extended phase still lacks of rigorous results. A Wigner-Dyson statistics is expected and numerically seen in the transport regime. The level statistics at the Anderson transition is of new type [Kra94][Zha97] (Ch.12).

The quantization of time-dependent classically chaotic systems brings in the phenomenon of dynamical localization, which is analogous to the disorder localization. In model systems such as Kicked Rotator the quantum system does not increase its energy as classically, but reaches a stationary state [Cas89][Haa]. This has been observed experimentally in optical systems, or in microwave ionization of Rydberg atoms.

Quantum chaos eventually revived in the early eighties the interest for *Band Random matrices*, for the transition in level statistics from Wigner-Dyson to Poissonian, or the semiclassical limit of quantum mechanics of chaotic systems [Fei89]. The study of quantum maps suggested that BRM have localized eigenstates and level statistics with scaling laws governed by the ratio b^2/n [Cas90][Haa], where $2b+1$ is the bandwidth. Analytic results became accessible by supersymmetric methods after the papers by Fyodorov and Mirlin [Fyo91]. BRM are now on stage to study the Anderson transition (Ch.23).

1.6.2 Hatano - Nelson transition

Hatano and Nelson [Hat96] proposed a model for the depinning of flux tubes in type II superconductors, which turned out to be useful for studying the delocalization transition, as well as phase transitions [Nak06]. In essence, it tests localization through eigenvalue sensitivity to b.c. that drive the model off Hermiticity. The HN deformation of 1D Anderson model with periodic b.c. is

$$e^{\xi}u_{k+1} + e^{-\xi}u_{k-1} + v_k u_k = \lambda u_k \tag{1.6.32}$$

By similarity it is equivalent to $\tilde{u}_{k+1} + \tilde{u}_{k-1} + v_k\tilde{u}_k = \lambda\tilde{u}_k$ with b.c. $\tilde{u}_{n+1} = e^{n\xi}\tilde{u}_1$ and $\tilde{u}_0 = e^{-n\xi}\tilde{u}_n$. For $\xi = i\varphi$ (Bloch phase), eigenvalues sweep n non-intersecting real bands; for ξ real they all enter the gaps. If $\xi > \xi_c$ they start to collide and enter the complex plane, where they fill a closed curve [Gol98] which encloses the depleted segment of real eigenvalues. The extrema of the segment are "mobility edges" $\pm\lambda_c(\xi)$ beyond which the eigenvalues are real and (numerically) unmodified by ξ; their eigenstates are localized enough to feel not the b.c. The equation of the spectral curve is $\xi = \gamma(\lambda)$, where γ is the Lyapunov exponent. The HN deformation has been applied to complex tridiagonal matrices [Mol09b].

1.6.3 Euclidean Random Matrix models

Euclidean Random Matrices were introduced by Mézard, Parisi and Zee[Mez99] to describe statistical properties of disordered systems such as harmonic vibrations in fluids, glasses, or electron hopping in amorphous semiconductors. Given n points $\{\vec{x}_i\}$ in R^D and a real function $F : R^D \to R$, one constructs the matrix $E_{ij} = F(\vec{x}_i - \vec{x}_j)$. As the n points are chosen randomly, an ensemble of Euclidean Random Matrices is constructed. In Distance matrices, F is just the distance of points [Bog03], with interesting connections between spectral properties and geometry. Another example are the Hessian matrices of some pair potential. For Lennard-Jones potential a mobility edge was found, i.e. a threshold frequency value between regimes of low energy localized modes (phonon-like) and a delocalized regime, with the critical exponents of Anderson's transition [Gri03][Hua09]. The transition may explain the excess in the density of vibrational states (Boson peak) with respect to Debye's ν^2 law observed in glasses.

References

[Ake97] G. Akemann, Nucl. Phys. B **507** (1997) 475.

[Ake99] G. Akemann, G. M. Cicuta, L. Molinari and G. Vernizzi, Phys. Rev. E **69** (1999) 1489, Phys. Rev. E **60** (1999) 5287; G. Akemann and G. Vernizzi, Nucl. Phys. B **583** (2000) 739.

[Ake02] G. Akemann and G. Vernizzi, Nucl. Phys. B **631** (2002) 471.

[Aiz08] M. Aizenman and S. Warzel, arXiv:0804.4231 [math-ph]

[Amb] J. Ambjorn, B. Durhuus, T. Jonsson, *Quantum Geometry*, Cambridge University Press (1997).;

[And91] A. Anderson, R. C. Myers and V. Periwal, Nucl. Phys. B **360** (1991) 463; R. C. Myers and V. Periwal, Nucl. Phys. B **390** (1993) 716.

[Gro92] D. J. Gross, T. Piran, S. Weinberg editors, *Two dimensional quantum gravity and random surfaces*, World Scientific (1992).

[And07] I. Andrić, L. Jonke and D. Jurman, Phys. Rev. D **77** (2008) 127701.

[Bac00] C. Bachas and P.M.S. Petropoulos, Phys. Lett. B **247** (1990) 363.

[Bah96] S. R. Bahcall, Phys. Rev. Lett. **77** (1996) 5276.

[Bee97] C. B. Beenakker, Rev. Mod. Phys. **69** (1997) 731.

[Bel04] J. Bellissard, J. Stat. Phys. **116** (2004) 739.

[Ble03] P. M. Bleher and B. Eynard, J. Phys. A: Math. Gen. **36** (2003) 3085.

[Ble08] P. M. Bleher, *Lectures on random Matrix Models. The Riemann-Hilbert Approach*, arXiv:0801-1858 [math-ph].

[Bog03] E. Bogomolny, O. Bohigas and C. Schmit, J. Phys. A: Math. Gen. **36** (2003) 3595.

[Bon00] G. Bonnet, F. David and B. Eynard, J. Phys. A: Math. Gen. **33** (2000) 6739.

[Bre78] E. Brézin, C. Itzykson, G. Parisi and J.-B. Zuber, Comm. Math. Phys. **59** (1978) 35.

[Bre80] E. Brézin and D. J. Gross, Phys. Lett. B **97** (1980) 120.

[BrH98] E. Brézin and S. Hikami, Phys. Rev. E **57** (1998) 4140.

[Car] R. Carmona and J. Lacroix, *Spectral theory of random Schroedinger operators*, Birkhauser (1990).

[Cas89] G. Casati, I. Guarneri and D. L. Shepelyansky, Phys. Rev. Lett. **62** (1989) 345.

[Cas90] G. Casati, F. Izrailev and L. Molinari, Phys. Rev. Lett. **64** (1990) 1851.

[Cic86] G. M. Cicuta, L. Molinari and E. Montaldi, Mod. Phys. Lett. A **1** (1986) 125.

[Cic87] G. M. Cicuta, L. Molinari and E. Montaldi, J. Phys. A: Math. Gen. **20** (1987) L67.

[Cic87b] G. M. Cicuta, L. Molinari, E. Montaldi and F. Riva J. Math. Phys. **28** (1987) 1716.

[Cla08] T. Claeys, Int. Math. Res. Notices 2008, ID rnm166.

[Cri88] H. J. Sommers, A. Crisanti, H. Sompolinsky and Y. Stein, Phys. Rev. Lett. **60** (1988) 1895.

[Cri93] A. Crisanti, G. Paladin, A. Vulpiani, *Products of random matrices*, Springer series in solid state sciences 104 (1993).

[Dav91] F. David, Nucl. Phys. B **348** (1991) 507; Phys. Lett. B **302** (1993) 403.

[Dei] P. Deift, *Orthogonal Polynomials and Random Matrices : A Riemann-Hilbert Approach*, Courant Institute of Mathematical Sciences (1999).

[Del00] R. Delannay, G. Le Caer, J. Physics A: Math. Gen. **33** (2000) 2611; G. Le Caer and R. Delannay, J. Physics A: Math. Gen. **36** (2003) 9885.

[Dem91] K. Demeterfi, C. I. Tan, Phys. Rev. D **43** (1991) 2622.

[Deo02] N. Deo, Phys. Rev. E **65** (2002) 056115.

[DiF95] P.Di Francesco, P.Ginsparg, J.Zinn-Justin, Phys. Rep. **254** (1995) 1.

[DiF02] P. Di Francesco, cond-mat/021159, arXiv:0406013 [math-ph].

[DiF03] P. Di Francesco, Nucl. Phys. B **648** (2003) 461.

[Dui08] M. Duits and A. B. J. Kuijlaars, Comm. Pure Appl. Math. **62** (2009) 1076.

[Eve08] F. Evers and A. D. Mirlin, Rev. Mod. Phys. **80** (2008) 1355.

[Eyn06] B. Eynard, J. Stat. Mech. (2006) P07005.

[Eyn07] B. Eynard and N. Orantin, Comm. Number Th. and Phys. **1** (2007) 347, arXiv:0702045 [math-ph].

[Fei97] J. Feinberg and A. Zee, Nucl. Phys. B **504** [FS] (1997) 579; J. Feinberg, R. Scalettar and A. Zee, J. Math. Phys. **42** (2001) 5718; J. Feinberg, J. Phys. A: Math. Gen. **39** (2006) 10029.

[Fei97b] J. Feinberg and A. Zee, J. Stat. Phys. **87** (1997) 473.

[Fei89] M. Feingold, D. M. Leitner and O. Piro, Phys. Rev. A **39** (1989) 6507.

[Fok91] A. S. Fokas, A. R. Its and A. V. Kitaev, Comm. Math. Phys. **142** (1991) 313; Comm. Math. Phys. **147** (1992) 395.

[Fro] H. Froese, Cycon and B. Simon, *Schroedinger operators*, Texts and Monographs in Physics, Springer (1987).

[Fyo91] Y. V. Fyodorov and A. D. Mirlin, Phys. Rev. Lett. **67** (1991) 2405.

[Fyo07] Y. Fyodorov and B. Khoruzhenko, Acta Phys. Pol. B **38** (2007) 4067.

[Gin65] J. Ginibre, J. Math. Phys. **6** (1965) 440.

[Gol98] I. Goldsheid and B. Khoruzhenko, Phys. Rev. Lett. **80** (1998) 2897.

[Got08] F. Gotze and M. Gordin, Comm. Math. Phys. **281** (2008) 203.

[Gri03] T. S. Grigera, V. Martin-Mayor, G. Parisi and P. Verrocchio, Nature **422** (2003) 289.

[Gro80] D. J. Gross and E. Witten, Phys. Rev. D **21** (1980) 446.

[Gro91] D. J. Gross and M. J. Newman, Phys. Lett. B **266** (1991) 291.

[Haa] F. Haake, *Quantum signatures of chaos*, 2nd ed. Springer (2000).

[Hat96] N. Hatano and D. R. Nelson, Phys. Rev. Lett. **77** (1996) 570.

[Hua09] B. J. Huang and T-M. Wu, Phys. Rev. E **79** (2009) 041105.

[Itz80] C. Itzykson and J. B. Zuber, J. Math. Phys. **21** (1980) 411.

[IHP] http://www.andersonlocalization.com/canonal50.php

[Jan97] R. A. Janik, M. A. Nowak, G. Papp and I. Zahed, Acta Phys. Pol. B **28** (1997) 2949; Phys. Rev. E **55** (1997) 4100; Nucl. Phys. B **501** (1997) 603.

[Jan02] R. A. Janik, Nucl. Phys. B **635** (2002) 492.

[Jan98] M. Janssen, Phys. Rep. **295** (1998) 1.

[Jar06] A. Jarosz and M. A. Nowak, J. Phys. A: Math. Gen. **39** (2006) 10107.

[Joh98] K. Johansson, Duke Math.J. **91** (1998) 151.

[Jur90] J. Jurkiewicz, Phys. Lett. B **245** (1990) 178.

[Jur91] J. Jurkiewicz, Phys. Lett. B **261** (1991) 260.

[Kaz90] V. A. Kazakov, Phys. Lett. B **237** (1990) 212.

[Kho96] B. Khoruzhenko, J. Phys. A: Math. Gen. **29** (1996) L165.

[Kon92] M. Kontsevich, Comm. Math. Phys. **147** (1992) 1.

[Kra93] B. Kramer and A. MacKinnon, Rep. Progr. Phys. **56** (1993) 1469.

[Kra94] V. E. Kravtsov, I. V. Lerner, B. L. Altshuler and A. G. Aronov, Phys. Rev. Lett. **72** (1994) 888.

[Kri06] G. S. Krishnaswami, JHEP 03 (2006) 067.

[LeC07] G. Le Caer and R. Delannay, J. Physics A: Math. Gen. **40** (2007) 1561.

[Lec91] O. Lechtenfeld, Int. J. Mod. Phys A **9** (1992) 2335.

[Liu09] Dang-Zheng Liu and Da-Sheng Zhou, arXiv:0905.4932.

[Mac81] A. MacKinnon and B. Kramer, Phys. Rev. Lett. **47** (1981) 1546.

[Man90] G. Mandal, Mod. Phys. Lett. A **5** (1990) 1147.

[Mar86] F. Martinelli, *A rigorous analysis of Anderson localization*, Lecture Notes in Physics 262, Springer (1986).

[Meh81] M. L. Mehta, Comm. Math. Phys. **70** (1981) 327.

[Mez99] M. Mézard, G. Parisi and A. Zee, Nucl. Phys. B **559** [FS] (1999) 689.

[Miz05] S. Mizoguchi, Nucl. Phys. B **716** (2005) 462.

[Mo07] M. Y. Mo, Int. Math. Res. Notices 2008, ID rnn042.

[Mo08] M. Y. Mo, arXiv:0811.0620 [math-ph].

[Mol88] L. Molinari, J. Phys. A: Math. Gen. **21** (1988) 1.

[Mol09a] L. G. Molinari, J. Phys. A: Math. Theor. **42** (2009) 265204.

[Mol09b] L. G. Molinari and G. Lacagnina, J. Phys. A: Math. Theor. **42** (2009) 395204.

[Moo90] G. Moore, Comm. Math. Phys. **133** (1990) 261; Progr. Theor. Phys. Supp. **102** (1990) 255.

[Nak06] Y. Nakamura and N. Hatano, J. Phys. Soc. Jpn. **75** (2006) 114001.

[NI] http://www.newton.ac.uk/webseminars/pg+ws/2008/mpa/mpaw01/

[Nis02] S. Nishigaki and A. Kamenev, J. Phys. A: Math. Gen. **35** (2002) 4571.

[Pas92] L. Pastur and A. Figotin, *Spectra of Random and Almost-Periodic Operators*, Springer (1992).

[Pas06] L. Pastur, J. Math. Phys. **47** (2006) 103303; L. Pastur and M. Shcherbina, J. Stat. Phys. **130** (2008) 205.

[Ros63] N. Rosenzweig, *Statistical Physics*, Brandeis Summer Inst. 1962, ed. by G. Uhlenbeck et al., Benjamin, New York (1963); B. V. Bronk, Ph.D Thesis, Princeton Univ. 1964.

[Sen92] D. Sénéchal, Int. J. Mod. Phys. A **7** (1992) 1491.

[Shk93] B. I. Shklovskii, B. Shapiro, B. R. Sears, P. Lambrianides and H. B. Shore, Phys. Rev. B **47** (1993) 11487.

[Shi82] Y. Shimamune, Phys. Lett. B **108** (1982) 407.

[Teo06] R. Teodorescu, J. Phys. A: Math. Gen. **39** (2006) 8921.

[Vol] D. Vollhardt and P. Wolfle, *Self consistent theory of Anderson localization*, in Electronic Phase Transitions, Ed. by W. Hanke and Yu. V. Kopaev, Elsevier (1992).

[vOp94] F. von Oppen, Phys. Rev. Lett. **73** (1994) 798.

[Wad81] S. R. Wadia, Phys. Rev. D **24** (1981) 970.

[Zak93] J. Zakrzewski and D.Delande, Phys. Rev. E **47** (1993) 1650.

[Zha97] I. K. Zharekeshev and B. Kramer, Phys. Rev. Lett. **79** (1997) 717.

[Zin98] P. Zinn-Justin, Comm. Math. Phys. **194** (1998) 631.

[Zyc94] K. Zyczkowski, L. Molinari and F. Izrailev, J. Physique I France **4** (1994) 1469.