Biextensions and 1-motives

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(joint work with Cristiana Bertolin)

The notion of biextension of two abelian groups by another abelian group is a classical one coming from [SGA7, VII], later extended by Deligne in [D, 10.2] to the case where K_0 and K_1 are two complexes of abelian sheaves and H is another abelian sheaf. His result $Biext^1(K_1, K_2; H) \cong Ext^1(K_1 \otimes^L K_2, H)$ suggests that the biextension of two 1-motives is related to their tensor product, which is outside the category of 1-motives for trivial reasons.

Cristiana Bertolin futher extended Deligne's work by defining the biextensions of two 1-motives (M_1 and M_2) by another 1-motive (M_3) and proved in [B] that

$$Biext^1(M_1, M_2; M_3) \cong Ext^1(M_1 \otimes^L M_2, M_3).$$

It was stated by Voevodsky, later proved by Orgogozo in [O], and generalized by Barbieri Viale and Kahn in [BVK], that there is a fully faithfull embedding Tot of the category of 1-motives into the category DM. Cristiana Bertolin and I proved in [BM] that, after tensoring with $\mathbb{Q}$, that the embedding realizes the connection between biextension and the tensor product of two 1-motives, i.e.,

$$Biext^1(M_1, M_2; M_3) \otimes \mathbb{Q} \cong Hom_{DM(\mathbb{Q})}(Tot(M_1) \otimes Tot(M_2), M_3).$$

This also answers a question in [BVK] becuse applying $LAlb$ on the right gives

$$Biext^1(M_1, M_2; M_3) \otimes \mathbb{Q} \cong Hom_{\text{1-isoMot}}(M_1 \otimes^1 M_2, M_3).$$

Here is a sketch of our proof: taking into account the different degree conventions by Deligne and Voevodsky, by [B] we get

$$Biext^1(M_1, M_2; M_3) \otimes \mathbb{Q} \cong Ext^1(M_1[1] \otimes^L M_2[1], M_3[1]) \otimes \mathbb{Q},$$

and by simple homological algebra we have

$$Ext^1(M_1[1] \otimes^L M_2[1], M_3[1]) \otimes \mathbb{Q} \cong Hom_{D(Sh)\otimes\mathbb{Q}}(M_1 \otimes^L M_2, M_3),$$

where $D(Sh)$ is the derived category of abelian sheaves. By [BVK, 4.4.1]:

$$Hom_{D(Sh)\otimes\mathbb{Q}}(M_1 \otimes^L M_2, M_3) \cong Hom_{DM(\mathbb{Q})}(Tot(M_1) \otimes Tot(M_2), M_3).$$

References

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