ESSAYS ON ECONOMIC GROWTH: TECHNICAL PROGRESS, POPULATION DYNAMICS AND THE ENVIRONMENT

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by

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To Dad and Mum,

and

Sonia

Declaration

This thesis is a presentation of my original research work. Wherever contributions of others are involved, every effort is made to indicate this clearly, with due reference to the literature, and acknowledgement of collaborative research and discussions. Some parts of chapters 2 and 3 come from joint works of Davide La Torre and myself. My personal contribution in these joint works is prevailing: it is not easy quantifying it, but we could say that it is around 70%.

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Simone Marsiglio

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Contents

Introduction

Economic growth studies the growth of the per-capita GDP observed in modern economies after the industrial revolution. Such an event marked the end of the so-called Malthusian era, characterized by stagnation and high population growth, and the start of the Solowian era, characterized instead by consistent output growth and low demographic change. The explanation and investigation of such a sudden break in recent economic history is not among the goals of economic growth theory (a new branch of this field, called unified growth theory, has recently arisen, with the aim of determining the causes and reasons which led to the switch between these two antithetic eras; see for example Galor and Weil (2000)). Standard growth theory wonders mainly why two different countries, with similar endowments and initial conditions, dramatically differ in their later growth performances (see for example Lucas (1993)). Among possible explanations, economists emphasize the role played by education and innovation, as the main source of discrepancies among countries. Another explanation, not often analyzed by growth economists, is the presence of indeterminate equilibria. In fact, we can define as *"indeterminate a situation in which there exists a continuum of distinct equilibrium paths sharing a common initial condition"* (Boldrin and Rustichini (1994)). If such a condition is verified, the economic dynamics is not unique in the sense that multiple paths lead the same economy to converge towards its long-run equilibrium. In growth theory, the possibility of indeterminacy has never played a crucial role in describing different developing trajectories, because in standard models such a situation can be verified only in the presence of increasing returns to scale. In this work, especially in chapter 3, we will show how even ruling out the strong assumption of increasing returns to scale, equilibria in multi-sector growth models can be indeterminate (in chapter 2 we show that this can happen by considering also the dynamics of population, while in chapter 3 by introducing endogenous technical progress). This is just a first step to underline the relationship between growth and fluctuations, and to promote the joint analysis of such issues.

This thesis analyzes economic growth and how this is related to different issues, namely technical progress, population change and environment. It studies each of these issues in a separate paper. The choice of these issues has been driven by their growing importance in the analysis of the development process of modern economies. Technological progress, jointly with the accumulation of human capital, is one of the most relevant causes of the consistent growth showed in the last century by industrialized countries; therefore, it is important to understand what are its features in order to promote further technical improvements. Demographic growth has dramatically changed during and after the transition from stagnation to growth (happened contemporaneously to the industrial revolution): fertility and mortality rates have dropped and many economies now show a rate of population growth just over the replacement one; studying the implications of population change for economic growth can be really important in order to understand whether population policies can be necessary or not for the economy. The environment is an important source of welfare services to people and just in the last decades such a fact has been widely recognized: this is due to the fact that as the economy reaches a certain level of development, its agents feel the importance of some aspects which in a previous phase they did not care about; human activity is the main source of environment degradation and an increasing need for the policy makers to understand how regulating it has arisen. As it may be clear, these topics are crucially interrelated: population growth affects technical progress (through the number of researchers employed in R&D activity) and the environment (through the necessity of satisfying the needs of a larger population, in terms of consumption demand and waste production), while technical progress affects the environment (switching from polluting to clean technologies such an effect can crucially change). However, such aspects are not analyzed in the present work, mainly for problems of tractability and time, but these issues are particularly important and interesting, therefore they are left for further studies.

The first chapter analyzes the relationship between environment, growth and population. It combines two different issues: that on the linkages between population and economic growth, and that on linkage between sustainable development and population. The literature on the relationship between economic performance and population growth has really ancient roots, but a unique view has not arisen yet. The empirical literature is critical on the existence of a relationship between population growth and economic (per-capita GDP) growth. The main conclusion of this theory concerns the presence of a different impact varying from country to country. Kelly and Schmidt (1995) conclude that the impact of population on the economy depends on the level of economic development: the relationship between population and economic growth is non-monotonic and if so, it is non-linear. The literature on sustainable development is based on the recognition that developments of human activity in the last centuries has dramatically changed the planet's climate, the biological mix and the natural resources. The main reasons of such impacts are related to the economy and population. Economic production uses energy, which is mainly obtained by fossil fuels, leading to carbon emissions. The emissions generated in the past century have constantly grown, irreversibly altering the planet climate.

Population has also constantly grown, leading to higher and higher demand for production (of food and other goods), waste and space, increasing the use of natural resources. The first definition of sustainable development is due to the Brundtland Commission (1997), which defines sustainable development as the *"[development that] satisfies the needs of the present without compromising the ability of future generations to meet their own needs"*. Standard macroeconomic theory adopt discounted total utilitarianism as a welfare criterion, but it attaches less weight to future generations, therefore it is not useful for the analysis of the sustainable matter. Different approaches have been proposed in the literature to deal with it: some suggest to modify the welfare criterion emphasizing the importance of asymptotic behavior (Chichilnisky et al. (1995); and Chichilnisky (1997)), while others suggest to impose some additional constraints to the optimization problem in order to ensure sustainability (Arrow et al. (2004); and Pezzey (1997)). Which is the most convenient approach in order to deal with sustainability is still an open question. We look for some minimal requirements for sustainability, combining these approaches: following Arrow et at. (2004) and Pezzey (1997), we try to impose some constraints to the standard optimal control problem in order to ensure sustainability, weakening their definition; following Chichilnisky et al. (1995) and Chichilnisky (1997), we rely on the importance of asymptotic behavior. We define a path as sustainable if it implies (strictly) positive values of all the economic variables, both in finite and infinite time.

The paper analyzes a growth model driven by natural resources and without production, where agents have jointly to determine consumption and fertility, taking into account the effect of their decisions on the dynamics of natural resources. We look for the existence of sustainable path, adopting the most optimistic view on natural resources (they ensure endogenous growth) and the weakest definition of sustainable path (all variables positive). In this framework the expected outcome is that sustainable paths exist. However, we show that this is not always true. In fact, even if the renewal capacity of natural resources is unbounded, not always a sustainable path, where both population and natural resources coexist, can be found: it depends on the stationary fertility level (whether it is higher of the mortality rate or not). With respect to the sustainability notions of Arrow et al. (2004) and Pezzey (1997), our definition of sustainability has the advantage of discriminating among different paths, labeling some of them as sustainable and some others as not (according to Arrow et al., no path will be sustainable in our model while according to Pezzey, all paths will be). Moreover, we show that the planner, through policies oriented to improve public attention to environment protection or to modify the intensity of the dilution effect, can affect the growth rate of population and of the overall economy: this is an indirect effect working through the interaction between fertility, population and economic growth. In particular, if the stationary fertility is lower than the mortality rate, public intervention can be necessary in order to address the economy along a sustainable path.

This can be simply done through policies affecting the fertility or the mortality rate.

The second chapter studies the impact of population growth on the economy. In the literature of optimal growth, most of the papers assume population growth is constant and exponential, implying that population (and labor force) asymptotically approaches infinity. However, most populations are constrained by limitations on resources, at least in the short run, and none is unconstrained forever. Several studies support the idea that human population growth is decreasing and tending towards zero (as Day (1996)): recently such a result has been used to claim that population dynamics follows a logistic law. Really recently, some papers try to relax the assumption of exponential population growth, studying the implications of other population growth functions. However, they also relax an important standard assumption of optimal growth theory, namely the social welfare function is founded on the Benthamite criterion. This criterion says that total welfare is the sum of per-capita welfare over population (the product between population size and average welfare). All the papers dealing with non exponential population growth instead assume the social welfare function is based on the Millian criterion: total welfare equals average welfare or per-capita utility. Such a criterion has been used in order to limit population size, but in an optimal growth framework this seems somehow reductive. The ethical implications of both criteria are discussed in Marsiglio (2010): for example, it is well known that the Benthamite criterion can lead to what Parfit (1984) defined as a repugnant conclusion.

This paper studies the impact of different population growth functions on optimal growth models, in which the social welfare function is based on the Benthamite criterion (or total utilitarianism) or the Millian one (average utilitarianism), according to the degree of agents' altruism. We analyze a Uzawa (1965) - Lucas (1988) type model and show that a unique non-trivial equilibrium exists and the economy converges towards it along a saddle path, independently of the shape of the population change function. What is affected by its shape is the dimension of the stable manifold, which can be one or two, and the timing when the equilibrium is reached, which can happen in finite or infinite time. It shows that if population growth shows some zeros, as in the logistic case, both the stable and unstable manifolds result to be two-dimensional loci and the steady state is only asymptotically approached. Notice that in this case we have equilibrium indeterminacy and multiple converging trajectories exist. Moreover, if population growth shows some zeros the choice of the Benthamite rather than the Millian criterion is completely irrelevant for the outcome of the model: even if the transitional dynamics changes adopting one or the other criterion, this difference completely vanishes as the equilibrium is approached. The paper also studies the situation in which population is subject to random shock driven by a geometric Brownian motion and shows that a closed form solution can be found if the altruism is impure, and in particular if it coincide with the capital share and the inverse of the intertemporal elasticity of substitution.

The third chapter analyzes endogenous technical progress in a multi sector growth model. In the literature, during the last fifty years different variables have been identified as possible sources of economic growth. Solow (1956) and others after him identified the accumulation of physical capital as crucial in explaining growing economies. In Lucas' (1988) view, instead, the evolution of human capital is the key feature driving growth in the long run. Recently, a literature stream, led mainly by Romer (1986, 1990), has started considering ideas as the relevant engine of growth. Ideas-based growth models assume that the creation of new ideas is the source of endogenous growth and in order to so, some kind of linearities have been introduced in the technology production function (see Jones (2005) for a survey). For example, Romer (1990) assumes creation of ideas is linear both in the stock of ideas and human capital employed in research. However, empirical works suggest there are significant decreasing returns of ideas at the aggregate level. Therefore, we relax this linearity assumption and consider that knowledge is created according to neoclassical technology, and we rely on Lucas view that human capital is force driven long-run growth.

This paper combines these different approaches in a unified model, and the natural candidate for this goal seems to be the Uzawa(1965)-Luca(1988) growth model. It presents an endogenous growth model driven by human capital, where human capital can be allocated across three sectors: the final one, the educational sector and that devoted to accumulating technological capital (in the form of knowledge or ideas). In the model, labor augmenting technical progress is endogenous and this enriches the transitional dynamics of the economy. With respect to the Uzawa-Lucas model, we introduce an additional sector, that creating new ideas, and we let human capital to be endogenously allocated across three sectors. With respect to ideas-based growth models, instead, we assume knowledge is produced according to a neoclassical technology, combining ideas and human capital. Such an assumption is motivated by empirical works showing the existence of significant decreasing returns in the creation of ideas at the aggregate level (as Kortum (1993); and Pessoa (2005)) and of the weak relationship between some inputs of the knowledge production process (as the number of researchers) and the total factor productivity growth rate (as Jones (2002)). Under some general conditions, the economy converges towards its equilibrium along a form of generalized saddle path, along which both the stable arm and the unstable manifold are multidimensional. This means that the equilibrium is indeterminate: there exists a continuum of path satisfying the initial conditions and converging to equilibrium. The paper also studies through numerical examples the resulting optimal allocation of human capital in steady state. Under general parameter values, the highest share of human capital is devoted to creation of new human capital, and the lowest share is allocated to knowledge production. Such an outcome is clear: the impact of human capital is more important in the educational sector, since it is the growth driven force, and in the physical one, since it produces the consumable good, which is the argument of agents' utility function, while it is lower in the technological sector. This ranking is clearly reflected by the optimal allocation of resources.

As it may seem clear from this quick outlook of this work, the issues tackled in this thesis are crucially interrelated. Technical progress, demography and environment are different aspects of the economy which should be jointly analyzed in order to allow us to understand the role of public intervention. In fact, whether and which policies are needed to promote growth is still not clear from the separate analysis of these aspects. This gives rise to the necessity of further investigations of these issues and their mutual implications. This requires time and resources, but an effort to fully answer the open questions is necessary, given the great importance these topics have for the development possibilities of modern economies. This is left for future research and it is currently on top of my personal research agenda.

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Chapter 1

Population Change and Sustainable Development

This paper[1] investigates the relationship between population growth and economic growth, through the study of fertility choices and their effects on natural resources. It aims at analyzing the interactions between endogenous fertility choices and the environment and their link to the sustainable matter. We analyze a growth model driven by natural resources and without production, where agents have jointly to determine consumption and fertility, taking into account the effects of their decisions on the dynamics of natural resources. We adopt the most optimistic view on natural capital (it generates endogenous growth) and the weakest notion of sustainable paths (all variables are positive): in such a framework we expect that sustainable paths exist. We instead show that this is not always true. In fact, even if the renewal capacity of natural resources is unbounded, not always a sustainable path can be found: this depends on the difference between the stationary fertility rate and the mortality rate. If the stationary fertility is lower than the mortality rate a sustainable path will not be found, and in such a case public intervention is necessary in order to address the economy along a sustainable path. This can be simply done through policies affecting public attention to environmental protection or the intensity of the dilution effect.

Keywords: Economic Growth, Sustainability, Intertemporal Welfare, Natural Resources, Population Change

JEL Classification: O40, O41, J13, Q20, Q56

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1.1 Introduction

The issue of the relationship between population and economic growth has really ancient roots in the economic literature: Adam Smith and Malthus were among the first discussing the importance of controlling population growth in order to promote economic performance. After them, several theoretical and empirical studies investigated the relationship between population change and economic growth both from the economic and the demographic viewpoint, but a shared view has not arisen yet. In fact, as Bloom et al. (2003) summarize: *"...Though countries with rapidly growing populations tend to have more slowly growing economies..., this negative correlation typically disappears (or even reverses direction) once other factors... are taken into account"*. Three approaches have been proposed in order to study the issue: an optimistic, a pessimistic and a neutral view[2] (see Bloom et al., 2003). The most probably spread is the pessimistic one (Solow (1956), Becker and Barro (1988) and Barro and Becker (1989)) and sees population as a threat for growth. This can be due to two different reasons: if the economy shows fixed resources and no sources of technical progress, in the long-run the (food) production activity will not be able to satisfy the pressure of population growth, leading per-capita resources to fall below a minimal subsistence level (Malthus, 1798); if the economy instead shows rapid population growth, then a large share of investment will be devoted to satisfy the needs of the increasing population (*"investment-diversion effect"* - Kelley, 1988), rather than to increase per-capita capital endowments. The proponents of this view base their argument on the idea that an increase in the population size leads to a dilution of available resources.

The topic of sustainable growth, instead, is a recent and growing issue in the economic growth literature. The possibility that deterioration of environmental quality, in particular caused by pollution, could inhibit economic growth was firstly suggested in the report to the Club of Rome entitled *'Limits to Growth'* by Meadows et al. (Meadows et al., 1972). The first recognition of the issue at international level was the creation by the UN General Assembly in 1985 of the Commission for Sustainable Development, chaired by the Prime Minister of Norway, Mrs Brundtland. The commission's report *'Our Common Future'* tried to emphasize that environmental protection is essential for economic development since the environment is an essential *"factor of production"* and source of important welfare services to people, even more in poor countries than in wealthy countries (World Commission on Environment and Development, 1987). There is wide agreement on the fact that sustainable development involves

[2]The optimistic view (Kuznets, 1960 and 1967, and Boserup, 1989; most recent analysis can be found in Jones, 2001, and Tamura, 2002) considers population as an important input to produce knowledge: the higher the population, the higher the probability new Isaac Newton were born. The neutral view (Bloom et al., 2003) instead has empirical foundation: there exists little cross-country evidence that population growth might either slow down or encourage economic growth

an integrated approach to economic, social and environmental processes; however, until now, the attention has been mainly focalized on the environmental and economic dimensions, addressing the social one (which can be mostly identified in the demographic one) only to a secondary role. In the current world, facing the uncertainties concerning the future of earth climate and environment, it is important to understand how finding a sustainable development path, where production, population and resources coexist without leading to an economic collapse. Such a situation in fact is not to be considered as unreal, as history can and should teach us. Typical examples are the collapse of the classical Maya civilization in the ninth century (Demarest, 2004), the dramatic decline of the Easter Island society (Flenley and Bahn, 2003) and the complete extinction of the Viking's colonies of Greenland (Diamond, 2005). However, what sustainable development really means is not clear: several definitions have been proposed and each of them underline different aspects of the matter; someone is too strong and someone else is too weak. The introduction of a clear notion of sustainability is still an open question and we propose an additional definition, which aim is at introducing minimal requirements for sustainability.

Population growth, as firstly Malthus (1798) noticed, is an important factor of environment depletion: consumption activities deteriorate environment and more people exert higher pressure on environmental stock. Therefore, since the interaction between natural resources and population can be really important, the aim of this paper is to build a bridge between these two different kinds of literature. In fact, economic growth can be considered the main goal of current economies: however, its link with population dynamics and natural resources has often been underestimated. Since there is still not a shared view on the relationship between population and economics (population growth is an important factor of environment depletion and environmental assets have a fundamental role for economic development), the interaction between these three factors deserves particular attention. Following Chinchilnisky et al. (1995), we analyze the problem of sustainability studying an optimal growth model, where environment is represented by the stock of natural resources. We therefore assume there exist a one to one correspondence between environment and natural resources. With respect to Chinchilnisky et al. (1995), we rely on discounted utilitarianism as a welfare criterion and we introduce population change and its linkage with environment (natural resources). Our paper is strictly related to Nerlove (1991), who studies the mutual relationship between population dynamics and the evolution of natural resources. With respect to him, we adopt an optimal growth framework (and not an OLG model) since sustainability issues have to be dealt with a long horizon approach; moreover, we explicitly model the population-environment relationship and we investigate under which conditions the economy is addressed along a sustainable path. We study the simplest model of endogenous growth, an AK type model, driven by natural resources, where economic agents jointly determine their consumption level and their fertility rate. Agents' decisions concerning consumption and fertility deplete the natural resources, which represent a source of utility. Therefore, they have to take into account the pressure their choices exert on the environment, trying to identify a possible sustainable development path, along which natural resources, population and growth coexist.

In section 2, we quickly review the issue of sustainability and its main implications for economic modeling. The attention is especially focused on the definition of sustainable development and on the choice of the welfare criterion to adopt in order to deal with such a matter. In section 3, we introduce the model economy in its general formulation, and derive the optimal paths for the control variables, consumption and fertility. Section 4 performs steady state analysis, studying a balanced growth path along which the fertility rate is constant and identifying the presence of possible sustainable paths (defined as a path along which population, natural resources and consumption are positive, also asymptotically), and develops a comparative statics analysis, focalizing on policy implications. We show that even if the renewal capacity of natural resources is unbounded, not always a sustainable path, where both population and natural resources coexist, can be found: this depends on the stationary fertility level. In particular, if it is lower than the mortality rate, the population will asymptotically disappear (implying that the economy will collapse) and the path followed by the economy is clearly not sustainable. We also show that with respect to other notions of sustainability (as Pezzey, 1997; and Arrow et al., 2004) our definition has the advantage of discriminating among different paths, labeling some of them as sustainable and some others as not. In section 5 we consider a special case of the model, that is an economy in which the stock of natural resources does not affect welfare, and we highlight the main differences. We show that also when utility depends only on consumption, the growth rate of population determines whether the path followed by the economy is sustainable or not, and the main results of the previous section still hold. Section 6, as usual, concludes.

1.2 Sustainable Development and Intertemporal Welfare

The developments of human activity in the last two centuries has dramatically changed the planet's climate, the biological mix and the natural resources. The main reasons of such impacts are related to the economy and population. Economic production uses energy, which is mainly obtained by fossil fuels, leading to carbon emissions. The emissions generated in the past century have consistently grown, irreversibly altering the climate of the planet. Population has also constantly grown, leading to higher and higher demand for production (of food and other consumption goods), waste and space, increasing the use of natural resources. These facts have remained without consideration for long time; only during the last decade of the XXI century the problem of ensuring a certain level of equity, among generations and among countries, emerged.

Probably the most important definition of sustainable development has been introduced by the Brundtland Commission, which labels sustainable development as development that *"satisfies the needs of the present without compromising the ability of future generations to meet their own needs"* (World Commission on Environment and Development, 1987). This notion has been widely accepted, probably because of its weakness: it does not impose any constraints to growth and neither any particular duties to current generation, and moreover it is not formal at all. However, it clearly implies that sustainability mostly concerns two different but interrelated issues: respect of natural resources and intergenerational equity. The respect of natural resources is crucial to ensure equity among generations, because if each generation determines its consumption level without taking into account the effect implied on the future one, all resources will be exhausted soon and probably no future on the earth could be ensured. The intergenerational equity, instead, plays a central role on the evaluation of intertemporal welfare[3], and therefore on the identification of optimal allocations. Such an issue in standard macroeconomic (and in particular in growth) theory has generally been dealt with discounted (total) utilitarianism, which defines the social welfare as:

$$W = \int_0^\infty u(c_t) N_t e^{-\rho t} dt.$$

This means that social welfare equals the average utility, $u(c_t)$ where $c_t = \frac{C_t}{N_t}$ is per-capita consumption, multiplied by the population size, N_t, discounted by rate of time preference, $e^{-\rho t}$ where ρ it the pure rate of time preference. Notice that the introduction of discounting is necessary only for mathematical reasons, that is to ensure the bounded-ness of the objective function. But this mathematical necessity implies economic consequences, that is we are attaching less weight to future generations. As Ramsey (1928) commented, *"discounting of future utilities is ethically indefensible and arises purely from a weakness of the imagination"*. Of course, such a criterion cannot be used to deal with intergenerational equity, and therefore with sustainable issues. In order to avoid this, several attempts have been done in the literature. Some of them require to adopt a different welfare criterion (as Ramsey, 1928; von Weizcker, 1967; Chichilnisky et al., 1995; and Chichilnisky, 1997) while some others to impose some additional constraints to the standard optimal control problem (as Arrow et al., 2004; and Pezzey, 1997).

Ramsey (1928), assuming (other than special assumptions to ensure the sum converge) that utility levels are bounded above, proposes to minimize the total difference over time between

[3] See Heal (2005) for a survey of the issue

maximal utility and actual utility levels: $\int_0^\infty [b - u(c_t)]dt$, where b is a bliss point, the upper bound of the utility function. von Weizcker (1967) and others after him try to develop the overtaking approach: *"the overtaking criterion ranks as best the consumption sequence, if any, whose cumulative utility sum eventually exceeds that on any other path"* (Heal, 2005). These approaches ensure that we assign equal weight to all generations, but they have never been used, at least to our knowledge, in the literature. Another idea instead suggests that the sustainability issue has to be somehow linked with steady state and asymptotic behavior. For example, Chichilnisky et al. (1995) propose the Green Golden Rule as a welfare criterion to take into account sustainable matters. The Green Golden Rule consists of maximizing:

$$\lim_{t \to \infty} u(c_t) N_t,$$

that is, it represents the allocation maximizing the asymptotic (steady state) utility level, in order to determine the highest indefinitely maintainable utility level (rather than the highest consumption level, as implied by the standard Golden Rule). Again, Chichilnisky (1997) proposes the objective function to be a weighted average of discounted utility and asymptotic one:

$$W = \pi \int_0^\infty u(c_t) N_t e^{-\rho t} dt + (1 - \pi) \lim_{t \to \infty} u(c_t) N_t,$$

where $\pi \in (0, 1)$ is the weight assigned to discounted integral of utilities and $(1 - \pi)$ it that of long-run utility level, representing the sustainable utility level. The importance of the introduction of an asymptotic term is due to the fact that discounted utilitarianism attaches less weight to future generations in order to ensure the bounded-ness of the objective function and the role of the asymptotic term is therefore taking into account also long-run generations' welfare.

A completely different approach has instead been introduced in Arrow et al. (2004), which defines a path as sustainable if it implies non-decreasing welfare. The authors do not try to modify the standard welfare criterion, which remains discounted utilitarianism,

$$W = \int_0^\infty u(c_t) N_t e^{-\rho t} dt,$$

but just look for the imposition of some constraint in order to ensure sustainability, that is $\frac{\partial W}{\partial t} \geq 0$. However, a drawback of this definition is that paths satisfying both non-decreasing welfare and Pontryagin necessary conditions for consumption do not exist. In fact, optimal paths in neoclassical models do not satisfy this additional constraint. This is probably the reason why such an approach has never been used in the following literature. A similar approach can be found in Pezzey's (1997) survivable criterion, which proposes that the welfare derived from the standard dynamic maximization problem has to be higher than a minimal welfare associated to the survival of current population in order for the economy to be sustainable (survivable).

As it may be clear from this quick survey on the most important attempts to deal with sustainability, which is the most convenient approach is still an open question. Following Arrow et at. (2004) and Pezzey (1997), we try to impose some constraints to the standard optimal control problem in order to ensure sustainability. With respect to them, we look for some minimal requirements for sustainability (therefore something much weaker than their notion), relying on the importance of asymptotic behavior, as suggested by Chichilnisky et al. (1995) and Chichilnisky (1997). In particular, we define a path as sustainable if it implies (strictly) positive values of all the economic variables, both in finite and infinite time (see Definition 2).

1.3 The Model

The economy is closed and composed of households that can only consume the unique good present in the economy (a natural good) and have to choose how much consumption and how many children to have. Therefore, population grows in accordance to household decisions[4]. There is no production and human choices of consumption and fertility decrease the stock of natural resources.

The representative household wants to maximize its lifetime utility function, which is the sum of its instantaneous utility function, depending both on per-capita consumption, c_t, and on the stock of natural resources, E_t, where $\frac{\partial u(\cdot)}{\partial c_t} > 0$, $\frac{\partial^2 u(\cdot)}{\partial c_t^2} < 0$ and also $\frac{\partial u(\cdot)}{\partial E_t} > 0$, $\frac{\partial^2 u(\cdot)}{\partial E_t^2} < 0$. In order to get a closed form solution, it is assumed to be iso-elastic:

$$u(c_t, E_t) = \frac{(c_t E_t^{\beta})^{1-\sigma}}{1-\sigma}, \tag{1.1}$$

where $\sigma \in (0, 1)$ and $\beta \geq 0$. The utility function depends on the individual consumption level (households are not interested in aggregate consumption, but only in per-capita consumption) and on the stock of natural resources and the term β represents the weight of environment in agents utility (the green preferences), but does not depend on the fertility rate (having children or not does not affect the utility level). Notice that $\beta = 0$ represents the case in which environment is not a source of utility (the utility function depends only on consumption) and such a case will be analyzed in Section 5.

[4]The analysis of the causes of household choices is out of the goal of this work. In our model, fertility choices are endogenous as a result of the non-linear relationship between population growth and the environment, while the trade-off concerning the fertility decision is not tackled for the sake of simplicity. According to distinction of Nerlove and Raut (1997), our model should be labeled as a model of endogenous population change, rather than as model of endogenous fertility, since *"no decision-making mechanism is presupposed"*

Population grows over time at a non constant rate, given by the difference between the endogenously determined birth rate, n_t, and the exogenous mortality rate, d:

$$\dot{N}_t = (n_t - d)N_t, \tag{1.2}$$

where both n_t and d are strictly positive.

The dynamics of natural resources depends on their renewal capacity, aggregate consumption and the dilution effect associated to population growth:

$$\dot{E}_t = R(E_t) - C_t - \phi(n_t)E_t \tag{1.3}$$

where $R(E_t)$ is the renewal capacity of the environment and $\phi(n_t)$ is the dilution function, related to population variations. We consider for simplicity the case in which renewal capacity is unbounded, that is $R'(E_t) > 0$, and we assume it is a linear function[5] of the stock of natural resources:

$$R(E_t) = rE_t. \tag{1.4}$$

The dilution effect in natural resources instead represents the pressure wielded by population growth on natural resources (and therefore economic growth). A diffused view on the relationship between population growth and economic growth sees population growth as detrimental for growth: in fact, the food production activity, which is mainly derived from natural resources, is overwhelmed by the pressures of population growth, and this can lead the available diet to fall below the subsistence level (Malthus, 1798). Therefore, it seems plausible to assume $\phi'(n_t) > 0$: as fertility increases, the pressure on natural resources increases too. Kelly and Schmidt (1995) shows that the impact of population on the economy depends on the level of economic development: the impact of population growth is negative for less developed countries, while it is positive for developed ones. Therefore, this impact can change over time as the development proceeds. According to this result, therefore, the relationship between population and economic growth is non-monotonic. Non-monotonicity implies that such a relationship is non-linear[6]. We

[5]The choice of a linear function is reductive, but it permits us to characterize the steady state of our economy as a balanced growth path, simplifying computational problems. However, choosing another kind of unbounded function should not lead to different results. Moreover, the linear specification represents the most optimistic view on environmental regeneration and therefore it is an interesting benchmark for our analysis. Of course, if we consider a bounded renewal capacity, as in a logistic function, the outcome of the model can dramatically change

[6]Since natural capital is the force driving endogenous growth, the $\phi(\cdot)$ function represents the pressure wielded by population change both on the environment and on the economic performance. Because no other linkage is present in our model between demography and economic growth, according to Kelly and Schmidt (1995) we assume such a function to be non-linear

therefore consider a function non-linear in the fertility rate:

$$\phi(n_t) \quad = \quad a n_t^b, \tag{1.5}$$

where $a > 0$ and $b > 0$ but $b \neq 1$. In particular, if $0 < b < 1$ ($b > 1$), an increase in the fertility rate (population size) decreases less (more) than proportionally the stock of natural resources. The presence of this non-linear function permits to have endogenous fertility even if fertility itself is not a source of utility. Notice that the environment is negatively affected by human choice through two different channels: consumption activity (human needs for life) requires the use of natural resources and fertility choices determine the size of population, which causes the weight of the pressure on environment.

The social planner maximizes the social welfare function of the economy under the economy resource constraint, the law of motion of demography and the initial conditions for natural resources and population:

$$\begin{aligned} \max_{c_t, n_t} \quad & W = \int_0^\infty u(c_t, E_t) N_t e^{-\rho t} dt \qquad (1.6) \\ s.t. \quad & \dot{E}_t = r E_t - N_t c_t - a n_t^b E_t \\ & \dot{N}_t = (n_t - d) N_t \\ & E_0, \; N_0 \text{ given} \end{aligned}$$

The planner objective function takes into account the size of current and future generations, showing inter-temporal altruism, represented by ρ, the rate of time preference (the lower the rate of time preference, the higher the planner's altruism towards later generations), and full intra-temporal altruism (it means that the weight assigned by the planner to each member of the same generation is the same: the weight of each individual is independent of the size of the generation).

1.3.1 Optimal Paths

From the social planner maximization problem, we can derive the Hamiltonian function:

$$\mathcal{H}_t(c_t, E_t) \quad = \quad \frac{(c_t E_t^\beta)^{1-\sigma}}{1-\sigma} N_t e^{-\rho t} + \lambda_t \left[r E_t - N_t c_t - a n_t^b E_t \right] + \mu_t (n_t - d) N_t$$

and the first order necessary conditions:

$$\frac{\partial \mathcal{H}_t(\cdot)}{\partial c_t} = 0 \quad \rightarrow \quad (c_t E_t^\beta)^{-\sigma} E_t^\beta N_t e^{-\rho t} = \lambda_t N_t \tag{1.7}$$

$$\frac{\partial \mathcal{H}_t(\cdot)}{\partial n_t} = 0 \quad \rightarrow \quad \mu_t N_t = b \lambda_t a n_t^{b-1} A_t \tag{1.8}$$

$$\frac{\partial \mathcal{H}_t(\cdot)}{\partial E_t} = -\dot{\lambda}_t \quad \rightarrow \quad \beta(c_t E_t^{\beta})^{-\sigma} c_t E_t^{\beta-1} N_t e^{-\rho t} + \lambda_t \left[r - a n_t^2 \right] = -\dot{\lambda}_t \tag{1.9}$$

$$\frac{\partial \mathcal{H}_t(\cdot)}{\partial N_t} = -\dot{\mu}_t \quad \rightarrow \quad \frac{(c_t E_t^{\beta})^{1-\sigma}}{1-\sigma} e^{-\rho t} - \lambda_t c_t + \mu_t (n_t - d) = -\dot{\mu}_t \tag{1.10}$$

together with the initial conditions E_0 and N_0, the state equations:

$$\dot{E}_t = r E_t - N_t c_t - a n_t^b E_t \tag{1.11}$$

$$\dot{N}_t = (n_t - d) N_t \tag{1.12}$$

and the transversality conditions:

$$\lim_{t \to \infty} \quad E_t \lambda_t = 0 \tag{1.13}$$

$$\lim_{t \to \infty} \quad N_t \mu_t = 0 \tag{1.14}$$

Solving the system of FOCs, we can obtain the optimal paths of consumption and fertility:

$$\frac{\dot{c}_t}{c_t} = \frac{1}{\sigma} \left[\sigma \beta \frac{c_t N_t}{E_t} + [1 + \beta(1-\sigma)](r - a n_t^b) - \rho \right] \tag{1.15}$$

$$\frac{\dot{n}_t}{n_t} = \frac{1}{b-1} \frac{c_t N_t}{E_t} \left[(1+\beta) - \frac{\sigma}{b(1-\sigma) a n_t^{b-1}} \right]. \tag{1.16}$$

Equation (1.15) depends positively on the ratio between aggregate consumption and stock of natural resources and negatively on the fertility rate. In particular, a rise in the fertility rate leads to a non-proportional reduction in the growth rate of per capita consumption and this is due to the dilution effect, which is non-linear. Notice that, if $b > 1$, the growth rate of consumption is a concave function of the fertility level and the fertility rate maximizing consumption growth results to be null.

Equation (1.16) instead depends on the ratio between aggregate consumption and environmental stock and on the rate of fertility. The signs of the former relation cannot be determined a priori: in fact, it crucially depends on b (if it is higher or lower than one) and on the term in the square brackets; also the sign of the latter one is undetermined, since it depends on b. It is interesting to notice that the choice of the consumption level, determining the size of the ratio term, affects the growth rate of fertility.

The TVC (1.13) implies that the growth rate of natural resources is bounded above:

$$\gamma_E < r - a n_t^b + \beta \frac{c_t N_t}{E_t} \tag{1.17}$$

while the TVC (1.14) implies that the ratio between aggregate consumption and natural resources is positive.

1.4 Steady State Analysis

The growth rates of the per-capita consumption, natural resources, population and fertility rate are:

$$
\begin{aligned}
\gamma_c &= \frac{1}{\sigma}\left[\sigma\beta\frac{c_t N_t}{E_t} + [1+\beta(1-\sigma)](r - a n_t^b) - \rho\right] &(1.18)\\
\gamma_E &= r - \frac{c_t N_t}{E_t} - a n_t^b &(1.19)\\
\gamma_N &= n_t - d &(1.20)\\
\gamma_n &= \frac{1}{b-1}\frac{c_t N_t}{E_t}\left[(1+\beta) - \frac{\sigma}{b(1-\sigma) a n_t^{b-1}}\right] &(1.21)
\end{aligned}
$$

We now analyze possible equilibrium paths considering a balanced growth path, along which the growth rate of fertility is null.

Definition 1: (*Balanced Growth Path, BGP*) a balanced growth path, BGP, or steady state equilibrium, $(\overline{c}, \overline{n}, \overline{E}, \overline{N}, \gamma_c, \gamma_n, \gamma_E, \gamma_N)$, is a sequence of time paths, $\{c_t, n_t, E_t, N_t\}_{t\geq 0}$, along which all economic variables grow at constant rates. A BGP is said non-degenerate if c_t and E_t grow at non negative rates.

Along the BGP, the fertility rate, n_t, must be constant, $n_t = \overline{n}$: this means that the growth rate of fertility is null:

$$\overline{n} = \left[\frac{\sigma}{b(1+\beta)(1-\sigma)a}\right]^{\frac{1}{b-1}} \tag{1.22}$$

and the growth rate of population can be positive, negative or null in accordance to the difference[7] between $\overline{n}$ and d:

$$\gamma_N = \overline{n} - d. \tag{1.23}$$

Consequently, the growth rate of per capita consumption is:

$$\gamma_c = \frac{1}{\sigma}\left[\sigma\beta\frac{\overline{c}\overline{N}}{\overline{E}} + [1+\beta(1-\sigma)](r - a\overline{n}^b) - \rho\right] \tag{1.24}$$

and that of environment is:

$$\gamma_E = r - \frac{\overline{c}\overline{N}}{\overline{E}} - a\overline{n}^b \tag{1.25}$$

[7] In fact, no condition a priori imposes restrictions on the gap between the stationary fertility and the exogenous mortality rate

since the growth rate of environment must equalize the growth rate of aggregate consumption, otherwise $\gamma_E = -\infty$ or it would violate the TVC (1.13):

$$\gamma_E = \gamma_C = \gamma_c + \gamma_N. \tag{1.26}$$

This implies that per-capita variables, consumption and natural resources, grow at the same rate $\gamma = \gamma_c$.

In order to have endogenous growth, we have a lower bound for r:

$$r > a\overline{n}^b + \frac{\rho}{1+\beta(1-\sigma)} - \frac{\sigma\beta}{1+\beta(1-\sigma)}\frac{\overline{c}\overline{N}}{\overline{E}}, \tag{1.27}$$

while, in order to ensure bounded objective function instead,we have an upper bound for r:

$$r < a\overline{n}^b + \frac{\rho - \sigma(\overline{n}-d)}{(1-\sigma)(1+\beta)}. \tag{1.28}$$

Therefore, along the BGP the stationary fertility rate is positive, the economic growth rate is positive as well, while that of population can be positive, negative or null:

$$\overline{n} = \left[\frac{\sigma}{(1+\beta)(1-\sigma)ab}\right]^{\frac{1}{b-1}} \tag{1.29}$$

$$\gamma = \frac{1}{\sigma}\left[\sigma\beta\frac{\overline{c}\overline{N}}{\overline{E}} + [1+\beta(1-\sigma)](r - a\overline{n}^b) - \rho\right] \tag{1.30}$$

$$\gamma_N = \overline{n} - d. \tag{1.31}$$

If the fertility is lower than mortality rate, the population growth is negative and in the long-run all individuals will disappear: the population will continue to decrease until its complete disappearance but this would lead to have an high rate of growth during the life of the economy (the lower $\overline{n}$, the higher γ). If natality and mortality rate perfectly offset, the population will reach a positive stationary equilibrium level. If, instead, the birth rate is higher than the death rate, the population size will continue to rise, leading to a lower growth rate.

Proposition 1: *along the BGP, the following results hold:*
(i) if $b > 1$ ($b < 1$), the stationary fertility level is a positive (negative) function of the elasticity of substitution, σ, while it is a negative (positive) function of the green preferences, β, and of the dilution effect parameter, a;
(ii) the growth rate of the economy depends positively on the consumption-natural resources ratio, $\frac{C_t}{E_t}$, and negatively on the stationary fertility rate, $\overline{n}$;
(iii) population growth is a positive function of the stationary fertility level, $\overline{n}$ and a decreasing function of the mortality rate, d.

Proof: The result just derives from the partial derivatives of (1.29), (1.30) and (1.31), respect to the main parameters. ■

Among all possible paths, we are interested in a sustainable one, along which population, natural resources and growth coexist. However, as previously discussed, how sustainable path has to be interpreted is still not clear, even if several definitions have been proposed in the literature. Following Arrow et at. (2004) and Pezzey (1997), we introduce a definition of sustainable paths, rather than modifying the welfare criterion to adopt. With respect to them, we try to introduce the weakest notion, which could represent a minimal requirement for sustainability. Following Chinchilnisky et al. (1995) and Chinchilnisky (1997), we rely on the importance of asymptotic behavior. We define as sustainable every path along which all economic variables are strictly positive, requiring that this condition is satisfied also asymptotically.

Definition 2: (*Sustainable Development Path*) a sustainable development path is a sequence of time paths, $\{c_t, n_t, E_t, N_t\}_{t \geq 0}$, along which all economic variables are (strictly) positive, $c_t, n_t, E_t, N_t > 0$, and also asymptotically (strictly) positive, $\lim_{t \to \infty} c_t, \lim_{t \to \infty} n_t, \lim_{t \to \infty} E_t$, $\lim_{t \to \infty} N_t > 0$.

This definition is particularly weak: it does not concern growth rates but requires only that the variables are not addressed along a collapsing path. We require that population, consumption and environment (not their growth rate) are positive along the time horizon, but also at steady state. This looks like a minimal requirement in order to consider a path as sustainable: paths violating our definition cannot be labeled as sustainable in a stronger sense. In fact, a path along which consumption, population size and/or natural resources collapses cannot ensure the ability of future generations of satisfying their own needs, which is the most diffused (Brundtland Commission) definition of sustainability.

Notice that since the renewal capacity of natural resources is unbounded we can find a sustainable paths, where both population and natural resources coexist and moreover the economy grows. However, such a path does not always exist: it depends on the difference between the stationary fertility level and the mortality rate. If the former is lower than the latter, population will asymptotically disappear and no path can be found where both population and natural resource coexist: this means that agents endogenously decide the collapse of the society. Therefore, we have just proved:

Proposition 2: *the development path along which the economy is addressed is sustainable if the stationary fertility rate is at least as high as the mortality rate, that is if* $\overline{n} \geq d$, *otherwise* *it is not, that is the case in which $\overline{n} < d$.*

This is quite surprising: the model is really optimistic in terms of natural resources regeneration capacities and really weak in the notion of sustainability. This means that even in the most optimistic framework, if we consider also the interaction between population and environment, the existence of a sustainable path does not have to be taken for granted. However, in the case $\overline{n} < d$, the planner can intervene affecting $\overline{n}$ or d in order to switch the economy to a sustainable path (see next subsection). Notice that our definition of sustainability leads to a more realistic result than those we would obtain adopting another notion introducing sustainability as an additional constraint to the dynamic maximization problem (i.e. Pezzey (1997); Arrow et al. (2004)). Remember that Arrow et al. (2004) defines as sustainable a path along which welfare is non-decreasing over time, while Pezzey's (1997) define as survivable a path characterized by a welfare level higher than the minimal welfare allowing the survival of the current population. It is straightforward verifying that along the BGP, the welfare is decreasing over time, because of the necessity of ensuring bounded-ness of objective function[8]: this means that according to Arrow et al. (2004) formulation, no path is sustainable. It is also easy to see that if the minimal welfare associated to the survival of population is sufficiently low[9], then the steady state welfare level will always be higher than this: according to Pezzey's (1997) notion, all paths are sustainable (survivable). With respect to these notions, our definition of sustainability has the advantage of discriminating among different paths, labeling some of them as sustainable and some others as not.

We can notice that along the BGP, where the fertility rate is constant, the dynamic behavior of the economy is the same as in a standard AK model. Moreover, as in standard AK model, the model does not show any transitional dynamics: the economy lies along its BGP since time 0 (a similar result is obtained in Palivos and Yip (1993), who show that an AK model with endogenous fertility does not show transitional dynamics; see Appendix A for more details). In fact, at time 0 if the fertility rate is chosen equal to $\overline{n}$, the growth rate of the economy, given is

[8]In fact, since the economy lies on the equilibrium path from time 0 (see Appendix A), the derivative of the (integrand in the) welfare function in steady state is:

$$[(1-\sigma)\gamma_c + \beta(1-\sigma)\gamma_E - (\rho - \overline{n})]\frac{Ae^{[(1-\sigma)\gamma_c+\beta(1-\sigma)\gamma_E-(\rho-\overline{n})]t}}{1-\sigma},$$

where $A = \overline{c}^{1-\sigma}\overline{E}^{\beta(1-\sigma)}\overline{N}$. Condition (1.28) which ensures the fact that the welfare function is bounded, essential for the integral in W to be well defined, also ensures that the term in the square brackets is negative and therefore the whole time derivative is negative as well

[9]The value to attribute to the minimal welfare permitting actual population to survive is almost arbitrary. If this is too high, the welfare level along the BGP will never be higher than this and we will obtain the same conclusion of Arrow et al. (2004): no path will be sustainable. If this is low instead, all paths will be sustainable

equation (1.30) is constant (in fact, the consumption-natural resources ratio has to be constant because of the TVC (1.13)); moreover, if the economy is not along its BGP from time 0, it will never converge to it.

1.4.1 Comparative Statics

We now perform an exercise of comparative statics in order to better understand the role of some parameters on the economic and population growth rates and identify the main policy[10] implications, underlying especially how policies affecting the stationary fertility level (or the mortality rate) should be introduced in order to address the economy along a sustainable path. In particular, we consider changes in the mortality rate, in the dilution effect parameter and in the green preferences.

Changes in the Mortality Rate

Suppose a new policy (like public expenditures in health or incentives to private health care, assuming such a policy can be achieved at zero cost) has just been introduced and its effect is to lower the mortality rate, d, affecting therefore the population growth. If the economy were initially along the BGP, this shock would have the effect[11] of shifting the economy from a BGP to another one. Along the new BGP the economic growth rate and the stationary fertility rate would remain unaffected while net population growth changes.

Suppose originally the stationary fertility were lower than the mortality rate[12]: this means, according to Proposition 2, that the path along which the economy is evolving is not sustainable. Then along the new BGP population growth would increase but it could be positive, negative or null, in accordance to the magnitude of the change. If such a change were strong enough, then population growth would be positive and as a result the economy is switched from a not sustainable path to a sustainable one. Therefore the introduction of a new policy aiming at

[10]Notice that in our model only a natural good is present, and economic activities, as production and public expenditures, cannot be easily encompassed in such a framework. Therefore, public intervention can be viewed as a mere exogenous shock affecting some parameters

[11]Short-run and long-run effects coincide, as in the standard AK model: every shock in the economy translates in a jump of the (economic and population) growth rates of the stationary fertility level or both. In fact, at any shock the fertility rate can be adjusted in order to lie directly on the new BGP

[12]Such a situation is consistent with several industrialized economies (as for example Italy), in which the growth rate of (domestic) population is lower than its replacement rate. The result that the overall demographic growth is positive is just due to increasing migration flows (not present in our model since economies are closed). Other examples of how this situation can be relevant for the economic system are given by the collapse of the Maya civilization (Demarest, 2004), the decline of the Easter Island society (Brander and Taylor, 1998) and the complete extinction of the Viking's colonies of Greenland (Diamond, 2005)

increase public or private spending in health care, through the effect of lowering the mortality rate, can be used in order to reach a desired level of population, permitting the planner to switch the economy to sustainable paths.

Changes in the Intensity of the Dilution Effect

Suppose a new policy, as the introduction of public expenditure to improve environment protection (assuming that it can be achieved at zero cost), has the effect of decreasing a, that is it lowers the cost for the natural resources to maintain the same stock of population. If the economy were initially along the BGP, along the new one the economic growth, population growth and the steady state fertility level change. If $b > 1$, a drop in a leads first of all to a shift from a fertility rate to an higher new one, while the fertility growth rate continues to be null. This implies an higher population growth and a lower economic growth. If instead $b < 1$, the new stationary fertility rate will be lower and also the economic growth rate will be lower.

Suppose, again, that originally the stationary fertility were lower than the mortality rate, such that the economy do not lie on a sustainable path. The planner, through this kind of policies, can increase $\overline{n}$ enough to permit the shift of the economy to a sustainable path: this can be done by decreasing (increasing) a, if $b > 1$ ($b < 1$).

Shifts in the Green Preferences

Suppose a policy oriented to awake households to environmental problem or to change the priority of agents is introduced. Its effect would be an increase in β. If the economy were initially along the BGP, this change would have the effect of shifting the economy from a BGP to another one. The stationary fertility level[13] and therefore population growth and economic growth rates will be different. If $b > 1$ an increase in β decreases the stationary fertility rate, leading to a lower population growth and an higher economic growth. If instead $b < 1$ fertility will increase and population growth will increase too while the effect of such a shock on economic growth is ambiguous.

Suppose that originally the stationary fertility were lower than the mortality rate, so that the dynamic path followed by the economy is not sustainable. Then, the green preferences can

[13]Notice that, considering the case where $b > 1$, a really high β could lead to reach the per-capita consumption growth maximizing null fertility level. In such a case, population will decrease along the BGP (implying that population asymptotically collapses) while economic growth will be maximal. Moreover, $\beta = 0$ represents the case where environment is not a source of utility. Therefore, β can be interpreted as a crucial policy parameter: promoting attention to environment, the planner can indirectly affect economic and population growth, through the direct effect of manipulating the stationary fertility rate

be influenced by pro-environment policies in such a way to lead the economy on a sustainable path: this can be done increasing (decreasing) β if $b > 1$ ($b < 1$).

1.5 Environment not Source of Utility

If $\beta = 0$, the stock of natural resources does not affect welfare and such a situation is an extreme case of the model. In this case the optimal paths of consumption and fertility simplify in:

$$\frac{\dot{c}_t}{c_t} = \frac{1}{\sigma}\left[r - an_t^b - \rho\right] \tag{1.32}$$

$$\frac{\dot{n}_t}{n_t} = \frac{1}{b-1}\frac{c_t N_t}{E_t}\left[1 - \frac{\sigma}{b(1-\sigma)an_t^{b-1}}\right]. \tag{1.33}$$

We can notice that the growth rate of per-capita consumption is constant unless the term representing the dilution effect. In this framework, along the BGP the stationary fertility rate is:

$$\overline{n} = \left[\frac{\sigma}{b(1-\sigma)a}\right]^{\frac{1}{b-1}} \tag{1.34}$$

while the economy growth rate is positive and that of population can be positive, negative or null:

$$\gamma = \frac{1}{\sigma}\left[r - a\overline{n}^b - \rho\right] \tag{1.35}$$

$$\gamma_N = \overline{n} - d. \tag{1.36}$$

We can notice that the stationary fertility level is higher if $b > 1$ (and lower if $b < 1$) in this case: this is due to the fact that agents, since environmental stock does not affect their welfare, do not take into account the interaction between fertility choices and natural resources dynamics. In fact, an higher fertility rate decreases more the stock of natural resources, leading to a lower economic growth and an higher population growth. Notice that also in this case, Proposition 2 holds: the development path is sustainable if the stationary fertility rate is at least as high as the mortality rate. This means that the introduction of the environment (natural stock) in the instantaneous utility function is not essential for such a result to hold. However, since in the case $b > 1$, the stationary fertility rate is higher, the possibility of the economy to lie on a non-sustainable path will be less likely. In fact, fixed the other parameter values, the mortality rate has to be higher for having a negative growth rate of population.

1.6 Conclusion

The Brundtland Commission defined sustainable development as development that *"satisfies the needs of the present without compromising the ability of future generations to meet their* *own needs"*. The goal of our paper is to study the situations under which a sustainable path can be reached, focalizing our attention also on the interaction between population growth, environment and economic growth. With respect to previous works, we introduce a formal definition of sustainable path which permits us to take into account long-run generations' welfare without modifying the standard welfare criterion (the utilitarian approach). We define a path as sustainable if it implies strictly positive values of all economic variables, both in finite and infinite time (see Definition 2). We analyze a growth model driven by natural resources and without production, where agents have jointly to determine consumption and fertility, taking into account the effect of their decisions on the dynamics of natural resources. In our model, even if the renewal capacity of natural resources is unbounded, not always a sustainable path, where both population and natural resources coexist, can be found: this depends on the difference between the stationary fertility level and the mortality rate. A sustainable path can be found only if the stationary fertility is higher or equal to the mortality rate.

Along the BGP, the growth rate of the economy and of population ultimately depends on the stationary fertility level. Suppose the stationary fertility is lower than the mortality rate; then public intervention can be necessary in order to address the economy along a sustainable path. In fact, the planner can directly intervene in the economy determining both population and economic growth through policies aiming at affecting the stationary fertility rate. This can be done mainly with two different kind of policies: one affecting the dilution effect parameter and one affecting green preferences. The outcome of these policies is similar: both changes in the dilution effect and variations in the green preferences modify the population growth and affect economic growth therefore they can be adopted in order to reach the desired population level and in order to alter economic growth. In particular, such policies can be used in order to reach a sustainable path: suppose the stationary fertility is lower than the mortality rate; the planner can affect fertility or mortality with the right policy[14] in order to cancel the gap and permit a sustainable path to exist.

For further research, we propose to focalize the attention on a more realistic framework, where also production and accumulation of physical capital play an active role in the economy. In particular, a two sector growth model, a-la Uzawa-Lucas, in which natural resources have to be allocated between physical and natural production can do it. Another interesting line of research can be represented by the introduction of bounded renewal function, as a logistic one: in such a case is not obvious that a sustainable path exists at all.

[14]For example, if $b > 1$, the fertility rate can be increased augmenting public expenditure to improve environmental protection or decreasing public care of environment; if $b < 1$, the policies have to be of the opposite sign. The mortality rate, instead, can be reduced through incentives to private health care

A. On the Transitional Dynamics

We can study the stability of the BGP, introducing the intensive variable $\chi_t = \frac{c_t N_t}{E_t}$ and studying the system in (χ_t, n_t). The law of motion of χ_t and n_t are:

$$\frac{\dot{\chi}_t}{\chi_t} = (1+\beta)\chi_t + \frac{(1-\sigma)(1+\beta)}{\sigma}(r - an_t^b) - \frac{\rho}{\sigma} + n_t - d \tag{1.37}$$

$$\frac{\dot{n}_t}{n_t} = \frac{1}{b-1}\chi_t \left[(1+\beta) - \frac{\sigma}{b(1-\sigma)an_t^{b-1}}\right]. \tag{1.38}$$

Therefore, the equilibrium of this system is given by $E = (\overline{\chi}, \overline{n})$, where:

$$\overline{\chi} = \frac{\rho}{\sigma(1+\beta)} - \frac{1-\sigma}{\sigma}(r - a\overline{n}^b) - \frac{\overline{n} - d}{1+\beta} \tag{1.39}$$

$$\overline{n} = \left[\frac{\sigma}{b(1+\beta)(1-\sigma)a}\right]^{\frac{1}{b-1}}. \tag{1.40}$$

Notice that $\overline{n} > 0$ if $0 < \sigma < 1$ while $\overline{\chi}$ is if $\overline{n} > \frac{b(1-\sigma)(\rho+\sigma d) - b(1-\sigma)^2(1+\beta)r}{\sigma^2 - b\sigma(1-\sigma)}$.

Linearizing the system, we obtain the Jacobian matrix, $J(\chi_t, n_t)$:

$$\begin{bmatrix} 2(1+\beta)\chi_t + \frac{(1-\sigma)(1+\beta)}{\sigma}(r - an_t^b) - \frac{\rho}{\sigma} + n_t - d & \chi_t[1 - \frac{b(1-\sigma)(1+\beta)an_t^{b-1}}{\sigma}] \\ \frac{1}{b-1}[(1+\beta)n_t - \frac{\sigma}{b(1-\sigma)an_t^{b-2}}] & \frac{1}{b-1}\chi_t[(1+\beta) + \frac{\sigma(b-2)}{b(1-\sigma)an_t^{b-1}}] \end{bmatrix}, \tag{1.41}$$

which evaluated at steady state is:

$$J(\overline{\chi}, \overline{n}) = \begin{bmatrix} (1+\beta)\overline{\chi} & 0 \\ 0 & (1+\beta)\overline{\chi} \end{bmatrix}. \tag{1.42}$$

Since $\overline{\chi}$ is positive, it is straightforward to notice that both eigenvalues are positive, implying that the system never reaches the steady state (namely its BGP), unless the initial choice for n_t is such that the fertility rate coincides with its stationary level from time 0.

Therefore, the economy lies immediately on the BGP, otherwise will never converge to it. If $n_0 = \overline{n}$, the model behaves as a standard AK model: there is no transitional dynamics and from $t = 0$ the model is in its steady state. If instead $n_0 \neq \overline{n}$, the economy does not converge to the BGP and the path followed by the economy does not show balanced growth. This means the economy, in order not to show diverging trajectories, lies on the BGP from time 0, meaning that the model does not show transitional dynamics.

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Chapter 2

Population Growth and Utilitarian Criteria in the Lucas-Uzawa Model

This paper[1] studies the introduction of population change in an optimal model of endogenous growth, analyzing the implications of the choice of the welfare criterion on the model's outcome. Traditional growth theory assumes population growth is exponential, but this is not a realistic assumption (as discussed in Brida and Accinelli (2007)). We model exogenous population change by a generic function of population size, which is later formalized as different cases. We analyze a Uzawa-Lucas type model and show that a unique non-trivial equilibrium exists and the economy converges towards it along a saddle path, independently of population dynamics. What is affected by the type of population dynamics is the dimension of the stable manifold, which can be one or two, and the timing when the equilibrium is reached, which can happen in finite time or asymptotically. Moreover, we show that the choice of the utilitarian criterion is irrelevant on the equilibrium of the model if the steady state growth rate of population is null, as in the case of logistic and bounded population growth. We discuss how several types of demographic dynamics introduced in the previous literature can be seen as special cases of our model. We also look for a closed form solution for the model when population is subject to random shocks, driven by a Brownian motion and show that it can be found for a certain value of the altruism parameter.
Keywords: Population Change, Endogenous Growth, Utilitarian Criteria, Stochastic Shocks, Closed Form Solution
JEL Classification: O40, O41, J13

[1]I wish to thank the participants of the workshop held at Monash University (Melbourne Graduate Student Conference in Economics 2010, July 2010) for helpful comments and suggestions. This paper is part of a joint work of La Torre Davide and myself, which is still in progress. With respect to that paper, this version does not include population (unskilled labor) as a distinct factor of production; this allows us to obtain the complete dynamics of physical and human capital in the case in which population is hit by random shocks

2.1 Introduction

In standard economic growth theory, population is assumed to grow at an exogenous and exponential rate. This assumption has been firstly introduced by the Solow-Swan model (1956) and it has been applied also to following models with optimizing behavior, as the single-sector Ramsey-Cass-Koopmans (1965) model and the two-sector Lucas-Uzawa (1988) model. Such an assumption however is not without consequences for the analysis of growing economies. In fact, exponential population growth models imply unconstrained growth of population size. However, most populations are constrained by limitations on resources, at least in the short run, and none is unconstrained forever. For this reason, firstly Malthus (1798) discusses about the inevitable dire consequences of exponential growth of the human population of the earth. Recently, Brida and Accinelli clearly state: "*The simple exponential growth model can provide an adequate approximation to such growth only for the initial period because, growing exponentially, as* $t \to \infty$*, labor force will approach infinity, which is clearly unrealistic. As labor force becomes large enough, crowding, food shortage and environmental effects come into play, so that population growth is naturally bounded. This limit for the population size is usually called the carrying capacity of the environment*".

Accinelli and Brida (2005) firstly introduce non-exponential population growth in a growth model, assuming that population dynamics is described by a logistic[2] function. After this work a growing literature studying how different demographic change functions modify standard growth models arises. For example, the Solow model has been extensively analyzed assuming different demographic dynamics. Guerrini (2006) and Brida and Pereyra (2008) introduce respectively bounded population growth (which represents a generalization of the logistic case) and a decreasing population growth in the Solow-Swan model; Bucci and Guerrini (2009) instead study its transitional dynamics in the case of AK technology and logistic population. Also the Ramsey model has been recently extended to encompass several types of population change functions. Brida and Accinelli (2007) study the case of logistic population growth while Guerrini (2010a) looks for a closed form solution to the same model; Guerrini (2009, 2010b)

[2]Some decades ago, Maynard Smith (1974) concluded that the growth of natural populations is more accurately depicted by a logistic law. This result has been recently used to claim that such a dynamics can be probably better describe also human population growth. In fact, several studies support the idea that human population growth is decreasing and tending towards zero (as Day (1996)). The main features of a logistic dynamics is that the implied growth rate is bounded and it represents a generalization of exponential dynamics. Even the Belgian mathematician Verhulst in the XIX century studies this idea; using data from the first five U.S. censuses, he makes a prediction in 1840 of the U.S. population in 1940 and was off by less than 1%. Moreover, based on the same idea, he predicts the upper limit of Belgian population; more than a century later, but for the effect of immigration, his prediction looks good (Verhulst (1838))

instead assumes that population growth is given by a bounded function, analyzing both the neoclassical and the endogenous case.

However, all these papers also relax an important standard assumption of optimal growth theory, namely the social welfare function is founded on the Benthamite criterion (total utilitarianism). This criterion says that total welfare is the sum of per-capita welfare across population (the product between population size and average welfare if no heterogeneity among agents is present). These papers[3] instead assume the social welfare function is based on the Millian criterion (average utilitarianism): total welfare equals average welfare or per-capita utility (see Marsiglio (2010) for a discussion of the implications of both criteria). Such a criterion has been used in order to limit population size and in an optimal theory of growth seems to be somehow reductive. In fact, the main difference in the model's outcome is the effect of population growth on the per-capita consumption dynamics: the Benthamite criterion implies that consumption growth is independent of population dynamics, while the opposite is true for the Millian criterion.

Some papers in the literature discuss how the choice of total rather than average utilitarianism affects the outcome of the model. Such an issue has always been studied in a context of exponential population change, where the general conclusion is that the Benthamite and Millian criteria lead to different effects of population growth on economic performance. This issue is quite popular in the framework of endogenous fertility, in which the steady state outcome is represented by exponential population growth. For example, Nerlove et al. (1982, 1985) and Barro and Becker (1989) analyze a neoclassical setup while Palivos and Yip (1993) an endogenous growth context. Barro and Becker (1989) show that according to the degree of altruism towards future generations, the social welfare function results to be a mix of the Benthamite and Millian criteria. Palivos and Yip (1993) show instead that the Benthamite criterion leads to an higher economic growth and a smaller population size. Few papers tackle the issue when population change is exogenous, namely Strulik (2005) and Bucci (2008). They both study the effect of exogenous population growth on the economic growth rate in an endogenous growth model driven by R&D activity, as the degree of agents' altruism towards future generations changes. They both show that the impact of demographic change on the economy varies as the magnitude of the altruism parameter does so. All these works assume population growth is exponential (at least in steady state) and suggest that different utilitarian criteria affect the economic growth rate.

[3]An exception is represented by La Torre and Marsiglio (2010). They introduce logistic population growth in a three sectors Uzawa (1965)-Lucas (1988) type growth model, in which the welfare function is defined according to the Benthamite criterion. However, since their goal is to focus on endogenous technical progress, they do not study population dynamics (because population size in steady state is constant, under the logistic assumption)

The aim of this paper is studying the introduction of not exponential population change in endogenous growth models, and analyzing the effect of different utilitarian criteria on the model's outcome. We formalize demographic growth as a generic function of population size, discussing how different shapes affect the model. We focalize our analysis on a two-sector model of endogenous growth, á-la Uzawa (1965) - Lucas (1988), since, it has never been analyzed in a framework of non-exponential population growth and, as claimed in Boucekkine and Ruiz-Tamarit (2008), it is one of the most studied and interesting endogenous growth models. We show that a unique non-trivial equilibrium exists and the economy converges towards it along a saddle path, independently of the shape of the population growth function. In section 2 we introduce the model in its general form, namely we assume population change depends on a generic function of population size and the social welfare function results to be of the Benthamite or Millian type according to the value of a parameter (representing the degree of altruism). Section 3 performs steady state analysis, which is characterized by a balanced growth path or an asymptotic balanced growth path, according to the features of the population growth function. However, we show that independently on the shape of such a function, the economy converges towards its equilibrium along a saddle path. What is affected by its shape is the dimension of the stable manifold, which can be one or two. We also show the adopted utilitarian criterion is irrelevant for the economic growth rate if in steady state population growth is null, as in the case in which population growth is logistic or is bounded. In section 4, instead, we show different examples of population growth function which represent particular cases of our general model. In Section 5 we study the global dynamics of the model under a particular parametric restriction concerning the altruism parameter; in section 6 instead we look for a closed form solution of the model, in the case in which population is subject to random shocks and show that this can be found under the same conditions on the value of the altruism parameter. Section 6 instead concludes.

2.2 The Model

The economy is closed and composed of households (that receive wages and interest income, purchase the consumption good and choose how much to save and how much to invest in education) and firms (that produce the consumption good). Population coincides with the available number of workers, so that there is no unemployment and the labor supply is inelastic (no leisure-work choice), and it grows exogenously in accordance to a generic function of population size. The economy is composed of two sectors, where physical and human capital are used to produce only one homogeneous final good, which can be consumed or invested in physical capital. Physical capital can be used only for producing the final good, while human capital can be allocated to physical production or to the educational sector.

The infinitely-lived representative household wants to maximize its lifetime utility function, which is the infinite discounted sum of its instantaneous utility function:

$$U = \int_0^\infty u(c_t) N_t^{1-\epsilon} e^{-\rho t} dt, \tag{2.1}$$

where $c_t \equiv \frac{C_t}{N_t}$ is per-capita consumption, $\rho > 0$ is the rate of time preference and N_t, the population size, is weighted by the degree of altruism towards future generations, $1-\epsilon$, where $\epsilon \in [0,1]$. The instantaneous utility function is assumed to be iso-elastic:

$$u(c_t) = \frac{c_t^{1-\sigma}}{1-\sigma}, \tag{2.2}$$

where $\sigma > 0$ is the inverse of the intertemporal elasticity of substitution.

The final good is produced combining physical capital, K_t, and the share of human capital allocated to final production, $u_t H_t$, according to a Cobb-Douglas technology:

$$Y_t = K_t^\alpha (u_t H_t)^{1-\alpha}, \tag{2.3}$$

where $0 < \alpha < 1$ and $u_t \in (0,1)$.

Physical capital accumulates over time in accordance to the difference between output, Y_t, and consumption, C_t:

$$\dot{K}_t = Y_t - C_t. \tag{2.4}$$

Human capital accumulation coincides with the production of new human capital, which depends only on the effort devoted to education, $1-u_t$, and on the existing human capital stock (the education sector is intensive in human capital), H_t:

$$\dot{H}_t = (1-u_t) H_t, \tag{2.5}$$

We assume for simplicity that physical and human capital do not depreciate over time[4].

Population coincides with the available number of workers so that there is no unemployment and the labor supply is inelastic (no leisure-work choice), and it grows according to the following function:

$$\dot{N}_t = N_t g(N_t), \tag{2.6}$$

where $g(\cdot)$ is a generic function of population size. The shape of such a function, as we shall later show, results to be irrelevant for the equilibrium of the model. The transitional dynamics instead is differently affected by the fact that $g(\cdot)$ shows or not a zero.

[4]Introducing non zero (but constant) depreciation rates would not affect the main outcome of the model. The difference would be simply represented by additional constants in the Euler and state equations

The social planner maximizes the social welfare function, that is it has to choose c_t and u_t, in order to maximize agents lifetime utility function subject to physical and human capital accumulation constraints, the demographic dynamics and the initial conditions:

$$\begin{aligned}
\max_{c_t,u_t} \quad & \int_0^\infty \frac{c_t^{1-\sigma}}{1-\sigma} N_t^{1-\epsilon} e^{-\rho t} dt \qquad (2.7)\\
s.t. \quad & \dot{K}_t = AK_t^\alpha (u_t H_t)^{1-\alpha} - N_t c_t \\
& \dot{H}_t = B(1-u_t)H_t \\
& \dot{N}_t = N_t g(N_t) \\
& K_0,\ H_0,\ N_0 \text{ given.}
\end{aligned}$$

Notice that the degree of altruism towards future generations, given by the term $1-\epsilon$, determines the type of social welfare function we are adopting. In fact, if $\epsilon = 0$ ($\epsilon = 1$), the social welfare is defined according to the Benthamite (Millian) criterion. In the former (latter) case, we are adopting total (average) utilitarianism.

2.2.1 Optimal Paths

Notice that the dynamical equation for population change is an auxiliary equation (not a state or a control variable), since it is completely exogenous. Therefore, from the maximization problem we can set the Hamiltonian function (considering population dynamics as auxiliary):

$$\mathcal{H}_t(\cdot) = \frac{c_t^{1-\sigma}}{1-\sigma} N_t^{1-\epsilon} e^{-\rho t} + \lambda_t [AK_t^\alpha (u_t H_t)^{1-\alpha} - N_t c_t] + \mu_t B(1-u_t)H_t, \qquad (2.8)$$

and derive the FOCs:

$$\frac{\partial \mathcal{H}_t(\cdot)}{\partial c_t} = 0 \quad \rightarrow \quad c_t^{-\sigma} N_t^{1-\epsilon} e^{-\rho t} = \lambda_t N_t \qquad (2.9)$$

$$\frac{\partial \mathcal{H}_t(\cdot)}{\partial u_t} = 0 \quad \rightarrow \quad (1-\alpha)AK_t^\alpha (u_t H_t)^{-\alpha} H_t \lambda_t = BH_t \mu_t \qquad (2.10)$$

$$\frac{\partial \mathcal{H}_t(\cdot)}{\partial K_t} = -\dot{\lambda}_t \quad \rightarrow \quad \lambda_t A \alpha K_t^{\alpha-1} (u_t H_t)^{1-\alpha} = -\dot{\lambda}_t \qquad (2.11)$$

$$\frac{\partial \mathcal{H}_t(\cdot)}{\partial H_t} = -\dot{\mu}_t \quad \rightarrow \quad (1-\alpha)AK_t^\alpha (u_t H_t)^{-\alpha} u_t \lambda_t + \mu_t B(1-u_t) = -\dot{\mu}_t. \qquad (2.12)$$

together with the initial conditions K_0 and H_0, the dynamic constraints:

$$\dot{K}_t = AK_t^\alpha (u_t H_t)^{1-\alpha} - N_t c_t \qquad (2.13)$$

$$\dot{H}_t = (1-u_t)H_t \qquad (2.14)$$

and the transversality conditions:

$$\lim_{t\to\infty} \quad \lambda_t K_t = 0 \qquad (2.15)$$

$$\lim_{t\to\infty} \quad \mu_t H_t = 0. \qquad (2.16)$$

Solving the resulting system we obtain the Euler equations for per-capita consumption and share of human capital to be allocated to physical production:

$$\frac{\dot{c}_t}{c_t} = \frac{1}{\sigma}\left[A\alpha K_t^{\alpha-1}(u_t H_t)^{1-\alpha} - \rho - \epsilon g(N_t)\right] \tag{2.17}$$

$$\frac{\dot{u}_t}{u_t} = \frac{B(1-\alpha)}{\alpha} + Bu_t - \frac{N_t c_t}{K_t}. \tag{2.18}$$

Equations (2.17) and (2.18) are standard, unless for the presence of the term $-\epsilon g(N_t)$ in the Euler equation of per-capita consumption. Population growth does not affect the path of human capital share allocated in the production sector while whether it does or not per-capita consumption growth depending on the adopted utilitarian criterion. If $\epsilon = 0$, we are adopting a classical or total utilitarianism approach and population change is completely irrelevant for the dynamics of consumption, as in the standard Ramsey model. If instead $\epsilon = 1$, welfare is based on average utilitarianism and population change has a negative impact on the dynamics of consumption, as for example in Brida and Accinelli (2007); the same is true for impure altruism values, that is $\epsilon \in (0, 1)$.

2.3 Steady State Analysis

The dynamic behavior of the economy is summarized by equations (2.17), (2.18), (2.4), (2.5) and (2.6). We now analyze the steady state of such an economy. We can study the dynamics of a simplified system, by introducing the intensive variables $\chi_t = \frac{N_t c_t}{K_t}$ and $\psi_t = (u_t \frac{H_t}{K_t})^{1-\alpha}$, representing respectively the consumption-capital ratio and the average product of capital:

$$\frac{\dot{\psi}_t}{\psi_t} = \frac{B(1-\alpha)}{\alpha} - (1-\alpha)A\psi_t \tag{2.19}$$

$$\frac{\dot{\chi}_t}{\chi_t} = \frac{\alpha-\sigma}{\sigma}A\psi_t - \frac{\rho}{\sigma} + \chi_t + \frac{\sigma-\epsilon}{\sigma}g(N_t) \tag{2.20}$$

$$\frac{\dot{u}_t}{u_t} = \frac{B(1-\alpha)}{\alpha} + Bu_t - \chi_t \tag{2.21}$$

$$\frac{\dot{N}_t}{N_t} = g(N_t). \tag{2.22}$$

Depending on the type of demographic dynamics, the equilibrium of the economy derives from a three-dimensional or a four-dimensional system. In fact, if the $g(\cdot)$ function does not show any zero[5], we have a three-dimensional system, since a stationary population size does

[5] A benchmark for such a case is represented by constant and exponential population growth. In the following discussion and analysis we focalize on constant population growth, for a matter of tractability and since it is probably the most relevant case of growth function not showing any zeros

not exist and equation (2.22) does not imply any equilibrium value; therefore, it can simply be deleted from the system of differential equations, which reduces to:

$$\frac{\dot{\psi}_t}{\psi_t} = \frac{B(1-\alpha)}{\alpha} - (1-\alpha)A\psi_t \tag{2.23}$$

$$\frac{\dot{\chi}_t}{\chi_t} = \frac{\alpha-\sigma}{\sigma}A\psi_t - \frac{\rho}{\sigma} + \chi_t + \frac{\sigma-\epsilon}{\sigma}g_N \tag{2.24}$$

$$\frac{\dot{u}_t}{u_t} = \frac{B(1-\alpha)}{\alpha} + Bu_t - \chi_t. \tag{2.25}$$

The equilibrium point of such a system is characterized by strictly positive values for all the variables if $\rho > B(1-\sigma) + (\sigma - \epsilon)g_N$, where g_N represents the steady state (therefore constant and non-negative) growth rate of population. Such a system converges to its steady state equilibrium through a saddle path, along which the stable arm has dimension one. Moreover, we can study the implications of the utilitarian criteria on economic growth, simply analyzing the steady state values of the variables: the steady state of the consumption-capital ratio and the share of human capital allocated to physical production are affected by ϵ, while the average product of capital is not. Therefore, we can conclude that total utilitarianism leads to higher economic growth than average utilitarianism. We can summarize the main results in the following proposition:

Proposition 1 *Assume $\rho > B(1-\sigma)+(\sigma-\epsilon)g_N$; if the population growth function is constant, then the following results hold:*
(i) the economy converges to its steady state equilibrium, along a saddle path, and the stable arm is a one-dimensional locus;
(ii) total utilitarianism ($\epsilon = 0$) implies an higher economic growth rate than average utilitarianism ($\epsilon = 1$) if $g_N > 0$.

Proof: Appendix A proves part (i). To prove part (ii) notice that the growth rate of per-capita consumption, from equation (2.17), in steady state is $\gamma = \frac{1}{\sigma}\left[A\alpha\psi^* - \rho - \epsilon g_N\right]$; it is straightforward to see that its derivative respect to ϵ is negative: $\frac{\partial\gamma}{\partial\epsilon} = -\frac{g_N}{\sigma}$. Therefore, the Benthamite criterion implies an higher economic growth than the Millian one. ■

If instead the $g(\cdot)$ function shows any zeros, we have a four dimensional system since a stationary population size exists and therefore equation (2.22) implies an equilibrium value. The system of differential equations is the following:

$$\frac{\dot{\psi}_t}{\psi_t} = \frac{B(1-\alpha)}{\alpha} - (1-\alpha)A\psi_t \tag{2.26}$$

$$\frac{\dot{\chi}_t}{\chi_t} = \frac{\alpha-\sigma}{\sigma}A\psi_t - \frac{\rho}{\sigma} + \chi_t + \frac{\sigma-\epsilon}{\sigma}g(N_t) \tag{2.27}$$

$$\frac{\dot{u}_t}{u_t} = \frac{B(1-\alpha)}{\alpha} + Bu_t - \chi_t \quad (2.28)$$

$$\frac{\dot{N}_t}{N_t} = g(N_t). \quad (2.29)$$

The equilibrium point of such a system is characterized by strictly positive values for all the variables if $\rho > B(1-\sigma)$. By equation (2.29), the existence of a stationary population size is ensured if $g(N_t) = 0$. Moreover, the equilibrium is saddle point stable in a generalized form, since the stable manifold has dimension one (two) and the unstable one has dimension three (two) if $\frac{\partial g(\cdot)}{\partial N_t}|_{N_t=N^*} > 0 (< 0)$. This means that equilibrium indeterminacy can arise in the Lucs-Uzawa model simply because of population dynamics: in such a case, a continuum of paths converging to equilibrium exists. As before, we can study the implications of average and total utilitarianism on the outcome of the model, simply analyzing the steady state values of the variables: the steady state values of all the economic variables are independent of ϵ. Therefore, the adopted utilitarian criterion affects only the transitional dynamics of the economy, but in steady state all the differences vanish. We can summarize this result in the following proposition:

Proposition 2 *Assume $\rho > B(1-\sigma)$; if the population growth function shows any zeros, then: (i) the economy converges to its steady state equilibrium, along a saddle path. The stable arm is a two-dimensional locus if $\frac{\partial g(\cdot)}{\partial N_t}|_{N_t=N^*} < 0$, while it has dimension one if $\frac{\partial g(\cdot)}{\partial N_t}|_{N_t=N^*} > 0$*
(ii) Whether the social welfare function is built on the Benthamite ($\epsilon = 0$) or the Millian ($\epsilon = 1$) criterion, the steady state growth rate of the economy does not change.

Proof: Appendix B proves part (i). To prove part (ii) notice that in such a framework, since population growth is null in equilibrium, the growth rate of per-capita consumption, from equation (2.17), in steady state is $\gamma = \frac{1}{\sigma}[A\alpha\psi^* - \rho]$. It is straightforward to see that its derivative respect to ϵ is null and therefore the Benthamite and the Millian criteria do not imply any difference for the economic growth rate. ■

We can notice that the equilibrium of such a model is only marginally affected by the shape of the population growth function. In fact, the economic variables (χ, ψ, u) converge to their equilibrium independently of the behavior of the demographic variable (N). The features of population dynamics affect mainly the timing where convergence is reached, which can happen in finite time or asymptotically, characterizing the equilibrium respectively as a balanced growth path (BGP), as in the growth models with constant exponential population growth, or as an asymptotically balanced growth path (ABGP), as in the case of logistic population growth. If the population growth function shows any zeros, then its shape determines the dimension of the stable arm. In fact, if $\frac{\partial g(\cdot)}{\partial N_t}|_{N_t=N^*} < 0$ the stable arm has dimension two (implying equilibrium indeterminacy) while if $\frac{\partial g(\cdot)}{\partial N_t}|_{N_t=N^*} > 0$ it has dimension one (implying uniqueness of the convergence path). We have just proved:

Proposition 3 *The economy described by (2.7) converges towards its unique (non-trivial) equilibrium independently of population dynamics. Demographic dynamics just determines the timing of convergence and the dimension of the stable arm.*

This results derive from the assumption that population growth is exogenous, and therefore it just represents an auxiliary variable in the optimal control problem (2.7). Under such an assumption, how we model this dynamics does not affect the main outcome of the model (clearly the equilibrium values of χ and u can change as we introduce a different law of motion for demography). What can change according to the features of the $g(\cdot)$ function is the timing when the equilibrium is reached (finite or infinite) and the dimension of the stable arm (one or two).

2.4 Some Examples of Demographic Change

In this section we discuss several examples of population dynamics introduced in the previous literature, showing how they are just particular cases of our general formulation. We consider the cases in which population is exponential, logistic and follows a von Bertalanffy function.

2.4.1 Exponential Population

The standard assumption of growth theory on demography is that population growth is exponential and constant over time (see Solow (1956)). This in our general formulation represents the case in which $g(\cdot)$ is simply a constant:

$$g(N_t) = n, \tag{2.30}$$

where n can be positive, negative or null. If it is negative, population size constantly decreases and asymptotically will completely disappear; if it is null, population size is constant and it does not show any dynamics over time; if it is positive instead population constantly increases and it will asymptotically approach infinity. This last case gives birth to the critique to exponential demographic change, since it implies the absence of any natural and environmental limits to population growth. This specification implies that demographic dynamics is monotonic and Proposition 1 holds: in such a case the stable arm is a one-dimensional locus and the Benthamite criterion leads to an higher economic growth rate than the Millian one.

2.4.2 Logistic Population

A first attempt to avoid the implications of exponential population growth has been the introduction of logistic demography (see Brida and Accinelli (2007)). This represents the case in which $g(\cdot)$ is:

$$g(N_t) = n - dN_t, \tag{2.31}$$

where n and d are both positive (if $d = 0$, we are driven back to exponential growth), and n represents the trend of population growth. Notice that population dynamics is given by a Bernoulli-type differential equation which can be explicitly solved obtaining:

$$N_t = \frac{n}{d + (\frac{n}{N_0} - d)e^{-nt}}. \tag{2.32}$$

Population size therefore is increasing over time and it reaches a stationary level only when $t \to \infty$; in fact, $\lim_{t\to\infty} N_t = \frac{n}{d}$. This formulation implies that population growth is null in steady state and therefore Proposition 2 holds: the stable arm is two-dimensional locus since $g'(\cdot) < 0$ and the economic growth rate is independent of the adopted utilitarian criterion. If population growth is logistic, the equilibrium is only asymptotically approached since population size converges to its steady state value in the very long-run.

2.4.3 von Bertalanffy Population

The von Bertalanffy population growth has been introduced by Accinelli and Brida (2007) to describe a population strictly increasing and bounded whose growth rate is strictly decreasing to zero. This function corresponds to:

$$g(N_t) = \frac{n(N_\infty - N_t)}{N_t}, \tag{2.33}$$

where N_∞ is the theoretical maximum population size and n determines the speed at which demography reaches its maximal level. The equation of population dynamics can be explicitly solved obtaining:

$$N_t = N_\infty - (N_\infty - N_0)e^{-nt}. \tag{2.34}$$

Population size therefore is increasing over time and it reaches a stationary level only when $t \to \infty$; in fact, $\lim_{t\to\infty} N_t = N_\infty$. Also in this case Proposition 2 holds: the stable arm has dimension two since $g'(\cdot) < 0$, and the utilitarian criteria do not imply any difference for the economic growth rate since the growth rate of population is null in steady state.

2.5 Transitional Dynamics in the Case $\sigma = \epsilon = \alpha$

We now study the transitional dynamics of the economy. Since population growth is exogenous its entire dynamics is the following:

$$\overline{N}_t = \int_0^t g(N_s)N_s ds. \tag{2.35}$$

Notice that all the cases discussed in the previous section show a closed form solution for their dynamic path, given by $N_t = N_0 e^{nt}$ for the constant growth case, equations (2.32) for the logistic case and (2.34) for the von-Bertalanffy one. The dynamics of χ, ψ and u are instead given by:

$$\overline{\psi}_t = \frac{e^{\frac{B(1-\alpha)}{\alpha}t}}{\psi_0^{-1} + \frac{\alpha A}{B} e^{\frac{B(1-\alpha)}{\alpha}t}} \tag{2.36}$$

$$\overline{\chi}_t = \frac{e^{\int_0^t [\frac{\alpha-\sigma}{\sigma} A\overline{\psi}_s - \frac{\rho}{\sigma} + \frac{\sigma-\epsilon}{\sigma} g(\overline{N}_s)]ds}}{\chi_0^{-1} - \int_0^t e^{\int_0^s [\frac{\alpha-\sigma}{\sigma} A\overline{\psi}_v - \frac{\rho}{\sigma} + \frac{\sigma-\epsilon}{\sigma} g(\overline{N}_v)]dv} ds} \tag{2.37}$$

$$\overline{u}_t = \frac{e^{\int_0^t [\frac{B(1-\alpha)}{\alpha} - \overline{\chi}_s]ds}}{u_0^{-1} - B\int_0^t e^{\int_0^s [\frac{B(1-\alpha)}{\alpha} - \overline{\chi}_v]dv} ds}. \tag{2.38}$$

From these we can obtain the entire dynamics of consumption, physical and human capital:

$$\overline{c}_t = c_0 e^{\int_0^t [\frac{\alpha}{\sigma} A\overline{\psi}_s - \frac{\rho}{\sigma} - \frac{\epsilon}{\sigma} g(\overline{N}_s)]ds} \tag{2.39}$$

$$\overline{K}_t = K_0 e^{\int_0^t (A\overline{\psi}_s - \bar{\chi}_s)s} \tag{2.40}$$

$$\overline{H}_t = H_0 e^{\int_0^t (1-\overline{u}_s)ds}, \tag{2.41}$$

where $\overline{\chi}, \overline{\psi}, \overline{u}$ and $\overline{N}$ are respectively given by equations (2.35), (2.36), (2.37) and (2.38).

Notice that in the case in which $\sigma = \epsilon = \alpha$, we can find a full closed-form solution for the transitional dynamics of c, u, K, H. In such a case, solving the integrals in equations (2.37) is straightforward, and the global dynamics of the control and state variables is the following:

$$\overline{u}_t = \frac{e^{\frac{B(1-\alpha)}{\alpha}t}\left(1 - \frac{\alpha\chi_0}{\rho}\right) + \frac{\alpha\chi_0}{\rho} e^{\frac{B(1-\alpha)-\rho}{\alpha}t}}{u_0^{-1} - \alpha\left[\left(1 - \frac{\alpha\chi_0}{\rho}\right)\left(e^{\frac{B(1-\alpha)}{\alpha}t} - 1\right)\frac{1}{1-\alpha} + \frac{\chi_0}{\rho}\left(e^{\frac{B(1-\alpha)-\rho}{\alpha}t} - 1\right)\frac{\alpha B}{B(1-\alpha)-\rho}\right]}$$

$$\overline{c}_t = c_0 \left[\frac{\psi_0^{-1} + \frac{\alpha A}{B}(e^{\frac{B(1-\alpha)}{\alpha}t} - 1)}{\psi_0^{-1} + \frac{\alpha A}{B}}\right]^{\frac{\alpha^2 A^2}{B^2(1-\alpha)}} e^{-\frac{\rho}{\alpha}t - \int_0^t g(N_s)ds}$$

$$\overline{K}_t = K_0 \left[\frac{\psi_0^{-1} + \frac{\alpha A}{B}(e^{\frac{B(1-\alpha)}{\alpha}t} - 1)}{\psi_0^{-1} + \frac{\alpha A}{B}}\right]^{\frac{\alpha^2 A^2}{B^2(1-\alpha)}} \left[1 + \frac{\alpha\chi_0}{\rho}(e^{-\frac{\rho}{\alpha}t} - 1)\right]$$

$$\overline{H}_t = H_0 \left[1 - \alpha u_0 \left[\left(1 - \frac{\alpha\chi_0}{\rho}\right)\left(e^{\frac{B(1-\alpha)}{\alpha}t} - 1\right)\frac{1}{1-\alpha} + \frac{\alpha\chi_0}{\rho}\left(e^{\frac{B(1-\alpha)-\rho}{\alpha}t} - 1\right)\frac{B}{B(1-\alpha)-\rho}\right]\right]^{\frac{1}{B}} e^t.$$

If the inverse of the intertemporal elasticity of substitution is equal both to the capital share and the altruism parameter, we can evaluate the dynamics of the control and state variables for all t, by solving the integrals in equations (2.38) - (2.41). Notice that these dynamics can be fully characterized in the case in which the social welfare function is built upon impure altruism.

2.6 A Closed Form Solution for Stochastic Population Shocks

Demographic shocks consist of changes in population growth rates or immigration policies, and their main effect is altering the size of labor force (population), without affecting per-capita stock of physical or human capital. They are thought to have important effects on macroeconomic variables such as growth rates and investment decisions. Robertson (2002) studies the transitional effects of demographic shocks in an Uzawa-Lucas model; however, he analyzes such shocks simply through a comparative statics exercise. In this section we instead consider the case in which population growth is stochastic, namely we assume population is subject to random shocks and suppose that it follows an exogenous stochastic differential equation driven by a Brownian motion.

We therefore replace equations (2.6) in our model by a geometric Brownian motion:

$$dN_t = \mu N_t dt + N_t \theta dW_t, \tag{2.42}$$

where μ is the drift and $\theta \geq 0$ the variance parameter, while dW_t is the increment of a Wiener process such that $E[dW_t] = 0$ and $\mathrm{var}(dW_t) = dt$. Since the presence of this random term, the objective function (2.1) has to rewritten as an expected term:

$$U = E\left[\int_0^\infty \frac{C_t^{1-\sigma}}{1-\sigma} N_t^{1-\epsilon} e^{-\rho t} dt\right]. \tag{2.43}$$

Notice first of all that the maximization problem is totally equivalent to the following[6]:

$$\max_{C_t, u_t} \quad U = E\left[\int_0^\infty \frac{C_t^{1-\sigma}}{1-\sigma} N_t^{\sigma-\epsilon} e^{-\rho t} dt\right] \tag{2.44}$$

$$s.t. \quad \dot{K}_t = K_t^\alpha (u_t H_t)^{1-\alpha} - C_t \tag{2.45}$$

$$\dot{H}_t = B(1-u_t)H_t \tag{2.46}$$

$$dN_t = \mu N_t dt + N_t \theta dW_t \tag{2.47}$$

$$K_0,\ H_0,\ N_0 \text{ given}, \tag{2.48}$$

[6]Notice that since there is no technical progress in the model, normalizing to 1 the total factor productivity is irrelevant

where $C_t = c_t N_t$ represents aggregate capital.

Define $J(H_t, K_t, N_t)$ as the maximum expected value associated with the stochastic optimization problem described above. The Hamilton-Jacobi-Bellman (HJB) equation is:

$$\rho J = \max_{C_t, u_t} \left\{ \frac{C_t^{1-\sigma}}{1-\sigma} N_t^{\sigma-\epsilon} + J_K \dot{K}_t + J_H \dot{H}_t + J_N \mu N_t + \frac{J_{NN}\theta^2 N_t^2}{2} \right\}, \tag{2.49}$$

where the differential equations for K_t and H_t are defined in (2.45) and (2.46) and subscripts denote partial derivatives of J with respect to the relevant variables of interest. Notice that if $\sigma = \epsilon$, the first term on the RHS of equation (2.49) becomes $\frac{C_t^{1-\sigma}}{1-\sigma}$ since the population term vanishes (from now onwards we continue as such an assumption holds). Dropping the ts for clarity, differentiating (2.49) with respect to the control variables gives:

$$c = {J_K}^{-\frac{1}{\sigma}}, \tag{2.50}$$

$$u = \frac{K}{H}\left[\frac{(1-\alpha)J_K}{BJ_H}\right]^{\frac{1}{\alpha}}, \tag{2.51}$$

which substituted back into (2.49) yield:

$$\begin{aligned} 0 \; = \; & \left(\frac{\sigma}{1-\sigma}\right) {J_K}^{1-\frac{1}{\sigma}} - \rho J + J_K \left[K \left[\frac{(1-\alpha)J_K}{BJ_H}\right]^{\frac{1}{\alpha}} \left[\frac{BJ_H}{(1-\alpha)J_K}\right]\right] + \\ & + J_H \left[BH - BK \left[\frac{(1-\alpha)J_K}{BJ_H}\right]^{\frac{1}{\alpha}}\right] + J_N \mu N + \frac{J_{NN}\theta^2 N^2}{2}. \end{aligned} \tag{2.52}$$

We postulate a value function separable in the state variables of the problem:

$$J(H, K, N) = T_H H^{\lambda_1} + T_K K^{\lambda_2} + T_N N^{\lambda_3}, \tag{2.53}$$

where T_H, T_K and T_N are constant parameters. We have the following result which shows that a closed form solution to the problem exists under a particular combination of parameter values.

Theorem 1 *Assume that $\sigma = \epsilon = \alpha$ and $B = \rho$. Then (2.49) has a solution given by:*

$$J(H, K, N) = T_H H + T_K K^{1-\alpha} + T_N, \tag{2.54}$$

where:

$$T_K = \left[\frac{\left(\frac{\alpha}{1-\alpha}\right)(1-\alpha)^{\frac{-(1-\alpha)}{\alpha}}}{\rho}\right]^{\alpha} \quad \text{and} \quad \frac{T_H^{\frac{\alpha-1}{\alpha}}}{T_N} = \frac{-\rho}{\frac{-B\alpha}{1-\alpha}\left[\frac{(1-\alpha)^2 T_K}{B}\right]^{\frac{1}{\alpha}}}$$

$$\frac{T_K}{T_H} = \frac{1}{(1-\alpha)}\left[\frac{H_0\rho}{B}\right]^{\alpha}, \tag{2.55}$$

Proof: See Appendix C. ■

Given this expression for J, we obtain the first derivatives with respect to K_t and H_t and substitute into (2.50) and (2.51), thereby deriving the optimal policy rules for c and u, which substituted into the constraints, give us the solutions for K_t and H_t.

Proposition 4 *Under the assumptions of theorem 1, the optimal rules for the levels of consumption and rate of investment in physical capital are given by:*

$$C_t = \frac{K_t}{[(1-\alpha)T_K]^{\frac{1}{\alpha}}}, \quad u_t = \frac{1}{H_t}\left[\frac{(1-\alpha)^2 T_K}{BT_H}\right]^{\frac{1}{\alpha}}. \tag{2.56}$$

Moreover, the optimal path of physical capital is:

$$\dot{K}_t = \Omega_1 K_t^{\alpha} - \Omega_2 K_t,$$

that is:

$$K_t = e^{-\Omega_2 t}\left[\frac{\Omega_1}{\Omega_2}\left[e^{\Omega_2(1-\alpha)t} - 1\right] + K_0^{1-\alpha}\right]^{\frac{1}{1-\alpha}}, \tag{2.57}$$

where:

$$\Omega_1 = \left[\frac{(1-\alpha)^2 T_K}{B}\right]^{\frac{1-\alpha}{\alpha}} \quad \text{and} \quad \Omega_2 = \frac{1}{[(1-\alpha)T_K]^{\frac{1}{\alpha}}},$$

while the optimal path of human capital is:

$$\dot{H}_t = BH_t - B\Omega_3,$$

that is:

$$H_t = e^{Bt}\left[H_0 + \Omega_3\left(e^{-Bt} - 1\right)\right], \tag{2.58}$$

where:

$$\Omega_3 = \left[\frac{(1-\alpha)^2 T_K}{BT_H}\right]^{\frac{1}{\alpha}}.$$

(2.56) provides a result similar to that of the AK-model since for all t, there exists a linear relationship between the optimal level of consumption and capital. Proposition 4 shows that the optimal levels of the state variables K_t and H_t are independent of N_t. The effects of population shocks is instead present in per-capita variables, $k_t = \frac{K_t}{N_t}$ and $h_t = \frac{H_t}{N_t}$. In fact, using Ito's lemma, the law of motion of per-capita physical and human capital are respectively given by:

$$\begin{aligned}\frac{dk_t}{k_t} &= \frac{dK_t}{K_t} - \frac{dN_t}{N_t} + \left(\frac{dN_t}{N_t}\right)^2 \\ &= \left[K_t^{\alpha-1}(u_tH_t)^{1-\alpha} - \frac{C_t}{K_t} - \mu + \theta^2\right] dt - \theta dW_t \qquad (2.59)\\ \frac{dh_t}{h_t} &= \frac{dH_t}{H_t} - \frac{dN_t}{N_t} + \left(\frac{dN_t}{N_t}\right)^2 \\ &= \left[B(1-u_t) - \mu + \theta^2\right] dt - \theta dW_t. \qquad (2.60)\end{aligned}$$

In order to understand the role of population shocks, we need to take expectations of per-capital physical and human capital. In order to do so, we can first find the distribution of $X_t = \frac{1}{N_t}$:

$$\begin{aligned}\frac{dX_t}{X_t} &= \left(\frac{dN_t}{N_t}\right)^2 - \frac{dN_t}{N_t} \\ &= \left[-\mu + \theta^2\right] dt - \theta dW_t. \qquad (2.61)\end{aligned}$$

It is easy to prove that:

$$X_t = X_0 e^{(\theta^2-\mu-\frac{\theta^2}{2})t+\theta W_t}, \qquad (2.62)$$

whose expectation is:

$$\begin{aligned}E[X_t] &= X_0 e^{(\theta^2-\mu-\frac{\theta^2}{2})t} E[e^{\theta W_t}] \\ &= X_0 e^{(\theta^2-\mu)t}. \qquad (2.63)\end{aligned}$$

Using such a result and since $E[k_t] = E[\frac{K_t}{N_t}] = E[K_tX_t]$ it is straightforward finding the expected value of k_t:

$$\begin{aligned}E[k_t] &= E[K_tX_t] \\ &= K_tE[X_t] \\ &= X_0 e^{-\Omega_2 t}\left[\frac{\Omega_1}{\Omega_2}\left[e^{\Omega_2(1-\alpha)t} - 1\right] + K_0^{1-\alpha}\right]^{\frac{1}{1-\alpha}} e^{(\theta^2-\mu)t}. \qquad (2.64)\end{aligned}$$

The same reasoning applies fot h_t:

$$\begin{aligned}E[h_t] &= E[H_tX_t] \\ &= H_tE[X_t] \\ &= X_0 e^{Bt}\left[H_0 + B\Omega_3\left(e^{-Bt} - 1\right)\right] e^{(\theta^2-\mu)t}. \qquad (2.65)\end{aligned}$$

Setting $\theta = 0$ in (2.64) and (2.65) yields the levels of the state per-capita variable in the deterministic version of the model. We can therefore contrast the results of the deterministic version of the model, which is indicated with a star $*$, with those of the stochastic version.

Proposition 5 *Under the assumptions of theorem 1, we have for all $t = 0, \ldots, \infty$:*

$$\begin{aligned} E[k_t] &\geq k_t^* \\ E[h_t] &\geq h_t^* \end{aligned}$$

According to this result, uncertainty of N increases on average the levels of physical and human capital. Since the evolution of population is highly uncertain, due to the sudden variations in the migration flows and in fertility and mortality rates, analyzing how demographic shocks affect the economy is important in order to understand their effects on the optimal policy rules. In this section we introduced random shocks driven by a geometric Brownian motion to the level of population in the Uzawa-Lucas model. We showed that if the altruism is impure and in particular it equals both the inverse of the intertemporal elasticity of substitution and the physical capital share, a closed-form solution can be found. Moreover, shocks on population increase both the per-capita human and physical capital stock.

2.7 Conclusion

A standard assumption in growth theory is that population change is constant and exponential. Recently, the idea that such a specification is unrealistic has been arisen. This is due to an implication of such a hypothesis: population size goes to infinity as time goes to infinity. This is clearly unrealistic, since it would deny the presence of an environmental and economic carrying capacity (Brida and Accinelli (2007)). As a result, several papers study the introduction of different population growth functions in canonical growth models.

In this paper we introduce a generic population change function in a two-sector endogenous model of growth, á-la Lucas-Uzawa and we show that the outcome of the model does not dependent on the choice of such a function. In fact, a unique non-trivial equilibrium exists and the economy converges towards it along a saddle path, independently of the shape of the population change function. What can be affected by its shape is the dimension of the stable transitional path, which can be one (if $\frac{\partial g(\cdot)}{\partial N_t}|_{N_t=N^*} > 0$) or two (if $\frac{\partial g(\cdot)}{\partial N_t}|_{N_t=N^*} < 0$), and the timing of convergence, which can happen in finite (the steady state is characterized by a BGP) or infinite (by an ABGP) time. Moreover, with respect to other works dealing with non exponential population change we do not relax one of the standard assumptions in economic growth theory, that is the social welfare function is founded on the Benthamite criterion. In fact, we consider a generic social welfare function which results to be based on the Benthamite or the Millian criterion according to the value of the altruism parameter. We show that if population growth is null in steady state (as in the case of logistic population growth), choosing one or the other criterion is irrelevant for the outcome of the model. Instead, if population growth is not null in equilibrium, the Benthamite criterion leads to higher economic growth than the Millian criterion.

Then, we formalize the population growth function by different functions which represent alternative demographic dynamics studied in the previous literature and show how our model is able to encompass all of them as particular cases. We look for a closed form solution of the model, showing that this can be fully characterized under a certain condition on the altruism parameter, namely when it coincides with the inverse of the intertemporal elasticity of substitution and the capital share. We also look for a closed form solution of the model when population is subject to random shocks and follows a geometric Brownian motion. We show that a closed form solution can be found under the same condition on the altruism parameter. In such a case, shocks on population increase the human and physical capital stock.

For further research, we suggest to study the dynamics of a two-sector economic growth model when population change is endogenous. Really few papers introduce endogenous population change in optimal models of growth and moreover they just analyze the case of a single sector economy (a really recent example is Marsiglio (2010)). It can be interesting to see whether endogenizing fertility can play a crucial role in determining the transitional dynamics of multi-sector growth models and whether the degree of intertemporal altruism affects the economic equilibrium.

A. Exponential Population Growth

The steady state of the quasi-linear three dimensional system can be found by setting equation (2.23), (2.24) and (2.25) equal to zero:

$$0 = \frac{B(1-\alpha)}{\alpha} - (1-\alpha)A\psi_t \tag{2.66}$$

$$0 = \frac{\alpha-\sigma}{\sigma}A\psi_t - \frac{\rho}{\sigma} + \chi_t + \frac{\sigma-\epsilon}{\sigma}g_N \tag{2.67}$$

$$0 = \frac{B(1-\alpha)}{\alpha} + Bu_t - \chi_t, \tag{2.68}$$

where g_N is the constant growth rate of population. From equation (2.66) we have:

$$\psi^* = \frac{B}{\alpha A}, \tag{2.69}$$

which substituted into equation (2.67) yields:

$$\chi^* = \frac{\rho\alpha - B(\alpha-\sigma) - \alpha(\sigma-\epsilon)g_N}{\alpha\sigma}. \tag{2.70}$$

Plugging this into equation (2.68) we have:

$$u^* = \frac{\rho - B(1-\sigma) - (\sigma-\epsilon)g_N}{B\sigma}. \tag{2.71}$$

The steady state value of all variables is strictly positive if $\rho > B(1-\sigma) + (\sigma - \epsilon)g_N$, because we need:

$$\begin{cases} \rho - B(1-\sigma) - (\sigma-\epsilon)g_N > 0 \\ \rho\alpha - B(\alpha-\sigma) - \alpha(\sigma-\epsilon)g_N > 0 \end{cases}$$

and since $\frac{\sigma}{\alpha} > \sigma$, $\rho > B(1-\sigma) + (\sigma-\epsilon)g_N$ is enough to ensure both inequalities are satisfied. Notice that if g_N is not too big, such a condition generally holds for reasonable values of σ; in fact, many studies (see Mehra and Prescott (1985); and more recently Obstfeld (1994). For example, Mullingan and Sala-i-Martin (1993) in their baseline specification set it equal to 2) find that the inverse of the intertemporal elasticity of substitution is higher than one and in this case the previous condition is automatically satisfied.

We can study the stability of the system by linearizing around the steady state. The Jacobian matrix, $J(\chi_t, \psi_t, u_t)$, is:

$$\begin{bmatrix} \frac{\alpha-\sigma}{\sigma}A\psi_t - \frac{\rho}{\sigma} + 2\chi_t + \frac{\sigma-\epsilon}{\sigma}g(N_t) & \frac{\alpha-\sigma}{\sigma}A\chi_t & 0 \\ 0 & \frac{B(1-\alpha)}{\alpha} - 2(1-\alpha)A\psi_t & 0 \\ -u_* & 0 & \frac{B(1-\alpha)}{\alpha} + 2Bu_t - \chi_t \end{bmatrix}, \tag{2.72}$$

which evaluated at steady state, $J(\chi^*, \psi^*, u^*)$, becomes:

$$\begin{bmatrix} \chi^* & \frac{\alpha-\sigma}{\sigma}A\chi^* & 0 \\ 0 & -(1-\alpha)\psi^* & 0 \\ -u^* & 0 & Bu^* \end{bmatrix}. \tag{2.73}$$

The eigenvalues results to be the elements on the main diagonal: therefore, we have two positive and one negative eigenvalues: the equilibrium is saddle-point stable. The system therefore converges to its steady state equilibrium through a saddle path, along which the stable arm is a one-dimensional locus while the unstable manifold has dimension two.

B. Non-Exponential Population Growth

The steady state of the quasi-linear four dimensional system can be found by setting equation (2.26), (2.27), (2.28) and (2.29) equal to zero:

$$0 = \frac{B(1-\alpha)}{\alpha} - (1-\alpha)A\psi_t \tag{2.74}$$

$$0 = \frac{\alpha-\sigma}{\sigma}A\psi_t - \frac{\rho}{\sigma} + \chi_t + \frac{\sigma-\epsilon}{\sigma}g(N_t) \tag{2.75}$$

$$0 = \frac{B(1-\alpha)}{\alpha} + Bu_t - \chi_t \tag{2.76}$$

$$0 = g(N_t) \tag{2.77}$$

From equation (2.77), we have:

$$g(N_t) = 0 \quad \rightarrow \quad N^*; \tag{2.78}$$

this means that the steady state growth rate of population is null, that is $g_N = 0$. Substituting this into the results of the previous section we obtain:

$$\psi^* = \frac{B}{\alpha A} \tag{2.79}$$

$$\chi^* = \frac{\rho\alpha - B(\alpha - \sigma)}{\alpha\sigma} \tag{2.80}$$

$$u^* = \frac{\rho - B(1 - \sigma)}{B\sigma}. \tag{2.81}$$

The steady state value of all variables is strictly positive if $\rho > B(1 - \sigma)$.

We can study the stability of the system by linearization. The Jacobian matrix, $J(\chi_t, \psi_t, u_t, N_t)$, is:

$$\begin{bmatrix} \frac{\alpha-\sigma}{\sigma}A\psi_t - \frac{\rho}{\sigma} + 2\chi_t + \frac{\sigma-\epsilon}{\sigma}g(N_t) & \frac{\alpha-\sigma}{\sigma}A\chi_t & 0 & \frac{\sigma-\epsilon}{\sigma}\frac{\partial g(\cdot)}{\partial N_t}\chi_t \\ 0 & \frac{B(1-\alpha)}{\alpha} - 2(1-\alpha)A\psi_t & 0 & 0 \\ -u_* & 0 & \frac{B(1-\alpha)}{\alpha} + 2Bu_t - \chi_t & 0 \\ 0 & 0 & 0 & \frac{\partial g(\cdot)}{\partial N_t}N_t + g(N_t) \end{bmatrix},$$

which evaluated at steady state, $J(\chi^*, \psi^*, u^*, N^*)$, becomes:

$$\begin{bmatrix} \chi^* & \frac{\alpha-\sigma}{\sigma}A\chi^* & 0 & \frac{\sigma-\epsilon}{\sigma}\frac{\partial g(\cdot)}{\partial N_t}|_{N_t=N^*}\chi^* \\ 0 & -(1-\alpha)\psi^* & 0 & 0 \\ -u^* & 0 & Bu^* & 0 \\ 0 & 0 & 0 & \frac{\partial g(\cdot)}{\partial N_t}|_{N_t=N^*}N^* \end{bmatrix}, \tag{2.82}$$

Also in this case, the eigenvalues results to be the elements on the main diagonal: therefore, we have two positive and one negative eigenvalues, independent of the $g(\cdot)$ function, while the last one crucially depends on it. However, the equilibrium is saddle-point stable and the system therefore converges to its steady state equilibrium through a saddle path. The shape of the $g(\cdot)$ function affects only the dimension of the stable and unstable transitional paths. In fact, if $\frac{\partial g(\cdot)}{\partial N_t}|_{N_t=N^*} > 0$ the stable arm has dimension one, while if $\frac{\partial g(\cdot)}{\partial N_t}|_{N_t=N^*} < 0$ it has dimension two.

C. Proof of theorem 1

From equation (2.54), we have:

$$J_H = \lambda_1 T_H H^{\lambda_1 - 1}, \; J_K = \lambda_2 T_K K^{\lambda_2 - 1}, \tag{2.83}$$

$$J_N = \lambda_3 T_N N^{\lambda_3 - 1}, J_{NN} = \lambda_3(\lambda_3 - 1) T_N N^{\lambda_3 - 2}, \tag{2.84}$$

which substituted into equation (2.49), yield:

$$0 = \left(\frac{\sigma}{1-\sigma}\right)\left[\lambda_2 T_K K^{\lambda_2-1}\right]^{1-\frac{1}{\sigma}} - \rho\left(T_H H^{\lambda_1} + T_K K^{\lambda_2} + T_N N^{\lambda_3}\right) + \\ +\left(\frac{B}{1-\alpha} - B\right) K \left[\frac{(1-\alpha)\lambda_2 T_K K^{\lambda_2-1}}{\lambda_1 B T_H H^{\lambda_1-1}}\right]^{\frac{1}{\alpha}} \lambda_1 T_H H^{\lambda_1-1} + \\ +\lambda_1 T_H H^{\lambda_1-1} BH + \lambda_3 N^{\lambda_3-1}\mu T_N N + \frac{\lambda_3(\lambda_3-1)T_N N^{\lambda_3-2}\theta^2 N^2}{2}, \tag{2.85}$$

that is:

$$0 = \left(\frac{\sigma}{1-\sigma}\right)[\lambda_2 T_K]^{1-\frac{1}{\sigma}} K^{\frac{(\lambda_2-1)(\sigma-1)}{\sigma}} - \rho\left(T_H H^{\lambda_1} + T_K K^{\lambda_2} + T_N N^{\lambda_3}\right) + \\ +\left(\frac{B}{1-\alpha} - B\right)\lambda_1 T_H \left[\frac{(1-\alpha)\lambda_2 T_K}{\lambda_1 B T_H}\right]^{\frac{1}{\alpha}} H^{\lambda_1-1} K K^{\frac{\lambda_2-1}{\alpha}} H^{\frac{1-\lambda_1}{\alpha}} + \\ +\lambda_1 T_H B H^{\lambda_1} + \lambda_3 \mu T_N N^{\lambda_3} + \frac{\lambda_3(\lambda_3-1)\theta^2}{2} T_N N^{\lambda_3}. \tag{2.86}$$

Let $\lambda_1 = 1$, $\lambda_2 = 1-\alpha$ and $\lambda_3 = 0$. Then we get:

$$0 = \left(\frac{\sigma}{1-\sigma}\right)[(1-\alpha)T_K]^{1-\frac{1}{\sigma}} K^{\frac{-\alpha(\sigma-1)}{\sigma}} - \rho\left(T_H H + T_K K^{1-\alpha} + T_N\right) + \\ +\left(\frac{B}{1-\alpha} - B\right) T_H \left[\frac{(1-\alpha)^2 T_K}{B T_H}\right]^{\frac{1}{\alpha}} + T_H B H. \tag{2.87}$$

If $\sigma = \alpha$, then

$$0 = \left[\left(\frac{\alpha}{1-\alpha}\right)(1-\alpha)^{\frac{-(1-\alpha)}{\alpha}} T_K^{-\frac{1}{\alpha}} - \rho\right] K^{1-\alpha} T_K + (B-\rho) T_H H + \\ \left[-\rho T_N + \left(\frac{B}{1-\alpha} - B\right) T_H \left(\frac{(1-\alpha)^2 T_K}{B T_H}\right)^{\frac{1}{\alpha}}\right] \tag{2.88}$$

Since $B - \rho = 0$, the second term in the summation disappears. Since this equation has to be satisfied for all values of K and N, the square brackets have to be zero. This implies the values of T_K and $\frac{T_H^{\frac{\alpha-1}{\alpha}}}{T_N}$ given by (2.55).

The verification theorem requires that the transversality condition is satisfied in order to have an optimal solution. The TVC implies that $\lim_{t\to\infty} E[e^{-\rho t} J(H, K, N)] = 0$, that is:

$$\lim_{t\to\infty} E[e^{-\rho t}(T_H H + T_K K^{1-\alpha} + T_N] = 0. \tag{2.89}$$

The third term,

$$\lim_{t\to\infty} T_N e^{-\rho t} \tag{2.90}$$

to automatically converge to zero. The second term instead,

$$\lim_{t\to\infty} T_K E\left[e^{-\rho t}e^{-\Omega_2(1-\alpha)t}\left[\frac{\Omega_1}{\Omega_2}(e^{\Omega_2(1-\alpha)t}-1)+K_0^{1-\alpha}\right]\right]$$

automatically converges to zero as well. The first term is given by:

$$\lim_{t\to\infty} T_H E\left[e^{-\rho t}e^{Bt}\left[H_0+\Omega_3(e^{-Bt}-1)\right]\right]=0,$$

which requires:

$$\lim_{t\to\infty} T_H E\left[H_0+\Omega_3(e^{-\rho t}-1)\right]=0;$$

the term in the square brackets has therefore to be null, leading to the value $\frac{T_K}{T_H}$ given by (2.55).

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Chapter 3

Endogenous Technical Progress in a Three-Sectors Growth Model

This paper[1] presents an endogenous growth model driven by human capital, where human capital can be allocated across three sectors: the production of the final consumption good, the educational sector and the production of technological capital (in the form of knowledge or ideas). Typical ideas-based growth models, such as Romer (1990), assume that the production of knowledge is linear in some variables, usually ideas or human capital. We assume that knowledge is produced according to a neoclassical technology, combining ideas and human capital. Such an assumption is motivated by empirical works showing the existence of significant decreasing returns in the creation of ideas at the aggregate level (as Kortum (1993); and Pessoa (2005)) and of weak relationship between some inputs of the knowledge production process (as the number of researchers) and the total factor productivity growth rate (as Jones (2002)). We show that the economy converges towards its steady state equilibrium along a form of generalized saddle path, along which the stable arm is multidimensional. This gives rise to equilibrium indeterminacy, which happens under the (fairly reasonable) conditions ensuring convergence to steady state.
Keywords: Economic Growth, Capital Accumulation, Technological Progress, Equilibrium Indeterminacy

JEL Classification: E32, O33, O40, O41

[1] The first part of this paper is a simplified version of a joint work of La Torre Davide and myself, recently published on Economic Modelling: La Torre, D., Marsiglio, S. (2010), Endogenous Technological Progress in a Multi Sector Growth Model, Economic Modelling 27(5), 1017-1028. With respect to that paper, this version does not include government expenditure and population change, but also studies a particular case of the model, in which the human and technical capital share coincide. Such a case allows us to further simplify the model and therefore to prove that the stable transitional path is multidimensional, under an additional general assumption; this is used to prove that also a multi-sector model with constant returns to scale in each sector can generate equilibrium indeterminacy

3.1 Introduction

During the last century, modern economies have been characterized by growth in many forms: output, productivity, average human capital and knowledge are all variables which have grown consistently over the past hundred years. This increased the interest in identifying the underlying causes of such improvements, and different variables have been identified as possible sources of growth in the literature during the last fifty years. Solow (1956) and others after him identified the accumulation of physical capital as crucial in explaining growing economies. In Lucas' (1988) view, instead, the evolution of human capital is the key feature driving growth in the long run. Recently, a literature stream, led mainly by Romer (1986, 1990), has started considering ideas as the relevant engine of growth.

In ideas-based growth models, the dynamics of knowledge is assumed to be the ultimate determinant of growth. This means that the creation of new ideas is enough to explain the features of growing economies over the last century. In order to model this, some kind of linearities have been introduced in the technology production function (see Jones (2005) for a survey). For example, Romer (1990) assumes creation of ideas is linear both in the stock of ideas and human capital employed in research. Empirical works suggest there are significant decreasing returns of ideas at the aggregate level (see for example, Kortum (1993) and Pessoa (2005)). Moreover, as Pessoa (2005) clearly summarizes: *"The ideas-driven model ... predicts that expansion in the number of researchers leads to a permanent increase in TFP growth rate. In contrast, the empirical evidence suggests that most OECD economies have increased the size of their R&D workforce, while experiencing (at best) constant TFP growth rates. This weak relationship between the number of researchers and TFP growth rate has led some to question the viability of ideas-driven growth for the long run"*. Therefore, we consider that the concept of creation of ideas has to be introduced in standard economic growth models, modeling its interaction with capital (in particular with human capital), in order to deepen their dynamics. In this paper, we introduce ideas in a multi-sector endogenous growth model. In particular, ideas affect the production of the consumption good and are used to create new ideas[2]. Therefore, our model economy is composed of three main interrelated sectors: the final

[2]The introduction of ideas in a growth model, as Jones (2005) emphasizes, raises the problem of introducing non- rivalry. Romer (1993) divides goods into two categories: ideas and objects. Ideas can be represented by instructions or blueprints, while objects are the standard rival goods, such as capital, labor, output... Ideas are merely instructions for combining the objects in order to produce utility. Accepting Romer's (1993) definition of ideas, we have to recognize that the use of ideas by one person does not diminish others' use and therefore ideas are non-rival goods. This of course implies the presence of increasing returns to scale in production possibilities, and consequently leads to a framework where the first fundamental theorem of welfare economics does not hold. As a result, the decentralized outcome may not coincide with the planned one, resulting in a

one, the educational sector and that devoted to accumulating knowledge. The basic model we use in order to study the linkages among ideas, human and physical capital is an extension of the Uzawa-Lucas model (see Uzawa (1965) and Lucas (1988)), which is one of the most studied and celebrated endogenous growth models. As Boucekkine and Ruiz-Tamarit (2008) have recently emphasized, *"the Lucas-Uzawa model, one of the most celebrated endogenous growth model, [...] has some interesting properties. [...] being a two- sector model, it gives rise to a sophisticated dynamic system with two controls, consumption (c) and the fraction of human capital operating in the final good sector (u), and two state variables, physical capital (K) and human capital (H)"*, and *"because it is mathematically appealing, it has been studied by many authors, using different approaches, allowing for a stimulating methodological discussion"*. In literature, several extensions of the classical Uzawa-Lucas model can be found (see for example Bucci et al. (2010); and Robertson (2002)). For instance, Robertson (2002) introduces unskilled labor as a distinct factor from human capital in the Uzawa-Lucas model, and uses this modified model in order to study the effects of demographic shocks on the economy. Bucci et al. (2010), instead, extend the Uzawa-Lucas model by assuming that the level of technology might be subject to random shocks and suppose that the level of technology follows an exogenous stochastic differential equation driven by a Brownian motion.

In this paper we extend the Uzawa-Lucas model in two different ways. Ideas affect the production of the final consumption good (derived from a technology combining ideas, physical and human capital) and are used to produce new ideas (therefore we have an additional differential equation, representing the accumulation of knowledge over time). Human capital, which still represents the engine of growth, is used also in the production of new ideas, and therefore has to be endogenously allocated across the three sectors. This means that our model economy is composed of three sectors, where physical, human and technological capital are used to produce only one homogeneous final good, that can be consumed or invested in physical capital. Physical capital can be used only for producing the final good, while human capital can be allocated in the production of the final good, in the education sector or in the production of ideas. Technological capital instead can be used as an input in the production of the final good or to create new technical capital. A benevolent social planner in this model has to decide where to allocate resources. The human capital decision is crucial, since it affects production of the final good, education, and ideas.

The paper proceeds as follow. Section 2 introduces the model and derives the optimal paths for the three control variables: consumption, share of human capital allocated to physical pro-

sub optimal allocation of resources. In this paper, we do not analyze the problems related to the decentralized allocation (postponing this issue to future research), but we aim at studying the optimal planned economy and the linkages between ideas, physical and human capital duction and to knowledge creation. Section 3 analyzes the steady state of our model economy, that is characterized by a balanced growth path, along which the fractions of human capital allocated in each sector are constant, and its transitional dynamics. We show the economy converges towards its steady state equilibrium along a saddle path. We moreover present some numerical examples, in order to underline the role of human capital in the three sectors. In Section 4 we present a special case of the model, that is when the human and technical capital share coincide in order to emphasize the features of the stable arm of the model; such a case allows us to prove that equilibrium indeterminacy can arise also in a growth model with constant returns to scale in each sector. Section 5 instead concludes.

3.2 The Model

The model is an extension of the Uzawa-Lucas model, in which we consider a three sectors economy, where physical, human and technological capital are used to produce only one homogeneous final good, that can be consumed or invested in physical capital. Physical capital can be used only for producing the final good, while the human capital can be allocated in the production of the final good, in the education sector or in the production of ideas. Technological capital or ideas instead can be used as an input in the production of the final good or to create ideas.

The economy is closed and composed of households (that receive wages and interest income, purchase the consumption good and choose how much to save and how much to invest both in education and in ideas) and firms (that produce the consumption good). The infinitely-lived representative household wants to maximize its lifetime utility function:

$$U = \int_0^\infty u(c_t) N_t e^{-\rho t} dt, \tag{3.1}$$

where $c_t \equiv \frac{C_t}{N_t}$ is per-capita consumption and $\rho > 0$ is the rate of time preference. The instantaneous utility function is assumed to be iso-elastic:

$$u(c_t) = \frac{c_t^{1-\sigma} - 1}{1-\sigma}, \tag{3.2}$$

where $\sigma > 0$ represents the inverse of the intertemporal elasticity of substitution in consumption.

The final good is produced, combining physical capital, K_t, the share of human capital allocated to final production, $u_t H_t$, and labor in efficiency units (the product of row labor and its efficiency, given by level of ideas in the economy), $A_t N_t$, according to the following technology:

$$Y_t = K_t^\alpha (u_t H_t)^\beta (A_t N_t)^{1-\alpha-\beta}, \tag{3.3}$$

where $0 < \alpha, \beta, (\alpha + \beta) < 1$ and $u_t \in (0, 1)$.

Notice that two inputs of production, human capital and ideas, are allocated across other sectors: other than the final one, the former is also assigned to production of new human capital and production of new ideas, while the latter is assigned only to the creation of knowledge. The main difference between these factors is found in their own nature: human capital is a rival good, meaning that its usage in a sector lowers its availability in the other ones; ideas instead are non-rival, meaning that being widespread in the economy, they can be freely and contemporaneously used in different sectors, without compromising their availability in each one. This means that human capital used in the production of the final good is only a share, u_t, of the total, while the remaining fraction is split between the production of new ideas (or blueprints), x_t and education, $1 - u_t - x_t$. Instead, all ideas available are independently used both in physical and knowledge production. Moreover, notice that the production function combines with constant returns to scale the rival inputs, physical, human capital and labor, while showing increasing returns to scale jointly to these inputs and ideas.

Population coincides with the available number of workers, N_t, so that there is no unemployment and the labor supply is inelastic (no leisure-work choice), and it grows at an exogenous constant rate. For the sake of simplicity we assume this constant rate is zero[3]. The constant level of population therefore can be normalized to one, $N_0 = N_t \equiv 1, \forall t$, implying no difference between aggregate and per-capita variables. Consequently, the production function of the consumption good can be simplified as:

$$y_t = k_t^{\alpha}(u_t h_t)^{\beta} a_t^{1-\alpha-\beta}. \tag{3.4}$$

Physical capital accumulates over time in accordance to the difference between output, y_t, and consumption, c_t:

$$\dot{k}_t = y_t - c_t. \tag{3.5}$$

Human capital accumulation coincides with the production of new human capital, which depends only on the effort devoted to the accumulation of human capital, $1 - u_t - x_t$, and on the already attained human capital stock (the education sector is intensive in human capital), h_t:

$$\dot{h}_t = (1 - u_t - x_t)h_t, \tag{3.6}$$

where $0 < u_t, x_t < 1$. Notice that new human capital is produced accordingly to a linear production function, meaning that human capital, as in the Uzawa-Lucas model, represents the force driving endogenous growth.

[3]The outcome of the model would not change if we relax this assumption: the only differences would be represented by an additional constant in the Euler and state equations

Also idea accumulation coincides with the production of new ideas, depending on the effort devoted to the accumulation of new ideas, $x_t H_t$, and on already existing stock of the knowledge, a_t:

$$\dot{a}_t = (x_t h_t)^{\phi} a_t^{1-\phi}, \tag{3.7}$$

where $0 < \phi < 1$.

Since $1-\phi > 0$, we rely on what Jones (2005) labels as a *"standing on the shoulders effect"*: the discovery of ideas in the past increases the possibilities of new discoveries[4]; however, we assume that this effect vanishes as the stock of ideas is sufficiently large. As in Jones (2005), we assume that the production of new ideas is Cobb-Douglas, combining ideas and human capital, but we assume that these factors are combined with constant returns to scale. This means that the marginal productivity of both inputs is increasing and concave. The specification of decreasing returns in ideas is suggested also in Romer (1990) and considered by Kortum (1993) and Jones (1995). Moreover, empirical works suggest there are significant decreasing returns of ideas at the aggregate level; for example, Kortum (1993) reports elasticity estimates of $1-\phi$ in the range between 0.1 and 0.6. Our assumption of constant returns to scale implies that ϕ lies in the interval of 0.4 to 0.9. Romer (1990) also suggests the specification of decreasing returns in human capital, considered by Stokey (1995) and Jones (2002).

We consider the possibility of duplication in technology: if we double the stock of ideas and the share of human capital used for producing new ideas, we will also double the production of new knowledge. However, in order to do so, we have to allow a decrease in the fraction of human capital allocated to the educational sector, meaning that the endogenous growth rate will be lower.

Notice that we have assumed that the depreciation rate of physical, human and technological capital is common and equal to zero[5].

The social planner maximizes the social welfare function, that is it has to choose c_t, u_t, and x_t in order to maximize agents' lifetime utility function subject to physical, human capital and idea accumulation constraints and the initial conditions:

$$\begin{aligned} \max_{\{c_t,u_t,x_t\}_0^\infty} \quad & U = \int_0^\infty \frac{c_t^{1-\sigma}}{1-\sigma} e^{-\rho t} dt, \ \rho > 0 \\ s.t. \quad & \dot{k}_t = k_t^{\alpha}(u_t h_t)^{\beta} a_t^{1-\alpha-\beta} - c_t \\ & \dot{h}_t = (1 - u_t - x_t) h_t \end{aligned} \tag{3.8}$$

[4]The opposite case, where $1-\phi < 0$, corresponds to what the productivity literature defines as the *"fishing out effect"*, in which the rate of innovation decreases with the level of knowledge

[5]This is of course a simplifying assumption, but the outcome of the model would not change even if we introduce non zero depreciation rates

$$\dot{a}_t = (x_t h_t)^\phi a_t^{1-\phi}$$
$$k_0, h_0, a_0 \text{ given.}$$

3.2.1 Optimal Paths

The Hamiltonian function, $\mathcal{H}_t(\cdot)$, associated to the maximization problem is:

$$\mathcal{H}_t(\cdot) = \frac{c_t^{1-\sigma}}{1-\sigma} e^{-\rho t} + \lambda_t [k_t^\alpha (u_t h_t)^\beta a_t^{1-\alpha-\beta} - c_t] + \mu_t [(x_t h_t)^\phi a_t^{1-\phi}] + \nu_t (1 - u_t - x_t) h_t. \quad (3.9)$$

The first order necessary conditions are:

$$\begin{aligned}
c_t) &\rightarrow c_t^{-\sigma} e^{-\rho t} = \lambda_t && (3.10)\\
u_t) &\rightarrow \lambda_t \beta k_t^\alpha (u_t h_t)^{\beta-1} a_t^{1-\alpha-\beta} h_t = \nu_t h_t && (3.11)\\
x_t) &\rightarrow \mu_t \phi (x_t h_t)^{\phi-1} a_t^{1-\phi} h_t = \nu_t h_t && (3.12)\\
k_t) &\rightarrow -\dot{\lambda}_t = \lambda_t \alpha k_t^{\alpha-1} (u_t h_t)^\beta a_t^{1-\alpha-\beta} && (3.13)\\
a_t) &\rightarrow -\dot{\mu}_t = \lambda_t (1-\alpha-\beta) k_t^\alpha (u_t h_t)^\beta a_t^{-\alpha-\beta} + \mu_t (1-\phi)(x_t h_t)^\phi a_t^{-\phi} && (3.14)\\
h_t) &\rightarrow -\dot{\nu}_t = \lambda_t \beta k_t^\alpha (u_t h_t)^{\beta-1} a_t^{1-\alpha-\beta} u_t + \mu_t \phi (x_t h_t)^{\phi-1} a_t^{1-\phi} x_t + \nu_t (1 - u_t - x_t) && (3.15)
\end{aligned}$$

together with the initial conditions k_0, h_0 and a_0, the dynamic constraints:

$$\begin{aligned}
\dot{K}_t &= k_t^\alpha (u_t h_t)^\beta a_t^{1-\alpha-\beta} - c_t && (3.16)\\
\dot{H}_t &= (1 - u_t - x_t) h_t && (3.17)\\
\dot{A}_t &= (x_t h_t)^\phi a_t^{1-\phi} && (3.18)
\end{aligned}$$

and the transversality conditions:

$$\begin{aligned}
\lim_{t\to\infty} \quad & \lambda_t k_t = 0 && (3.19)\\
\lim_{t\to\infty} \quad & \nu_t h_t = 0 && (3.20)\\
\lim_{t\to\infty} \quad & \mu_t a_t = 0. && (3.21)
\end{aligned}$$

Solving the system, the optimal paths of consumption, share of human capital allocated to production and to new ideas are:

$$\begin{aligned}
\frac{\dot{c}_t}{c_t} &= \frac{1}{\sigma} \left[\alpha k_t^{\alpha-1} (u_t h_t)^\beta a_t^{1-\alpha-\beta} - \rho \right] && (3.22)\\
\frac{\dot{u}_t}{u_t} &= \frac{1}{\beta-1} \left[\alpha \frac{c_t}{k_t} - \beta - (1-\beta)(u_t + x_t) - (1-\alpha-\beta)(x_t h_t)^\phi a_t^{-\phi} \right] && (3.23)\\
\frac{\dot{x}_t}{x_t} &= \frac{1}{\phi-1} \left[\frac{\phi(1-\alpha-\beta)}{\beta} u_t x_t^{\phi-1} h_t^\phi a_t^{-\phi} - \phi - (1-\phi)(u_t + x_t) \right] && (3.24)
\end{aligned}$$

Equation (3.22) is the standard Keynes-Ramsey rule, showing that the growth rate of consumption is an increasing function of the marginal productivity of physical capital in the final sector. Notice that, as usual, it is positive if the marginal productivity of physical capital is higher than the rate of time preference, ρ. Equation (3.23) says that the change of the share of human capital allocated to the production of the final good is a negative function of the consumption-capital ratio and a positive function of the average productivity of ideas in the research sector and of the shares of human capital not allocated to the educational sector. Equation (3.24) instead relates the share of human capital devoted to the production of ideas positively with the shares of human capital not allocated to the educational sector and with the $\frac{u_t}{x_t}$ ratio while negatively with the human capital-ideas ratio.

3.3 Steady State Analysis

We now analyze the steady state of our model economy, which is characterized by a balanced growth path equilibrium, that is a situation where all economic variables grow at constant and finite rates.

Definition 1: (*Balanced Growth Path, BGP*) a balanced growth path, BGP, or steady state equilibrium, $(\overline{c}, \overline{h}, \overline{k}, \overline{a}, \overline{u}, \overline{x}, \gamma_c, \gamma_h, \gamma_k, \gamma_a, \gamma_u, \gamma_x)$, is a sequence of time paths, $\{c_t, h_t, k_t, a_t, u_t, x_t\}_{t\geq 0}$, along which all economic variables grow at constant rates. A BGP is said to be non-degenerate if c_t, h_t, k_t and a_t grow at non negative rates.

First of all, notice that the growth rate of consumption must equalize that of physical capital, in order to have endogenous growth and not to violate TVC (3.19). Moreover, along the BGP, the share of human capital allocated in the production of the consumption good and in the creation of new ideas have both to be constant, otherwise the growth rate of human capital cannot be constant.

Therefore, we can study the dynamics of a simplified system, where the variables do not asymptotically grow, and the BGP of the original system is represented by the equilibrium point of such a simplified system. In fact, by introducing the intensive variables: $\chi_t \equiv \frac{c_t}{k_t}$, $\varphi_t \equiv \frac{h_t}{k_t}$ and $\psi_t \equiv \frac{a_t}{k_t}$, we obtain the following system of five nonlinear differential equations:

$$\frac{\dot{\chi}_t}{\chi_t} = \frac{\alpha - \sigma}{\sigma}(u_t\varphi_t)^{\beta}\psi_t^{1-\alpha-\beta} - \frac{\rho}{\sigma} + \chi_t \tag{3.25}$$

$$\frac{\dot{\psi}_t}{\psi_t} = (x_t\varphi_t)^{\phi}\psi_t^{-\phi} - (u_t\varphi_t)^{\beta}\psi_t^{1-\alpha-\beta} + \chi_t \tag{3.26}$$

$$\frac{\dot{\varphi}_t}{\varphi_t} = (1 - u_t - x_t) - (u_t\varphi_t)^{\beta}\psi_t^{1-\alpha-\beta} + \chi_t \tag{3.27}$$

$$\frac{\dot{u}_t}{u_t} = \frac{1}{\beta - 1}\left[\alpha\chi_t - \beta - (1-\beta)(u_t + x_t) - (1-\alpha-\beta)(x_t\varphi_t)^{\phi}\psi_t^{-\phi}\right] \quad (3.28)$$

$$\frac{\dot{x}_t}{x_t} = \frac{1}{\phi - 1}\left[\frac{\phi(1-\alpha-\beta)}{\beta}\frac{u_t}{x_t}(x_t\varphi_t)^{\phi}\psi_t^{-\phi} - \phi - (1-\phi)(u_t + x_t)\right]. \quad (3.29)$$

Moreover, by introducing the variables $Z_t = (u_t\varphi_t)^{\beta}\psi_t^{1-\alpha-\beta}$ and $M_t = (x_t\varphi_t)^{\phi}\psi_t^{-\phi}$, we can recast the system in a quasi linear form:

$$\frac{\dot{\chi}_t}{\chi_t} = \frac{\alpha-\sigma}{\sigma}Z_t - \frac{\rho}{\sigma} + \chi_t \quad (3.30)$$

$$\frac{\dot{Z}_t}{Z_t} = \frac{\alpha+\beta-1}{\beta-1}\chi_t - \frac{\beta}{\beta-1} - (1-\alpha)Z_t - \frac{1-\alpha-\beta}{\beta-1}M_t \quad (3.31)$$

$$\frac{\dot{M}_t}{M_t} = \frac{\phi^2(1-\alpha-\beta)}{\beta(\phi-1)}\frac{u_t}{x_t}M_t - \phi M_t - \frac{\phi}{\phi-1} \quad (3.32)$$

$$\frac{\dot{u}_t}{u_t} = \frac{1}{\beta-1}\left[\alpha\chi_t - \beta - (1-\beta)(u_t+x_t) - (1-\alpha-\beta)M_t\right] \quad (3.33)$$

$$\frac{\dot{x}_t}{x_t} = \frac{1}{\phi-1}\left[\frac{\phi(1-\alpha-\beta)}{\beta}\frac{u_t}{x_t}M_t - \phi - (1-\phi)(u_t+x_t)\right]. \quad (3.34)$$

The equilibrium point of this system is represented by a point where equations (3.30), (3.31), (3.32), (3.33) and (3.34) are null. Such a point is characterized by a strictly positive level of all variables if $\sigma > 1-\rho > 0$. Moreover, under the same assumption, the equilibrium shows a multidimensional unstable manifold. Notice that such a parametric condition is usually satisfied: it only requires the inverse of the intertemporal elasticity of substitution to be sufficiently high. Several empirical works suggest the elasticity of substitution is lower than one, meaning that its inverse (the relative risk aversion parameter) is higher than one (see for example, Mehra and Prescott (1985); and Hall (1988)). Therefore, we can summarize these results in the following propositions.

Proposition 1: *if the following parameter restrictions apply:*

$$\sigma > 1-\rho > 0, \quad (3.35)$$

then the following results hold:
(i) the BGP equilibrium is characterized by a strictly positive level of consumption, physical, human and technological capital, shares of human capital allocated to the educational and research sectors;
(ii) the BGP equilibrium shows a multidimensional unstable manifold.

Proof: see Appendixes A and B. Appendix A proves part (i) of the proposition while Appendix B part (ii). ■

Moreover, under standard parameter values, it is possible to show that the BGP equilibrium is saddle-point stable. This is shown by the next numerical examples.

Example 3.3.1 *Let us do a numerical simulation using the following parameter values:* $\alpha = 0.3333$, $\beta = 0.2$, $\phi = 0.2$, $\sigma = 2$, $\rho = 0.04$. *The equilibrium is:* $M^* = 0.48$, $\chi^* = 2.52$, $Z^* = 3$, $u^* = 0.3813$, $x^* = 0.1386$. *The linearized Jacobian is:*

$$\begin{bmatrix} 2.520000000 & -2.100000000 & 0 & 0 & 0 \\ 1.750000000 & -2.0 & 1.750000000 & 0 & 0 \\ 0 & 0 & -0.2499999999 & -0.1938461539 & 0.5330769228 \\ -0.1588888888 & 0 & 0.2224444445 & 0.3813333333 & 0.3813333333 \\ 0 & 0 & -0.2224444445 & -0.1413333333 & 0.9086666667 \end{bmatrix}$$

The eigenvalues of the linearized model are $\lambda_1 = -.8982525480$, $\lambda_2 = 1.418252548$, $\lambda_3 = .5199999998$, $\lambda_4 = .6952597330$, $\lambda_5 = -.1752597328$ *with eigenvectors*

$$v_1 = \begin{bmatrix} -0.5183670708 & -0.8437664579 & -0.01284295470 & -0.06025831199 & -0.006294333392 \end{bmatrix}$$

$$v_2 = \begin{bmatrix} 1.615847878 & 0.8477410890 & 0.04003391112 & -0.2227207860 & 0.04429587035 \end{bmatrix}$$

$$v_3 = \begin{bmatrix} 0.000000007300000000 & 0.000000007700000000 & 0.000000002097000000 & -1.005291378 & -0.3655604976 \end{bmatrix}$$

$$v_4 = \begin{bmatrix} -2.495952980 & -2.168793288 & -0.8443105758 & -2.066326458 & -2.248535201 \end{bmatrix}$$

$$v_5 = \begin{bmatrix} 2.687615181 & 3.449438562 & 0.9091444981 & 0.2534170105 & 0.2196186132 \end{bmatrix}$$

The unstable manifold M_u *is generated by the vectors* $\langle v_2, v_3, v_4 \rangle$ *while the stable manifold* M_s *by the vector* $\langle v_1, v_5 \rangle$.

Example 3.3.2 *Let us do a numerical simulation using the following parameter values:* $\alpha = 0.3333$, $\beta = 0.2$, $\phi = 0.8$, $\sigma = 2$, $\rho = 0.04$. *The equilibrium is:* $M^* = 0.48$, $\chi^* = 2.52$, $Z^* = 3$, $u^* = 0.2611555556$, $x^* = 0.2588444444$. *The linearized Jacobian is:*

$$\begin{bmatrix} 2.520000000 & -2.100000000 & 0 & 0 & 0 \\ 1.750000000 & -2.0 & 1.750000000 & 0 & 0 \\ 0 & 0 & -4.000000003 & -6.646153850 & 6.705494512 \\ -0.1088148148 & 0 & 0.1523407408 & 0.2611555556 & 0.2611555556 \\ 0 & 0 & -2.437451853 & -4.221155557 & 4.778844445 \end{bmatrix}$$

The eigenvalues of the linearized model are $\lambda_1 = 1.496661179 + .6462436692i$, $\lambda_2 = 1.496661179 - .6462436692i$, $\lambda_3 = -.9766611807 + .6462436678i$, $\lambda_4 = -.9766611807 - .6462436678i$, $\lambda_5 = .5200000002$, *with eigenvectors*

$$v_1 = \left[\begin{array}{ccccc} -0.662428e-1+.820589i & .220243+.420261i & .351114+.100464i & 0.743842e-1-0.939052e-1i & .351861+0.23117e-1i \end{array}\right]$$

$$v_2 = \left[\begin{array}{ccccc} -0.662428e-1-.820589i & .220243-.420261i & .351114-.100464i & 0.743842e-1+0.939052e-1i & .351861-0.23117e-1i \end{array}\right]$$

$$v_3 = \left[\begin{array}{ccccc} 0.54299e-1-2.51137i & -.682426-4.19834i & 1.09701-.195676i & -.105991-.211491i & .408421-.19212i \end{array}\right]$$

$$v_4 = \left[\begin{array}{ccccc} 0.54299e-1+2.51137i & -.682426+4.19834i & 1.09701+.195676i & -.105991+.211491i & .408421+.19212i \end{array}\right]$$

$$v_5 = \left[\begin{array}{ccccc} 0.213000e-7 & 0.197000e-7 & 0.210000e-8 & .368319 & .365059 \end{array}\right]$$

The unstable manifold M_u is generated by the vectors $\langle v_1, v_2, v_5 \rangle$ while the stable manifold M_s by the vector $\langle v_3, v_4 \rangle$.

The previous two examples have been implemented using MAPLE 13. They show that the economy converges to its steady state through a saddle path and the unstable transitional manifold has dimension three while the stable manifold has dimension two.

Along the BGP, the consumption-capital ratio is an increasing function of the inverse of the elasticity of substitution, σ, and of the rate of time preference, ρ; it, instead, depends negatively on the capital share, α. Notice that it is independent of the human capital share, β. The stationary share of human capital allocated to physical and knowledge production instead are complicated functions of the physical and human capital shares, α and β, of the elasticity of human capital in the knowledge production process, ϕ, of the rate of time preference, ρ and of the inverse of the intertemporal elasticity of substitution in consumption, σ. In particular, the share of human capital allocated to knowledge (physical) production is a decreasing (increasing) function of β and an increasing (decreasing) function of ϕ; the share allocated to the educational sector instead is independent of both σ and ϕ.

3.3.1 The Allocation of Human Capital

We now illustrate the behavior of the allocation of human capital among the three sectors implied by the model, under a given set of parameter values. In choosing such values we rely on existing empirical estimates or on baseline specifications coming from previous works.

The physical capital share has been traditionally considered to be around one third (see Denison (1962); Maddison (1982); Jorgenson et al. (1987); and Mankiw et al. (1992)) while the human capital share has been estimated by Mankiw et al. (1992) to vary in the range (0.333, 0.5). The elasticity of ideas in the production function of technology has been estimated by Kortum (1993) in the range (0.1, 0.6), implying that $\phi \in (0.4, 0.9)$. We set $\alpha = 0.33$, and following Mulligan and Sala-i-Martin (1993), $\rho = 0.04$ and $\sigma = 2$. Firstly, we fix $\beta = 0.42$ (the median value of the interval estimated by Mankiw et al. (1992)) and let ϕ vary in the interval (0.4,0.9); then we fix $\phi = 0.65$ (the median value of the interval estimated by Kortum (1993)) and let β vary in the range (0.333,0.5).

This set of parameters implies that in steady state the highest share of human capital is devoted to creation of new human capital, and the lowest share is allocated to knowledge production. Such an outcome is clear: the impact of human capital is more important in the educational sector, since it is the growth driven force, and in the physical one, since it produces the consumable good, which is the argument of agents' utility function, while it is lower in the technological sector. This ranking is clearly reflected by the optimal allocation of resources. In fact, the share of human capital in the technological sector is always particularly low, while that in physical production can reach a value close to that in education.

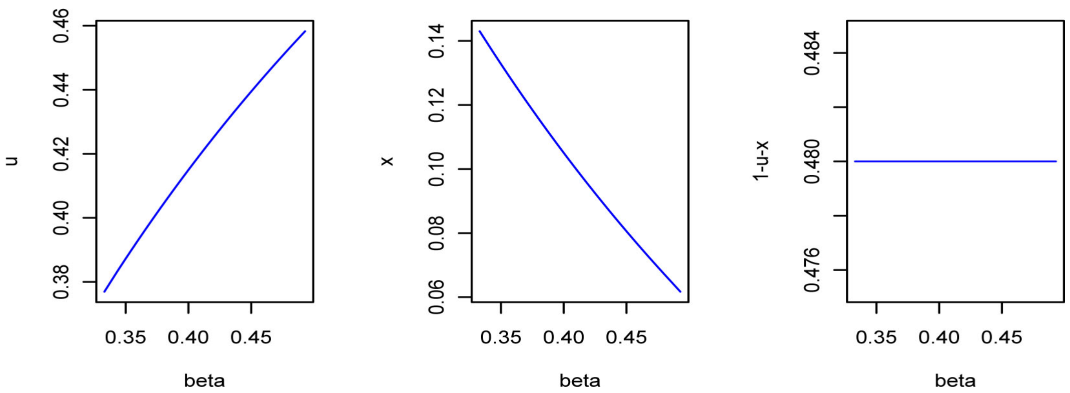

Figure 3.1: Optimal allocation of human capital in steady state. The figure shows the values of u, x and $1 - u - x$, as β changes

Notice that even if we let β and ϕ be in the ranges empirically estimated, the same result holds. Of course, as one of these parameters changes, the steady state values of u and x changes, while that of $1 - u - x$ does not. If we increase β, u increases (and x decreases) but its steady state value is always lower than the share allocated to the educational sector; in fact, when $\beta = 0.5$, that is the upper bound of its estimated range, the optimal allocation of human capital is the following: $\overline{u} = 0.46$, $\overline{x} = 0.06$ and $1 - \overline{u} - \overline{x} = 0.48$. Only if the human capital share gets particularly large (higher than 0.56), the fraction of human capital allocated to physical production will be higher than that in education.

If we increase ϕ, instead, x increases (and u decreases) but, again, the steady state value of u and x are always lower than the share allocated to educational sector; in fact, when $\phi = 0.4$, the lower bound of its estimated range, the optimal allocation of human capital is: $\overline{u} = 0.44$, $\overline{x} = 0.07$ and $1 - \overline{u} - \overline{x} = 0.48$. If ϕ gets particularly small (lower than 0.2), then the fraction of human capital allocated to physical production will be higher than that in education.

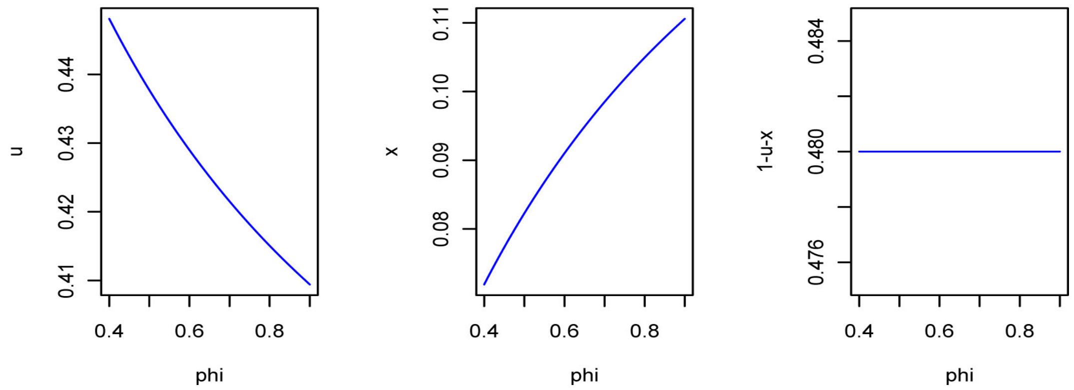

Figure 3.2: Optimal allocation of human capital in steady state. The figure shows the values of u, x and $1-u-x$, as ϕ changes

3.4 A Special Case of the Model: $\beta = \phi$. Transitional Dynamics and Equilibrium Indeterminacy

In the previous section, we show that the steady state equilibrium of the model is characterized by a balanced growth path and, under a general condition, the unstable manifold has at least dimension one while we cannot determine the properties of the stable arm. We can just show through general numerical examples that the stable arm is a multi-dimensional locus. Therefore, in order to shed some light on the transitional dynamics of the economy, it can be convenient to study a particular case of the model.

In fact, in order to simplify the system of equations (3.30) - (3.34) and be able to prove the properties of the stable and unstable transitional paths, we concentrate on the case $\beta = \phi$. We suppose that human capital share, β, and the technical share, ϕ coincide. Such a case is consistent with empirical studies discussed in the previous section (see Mankiw et al. (1992) and Kortum (1993)), which estimate the following parameter ranges: $\beta \in (0.333, 0.5)$ and $\phi \in (0.4, 0.9)$. Under the assumption $\phi = \beta$, by introducing the variable $\eta_t = \frac{u_t}{x_t}$, the system of differential equations (3.30) - (3.34) can be simplified into a four dimensional system:

$$\frac{\dot{\chi}_t}{\chi_t} = \frac{\alpha-\sigma}{\sigma} Z_t - \frac{\rho}{\sigma} + \chi_t \tag{3.36}$$

$$\frac{\dot{Z}_t}{Z_t} = \frac{\alpha+\beta-1}{\beta-1}\chi_t - \frac{\beta}{\beta-1} - (1-\alpha)Z_t - \frac{1-\alpha-\beta}{\beta-1}M_t \tag{3.37}$$

$$\frac{\dot{M}_t}{M_t} = \frac{\beta(1-\alpha-\beta)}{\beta-1}\eta_t M_t - \beta M_t - \frac{\beta}{\beta-1} \tag{3.38}$$

$$\frac{\dot{\eta}_t}{\eta_t} = \frac{\alpha}{\beta-1}\chi_t - \frac{1-\alpha-\beta}{\beta-1}(1+\eta_t)M_t. \tag{3.39}$$

The equilibrium point of such a system is characterized by a strictly positive level of all variables if $\sigma > 1 - \rho > 0$, as before. Moreover, under a general assumption, that is if $\sigma < 2(1 - \rho)$, the equilibrium is stable. This means that, considering the standard empirical value of ρ, about 0.04, σ has to be higher than 0.96 and lower than 1.92. This does not represent a unreasonable range for the inverse of the elasticity of substitution in consumption. So we can summarize this result in the following proposition:

Proposition 2: *suppose the following parameter restrictions hold, $\beta = \phi$ and $1 - \rho < \sigma < 2(1 - \rho)$; then the BGP equilibrium is stable and the stable arm is a multi-dimensional locus.*

Proof: see Appendix C. ■

Notice that the stability property in such a framework has to be interpreted in a generalized sense. In fact, the stable manifold is multi-dimensional: it can have dimension two or four, meaning that the unstable manifold has respectively dimension two or zero. Therefore, in this case stability does not imply uniqueness of the stable arm: it means that there exists a multiplicity of paths converging to the equilibrium: such a situation is called equilibrium indeterminacy.

Equilibria (in)determinacy in macroeconomics models is a well-known and debated problem. We can define *"indeterminate a situation in which there exists a continuum of distinct equilibrium paths sharing a common initial condition"* (Boldrin and Rustichini, 1994). If such a condition is verified, the economic dynamics is not unique in the sense that multiple paths lead the same economy to converge towards its long-run equilibrium. This would explain why pretty similar economies choose different developing trajectories, but would also make difficult determining the effects of alternative policies for the future developments of the economy. Such an issue is particularly relevant for growth economists, whose main goal is understanding how promoting improvements in (per-capita) output across countries. As Palivos et al. (2003) clearly underline, indeterminacy *"can potentially explain an important question posed by, among others, Lucas (1993): Why would two different countries, such as South Korea and the Philippines, whose initial conditions were so close, differ so much in their later growth performance?"*. Several papers (Benhabib and Perli (1994); Xie (1994); Boldrin and Rustichini (1994); and Palivos et al. (2003)) analyze the issue in optimal growth models. They all show that indeterminacy can quite easily arise in the Uzawa (1965) - Lucas (1988) model, if the production function generates some external effects on capital accumulation. In such a framework, equilibrium indeterminacy is not only a theoretical possibility, but a concrete result of the model, since it will be the outcome for reasonable parameter values. However, as Boldrin and Rustichini (1994) clearly underline: *"Quite naturally an issue of 'realism' can be made with* *regard to the parameter values at which these more complicated phenomena arise. While they do not appear as far away from reality as those previously encountered in the optimal growth brand of the chaotic dynamics literature, they do rely on particularly strong externalities. For this reason and for the lack of reliable empirical evidence about the external effects consistent with this type of technology, we refrain from speculating on the positive implications of our findings".*

What we have just shown is that in a multi-sector model of endogenous growth, without any external effect (the production functions in each sector show constant returns to scale), indeterminacy can arise. Proposition 2 tells us that there exists a continuum of balanced growth paths satisfying the same initial conditions and converging to the same equilibrium. This happens for really plausible values of the intertemporal elasticity of substitution. Moreover, such a feature of the economy is ensured by the same conditions characterizing the convergence to equilibrium: stability implies indeterminacy. This can be reinterpreted as follows. In order for the economy to converge to its steady state, the equilibrium to which the economy is converging has to be indeterminate.

3.5 Conclusion

During the last century, knowledge (technology) has increased consistently in most of the industrialized countries. Economic growth theory introduces endogenous technical change in order to describe this fact. As a result, a new literature stream has recently arisen, leading to ideas-based growth models. Our goal is to introduce ideas in a standard multi-sector endogenous growth model and the natural candidate for such an aim seems to be the Uzawa-Lucas model. Therefore, we extend it along different lines: we formally introduce ideas, which are used for producing the final physical good and are created in a particular separate sector, and we emphasize the importance of education in the process generating ideas, also considering the endogenous allocation of human capital in this technical sector. The interaction between human capital and ideas rules the economy. Since human capital is a rival good, while ideas are not, the allocation of human capital across sectors is crucial. The stock of ideas can be contemporaneously exploited in the final and technical sectors, while human capital stock has to be shared across the three sectors. Therefore, the planner has to determine how to optimize the trade-off arising from the allocation of resources.

We show that the economy converges towards its steady state equilibrium, along which the consumption-capital ratio is independent of the human capital share. The convergence paths take the form of a generalized saddle path, along which both the stable arm and the unstable manifold are multidimensional, meaning equilibrium indeterminacy. We show that under the (fairly plausible) conditions ensuring convergence to steady state, the equilibrium is indeterminate, namely a continuum of different and converging trajectories exist. The fact that distinct, identically endowed economies choose different converging paths is not only a possibility, but an outcome of the model. It remains an open problem if it is possible to obtain, maybe for a certain combination of parameters, a closed-form solution to this model.

For further research, we suggest to focus on the decentralized outcome, investigating how and whether this allocation differs from the optimal planned one and how it can be possible to decentralize the optimal allocation of resources. Another aspect which could deserve some attention is the production function of ideas: we assume it is neoclassical; however, it could be interesting to analyze the case in which it is somehow linear, meaning that ideas represent an additional source of endogenous growth.

A. Steady State

The steady state of the five dynamical equations system is characterized by setting equations (3.25), (3.26), (3.27), (3.28) and (3.29) equal to zero:

$$0 = \frac{\alpha-\sigma}{\sigma}\overline{Z} - \frac{\rho}{\sigma} + \overline{\chi} \tag{3.40}$$

$$0 = \overline{M} - \overline{Z} + \overline{\chi} \tag{3.41}$$

$$0 = 1 - \overline{u} - \overline{x} - \overline{Z} + \overline{\chi} \tag{3.42}$$

$$0 = \frac{1}{\beta-1}\left[\alpha\overline{\chi} - \beta - (1-\beta)(\overline{u}+\overline{x}) - (1-\alpha-\beta)\overline{M}\right] \tag{3.43}$$

$$0 = \frac{1}{\phi-1}\left[\frac{\phi(1-\alpha-\beta)}{\beta}\frac{\overline{u}}{\overline{x}}\overline{M} - \phi - (1-\phi)(\overline{u}+\overline{x})\right]. \tag{3.44}$$

Plugging equation (3.42) into (3.41) we get:

$$1 - \overline{u} - \overline{x} = \overline{M}; \tag{3.45}$$

substituting equation (3.45) into equation (3.43), we obtain:

$$\overline{\chi} = \frac{1}{\alpha} - \overline{M}. \tag{3.46}$$

Substituting equation (3.46) into (3.40) instead:

$$\overline{Z} = \frac{\sigma}{\alpha}\overline{M} + \frac{\rho}{\alpha}. \tag{3.47}$$

Then, from equations (3.41) and (3.47) we obtain the steady state value of M:

$$\overline{M} = \frac{1-\rho}{\sigma}. \tag{3.48}$$

Therefore those of χ and Z are:

$$\overline{\chi} = \frac{\sigma - \alpha(1-\rho)}{\alpha\sigma} \tag{3.49}$$

$$\overline{Z} = \frac{1}{\alpha}. \tag{3.50}$$

From equation (3.44), we have:

$$\overline{u} = \frac{\beta\phi(1-\rho) + \beta(\sigma+\rho-1)}{\phi(1-\alpha-\beta)(1-\rho)}\overline{x}, \tag{3.51}$$

which substituted in equation (3.45) yields:

$$\overline{x} = \frac{\phi(1-\alpha-\beta)(1-\rho)(\sigma+\rho-1)}{\sigma[\phi(1-\alpha)(1-\rho) + \beta(\sigma+\rho-1)]} \tag{3.52}$$

and therefore:

$$\overline{u} = \frac{(\sigma+\rho-1)[\beta\phi(1-\rho) + \beta(\sigma+\rho-1)]}{\sigma[\phi(1-\alpha)(1-\rho) + \beta(\sigma+\rho-1)]} \tag{3.53}$$

Notice that the steady state values of the five variables, given by expressions (3.48), (3.49), (3.50), (3.52) and (3.53), are positive if the following conditions hold:

$$1-\rho > 0 \rightarrow \rho < 1 \tag{3.54}$$

$$\left.\begin{array}{c} \sigma - \alpha(1-\rho) > 0 \\ \sigma + \rho - 1 > 0 \end{array}\right\} \rightarrow \sigma > (1-\rho) > \alpha(1-\rho) \tag{3.55}$$

B. Local Stability

We can study the stability of the steady state, by linearizing the system of differential equations. The Jacobian matrix, $J(\chi_t, Z_t, M_t, u_t, x_t)$, is:

$$\left[\begin{array}{cc} \frac{\alpha-\sigma}{\sigma}Z_t - \frac{\rho}{\sigma} + 2\chi_t & \frac{\alpha-\sigma}{\sigma}\chi_t \\ \frac{\alpha+\beta-1}{\beta-1}Z_t & \frac{\alpha+\beta-1}{\beta-1}\chi_t - \frac{\beta}{\beta-1} - 2(1-\alpha)Z_t - \frac{1-\alpha-\beta}{\beta-1}M_t \\ 0 & 0 \\ \frac{\alpha}{\beta-1}u_t & 0 \\ 0 & 0 \end{array}\right.$$

$$\left.\begin{array}{cc} 0 & 0 \\ -\frac{1-\alpha-\beta}{\beta-1}Z_t & 0 \\ \frac{2\phi^2(1-\alpha-\beta)}{\beta(\phi-1)}\frac{u_t}{x_t}M_t - 2\phi M_t - \frac{\phi}{\phi-1} & \frac{\phi^2(1-\alpha-\beta)}{\beta(\phi-1)}\frac{M_t^2}{x_t} \\ -\frac{1-\alpha-\beta}{\beta-1}u_t & \frac{1}{\beta-1}[\chi_t - \beta - (1-\beta)(2u_t + x_t) - (1-\alpha-\beta)M_t] \\ \frac{\phi(1-\alpha-\beta)}{\beta(\phi-1)}u_t & \frac{\phi(1-\alpha-\beta)}{\beta(\phi-1)}M_t + x_t \end{array}\right]$$

$$\begin{bmatrix} 0 \\ 0 \\ -\frac{\phi^2(1-\alpha-\beta)}{\beta(\phi-1)}\frac{u_t}{x_t^2}M_t^2 \\ u_t \\ \frac{1}{\phi-1}[-\phi-(1-\phi)(u_t+2x_t)] \end{bmatrix},$$

and, evaluated at the steady state, $J(\overline{\chi}, \overline{Z}, \overline{M}, \overline{u}, \overline{x})$, it can also be written as:

$$\begin{bmatrix} \overline{\chi} & \frac{\alpha-\sigma}{\sigma}\overline{\chi} & 0 & 0 & 0 \\ \frac{\alpha+\beta-1}{\beta-1}\overline{Z} & -(1-\alpha)\overline{Z} & -\frac{1-\alpha-\beta}{\beta-1}\overline{Z} & 0 & 0 \\ 0 & 0 & \frac{\phi^2(1-\alpha-\beta)}{\beta(\phi-1)}\frac{\overline{u}}{\overline{x}}\overline{M}-\phi\overline{M} & \frac{\phi^2(1-\alpha-\beta)}{\beta(\phi-1)}\frac{\overline{M}^2}{\overline{x}} & -\frac{\phi^2(1-\alpha-\beta)}{\beta(\phi-1)}\frac{\overline{u}}{\overline{x}^2}\overline{M}^2 \\ \frac{\alpha}{\beta-1}\overline{u} & 0 & -\frac{1-\alpha-\beta}{\beta-1}\overline{u} & \overline{u} & \overline{u} \\ 0 & 0 & \frac{\phi(1-\alpha-\beta)}{\beta(\phi-1)}\overline{u} & \frac{\phi(1-\alpha-\beta)}{\beta(\phi-1)}\overline{M}+\overline{x} & \overline{x}-\frac{\phi(1-\alpha-\beta)}{\beta(\phi-1)}\frac{\overline{u}}{\overline{x}}\overline{M} \end{bmatrix}$$

Remembering that the sum of the eigenvalues of a matrix is equal to the trace of the matrix, we can study the sign of the trace of the Jacobian evaluated at steady state: if it is positive, this would mean that at least one eigenvalue is positive. The trace of $J(\overline{\chi}, \overline{Z}, \overline{M}, \overline{u}, \overline{x})$ is:

$$tr(J(\overline{\chi}, \overline{Z}, \overline{M}, \overline{u}, \overline{x})) = \frac{3(\sigma+\rho-1)}{\sigma}, \tag{3.56}$$

which is positive if:

$$\sigma > (1-\rho). \tag{3.57}$$

Notice that the same condition ensures that the steady state values of the variables are positive. This means that, considering the standard empirical value of ρ, about 0.04, σ has to be higher than 0.96. This does not represent an unreasonable value for the inverse of the elasticity of substitution in consumption, as empirical works suggest (see for example, Mehra and Prescott (1985); Hall (1988)). If condition (3.57) holds, there exists at least one positive eigenvalue.

Moreover, it is possible to show that the determinant of the Jacobian, $\det(J(\overline{\chi}, \overline{Z}, \overline{M}, \overline{u}, \overline{x}))$, which equals the product of the eigenvalues, is:

$$\det(J(\overline{\chi}, \overline{Z}, \overline{M}, \overline{u}, \overline{x})) = \frac{(1-\alpha)(1-\alpha-\beta)\alpha\phi^2\overline{M}^2\overline{u}\overline{\chi}\overline{Z}(\overline{x}+\overline{u})}{(\phi-1)(\beta-1)\beta\overline{x}}. \tag{3.58}$$

Under the condition $\sigma > (1-\rho)$, which ensures all the variables are positive in steady state, it is easy to show that this determinant is positive. Since the product of the eigenvalues is positive and their sum is positive (under the condition $\sigma > 1-\rho$), if the eigenvalues are real numbers then the number of positive eigenvalues has to be odd, that is there can be one or three or five positive eigenvalues (and consequently, four or two or zero negative eigenvalues). Therefore an unstable transitional manifold of at least dimension one exists.

C. Transitional Dynamics in the Case $\beta = \phi$

The steady state values of the system (3.36) - (3.39) are given in Appendix A, by equations (3.48), (3.49), (3.50) and (3.51). These respectively represent $\overline{M}$, $\overline{\chi}$, $\overline{Z}$ and $\overline{\eta}$. In fact, equation (3.51) implies that:

$$\overline{\eta} \quad = \quad \frac{\beta(1-\rho)+\sigma+\rho-1}{(1-\alpha-\beta)(1-\rho)}. \tag{3.59}$$

These values are positive if $\sigma > 1-\rho$, as discussed in Appendix A. The stability properties of the steady state can be studied evaluating the Jacobian matrix at the steady state, $J(\overline{\chi}, \overline{Z}, \overline{M}, \overline{\eta})$:

$$\begin{bmatrix} \overline{\chi} & \frac{\alpha-\sigma}{\sigma}\overline{\chi} & 0 & 0 \\ -\frac{1-\alpha-\beta}{\beta-1}\overline{Z} & -(1-\alpha)\overline{Z} & -\frac{1-\alpha-\beta}{\beta-1}\overline{Z} & 0 \\ 0 & 0 & \frac{\beta(1-\alpha-\beta)}{(\beta-1)}\overline{\eta}\overline{M} - \beta\overline{M} & \frac{\beta(1-\alpha-\beta)}{(\beta-1)}\overline{M}^2 \\ \frac{\alpha}{\beta-1}\overline{\eta} & 0 & -\frac{1-\alpha-\beta}{\beta-1}(1+\overline{\eta})\overline{\eta} & -\frac{1-\alpha-\beta}{\beta-1}\overline{\eta}\overline{M} \end{bmatrix}$$

Remembering that the sum of the eigenvalues of a matrix is equal to the trace of the matrix, and their product is equal to its determinant, we can show the stability properties of the equilibrium. In fact, the trace of $J(\overline{\chi}, \overline{Z}, \overline{M}, \overline{\eta})$ results to be:

$$tr(J(\overline{\chi}, \overline{Z}, \overline{M}, \overline{\eta})) \quad = \quad \frac{\sigma - 2 + 2\rho}{\sigma}, \tag{3.60}$$

which is negative if the following condition holds:

$$\sigma < 2(1-\rho). \tag{3.61}$$

Notice that such a condition, jointly with that ensuring the positivity of the steady state values of the variables, implies:

$$1-\rho < \sigma < 2(1-\rho). \tag{3.62}$$

This means that, considering the standard empirical value of ρ, about 0.04, σ has to be higher than 0.96 and lower than 1.92. This does not represent a unreasonable range for the inverse of the elasticity of substitution in consumption.

Moreover, it is possible to show that the determinant corresponds to the following expression:

$$det(J(\overline{\chi}, \overline{Z}, \overline{M}, \overline{\eta})) \quad = \quad \frac{(1-\alpha)(1-\alpha-\beta)\alpha\beta\overline{M}^2\overline{\chi}\overline{\eta}\overline{Z}}{(\beta-1)^2}, \tag{3.63}$$

which is clearly positive.

These two results jointly mean that there exists at least one negative eigenvalue and the number of negative eigenvalues has to be even. Since the number of eigenvalues is four, and we need their product to be positive and their sum to be negative, they also mean that the Jacobian matrix is characterized by two or four negative eigenvalues and (two or zero) positive eigenvalues. Therefore, the stable manifold is multidimensional and the steady state equilibrium results to be stable. Notice that in the case both the positive and negative eigenvalues are two, the equilibrium is characterized by saddle-point stability.

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