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**A Mathematical Analysis of Conflicts in
Voting Systems**

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Gennaio 2013,
Michela CHESSA

Preface

My career as PhD student at University of Milan started 3 years ago, in January 2010. After a bachelor and a master thesis on Game Theory, I was really interested in continuing my research on this subject, when my PhD supervisor, Prof. Fragnelli, proposed me to start working on voting systems. I immediately found the topic actual and fascinating, particularly because of all the problems about the adopted electoral system and the political scenario my country, Italy, has in these years. It was a very convincing opportunity of working on Game Theory and on Game Practice in the same time, because of the concreteness of the problems I was going to deal with.

Some of the results included in this thesis have been taken from some articles I previously published: the work presented in Chapter 3 contains some results included in "Chessa M., Fragnelli V., *Embedding classical indices in the FP family*, Czech Economic Review, vol.5, 2011, pp. 289-305", while the work in Chapter 4 has been presented as invited talk at the workshop "Models of Collusion, Games and Decisions for Applications to Judging, Selling and Voting" in Monte Isola (BS), Italy, on June 2012 with the title "*The Bargaining Set for Sharing the Power*".

Chapter 5 contains part of analysis presented in "Chessa M., Fragnelli V., *A quantitative evaluation of veto power*, Operations Research and Decisions, Vol.21, 2011" and in "Chessa M., Fragnelli V., *Open Problems in Veto Theory*, To appear in: International Game Theory Review"; Chapter 6 presents some results included in "Chessa M., Fragnelli V., *A note on "Measurement of disproportionality in proportional representation systems"*, Mathematical and Computer Modelling, vol.55, 2012, pp. 1655-1660".

Chapter 7 contains the work presented as invited talk at the workshop "Models of Collusion, Games and Decisions for Applications to Judging, Selling and Voting" in Monte Isola (BS), Italy, on June 2012 with the title "*A Generating Functions Approach for Computing Holler Index Efficiently*".

In these years I investigated also other branches in which Game Theory could be useful for; this is the reason why I dealt also with the work on the problem of how to enhance the research on rare diseases, guaranteeing that every patient, affected by a rare or by an ordinary disease, could be treated with fairness. The two works I published on the topic, "Chessa M., Gagliardo S., *Where is the Profit in Rare Diseases Research?*, Operations research for patients - Centered health care delivery, 2010, FrancoAngeli, pp. 51-57" and "Chessa M., Fragnelli V., Gagliardo S., *A coordination model for enhancing research on rare diseases*, Tanfani E. and Testi A. (Eds), Advanced Decision Making Methods Applied to Health Care, International Series in Operations Research & Management Science, Vol.173, 2012, pp. 51-66", even if still studied through a game theoretical point of view, are not included in this thesis, as I preferred to select a common and central topic for this dissertation: the voting games.

October 2012,
Michela CHESSA

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Introduction

Many different kinds of voting situations have been studied by social choice theorists and by game theorists, but one can say that the theory is clearly divided into two main branches. The first branch is the evaluation of the voting system, i.e. of the Parliament resulting after an election and of the power share inside it (to which we dedicate the first part of this thesis). The second branch is the evaluation of the criteria for the assessment of the voters' preferences, starting from their preferences profile (to which we dedicate the second part). The third part of the thesis is dedicated to the definition of a new algorithm to deal with a few computational problems we encountered during the work on the previous parts. In this thesis we deal with all these problems using the instruments provided by Game Theory.

The modern Game Theory as interactive decision-making has been originated in 1944, with the publication of the book, *Theory of Games and Economic Behavior* by von Neumann and Morgenstern [95]. Previous works, in fact, were fragmentary and did not attract much attention, even if they provided some general ideas which found a concrete organization only with this volume. It was somehow a recompilation of previous work, but this book provided some new important developments, like the introduction of information sets and the formal definition of strategy. The authors treated with coalitions dealing with cooperative games, and introducing the characteristic form, gave a formal definition to the concept of imputation; however, these are only some of the wide range of concepts the authors presented. Game Theory as a structured science is quite "young", having been developed 70 years ago, more or less. We find an excellent introduction to Game Theory and its history in the preliminary part of *Essays on Cooperative Games* edited by Gambarelli [39].

The year 1944 is also very important because of the appearance of another important tool which changed our lives in the ensuing 70 years. This tool was the first computer introduced to the general public, after some precedents in which it had remained a "military secret". Also Game Theory, at the beginning, was used for military purposes and this is one of the common point this science has with Computer Science. For example, the first studies in linear programming were developed in order to solve problems in two-person Game Theory. Later, games required the application of general mathematics, and this subject became a more mathematical science, using and requiring concepts coming from general topology, probability, theory of sets, and many other fields. But Game Theory is nowadays known as a subject which gave some of the major contributions to the field of economics.

John Harsanyi, John Nash, Reinhard Selten, Robert Aumann, Thomas Schelling,

Leonid Hurwicz, Eric Maskin, Roger Myerson, and lately, Alvin Roth and Lloyd Shapley were awarded a Nobel Prize in Economic Sciences for their results on economic studies using Game Theory.

According to Osborne and Rubinstein [78], who wrote one of the most important surveys on the subject, *Game Theory is a bag of analytical tools designed to help us understand the phenomena that we observe when decision-makers interact. The basic assumptions that underlie the theory are that decision-makers pursue well-defined exogenous objectives (they are “rational”) and take into account their knowledge or expectations of other decision-makers’ behavior (they “reason strategically”).*

In Game Theory, real-life situations are represented through abstract models and this allows one to study a wide range of phenomena. A *player* is the basic entity of any such theoretical model. He or she may be interpreted as an individual or a group of individuals making a decision. There are two types of models: those in which the primitives are the sets of possible actions of individual players, and those in which they are the sets of possible joint actions of groups of players. Models of the first type are referred to as *noncooperative* (and historically, most research has been devoted to these games), while those of the second type as *cooperative*. The most commonly used solution concept for noncooperative games is the *Nash equilibrium*, while the most famous one for cooperative games is the *Shapley value*. We have three ways of representing a game, *strategic*, *extensive* and *characteristic* form. Characteristic form can be adopted only to represent cooperative situations.

Another important distinction in games is that in some models the participants are fully informed about each others’ moves and we call these games with *perfect information*. When the players may be imperfectly informed we speak of these games with *imperfect information*.

In this thesis we study the model of a cooperative game (even if we will refer occasionally to a noncooperative model). Cooperative games (as suggested by the name) allow us to represent situations in which the players can cooperate, establishing binding agreements among themselves and forming *coalitions* in order to coordinate their strategies and share their joint payoff. Although actions are taken by coalitions, the theory is based on the individuals’ preferences over the set of all possible outcomes. A particular class of cooperative games is given by *simple games*. Simple games represent conflicts in which the objective is winning and one of the main applications of this model is given by *voting games*. Then, after Computer Science and Economy in the last years’ Game Theory, it is starting to serve other important disciplines, in this case, Political Science.

Voting is a very common way of resolving disagreements, determining common opinions, choosing public policies and finding other solutions to the problem of aggregating a set of individual opinions in a democratic society, transforming many preferences into one. The most typical voting system may be represented by a simple game: in order to change policies (the legislative *status quo*) a certain number of

individual or collective actors have to vote in order to promote the outcome that they prefer, voting in favor (Y) or against the proposal (N). In the classical model each member is called *party* and party has a *voting weight* (number of votes, shares etc.) and a voting rule is defined by a minimal number of weights required for passing a proposal, called *quota*. Given a voting rule, voting weights provide members with voting power. Voting power means the ability to influence the outcome of voting sessions and it is quantified using power indices. Several power indices were proposed in order to account for different features of the possible situations. For instance the indices may emphasize the possibility to form different majorities, such as the Banzhaf index [12], or they may emphasize the importance of the ordering in the majority formation process. This is the case of the one which is probably the most famous power index: the Shapley-Shubik index [88]. Other indices may take into account the role played by the majorities with minimal number of agents, like the Deegan-Packel index [27]. Also, the Public Good index [48], which has been defined by Holler starting from the idea that the coalition value is a public good, reduces to taking into account only minimal winning coalitions in its version for voting games.

In dealing with a voting game, anyway, it is important to take into account that the ideological position of each party does not allow every coalition forming with the same probability, even if it is winning. In order to include this kind of information other indices have been defined, introducing the concept of *a priori unions* (Owen [81]) for accounting the existing agreements between parties or using a graph (Myerson [74]) for representing possible communications among parties. Another common way to represent a political scenario is via a *left-right axis* where the parties are ordered according to their ideological position. Assuming that the negotiations take place uniquely between adjacent parties, the feasible coalitions include only contiguous parties and on this idea the *FP family* was defined by Fragnelli et al. [36]. In this family only contiguous winning coalitions have a non zero probability to form. Another parameter is specified, such as the power share among the members of each coalition. Defining these parameters in a proper way we are able to deal with different situations and to better describe different real Parliaments. In Chapter 3, we analyze the way to select the appropriate parameters with the aim of representing classical power indices from a FP family point of view. We start by relaxing the hypothesis of contiguity, assigning a non zero probability to form to every winning coalition. Then we reduce the relevance of noncontiguous coalitions, defining a sequence of indices that converges to a modified version of the classical indices. When this method is applied to the Italian lower chamber of 2008, we give a concrete idea of the improvement in the evaluation of the power share we can obtain modifying classical power indices through the FP indices point of view. Finally, we extend our approach to situations in which the parties are not necessarily ordered according to the left-right axis, expressing their relations by a graph, following the idea of Myerson (but showing how the two approaches are completely different). The idea of defining a sequence of indices, moreover, suggests to us the possibility of considering an intermediate one, in case deleting all the noncontiguous coalitions looks too strong and we just want to

consider them less probable.

Chapter 4 of this thesis is devoted to another problem in the evaluation of the power share of a Parliament. The solutions are usually carried out following a static approach, while a dynamic model in which parties may blackmail each other in order to increase their power at expenses of the others would be, in our opinion, more close to real-world situations. This model may be solved referring to the *bargaining set* [10]. Verifying if a vector belongs to the bargaining set may be used to check the robustness of a power sharing in order to avoid blackmailing behaviors. One of the most negative features of this solution is its computational complexity, because we need to determine a sequence of inequalities that represent the conditions under which each player may raise an objection against each other player forming a different winning coalition, and another sequence of inequalities that represents the conditions for the existence of suitable counterobjections. This system of inequalities defines the bargaining set. Despite the computational complexity, we manage to show a real-world example, the German *Bundestag*, in which we are able to calculate the bargaining set due to the particular structure of its winning coalitions. Moreover, remembering the statement that in a voting situation just a few coalitions are feasible, we observe that the computation can become much easier while considering only these coalitions, removing the other possible configurations.

After dealing with the problem of representing in a proper way the power of the parties in a Parliament, in Chapter 5 we deal with its natural counterpart, the *veto power*. In Game Theory the concept of veto is mainly associated with the concept of *veto player*. According to the classical definition, in a simple game a veto player is a player whose approval is fundamental to pass a proposal. The most popular example of veto is given by the United Nations Security Council (UNSC) where the five permanent countries are able alone to block a proposal. The notion of veto player may be generalized, for several players, to the notion of *blocking coalition*, which is a coalition such that the other players outside the coalition are not able to pass a proposal. The power of veto represents a central topic in politics, and then it is natural to ask, "how is it evaluated?" This question brought in the last years, an increasing number of papers and surveys on the topic, but the attention to veto power indices (power indices which are used to evaluate the power to block instead of the power to win) in the literature is still less than that devoted to power indices. Two questions arise at first: are veto power and power analogous concepts? May we evaluate them with the same instruments?

Our idea is that some different features have to be considered in order to define an index suitable for analyzing the power of veto. A party, for example, can be able alone to block a proposal voting against it, but it may not have the possibility to make an opposite law being approved without the support of other parties. This happens, for example, in the already cited example of the UNSC, where a permanent member has full veto power, but not full power according to the classical indices. Moreover, the concepts of a priori unions and/or of connected coalitions, which have been introduced

to better represent the relations between parties, are no longer relevant while speaking about the power of a party which is against the approval of a proposal. In fact, to block a proposal it is not necessary anymore to have a common ideological position. Two parties very far from each other can be both because of opposite reasons, decide to vote against a law, even if this does not mean that they would agree in approving a common different proposal. We define a new veto power index according to which a veto player has veto power equal to one, while each other player has a fraction according to his possibility to block a given proposal. Such an index is quantitative. In fact we observe that it is no longer necessary that the power of the agents sum up to a given fixed number, which is normally assumed to be equal to 1, as one, two, or all the agents of a voting procedure may have full power to block a proposal. This index that we call the *loose protectionism index of player i* coincides with the expected payoff at the Bayesian equilibrium of a suitable noncooperative *Bayesian game* ([45], [46] and [47]), which catches the noncooperative point of view of a decision-making mechanism.

If Chapters 3, 4 and 5 are devoted to the evaluation of the power share inside a Parliament, then in Chapter 6 we present the second part of this thesis, the evaluation of the criteria for the assessment of the voters' preferences.

In a representative democracy different committees, for example Parliaments, are elected to make decisions on the behalf of the voters. The basis of a democracy lies then in its electoral system. An *electoral system* is a set of rules and norms (i.e. a mechanism) that starting from the preferences of the voting body produces a Parliament. The electoral system has a great impact on the functioning of democracy so it is fundamental to select a good one. The choice of the "best" Parliament may be affected by a lot of facets of the political process, but two of them may be considered more relevant than the others. The first being *representativeness*, that depends on the efficiency of the system in representing electors' preferences; and the second, *governability* that measures the effect on the efficiency of the resulting government. These two dimensions may be evaluated through the assessment of plausible numerical indicators. In the 20th century, the wide appearance of proportional systems was one of the reasons for the development of various methods for measuring the quality of electoral systems, mainly due to evaluate the representativeness, often called *proportionality* (or, evaluating the lack of proportionality, called *disproportionality*), of a Parliament.

Some scholars proved the impossibility of constructing a proportional system that allocates seats in an exactly proportional way. The problems regarding proportional systems are mainly due to two reasons. The first is that the so-called proportional systems often introduce some modifications in order to enhance other good features, *in primis* the governability, excluding the smallest parties, via a threshold, and/or strengthening the largest parties, via a majority prize. The second is that even with a perfect proportional system it is necessary to assign an integer number of seats through some rounding methods. The impossibility of creating an ideal proportional

electoral system forced researchers to search for quantitative indices that would reflect the degree to which the system satisfies certain conditions. Such indices contain quantitative information and allow researchers to conduct empirical research and compare various electoral systems. Then, in Chapter 6, starting from the work of Karpov [55], who studied the properties of nineteen disproportionality indices applying them to four electoral sessions in Russia, we introduce the indices proposed by Ortona [34], Fragnelli [32] and Gambarelli and Biella [40]. In particular, the last two indices account for the issue of power for measuring the disproportionality. In our mind, the power, such as the influence of each party on the decision of passing a law, should play a more relevant role in evaluating the characteristics of a Parliament, and, as suggested by Fragnelli et al. [34], the notion of power has a lot to do with the choice of the electoral system, both with governability and with representativeness. Doing simulations, they use a power-based index to define governability, similarly, in this thesis we propose to analyze what happens in the evaluation of representativeness while using power-based indices. It is possible that an apparently unfair distribution of seats with respect to votes may provide the parties the same power of their voters, so we can conclude that the voting body is well represented by the Parliament if the representativeness (the disproportionality) is measured by an index accounting the issue of power. In this case, the classical indices listed by Karpov may assign a high level of disproportionality to the system.

As the idea of assigning the seats by only looking at the power share, one can look hard to be accepted, as it can sometimes provide counterintuitive results. In the last part we suggest the idea of adopting the issue of power in order to define a rounding method, which minimizes the differences of the power share.

With the aim of computing the power on the vote share after the election of the Russian Parliament (State Duma), in order to evaluate the representativeness using a power-based index, we had the necessity to solve another problem: to find an efficient way to evaluate the Public Good index, which was introduced by Holler in 1982 [48].

In the third and last part of this thesis in Chapter 7, we deal with another important problem; the computation of power indices can be very hard, especially when the number of players is high.

The problem of calculating power indices in a reasonable amount of time is of great interest since the first power indices have been defined. Some classical indices require the enumeration of all coalitions and this becomes computationally complex as soon as the number of players increases. This can happen also for those indices, e.g. the Public Good index, which takes into account only minimal winning coalitions. Also, selecting the minimal winning ones may require in the worst case running the enumeration of all of them.

The big number of players did not allow us to make the computation through an ordinary algorithm. To evaluate the power through the Holler index is very interesting as it considers only minimal winning coalitions and it proposes, then, a different approach compared, for example to the Shapley-Shubik index. Differently from the

classical indices, moreover, it does not have the property of monotonicity. In 1962, Mann and Shapley [63] proposed an exact calculation of the Shapley-Shubik index, following an idea due to David G. Cantor, to evaluate this index for large voting games, i.e. games with a high number of players. This idea, together with ramifications, has made it possible to calculate the exact power in a reasonable amount of time and it has been adopted, in the following years, to exactly calculate many other indices. We use a generating functions approach in order to compute the Public Good index, following the previous works but facing many problems because of the minimality of the coalitions, which are taken into account by this index. As it is not possible to do this computation using a generating function which is similar to the ones used to compute, for example, the Shapley-Shubik, we introduce some recursive generating functions, using a noncommutative operator, which allows us computing the Public Good index exactly and efficiently. We then provide an analysis of the computational complexity of the proposed algorithm.

The first part of this thesis, including Chapters 3, 4 and 5 is dedicated to voting systems once they have already been formed. The second part, including Chapter 6, deals with the evaluation of the criteria for assessment of the voters' preferences based on proportionality. The last part, including Chapter 7, shows a computational algorithm to evaluate a specific power index, the Public Good index, for which an exact and efficient program had not been implemented yet.

Preliminaries

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2.1 TU Games

This section is devoted to illustrate some preliminaries in Game Theory, in particular in the field of cooperative games. We refer to Osborne and Rubinstein [78] for a more accurate illustration of the following concepts.

A *cooperative game with transferable utility*, or *TU game*, consists of a pair (N, v) where

- i $N = \{1, \dots, n\}$ denotes the *set of players*;
- ii $v : 2^N \rightarrow \mathbb{R}$ is the *characteristic function* that associates with every nonempty subset S of N , a *coalition*, a real number $v(S)$, the *worth* of S , representing the total payoff to this coalition of players when they cooperate, whatever the remaining players do. By convention, we assume $v(\emptyset) = 0$.

We denote \mathcal{GN} the class of games with players set N and we call N , the coalition containing all the players, the *grand coalition*.

The *subgame* of (N, v) with respect to coalition $S \subset N$, $S \neq \emptyset$ is defined as the TU game (S, v^S) with $v^S(T) = v(T)$ for all $T \subseteq S$.

A game in \mathcal{GN} is called

- *monotonic*, if $S \subseteq T \Rightarrow v(S) \leq v(T)$;
- *superadditive*, if $v(S) + v(T) \leq v(S \cup T)$ for each $S, T \subseteq N$ s.t. $S \cap T = \emptyset$;
- *cohesive*, if for each partition $\{S_1, \dots, S_k\}$ of N , $\sum_{i=1 \dots k} v(S_i) \leq v(N)$;

- *convex*, if $v(S) + v(T) \leq v(S \cup T) + v(S \cap T)$ for each $S, T \subseteq N$.

A convex game is superadditive and a superadditive game is cohesive. Obviously, a convex game is cohesive.

A *simple* game is a monotonic game with characteristic function such that $v : 2^N \rightarrow \{0, 1\}$ and $v(N) = 1$. When a simple game is adopted to describe a voting situation, we can refer to a player as a *party*. In a simple game a coalition S is *winning* if $v(S) = 1$, *losing* if $v(S) = 0$. Given a winning coalition S , if $v(T) = 0$ for each $T \subset S$, we say that S is a *minimal winning* coalition, if there exists at least one player $i \in S$ s.t. $v(S \setminus \{i\}) = 0$, S is a *quasi-minimal winning* coalition. We denote with W the set of all winning coalitions, with W^m the set of all minimal winning coalitions and with W^{qm} the set of all quasi-minimal winning coalitions. We use indifferently (N, v) , (N, W) or (N, W^m) to denote a simple TU game, as listing the set of winning or of minimal winning coalitions is sufficient to uniquely define the characteristic function. A *veto player* is a player i that belongs to all winning coalitions, i.e. for each $S \in W$, then $i \in S$. A *dictator* is a player that is winning without any support, i.e. $\{i\} \in W$. Given a coalition $S \in W$, a player $i \in S$ is *critical* for S if $S \setminus \{i\} \notin W$. The quantity $v(S) - v(S \setminus \{i\})$ is called the *marginal contribution* of player i w.r.t. S .

Given a game (N, W) , the set of coalitions 2^N splits into four classes (from Carreras [22]), namely:

- D (*decisive winning*): $S \in W$ such that $N \setminus S \notin W$;
- C (*conflictive winning*): $S \in W$ such that $N \setminus S \in W$;
- Q (*blocking*): $S \notin W$ such that $N \setminus S \notin W$;
- P (*strictly losing*): $S \notin W$ such that $N \setminus S \in W$.

Thus, $W = D \cup C$. The family Q is called the *blocking family*; the game is *strong* if $Q = \emptyset$ and *weak* otherwise. The game is *proper* if $C = \emptyset$ and *improper* otherwise. When a game is proper and strong, it is called *decisive*.

A particular class of simple games is represented by the weighted majority games. Given the set of players N we can consider the *weighted majority situation* $[q; w_1, \dots, w_n]$, where q is the *majority quota* and w_1, \dots, w_n are the *weights* with $0 < q \leq w(N) = \sum_{i=1}^n w_i$. For each $S \subseteq N$ we denote $w(S) = \sum_{i \in S} w_i$. The corresponding *weighted majority game* (N, v) has characteristic function defined as

$$v(S) = \begin{cases} 1 & \text{if } w(S) \geq q \\ 0 & \text{otherwise} \end{cases} \quad \forall S \subseteq N.$$

Usually we ask that the game is proper; for this aim it is sufficient to choose $q > \frac{1}{2}w(N)$. Note that a weighted majority situation always corresponds to a simple game, while the opposite is not true.

The *dual game* of (N, W) is the game (N, W^*) where $W^* = \{S \subseteq N : N \setminus S \notin W\}$. Notice that $W^* = D \cup Q$, $D^* = D$, $C^* = Q$, $Q^* = C$ and $P^* = P$.

2.2 Solutions of TU Games

We introduce now the basic concepts of solutions, which are divided between set valued and pointwise solutions. They are usually defined only for cohesive games, but we will give the definition in general for every game, as, particularly in voting situations, it is sometimes interesting to consider games which do not respect this property, like, for example, the dual of a simple proper game.

2.2.1 Set Valued Solutions

Given a game (N, v) , an *allocation* is a n -dimensional vector $(x_1, \dots, x_n) \in \mathbb{R}^N$ assigning to player $i \in N$ the amount x_i . For each $S \subseteq N$, we denote $x(S) = \sum_{i \in S} x_i$. The *imputation set* is defined by

$$I(v) = \{x \in \mathbb{R}^N \mid x(N) = v(N) \text{ and } x_i \geq v(i) \forall i \in N\};$$

i.e. by those allocations which are *efficient* ($x(N) = v(N)$) and *individually rational* ($x_i \geq v(i) \forall i \in N$).

The *core* is the set of imputations which are also *coalitionally rational*, i.e.

$$C(v) = \{x \in I(v) \mid x(S) \geq v(S) \forall S \subseteq N\};$$

thus the core is the set of outcomes satisfying a system of weak linear inequalities and hence is closed and convex. A core element is stable in the sense that if such a vector is proposed as allocation for the grand coalition, no coalition will have an incentive to split off and cooperate on its own. The idea behind the core is analogous to that behind a Nash equilibrium of a noncooperative game: an outcome is stable if no deviation is profitable. For the Nash equilibrium the possible deviation is for a single player, while in the core we speak about deviations of groups of players.

The imputation set and the core can be empty. A game is called *balanced* if its core is nonempty and *totally balanced* if the core of each of its subgames is nonempty. When the game is simple we know that it is balanced if and only if there is at least one veto player.

2.2.2 Solutions and Power Indices

A *solution* is a function $\psi : \mathcal{GN} \rightarrow \mathbb{R}^N$ that assigns an allocation $\psi(v)$ to every TU game belonging to \mathcal{GN} . Quite often a solution is required to be efficient, but we have some examples of solutions which do not respect this property.

For simple games, and in particular for weighted majority games, an efficient solution is often called a *power index*, as each component x_i may be interpreted as the power assigned to player $i \in N$.

Some concepts of solution take into account the marginal contribution $m_i(S) = v(S) - v(S \setminus \{i\})$ that player i provides to any coalition S . Between them the *Shapley*

value [86] is an efficient solution defined as the average of the marginal contributions of player i w.r.t. all the possible orderings

$$\phi_i(v) = \sum_{S \subseteq N, S \ni i} \frac{(|S| - 1)!(n - |S|)!}{n!} m_i(S) \quad \forall i \in N; \quad (2.1)$$

where $n = |N|$. Its equivalent form for simple games is the *Shapley-Shubik index* [88], defined as

$$\phi_i(v) = \sum_{S \in W: S \setminus \{i\} \notin W} \frac{(|S| - 1)!(n - |S|)!}{n!} \quad \forall i \in N. \quad (2.2)$$

If the game is superadditive, the Shapley value is an imputation, if the game is convex, it belongs to the core.

The *Banzhaf value* [12] considers all the marginal contributions of a player to all possible coalitions, independently from the order of the players

$$\beta_i(v) = \frac{1}{2^{n-1}} \sum_{S \subseteq N, S \ni i} m_i(S), \quad \forall i \in N. \quad (2.3)$$

It is often called the *Penrose-Banzhaf value* or the *Banzhaf-Coleman value* as in a previous work in 1946 Penrose [84] and in a later one in 1971 Coleman [26] defined a value which is formally identical. For simplicity, we will refer to it only referring to the name of Banzhaf. This value is not efficient, but its normalized version for simple games, the *normalized Banzhaf index*, is and it can be written as

$$\beta_i(v) = \frac{\beta_i^*(v)}{\sum_{j \in N} \beta_j^*(v)}, \quad \forall i \in N, \quad (2.4)$$

where

$$\beta_i^*(v) = \sum_{S \in W: S \setminus \{i\} \notin W} 1, \quad \forall i \in N.$$

We list now three more power indices not based on the marginal contributions. The formula of the *Deegan-Packel index* [27] is based only on minimal winning coalitions; the power is firstly equally divided among them and then the power of each is equally divided among its members. We denote $W_i^m = \{S \in W^m : i \in S\}$, the index is defined as

$$\delta_i(v) = \sum_{S \in W_i^m} \frac{1}{|W^m|} \frac{1}{|S|}, \quad \forall i \in N. \quad (2.5)$$

The *Johnston index* [53], instead, considers only the quasi-minimal winning coalitions; the power is firstly equally divided among them and then the power of each is equally divided among its critical players. Defined $W_i^{qm} = \{S \in W^{qm} : S \setminus \{i\} \notin W\}$ and $S^c = \{i \in S : S \in W, S \setminus \{i\} \notin W\}$

$$\gamma_i(v) = \sum_{S \in W_i^{qm}} \frac{1}{|W^{qm}|} \frac{1}{|S^c|}, \quad \forall i \in N. \quad (2.6)$$

The *Public Good index* has been defined by Holler [48] and it counts how many minimal winning coalitions a player i belongs to. It is defined as

$$H_i(v) = \frac{|W_i^m|}{\sum_{j \in N} |W_j^m|}, \quad \forall i \in N. \quad (2.7)$$

As it is important in the work of the following chapters, we remark that the Shapley value, the Banzhaf value and the Johnston index are *monotonic* with respect to the weights when we are evaluating the power in a weighted majority game, i.e. given $i, j \in N$, $\phi_i(v) = \phi_j(v)$ when $w_i = w_j$ and $\phi_i(v) \geq \phi_j(v)$ when $w_i > w_j$. This is not true for the Deegan-Packel and the Public Good indices.

Power Indices and the Issue of Contiguity/Connectedness

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3.1 Introduction

In this chapter and in the two following ones we consider a real-world voting situation in which a set of agents has to decide in favour of an issue or against it; we assume as a model a simple game. Some subsets of agents are able to reach an agreement that makes the issue approved, while some other subsets may at most decide against it, but they are not able to pass a counterproposal. We may think to the parties of a Parliament that have to pass a law, or to a council whose members have to take a decision. A relevant role is played by those agents that mostly can influence the final outcome.

Power indices are a relevant tool to measure the influence of each member on decisions and to allow evaluating the role played by each agent in the process ending with the formation of a majority. In the last decade they have received increasing attention in political science, mostly because of the necessity of both to study the voting power among EU member states and to analyze the effects of institutional reforms. However, some features of power indices lead some scholars to call their exactness in question. Several power indices were proposed in order to account different

features of the possible situations; we have already listed some of them in Chapter 2, like the Shapley-Shubik [88], the normalized Banzhaf [12], the Deegan-Packel [27], the Johnston [53] and the Public Good [48] indices. Many other indices have been defined in the following years in order to better represent different situations.

The main critics to the traditional indices are due to the fact that there is no restriction on the relations between parties and they only aim at representing the gains that any group of agents obtains from cooperation. In many real situations, however, before starting the analysis we have information about the behavior of the players and we know that only partial cooperation may occur. Several models have been used to represent these situations, including other information in the game: Owen [81] introduced the concept of *a priori unions* for accounting the existing agreements between parties, idea which has been extended by Winter [96] who required that the different unions may join only according to a predefined scheme, the *levels structure*; Myerson [74] proposed to use a graph for representing possible connections among parties that are the basis for negotiations; Kalai and Samet [54] allowed the possibility of different roles for the players assigning to each of them a suitable weight.

More recently, the *FP* family of power indices has been introduced in Fragnelli et al. [36]. It focuses on the contiguity of the parties of a Parliament ordered on a left-right axis and the basic idea is that a coalition may form after a negotiation that includes all the intermediate parties. A simple example of the philosophy behind this proposal may be given by a party that is never able to change the decision taken by any other coalition (the so-called null player) but may play an important role due to its intermediate position that makes it necessary for a positive conclusion of the negotiation. The indices in the *FP* family depend on the setting of some parameters, namely the set of majorities with contiguous parties that are relevant in the situation at hand, their probabilities to form and the relevance that each member has in each majority.

Taking into account a restricted set of feasible coalitions, the indices of these family allow evaluating the voting power of each party starting from the possible negotiation processes. The resulting power will be influenced by the ideological connections among the parties in a Parliament and not only by the seat share, as it happens with the classical indices.

Exploiting the degrees of freedom of the *FP* family, in this chapter we select the parameters in order to embed the classical power indices by Shapley-Shubik, Banzhaf (in the normalized version), Deegan-Packel and Holler, in the new family. The motivation is that the modified indices may profit of some features of the classical ones, adding the relevance assigned to intermediate parties in the new family. Clearly, the characteristic of assigning a null power to a null player, that is satisfied by the four classical indices we mentioned, still holds after the embedding; nevertheless some parties that are relevant in the negotiation process increase their power, while less important parties decrease their own. The idea of the *FP* indices remembers the structure proposed by Myerson, but the two approaches are very different and in this chapter we propose a comparison, in order to underline similarities and deep differences.

The chapter is structured as follow: in Section 3.2 we recall the two main contributions to the philosophy of adding more information to the model of a voting situation, the Myerson value and the Owen value. In Section 3.3 we present the notion of contiguity and the *FP* family of power indices; then, in Section 3.4, we present the procedure to embed the classical indices into this family. The real-world example of the Italian “Camera dei Deputati” is presented in Section 3.5; two possible extensions to the model are finally shown in Section 3.6 and in Section 3.7.

3.2 Graph Structure and A Priori Unions

3.2.1 Myerson’s Graph Structure

The way to describe cooperation structures adopted by Myerson [74] in 1977 is by the concept of a *cooperation graph*, with which it is possible to model a much greater variety of cooperation structures than we could with only the concept of coalitions.

We consider a nonoriented graph whose vertices are the parties and whose edges represent the willingness of the parties corresponding to the vertices to reach an agreement taking into account their ideological positions. We denote an edge between parties k and h by $k : h$. Let $g^N = \{k : h \mid k \in N, h \in N, k \neq h\}$ be the complete graph and let $G^N = \{g \mid g \subseteq g^N\}$ be the set of all graphs on N , each one representing a possible political situation involving the parties at hand.

Given a subset of parties $S \subseteq N$ and a graph $g \in G^N$, we say that $k, h \in S$ are *connected in S by g* if there exists a path in g from k to h , i.e. a sequence (k^0, \dots, k^i) such that $k^0 = k$, $k^i = h$ and $k^{j-1} : k^j \in g$ and $k^j \in S$ for $j = 1, \dots, i$.

A coalition $S \subseteq N$ is *connected by g* if all pairs $k, h \in S$ are connected in S by g . Given $g \in G^N$ and $S \subseteq N$, there is a unique partition of S which groups players together iff they are connected in S by g , and we will denote this partition by S/g (read *S divided by g*). That is

$$S/g = \{\{k \mid k \text{ and } h \text{ are connected in } S \text{ by } g\} \mid h \in S\}$$

We can interpret S/g as the collection of smaller coalitions into which S would break up, if players could only coordinate along links in g .

For any game $v \in \mathcal{GN}$ and any graph $g \in G^N$, define $v/g \in \mathcal{GN}$ so that

$$(v/g)(S) = \sum_{T \subseteq S/g} v(T) \quad \forall S \subseteq N.$$

So v/g can be interpreted as the TU game which would result if we altered the situation represented by v , by requiring that players can only coordinate along links in g . Notice that $v/g^N = v$.

For any game $v \in \mathcal{GN}$ and any graph $g \in G^N$, the *Myerson value* is defined as

$$M(v, g) = \phi(v/g) \tag{3.1}$$

where ϕ denotes the Shapley value.

The idea behind the Myerson value is then to reduce the game, according to a cooperation graph, and to evaluate the Shapley value of the reduced game. When considering a simple game, the reduced game is a game in which the losing coalitions of the original game remain losing, while the winning coalitions remain winning if they contain a connected component which is winning. We show in Example 3.2.1 how the game can change introducing a connection graph and how the Myerson value can give a totally different share of the power from the Shapley value.

Example 3.2.1. Consider the weighted majority situation $[6; 4, 2, 2, 2]$ and the associated weighted majority game. The set of winning coalition is $W = \{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}, \{1, 2, 3, 4\}\}$ and the Shapley value is given by the vector $\phi(v) = (\frac{3}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6})$.

We suppose now that the possible connections between the players are the ones given by graph g in Figure 3.1. The winning coalitions for the reduced game v/g are

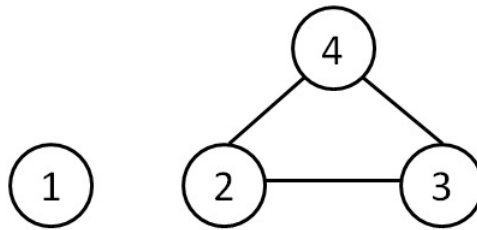


Figure 3.1: Connection graph g

only $\{2, 3, 4\}$ and $\{1, 2, 3, 4\}$, as for example $(v/g)(\{1, 2\}) = v(\{1\}) + v(\{2\}) = 0$ or $(v/g)(\{1, 2, 4\}) = v(\{1\}) + v(\{2, 4\}) = 0$. The resulting Myerson value of the game is given by the vector $M(v, g) = (0, \frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. We simply notice that player 1, who had the highest power according to the Shapley value, has now power zero, as he is not connected to any other player and he is not able to reach the majority quota alone. This simple example shows also the nonmonotonicity of the Myerson value (player 1 is the one with the highest weight), which is a property which is always satisfied by the Shapley value.

3.2.2 Owen’s A Priory Unions

With the same aim of including new information in the description of a game, but adopting a different approach, in the same year, 1977, Owen [81] defined a value based on the idea of *a priori unions*. It is a modification of the Shapley value so as to take into account the possibility that some players, because of personal affinities, may be more likely to act together than others.

Given the set of players N , we denote by \mathcal{P}^N the set of all partitions of N . An element $P \in \mathcal{P}^N$ is called a *coalition structure* and it describes the *a priori unions* on

N . We denote as (N, v, P) the game (N, v) with the coalition structure P . For any game (N, v, P) with $P = \{P_1, \dots, P_u\}$, the *quotient game* is the simple game (U, w) , where the set of players $U = \{1, \dots, u\}$ is given by the unions and a set $R \subseteq U$ has value $w(R) = v(\cup_{k \in R} P_k)$; in particular, in a simple game R is a winning coalition in (U, w) if $\cup_{k \in R} P_k$ is a winning coalition in (N, v) . The quotient game represents the game played among the coalitions of the partition. The Shapley value of the game (U, w) assigns a part of the total value to each of the coalitions P_k . The Owen value allocates the value assigned to the coalitions among its members according to the philosophy of the Shapley value. Hence, the share of the value that each member of the coalition gets is determined using marginal contributions. Formally, the *Owen value* of the game (N, v, P) is given by the following formula. Take $i \in N$ and let P_k be the unique coalition of the partition P to which i belongs, i.e. $i \in P_k \in P$. Then

$$\psi_i(v, P) = \sum_{Q \subseteq P, Q \ni P_k} \sum_{S \subseteq P_k, S \ni i} \frac{(|S| - 1)! (|P_k| - |S|)! (|Q| - 1)! (|P| - |Q|)!}{|P_k|! |P|!} \cdot (v(\cup_{P_h \in Q} P_h \cup S) - v((\cup_{P_h \in Q} P_h \cup S) \setminus \{i\})) \quad \forall i \in N. \quad (3.2)$$

3.3 The FP Family

The indices of Myerson and Owen deal with the problem of taking into account the affinities between the players; in simple games, and in particular in voting games, this allows giving a more realistic representation of the power share. Later, different authors dealt with the problem of describing more sophisticated models; we can mention, between the others, Vázquez-Brage et al. [94] and Alonso-Mejide et al. [4], who studied games with both the generalization jointly, i.e. games with a priori unions and graph restricted communication.

The work of this chapter follows the idea of Fragnelli et al. [36], who introduced the *FP* family of indices to evaluate the power in a voting situation.

We consider a weighted majority situation $[q; w_1, \dots, w_n]$ and the associated weighted majority game (N, v) . As in a realistic situation, the ideological position of each party does not allow every coalition forming with the same probability, even if it is winning. A common scheme to describe a political scenario is to represent the parties on a left-right axis, via a suitable analysis of their ideologies, and we suppose that the players are ordered according to their position on this axis. Usually, the axis is represented by the segment 0-1 and the locations of the parties represent their ideology, where 0 is the extreme left and 1 the extreme right. Assuming that the negotiations take place uniquely between adjacent parties, the feasible coalitions include only contiguous parties. A coalition $S \subseteq N$ is *contiguous* if for all $i, j \in S$ if there exists $k \in N$ with $i < k < j$ then $k \in S$. An example of contiguous coalition and of noncontiguous coalition is given in Figure 3.2 and in Figure 3.3, respectively. Formally, let W^c be the set of contiguous winning coalitions, the general formula of

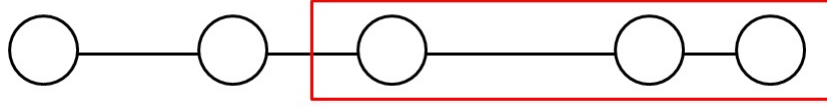


Figure 3.2: Contiguous coalition on a left-right axis

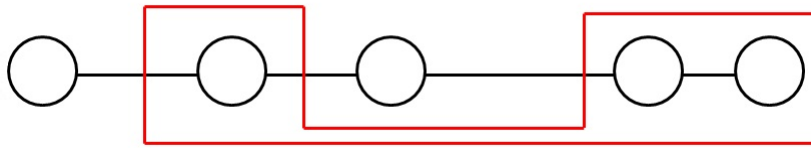


Figure 3.3: Noncontiguous coalition on a left-right axis

an *FP index* is

$$FP_i = \sum_{S \in W^c, S \ni i} \alpha_S \beta_{S_i} \quad \forall i \in N, \quad (3.3)$$

where $\alpha_S \geq 0$ represents the relative probability of coalition S to form, with the condition

$$\sum_{S \in W^c} \alpha_S = 1 \quad (3.4)$$

and $\beta_{S_i} \geq 0$ is the power share assigned to player $i \in S$, with the condition

$$\sum_{i \in S} \beta_{S_i} = 1 \quad \forall S \in W^c. \quad (3.5)$$

The choice of parameters α_S differentiates the power of the coalitions and the choice of parameters β_{S_i} differentiates the role of the parties inside coalitions. We address to Fragnelli et al. [36] for possible methods to compute the values of the parameters that account for ideological distances (assuming for example that the higher the distance between the extreme parties of a coalition, the less the probability of such a coalition to form), number of parties in the majority (minimal winning coalitions can be considered the only ones with a nonzero probability to form), their number of seats or via a suitable analysis of historical data. In general we can notice that only contiguous coalitions are given a probability to form. In particular, we remark that the definition of the *FP* family allows considering even a subset of contiguous winning coalitions, but this is equivalent to assigning a null probability to the remaining ones. Fragnelli et al. proposed a possible choice of the parameters defining the following index belonging

to the FP family, of which it is the “archetype”

$$FP_i^{Arc} = \sum_{S \in W^c, S \ni i} \frac{1}{|W^c|} \frac{1}{|S|} \quad \forall i \in N; \quad (3.6)$$

in this case the unitary power is equally shared among the coalitions in W^c and in the second step the quota assigned to each coalition S is equally shared among its members.

3.4 Embedding Classical Indices into the FP Family

In this section we want to embed classical indices in the general structure of the FP family. Our work is focused on weighted majority games and we start noticing that in general, classical indices do not take into account only the contiguous coalitions, as they are not based on a left-right axis structure. In order to embed them in the FP family, we need an extension of the formula (3.3) which allows the winning but noncontiguous coalitions to have a probability to form and only in a following step we will look for a standard FP index, summing only on the contiguous winning coalitions. We define the extended family \overline{FP} as

$$\overline{FP}_i = \sum_{S \in W, S \ni i} \alpha_S \beta_{S_i} \quad \forall i \in N \quad (3.7)$$

where $\alpha_S \geq 0$ and $\beta_{S_i} \geq 0$ have the same interpretation as above, with the conditions

$$\sum_{S \in W} \alpha_S = 1 \quad (3.8)$$

and

$$\sum_{i \in S} \beta_{S_i} = 1 \quad \forall S \in W. \quad (3.9)$$

This is an extension of the previous family because it is sufficient to assign $\alpha_S = 0$ for each $S \in W \setminus W^c$ to obtain an FP index.

In order to embed a generic index ψ into the \overline{FP} family, we have to impose the relations

$$\sum_{S \in W, S \ni i} \alpha_S \beta_{S_i} = \psi_i(v) \quad \forall i \in N \quad (3.10)$$

and to make a suitable choice of the parameters in order to have this relation verified.

In particular, we will analyze the way to embed the Shapley-Shubik, the normalized Banzhaf, the Deegan-Packel and the Holler indices. We can observe that there are several choices of the parameters in order to satisfy relations (3.10) as the system is overdetermined. For instance, a trivial solution is given by $\alpha_N = 1$ for the grand coalition and zero for the others $\alpha_S, S \neq N$ and $\beta_{S_i} = \psi_i$. This solution is not very interesting as it allows only the grand coalition forming and it assumes we already know the value of the index in order to evaluate the parameters β_{S_i} . We look now

for a non trivial solution, at first for the Shapley-Shubik index and then for the other indices.

We just want to remember that the final aim of this chapter is to combine the issue of contiguity with the philosophy of the classical indices (for instance the marginal contribution for the Shapley-Shubik and the normalized Banzhaf indices and the minimal winning coalitions for the Holler and the Deegan-Packel indices). After defining the appropriate parameters to write as \overline{FP} indices the classical ones, we will introduce the idea of contiguity.

3.4.1 Embedding the Shapley-Shubik Index

Following the purpose of embedding classical indices in the FP family, we start from the most common one, the Shapley-Shubik index. In particular, the first aim is to determine suitable values for the parameters in (3.7) in order to describe the formula given in (2.1). We start by imposing that

$$\alpha_S \beta_{Si} = p(S)[v(S) - v(S \setminus \{i\})] \quad \forall S \in W, \forall i \in S \quad (3.11)$$

where

$$p(S) = \frac{(|S| - 1)!(n - |S|)!}{n!}. \quad (3.12)$$

Summing on $i \in S$ and because of the condition (3.9) this is equal to

$$\alpha_S = p(S) \sum_{i \in S} [v(S) - v(S \setminus \{i\})] \quad \forall S \in W.$$

We remember that we denoted the set of the critical players of S as S^c , we call $c_S = |S^c|$ and we can write the parameters α_S as

$$\alpha_S = p(S)c_S \quad \forall S \in W. \quad (3.13)$$

Condition (3.8) holds as

$$\sum_{S \in W} \sum_{i \in S} p(S)[v(S) - v(S \setminus \{i\})] = \sum_{i \in N} \sum_{S \in W, S \ni i} p(S)[v(S) - v(S \setminus \{i\})] = 1$$

because of the efficiency of the Shapley-Shubik index.

By relations (3.11) and (3.13) we obtain

$$\beta_{Si} p(S) c_S = p(S)[v(S) - v(S \setminus \{i\})] \quad \forall S \in W, \forall i \in S.$$

It is sufficient to observe that if player i is critical for S then $v(S) - v(S \setminus \{i\}) = 1$, otherwise $v(S) - v(S \setminus \{i\}) = 0$, from which we get

$$\beta_{Si} = \begin{cases} \frac{1}{c_S} & \text{if } i \in S^c \\ 0 & \text{otherwise} \end{cases} \quad \forall S \in W \quad (3.14)$$

Relations (3.13) and (3.14) provide the parameters α_S and β_{S_i} , respectively, that enable us to write the Shapley-Shubik index as an \overline{FP} index.

In the definition of the \overline{FP} indices family, the order of the parties is not important and we do not refer to the left-right axis. We want now to come back to the idea of contiguous coalitions as the only alliances which are allowed forming, so the power of a party depends only on the coalitions in W^c it belongs to. In order to obtain an FP index, we decrease the probability to form of the noncontiguous coalitions modifying the parameters α_S given in (3.13). For each coalition $S \in W$ we introduce a sequence of parameters $((\gamma_S)_t)_{t \in \mathbb{N}}$ defined as

$$(\gamma_S)_t = \begin{cases} p(S)c_S & \text{if } S \in W^c \\ (p(S)c_S)^t & \text{if } S \in W \setminus W^c \end{cases}$$

from which we get a sequence of normalized parameters $((\alpha_S)_t)_{t \in \mathbb{N}}$

$$(\alpha_S)_t = \frac{(\gamma_S)_t}{\sum_{T \in W} (\gamma_T)_t} \quad \forall S \in W$$

It is easy to check that $(\alpha_S)_1 = \alpha_S$ for each $S \in W$.

In order to decrease the probability of noncontiguous coalitions to form, we take the limit for $t \rightarrow +\infty$ which gives us

$$(\gamma_S)_t \rightarrow \gamma_S^*$$

where

$$\gamma_S^* = \begin{cases} p(S)c_S & \text{if } S \in W^c \\ 0 & \text{if } S \in W \setminus W^c \end{cases}$$

as $p(S)c_S < 1$ if $S \neq N$, while $N \in W^c$.

Consequently $(\alpha_S)_t$ converges to $\alpha_S^* = \frac{\gamma_S^*}{\sum_{T \in W} \gamma_T^*}$ that, using the data of the problem, can be written as

$$\alpha_S^* = \begin{cases} \frac{p(S)c_S}{\sum_{T \in W^c} p(T)c_T} & \text{if } S \in W^c \\ 0 & \text{if } S \in W \setminus W^c \end{cases} \quad (3.15)$$

These values respect condition (3.4) by definition. Note that the sum in (3.15) does not consider the values $p(T)c_T$ for noncontiguous coalitions for which $\gamma_k^* = 0$.

We can assume that the definition of β_{S_i} does not depend on t , so $\beta_{S_i}^* = (\beta_{S_i})_t = \beta_{S_i}$ for each $t \geq 1$.

The values of parameters α_S^* and $\beta_{S_i}^*$ allow us embedding the Shapley-Shubik index in the FP family defining a new index ϕ^{FP}

$$\phi_i^{FP}(v) = \sum_{S \in W^c, S^c \ni i} \left(\frac{p(S)c_S}{\sum_{T \in W^c} p(T)c_T} \frac{1}{c_S} \right) \quad \forall i \in N$$

and, by formula (3.12) we get

$$\phi_i^{FP}(v) = \sum_{S \in W^c, S^c \ni i} \left(\frac{\frac{(|S|-1)!(n-|S|)!}{n!} c_S}{\sum_{T \in W^c} \frac{(|T|-1)!(n-|T|)!}{n!} c_T} \frac{1}{c_S} \right) \quad \forall i \in N \quad (3.16)$$

3.4.2 Embedding Other Power Indices

The procedure we used for the Shapley-Shubik index may be applied to any power index in the family \overline{FP} . Let us assume we have an \overline{FP} index given by (3.7) which respects the conditions (3.8) and (3.9), with the additional hypotheses that $\alpha_S < 1$ for each noncontiguous coalition $S \in W \setminus W^c$ and $\alpha_S > 0$ for at least one contiguous coalition $S \in W^c$. We may decrease the probability of the noncontiguous coalitions to form by defining

$$(\gamma_S)_t = \begin{cases} \alpha_S & \text{if } S \in W^c \\ (\alpha_S)^t & \text{if } S \in W \setminus W^c \end{cases}$$

and

$$(\alpha_S)_t = \frac{(\gamma_S)_t}{\sum_{T \in W} (\gamma_T)_t} \quad \forall S \in W \quad (3.17)$$

whose limit value is

$$\alpha_S^* = \begin{cases} \frac{\alpha_S}{\sum_{T \in W^c} \alpha_T} & \text{if } S \in W^c \\ 0 & \text{if } S \in W \setminus W^c \end{cases} \quad (3.18)$$

Again, we assume that the parameters β_{S_i} do not depend on t , i.e. $\beta_{S_i}^* = (\beta_{S_i})_t = \beta_{S_i}$ for each $t \geq 1$.

We notice that for each positive $t \in \mathbb{N}_>$ the vector $(\overline{FP})_t$, defined as $(\overline{FP}_i)_t = \sum_{S \in W} ((\alpha_S)_t \beta_{S_i})$, is a power index that assigns a reduced probability to form to the noncontiguous winning coalitions, as stated in the following proposition.

Proposition 3.4.1. *For each power index \overline{FP} and for each $t \in \mathbb{N}_>$ we have that $(\overline{FP})_t = ((\overline{FP}_1)_t, \dots, (\overline{FP}_n)_t)$ is a power index, i.e. $(\overline{FP}_i)_t \geq 0$ for each $i \in N$ and $\sum_{i \in N} (\overline{FP}_i)_t = 1$.*

Proof. $(\overline{FP}_i)_t \geq 0$ for each $i \in N$ and for each $t \in \mathbb{N}_>$ by definition. The value of $(\overline{FP}_i)_t$ for each $t \in \mathbb{N}_>$ is

$$\begin{aligned} (\overline{FP}_i)_t &= \sum_{S \in W} ((\alpha_S)_t \beta_{S_i}) \\ &= \sum_{S \in W^c} \left(\frac{(\gamma_S)_t}{\sum_{T \in W} (\gamma_T)_t} \beta_{S_i} \right) + \sum_{S \in W \setminus W^c} \left(\frac{(\gamma_S)_t}{\sum_{T \in W} (\gamma_T)_t} \beta_{S_i} \right) \end{aligned}$$

So

$$\begin{aligned}
\sum_{i \in N} (\overline{FP}_i)_t &= \sum_{i \in N} \sum_{S \in W^c} \left(\frac{(\gamma_S)_t}{\sum_{T \in W} (\gamma_T)_t} \beta_{Si} \right) + \sum_{i \in N} \sum_{S \in W \setminus W^c} \left(\frac{(\gamma_S)_t}{\sum_{T \in W} (\gamma_T)_t} \beta_{Si} \right) \\
&= \sum_{S \in W^c} \sum_{i \in N} \left(\frac{(\gamma_S)_t}{\sum_{T \in W} (\gamma_T)_t} \beta_{Si} \right) + \sum_{S \in W \setminus W^c} \sum_{i \in N} \left(\frac{(\gamma_S)_t}{\sum_{T \in W} (\gamma_T)_t} \beta_{Si} \right) \\
&= \frac{\sum_{S \in W^c} (\gamma_S)_t}{\sum_{T \in W} (\gamma_T)_t} \sum_{i \in N} \beta_{Si} + \frac{\sum_{S \in W \setminus W^c} (\gamma_S)_t}{\sum_{T \in W} (\gamma_T)_t} \sum_{i \in N} \beta_{Si} \\
&= \frac{\sum_{S \in W} (\gamma_S)_t}{\sum_{T \in W} (\gamma_T)_t} = 1
\end{aligned}$$

□

To embed the other classical power indices of Banzhaf (in the normalized version), Deegan-Packel and Holler in the \overline{FP} family and, consequently, to obtain the corresponding FP indices, is now simply a matter of suitably defining the parameters α_S and β_{Si} .

For the normalized Banzhaf index, the probability of a coalition to form is proportional to the number of critical players. The same holds for the Public Good index, with the difference that non minimal coalitions have null probability to make an agreement and for each minimal one the number of critical players is equal to the cardinality of the coalition itself. Differently, the Deegan-Packel index takes into account only minimal coalitions but it assumes they have all the same probability to create an agreement. The sharing of the power between players inside a given winning coalition is always given by an equal division between critical players.

By the procedure presented in Section 3.4.1 we obtain the following formulas. For the normalized Banzhaf index we get

$$\beta_i^{FP}(v) = \frac{h_i^c}{\sum_{k \in N} h_k^c} \quad \forall i \in N \quad (3.19)$$

where h_i^c counts how many times player i is critical in a contiguous winning coalition.

Embedding the Deegan-Packel index in the FP family, we obtain an index where the probability to form is the same for all the coalitions in the set W^{mc} of contiguous minimal winning coalitions and the power of each coalition is equally divided among its members

$$\delta_i^{FP}(v) = \sum_{S \in W^{mc}} \frac{1}{|W^{mc}|} \frac{1}{|S|} \quad \forall i \in N \quad (3.20)$$

Finally, the Holler index adapted to the FP family is given by

$$H_i^{FP}(v) = \frac{h_i^{mc}}{\sum_{k \in N} h_k^{mc}} \quad \forall i \in N \quad (3.21)$$

where h_i^{mc} counts how many times player i belongs to a contiguous minimal winning coalition.

In Table 3.1 we summarize the choice of the parameters for the normalized Banzhaf, the Deegan-Packel and the Holler indices, respectively, in order to write them as \overline{FP} indices.

Parameters	α_S	β_{Si}
normalized Banzhaf	$\frac{c_S}{\sum_{T \in W} c_T}, S \in W$	$\begin{cases} \frac{1}{c_S} & \text{if } i \in S^c \\ 0 & \text{otherwise} \end{cases}$
Deegan-Packel	$\begin{cases} \frac{1}{ W^m } & \text{if } S \in W^m \\ 0 & \text{otherwise} \end{cases}$	$\begin{cases} \frac{1}{c_S} & \text{if } i \in S^c \\ 0 & \text{otherwise} \end{cases}$
Holler	$\begin{cases} \frac{c_S}{\sum_{T \in W^m} c_T} & \text{if } S \in W^m \\ 0 & \text{otherwise} \end{cases}$	$\begin{cases} \frac{1}{c_S} & \text{if } i \in S^c \\ 0 & \text{otherwise} \end{cases}$

Table 3.1: Parameters to embed some classical indices in the \overline{FP} family

3.5 The Italian “Camera dei Deputati”: a Real-World Example

In this section we apply the results presented in the previous ones to a real Parliament, the Italian lower chamber, *Camera dei Deputati* (or *Camera*). It is formed by 630 seats and the majority quota is $\lfloor \frac{v}{2} + 1 \rfloor$, where v is the number of voters, excluding absences and abstentions. The required quorum during a legislative vote is 316 Deputies.

Parties	<i>IdV</i>	<i>PD</i>	<i>UDC</i>	<i>PDL</i>	<i>LN</i>
Seats	28	218	34	272	60

Table 3.2: Seats allocation in the Italian Camera (April 2008) for the five main parties

The data used in the example are taken from the general election of April 2008; for sake of simplicity we decided of not considering 18 seats belonging to very small parties which, historically, have no practical influence on the decisions of the Camera even if, in theory, they could change the outcome. The remaining 612 seats are assigned as in Table 3.2 to five parties, from left to right, namely:

- Italia dei Valori (*IdV*) - Italy for Values
- Partito Democratico (*PD*) - Democratic Party
- Unione di Centro (*UDC*) - Centre Union
- Popolo della Libertà (*PDL*) - People for Freedom

- Lega Nord (*LN*) - Northern League

The ordering of the parties is assigned according to their willingness to form a coalition in the recent political history and it is represented by the following left-right axis.

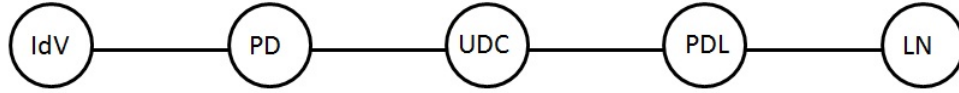


Figure 3.4: Italian Camera on a left-right axis

We suppose that all the Deputies of the five main parties are present and vote (we ignore situations of absences and abstentions). The majority quota is 307 and we may represent the Camera as the weighted majority situation $[307; 28, 218, 34, 272, 60]$. In order to compute the Shapley-Shubik index using the relations in (3.13) and (3.14), we need the data in Table 3.3 (for each coalition the critical parties are underlined and $\beta_{Si} = \frac{1}{c_S}$ for each critical party i in each coalition $S \in W$). Looking

S	<u>24</u>	<u>45</u>	<u>124</u>	<u>134</u>	<u>145</u>	<u>234</u>	<u>235</u>	<u>245</u>	<u>345</u>	1234	1235	1245	1345	2345	12345
$p(S)$	$\frac{1}{20}$	$\frac{1}{20}$	$\frac{1}{30}$	$\frac{1}{30}$	$\frac{1}{30}$	$\frac{1}{30}$	$\frac{1}{30}$	$\frac{1}{30}$	$\frac{1}{30}$	$\frac{1}{20}$	$\frac{1}{20}$	$\frac{1}{20}$	$\frac{1}{20}$	$\frac{1}{20}$	$\frac{1}{5}$
α_S	$\frac{6}{60}$	$\frac{6}{60}$	$\frac{4}{60}$	$\frac{6}{60}$	$\frac{4}{60}$	$\frac{4}{60}$	$\frac{6}{60}$	$\frac{2}{60}$	$\frac{4}{60}$	$\frac{3}{60}$	$\frac{9}{60}$	$\frac{3}{60}$	$\frac{3}{60}$	0	0
β_{Si}	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{3}$	1	$\frac{1}{2}$	1	$\frac{1}{3}$	1	1	-	-

Table 3.3: Parameters assigned to the winning coalitions to compute the Shapley-Shubik index

at the parameters α_S , we remark how, according to the Shapley-Shubik index, the real majority governing coalition $\{4, 5\}$ has not the highest probability to form, while the most probable coalition is $\{1, 2, 3, 5\}$, that includes the leftmost and rightmost parties (*IdV* and *LN*) and excludes the relative majority party (*PDL*).

The Shapley-Shubik index of this voting game is

$$\phi(v) = \left(\frac{2}{60}, \frac{12}{60}, \frac{7}{60}, \frac{27}{60}, \frac{12}{60} \right).$$

Using the procedure previously described, we modify the parameters α_S according to (3.17) and (3.18) and compute the relative *FP* index given by (3.16)

$$\phi^{FP}(v) = \left(0, \frac{2}{17}, 0, \frac{10}{17}, \frac{5}{17} \right).$$

Figure 3.5 shows the first steps of the procedure of reducing the probability to form of the noncontiguous coalitions

We complete the example computing the power given to the five parties by the other classical indices and the modified power obtained with the *FP* versions. We summarize the results in Table 3.4.

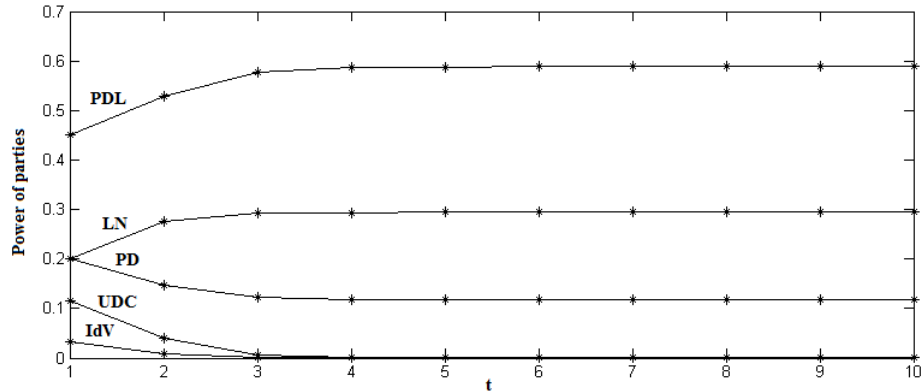


Figure 3.5: First 10 steps of the procedure for reducing the probability to form of the noncontiguous coalitions for the Shapley-Shubik index

Parties	IdV	PD	UDC	PDL	LN
$\phi(v)$	$\frac{2}{60}$	$\frac{12}{60}$	$\frac{7}{60}$	$\frac{27}{60}$	$\frac{12}{60}$
$\phi^{FP}(v)$	0	$\frac{2}{17}$	0	$\frac{10}{17}$	$\frac{5}{17}$
$\beta(v)$	$\frac{1}{25}$	$\frac{5}{25}$	$\frac{3}{25}$	$\frac{11}{25}$	$\frac{5}{25}$
$\beta^{FP}(v)$	0	$\frac{1}{7}$	0	$\frac{4}{7}$	$\frac{2}{7}$
$\delta(v)$	$\frac{2}{24}$	$\frac{5}{24}$	$\frac{4}{24}$	$\frac{8}{24}$	$\frac{5}{24}$
$\delta^{FP}(v)$	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$
$H(v)$	$\frac{1}{10}$	$\frac{2}{10}$	$\frac{2}{10}$	$\frac{3}{10}$	$\frac{2}{10}$
$H^{FP}(v)$	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$

Table 3.4: Main power indices in the classical and in the modified version

The Shapley-Shubik and the normalized Banzhaf indices assign a positive power to the small parties, *IdV* and *UDC*, as they are critical for some winning coalitions ($\{1, 3, 4\}$, $\{2, 3, 5\}$ and $\{1, 2, 3, 5\}$), none of which is contiguous. Consequently, the power of these two parties, with the modified indices which take into account only when a player is critical for a contiguous winning coalition, goes down to zero, even if one of them, *UDC*, has an intermediate position on the left-right axis. Allowing only connected coalitions to form, we expect that this should be beneficial for the centrist parties, so it is not surprising to assign zero to *IdV*, as it does not have a central position in the Parliament, but it seems strange, at a first look, the zero given to *UDC*. The reason why of this result is given by the fact that the dummy property filters through contiguity: a dummy player will remain so, but it is possible that a non dummy player, as *UDC*, which has non zero marginal contribution only in noncontiguous coalitions,

will become dummy once we evaluate the embedded Shapley-Shubik and normalized Banzhaf indices. This is not true anymore for a generic FP index, which does not account the philosophy of the marginal contribution.

We say that a party is *contiguous-critical* for a contiguous winning coalition S if $S \setminus \{i\}$ is losing or is no longer contiguous. If we adopt the definition of contiguous-critical instead of the definition of critical player, it is possible that a FP index based on this concept creates power (i.e. it assigns a positive power to a dummy player).

It is interesting to observe how the indices $\phi^{FP}(v)$ and $\beta^{FP}(v)$ give a higher power to the two parties of the actual majority coalition, PDL and LN (greater for PDL , the party with the relative majority quota of seats), guaranteeing a positive power to the second party of the Camera for number of seats, PD , which is critical for the contiguous coalition $\{2, 3, 4\}$.

Once we take into account only minimal winning coalitions, as we do when we evaluate the Deegan-Packel and the Public Good indices, we notice that every party belongs to at least one of them. But focusing on the contiguous ones, the unique contiguous minimal winning coalition is the real majority coalition formed by PDL and LN and the power is equally shared between these two parties, as they are in a symmetric position.

After the embedding in the FP family, classical indices look like better predicting the real power of the parties in the formation of a governing coalition, respecting the ideologies relations among them, which do not allow some coalitions forming.

3.6 From Contiguity to Connectedness

The well known left-right political dimension has been used to characterize key political differences since the era of the French Revolution. However, it is common to think that to measure policy spaces requires more than a single dimension (we refer for example to Benoit and Laver [14]). In order to extend what we have done in the previous section to a multidimensional space, we notice that the idea of a left-right axis and of the contiguity of the players naturally remembers the cooperation structure defined by Myerson, whose model has already been clarified in Section 3.2.1, as the left-right axis is a particular example of graph in G^N , which we can denote as g' and in which the notions of contiguity and of connectedness coincide. This can suggest the possibility of obtaining the embedded power indices by defining a new game in which the characteristic function v is modified as v' s.t. $v'(S) = v(S)$ if $S \in W^c$ and $v'(S) = 0$ otherwise and evaluating the classical power indices on this new game. We underline that this is not the definition of reduced game given by Myerson, in which a coalition remains winning if it contains a connected winning subcoalition, because with this new model every nonconnected coalition becomes losing. However, this procedure produces completely different results from the embedding procedure as we can immediately notice that (N, v') is not a simple game anymore, as it loses the property of monotonicity. Evaluating the Shapley-Shubik index on this new game, for example, we may obtain negative values for some parties, so that it may hardly be

considered as a measure of relevance. Differently from the Myerson value, then, the procedure of embedding some classical indices in the new family cannot be seen as the evaluation of these indices on a reduced game.

Taking inspiration from the model of Myerson, we can make an extension of the FP family applying the results to a generic graph $g \in G^N$. The central role played by the contiguous coalitions is now assigned to the connected ones, then we assume that the sum is defined on \widetilde{W}^c , which represents the set of winning coalitions that are connected in a given graph. Given a graph $g \in G^N$, we extend the FP family defining a new family, denoted by \widetilde{FP} , that is based on the set of the coalitions connected in g , their relative probability to form and a rule for sharing the power inside each coalition, then

$$\widetilde{FP}_i = \sum_{S \in \widetilde{W}^c, S \ni i} \alpha_S \beta_{Si} \quad \forall i \in N, \tag{3.22}$$

where $\alpha_S \geq 0$ represents the relative probability of coalition S to form, with the condition

$$\sum_{S \in \widetilde{W}^c} \alpha_S = 1 \tag{3.23}$$

and $\beta_{Si} \geq 0$ is the power share assigned to player $i \in S$, with the condition

$$\sum_{i \in S} \beta_{Si} = 1 \quad \forall S \in \widetilde{W}^c. \tag{3.24}$$

In order to obtain the embedded version of the classical power indices into this new family, Formula (3.18) is then modified simply replacing W^c with \widetilde{W}^c , i.e. reducing to zero the power of the nonconnected coalitions instead of the one of the noncontiguous coalitions. This approach allows us applying the theoretical results to situation in which the possible communications between parties are larger.

We may apply the modified procedure to compute the power assigned to the parties of the example in Section 3.5, when we add an edge connecting PD and LN . We remark that in the actual Italian political situation these two parties have far ideologies, but such an agreement took place in 1996 so it is practically feasible. This cooperation structure is represented by the following graph, denoted by g''

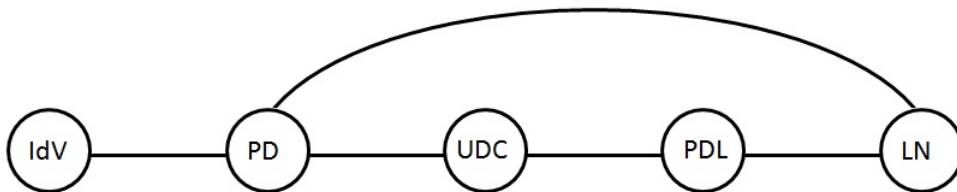


Figure 3.6: Italian Camera represented through the graph g''

The computation of the new limit values of the embedded power indices is given in Table 3.5.

Parties	IdV	PD	UDC	PDL	LN
$\phi^{\widetilde{FP}}$	0	$\frac{7}{40}$	$\frac{5}{40}$	$\frac{18}{40}$	$\frac{10}{40}$
$\beta^{\widetilde{FP}}$	0	$\frac{3}{16}$	$\frac{2}{16}$	$\frac{7}{16}$	$\frac{4}{16}$
$\delta^{\widetilde{FP}}$	0	$\frac{2}{12}$	$\frac{2}{12}$	$\frac{3}{12}$	$\frac{5}{12}$
$H^{\widetilde{FP}}$	0	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{2}{5}$

Table 3.5: Modified indices related to the connection structure given by graph g''

Comparing the results in Table 3.4, referred to graph g' , with those in Table 3.5, related to graph g'' , we may notice that all the indices related to g'' reduce the power of the main party, PDL, after the introduction of the new edge. In the new situation also coalitions that do not include PDL may form.

For a complete analysis of the results we obtained, as we have largely used the idea of a cooperation structure by Myerson, we compute the Myerson index for both the situations g' and g'' in order to underline some important differences between the two ideas of solution. The results are shown in Table 3.6.

Parties	IdV	PD	UDC	PDL	LN
$M(v, g')$	0	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{7}{12}$	$\frac{3}{12}$
$M(v, g'')$	0	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{3}$

Table 3.6: The Myerson indices for the game (N, v) of the example of Section 3.5 referred to the graphs g' and g''

As expected, the FP index and the Myerson index give different vectors of power. In particular, we can underline how in our model not necessarily both the parties, PD and LN , should have the same advantage by allowing a possible agreement between them. We notice that in this case PD has a higher power when the cooperation structure is given by g'' , while LN should still prefer the g' situation. This could not have been happened using Myerson index as the property of *equity*¹ is always guaranteed. The lost of power of LN shows that even the property of *total stability*² no longer holds.

We want to focus our attention on another deep difference between the model of the \widetilde{FP} family of power indices and the idea of simply reducing the game through a graph, like Myerson does. We decided to embed classical power indices into the new family in order to underline how the obtained results can be corrected introducing the

¹ According to equity, introducing (or removing) an edge, both the parties corresponding to its extreme vertices have the same variation of power.

² According to stability, after introducing an edge, the variation of power is non negative for both its extreme vertices.

notion of connectedness (or of contiguity, if we speak about the FP family). But the general idea behind the definition of this new family is also the freedom in the choice of the parameters to better catch some particular situations of incompatibilities of the agents which cannot be described through other models. Let now consider the following example, which has been introduced by Fragnelli [33]:

Example 3.6.1. Let (N, v) be the weighted majority game defined starting from the following weighted majority situation [51; 35, 30, 25, 10]. The set of winning coalitions is $W = \{\{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}, \{1, 2, 3, 4\}\}$. When the graph representing the possible connections between players is the one shown in the following figure the set of connected winning coalitions is

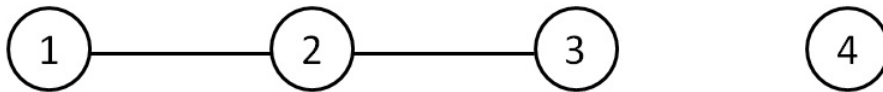


Figure 3.7: Example of graph

$$\widetilde{W}^c = \{\{1, 2\}, \{2, 3\}, \{1, 2, 3\}\}.$$

Myerson would assume coalition $\{1, 2, 3, 4\}$ as possible, but we can immediately notice that player 4 does not want to cooperate with any of the other players, making such a coalition not likely at all. If we just simply decide to reduce the game according to the set of connected winning coalitions, we should assume that all the three coalitions in \widetilde{W}^c have a positive probability of forming. But let us suppose that parties 1 and 3 are both available for forming a two-party coalition with party 2 but they do not want to stay in the same coalition (because of some incompatibilities), so that also coalition $\{1, 2, 3\}$ has a zero probability of forming. Then the idea of reducing the game according to the connectedness would not catch such a situation. We can modify the definition saying that a coalition may form if the corresponding subgraph is complete. In this way the graph represents now the situation in which coalitions $\{1, 2\}$ and $\{2, 3\}$ have a positive probability of forming while coalition $\{1, 2, 3\}$ cannot form. But if we want now to catch the situation in which, because of the presence of player 2, players 1 and 3 accept to cooperate, we may add an edge connecting 1 and 3, with the consequence that now also coalition $\{1, 3\}$ has a strictly positive probability of forming. Then, the model would one more time not perfectly represent the compatibilities between the players.

The general definition of the \widetilde{FP} family allows us to take into account all the connected coalitions, and then to assign a probability to each one of them according to some preliminary information we have about the parties and their wish to cooperate.

3.7 Intermediate FP Indices

We may observe the loss of monotonicity during the embedding procedure: the issue of contiguity is “against” the monotonicity, as the connection degree may be more relevant than the number of seats. The nonmonotonicity can be simply seen in Example 3.2.1, where we have already noticed the nonmonotonicity of the Myerson value. In this example, player 1, the one with the highest weight, has zero power according to the Myerson value and has zero power with every FP index, as he does not belong to any contiguous winning coalition. In particular he has zero power with the embedded Shapley-Shubik and normalized Banzhaf indices, whose original versions have the property of monotonicity.

The idea of giving zero probability to form to the noncontiguous coalitions can be a strong assumption. It can be observed that, even if it is not very common that parties with quite different political ideologies can decide to cooperate, it is still possible they have the necessity to negotiate and make an agreement in some particular situations. The procedure we showed to obtain an FP index, starting from an \overline{FP} one, is based on the idea of putting down to zero the probability of the noncontiguous coalitions to form, using a sequence of vectors, $(\overline{FP})_t$, that for each value of $t \in \mathbb{N}_>$ provides a power index, where the noncontiguous coalitions have a reduced, but positive, probability.

It should be clear that the main point is to have a sequence of probability distributions that reduces to zero the probability of noncontiguous coalitions, leaving a positive probability to some contiguous coalitions. Consequently, we could make use of any sequence that satisfies these requirements. The one we proposed in Section 3.4.1 is only a simple way to accomplish the requests. For instance, it is possible to use different criteria of convergence to zero of noncontiguous coalitions, taking into account the ideological distance among the parties and to assign different probabilities to form to the contiguous ones. In particular, via a suitable analysis of real data, we can choose any vector $(\overline{FP})_t$ selecting an appropriate value for t . Of course, the approach of a sequence of values may be replaced by defining a distribution of probabilities that directly assigns zero to noncontiguous coalitions. This idea is very simple but does not provide us a sequence of power indices. Moreover, the probabilities to form of contiguous coalitions may result in an index that no longer embeds the original one.

Finally, a different sequence of probability distributions may give rise to questions about the sensitivity analysis of the resulting indices, i.e. about how small differences on the weights of the parties may influence the resulting indices. Some noncontiguous coalitions, for example, may become winning by varying a bit their weight (or the opposite can happen) and this can influence the indices obtained by simply reducing the probability to form of these coalitions.

The Bargaining Set as “Nonstatic” Power Index

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4.1 Introduction

The indices of power are used to evaluate the relevance of the agents in a decisional situation, e.g. a parliamentary voting session in which the agents are the different parties. The main issue they are based on is the capacity of a party to influence the approval of a proposal. We may think of this as a *static power*, i.e. what happens referring to a precise proposal under discussion. Many indices of power have been defined and we refer to Section 2.2.2 for a summary of the most famous ones. In Chapter 3 we analyzed some new solution concepts based on the notions of contiguity and connectness. Power indices can also be based on the capacity of a party to influence the rejection of a proposal, so we may account not only the possibility that the favorable vote casted by a party results in passing a law, but also that a negative vote leads to the rejection of the law; this situation will be widely studied in Chapter 5 where we will call these indices *veto power indices*.

On the other hand, we may consider the index of power as a measure of the influence of the parties on the formation of a government majority. In this chapter, we account the possibility that a party belonging to the majority coalition may decide to join to other parties and form a different majority. This may be viewed as a *blackmailing power* in the sense that a party may ask for more power when it has an

effective opportunity for joining to a more profitable coalition and naturally leads to the evaluation of a *dynamic power*.

We may think, for example, of a situation in which we have four parties, namely $N = \{1, 2, 3, 4\}$. The only winning coalitions are $\{1, 2, 3\}$, $\{2, 3, 4\}$ and $\{1, 2, 3, 4\}$. We can observe that parties 1 and 4 cannot reject any amount of power assigned to them, because they do not have any credible alternative to coalitions $\{1, 2, 3\}$ and $\{2, 3, 4\}$ respectively, while party 2 and 3 may blackmail parties 1 and 4 by entering a more profitable coalition, but they cannot blackmail each other, as they need to be together to form a majority.

The classical power indices are not designed for accounting this blackmailing behavior, that, on the other hand, may be caught by the concept of Bargaining Set by Aumann and Maschler [10], that is based on the idea that no agent may reject a division of the value of the game in favor of a better one (objection) if another agent may reject the new division in favor of a third one (counterobjection).

Motivated by this remark, we propose to refer to the Bargaining Set as a tool for assigning each party a quota of power that cannot be rationally objected by the other parties. Unfortunately, the Bargaining Set may be very hard to compute, as already a result of Maschler himself [64] shows that it is the union of a high number of convex polyhedra also for games with a small number of players. But, as we have already discussed in the previous chapter, not every coalition is feasible and the set of minimal winning coalitions can be restricted in order to make the computation easier, even if still very hard. Moreover, we suggest to use the concept of Bargaining Set to verify if a given solution, for example the Shapley-Shubik index, belongs to the Bargaining Set and is in this sense “stable”. This problem, even if easier, has still a very high computational complexity.

The chapter is organized as follows: in Section 4.2, we provide some more theoretical notions and notations and the results of Maschler on the inequalities to evaluate the Bargaining Set; in Section 4.3, we describe our proposal; in Section 4.4, we apply the model to the German Bundestag; in Section 4.5 we discuss how to reduce the computational complexity in real-world situations, by reducing the number of minimal winning coalitions; Section 4.6 concludes.

4.2 The Bargaining Set: a Solution Based on a Blackmailing Behaviour

The definition of the core given in Section 2.2.1 does not restrict a coalition’s credible deviations, beyond imposing a feasibility constraint. In particular it assumes that any deviation is the end of the story and ignores the fact that a deviation may trigger a reaction that leads to a different final outcome. Now, we introduce the Bargaining Set, a solution obtained by considering the discussion that may actually take place during a play of a game. Thus it considers the possible threats and counterthreats made by the several players.

The Bargaining Set was introduced by Aumann and Maschler [10] in 1964 and it is based on the following concepts of objection and counterobjection.

Given a game (N, v) , let x be an imputation and let i and j be two distinct players in N . An *objection of i against j at x* is a pair (C, y) satisfying:

- i** $C \subset N$ $i \in C$, $j \notin C$;
- ii** $y \in \mathbb{R}^n$ $y(C) = v(C)$;
- iii** $y_k > x_k$ if $k \in C$.

Given a game (N, v) and an imputation x , let (C, y) be an objection of i against j at x , $i, j \in N$. A *counterobjection of j against i at y* is a pair (D, z) satisfying:

- i** $D \subset N$ $j \in D$, $i \notin D$;
- ii** $z \in \mathbb{R}^n$ $z(D) = v(D)$;
- iii** $z_k \geq y_k$ if $k \in C \cap D$;
- iv** $z_k \geq x_k$ if $k \in D \setminus C$.

Given a game (N, v) , the *Bargaining Set* is the set \mathcal{M} of imputations that have no justified objection, i.e. whose objections have counterobjections. It is clear that imputations that have no objections belong to the Bargaining Set. The Bargaining Set is nonempty for proper simple games, differently from the core.

The complexity of the evaluation of the Bargaining Set is well-known. The payoffs of the Bargaining Set are a finite unions of closed convex polyhedra and already in 1966 Maschler [64] provided the system of inequalities that determines these polyhedra in explicit form, showing how complicated can be the problem of finding the vectors of the Bargaining Set.

Let x be an imputation for a game (N, v) ; for any coalition S call $e(S, x) = v(S) - x(S)$ the *excess of S* . If the excess of the coalition S is positive, then it measures the amount that S has to forgo in order for the imputation x to be implemented: it is the sacrifice that S makes to maintain the social order. If the excess of S is negative then its absolute value measures the amount over and above the worth of S that S obtains when the imputation x is implemented: it is the surplus of S in the social order. We start recalling a criterion for a player having an objection.

Lemma 4.2.1. *Let x be an imputation for a game (N, v) . Let i and j be two distinct players in N . Let $C \subseteq N$ s.t. $i \in C$ and $j \notin C$. In order that i has an objection against j at x using C , it is necessary and sufficient that $e(C, x) > 0$.*

In order to find a criterion for i having a justified objection against j by using coalition C , Maschler constructs the $(C; i, j; x)$ -game.

Let x be an imputation and (i, j) be an ordered pair of players in N . Let $C \subseteq N$ s.t. $i \in C$ and $j \notin C$ which contains at least two members. The $(C; i, j; x)$ -game is the game $(C \setminus \{i\}, v_C)$ where v_C is defined as follows

$$v_C(S) = \text{Max} (0, \text{Max}_{D \subseteq N, j \in D, i \notin D, D \cap C = S} e(D, x)) \quad \forall S \subseteq C \setminus \{i\}. \quad (4.1)$$

We recall the definition of Bondareva and Shapley ([16] and [87]), according to whom a collection $\mathcal{S} = \{S_1, \dots, S_q\}$ of coalitions of N is called *balanced* if there exist $\lambda_1, \dots, \lambda_q$, s.t. $\lambda_i > 0$ for each $i = 1, \dots, q$, called *weights* and

$$\sum_{k|i \in S_k} \lambda_k = 1 \quad \forall i \in N.$$

A balanced collection is *minimal* if no proper subcollection is balanced. The weight vector is unique iff \mathcal{S} is a minimal balanced collection. We can now present the important result of Maschler, which describes the system of inequalities which determine the Bargaining Set of a game:

Theorem 4.2.2. *Let (N, v) be a cooperative game s.t. $v(S) \geq \sum_{i \in S} v(i)$ for each $S \subseteq N$. A necessary and sufficient condition that $x = (x_1, \dots, x_n)$ belongs to the Bargaining Set \mathcal{M} is*

- i $\sum_{i \in N} x_i = v(N)$;
- ii $x_i \geq v(i) \quad \forall i \in N$;
- iii for each ordered pair of distinct players (i, j) and for each coalition $C \subseteq N$ s.t. $i \in C$ and $j \notin C$ which contains at least two members, either there exists a coalition D s.t. $i \notin D, j \in D, D \cap C = \emptyset$ and $e(D, x) \geq 0$; or

$$e(C, x) \leq \text{Max}_{\mathcal{S} \in R} \sum_{k|S_k \in \mathcal{S}} \lambda_k(\mathcal{S}) v_C(S_k),$$

where $v_C(S_k)$ is defined by the Formula (4.1), R is the set of all minimal balanced collections of $C \setminus \{i\}$ and $\lambda_k(\mathcal{S})$ is the weight of S_k for \mathcal{S} , $S_k \in \mathcal{S}$

This theorem provides a set of inequalities connected by the words “and” and “or”. Maschler provides an easy example of a simple 4-player game, showing that also in this case the Bargaining Set is the union of 150^{12} convex polyhedra. On the other side, in order to check if a particular payoff is in the Bargaining Set, “only” 197 inequalities need to be checked.

4.2.1 A Particular Vector in the Bargaining Set: the Nucleolus

We now describe another solution that, like the Bargaining Set, is defined by the condition that to every objection there is a counterobjection; it differs from the previous one in the nature of objections and counterobjections that are considered effective.

An *objection to x* is a pair (S, y) s.t. $e(S, x) > e(S, y)$ (i.e. $y(S) > x(S)$). A *counterobjection to the objection (S, y)* is a coalition T s.t. $(e(T, y) > e(T, x)$ (i.e. $x(T) > y(T)$ and $e(T, y) \geq e(S, x)$).

Given a game (N, v) , the *nucleolus* is the set of all imputations x with the property that for every objection (S, y) to x there is a counterobjection to (S, y) .

The Nucleolus is always a vector in the Bargaining Set. Montero [73] suggested to use it as a power index. Her motivation is due to the solid noncooperative foundations this solution has when applied to weighted majority games. We simply suggest the possibility of using the nucleolus as a measure of power as it is a vector in the Bargaining Set which is, in general, easier to evaluate.

4.3 The Bargaining Set and the Issue of Power

As we have already said in the Introduction of this chapter, the main motivation of this work is to model a voting situation through a dynamic approach, accounting the possibility that after the division of the power a bargaining process starts, and parties make use also of blackmailing in order to increase their power share. The Bargaining Set includes those power allocations that have no objection at all or that have no justified objection, so that these allocations cannot be rejected by the parties using reasonable arguments, i.e. they are stable. It is easy to see that the two concepts have a lot of common elements, so when a power index belongs to the Bargaining Set, it adds the property of avoiding blackmailing behaviors to its own features. Of course it is possible that no power index, at least one of the most popular, e.g. those recalled in the Introduction of this thesis, belongs to the Bargaining Set, then in this case the idea of sharing the power using one of the points in it is a very good way to protect against possible blackmailing when this issue is considered relevant, as no party has incentives to deviate from the *status quo*.

One of the most negative features of the Bargaining Set is its computational complexity, because we need to determine a sequence of inequalities that represent the conditions under which each player may raise an objection against each other player forming a different winning coalition, and another sequence of inequalities that represents the conditions for the existence of suitable counterobjections. This system of inequalities defines the Bargaining Set.

On the other hand, the complexity may reduce. In fact, we may remark that in general the number of parties is seldom larger than 10 and often the parties are in a symmetric position, so the number of inequalities results to be more tractable. We can add also a result due to Einy and Wettstein [30] according to which in a simple game the Bargaining Set coincides with the core when this last one is nonempty; moreover, it is very easy to check this case as the core of a simple game is nonempty if and only if there exist veto players, i.e. players that belong to every winning coalition, and the core is made up by those allocations that divide the unitary power only among the veto players. In general, even if the evaluation of the Bargaining Set may be very

difficult, in some real-world situations it is possible to exploit the particular structure of the winning coalitions, or to reduce the set of feasible minimal winning coalitions, taking into account the political affinities of the parties. These two approaches allow us to evaluate the Bargaining Set in the following two examples.

4.4 The German “Bundestag”: a Real-World Example

In this section we illustrate the previous concepts applying them to the 17. *Deutscher Bundestag* (17th German Bundestag, German Lower Chamber) that at 3 March 2011 counted 620 Members, divided in 5 parliamentary groups, namely:

- CDU/CSU-Bundestagsfraktion (*CDU/CSU*) - Christian Democratic Union/Christian Social Union
- Sozialdemokratische Partei Deutschlands (*SPD*) - Social Democratic Party of Germany
- Freie Demokratische Partei (*FDP*) - Free Democratic Party
- Die Linke (*Die Linke*) - Left Party
- Bündnis 90/Die Grünen (*Die Grünen*) - Alliance 90/Greens

Party	Seats
<i>CDU/CSU</i>	237
<i>SPD</i>	146
<i>FDP</i>	93
<i>Die Linke</i>	76
<i>Die Grünen</i>	68

Table 4.1: German Lower Chamber (2011)

The seat share is shown in Table 4.1 and then the German Bundestag may be represented by the weighted majority situation $[311; 237, 146, 93, 76, 68]$; in the related weighted majority game $N = \{CDU/CSU, SPD, FDP, Die Linke, Die Grünen\}$ and the minimal winning coalitions are $\{CDU/CSU, SPD\}$, $\{CDU/CSU, FDP\}$, $\{CDU/CSU, Die Linke\}$ and $\{SPD, FDP, Die Linke\}$. *Die Grünen* is critical in no coalition, so it has always a null payoff according to all the classical solutions. We show in Table 4.2 the power share in the German Bundestag according to the Shapley-Shubik, the normalized Banzhaf, the Deegan Packel, the Johnston and the Public Good indices.

Also according to the Bargaining Set, *Die Grünen* has zero power, because it does not belong to any minimal winning coalitions. We have already said that the computational complexity of the evaluation of the vectors in the Bargaining Set is very high. We just notice that the other four parties are in the situation of a famous

Parties	CDU/CSU	SPD	FDP	Die Linke	Die Grünen
$\phi(v)$	$\frac{3}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	0
$\beta(v)$	$\frac{3}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	0
$\delta(v)$	$\frac{9}{24}$	$\frac{5}{24}$	$\frac{5}{24}$	$\frac{5}{24}$	0
$\gamma(v)$	$\frac{27}{42}$	$\frac{5}{42}$	$\frac{5}{42}$	$\frac{5}{42}$	0
$H(v)$	$\frac{3}{9}$	$\frac{2}{9}$	$\frac{2}{9}$	$\frac{2}{9}$	0

Table 4.2: Power share in the German Bundestag according to the main power indices

example, called *My aunt and I*, written by Osborne and Rubinstein [78], and from which we get that the Bargaining Set of the German Bundestag is:

$$\mathcal{M} = \left\{ (1 - 3\alpha, \alpha, \alpha, \alpha, 0), \frac{1}{7} \leq \alpha \leq \frac{1}{5} \right\}.$$

We can notice that the Shapley-Shubik index and the normalized Banzhaf index belong to \mathcal{M} , while the indices of Deegan-Packel and Holler do not belong to \mathcal{M} because they assign a larger power to *SPD*, *FDP* and *Die Linke*. This means that *CDU/CSU* could blackmail the other three parties. Also the index of Johnston does not belong to \mathcal{M} , but this time the problem is that it assigns a larger power to *CDU/CSU*, that could be blackmailed by *SPD*, *FDP* and *Die Linke*.

4.5 The Catalan Parliament: a Reduced Game

As we noticed in Sections 4.2 and 4.3, one of the most negative aspects of the Bargaining Set is the computational complexity; on the other hand, we already remarked that in a voting situation the complexity may be reduced, as it is possible to consider a smaller set of minimal winning coalitions to be taken into account, due to different ideological affinities of the parties.

A simple real-world example of this situation is the Catalan Parliament we present in Example 4.5.1.

Example 4.5.1. *In the Legislature 2003-2007 (see Alonso-Meijide and Carreras [7]), the Catalan Parliament counted 135 members, divided in five parties, namely:*

- *Convergència i Unió (CiU) - Catalan nationalist*
- *Partit dels Socialistes de Catalunya (PSC) - moderate left-wing socialist*
- *Esquerra Republicana de Catalunya (ERC) - radical Catalan left-wing nationalist*
- *Partit Popular de Catalunya (PPC) - conservative*
- *Iniciativa per Catalunya-Verds (ICV) - Catalan eurocommunist and ecologist groups*

Party	Seats
<i>CiU</i>	46
<i>PSC</i>	42
<i>ERC</i>	23
<i>PPC</i>	15
<i>ICV</i>	9

Table 4.3: Catalan Parliament (2003)

The seat share is shown in Table 4.3.

The Catalan Parliament corresponds to the following weighted majority situation: $[68; 46, 42, 23, 15, 9]$. There are five minimal winning coalitions: $\{CiU, PSC\}$, $\{CiU, ERC\}$, $\{CiU, PPC, ICV\}$, $\{PSC, ERC, PPC\}$, $\{PSC, ERC, ICV\}$. Due to ideological incompatibilities only three minimal coalitions are feasible: $\{CiU, PSC\}$, $\{CiU, ERC\}$, and $\{PSC, ERC, ICV\}$. This means that *PPC* cannot claim for power as it becomes a null player in the reduced game.

We now apply the definition of objection and counterobjection in order to evaluate the Bargaining Set of the reduced game, referring to the parties as 1, 2, 3, 4 and 5 instead of *CiU*, *PSC*, *ERC*, *PPC* and *ICV*.

Then, we consider the game (N, W^m) with $W^m = \{\{1, 2\}, \{1, 3\}, \{2, 3, 5\}\}$. Let $x = (x_1, x_2, x_3, x_4, x_5)$ be an imputation. Player 4 takes necessarily zero. We write now the inequalities that define the Bargaining Set.

1. **Objection of 1 against 2 at x :** $\{1, 3\}$, with (y_1, y_3) , $y_1 > x_1$, $y_3 > x_3$ and $y_1 + y_3 = 1$. Then

$$y_3 = 1 - y_1 < 1 - x_1. \quad (4.2)$$

- Counterobjection of 2 against 1 at y :** $\{2, 3, 5\}$ with (z_2, z_3, z_5) , $z_2 \geq x_2$, $z_3 \geq y_3$, $z_5 \geq x_5$ and $z_2 + z_3 + z_5 = 1$ if $x_2 + y_3 + x_5 \leq 1$, then if

$$y_3 \leq 1 - x_2 - x_5. \quad (4.3)$$

From (4.2) and (4.3) we get

$$x_2 + x_5 \leq x_1. \quad (4.4)$$

2. **Objection of 1 against 3 at x :** the counts are like at step 1. with player 3 instead of player 2. We get the inequality

$$x_3 + x_5 \leq x_1. \quad (4.5)$$

3. **Objection of 1 against 5 at x :**

- $\{1, 2\}$ with (y_1, y_2) , $y_1 > x_1$, $y_2 > x_2$ and $y_1 + y_2 = 1$. Then

$$y_2 < 1 - x_1. \quad (4.6)$$

Counterobjection of 5 against 1 at y : $\{2, 3, 5\}$ with (z_2, z_3, z_5) , $z_2 \geq y_2$, $z_3 \geq x_3$, $z_5 \geq x_5$ and $z_2 + z_3 + z_5 = 1$ if $y_2 + x_3 + x_5 \leq 1$, then if

$$y_2 \leq 1 - x_3 - x_5. \quad (4.7)$$

From (4.6) and (4.7) we get again the inequality in Formula (4.5).

- $\{1, 3\}$. This situation is similar to the previous one and leads to the inequality already seen in Formula (4.4).

4. **Objection of 2 against 1 at x :** $\{2, 3, 5\}$ with (y_2, y_3, y_5) , $y_2 > x_2$, $y_3 > x_3$, $y_5 > x_5$ and $y_2 + y_3 + y_5 = 1$. Then

$$y_3 < 1 - x_2 - x_5. \quad (4.8)$$

Counterobjection of 1 against 2 at y : $\{1, 3\}$ with (z_1, z_3) , $z_1 \geq x_1$ and $z_3 \geq y_3$ if $x_1 + y_3 \leq 1$, then if

$$y_3 \leq 1 - x_1. \quad (4.9)$$

From (4.8) and (4.9) we get

$$x_1 \leq x_2 + x_5. \quad (4.10)$$

5. **Objection of 2 against 3 at x :** $\{1, 2\}$ with (y_1, y_2) , $y_1 > x_1$, $y_2 > x_2$ and $y_1 + y_2 = 1$. Then

$$y_1 < 1 - x_2. \quad (4.11)$$

Counterobjection of 3 against 2 at y : $\{1, 3\}$ with (z_1, z_3) , $z_1 \geq y_1$, $z_3 \geq x_3$ if $y_1 + x_3 \leq 1$, then if

$$y_1 \leq 1 - x_3. \quad (4.12)$$

From (4.11) and (4.12) we get

$$x_3 \leq x_2. \quad (4.13)$$

6. **Objection of 2 against 5 at x :** $\{1, 2\}$ with (y_1, y_2) , $y_1 > x_1$, $y_2 > x_2$ and $y_1 + y_2 = 1$. Then

$$y_1 < 1 - x_2. \quad (4.14)$$

Counterobjection of 5 against 2 at y : $\{1, 3, 5\}$ with (z_1, z_3, z_5) , $z_1 \geq y_1$, $z_3 \geq x_3$ and $z_5 \geq x_5$ if $y_1 + x_3 + x_5 \leq 1$, then if

$$y_1 \leq 1 - x_3 - x_5. \quad (4.15)$$

From (4.14) and (4.15) we get

$$x_3 + x_5 \leq x_2. \quad (4.16)$$

7. **Objection of 3 against 1 at x:** The situation is similar to the one at point 4. and then we easily get the inequality

$$x_1 \leq x_3 + x_5. \quad (4.17)$$

8. **Objection of 3 against 2 at x:** The situation is similar to the one at point 5. and then we easily get the inequality

$$x_2 \leq x_3. \quad (4.18)$$

9. **Objection of 3 against 5 at x:** The situation is similar to the one at point 6. and then we easily get the inequality

$$x_2 + x_5 \leq x_3. \quad (4.19)$$

10. **Objection of 5 against 1 at x:** $\{2, 3, 5\}$ with (y_2, y_3, y_5) , $y_2 > x_2$, $y_3 > x_3$, $y_5 > x_5$ and $y_2 + y_3 + y_5 = 1$. Then

$$y_2 < 1 - x_3 - x_5 \quad (4.20)$$

Counterobjection of 1 against 5 at y:

- $\{1, 2\}$ with (z_1, z_2) , $z_1 \geq x_1$, $z_2 \geq y_2$ if $x_1 + y_2 \leq 1$, then if

$$y_2 \leq 1 - x_1. \quad (4.21)$$

From (4.20) and (4.21) we get again the inequality in Formula (4.17).

- $\{1, 3\}$. This situation is similar to the previous one and leads to the inequality already seen in Formula (4.10).

11. **Objection of 5 against 2 at x:** $\{1, 3, 5\}$ with (y_1, y_3, y_5) , $y_1 > x_1$, $y_3 > x_3$, $y_5 > x_5$ and $y_1 + y_3 + y_5 = 1$. Then

$$y_1 < 1 - x_3 - x_5. \quad (4.22)$$

Counterobjection of 2 against 5 at y: $\{1, 2\}$ with (z_1, z_2) , $z_1 \geq y_1$, $z_2 \geq x_2$ and $z_1 + z_2 = 1$ if $y_1 + x_2 \leq 1$, then if

$$y_1 \leq 1 - x_2. \quad (4.23)$$

From (4.22) and (4.23) we get

$$x_2 \leq x_3 + x_5. \quad (4.24)$$

12. **Objection of 5 against 3 at x:** The situation is similar to the one at point 11. and then we easily get the inequality

$$x_3 \leq x_2 + x_5. \quad (4.25)$$

The Bargaining Set is given by the vectors $(x_1, x_2, x_3, 0, x_5)$ which satisfy the inequalities (4.4), (4.5), (4.10), (4.13), (4.16), (4.17), (4.18), (4.19), (4.24) and (4.25), i.e. the following system

$$\begin{cases} x_2 + x_5 \leq x_1 \\ x_3 + x_5 \leq x_1 \\ x_1 \leq x_2 + x_5 \\ x_3 \leq x_2 \\ x_3 + x_5 \leq x_2 \\ x_1 \leq x_3 + x_5 \\ x_2 \leq x_3 \\ x_2 + x_5 \leq x_3 \\ x_2 \leq x_3 + x_5 \\ x_3 \leq x_2 + x_5 \end{cases}$$

It is now easy to verify that the only solution of this system is given by the vector $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0, 0)$.

We show in Table 4.4 the power share in the Catalan Parliament according to the Shapley-Shubik, the normalized Banzhaf, the Deegan-Packel, the Johnston and the Public Good indices. We can simply notice that these vectors do not belong to the Bargaining Set.

Parties	<i>CiU</i>	<i>PSC</i>	<i>ERC</i>	<i>PPC</i>	<i>ICV</i>
$\phi(v)$	$\frac{25}{60}$	$\frac{15}{60}$	$\frac{15}{60}$	0	$\frac{5}{60}$
$\beta(v)$	$\frac{5}{12}$	$\frac{3}{12}$	$\frac{3}{12}$	0	$\frac{1}{12}$
$\delta(v)$	$\frac{6}{18}$	$\frac{5}{18}$	$\frac{5}{18}$	0	$\frac{2}{18}$
$\gamma(v)$	$\frac{18}{36}$	$\frac{8}{36}$	$\frac{8}{36}$	0	$\frac{2}{36}$
$H(v)$	$\frac{2}{7}$	$\frac{2}{7}$	$\frac{2}{7}$	0	$\frac{1}{7}$

Table 4.4: Power share in the Catalan Parliament according to the main power indices

4.6 Concluding Remarks

The examples of the last two sections show that in the real-world situations the computational effort for determining the Bargaining Set may strongly reduce.

The situation described by Alonso-Mejide and Carreras was a bit different from the one we analyzed in the previous section, as the three minimal coalitions we considered in this example were assumed to be the only feasible coalitions. Anyway, trying to evaluate the Bargaining Set considering only these coalitions as feasible in order to make an objection or a counterobjection, the Bargaining Set turns out to be empty (this is possible as the game does not have all the good properties a usual voting game

has, such as, for example, the monotonicity). We had then to consider the three feasible coalitions as the minimal ones and to allow the possibility of other coalitions, containing one of these ones, to form.

In this chapter we propose to account not only the static power of a party in a Parliament, but also its dynamic power of blackmailing the others; this reasoning results in the well-known concept of the Bargaining Set. A negative aspect is its computational complexity, but in the case of political situations it may be not necessary to account all the winning coalitions, but only the feasible ones and even if the computation is still hard, it may be possible and we could provide the Bargaining Set for two real-world examples whose structure was good enough for the computation. Another important point is if it is possible to check when the other measures of power belong or not to the Bargaining Set, in order to verify their stability. This can be computationally difficult, but still easier than the evaluation of the totality of the stable solutions. In particular, when it is possible, like in the Example 4.5.1, to evaluate the Bargaining Set, it can be also interesting to define which of the classical concepts of solution is the closest to the stability and may better avoid some blackmailing behaviours: in the example of the Catalan Parliament the Deegan-Packel index is the most stable one between the proposed indices (using the Euclidean distance).

The Power of Veto and a New Quantitative Measure

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5.1 Introduction

This chapter deals with the problem of the evaluation of the power of veto, i.e. the capacity of a party to influence the rejection of a proposal.

The most popular real situation which provides an example of veto is given by the United Nations Security Council (UNSC). It is composed of 5 permanent members (a protecting philosophy during the postwar period brought to the designation of the five winner countries of the World War II) and 10 nonpermanent members for two-year terms starting on 1 January, with five replaced each year. We list in Table 5.1 in alphabetic order the composition of the Security Council in January 2012 and the year of term's end or the permanent member status.

Each Council member has one vote and decisions on procedural matters are made by an affirmative vote of at least 9 of the 15 members. On the other side, decisions on substantive matters require 9 votes, including the concurring votes of all 5 permanent members. This rule, called *great Power unanimity*, is often simply referred to as the *veto power* and the 5 permanent members are called *veto players*¹.

¹For further details on the UNSC see <http://www.un.org/Docs/sc/index.html>, [Online: accessed 27 October 2012]

Membership	year of term's end
Azerbaijan	2013
China	permanent member
Colombia	2012
France	permanent member
Germany	2012
Guatemala	2013
India	2012
Morocco	2013
Pakistan	2013
Portugal	2012
Russian Federation	permanent member
South Africa	2012
Togo	2013
United Kingdom	permanent member
United States	permanent member

Table 5.1: Members of the UNSC in January 2012

Among others, functions and powers of the UNSC under the Charter are to maintain international peace and security in accordance with the principles and purposes of the United Nations, which is the primary responsibility, but also to investigate any dispute or situation which might lead to international friction, to take military action against an aggressor, etc. The UNSC alone has the power to take decisions which Member States are compelled to carry out under the Charter.

As observed by Mercik [70], quite intuitively the right of veto will increase the power of a player in most cases. This is exactly the case of the UNSC, where the five permanent members have a large power in the decision process of substantive matters, not only because of the absence of a term-period, but mainly because of their right to veto.

In Game Theory the concept of *veto* is mainly associated to the concept of *veto player*. According to the classical definition (e.g. Osborne and Rubinstein [78]), in a simple game a veto player belongs to all winning coalitions. The analysis of the presence of veto players is fundamental in the study of a simple game, as they are related to some properties of the game itself. In particular

- if there exists at least one veto player the game is proper;
- there is no veto player if and only if the core is empty;
- if the set of veto players is nonempty then the core is the set of all nonnegative feasible payoff profiles that share the income only between the veto players.

These simple observations show how veto players have a central role in a simple game.

The notion of veto player may be generalized, for several players, to the notion of *blocking coalition*, that is a coalition such that the other players outside the coalition are not able to pass a proposal. A more restrictive definition says that a coalition is blocking if not only its complementary, but also the coalition itself is losing; this is the definition of the set Q given in Section 2.1.

It is common sense that a veto player has full veto power, but it is interesting to give an estimation of the intermediate power of those players which are not of veto, but can block a proposal by forming suitable coalitions. It should be clear that veto power is larger for larger parties, or agents with higher weights, so it is possible to increase the veto power of smaller parties introducing a second parameter, usually imposing not only a higher majority quota, so that the weight of winning coalitions has to be high enough, but also a minimum number of players that are in favor of the proposal; in this way the number of blocking coalitions raises (see Peleg [83]). For instance, according to the Treaty of Lisbon, the qualified majority in the European Council requires the approval of at least 55 % of the member States that have to represent at least 65 % of the population of the European Union. Another possibility for increasing the number of blocking coalitions is to have multicameral systems (see Gambarelli and Uristani [42]) or endowing the President of the state with the right to ask for a revision of the proposals.

In a chapter devoted to present the power of veto, it is important to remember that in Game Theory applied to political studies there is a large literature on the topic. Tsebelis in his veto players' theory [93] states that a veto player is an individual or collective actor whose agreement is necessary for policy changes. The *policy stability*, i.e. the impossibility of significant changes of the *status quo*, is strictly related to the role of veto players, as a significant policy change has to be approved by all of them. In his large literature on the topic, Tsebelis analyzes the connections between veto players and other important features: the agenda control or the production of significant laws, for example, are strictly related to this topic and veto players' theory represents for Tsebelis a way to unify the understanding of politics. Moreover the work of Tsebelis reminds us to specify another important problem in veto theory: next to the classical definition of veto player, there exist many others which can provide a totally different approach to the problem: Tsebelis, for example, assumes it is sufficient to be critical in the actual winning coalition, i.e. to be a crucial cabinet party, but his opinion has been largely criticized. The problem of how to define and identify veto players is a controversial point in the studies of a theory of veto.

Stated that the power of veto represents a central topic in politics, it is natural to ask: how to evaluate it? This question brought, in the last years, to an increasing in the number of papers and surveys on the topic, but the attention to veto power indices in the literature is still less than that devoted to power indices. Two questions arise at first: are veto power and power analogous concepts? May we evaluate them with

the same instruments? The wide range of existing power indices takes into account different features. In order to better study political situations, the possible ideological affinities have been analyzed, accounting possible existing relations among the players. We dedicated Chapter 3 to this topic.

In this chapter we want to support our idea that some different features have to be considered in order to define an index suitable for analyzing the power of veto and the power of blocking (the two concepts will be assumed to be strictly related and a common theory will be developed). A party, for example, can be able alone to block a proposal voting against it, but it may not have the possibility to make an opposite law being approved without the support of other parties; this happens, for example, to a permanent member of the UNSC, which has full veto power, but not full power according to the classical indices. Moreover, the concepts of *a priori unions* (Owen [81] and [82]), of *connected coalitions* (Myerson [74]) and of *contiguity* (Fagnelli et al. [36], Chessa and Fagnelli [25]), which have been introduced to better represent the relations between parties, are no longer relevant while speaking about the power of a party which is against the approval of a proposal. In fact, in order to block a proposal it is not necessary anymore to have a common ideological position: two parties very far from each other can be both, because of opposite reasons, decide to vote against a law, even if this does not mean that they would agree in making a common different proposal being approved.

After proposing the work of Tsebelis, who analyzed the veto from a politological point of view, then we present some works which all dealt, in many different ways, with the same problem: finding a suitable way for analyzing the power of block a proposal. We start from the work of Carreras ([22] and [23]), where there is a large analysis of the blocking power in a simple game, both from a collective and from an individual point of view. The *strict protectionism index* and the *Banzhaf strict blocking power index* are proposed as suitable tools to deal with the problem. After we present the *blockability relation*, which has been formalized by Ishikawa and Inohara [52] and the work of Kitamura and Inohara [57], that just a couple of years after defined the *blockability index*, a power index for coalitions. Another important contribution to the problem is given by Mercik [71], who studied the problem of evaluating the power of veto using the Johnston power index, suggested after defining some suitable axioms that an index for veto should satisfy.

In this chapter, after presenting and partially analyzing the cited literature about veto maintaining an informal tone, we propose a quantitative approach to the problem of evaluating the power of veto, which takes inspiration from the previous work of Carreras. We start from the observation that it is not necessary anymore that the power of the agents sum up to a given fixed number, which is normally assumed to be equal to 1, as one, two, or all the agents of a voting procedure may have full power to block a proposal. We will define an index which assigns full veto power (for simplicity equal to 1) to all those agents who are able alone to veto a proposal; all

the other players will be given a nonnegative veto power smaller than 1 according to their possibility to stop an approval joining other players. The definition of this new veto power index is parallel to the definition of a new power index, that is introduced in order to give a more complete scenario.

The chapter is organized as follows: in Section 5.2 we present Tsebelis' veto players' theory, as main contribution to the topic from a politological point of view. In Section 5.3 we present the contribution of Ishikawa, Kitamura and Inohara, in Section 5.4 the on going studies of Mercik and in Section 5.5 the work of Carreras. In Section 5.6 we present our contribution to the topic, with the definition of two new quantitative indices, one for measuring the power and the other one for measuring the power of veto; some natural relations with the definition of success and decisiveness of Laruelle and Carreras are presented in Section 5.7. Finally, the correspondence of these indices with the Nash equilibria in a Bayesian game is presented in Section 5.8.

5.2 Tsebelis: The Veto Players' Theory

This section presents one of the most influential conceptions of veto players under a politological point of view, developed by Tsebelis [93]. In order to change policies (the legislative *status quo*) a certain number of individual or collective actors have to vote in order to promote the outcomes that they prefer. In his *veto players' theory*, he defines a veto player as an actor whose agreement is necessary for a change of the status quo. According to Ganghof [43], this theory *relies on the concept of veto player in order to unify the comparative analysis of political systems and aims at unifying our causal understanding of politics*. In fact, the configuration of the veto players and the sequence of them in order to make policy decisions affect the set of outcomes that can be approved and this common argument is applied to study the main characteristics of a polity.

Tsebelis distinguishes two types of veto players: *institutional* and *partisan*. Institutional veto players are established by a country's constitution. This happens, for example, in the US, where the constitution specifies that legislation requires the approval by the President, the House of Representatives, and the Senate (ignoring veto overrule). Then, the three actors are the institutional veto players in the US. By a game theoretical point of view, an institutional veto player is established to be part of every winning coalition in a game, independently from the possible weighted majority situation corresponding to the game. Partisan veto players are determined by the political competition inside a country, i.e. by the political game itself. It may be, for example, that the US House of Representatives is controlled by a single cohesive party, and the only successful pieces of legislation are the ones supported by this party. While the House of Representatives is the institutional veto player, the majority party is the real partisan veto player. Referring to the classical definition of simple game, the corresponding weighted majority situation, if it exists, establishes the partisan veto

players depending on their weight.

Classifying the possible actors endowed with veto power in a parliamentary system, both institutional and partisan, Tsebelis lists at first the upper house, that, in most the cases in which it has veto power, is controlled by the same coalition as the government (but there are some exceptions). Then the head of state, which can be endowed with veto power, even if this is not very common in West European countries. Lastly, most of the times veto players are the government partners in a parliamentary system. Even if veto players can be individual or collective agents, for sake of simplicity, we refer now to individual veto players. Moving from individual to collective veto players may generate serious problems for the analysis because they cannot necessarily decide on what they want. A realistic way to eliminate the problem and to calculate the outcome of collective choices is to consider the situations in which the decisions of veto players are taken, for example, by simple or by qualified majority, but we do not enter in details in this thesis and we address the reader to Tsebelis [93] for a complete analysis of the problem.

An individual veto player is identified by his ideal point in an n -dimensional *policy space*. Every veto player has circular indifference curves and he is indifferent between alternatives that have the same distance from his ideal point. Given a veto player located in A and the status quo SQ , he will prefer (and then he will vote in favor of) anything inside the circle centered in A and with radius $|SQ - A|$. Each agent has an ideal point and his main goal is to move the collective decision as close as possible to this point.

The *winset of the status quo* ($W(SQ)$) is the set of points that are preferred over the status quo by all the veto players, i.e. the set of policies that can replace the existing one. Obviously, anything out of $W(SQ)$ cannot be approved as at least one veto player prefers the status quo to the proposal and he will vote against it.

The size of the winset of the status quo has specific consequences on policymaking: significant departures from the status quo are almost impossible when the winset is small, that is, when veto players are many or when they have significant ideological distances among them. Tsebelis calls this impossibility for significant departures from the status quo *policy stability*. He affirms that $W(SQ)$ does not enlarge with the number of veto players or with the distance among veto players along a given line. These two results are then unified affirming that, in one dimension, policy stability depends on the maximum ideological distance among veto players, not directly on their number. Obviously, adding a new veto player can increase the ideological distance and, consequently, the policy stability, by reducing the winset of the status quo.

We have already mentioned that Tsebelis' veto players' theory aims at unifying the understanding of politics. Starting from the concept of veto player, Tsebelis analyzes a wide range of topics: for example, he makes a direct and cross-national test of the prediction that the number of significant laws produced by a coalition government, particularly if there are wide ideological differences among government partners, is

significantly lower than the number of important laws produced by single-party government or by coalitions with partners that agree (see [91]). In the same paper he observes also that more veto players mean less government control of parliamentary agenda (which is a very important tool in all collective decision-making bodies, for further details we address to Nurmi [77]); even if he notices it is not simply a mere correlation, which of the explanations he provides is closer to the truth is left as an open question for further investigation. Hug and Tsebelis [51] see in the possibility of a referendum the introduction of one additional veto player in each country: the population. As a result, the political outcomes are moved closer to the preferences of the median voter, but strong policy changes are made more difficult. In a work of Yataganas and Tsebelis [97] the consequences of the Treaty of Nice, such as the shifting of the balance of legislative power in favor of the European Council and the more bureaucratic policy making, are explained using veto players' theory. The opinion of the authors is that after the Treaty of Nice the EU's decision-making process has become more complex, opaque and difficult.

These ones are just some of the topics that Tsebelis analyzed through veto players' theory, but the existing literature he provided is much wider (see, for example, [90] and [92] or the work of Chang and Tsebelis [24]).

Even if Tsebelis' veto players' theory is recognized as one of the most influential in the topic, many competing assumptions have been developed. In particular, we refer to McGann [67] who affirms that a true veto player must be a member of every possible winning coalition, according to the classical definition of veto player given for a simple game, and not only a member of the actual winning coalition. Tsebelis' theory allows only cabinet parties being veto players, but there are some critics to this point also by Ganghof and Bräuninger [44]. More generally, the debate on how to identify veto players shows that how to give a unique interpretation is still an open problem.

Another problem raised by Ganghof [43] as a critic to Tsebelis' veto players' theory is about how measuring preferences, as they often come from empirical estimates which may largely differ from the final policy preferences, bringing to a gap between theory and measurement. A wrong estimate may lead to a failure of the veto players' theory and of its potential in analyzing political systems, for example in predicting policy stability.

5.3 Inohara, Ishikawa and Kitamura: The Blockability and Desirability Relations on a Simple Game

In this section we want to present an important instrument which has been defined to describe the power to block of the coalitions, opposed to the power to win. As in our opinion, the authors are convinced that they are deeply different from each other and we think it is important to dedicate a section to the instruments and the arguments they illustrate.

Ishikawa and Inohara [52] propose a new method to compare nonwinning coalitions in the framework of simple games, the *blockability relation*. At first they recall the already existing *desirability relation on (N, W)* (c.f. [29]), denoted by \succeq^d and defined as: given $S, S' \subseteq N$, $S \succeq^d S' \Leftrightarrow$ for all $B \subseteq N$ such that $B \cap (S \cup S') = \emptyset$, if $B \cup S' \in W$ then $B \cup S \in W$. Note that, if $S' \subseteq S$, or if S is winning and S' is arbitrary, then $S \succeq^d S'$. Thus, the “top” of this ranking (preordering) is occupied by the grand coalition N .

Then, they propose the following *blockability relation on (N, W)* , denoted by \succeq^b and defined as: given $S, S' \subseteq N$, $S \succeq^b S' \Leftrightarrow$ for all $T \in W$, if $T \setminus S' \notin W$ then $T \setminus S \notin W$. Note that, if $S' \subseteq S$, or if S is loosely blocking and S' is arbitrary, then $S \succeq^b S'$. In this case, the “top” of this ranking (preordering) is occupied by the maximal blocking coalitions (in loose sense).

These two definitions are on the set of all coalitions. For both the desirability relation and the blockability relation, there exist versions on the set of all feasible coalitions (this is why they are often referred to as the desirability relations and the blockability relations), but we do not enter into details and we just consider the easiest case.

Ishikawa and Inohara notice that the blockability relation is definitely different from the desirability relation: being a winning coalition is regarded as the influence in the desirability relation, whereas the influence is expressed as the capability of a coalition to block other coalitions to be winning in the blockability relation. Desirability relation compares coalitions with respect to how close the coalitions are to be winning coalitions, that is how close the coalitions are to have enough power to completely control the decision of the situations. The blockability relation compares coalitions in terms of how much the coalitions can make winning coalitions non winning, that is, how much the coalitions can make other coalitions not have enough power to completely control the decision of the situation. Then, it turns out that these relations are totally different from each other.

Differently from the desirability relation, the blockability relation is always transitive. The authors focus on this point to affirm that the blockability relation is more appropriate than the desirability relation for the purpose of comparison of influence of coalitions on group decision-making, then once more the attention is focused on the power to block versus the power to win in the analysis of a game. They take the transitivity of the blockability relation as a convenient feature for defining a blocking power index.

Kitamura and Inohara [56] provide a necessary and sufficient condition for having a complete blockability relation on a simple game. They prove that given a game (N, W) and the blockability relation \succeq^b on (N, W) , such a relation is complete iff the simple game is *S-unanimous* for a coalition S , i.e. iff $W^m = \{S\}$. They propose to employ the *S-unanimous* simple games to provide a new power index which is consistent with the blockability relation. Such an index should indicate the power of each coalition as well as that of each member of N , in spite of the traditional ones, which are not convenient to compare power of coalitions; this is why we refer to such

an index as a *coalitional power index*, as it is a function $\psi : 2^N \rightarrow \mathbb{R}$ which assigns to each coalition a real number that indicates its power. Kitamura and Inohara [57] propose the *blockability index* as an example of coalitional power index. It is defined as

$$b(S) = \frac{|B(S)|}{|W|}, \quad (5.1)$$

where $B(S) = \{T \in W \mid T \setminus S \notin W\}$ is the set of all the winning coalitions that are blocked by coalition S . The larger the number $|B(S)|$ of winning coalitions that S can block is, the more blocking power coalition S has as a whole. This index measures the power of each coalition with respect to how many winning coalitions it can turn losing by withdrawing. It is also confirmed that the blockability index coincides with the Banzhaf value except their constant coefficients on oneplayer coalitions, so it can be seen as an extension of it to evaluate each coalition of parties.

5.4 Mercik: Measuring and Axiomatizing the Power of Veto

In the UNSC, the veto of one of the permanent members on substantive matters does not allow the proposal passing under any condition. In the Congress of the United States of America decisions are taken by approval of the Senate and of the House of Representatives using the straight majority rule in each one of them; the President is endowed with veto power and he can reject a bill. This situation is much different from the previous example, as the presidential veto right can be overruled by a 2/3 majorities in both the chambers. The standard definition of veto player is no longer valid and we have to deal with a bit different situation, as the President does not belong to all winning coalitions, but he is endowed anyway with a kind of veto right.

Mercik [70] describes the two different situations defining the *veto of the first degree* as the one which cannot be overruled, as it happens in the UNSC, and the *veto of the second degree*, on the other hand, as the one which can be overruled, as for the President of the United States.

Similarly to what happens in the United States, in Poland, according to the Constitutional Act, after a bill has been approved by the *Sejm* (the Polish lower chamber) is considered by the Senate, which may accept, amend or reject it. If a bill is amended or rejected by the Senate, then it goes back to the Sejm. The Sejm may, by absolute majority, reject the Senate's objection. After that, a bill accepted by the Sejm goes to the President who signs and declares the bill in the official monitor (*gazette*). It may happen that in the case of important state interests or poor quality of constituted law, the president may reject the bill, using his power of veto. The *Sejm* may accept the bill one more time by a majority of 3/5 of votes in the presence of at least half of the representatives and in this case the President's veto is overruled and he has to sign the bill. Differently from the United States case, in Poland any Senate's objection can always be rejected.

The two mentioned examples do not represent exceptional cases, but they give an idea of a very common political tool, the veto of the second degree, that we can find

in many different political situations.

After defining two different degrees of veto, Mercik poses a question: is blockability equivalent to veto?

The veto right awarded to one or more players is often taken in the literature as an example in the study of the blocking coalitions and in the definition of blocking power indices. Referring to the blockability principle of Ishikawa and Inohara (see Section 5.3), Mercik affirms that it is fulfilled only for veto of the first degree, while veto of the second degree may not fulfill it, concluding that blockability may be not equivalent to veto; in particular, blockability principle is stronger than veto.

Mercik [70] proposes to use the Johnston power index [53] to study the problem of evaluating the power of veto, comparing the power of players with and without the right of veto. He observes, for example, that using the mentioned index the power of a permanent member in the UNSC is 103 times the power of a nonpermanent member (!). Quite intuitively, the right of veto will increase the power of a player in most the real examples.

A measure of veto power by comparing the power of players with and without veto is always possible when studying an example of veto of the first degree. The situation is a bit more complicated when analyzing a case of veto of the second degree, as we will see just after.

In Mercik [69] we have an evaluation of the power of the members of a legislative process in Poland, i.e. of the President, the *Sejm* and the Senate. He analyzes and evaluate the Johnston power index in three different cases, assuming at first that there are no party structures (the representatives vote independently) in the Parliament, then that just the *Sejm* has a party structure (the representatives vote according to the party) and finally that the *Sejm* has still a party structure, but the President favors one of the opposition parties (the case in which the President and the government represent the same political faction is not analyzed, as in this case the President would not veto a bill supported by the government). In the last two examples the power is evaluated for the parties which are in the government.

In the first case the President of Poland is more than 12 times stronger than the *Sejm* as a whole (which is assumed to have a power equal to the sum of the power of the representatives). The position of the President changes radically introducing a party structure. The power of the President is decreased deeply, even when he is attributed with veto. We may say that the power of the President in this case consists of power of veto, and in this situation the Johnston power index for the President measures only the power of veto directly.

In the more complex situations of veto of the second degree, strategic thinking plays a relevant role: veto may become conditional and depending on other decision-makers. The legislative way a bill goes through the Polish parliamentary system may be different from time to time (e.g. the *Sejm* may be in action one, two or three times depending on the restrictions of the Senate and of the President). The estimation of veto power is connected with the President only but depends on the power of other players, which must look ahead and reason back. Mercik proposes to use a strategic

type power index for the power evaluation, suggesting, for example, the *strategic power index* defined by Steunenbergh et al. [89]. Employing the results of noncooperative sequential games in which players decide on their actions at different stages, it allows players acting strategically integrating actors preferences, as well as the rules of the decision-making process.

According to the opinion of Mercik, the estimation of veto right, in particular the one of the second degree, cannot be always done so directly as in the example of Poland and a new index remains to be defined for complicated games (compound games), with more general structure. Mercik [71] proposes a possible starting point for the solution of this open problem: to define a partition structure including the issue of veto (the first or the second degree) and then to look for the best power index to apply to this problem. As in the classical literature, he proposes some suitable axioms that such an index should satisfy. He shows that the Johnston power index does not look so bad for games with veto (also from an axiomatic point of view), but that it represents the optimal index for this kind of evaluation has to be discussed and proved yet.

5.5 Carreras: Protectionism and Blocking Power Indices

In this section we recall two works of Carreras in 2005 [22] and in 2009 [23]. Our work, provided in the next section, is mainly based on these two articles. The definition of veto player is the classical one which corresponds to the veto of first degree of the previous section. In this case, as observed by Mercik, the concepts of veto and of blockability coincide.

First, we recall the work of Carreras [22] mainly based on the idea of providing a numerical measure of the agility of the collective decision-making mechanism. Carreras defines the *decisiveness index* of the game (N, W) as

$$\delta(N, W) = \frac{|W|}{2^n} \quad (5.2)$$

where $n = |N|$. It gives the probability that an abstract proposal will pass in (N, W) , where each agent $i \in N$ has only two options: voting for the proposal (Y) or voting against it (N), with probability 1/2 (the abstention is allowed, but it counts for "against"). We remark that the probabilities of the agents are assumed to be independent. The motion will pass if and only if the set of agents that vote for Y is a winning coalition $S \in W$. Obviously $0 < \delta(N, W) < 1$ as $\emptyset \notin W$ and $N \in W$. Given two simple games (N, W) and (N, W') with the same player set, $|W| < |W'|$ implies $\delta(N, W) < \delta(N, W')$. If a game is decisive, then $\delta(N, W) = 1/2$ independently of the number of involved players. Since no improper and weak weighted majority game exists, in this subclass the index 1/2 characterizes the decisive games. Carreras also proves that when a game is weak and proper, the decisiveness index is smaller than 1/2 and when it is improper and strong the index is greater than 1/2.

The next work of Carreras [23] is about the concept and the role of blocking power in a simple game and it is drawn near the strictly related idea of veto. In that paper,

starting from the description of a simple game through the set of winning coalitions, Carreras specifies that, sometimes, some additional restrictions may decrease the number of them. He refers to a protecting philosophy which may bring, for example, to the designation of some veto players and he cites the classical example of the UNSC already mentioned in this chapter. In this contest, the so called *blocking coalitions*, i.e. those coalitions that, even if not winning, are powerful enough to prevent a proposal to pass, become relevant. We refer now to the definition of decisive winning, conflictive winning, blocking and strictly losing coalitions given in Section 2.1, as they are the ones adopted by Carreras; in particular according to him a blocking coalition is a losing coalition whose complementary is losing too.

Applying the decisiveness index to the dual game (N, W^*) , it is easy to verify that

- $\delta(N, W^*) + \delta(N, W) = 1$
- $\delta(N, W^*) - \delta(N, W) = \frac{|Q| - |C|}{2^n}$

Then, focusing on the dual game, he states that it allows defining an obvious protectionism index, which Carreras calls the *loose protectionism index*, based on the idea of providing a numerical measure of the inertia of any decision-making mechanism; in formula

$$\delta^*(N, W) = \delta(N, W^*) = 1 - \delta(N, W). \quad (5.3)$$

It gives the probability that a proposal will not pass in (N, W) , where again each agent has the options to vote for or against the proposal, with probability 1/2. This index is suggested as a possible choice to define a collective blocking index for simple games to measure the blocking capability at collective level, analyzing the game by a protectionism viewpoint. But as $\delta^*(N, W) = 1 - \delta(N, W)$, Carreras observes that it does not provide any new information. Analogously, he mentions the idea of defining an individual blocking index, which he calls the *Banzhaf loose blocking power index*, by

$$\beta_i^*(N, W) = \beta_i(N, W^*). \quad (5.4)$$

The reason why of this possible definition is that Carreras observes that in some manner the Banzhaf index measures the decisiveness of a game from a *local* viewpoint i.e., from the perspective of each player, mainly because of the following result

$$\beta_i(N, W) = 2\delta(N, W) - 2\delta(N_{-\{i\}}, W_{-\{i\}}), \quad (5.5)$$

where $(N_{-\{i\}}, W_{-\{i\}})$ denotes the residual game that arises when player i leaves, i.e. the subgame with players set $N \setminus \{i\}$. Unfortunately, neither this choice would bring any additional information to the study of the structure of the game, as $\beta_i^*(N, W) = \beta_i(N, W)$.

In [23] Carreras aims at defining some blocking indices, both from a collective and from an individual point of view, i.e. indices that, differently from the classical power indices which represent the power to win, can represent the power to block and that can add new information about the game. Starting from the observation that

when the set of winning coalitions reduces, the set of blocking coalitions necessarily increases, he conjectures a relation between the two families and the possibility of defining a game from the set of blocking coalitions. He concludes that a blocking set $Q \subseteq 2^N$ determines univocally a game iff it is *separating*, i.e. iff for each $S \subseteq N$, there exists some $T \in Q$ s.t. $S \subseteq T$ or $T \subseteq S$.

He defines a collective blocking index for simple games, which he calls the *strict protectionism index*, as

$$\pi(N, W) = \frac{|Q|}{2^n}, \quad (5.6)$$

in order to represent the power of the collectivity to block. Then, he defines a *blocking swing* for player $i \in N$ as a pair $(S, S \setminus \{i\})$ s.t. $S \in Q$ and $S \setminus \{i\} \notin Q$ and he call the *Banzhaf strict blocking power index* an individual blocking power index for the players involved defined as

$$\rho_i(N, W) = \frac{\xi_i(N, W)}{2^{n-1}}, \quad (5.7)$$

where $\xi_i(N, W)$ is the number of blocking swings for player i . The strict protectionism index is clearly somehow close to the definition of decisiveness index, given in Formula (5.2), as it gives the probability that an abstract proposal will be blocked (instead of the probability that it will pass, as for the decisiveness index) in (N, W) where each agent $i \in N$ votes for the proposal (Y) or against it (N) with probability 1/2.

Differently from the indices in (5.3) and in (5.4), these two new indices, π and ρ , add information in the description of the game. For example, nonequivalent players for a power index, as β , can be equivalent for a blocking power index, as ρ , underlining the natural fact that making a coalition winning by joining the coalition itself is not equivalent to making a coalition losing by leaving. In view of this, it is interesting to notice that, somehow, the Banzhaf strict blocking power index and the classical Banzhaf value have a relation: even if they are different, in the games with high number of blocking coalitions they can be very close each other and we obtain that $\rho_i(N, W) \approx \beta_i(N, W)$ for almost all $i \in N$. The UNSC is an example of a system where the power to win and the power to block perfectly coincide following the definitions proposed by Carreras, as $\rho_i(N, W) = \beta_i(N, W)$ for every $i \in N$.

Unfortunately, axiomatic characterizations for π and ρ are not yet available. Carreras proposes also to generalize π and ρ , in the same way δ and β have been generalized in [21] to the α -decisiveness index and the *Banzhaf α -index*. These new indices are built starting from the observation that the decisiveness index can be calculated in terms of the multilinear extension of the game (see Owen [79]) by replacing each variable with 1/2. Then, this suggests that, in fact, any values of the variables might make sense and this leads to a new generalized voting model and its corresponding decisiveness and Banzhaf indices. Thus, by assuming α is the vector which assigns to each player the probability to be in favor of a law, Carreras defines the α -decisiveness index and checks that it can be computed by means of the multilinear extension of the simple game to which it applies. Moreover, the analogue of the relationship between decisiveness and the Banzhaf value gives rise to a definition of a Banzhaf α -index, and

its computation in terms of multilinear extension is also provided, always generalizing the formal case.

5.6 A New Quantitative Index for Evaluating Veto Power

In this section we directly refer to the work of Carreras presented in Section 5.5. He defined the decisiveness index and the loose protectionism index of a game here presented in Formula (5.2) and in Formula (5.3) respectively. These indices describe the game entirely and only the first one has been adopted by Carreras as really useful, as they both give the same information in the description of the game. However, starting from this idea, we propose two new indices which are defined for a single player in order to measure the decisiveness and the protectionism of the game from the perspective of each player.

Let W_i be the set of winning coalitions including player i , i.e. $W_i = \{S \in W : i \in S\}$. The *decisiveness index of player i* is defined as

$$\delta_i(N, W) = \frac{|W_i|}{2^{n-1}} \quad (5.8)$$

where $n = |N|$. It gives the probability that a proposal will pass in (N, W) when we already know that player i votes in favor of the proposal (Y) and the other players vote in favor or against with probability 1/2.

Obviously, $0 < \delta_i(N, W) \leq 1$ as $N \in W_i$. If the outcome A stands for “the proposal passes” and B for “player i votes in favor”, the index corresponds to the conditional probability

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Proposition 5.6.1. *When i is a veto player, $\delta_i(N, W) = 2\delta(N, W)$*

Proof. Writing the conditional probability as $P(A|B) = \frac{P(B|A)P(A)}{P(B)}$ and considering that $P(B) = 1/2$, $P(A|B) = \delta_i(N, W)$ and $P(A) = \delta(N, W)$, we obtain $\delta_i(N, W) = 2P(B|A)\delta(N, W)$. When i is a veto player $P(B|A) = 1$ then $\delta_i(N, W) = 2\delta(N, W)$. \square

The *loose protectionism index of player i* is defined as

$$\delta_i^*(N, W) = \frac{2^{n-1} - |W| + |W_i|}{2^{n-1}} = 1 - 2\delta(N, W) + \delta_i(N, W). \quad (5.9)$$

It defines the probability that a proposal does not pass in (N, W) when we know that player i votes against the proposal (N) and the others vote in favor or against with probability 1/2.

Obviously, $0 < \delta_i^*(N, W) \leq 1$ as $\emptyset \notin W$. The numerator counts the number of losing coalitions which do not include player i , i.e. the number of coalitions without player i voting in favor of the proposal but not being able to make it approved.

We may observe that player i is a dictator iff $\delta_i(N, W) = 1$ and is of veto iff $\delta_i^*(N, W) = 1$. The indices $\delta_i(N, W)$ and $\delta_i^*(N, W)$ are strictly related and this relation depends on the decisiveness index of the game. In particular

- if the game is weak and proper, as $\delta(N, W) < 1/2$, we get that $\delta_i^*(N, W) > \delta_i(N, W)$ for each $i \in N$;
- if the game is strong and improper, by duality $\delta(N, W) > 1/2$ and $\delta_i^*(N, W) < \delta_i(N, W)$ for each $i \in N$;
- if the game is decisive, then a player is of veto ($\delta_i^*(N, W) = 1$) iff it is a dictator ($\delta_i(N, W) = 1$), in general when the game is decisive, $\delta_i(N, W) = \delta_i^*(N, W)$ for each $i \in N$. In a weighted majority game, as there are no weak and improper games, we are always able to say if the players will have a higher power or a higher power of veto, or if they are equivalent.

The two indices are also directly related as stated in the following proposition.

Proposition 5.6.2. $\delta_i^*(N, W) = \delta_i(N, W^*)$, for every $i \in N$.

Proof. We have that $\delta_i(N, W^*) = \frac{|W_i^*|}{2^{n-1}}$ and $\delta_i^*(N, W) = \frac{2^{n-1} - |W| + |W_i|}{2^{n-1}}$.

Let $D_i = \{S \in D : i \in S\}$, $C_i = \{S \in C : i \in S\}$, $Q_i = \{S \in Q : i \in S\}$ and $P_i = \{S \in P : i \in S\}$. We want to show that

$$2^{n-1} - |W| + |W_i| = |W_i^*|$$

i.e.

$$|D_i| + |C_i| + |Q_i| + |P_i| - |D| - |C| + |D_i| + |C_i| = |D_i| + |Q_i|$$

$$|D_i| + 2|C_i| + |P_i| = |D| + |C|$$

and this is true as $|D_i| + |P_i| = |D|$ and $2|C_i| = |C|$. In fact

$$\begin{aligned} |D_i| + |P_i| &= |\{S \in W, i \in S : N \setminus S \notin W\}| + |\{S \notin W, i \in S : N \setminus S \in W\}| \\ &= |\{S \in W : N \setminus S \notin W\}| = |D| \end{aligned}$$

and

$$|C_i| = |\{S \in W, i \in S : N \setminus S \in W\}| = |\{S \in W, i \notin S : N \setminus S \in W\}| = |C \setminus C_i| = |C| - |C_i|$$

□

Proposition 5.6.2 gives the possibility of defining the loose protectionism index of player i simply as the decisiveness index of player i evaluated on the dual game.

As observed by Carreras [22], the Banzhaf index [12] also measures the decisiveness of a game from the perspective of each player. The basic relationship between the decisiveness index and the Banzhaf index in Formula (5.5) suggests us to look for a possible relation between the Banzhaf index and the indices defined in (5.8) and (5.9). When $i \in N$ is a veto player, as Carreras noticed $\beta_i(N, W) = 2\delta(N, W)$, then

by Proposition 5.6.1 we simply get that when i is a veto player, $\beta_i(N, W) = \delta_i(N, W)$.

The indices in (5.8) and (5.9) are quantitative indices and we adopt them as a measure to evaluate the power of a player in making a proposal been accepted (δ_i) or rejected (δ_i^*).

In Example 5.6.1 we compute the previous indices for a simple theoretical situation. For sake of completeness, we add the Banzhaf strict blocking power index in Formula (5.7) and the Johnston index in Formula (2.6), J , as suggested by Mercik [70]. The comparison is carried out using the ratios of the indices among the players, as not all the indices sum up to one.

Example 5.6.1. Consider the simple weighted majority situation $[6; 2, 3, 5]$ representing a Parliament with only three parties, then $N = \{1, 2, 3\}$. The winning coalitions are $\{1, 3\}$, $\{2, 3\}$ and $\{1, 2, 3\}$.

The decisiveness index of the game is

$$\delta(N, W) = \frac{3}{8}$$

and the loose protectionism index is

$$\delta^*(N, W) = \frac{5}{8}.$$

We evaluate now the decisiveness index and the loose protectionism index of the parties

$$\delta_1(N, W) = \frac{1}{2} \quad \delta_2(N, W) = \frac{1}{2} \quad \delta_3(N, W) = \frac{3}{4}$$

$$\delta_1^*(N, W) = \frac{3}{4} \quad \delta_2^*(N, W) = \frac{3}{4} \quad \delta_3^*(N, W) = 1$$

We observe that player 1 has full veto power being a veto player, while no player is a dictator. This is an example of a weak and proper game, then the loose protectionism indices of the players are greater than their decisiveness indices.

Evaluating now the Banzhaf strict protectionism index and the Johnston index, we obtain

$$\rho_1(N, W) = \frac{1}{4} \quad \rho_2(N, W) = \frac{1}{4} \quad \rho_3(N, W) = \frac{1}{4}$$

$$J_1(N, W) = \frac{1}{6} \quad J_2(N, W) = \frac{1}{6} \quad J_3(N, W) = \frac{2}{3}$$

The Banzhaf strict protectionism index assigns the same power of blocking to every party, in particular also to party 3, which is a veto player. The Johnston index assigns to party 3 four times the power given to the others, while for our index of veto it has only four thirds of the power of the other parties.

The sum of the veto power of the agents may be lower than 1, but this requires that there is a large number of winning coalitions, and consequently a small number of blocking coalitions. A simple situation is represented by a restricted committee that have to decide which proposal may be admitted to a large assembly examination (e.g.

a parliamentary commission that have to decide which laws can be discussed by the Parliament). Usually, a very low majority is required, just to avoid to waste time on proposals that are of no interest for anybody. If we suppose that the committee is formed by 7 representatives and it is necessary that at least two of them vote in favor of discussing it we have that the blocking coalitions are those of 6 or 7 players. So the veto power of each person is $7/64$ and the total is $49/64$.

5.7 The New Indices as a Measure of “Success”

We now introduce some basic notions which are behind the definition of voting power in order to apply them to the two new indices introduced in the previous section. We initially refer to the work of Laruelle and Valenciano [58], in which the authors propose a simple model for measuring success or decisiveness in voting situations. We start by the following two definitions *ex post*, i.e. after a decision is made. Once a decision is taken, if S is the set of players who voted in favor of the proposal (Y):

- voter i is said to have been *successful* if the decision coincides with voter i 's vote, that is, iff

$$(i \in S \in W) \text{ or } (i \notin S \notin W);$$

- voter i is said to have been *decisive* if voter i was successful and i 's vote was critical for it, that is, iff

$$(i \in S \in W \text{ and } S \setminus \{i\} \notin W) \text{ or } (i \notin S \notin W \text{ and } S \cup \{i\} \in W).$$

In order to give the analogous definition *ex ante*, it is necessary to make an estimation of the likelihood of different vote configurations from the available information. We assume that for any coalition S we know, or at least have an estimate of, $p(S)$, i.e. the probability that S is the set of players in favor of a proposal and $N \setminus S$ the set against it. We can represent any such probability distribution by a function $p : 2^N \rightarrow \mathbb{R}$. Of course, $0 \leq p(S) \leq 1$ for each $S \subseteq N$ and $\sum_{S \subseteq N} p(S) = 1$.

Let (N, W) be a voting game and p the probability distribution over 2^N :

- voter i 's (*ex ante*) *success* is the probability that i is successful

$$\Omega_i(W, p) = P(i \text{ is successful}) = \sum_{S: i \in S \in W} p(S) + \sum_{S: i \notin S \notin W} p(S);$$

- voter i 's (*ex ante*) *decisiveness* is the probability that i is decisive

$$\Phi_i(W, p) = P(i \text{ is decisive}) = \sum_{S: i \in S \in W, S \setminus \{i\} \notin W} p(S) + \sum_{S: i \notin S \notin W, S \cup \{i\} \in W} p(S).$$

If we assume that voter i is sure to vote in favor (or against) the proposal, the conditional probabilities of success and decisiveness can be evaluated. In particular, denoting

$$\gamma_i(p) = P(i \text{ votes yes } (Y)) = \sum_{S: i \in S} p(S),$$

voter i 's conditional probability of being decisive given that voter i votes in favor of the proposal is given by

$$\Phi_i^{i+}(W, p) = P(i \text{ is decisive } | i \text{ votes yes (Y)}) = \frac{\sum_{S: i \in S \in W, S \setminus \{i\} \notin W} p(S)}{\gamma_i(p)},$$

voter i 's conditional probability of being decisive given that voter i votes against proposal is given by

$$\Phi_i^{i-}(W, p) = P(i \text{ is decisive } | i \text{ votes no (N)}) = \frac{\sum_{S: i \notin S \notin W, S \cup \{i\} \in W} p(S)}{1 - \gamma_i(p)},$$

voter i 's conditional probability of being successful given that voter i votes in favor of the proposal is given by

$$\Omega_i^{i+}(W, p) = P(i \text{ is successful } | i \text{ votes yes (Y)}) = \frac{\sum_{S: i \in S \in W} p(S)}{\gamma_i(p)},$$

and voter i 's conditional probability of being successful given that voter i votes against the proposal is given by

$$\Omega_i^{i-}(W, p) = P(i \text{ is successful } | i \text{ votes no (N)}) = \frac{\sum_{S: i \notin S \notin W} p(S)}{1 - \gamma_i(p)}.$$

We can now consider as equally probable all the configurations of votes, then we assume that the probability is given by

$$p^*(S) = \frac{1}{2^n} \quad \forall S \subseteq N.$$

That is equivalent to the assumption that each voter, independently from the others, will vote yes (Y) with probability $\frac{1}{2}$. This extreme case, which we have adopted also in the previous section, in the opinion of Laruelle and Valenciano makes sense when the objective is not to assess a particular voting situation, but the voting rule itself. Then, the authors show that some power indices can be seen as the particularization of some of the measures introduced in their paper, in particular the Banzhaf value can be seen as

$$\beta_i(W) = \frac{\text{number of winning coalitions in which } i \text{ is decisive}}{\text{total number of coalitions containing } i}.$$

As a coalition containing i means a coalition in which i votes yes, it can be easily seen that $\beta_i(W) = \Phi_i^{i+}(W, p^*)$. In particular, the authors show that

$$\beta_i(W) = \Phi_i^{i+}(W, p^*) = \Phi_i^{i-}(W, p^*) = \Phi_i(W, p^*). \quad (5.10)$$

Now, we add the observation that the Banzhaf loose blocking power index can be seen as

$$\beta_i^*(W) = \frac{\text{number of losing coalition in which } i \text{ is decisive}}{\text{total number of coalitions non containing } i}$$

and then

$$\beta_i^*(W) = \Phi_i^{i-}(W, p^*). \quad (5.11)$$

As a direct consequence of (5.10) and (5.11), the Banzhaf value and the Banzhaf loose blocking power index, defined in (2.3) and in (5.4), provide always the same measure of power, as it has already been shown in Section 5.5.

Now, we want to point out that also the decisiveness index of player i and the loose protectionism index of player i , defined in (5.8) and (5.9) respectively, can be seen as the particularization of some of the previous measures, in particular

$$\delta_i(N, W) = \Omega_i^{i+}(W, p) \quad (5.12)$$

and

$$\delta_i^*(N, W) = \Omega_i^{i-}(W, p). \quad (5.13)$$

As $\Omega_i^{i+} \neq \Omega_i^{i-}$, the two indices, as we have already seen, do not coincide and they give different information about the game.

5.8 A Bayesian Model

We may represent the situation through a noncooperative game, at first using its strategic form and then giving also an idea of its extensive form. Differently from the usual model of a voting game, we are assuming now that the agents cannot make binding agreements, as we are dealing with a noncooperative model. A *game in strategic form* consists of a 3-tuple $\langle N, (A_i), (\succsim_i) \rangle$ where

- i $N = \{1, \dots, n\}$ denotes the *set of players*;
- ii for each player $i \in N$ a nonempty set A_i (the set of *strategies* available to player i);
- iii for each player $i \in N$ a preference relation \succsim_i on $A = \times_{j \in N} A_j$ (the *preference relation* of player i).

A *Nash equilibrium of a game in strategic form* $\langle N, (A_i), (\succsim_i) \rangle$ is a profile $a^* \in A$ of strategies with the property that for every player $i \in N$ we have

$$(a_{-i}^*, a_i^*) \succsim_i (a_{-i}^*, a_i) \quad \forall a_i \in A_i,$$

where a_{-i} are the strategies of the players in $N \setminus i$. Then, in a Nash equilibrium no player can profitably deviate, given the strategies of the other players: this is what Nash defines an *equilibrium point* in *Equilibrium Points in n-Person Games* [75] and it is the most widely used solution for noncooperative games.

In this section we present a bayesian model, following the idea of Harsanyi ([45], [46] and [47]) of a noncooperative game in which the players do not have complete information of the game itself. This model is closely related to that of a game in

strategic form, and so its famous solution concept: the *Bayesian equilibrium* (or *Nash equilibrium of a Bayesian game*). Through this model we propose a noncooperative interpretation of the two indices defined in Section 5.6 and we give the possibility of extending the model to the situation in which the agents do not vote in favor or against a proposal with probability 1/2, but with a different probability distribution. The lacking of information due to the fact that the agents can predict the preferences and, consequently, the behavior of the other players, but they cannot be sure about it and the fact that they can only evaluate an expected payoff, trying to maximize it, suggested us the possibility to read the problem through a game with incomplete information.

A game with incomplete information played by bayesian players, or simply a *Bayesian game*, is a 5-tuple $(N, \{C_i\}_{i \in N}, \{T_i\}_{i \in N}, \{p_{ik}\}_{i \in N, k \in T_i}, \{u_i\}_{i \in N})$ where

- i N is the set of players;
- ii C_i is the set of the actions of player i ;
- iii T_i is the set of types of player i ;
- iv p_{ik} is the probability of player i of being of type k , with $k \in T_i$, $\sum_{k \in T_i} p_{ik} = 1$;
- v $u_i : \prod_{j \in N} C_j \times \prod_{j \in N} T_j \rightarrow \mathbb{R}$ is the utility function of player i .

A pure strategy for player i is a function $s_i : T_i \rightarrow C_i$ and Σ_i is the set of all the pure strategies of i . A mixed strategy for player i is a function $\sigma_i : C_i \times T_i \rightarrow [0, 1]$ with $\sum_{c \in C_i} \sigma_i(c, t) = 1$ for each $t \in T_i$.

In a *Bayesian equilibrium* each player chooses the best action available to him given the signal that he receives and his belief about the state and the other players' actions that he deduces from the signal.

In Example 5.6.1, $N = \{1, 2, 3\}$ is the set of players, i.e. the parties of the Parliament. Adopting a noncooperative approach, we assume that the parties, instead of cooperating, vote independently. Each one has two choices: voting yes (Y) or voting no (N), then $C_i = \{Y, N\}$ for each $i \in N$. The types of the parties can be identified with the ideological position: being in favor of the proposal (P) or in favor of the status quo (Q), then $T_i = \{P, Q\}$ for each $i \in N$. A given probability is assigned to the types of the players, in our model equal to 1/2, then $p_{ik} = 1/2$ for each $i \in N$, $k \in T_i$. These probabilities may represent, more in general, the prediction each type of each player does on the possibility of the other players of being of a certain type, but we take a simplified situation in which they all are equal; in our example, every voter knows that every other player can be of type P or Q with probability 1/2. The outcome of the game is given by "the law is approved", if the parties which voted Y have total number of seats greater than or equal to the majority quota, "the law is not approved" otherwise. The payoff of each party is 1 if it is of type P and the law

is approved or if it is of type Q and the law is not approved, 0 otherwise. Formally

$$u_i(s_1, \dots, s_n) = \begin{cases} 1 & \text{if } T_i = P \text{ and } \sum_{j \in N: s_j(T_j)=Y} w_j \geq q \\ 1 & \text{if } T_i = Q \text{ and } \sum_{j \in N: s_j(T_j)=Y} w_j < q \\ 0 & \text{otherwise} \end{cases}$$

In Table 5.2 we represent the game in strategic form, where the payoffs of the parties are shown in the 8 different configurations, starting from when they are all of type P, in favor of the proposal, finishing with the case of when they are all of type Q, in favor of the status quo. In every situation, that we call *state of nature*, we assume that party 1 chooses the row, party 2 the column and party 3 the matrix. The first choice for all of them corresponds to voting Y, the second one to voting N. Each player has a utility of 1 when the outcome is consistent with the type of the player who has been selected, 0 otherwise.

$(1_P, 2_P, 3_P)$	$\begin{pmatrix} (1, 1, 1) & (1, 1, 1) \\ (1, 1, 1) & (0, 0, 0) \end{pmatrix}$	$\begin{pmatrix} (0, 0, 0) & (0, 0, 0) \\ (0, 0, 0) & (0, 0, 0) \end{pmatrix}$
$(1_P, 2_P, 3_Q)$	$\begin{pmatrix} (1, 1, 0) & (1, 1, 0) \\ (1, 1, 0) & (0, 0, 1) \end{pmatrix}$	$\begin{pmatrix} (0, 0, 1) & (0, 0, 1) \\ (0, 0, 1) & (0, 0, 1) \end{pmatrix}$
$(1_P, 2_Q, 3_P)$	$\begin{pmatrix} (1, 0, 1) & (1, 0, 1) \\ (1, 0, 1) & (0, 1, 0) \end{pmatrix}$	$\begin{pmatrix} (0, 1, 0) & (0, 1, 0) \\ (0, 1, 0) & (0, 1, 0) \end{pmatrix}$
$(1_P, 2_Q, 3_Q)$	$\begin{pmatrix} (1, 0, 0) & (1, 0, 0) \\ (1, 0, 0) & (0, 1, 1) \end{pmatrix}$	$\begin{pmatrix} (0, 1, 1) & (0, 1, 1) \\ (0, 1, 1) & (0, 1, 1) \end{pmatrix}$
$(1_Q, 2_P, 3_P)$	$\begin{pmatrix} (0, 1, 1) & (0, 1, 1) \\ (0, 1, 1) & (1, 0, 0) \end{pmatrix}$	$\begin{pmatrix} (1, 0, 0) & (1, 0, 0) \\ (1, 0, 0) & (1, 0, 0) \end{pmatrix}$
$(1_Q, 2_P, 3_Q)$	$\begin{pmatrix} (0, 1, 0) & (0, 1, 0) \\ (0, 1, 0) & (1, 0, 1) \end{pmatrix}$	$\begin{pmatrix} (1, 0, 1) & (1, 0, 1) \\ (1, 0, 1) & (1, 0, 1) \end{pmatrix}$
$(1_Q, 2_Q, 3_P)$	$\begin{pmatrix} (0, 0, 1) & (0, 0, 1) \\ (0, 0, 1) & (1, 1, 0) \end{pmatrix}$	$\begin{pmatrix} (1, 1, 0) & (1, 1, 0) \\ (1, 1, 0) & (1, 1, 0) \end{pmatrix}$
$(1_Q, 2_Q, 3_Q)$	$\begin{pmatrix} (0, 0, 0) & (0, 0, 0) \\ (0, 0, 0) & (1, 1, 1) \end{pmatrix}$	$\begin{pmatrix} (1, 1, 1) & (1, 1, 1) \\ (1, 1, 1) & (1, 1, 1) \end{pmatrix}$

Table 5.2: Strategic form of the game

In order to show the complexity of the problem, we represent the situation in extensive form in Figure 5.1, where the black dots represent the choices of the nature which selects the type of each party with probability 1/2. Then each player (the white dots)

has to take its own decision, selecting an action between Y and N, finally one of the 64 outcomes is selected.

Every party knows its own type, but not the types of the other two parties. As it gives probability 1/2 to every type of the other parties, it assigns probability 1/4 to be in a given state of nature. If party 1, for example, is of type P, it will give probability 1/4 to each one of the first four states of nature shown in Table 5.2. Assuming that party 2 will play $(p, 1 - p)$ and party 3 $(q, 1 - q)$, the expected payoff for party 1 when it plays Y is

$$\frac{1}{4}[pq + (1 - p)q] + \frac{1}{4}[pq + (1 - p)q] + \frac{1}{4}[pq + (1 - p)q] + \frac{1}{4}[pq + (1 - p)q] = q$$

and when it plays N is

$$\frac{1}{4}[pq] + \frac{1}{4}[pq] + \frac{1}{4}[pq] + \frac{1}{4}[pq] = pq$$

then the best choice for party 1 of type P is to play $(t, 1 - t)$ with $t = 1$ if $p < 1$ and $t \in [0, 1]$ if $p = 1$.

Writing the best reply for every type of every player, we obtain the obvious result that the optimal strategy for the players is to choose Y if they are of type P and N if they are of type Q. This is the *Bayesian pure equilibrium* of the game.

The interest of the result is that, when the probabilities of the types are all equal to 1/2, playing the equilibrium strategy every party of type P can obtain an expected utility equal to its decisiveness index and every party of type Q an expected utility equal to its loose protectionism index.

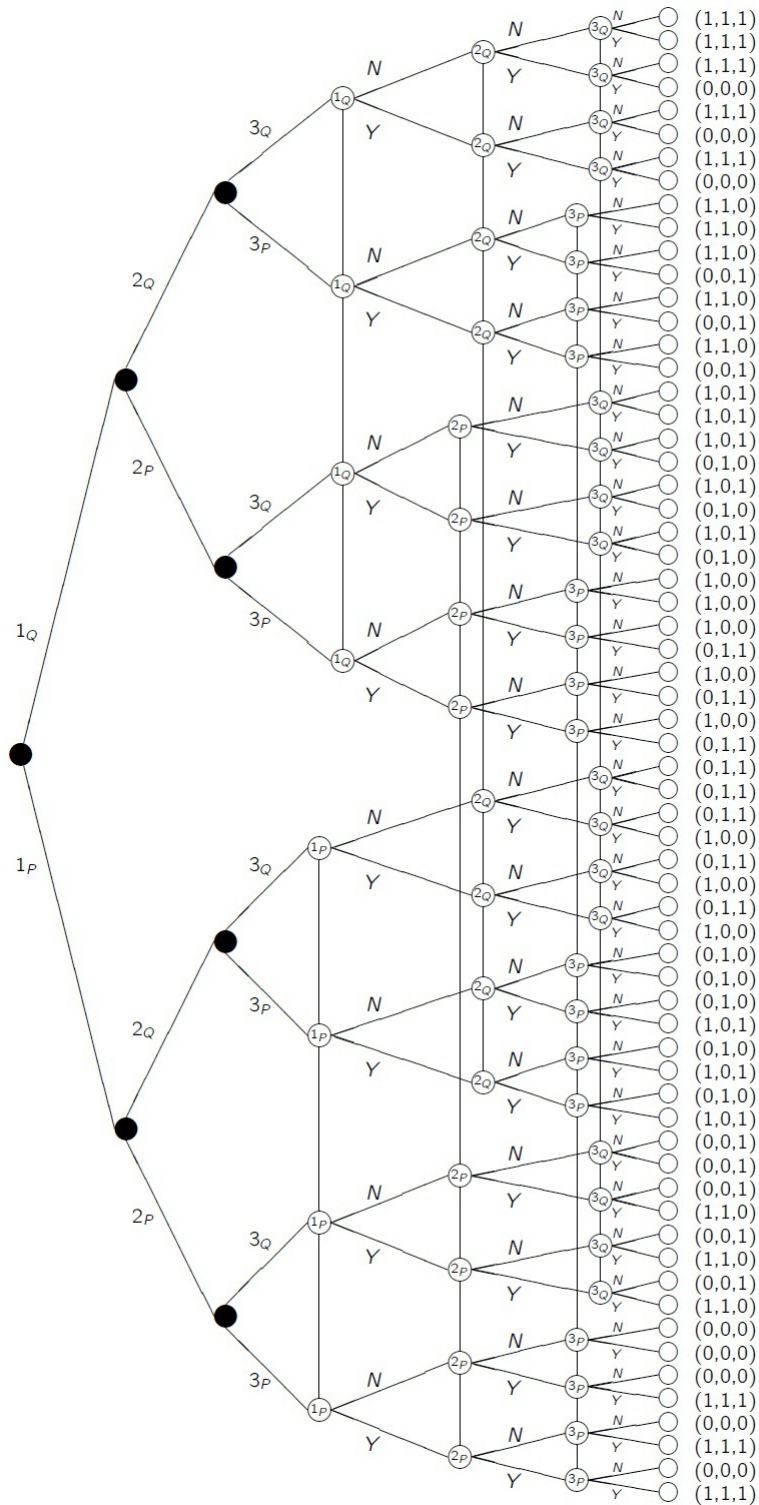


Figure 5.1: Extensive form of the game

Proportionality and the “Best” Electoral System

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6.1 Introduction

The second part of this thesis deals with a different branch of Game Theory applied to voting systems. We spent the previous three chapters evaluating the power share in a Parliament. Now we study a foregoing step of a democracy structure: we try to evaluate the “goodness” of an electoral system in providing a Parliament starting from the electors’ preferences expressed during an election.

Electoral systems and their features are widely studied in literature. We just mention the contributions on electoral systems by Brams and Fishburn [20], Hołubiec and Mercik [50], Nurmi [76] and Lijphart [60]. At first sight the problem of choosing the best electoral system for a Parliament cannot be solved: Arrow’s theorem [9] in 1950 and McKelvey’s theorem [68] in 1976 excluded the very possibility of finding out the optimal rule, but no theorem prohibits finding out a criterion to establish whether a rule is better or worse than another. Moreover, we may remark that already in 1952 May’s Theorem [66] disqualifies the theorem of Arrow, as it affirms that in some

particular situations (but the result can be generalized) the majority rule respects all the requirements of Arrow’s theorem, but transitivity. This is not a practical problem as intransitivity is extremely rare in parliamentary elections.

The choice of the “best” Parliament may be affected by a lot of facets of the political process, but two of them may be considered more relevant than the others: *representativeness* (R) that depends on the capacity of the system in representing electors’ preferences and *governability* (G) that measures the effect on the efficiency of the resulting government. In the 20th century, the wide appearance of proportional systems was one of the reasons for the development of various methods for measuring the quality of electoral systems, mainly due to evaluate the representativeness, often called *proportionality* (or, evaluating the lack of proportionality, called *disproportionality*), of a Parliament. In 1982 Balinski and Young [11] gained one of the main results: they proved the impossibility of constructing a proportional system that allocates seats in an exactly proportional way. The problems regarding proportional systems are mainly due to two reasons: the first is that the so-called proportional systems often introduce some modifications in order to enhance other good features, *in primis* the governability, excluding the smallest parties, via a threshold, and/or strengthening the largest parties, via a majority prize; the second is that even with a perfect proportional system it is necessary to assign an integer number of seats through some rounding methods, we will refer to the problem of the distribution of the rests as the *apportionment problem*. The impossibility of creating an ideal proportional electoral system forced researchers to define quantitative indices that would reflect the degree to which the system satisfies certain conditions. Such indices contain quantitative information and allow researchers to conduct empirical research and compare various electoral systems. Then proportionality and the related problem of apportionment became a largely studied topic in the last years. More specific on proportionality are the papers by Gallagher [38] and Monroe [72] and the analysis of apportionment methods by Gambarelli and Palestini [41].

In this chapter we mainly focus on the work of Karpov [55], who studied the properties of some of the most famous disproportionality indices (plus some modifications), which are presented and axiomatically analyzed; then, they are applied to four electoral sessions in Russia for studying the features of these indices when the number of parties varies. We complete his analysis introducing the indices proposed by Ortona [34], Fragnelli [32] and Gambarelli and Biella [40]. In particular, the last two indices account the issue of power for measuring the disproportionality. In our mind, the power, i.e. the influence of each party on the decision of passing a law, should play a more relevant role in evaluating the characteristics of a Parliament, including the proportionality. As we point out in Section 6.5 with suitably designed academic examples, it is possible that an apparently unfair distribution of seats w.r.t. votes may provide the parties the same power of their voters, so we can conclude that the voting body is well represented by the Parliament if the representativeness (the disproportionality) is measured by an index accounting the issue of power. In this case, the classical indices listed by Karpov assign a high level of disproportionality to the

system.

Proportional systems are largely adopted all over the world, so it may be difficult the idea to accept an electoral system entirely based on the issue of power. Because of that, in the last part of this chapter we propose to use this method to solve only the problem of the apportionment and not the entire problem of assigning the total amount of seats; in this way we use the issue of power keeping the idea of a proportional based allocation.

The chapter is organized as follows: in Section 6.2 we recall the disproportionality indices listed by Karpov; in Section 6.3 we present the three added indices; Section 6.4 presents the axioms a good index, according to Karpov, should satisfies, provides the result of Karpov on the properties those indices satisfy and add the analysis on the three new indices. In Section 6.5 we present two academic examples to show the relevance of the issue of power; Section 6.6 contains the value assigned by the three indices to the Russian electoral sessions used by Karpov and a discussion on the comparison with the indices in [55]; Section 6.7 illustrates the idea of using the issue of power for building a new apportionment rule .

6.2 Measurement of Disproportionality in Proportional Systems

Many indices of disproportionality have been constructed; some of them were developed for studying particular electoral systems, while others were adapted from other areas of science. Thus, there was and there is no agreement concerning what indices are better in a particular situation. In his work, Karpov [55] studies the properties of some of these disproportionality indices. He considers a proportional election where $N = \{1, \dots, n\}$ is the set of parties participating, with $n = |N|$. Let (V_1, \dots, V_n) be the vector of votes and (S_1, \dots, S_n) be the vector of seats each party receives. Then $\sum_{i=1}^n V_i = V$ and $\sum_{i=1}^n S_i = S$. We underline that these indices are mainly defined for a proportional election, but it is possible to adopt them to evaluate the proportionality of any election, even one based on a majoritarian system. The purpose of the election is to represent voters' preferences as closely as possible. According to the principle *one person - one vote*, each ballot should have "equal force" in the sense of the share of seats in the Parliament:

$$\frac{S_i}{V_i} = \frac{S}{V} \quad \forall i = 1, \dots, n \quad (6.1)$$

Let $v_i = V_i/V$ and $s_i = S_i/S$ be the vote and seat shares that party i receives and $y_i = S_i/V_i$ the *representation* of party i . Party i is *overrepresented* when $y_i > S/V$ and party i is *underrepresented* when the opposite inequality holds. We require that the apportionment respects the property of *monotonicity*, i.e. if $v_1 \geq \dots \geq v_n$, then $s_1 \geq \dots \geq s_n$. The ideal case is when each vote has equal force and each party obtains a share of seats equal to the share of votes, then $v_i = s_i$ for each $i = 1 \dots n$. The deviation from the exact equality is not only a mathematical problem, but also a political one, because it is a distortion of citizens' true preferences. As already

said, a real political system cannot achieve equality, first of all because the number of seats is an integer. Disproportionality indices measure the deviation between the real assignment and the exact one; we list now the various approaches for measuring the quality of an electoral system which have been analyzed by Karpov, divided into several groups.

6.2.1 Absolute Deviation Indices

These indices characterize apportionment by means of absolute deviations between vote share and seat share. Indices are equal to zero if and only if $v_i = s_i$ for each party: this corresponds to the ideal representation.

- The *Maximum Deviation index* shows the size of distortion of the most inaccurately represented party,

$$MD = \max_{i=1\dots n} |s_i - v_i|.$$

- The *Rae index* is the arithmetic mean of absolute deviations,

$$I_{Rae} = \frac{1}{n} \sum_{i=1}^n |s_i - v_i|.$$

It has a clear interpretation: it is the average of the deviation of the parties from the exact representation.

- The *Loosmore-Hanby index* seems to be similar to the Rae index, but it has a completely different meaning. The value of the Loosmore-Hanby index gives the total excess of seat shares of overrepresented parties over the exact quota and the total shortage accruing to other parties,

$$I_{LH} = \frac{1}{2} \sum_{i=1}^n |s_i - v_i|.$$

- The *Grofman index* implies calculating the mean of absolute deviations, but their sum is divided by the effective number of parties E rather than by the total number of parties,

$$I_{Gr} = \frac{1}{E} \sum_{i=1}^n |s_i - v_i|,$$

where $E = (\sum_{i=1}^n v_i^2)^{-1}$.

- The *Lijphart index* is calculated in the same way as the Rae index, but only the two largest parties are considered,

$$I_L = \frac{|s_k - v_k| + |s_h - v_h|}{2},$$

where k and h are the two largest parties. Since the largest parties usually have the most significant deviations from their exact quota, this measure can be used to evaluate the disproportionality of the whole system.

6.2.2 Quadratic Indices

Quadratic indices can be used to compare distributions with an equal sum of absolute deviations.

- The *Gallagher index*, often called the *least squares index*, has a different sensitivity to large and small deviations between vote and seat shares. Small differences have less influence on the index than big ones, which increase the index significantly,

$$Lsq = \sqrt{\frac{1}{2} \sum_{i=1}^n (s_i - v_i)^2}.$$

- The *Monroe index* is a small modification of the Gallagher index,

$$I_M = \sqrt{\frac{\sum_{i=1}^n (s_i - v_i)^2}{1 + \sum_{i=1}^n v_i^2}},$$

The sum of the squares of vote shares characterizes the number of parties. The denominator decreases as the number of parties increases.

- The *Gatev index* is higher when parties are approximately equal in size than if there is significant inequality between parties or if there is a higher number of parties. Thus, this index is more sensitive to small parties than the Gallagher index,

$$I_{Ga} = \sqrt{\frac{\sum_{i=1}^n (s_i - v_i)^2}{\sum_{i=1}^n (s_i^2 + v_i^2)}}.$$

- The *Ryabtsev index* insignificantly differs from the Gatev index, and it has lower values,

$$I_R = \sqrt{\frac{\sum_{i=1}^n (s_i - v_i)^2}{\sum_{i=1}^n (s_i + v_i)^2}}.$$

- The *Szalai index* is used in time-use research (the study dedicated to knowing how people allocate their time during an average day) for comparing activity profiles,

$$I_S = \sqrt{\frac{\sum_{i=1}^n \left(\frac{s_i - v_i}{s_i + v_i}\right)^2}{n}}.$$

6.2.3 Aleskerov-Platonov Index

- The *Aleskerov-Platonov index* is calculated only for overrepresented parties:

$$R = \frac{1}{k} \sum_{i=1}^k \frac{s_i}{v_i}.$$

When some parties are not represented, the other parties will obtain on average more than one percent of seats for each percent of votes. This index shows the average excess of seat share over the vote share for overrepresented parties.

6.2.4 Inequality Indices

The following indices are borrow from welfare economics, which faced the problem of measuring inequality, a problem that is quite similar to the problem of measuring disproportionality.

- The *Gini index* is calculated using the Lorenz curve. The curve passes through the points with cumulative shares of the income. If wealth is distributed equally among individuals, then the curve is a straight line. If not, the curve will lie under this line and be convex. The cumulative shares of electoral income are calculated by the following formula:

$$T_h = \frac{\sum_{i=1}^h \frac{S_i}{v_i}}{\sum_{j=1}^n \frac{S_j}{v_j}}$$

where $\frac{S_1}{v_1}, \dots, \frac{S_n}{v_n}$ are given in increasing order. The Gini index(G) is the ratio of the area between the Lorenz curve and the perfect equality line to the area of the triangle under the straight line.

- The *Atkinson index* is defined as

$$A = 1 - \left[\sum_{i=1}^n v_i \left(\frac{S_i}{v_i} \right)^{1-\epsilon} \right]^{\frac{1}{1-\epsilon}},$$

where ϵ characterizes the attitude of a society to inequality: a negative attitude to inequality is strengthened by an increase in ϵ .

- The *Generalized entropy* is given by the following formula:

$$GE = \frac{1}{\alpha^2 - \alpha} \left[\sum_{i=1}^n v_i \left(\frac{S_i}{v_i} \right)^\alpha - 1 \right],$$

where varying the value of the parameter α we have a class of indices with similar properties

6.2.5 Objective Functions

For each apportionment method, an objective function can be defined such that its optimization gives the best seat allocation. This very function can be used to measure disproportionality.

- The *d'Hondt index* equals the maximum excess of seat share over vote share,

$$Ho = \max_{i=1 \dots n} \frac{S_i}{v_i}.$$

- The *Sainte-Lague index* is a weighted sum of squares of relative deviation,

$$SL = \sum_{i=1}^n v_i \cdot \left(\frac{S_i}{v_i} - 1 \right)^2.$$

6.3 The Added Indices of Disproportionality

For completing the analysis of the disproportionality indices and of their properties, we add three indices: the Ortona index, the Fragnelli index and the Gambarelli-Biella index. The first two ones are defined as representativeness indices, but both of them are obtained as one minus a disproportionality index, so we present them as disproportionality indices, for consistency with the paper of Karpov.

- The *Ortona index* was proposed in [34] and it is based on the difference between seats assigned by a given electoral system and seats assigned by a perfect proportional system, *PP*, i.e. supposing a unique nation-wide proportional district and assigning the rest to the largest decimals. The idea is to avoid combining votes and seats as most of the indices of disproportionality commonly do, taking S_i^{PP} as the best approximation of V_i . The formula is:

$$r = \frac{\sum_{i=1}^n |S_i - S_i^{PP}|}{\sum_{i=1}^n |S_i^u - S_i^{PP}|} \quad (6.2)$$

where S_i is the number of seats of party i with the system under consideration, S_i^{PP} is the number of seats of party i with the perfect proportional system and S_i^u is the total number of seats for the relative majority party according to the seat share and 0 otherwise.

The index reads as follows. For the sum at the numerator, we assume that the disproportionality is minimal under perfect proportionality rule. Hence, the loss of disproportionality incurred by party i is the (absolute) difference between the seats actually obtained and those it would get under *PP*. Summing this absolute difference across all the parties we obtain the total disproportionality. The sum at the denominator is introduced to normalize this value. It is the maximum possible disproportionality obtained when the relative majority party, according to the selected system, takes all the seats instead of just its quota. Remembering that S is the total number of seats let, without loss of generality, party 1 be the relative majority party; then $\sum_{i=1}^n |S_i^u - S_i^{PP}| = |S_1^u - S_1^{PP}| + \sum_{i=2}^n |S_i^u - S_i^{PP}| = S - S_1^{PP} + \sum_{i=2}^n | - S_i^{PP}| = 2(S - S_1^{PP})$.

The ratio of the sums is a disproportionality index, normalized in the range $[0, 1]$.

Note that this index cannot be employed starting from real-world votes in a nonproportional system, due to strategic voting. However, some ingenuity could allow for using it starting from survey data.

- The *Fragnelli index* was introduced in [32]. We switch our attention to the power of the parties, following the idea that it plays a relevant role in evaluating the representativeness of a Parliament, and look for the relationship among the power they have according to the distribution of votes and according to the distribution of seats.

In order to deal with the concept of power, starting from the vote share (v_1, \dots, v_n) and the seat share (s_1, \dots, s_n) , we define two simple games (N, w) and (N, u) , where $N = \{1, \dots, n\}$ is the set of parties and w and u are the two characteristic functions $w, u : 2^N \rightarrow [0, 1]$ respectively defined by the following weighted majority situations $[q'; v_1, \dots, v_n]$ and $[q''; s_1, \dots, s_n]$, where $q' = \lfloor \frac{V}{2} + 1 \rfloor / V$ and $q'' = \lfloor \frac{S}{2} + 1 \rfloor / S$; then w is valued 1 for the winning coalitions of parties, i.e. coalitions with a total vote share greater than 0.5, sufficient to pass a law, and 0 for the loosing coalitions; similarly for the function u referred to the seat share.

This disproportionality index measures the distance of the distribution of power on the votes and on the seats, i.e. $\sum_{i \in N} |\psi_i(w) - \psi_i(u)|$. It is normalized, simply dividing $\sum_{i \in N} |\psi_i(w) - \psi_i(u)|$ by 2, as in the worst case the two distributions of power may assign complementary values¹. So, we have:

$$r^\Omega = \frac{\sum_{i \in N} |\psi_i(w) - \psi_i(u)|}{2}. \quad (6.3)$$

The distance is zero when the power of each party is identical in the two distributions. The idea is that a good system should provide a Parliament which respects the share of power given by the electors' votes. The principle one person - one vote, according to which each ballot should have “equal force” in the sense of the share of seats in the Parliament, it is now replaced by the principle that each ballot should have “equal power”.

- The last index, proposed in [40], is *Gambarelli-Biella index*. The Fragnelli index is not the first case in which power is used for evaluating representativeness (or disproportionality). In fact, Gambarelli-Biella index, which has been defined some years before, is a combination of the traditional approach, which considers vote and seat shares, with the idea of measuring the distance of the distributions of power related to the votes and to the seats. It is given by:

$$\Delta = \max_{i \in N} \{|v_i - s_i|, |\psi_i(w) - \psi_i(u)|\} \quad (6.4)$$

Remark 6.3.1. Note that r^Ω is based on norm 1, while Δ is based on norm ∞ , so it is possible to define other indices based on other norms.

6.4 Axiomatic Approach

Disproportionality indices must have certain properties. We refer now to the following four principles listed by Karpov

1. *Anonymity*. Any permutation of party labels does not change the value of the index.

¹ Two vectors are complementary when each nonzero component in a vector corresponds to a zero component in the other vector.

2. *Principle of transfers.* If we transfer seats from an overrepresented party to an underrepresented party the value of the index should not increase.
3. *Independence from split.* Suppose there are many parties with equal vote and seat shares, and these parties are grouped into one. If the value of the index calculated for all the parties in the group is equal to the value of the index for the group considered as a whole, then the property of independence from split holds.
4. *Scale invariance.* The index should not depend on any proportional change in the number of votes or seats in the parliament.

In Table 6.1 we summarize the analysis of Karpov about the properties each index satisfies. The “+” sign means that the index satisfies the property, the “-” sign that it does not. All the indices satisfy property 1 and 4 (then the two properties are not included in the table).

Indices	Principle of transfers	Independence from split
Maximum deviation	+	-
Rae index	+	-
LH index	+	+
Grofman index	+	-
Lijphart index	+	-
Gallagher index	+	-
Monroe index	+	-
Gatev index	-	+
Ryabtsev index	-	+
Szalai index	-	+
Aleskerov-Platanov index	+	+
Gini index	+	+
Atkinson index	+	+
Generalized entropy	+	+
D’Hondt index	+	+
Sainte-Lague index	+	+

Table 6.1: Axiomatic properties of the indices

We add now in Table 6.2 the analysis of the properties that the three indices we presented in Section 6.3 satisfy. They all satisfy property 1 and 4 as the previous ones.

The reason why the two disproportionality indices based on power do not respect Property 2 and 3 is due to the fact that the indices of power, in general, do not respect the similar corresponding properties. Karpov underlines the fact that violation of

Indices	Principle of transfers	Independence from split
Ortona	+	+
Fagnelli	–	–
Gambarelli-Biella	–	–

Table 6.2: Axiomatic properties of the added indices

property 3 means that the index depends on the number of parties. In our analysis, the violation of this property is more related to a variation of the distribution of the power inside a Parliament; analogously, the transfer of seats from an overrepresented party to an underrepresented one (Property 2), on the one hand, may have a positive effect on the distribution of seats w.r.t. to votes, but on the other hand may dramatically change the power distributions.

6.5 Academic Examples

The situations we are going to analyze, even if they are not very realistic, mainly for the small number of seats and the distribution of votes, are useful to show how the indices based on power focus on a totally different point, compared with the traditional ones. Parliaments which are very good, evaluated by the classical indices, can be very bad from the point of view of representing the power and vice versa. In our analysis we refer to the the disproportionality indices listed by Karpov and here recalled in Section 6.2 and to the three ones we added in Section 6.3. We decide not to take into account the indices which require to assign a particular parameter, as in an academic situation this is not possible. This does not affect our analysis, as the remaining indices are sufficient in number to perform a comparison with the three added indices.

The indices have different ranges of values and this does not allow for a cross-analysis, but the following situations are designed in such a way that all the above-mentioned indices and the index by Ortona produce a result in contrast with the two power-based indices, and to underline this difference is the aim of this analysis. For computing the last two indices we have to decide which power index to use. Between all the indices we listed in Section 2.2.2, we chose two of the most popular ones, the Shapley-Shubik index and the Public Good index. Our choice depends on their different behavior; in fact, the former has the local monotonicity property, so that it assigns larger power to parties with greater number of seats, while the latter does not respect monotonicity and the difference of power are often very small w.r.t other non monotonic indices (see Hollera and Napel [49]).

Example 6.5.1. *We start revisiting the classical situation in Example 1 in [32]. Three parties A, B, C receive a percentage of votes of 49.5%, 48.5%, 2.0%, respectively and 6 seats have to be assigned. We consider three different seats assignments:*

	A	B	C
s_{PP}	3	3	0
s	3	2	1
s_{PO}	2	2	2

Table 6.3: Seat distribution for academic Example 6.5.1

s_{PP} assigns the seats according to the perfect proportional system; s guarantees one seat to party C, with the consequence that party A receives one seat more than party B; finally, s_{PO} (PO after Power Oriented) guarantees a distribution of power on the seat share equal to the distribution of power on the vote share, even if it does not seem as reasonable as the other two.

Computing the abovementioned disproportionality indices in Example 6.5.1 we obtain the following table:

	MD	I_{LH}	I_L	Lsq	G	I_{Rae}	I_{Gr}	I_M	I_{Ga}	I_R	I_S	SL	R	Ho
s_{PP}	0.020	0.020	0.010	0.018	0.020	0.013	0.019	0.021	0.026	0.018	0.577	0.021	1.021	1.031
s	0.152	0.152	0.078	0.149	0.228	0.101	0.146	0.173	0.226	0.162	0.587	1.123	1.010	8.333
s_{PO}	0.313	0.313	0.157	0.271	0.313	0.209	0.301	0.315	0.425	0.315	0.598	5.009	16.667	16.667

Table 6.4: Disproportionality indices analyzed by Karpov [55] for Example 6.5.1

The indices consider the first assignment as the best one (with the only exception of the Aleskerov-Platonov index R) but the values confirm that the assignment s_{PO} is the worst under all the indices, having always the largest value in each column.

Next, we compute the Shapley-Shubik index and the Public Good index referring to the three seat distributions and to the vote share, v :

	A	B	C
$\phi(v)$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
$\phi(s_{PP})$	$\frac{1}{2}$	$\frac{1}{2}$	0
$\phi(s)$	$\frac{4}{6}$	$\frac{1}{6}$	$\frac{1}{6}$
$\phi(s_{PO})$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
$H(v)$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
$H(s_{PP})$	$\frac{1}{2}$	$\frac{1}{2}$	0
$H(s)$	$\frac{2}{4}$	$\frac{1}{4}$	$\frac{1}{4}$
$H(s_{PO})$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$

Table 6.5: Power indices for Example 6.5.1

Computing the indices by Ortona, Fragnelli and Gambarelli-Biella we obtain the following table:

	r	$r^{\Omega}(\phi)$	$r^{\Omega}(H)$	$\Delta(\phi)$	$\Delta(H)$
s_{PP}	0	0.333	0.333	0.333	0.333
s	0.333	0.333	0.167	0.333	0.167
s_{PO}	0.667	0	0	0.313	0.313

Table 6.6: Added disproportionality indices for Example 6.5.1

The index by Ortona that does not account the power assigns the worst value to assignment s_{PO} . The behaviour of this index is similar to the ones of the previous indices and s_{PP} is the best one, even having assigned a value 0 of disproportionality. Assignment s_{PO} , on the other hand, receives the best score under the index by Fragnelli that is strongly power-oriented; the index by Gambarelli-Biella is influenced by the seat assignment and provides an intermediate result.

Example 6.5.2. *In this second situation, we consider again three parties A, B, C that receive a percentage of votes of 49.9%, 25.2%, 24.9%; we analyze two different Parliaments with 9 and 10 seats, respectively; finally we compute the assignment of the seats under PP (s_{PP}) and looking at the power distribution (s_{PO}):*

		A	B	C
9	s_{PP}	5	2	2
	s_{PO}	4	3	2
10	s_{PP}	5	3	2
	s_{PO}	4	3	3

Table 6.7: Seat distribution for Example 6.5.2

Note that assignment s_{PP} gives to party A the absolute majority in the Parliament with 9 seats and the veto power with 10 seats, differently from the distribution of votes and assignment s_{PO} .

Computing the abovementioned disproportionality indices in Example 6.5.2 we obtain the following table:

		MD	I_{LH}	I_L	Lsq	G	I_{Rae}	I_{Gr}	I_M	I_{Ga}	I_R	I_S	SL	R	Ho
9	s_{PP}	0.057	0.057	0.043	0.049	0.057	0.038	0.042	0.059	0.078	0.056	0.579	0.013	1.113	1.113
	s_{PO}	0.081	0.081	0.068	0.072	0.082	0.054	0.061	0.087	0.119	0.084	0.584	0.035	1.323	1.323
10	s_{PP}	0.049	0.049	0.025	0.049	0.073	0.033	0.037	0.059	0.079	0.056	0.580	0.019	1.096	1.190
	s_{PO}	0.099	0.099	0.074	0.086	0.100	0.066	0.074	0.103	0.143	0.102	0.583	0.039	1.198	1.205

Table 6.8: Disproportionality indices analyzed by Karpov [55] for Example 6.5.2

Again, the assignment s_{PO} is the worst under all the indices, with both 9 and 10 seats, even if the absolute majority given to party A when there are 9 seats and the veto power when there are 10 seats make the system not exactly corresponding to the will of the voters. We decide to evaluate the disproportionality using the three added

disproportionality indices. The values of the Shapley-Shubik index and of the Public Good index are shown in Table 6.9.

		A	B	C
$\phi(v)$		$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
9	$\phi(s_{PP})$	1	0	0
	$\phi(s_{PO})$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
10	$\phi(s_{PP})$	$\frac{4}{6}$	$\frac{1}{6}$	$\frac{1}{6}$
	$\phi(s_{PO})$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
$H(v)$		$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
9	$H(s_{PP})$	1	0	0
	$H(s_{PO})$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
10	$H(s_{PP})$	$\frac{2}{4}$	$\frac{1}{4}$	$\frac{1}{4}$
	$H(s_{PO})$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$

Table 6.9: Power indices for Example 6.5.2

Computing the indices by Ortona, Fragnelli and Gambarelli-Biella we obtain the results shown in Table 6.10.

		r	$r^\Omega(\phi)$	$r^\Omega(H)$	$\Delta(\phi)$	$\Delta(H)$
9	s_{PP}	0	0.667	0.667	0.667	0.667
	s_{PO}	0.250	0	0	0.081	0.081
10	s_{PP}	0	0.333	0.167	0.333	0.167
	s_{PO}	0.200	0	0	0.099	0.099

Table 6.10: Added disproportionality indices for Example 6.5.2

Again, the index by Ortona assigns the worst value to assignments s_{PO} , that receives the best score under the two power-oriented indices. Moreover, Fragnelli index, which is based only on the power issue, considers the assignment s_{PO} as “perfect”, as it gives 0 as disproportionality value. This is simply due to the fact that the power distribution in the seat and in the vote shares are identical. Differently from Example 6.5.1, in which the s_{PO} assignment was very far from the s_{PP} assignment, in this case there is only a small variation in the seat distribution, allowing the power to be better shared between parties. s_{PO} has been obtained simply through a different apportionment: the rests have been given in order to have the share of power the most similar to the one of the votes.

6.6 Application to the Russian Parliament

In the work of Karpov the proposed disproportionality indices are computed for the elections to the *State Duma* (Russian Parliament) referring to four electoral sessions: 1995, 1999, 2003 and 2007. The results are shown in Tables 10.1 and 10.2 of the cited paper. In Table 6.11 we show the results for the three added indices, completing the analysis. The values of the Shapley-Shubik index and of the Public Good index will be shown in Chapter 7, where we analyze in details the algorithms we used for the computation. In particular, in evaluating the power on the vote share of the former elections, we had the problem of finding exact and efficient algorithms able to deal with such a high number of players. Such an algorithm already existed to compute the Shapley-Shubik index and it was based on generating functions. From that we had the idea of building a new algorithm, based again on generating functions, to evaluate the Public Good index exactly and efficiently. This brought to a new research work, which is presented in the next chapter and which constitutes the last part of this thesis.

	r	$r^{\Omega}(\phi)$	$r^{\Omega}(H)$	$\Delta(\phi)$	$\Delta(H)$
1995	0.6129	0.4298	0.9063	0.2178	0.3098
1999	0.1843	0.0624	0.3818	0.0503	0.1741
2003	0.4170	0.4213	0.9548	0.4213	0.9548
2007	0.2038	0	0	0.0498	0.0498

Table 6.11: Added disproportionality indices for Duma

Analyzing the indices, Karpov observed that almost all of them create the same ordering, that the Parliament of 1999 is the most proportional and the Parliament of 1995 is the least one. This is still true for the results obtained evaluating the Ortona index, and we can notice that, also computing the other two indices that take into account the role of the power, the Parliament of 1999 has a very low disproportionality value, but the best score is given to the Parliament of 2007. Fragnelli index accounts it for perfect proportionality; this is due to the particular situation that *United Russia* got the absolute majority both on the vote share (64.30%) and on the seat share (70.00%), so it got the whole power and the differences on vote and seat shares have no relevance at all.

This fact underlines that, even if from the academic examples we have shown that to evaluate disproportionality taking into account the power can give very different results from evaluating it with classical indices, it is possible that a Parliament has a good evaluation from both points of view, enforcing the approaches and showing that they are not necessarily conflicting.

6.7 A Possible New Apportionment Rule

The different values of the indices for the different Parliaments suggest that they are suitable for measuring the disproportionality more of a Parliament than of the adopted electoral system.

The issue of power allows new perspective for analyzing the disproportionality. The situation in Example 6.5.1 is designed in order to have the maximal impact on the reader, using a very limited number of seats. We want to make clear that with a higher number of seats it is possible to reduce the differences in the values of the indices for the three assignment systems. Moreover, the equal distribution of power on the votes and on the seats could be obtained with a very reasonable distribution of seats. For instance, using the same percentages of votes with 15 seats, the assignment rules s_{PP} and s correspond to $(8, 7, 0)$ and $(7, 7, 1)$, respectively, and the first is easily rejected because Party A would have the absolute majority. If we want to guarantee the same distribution of power on the votes and on the seats it is sufficient to assign to each party a number of seats greater than or equal to 1 and less than or equal to 7, i.e. s_{PO} can be selected equal to s . This remark may be exploited for designing a rule for assigning the rests in a proportional system (pure proportional, threshold proportional, prized proportional, etc.) in order to minimize the differences of the distribution of power on the votes and on the seats.

In this regard, we consider the situation in Example 6.7.1

Example 6.7.1. Five parties A, B, C, D, E receive a percentage of votes of 37.58%, 22.35%, 20.35%, 11.38%, 8.34%, respectively and 100 seats have to be assigned. In Table 6.12 we show the power on the vote share according to the Shapley-Shubik index. If we assign respectively 37, 22, 20, 11 and 8 seats, corresponding to

	A	B	C	D	E
$\phi(v)$	$\frac{12}{30}$	$\frac{7}{30}$	$\frac{7}{30}$	$\frac{2}{30}$	$\frac{2}{30}$

Table 6.12: The Shapley-Shubik index referred to the vote share for Example 6.7.1

the integer part of the percentages, as we are allocating 100 seats, the problem of the apportionment consists in assigning the last two remaining seats. Assuming that at most one additional seat can be assigned to each party (this is normally accepted by most of the allocation rules based on proportionality), we have 10 different possible apportionments as shown in Table 6.13.

The corresponding power on the seat share is shown in Table 6.14.

We can immediately notice that in the previous example, against any intuition, the perfect proportional system is the only one which does guarantees an exact distribution of power between the vote and the seat share, i.e. the only one in which the Fragnelli index (see Section 6.3) has value strictly bigger than zero. The interesting conclusion after analyzing this simple example is that it may be possible to maintain the same

	<i>A</i>	<i>B</i>	<i>C</i>	<i>D</i>	<i>E</i>
s_1	38	23	20	11	8
s_2	38	22	21	11	8
$s_3 \equiv s_{PP}$	38	22	20	12	8
s_4	38	22	20	11	9
s_5	37	23	21	11	8
s_6	37	23	20	12	8
s_7	37	23	20	11	9
s_8	37	22	21	12	8
s_9	37	22	21	11	9
s_{10}	37	22	20	12	9

Table 6.13: Different apportionments for Example 6.7.1

power share simply assigning the rests, without totally upsetting the seat distribution. This suggests that also in real cases, assigning the seats proportionally and solving the problem of the apportionment of the rests via a minimization of a power-based disproportionality index, like Fragnelli index, may provide interesting results, reducing the differences of power while using a strictly proportional system.

A more challenging proposal is to design an apportionment rule that, starting with a fixed number of seats (or better with a range for the number of seats), assigns the seats to the parties in order to minimize the difference in the two power distributions. We may go further, supposing to assign a different weight to the members of the different parties in order to have the same distribution of power on the votes and on the seats. A similar idea was already used for designing the VAP (Voting A Posteriori) system (see Fragnelli and Ortona [35]), where the different weights of the votes were introduced to increase the governability of the Parliament.

	<i>A</i>	<i>B</i>	<i>C</i>	<i>D</i>	<i>E</i>
$\phi(s_1)$	$\frac{12}{30}$	$\frac{7}{30}$	$\frac{7}{30}$	$\frac{2}{30}$	$\frac{2}{30}$
$\phi(s_2)$	$\frac{12}{30}$	$\frac{7}{30}$	$\frac{7}{30}$	$\frac{2}{30}$	$\frac{2}{30}$
$\phi(s_{PP})$	$\frac{27}{60}$	$\frac{12}{60}$	$\frac{12}{60}$	$\frac{7}{60}$	$\frac{2}{60}$
$\phi(s_4)$	$\frac{12}{30}$	$\frac{7}{30}$	$\frac{7}{30}$	$\frac{2}{30}$	$\frac{2}{30}$
$\phi(s_5)$	$\frac{12}{30}$	$\frac{7}{30}$	$\frac{7}{30}$	$\frac{2}{30}$	$\frac{2}{30}$
$\phi(s_6)$	$\frac{12}{30}$	$\frac{7}{30}$	$\frac{7}{30}$	$\frac{2}{30}$	$\frac{2}{30}$
$\phi(s_7)$	$\frac{12}{30}$	$\frac{7}{30}$	$\frac{7}{30}$	$\frac{2}{30}$	$\frac{2}{30}$
$\phi(s_8)$	$\frac{12}{30}$	$\frac{7}{30}$	$\frac{7}{30}$	$\frac{2}{30}$	$\frac{2}{30}$
$\phi(s_9)$	$\frac{12}{30}$	$\frac{7}{30}$	$\frac{7}{30}$	$\frac{2}{30}$	$\frac{2}{30}$
$\phi(s_{10})$	$\frac{12}{30}$	$\frac{7}{30}$	$\frac{7}{30}$	$\frac{2}{30}$	$\frac{2}{30}$

Table 6.14: The Shapley-Shubik index referred to the different seat shares for example 6.7.1

A Generating Functions Approach for Computing the Public Good Index Efficiently

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7.1 Introduction

This chapter is devoted to find an efficient way to evaluate the Public Good index, also called the Holler index [48] (see Section 2.2.2, Formula (2.7)). This necessity came while working on the research presented in Chapter 6, where we had to evaluate the disproportionality of the Russian Duma through power-based indices, then computing the power on the votes share. In this example, we had to deal with a very large number of players and traditional algorithms did not allow us to make the computation simply following the definition of the Public Good index. Differently from the best-known measures of power, proposed by Shapley and Shubik [88] and Banzhaf ([12] and [84]), the Holler index takes into account only the role of minimal winning coalitions. Moreover, the Shapley-Shubik and the Banzhaf indices agree that larger players (with a high weight in weighted majority games) never have less power than smaller players¹, while this does not happen computing the power using the Holler index: *situations where minor players possess greater potential for power are not anomalous, but occur rather frequently in real-world situations*, as stated by Deegan and Packel [28].

¹This property is called *monotonicity*

The importance of minimal winning coalitions was largely studied and motivated by Brams and Fishburn ([18] and [19]). Their work is based on Riker's *size principle* [85]. Riker shows that there are no circumstances wherein an incentive exists for coalitions of greater than minimal winning size to form and, on the other hand, there are sufficient incentives for such coalitions to form. His principle affirms that the realization of the goal of winning takes form in the creation only of minimal winning size coalitions, i.e. minimal winning coalitions with minimal weight. The ejection of the superfluous members from one of them means that the same total amount of value can be divided among the fewer members of the minimal winning coalition. Thus, each of the members can derive more profit from it. Moreover, if the coalition payoff is split between the members of the winning coalition according to their respective weights, each member's share will be maximized through the minimizing of the coalition partners' voting weights. A similar argument was adopted by Deegan and Packel [27], who proposed a nonmonotonic index to evaluate the power which takes into account only minimal winning coalitions and which assumes that each such coalition has an equal probability of forming and that players inside a minimal coalition divide the spoil equally.

Even if it is still focused on the minimality, the approach of Holler is totally different. He considers the coalition value to be a public good and any member of the voting body whose preferences correspond with the outcome of the winning coalition is considered as a member of the specific coalition. Some of these members are not essential, because they do not influence the winning coalition, and they are said to be merely lucky (according to Barry's definition of *luck* [13]) and not able to exert power. Then, he still considers only minimal winning coalitions, but this does not imply that only minimal winning coalitions will form, but simply suggests that only these coalitions should be considered for measuring *a priori* voting power. The proposed index is a normalized measure of the number of times a player is a member of a minimal winning coalition.

The problem of calculating power indices in a reasonable amount of time is of great interest since the first power indices have been defined. Some classical indices require the enumeration of all coalitions and this becomes computationally complex as soon as the number of players increases. This can happen also for those indices, e.g. the Deegan-Packel and the Public Good indices, which take into account only minimal winning coalitions: selecting the minimal winning ones may require, in the worst case running, the enumeration of all of them.

Already in 1960, Mann and Shapley [62] faced the problem, proposing some variations of Monte Carlo sampling methods and obtaining an approximation of the Shapley value [86]. Two years later they proposed an exact calculation [63], following an idea due to David G. Cantor, to evaluate this index for large voting games (in the literature and in this paper, *large voting games* are voting games with, generically, a high number of players) when they can be written through a weighted majority situation. This idea has made possible to calculate the exact power in a reasonable amount of

time and it is based on *generating functions*. Brams and Affuso [17] used a similar approach for computing the Banzhaf index. Bilbao et al. [15] applied this generating functions approach and obtained the complexity bound for the algorithms to evaluate the Shapley-Shubik and the Banzhaf indices. Due to the power of this tool, the Spanish scholars dedicated several works to the development of efficient algorithms based on generating functions: Fernández et al. [31] computed the Myerson value [74] in weighted majority games restricted by a tree, Algaba et al. [2] dealt with the problem of computing the Shapley-Shubik and the Banzhaf indices for weighted multiple majority games and Alonso-Meijide and Bowles [6] applied this method to the coalitional power indices, e.g. Owen index [81]. Algaba et al. [1] used generating functions to analyze the distribution of voting power in the Constitution for the enlarged European Union, while Alonso-Meijide et al. [5] computed coalitional power indices in weighted multiple majority games and compared these indices for new decision rules proposed by the Council of the European Union. The computational implementation of this method is efficient and simple: once the generating function has been defined and the algorithm provided, a program to implement it requires a basic knowledge of programming. This chapter focuses on generating functions method with the aim to apply it to the computation of the Public Good index, and our work is to provide the generating function and the algorithm for this purpose. We underline that this method can be applied only to weighted majority games. In all the other cases, it is not possible to use a generating functions approach to the computation of any value.

Anyway, for sake of completeness, we mention other approaches: the work of Owen ([79] and [80]), who proposed approximation algorithms based on multilinear extensions for calculating the Shapley-Shubik and the Banzhaf indices, later modified by Leech [59], who proposed a hybrid with the direct application of the definitions of the indices, and the work of Matsui and Matsui [65], who showed enumeration algorithms for calculating the two above indices and the Deegan-Packel index.

Using generating functions, the main difference between the Banzhaf index and the Shapley-Shubik index is that the first one needs only to take into account the weight of a given coalition, in order to verify if a player is critical in it, while the second one results a bit more complicated, as it has to keep the information regarding the cardinality of the coalition.

The Public Good index takes into account only minimal winning coalitions and counts how many of them a player belongs to. Basically, it counts the number of times a player is critical, but only related to the minimal winning coalitions. Then, it is natural for us to use what it has already been done for the computation of the Banzhaf index and to try to modify it in order to obtain an exact and efficient computation of the Holler index. Unfortunately, doing this a lot of problems arise. This requires building a generating function to evaluate only the numbers of minimal winning coalitions, but it is also fundamental to know when a given player belongs to one of the nonminimal ones or to one of the minimal ones and then it remains critical,

precisely because of the minimality. As it is not possible to do that defining a generating function which is similar to the ones used to compute the Shapley-Shubik and the Banzhaf indices, in this chapter we introduce some recursive generating functions, using a noncommutative operator, which allow us to compute the Public Good index exactly and efficiently.

The chapter is organized as follows: in Section 7.2 we present the generating functions approach to compute the Shapley-Shubik and the Banzhaf indices, while in Section 7.3 we define the new generating function to compute the Public Good index, proving it is suitable for our purpose. An example of how algorithm works, in which we explain all the steps in detail, is provided in Section 7.4. Some computational complexity results are shown in Section 7.5 and the algorithm is applied to evaluate the power in the Russian Duma in Section 7.6. Section 7.7 concludes proposing some possible extensions of the results.

7.2 Generating Functions for Computing the Shapley-Shubik and the Banzhaf Indices

Bilbao et al. [15] recalled a combinatorial method based on generating functions for computing the Shapley-Shubik and the Banzhaf indices exactly and efficiently in the case of weighted majority games. The number of elements $f(k)$ of a finite set can be determined by its generating function. The *ordinary generating function* of $f(k)$ is the formal power series

$$\sum_{k \in \mathbb{N}} f(k)x^k.$$

We call it formal because we ignore evaluation on particular values of x and problems on convergence and we pay attention only to the coefficient of the polynomial. For example, if we are interested in the number of coalitions of weight h , given the appropriate formal series $\sum_{k \in \mathbb{N}} b(k)x^k$, the coefficient $b(h)$ will give us the cardinality of the set of coalitions of weight h .

We can work with generating functions of several variables, as

$$\sum_{k \in \mathbb{N}} \sum_{j \in \mathbb{N}} f(k, j)x^k x^j.$$

Example 7.2.1. For each $n \in \mathbb{N}$, the number of subsets of k elements of the set $N = \{1, \dots, n\}$ is given by the explicit formula of the binomial coefficients

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n(n-1) \cdots (n-k+1)}{k!}.$$

A generating function approach to binomial coefficients may be obtained as follows. Let $S = \{x_1, \dots, x_n\}$ be a set of n elements. Regard the elements x_1, \dots, x_n as

independent indeterminates. It is an immediate consequence of the process of multiplication that

$$(1 + x_1) \cdots (1 + x_n) = \sum_{T \subseteq S} \prod_{x_i \in T} x_i.$$

Note that if $T = \emptyset$, then we obtain 1. If we put each $x_i = x$, we obtain

$$(1 + x)^n = \sum_{T \subseteq S} \prod_{x \in T} x = \sum_{T \subseteq S} x^{|T|} = \sum_{k \in \mathbb{N}} \binom{n}{k} x^k.$$

Then the generating function of the number $\binom{n}{k}$ is $(1 + x)^n$.

The Shapley-Shubik index for large voting games was computed exactly using generating functions by David G. Cantor (Mann and Shapley [63] and Lucas [61]). He observed that the Shapley-Shubik index of player $i \in N$ satisfies

$$\begin{aligned} \Phi_i(v) &= \sum_{\{S \notin W : S \cup \{i\} \in W\}} \frac{s!(n-s-1)!}{n!} \\ &= \sum_{j=0}^{n-1} \frac{j!(n-j-1)!}{n!} \left(\sum_{k=q-w_i}^{q-1} A_{-i}(k, j) \right), \end{aligned}$$

where $A_{-i}(k, j)$ is the number of coalitions S of j players with $i \in S$ and $w(S) = k$, i.e. the number of ways in which j players, other than i , can have a sum of weights equal to k . It is possible now to build the generating function of two variables of the number $A_{-i}(k, j)$, and to easily evaluate the Shapley-Shubik index.

Proposition 7.2.1. (Cantor) *Let $[q; w_1, \dots, w_n]$ be a weighted majority situation. Then the generating function of the number $A_{-i}(k, j)$ is given by $ShG_{-i}(x, z) = \prod_{j=1, \dots, n, j \neq i} (1 + zw^{x_j})$.*

A similar approach was applied by Brams and Affuso [17] for computing the normalized Banzhaf index, noticing that the number of times a player i is critical satisfies

$$\begin{aligned} \beta_i^*(v) &= |\{S \notin W : S \cup \{i\} \in W\}| \\ &= \sum_{k=q-w_i}^{q-1} b_{-i}(k), \end{aligned}$$

where $b_{-i}(k)$ is the number of coalitions that do not include i with weight k . Now we need to write the generating function of one variable of the number $b_{-i}(k)$ in order to evaluate the Banzhaf index.

Proposition 7.2.2. (Brams-Affuso) *Let $[q; w_1, \dots, w_n]$ be a weighted majority situation. Then the generating function of the number $b_{-i}(k)$ of coalitions S such that $i \notin S$, and $w(s) = k$, is given by $BG_{-i}(v) = \prod_{j=1, \dots, n, j \neq i} (1 + x^{w_j})$.*

7.3 Generating Function for Computing the Public Good Index

In this section, taking inspiration from the results proposed in the previous one, we present a combinatorial method based on generating functions for computing the Holler index exactly and efficiently. $BG_{-i}(v)$ is derived by the following generating function $\prod_{i=1}^n (1 + x^{w_i}) = \sum_{k \in \mathbb{N}} b(k)x^k$, which permits to obtain the number $b(k)$ of coalitions with weight k . To obtain the number $b_{-i}(k)$, we delete the factor $(1 + x^{w_i})$. In order to evaluate the Holler index, however, it is still necessary to have the number of coalitions with weight k , but with the requirement of being minimal in case they are winning.

We start defining the *reduced weighted majority game when player i leaves* as the game obtained by the weighted majority situation $[q_{-i}; w'_1, \dots, w'_{n-1}]$, with $q_{-i} = q - w_i$ and $w'_j = w_j$ if $j < i$ and $w'_j = w_{j+1}$ if $j > i$. The sets of all winning coalitions and of all minimal winning coalitions of the reduced game are denoted by W_{-i} and W_{-i}^m respectively and the reduced game can be simply denoted as (N, W_{-i}) or as (N, W_{-i}^m) .

Lemma 7.3.1. $S \in W_{-i}^m$ and $w(S) < q$ iff $S \cup \{i\} \in W^m$.

Proof. It is trivial to observe that S is winning in the reduced game iff $S \cup \{i\}$ is winning in the original game. We prove now that the minimality of S in (N, W_{-i}) together with the condition $\sum_{j \in S} w_j < q$ is equivalent to the minimality of $S \cup \{i\}$ in (N, W) :

" \Rightarrow " As S is minimal in (N, W_{-i}) , $\sum_{j \in S \setminus \{k\}} w_j < q_{-i}$ for each $k \in S$ and then $\sum_{j \in S \cup \{i\} \setminus \{k\}} w_j = \sum_{j \in S \setminus \{k\}} w_j + w_i < q_{-i} + w_i = q$ for each $k \in S$. Moreover, as $\sum_{j \in S} w_j < q$, the players in S have insufficient votes to win without the help of i when the majority quota is q , i.e. also i is critical in $S \cup \{i\}$.

" \Leftarrow " As $S \cup \{i\}$ is minimal in (N, W) , $\sum_{j \in S \cup \{i\} \setminus \{k\}} w_j < q$ for each $k \in S$ and then $\sum_{j \in S \setminus \{k\}} w_j < q - w_i = q_{-i}$. \square

In order to evaluate the Public Good index (Formula 2.7), from Lemma 7.3.1 we get that the number of minimal winning coalitions player i belongs to $h_i(v) = |W_{-i}^m|$ is given by

$$h_i(v) = \sum_{k=q-w_i}^{q-1} b_{-i}^m(k), \quad (7.1)$$

where $b_{-i}^m(k)$ is the number of coalitions that do not include i with weight k and that are minimal winning for the reduced game (N, W_{-i}) .

In order to compute the Holler index, we need then to define the generating function of the numbers $b_{-i}^m(k)$ for each player $i \in N$. We should notice that, in the above formula, we sum only for $k \geq q - w_i$ and then the coalitions are minimal winning for the reduced game. In order to obtain the generating function, we have to obtain values of $b_{-i}^m(k)$ for every $k \geq 0$, and then we will obtain also the number of coalitions

of weight k which are losing for the reduced game.

Before doing that, we recall some basic notions of algebra. Every polynomial $P(x)$ can be written in the *divisor-quotient form*, this means that considering the *dividend* and the *divisor* polynomials $P(x)$ and $D(x)$ with $\deg(D) < \deg(P)$, where we denote with $\deg(A)$ the degree of the polynomial A , then for some *quotient* and *remainder* polynomials $Q(x)$ and $R(x)$ with $\deg(R) < \deg(D)$ we can write $P(x) = R(x) + Q(x) \cdot D(x)$.

Given $q \in \mathbb{N}$, denoting as $\mathbb{N}[x]$ the set of polynomials with coefficients in \mathbb{N} , we define the following noncommutative operator

$$\begin{aligned} \otimes_q : \mathbb{N}[x] \times \mathbb{N}[x] &\longrightarrow \mathbb{N}[x] \\ P(x) \otimes_q t(x) &\longmapsto R(x) \cdot t(x) + Q(x) \cdot x^q \end{aligned}$$

where $R(x)$ and $Q(x)$ are respectively the remainder and the quotient polynomials of the divisor-quotient form $P(x) = R(x) + Q(x) \cdot x^q$.

Given the weighted majority situation $[q; w_1, \dots, w_n]$ with the additional condition that the players are ordered such that $w_1 \geq \dots \geq w_n$ we denote

$$\bigotimes_{i=1}^n q(1 + x^{w_i}) = (1 + x^{w_1}) \otimes_q \dots \otimes_q (1 + x^{w_n}).$$

Given the weighted majority situation $[q; w_1, \dots, w_n]$ with the additional condition that the players are ordered such that $w_1 \geq \dots \geq w_n$, we define the *associated Holler recursive functions* $\{HG^r(x)\}_{r=1, \dots, n}$ as

$$\begin{cases} HG^1(x) = 1 + x^{w_1} \\ HG^r(x) = HG^{r-1}(x) \otimes_q (1 + x^{w_r}) \quad r = 2 \dots n \end{cases} \quad (7.2)$$

Given the reduced game $[q_{-i}; w'_1, \dots, w'_{n-1}]$, we denote as $\{HG^r_{-i}(x)\}_{r=1, \dots, n-1}$ the associated Holler recursive functions and we note that, by definition

$$HG^{n-1}_{-i}(x) = \bigotimes_{r=1}^{n-1} q_{-i}(1 + x^{w'_r}).$$

Due to the noncommutativity of the operator, it is necessary to give an ordering of the players to have the uniqueness of the definition of the recursive functions; in particular, the reasons of the assumption of a weakly decreasing order will be discussed in the proof of Proposition 7.3.2.

Proposition 7.3.2. *Let $[q; w_1, \dots, w_n]$ be a weighted majority situation with $w_1 \geq \dots \geq w_n$. Then the generating function of the number $b^m_i(k)$ of coalitions S such that $i \notin S$, $w(S) = k$ and S is either losing (if $k < q$) or minimal winning (if $k \geq q$) for the reduced game (N, W_{-i}) , is given by $HG^{n-1}_{-i}(x) = \bigotimes_{r=1}^{n-1} q_{-i}(1 + x^{w'_r})$.*

Proof. We need to prove that considering the weighted majority situation $[q-i; w'_1, \dots, w'_{n-1}]$, at step r , $HG_{-i}^r(x)$ generates the number of either losing (if $k < q$) or minimal winning (if $k \geq q$) coalitions of weight k containing at most the first r players. By induction, when $r = 1$, $HG_{-i}^1(x) = 1 + x^{w'_1} = x^0 + x^{w'_1}$, that is a coalition of weight 0 and a coalition of weight w'_1 . The first one is the losing empty coalition, the second one is losing if $w'_1 < q-i$ and winning otherwise, also minimal as it contains just one player.

Suppose now that $HG_{-i}^{r-1}(x)$ generates the losing and the minimal winning coalitions of weight k formed by the first $r-1$ players, then we can write

$$HG_{-i}^{r-1}(x) = R_{-i}^{r-1}(x) + Q_{-i}^{r-1}(x) \cdot x^{q-i}.$$

The coefficients of $Q_{-i}^{r-1}(x) \cdot x^{q-i}$ are the number of minimal winning coalitions of weight k , while the coefficients of $R_{-i}^{r-1}(x)$ are the number of losing coalitions of weight k , both considering only the first $r-1$ players. Then, at step r we get

$$HG_{-i}^r(x) = HG_{-i}^{r-1}(x) \otimes_q (1 + x^{w_r}) = R_{-i}^{r-1}(x) \cdot (1 + x^{w_r}) + Q_{-i}^{r-1}(x) \cdot x^{q-i},$$

the coefficients of $R_{-i}^{r-1}(x) \cdot (1 + x^{w_r})$ are the number of losing subcoalitions of weight k (if $k < q$) considering the first r players including or not player r , or the number of winning subcoalitions of weight k (if $k \geq q$), considering the first r players, including player r and that were losing without player r . As $w_1 \geq \dots \geq w_r$, if these coalitions are winning and become losing when player r leaves, they become losing when any player of the coalition leaves, as they are formed by the first r players and the first $r-1$ players have weight larger than or equal to w_r . Then, the decreasing ordering of the weights of the players provides the minimality of the coalitions formed during the procedure. \square

It is important to remark that, obviously, an implementation of the procedure does not need to operate divisions between polynomials. As the divisor is simply a monic monomial, a check on the degree of the monomials of the dividend polynomial is sufficient to obtain the quotient and the remainder polynomials.

7.4 Building the Generating Functions for the German Bundestag

In order to provide an example of the procedure, we recall Section 4.4, where we illustrated the situation of the German Parliament on 3 March 2011. The 17th German Bundestag counted 620 Members, divided in 5 parliamentary groups, and can be represented through the following weighted majority situation $[311; 237, 146, 93, 76, 68]$. From now on, we will refer to the parties as 1, 2, 3, 4 and 5, instead of *CDU/CSU*, *SPD*, *FDP*, *Die Linke* and *Die Grünen*. We go over again the algorithm to calculate the Public Good index explaining all the steps in detail.

The reduced weighted majority game when player 1 leaves is the one associated to the following weighted majority situation [74; 146, 93, 76, 68]. The associated Holler recursive function (Formula 7.2) is given by

$$\left\{ \begin{array}{l} HG_{-1}^1(x) = 1 + x^{146} \\ HG_{-1}^2(x) = HG_{-1}^1(x) \otimes_{74} (1 + x^{93}) \\ \quad = 1 \cdot (1 + x^{93}) + x^{146} = 1 + x^{93} + x^{146} \\ HG_{-1}^3(x) = HG_{-1}^2(x) \otimes_{74} (1 + x^{76}) \\ \quad = 1 \cdot (1 + x^{76}) + x^{93} + x^{146} \\ \quad = 1 + x^{76} + x^{93} + x^{146} \\ HG_{-1}^4(x) = HG_{-1}^3(x) \otimes_{74} (1 + x^{68}) \\ \quad = 1 \cdot (1 + x^{68}) + x^{76} + x^{93} + x^{146} \\ \quad = 1 + x^{68} + x^{76} + x^{93} + x^{146} \end{array} \right.$$

According to Proposition 7.3.2, $HG_{-1}^4(x)$ is the generating function of the number $b_{-1}^m(k)$ of coalitions S such that $1 \notin S$, $w(S) = k$ and S is either losing (if $k < 74$) or minimal winning (if $k \geq 74$) for the reduced game (N, W_{-1}) , then from Formula (7.1) we get

$$h_1(v) = \sum_{k=74}^{310} b_{-1}^m(k) = 3.$$

The reduced weighted majority game when player 2 leaves is the one associated to the following weighted majority situation [165; 237, 93, 76, 68]. The associated Holler recursive function is given by

$$\left\{ \begin{array}{l} HG^1(x) = 1 + x^{237} \\ HG_{-2}^2(x) = HG_{-2}^1(x) \otimes_{165} (1 + x^{93}) \\ \quad = 1 \cdot (1 + x^{93}) + x^{237} = 1 + x^{93} + x^{237} \\ HG_{-2}^3(x) = HG_{-2}^2(x) \otimes_{165} (1 + x^{76}) \\ \quad = (1 + x^{93}) \cdot (1 + x^{76}) + x^{237} \\ \quad = 1 + x^{76} + x^{93} + x^{169} + x^{237} \\ HG_{-2}^4(x) = HG_{-2}^3(x) \otimes_{165} (1 + x^{68}) \\ \quad = (1 + x^{76} + x^{93}) \cdot (1 + x^{68}) + x^{169} + x^{237} \\ \quad = 1 + x^{68} + x^{76} + x^{93} + x^{144} + x^{161} + x^{169} + x^{237} \end{array} \right.$$

then

$$h_2(v) = \sum_{k=165}^{310} b_{-2}^m(k) = 2.$$

The reduced weighted majority game when player 3 leaves is the one associated to the following weighted majority situation [218; 237, 146, 76, 68]. The associated

Holler recursive function is given by

$$\left\{ \begin{array}{l} HG_{-3}^1(x) = 1 + x^{237} \\ HG_{-3}^2(x) = HG_{-3}^1(x) \otimes_{218} (1 + x^{146}) \\ \quad = 1 \cdot (1 + x^{146}) + x^{237} = 1 + x^{146} + x^{237} \\ HG_{-3}^3(x) = HG_{-3}^2(x) \otimes_{218} (1 + x^{76}) = (1 + x^{146}) \cdot (1 + x^{76}) + x^{237} \\ \quad = 1 + x^{76} + x^{146} + x^{222} + x^{237} \\ HG_{-3}^4(x) = HG_{-3}^3(x) \otimes_{218} (1 + x^{68}) \\ \quad = (1 + x^{76} + x^{146}) \cdot (1 + x^{68}) + x^{222} + x^{237} \\ \quad = 1 + x^{68} + x^{76} + x^{93} + x^{144} + x^{146} + x^{214} + x^{222} + x^{237} \end{array} \right.$$

then

$$h_3(v) = \sum_{k=218}^{310} b_{-3}^m(k) = 2.$$

The reduced weighted majority game when player 4 leaves is the one associated to the following weighted majority situation [235; 237, 146, 93, 68]. The associated Holler recursive function is given by

$$\left\{ \begin{array}{l} HG_{-4}^1(x) = 1 + x^{237} \\ HG_{-4}^2(x) = HG_{-4}^1(x) \otimes_{235} (1 + x^{146}) \\ \quad = 1 \cdot (1 + x^{146}) + x^{237} = 1 + x^{146} + x^{237} \\ HG_{-4}^3(x) = HG_{-4}^2(x) \otimes_{235} (1 + x^{93}) \\ \quad = (1 + x^{146}) \cdot (1 + x^{93}) + x^{237} \\ \quad = 1 + x^{93} + x^{146} + x^{237} + x^{239} \\ HG_{-4}^4(x) = HG_{-4}^3(x) \otimes_{235} (1 + x^{68}) \\ \quad = (1 + x^{93} + x^{146}) \cdot (1 + x^{68}) + x^{237} + x^{239} \\ \quad = 1 + x^{68} + x^{93} + x^{146} + x^{161} + x^{214} + x^{237} + x^{239} \end{array} \right.$$

then

$$h_4(v) = \sum_{k=235}^{310} b_{-4}^m(k) = 2.$$

Finally, the reduced weighted majority game when player 5 leaves is the one associated to the following weighted majority situation [243; 237, 146, 93, 76]. The asso-

ciated Holler recursive function is given by

$$\left\{ \begin{array}{l} HG_{-5}^1(x) = 1 + x^{237} \\ HG_{-5}^2(x) = HG_{-5}^1(x) \otimes_{243} (1 + x^{146}) \\ \quad = (1 + x^{237}) \cdot (1 + x^{146}) \\ \quad = 1 + x^{146} + x^{237} + x^{383} \\ HG_{-5}^3(x) = HG_{-5}^2(x) \otimes_{243} (1 + x^{93}) \\ \quad = (1 + x^{146} + x^{237}) \cdot (1 + x^{93}) + x^{383} \\ \quad = 1 + x^{93} + x^{146} + x^{237} + x^{239} + x^{330} + x^{383} \\ HG_{-5}^4(x) = HG_{-5}^3(x) \otimes_{243} (1 + x^{76}) \\ \quad = (1 + x^{93} + x^{146} + x^{237} + x^{239}) \cdot (1 + x^{76}) + x^{330} + x^{383} \\ \quad = 1 + x^{76} + x^{93} + x^{146} + x^{169} + x^{222} + x^{237} + x^{239} + x^{313} + x^{315} \\ \quad \quad + x^{330} + x^{383} \end{array} \right.$$

then

$$h_5(v) = \sum_{k=243}^{310} b_{-5}^m(k) = 0.$$

Because of Formula (2.7), the Public Good index of the game is given by the normalized vector of the $h_i(v)$, then

$$H(v) = \left(\frac{3}{9}, \frac{2}{9}, \frac{2}{9}, \frac{2}{9}, 0 \right).$$

7.5 Computational Complexity

Bilbao et al. [15] analyzed the computational complexity of the algorithms based on generating functions to evaluate the Shapley-Shubik and the Banzhaf indices, observing that in a classical procedure, if the input size of the problem is n , the function which measures the worst case running time for computing the indices is $O(2^n)$. They assumed a logarithmic cost model. In this model, if we perform only a polynomial number of operations on numbers with at most a polynomial number of digits, then the algorithm will be polynomial (Gács and Lovász [37]). Given $f(n)$ a function from \mathbb{Z}_+ to \mathbb{Z}_+ , we denote $O(f(n))$ for the set of all functions g such that $f(n) \leq cg(n)$ for $n \geq n_0$. Then a polynomial of degree d is in $O(n^d)$, this means that only asymptotic behavior of the function as $n \rightarrow +\infty$ is being considered.

Bilbao et al. proved the following two results:

Theorem 7.5.1. *Let $[q; w_1, \dots, w_n]$ be a weighted majority situation. If C is the number of nonzero coefficients of $BG(x)$, then the time complexity of the generating algorithm for the Banzhaf index is $O(n^2C)$.*

Theorem 7.5.2. *Let $[q; w_1, \dots, w_n]$ be a weighted majority situation. If C is the number of nonzero coefficients of $ShG(x, z)$, then the time complexity of the generating algorithm for the Shapley-Shubik index is $O(n^2C)$.*

We assume the same logarithmic cost model and we can now enunciate the following theorem. The proof traces the proof of Theorem 7.5 given by Bilbao et al. [15].

Theorem 7.5.3. *Let $[q; w_1, \dots, w_n]$ be a weighted majority situation. If C is the number of nonzero coefficients of $HG_{-i}^{n-1}(x)$, then the time complexity of the generating algorithm for the Holler index is $O(n^2C)$.*

Proof. Let i be a player, q_{-i} the majority quota of the reduced game (N, W_{-i}) associated to the weighted majority situation $[q_{-i}, w'_1, \dots, w'_{n-1}]$, the function $HG_{-i}^{n-1}(x)$ is given by

```

Q ← q-i
HG(x) ← 1
for j ∈ {1, ..., n} with j ≠ i do
  HHG(x) ← polynomial given by the monomials of HG(x) of deg < Q
  HG(x) ← HG(x) + HHG(x)xwj
endfor

```

For every player the time to compute the line in the loop is in $O(C)$, then the time to compute the complete function is $O(nC)$. To compute the Holler index we consider $HG_i^{n-1}(x) = \sum_{k \in \mathbb{N}} b_{-i}^m(k)x^k$ for player $i \in N$ and the **for** loop

```

w ← wi
s ← 0
for k ∈ {q - w, ..., q - 1} do
  s ← s + b-im(k)
endfor

```

The time spent in the above loop is $O(C)$. Then, the total time in the procedure for each player is $O(nC)$, executed n times we get a time complexity of the generating algorithm for the Public Good index of $O(n^2C)$. □

7.6 The Russian State Duma: a Real-World Example

In real-world situations, examples with a high number of parties can require fast algorithms to evaluate power indices efficiently. In the previous chapter we dealt with the problem of evaluating the power given by the vote share after the election of the Russian Parliament (State Duma). In particular, referring to the situation of 1995, which counted 43 parties submitted to the vote, in Table 7.1 we have a summary of parties and coalitions which participated in this election, with the vote share in percentage in the first column and the Shapley-Shubik, the normalized Banzhaf and the Public Good indices in the other three ones. The indices have been evaluated using generating functions algorithms; for the first two ones we implemented two algorithms following the generating function approach shown in Section 7.2, for the Public Good index we implemented the new algorithm following the new generating function approach proposed in Section 7.3. All these algorithms have been implemented in Matlab. We should remark that, as the vote share is not given by integer numbers, the weights

and the quota have been multiplied by one hundred. These algorithms provide a result in few seconds, while other algorithms not based on generating functions cannot even deal with such a big number of parties.

The vote share of “Against all” and of “Invalid ballots” has been included, in order to have a total vote share summing up to 1, but this two lines have been obviously eliminated by the evaluation of power and the quota adapted to the majority reachable by the other parties. The resulting game is then given by the weighted majority situation with $n = 43$ players and a majority quota of $q = 4765$. We may notice that the power obtained using, for example, the Shapley-Shubik index differs a lot from the power obtained evaluating the Holler index. In particular the Holler index gives 1/10 of power to the biggest party KPRF.

7.7 Concluding Remarks

The Public Good index represents a different approach in evaluating the power. Then, it is important to have a powerful instrument for being able to efficiently compute this index. A very famous example to show how the Holler index can provide some results which are very different from the ones provided by classical indices is the UNSC (see Section 5.1). Referring to the decisions on substantive matters, we represent the UNSC by the following weighted majority situation [39; 7, 7, 7, 7, 7, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1]. We compute the voting power in it, remembering one more time that the Holler index is based on a totally different approach which takes into account only minimal winning coalitions. As a result, the power of smaller players (the nonpermanent members) is bigger compared to the power assigned to them by the other measures of power. We show the results in Table 7.2 where P stands for permanent and NP for non permanent. We can notice that the Shapley-Shubik index evaluates the power of a permanent member more than 100 times the power of a nonpermanent member, the Banzhaf index more than 10 times while the Holler index suggests a ratio of only 5/2.

In our opinion, generating functions methods are, without any doubt, the most powerful instrument to compute power indices in weighted majority games, and this idea is confirmed by the interest that many researchers, mainly from Spain, showed for this topic. Our work provides a new approach to deal with an index based on minimal winning coalitions. The same approach, together with the idea proposed to evaluate the Shapley-Shubik index, which permits to take into account the cardinality of a coalition, may be applied to obtain a generating function to compute the Deegan-Packel index. This index has already been studied according to a different generating function by Alonso-Meijide [3]. Then, it can be interesting to make a comparison of the two methods in dealing with minimal winning coalitions. Moreover, some variations of the Public Good index were proposed, like for example the ones proposed by Alonso-Meijide et al. [8], and our algorithm can be modified and applied to their implementation.

Chapter 7. A Generating Functions Approach for Computing the Public Good Index Efficiently

Parties	Votes %	ϕ	β	H
KPRF - Comm. Party of the Russian Fed.	22.3	0.2822	0.3053	0.0235
LDPR - Liberal Democratic Party of Russia	11.18	0.1157	0.1041	0.0234
NDR - Our Home Russia	10.13	0.1039	0.0969	0.0234
Yabloko	6.89	0.0684	0.0659	0.0234
ZhR - Women of Russia	4.61	0.0448	0.0440	0.0234
Communist and Working Russia	4.53	0.0440	0.0432	0.0234
KRO - Congress of Russian Communities	4.31	0.0418	0.0411	0.0234
Party of Workers' Self-Government	3.98	0.0385	0.0380	0.0234
Democratic Russia's Choice	3.86	0.0373	0.0369	0.0234
APR - Agrarian Party of Russia	3.78	0.0365	0.0361	0.0234
Strong State	2.57	0.0246	0.0245	0.0234
Forward Russia	1.94	0.0185	0.0185	0.0234
Power to the People	1.61	0.0153	0.0154	0.0234
Pamfilova - Gurov-V, Lysenko	1.6	0.0152	0.0153	0.0234
Trade Unions and Industrialists	1.55	0.0147	0.0148	0.0234
Environmental Party of Russia "Kedr"	1.39	0.0132	0.0133	0.0234
Bloc of Ivan Rybkin	1.11	0.0105	0.0106	0.0234
Bloc of Stanislav	0.99	0.0093	0.0095	0.0234
My Fatherland	0.72	0.0068	0.0069	0.0234
Common Cause	0.68	0.0064	0.0065	0.0234
Beer Lovers' Party	0.62	0.0058	0.0059	0.0234
All Russian Muslim Public Movement "Nur"	0.57	0.0054	0.0054	0.0234
Transformation of the Fatherland	0.49	0.0046	0.0047	0.0234
Transformation of the Fatherland	0.49	0.0046	0.0047	0.0234
National Republican Party of Russia	0.48	0.0045	0.0046	0.0234
Electoral Bloc 1	0.47	0.0044	0.0045	0.0234
PRES - Party of Russian Unity and Accord	0.36	0.0034	0.0034	0.0234
Russian Lawyers' Association	0.35	0.0033	0.0033	0.0234
For Motherland!	0.28	0.0026	0.0027	0.0234
Christian-Democratic Union - Christians o Russia	0.28	0.0026	0.0027	0.0234
Electoral Bloc 2	0.21	0.0020	0.0020	0.0234
People's Union	0.19	0.0018	0.0018	0.0234
Bloc "Tikhonov-Tupolev-Tikhonov"	0.15	0.0014	0.0014	0.0234
Social Democrats	0.13	0.0012	0.0012	0.0234
Party of Economic Freedom	0.13	0.0012	0.0012	0.0234

ROD - Russian All-People's Movement	0.12	0.0011	0.0011	0.0234
Bloc of Independents	0.12	0.0011	0.0011	0.0234
FDD - Federal Democratic Movement	0.12	0.0011	0.0011	0.0234
Socio-political Movement "Stable Russia"	0.12	0.0011	0.0011	0.0234
Duma - 96	0.08	0.0007	0.0008	0.0225
Frontier Generations	0.06	0.0006	0.0006	0.0216
89	0.06	0.0006	0.0006	0.0216
Interethnic Union	0.06	0.0006	0.0006	0.0216
Against all	2.77	-	-	-
Invalid ballots	1.91	-	-	-
Total	100	1	1	1

Table 7.1: December 17, 1995 Russian State Duma, vote share and power evaluated using the Shapley-Shubik, the normalized Banzhaf and the Holler indices

	P	NP
$\phi(v)$	$\frac{421}{2145}$	$\frac{4}{2145}$
$\beta(v)$	$\frac{212}{4096}$	$\frac{21}{4096}$
$H(v)$	$\frac{5}{45}$	$\frac{2}{45}$

Table 7.2: Power share in the UNSC

General Conclusions

Game Theory applied to voting may bring rise to the study of many different topics. In this thesis we presented some results in order to evaluate the voting systems, to analyze the criteria for the assessment of the voters' preferences and to provide more functional ways to compute the existing instruments.

The goal of perfectly describing and analyzing voting situations is not in the near future, but several researchers all over the world provided some very useful progresses for this purpose. The bibliography of this thesis lists some of the major results in this field and we hope that the research work contained in the thesis itself can provide some small new steps and some suggestions for future analysis. We have already listed, chapter by chapter, what we thought the possible development of the research was, but we want to summarize here the points which are, in our opinion, crucial.

In Chapter 3, we investigated the way of combining a communication structure with the already existing indices of power. We are totally convinced that the instruments provided by the literature have to be taken into account as a valid starting point in the description of a voting system, but, as it often happens, the models have to aim at becoming more and more sophisticated, in order to catch the different facets of the real-world problems. We remarked how the communication structures are not able yet to represent every possible situation in which the set of feasible coalitions is reduced, due to the different ideologies of the agents; moreover we presented the possibility of assuming some coalitions as less probable, but not infeasible, as the political scenario is complex enough to admit the possibility of very unlikely coalitions to form. A general model which is able to include and analyze every possible situation, even if very difficult to find, should be the goal of every future research.

In Chapter 4, we added another important aspect to the problem; the alliances inside a decisional situation are not stable, but may evolve with time, mainly due to the fact that each agent aims at getting a higher power. We provided an instrument which, theoretically, may perfectly describe this situation; unfortunately, the computational complexity does not permit to adopt it in many real cases. At least in those situations in which it is possible to evaluate it, or at least to know if a given solution belongs to this set, we think it can be adopted for a good evaluation of the stability of a power share.

Chapter 5 analyzes the problem of evaluating the power by another point of view, the power to block instead of the power to win. We provided an index which evaluates the veto power, but it can be extended in order to catch other characteristics of the game, for example, the probability that a party takes a particular decision, or that not

every member of a party is present at the vote.

A very high goal every research should aim for is the definition of a general model for the evaluation of the power which takes into account all these aspects: the communication structure, the dynamic configuration of a decisional model and the important role of a player in blocking a proposal and not only in making it approved. A model which can include these (and possibly other) aspects may provide a better realistic evaluation of the power in a complex and real decisional situation.

Chapter 6 considers a previous step of the formation of a democracy: the evaluation of how good the resulting Parliament is after the electors have expressed their preferences. This is probably the most actual and discussed aspect of the research in voting theory, as many countries have to deal with a mechanism which does not provide convincing results. But one more time, we think that the goodness of a Parliament mainly depends on the power share between the parties and not only on the way the seats are allocated. Then, a good instrument for the evaluation of the power can bring to a good evaluation of the representativeness of a Parliament.

Finally, in Chapter 7 we provided a new method to evaluate one of the existing indices of power. In fact, even a possibly “perfect” power index can be very hard to compute and it may turn out to be useless if we are not able to evaluate it in real-world situations. This was the problem, for example, of the solution proposed in Chapter 4. It is important that the research continues in trying to find efficient algorithms to make the computation of this and of other indices possible, at least in the near future.

We think that a mathematical approach to the study of voting systems can provide many improvements and many interesting results for a better organization of a democracy. Optimization, simulation, Game Theory and many other branches of the mathematical research can be very useful in order to better understand how this form of government works and how it can evolve.

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