A second-order identity for the Riemann tensor and applications

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Abstract

A second-order differential identity for the Riemann tensor is obtained, on a manifold with a symmetric connection. Several old and some new differential identities for the Riemann and Ricci tensors are derived from it. Applications to manifolds with recurrent or symmetric structures are discussed. The new structure of K-recurrency naturally emerges from an invariance property of an old identity due to Lovelock.

1 Introduction

Given a symmetric connection on a smooth manifold, one introduces the covariant derivative and the Riemann curvature tensor $R_{abc}{}^d = \partial_b \Gamma^d_{ac} - \partial_a \Gamma^d_{bc} + \Gamma^k_{ac}\Gamma^d_{bk} - \Gamma^d_{ak}\Gamma^k_{bc}$. The tensor is antisymmetric in a, b and satisfies the two Bianchi identities, $R_{(abc)}{}^d = 0$ and $\nabla_{(a} R_{bc)d}{}^e = 0$ [1]. The latter represents the closure property of the curvature 2-form associated to the Riemann tensor $\Omega_c{}^d = -\frac{1}{2} R_{abc}{}^d dx^a \wedge dx^b$ [BE, LO] in absence of torsion.
From the Bianchi identities various others for the Riemann tensor and the

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[1]Hereafter the symbol $(\cdots)$ denotes a summation over cyclic permutations of tensor indices; for example, $K_{(abc)} = K_{abc} + K_{bca} + K_{cab}$ [PO]. In particular, if $K_{abc} = -K_{bac}$ the cyclic summation (abc) equals, up to a factor, the complete antisymmetrization $[abc]$.

Ricci tensor $R_{ac} = R_{abc}{}^b$ can be derived. The following first-order one is due to Oswald Veblen [EI, LO]:

$$\nabla_a R_{bcd}{}^e - \nabla_b R_{adc}{}^e + \nabla_c R_{adb}{}^e - \nabla_d R_{bca}{}^e = 0. \tag{1.1}$$

If the connection comes from a metric, Walker's identity of the second order holds [WA, SH]

$$[\nabla_a, \nabla_b] R_{cdef} + [\nabla_c, \nabla_d] R_{abef} + [\nabla_e, \nabla_f] R_{abcd} = 0 \tag{1.2}$$

and, if the Ricci tensor vanishes, Lichnerowicz's nonlinear wave equation holds [HU, MT]

$$\nabla^e \nabla_e R_{abcd} + R_{ab}{}^{ef} R_{efcd} - 2R^e{}_{ac}{}^f R_{ebdf} + 2R_{ead}{}^f R^e{}_{bcf} = 0. \tag{1.3}$$

In this paper we derive, with the only requirement that the connection be symmetric, a useful identity for the cyclic combination $\nabla_{(a} \nabla_b R_{cd)e}{}^f$. An identity due to Lovelock for the divergence of the Riemann tensor follows from it. We show that it holds more generally for curvature tensors K originating from the Riemann tensor (Weyl, concircular etc.).

The main identity and Lovelock's enable us to obtain in a unified manner various known identities, and some new ones, that apply in Riemannian spaces with structures. In Section 3 we show that various properties of the Riemannian metric, such as being a) locally symmetric, b) nearly conformally symmetric, c) semisymmetric, d) pseudosymmetric, e) generalized recurrent, lead to the same set of algebraic identities for the Riemann tensor. We then introduce the new structures, K-harmonic and K-recurrent, that also yield the set of identities, and include cases a), b), e) and others, that arise from different choices of the tensor K.

In Section 4, weakly Ricci symmetric structures are considered, with their associated covectors A,B,D. We show that one of the above algebraic identities holds iff the vector field $A - B$ is closed. We end with Section 5, where we derive Lichnerowicz's wave equation for the Riemann tensor from the main equation.

2 A second order identity

We begin with the main identity; as a corollary we derive an identity due to Lovelock which is used throughout the paper, and a generalization of it, for various curvature tensors K.

Main Theorem 2.1. (The second order identity)
In a smooth differentiable manifold with a symmetric connection:

$$\nabla_{(a}\nabla_b R_{cd)e}{}^f = -R_{(abc}{}^m R_{d)me}{}^f + R_{ace}{}^m R_{bdm}{}^f - R_{acm}{}^f R_{bde}{}^m \tag{2.1}$$

Proof. Take the covariant derivative of the second Bianchi identity, and sum over cyclic permutations of the four indices $abcd$:

$$\begin{aligned} 0 &= \nabla_a\nabla_{(b}R_{cd)e}{}^f + \nabla_b\nabla_{(c}R_{da)e}{}^f + \nabla_c\nabla_{(d}R_{ab)e}{}^f + \nabla_d\nabla_{(a}R_{bc)e}{}^f \\ &= 2\nabla_{(a}\nabla_b R_{cd)e}{}^f + [\nabla_b,\nabla_a]R_{cde}{}^f + [\nabla_c,\nabla_b]R_{dae}{}^f \\ &+ [\nabla_d,\nabla_c]R_{abe}{}^f + [\nabla_a,\nabla_d]R_{bce}{}^f + [\nabla_a,\nabla_c]R_{dbe}{}^f + [\nabla_b,\nabla_d]R_{ace}{}^f \end{aligned} \tag{2.2}$$

The action of a commutator on the curvature tensor gives quadratic terms

$$[\nabla_a,\nabla_b]R_{cde}{}^f = R_{abc}{}^k R_{kde}{}^f + R_{abd}{}^k R_{cke}{}^f + R_{abe}{}^k R_{cdk}{}^f - R_{abk}{}^f R_{cde}{}^k$$

that produce 24 quadratic terms. Eight of them cancel because of the antisymmetry of R and the remaining ones can be grouped as follows:

$$\begin{aligned} &= 2\nabla_{(a}\nabla_b R_{cd)e}{}^f - 2R_{ace}{}^s R_{bds}{}^f - 2R_{acs}{}^f R_{dbe}{}^s \\ &+ R_{sce}{}^f(R_{adb}{}^s + R_{bda}{}^s + R_{abd}{}^s) + R_{sbe}{}^f(R_{dca}{}^s + R_{acd}{}^s + R_{dac}{}^s) \\ &+ R_{ase}{}^f(R_{dcb}{}^s + R_{bdc}{}^s + R_{bcd}{}^s) + R_{dse}{}^f(R_{cba}{}^s + R_{acb}{}^s + R_{abc}{}^s). \end{aligned}$$

The last two lines simplify by the first Bianchi identity,

$$\begin{aligned} &= 2\nabla_{(a}\nabla_b R_{cd)e}{}^f - 2R_{ace}{}^s R_{bds}{}^f - 2R_{acs}{}^f R_{dbe}{}^s + 2R_{sce}{}^f R_{adb}{}^s - 2R_{sbe}{}^f R_{cda}{}^s \\ &+ 2R_{ase}{}^f R_{bcd}{}^s - 2R_{sde}{}^f R_{abc}{}^s \end{aligned}$$

Four terms are seen to be a cyclic summation $(abcd)$. □

The contraction of f with the index e gives an equation for the antisymmetric part of the Ricci tensor (which, essentially, coincides with $R_{abc}{}^c$ by the first Bianchi identity):

Corollary 2.2. *If* $U_{ab} = R_{ab} - R_{ba}$ *then* $\nabla_{(a}\nabla_b U_{cd)} = -R_{(abc}{}^m U_{d)m}$.

The contraction of f with the index a leads to an identity for the *divergence* of the Riemann tensor ([LO] ch. 7), which will be used extensively. We refer to it as

Corollary 2.3. *(Lovelock's differential identity)*

$$\begin{aligned} &\nabla_a\nabla_m R_{bce}{}^m + \nabla_b\nabla_m R_{cae}{}^m + \nabla_c\nabla_m R_{abe}{}^m \\ &\quad = -R_{am}R_{bce}{}^m - R_{bm}R_{cae}{}^m - R_{cm}R_{abe}{}^m \end{aligned} \tag{2.3}$$

Proof. The contraction in (2.1) gives $\nabla_{(a}\nabla_b R_{cd)e}{}^a = -R_{(abc}{}^m R_{d)me}{}^a + R_{ace}{}^m R_{bdm}{}^a$ $+ R_{cm} R_{bde}{}^m$. The two cyclic sums are now written explicitly:

$$\nabla_a\nabla_b R_{cde}{}^a + \nabla_b\nabla_c R_{de} - \nabla_c\nabla_d R_{be} + \nabla_d\nabla_a R_{bce}{}^a = -R_{abc}{}^m R_{dme}{}^a$$
$$+R_{bcd}{}^m R_{me} - R_{cda}{}^m R_{bme}{}^a - R_{dab}{}^m R_{cme}{}^a + R_{ace}{}^m R_{bdm}{}^a + R_{cm} R_{bde}{}^m.$$

Next the order of covariant derivatives is exchanged in the first term of the l.h.s. Some terms just cancel and a triplet vanishes for a Bianchi identity. One gets $\nabla_b\nabla_a R_{cde}{}^a + \nabla_b\nabla_c R_{de} - \nabla_c\nabla_d R_{be} + \nabla_d\nabla_a R_{bce}{}^a = -R_{ba} R_{cde}{}^a + R_{ae} R_{bcd}{}^a + R_{ca} R_{bde}{}^a$. A Ricci term in the l.h.s. is replaced with the identity $\nabla_c\nabla_d R_{be} = \nabla_c(\nabla_b R_{de} - \nabla_a R_{dbe}{}^a)$. The l.h.s. becomes: $\nabla_b\nabla_a R_{cde}{}^a + \nabla_c\nabla_a R_{dbe}{}^a + \nabla_d\nabla_a R_{bce}{}^a + [\nabla_b, \nabla_c] R_{de}$ It is a cyclic sum on (bcd) plus a commutator. The latter is moved to the r.h.s. and evaluated. A cancellation of two terms occurs and the r.h.s. ends as a cyclic sum too: $-R_{bce}{}^a R_{da} - R_{cde}{}^a R_{ba} - R_{dbe}{}^a R_{ca}$. □

Remark. In the Riemannian case, the left-hand side of the identity (2.3) is the exterior covariant derivative $D\Pi_c$ [LO] of the 2-form associated to the divergence of the curvature tensor, $\Pi_c = \nabla_d R_{abc}{}^d dx^a \wedge dx^b$. This identity and its various generalizations are referred to as "the Weitzenböck formula" (for curvature-like tensors) (see eq. (4.2) in [BOU]).

Note that Lovelock's identity implies that the closure condition $D\Pi_c = 0$ is equivalent to the algebraic relation

$$R_{am} R_{bce}{}^m + R_{bm} R_{cae}{}^m + R_{cm} R_{abe}{}^m = 0. \tag{2.4}$$

Lovelock's identity is left unchanged if the divergence of the Riemann tensor in the l.h.s. is replaced by the divergence of any curvature tensor K with the property

$$\nabla_m K_{bce}{}^m = A\,\nabla_m R_{bce}{}^m + B\,(a_{be}\nabla_c\varphi - a_{ce}\nabla_b\varphi), \tag{2.5}$$

where A and B are nonzero constants, φ is a real scalar function and a_{bc} is a symmetric (0,2) Codazzi tensor, i.e. $\nabla_b a_{cd} = \nabla_c a_{bd}$ [DZ2]. This conclusion also follows from formula (4.8) in Bourguignon's paper, cited above.
Some curvature tensors K with the property (2.5) and trivial Codazzi tensor (i.e. constant multiple of the metric) are well known: Weyl's conformal tensor C [PO], the projective curvature tensor P [EI], the concircular tensor $\tilde{C}$ [YA1, SH], the conharmonic tensor N [MI, SI1] and the quasi conformal curvature tensor W [YA2]. Their definitions and some identities used in this paper are collected in the appendix. Since in the next section we introduce the concept of *K-recurrency*, and Weyl's tensor will be considered in section 4, we give a proof of this statement:

Proposition 2.4.

$$\begin{aligned} &\nabla_a\nabla_m K_{bce}{}^m + \nabla_b\nabla_m K_{cae}{}^m + \nabla_c\nabla_m K_{abe}{}^m \\ &= -A[R_{am}R_{bce}{}^m + R_{bm}R_{cae}{}^m + R_{cm}R_{abe}{}^m] \end{aligned} \tag{2.6}$$

Proof. The covariant derivative ∇_a of (2.5) is evaluated and then summed with indices chosen as in Lovelock's identity. Since a symmetric connection is assumed, we obtain:

$$\begin{aligned} &\nabla_a\nabla_m K_{bce}{}^m + \nabla_b\nabla_m K_{cae}{}^m + \nabla_c\nabla_m K_{abe}{}^m \\ &= A[\nabla_a\nabla_m R_{bce}{}^m + \nabla_b\nabla_m R_{cae}{}^m + \nabla_c\nabla_m R_{abe}{}^m] \\ &+B[(\nabla_b a_{ce} - \nabla_c a_{be})\nabla_a\varphi + (\nabla_c a_{ae} - \nabla_a a_{ce})\nabla_b\varphi + (\nabla_a a_{be} - \nabla_b a_{ae})\nabla_c\varphi]. \end{aligned}$$

The last line is zero if a_{bc} is a Codazzi tensor. Lovelock's identity is then used to write the r.h.s. as in (2.6). □

An apparently new Veblen-type identity for the divergence of the Riemann tensor can be obtained by summing Lovelock's identity with indices cycled:

Corollary 2.5.

$$\begin{aligned} &\nabla_a\nabla_m R_{bec}{}^m - \nabla_b\nabla_m R_{ace}{}^m + \nabla_c\nabla_m R_{eba}{}^m - \nabla_e\nabla_m R_{cab}{}^m \\ &= -R_{am}R_{bec}{}^m + R_{bm}R_{ace}{}^m - R_{cm}R_{eba}{}^m + R_{em}R_{cab}{}^m \end{aligned} \tag{2.7}$$

Proof. Write Lovelock's identity (2.3) for all cyclic permutations of (a, b, c, e) and sum them. Simplify by using the first Bianchi identity. □

We note that an analogous Veblen-type identity can be obtained for a tensor K, starting from Proposition 2.4.

Corollary 2.6. *In a manifold with a Levi-Civita connection*

$$\nabla_m\nabla_n R_{ab}{}^{mn} = 0 \tag{2.8}$$

Proof. Eq.(2.3) is contracted with g^{ce}. The formula is reported in Lovelock's handbook [LO]. □

3 Symmetric and recurrent structures

From now on, we restrict to Riemannian manifolds $(\mathcal{M}^n, g)$. If additional differential structures are present, the differential identities (2.1), (2.3) and (2.7) simplify to interesting algebraic constraints.

A simple example is given by a *locally symmetric space* [KO], i.e. a Riemannian manifold such that $\nabla_a R_{bcd}{}^e = 0$. Then the aforementioned identities imply straightforwardly the algebraic ones

$$R_{(abc}{}^m R_{d)me}{}^f - R_{ace}{}^m R_{bdm}{}^f + R_{acm}{}^f R_{bde}{}^m = 0 \tag{3.1}$$

$$R_{am} R_{bce}{}^m + R_{bm} R_{cae}{}^m + R_{cm} R_{abe}{}^m = 0 \tag{3.2}$$

$$R_{am} R_{bec}{}^m - R_{bm} R_{ace}{}^m + R_{cm} R_{eba}{}^m - R_{em} R_{cab}{}^m = 0 \tag{3.3}$$

We show that these identities hold in several circumstances. An example is a manifold with harmonic curvature [BE], $\nabla_a R_{bcd}{}^a = 0$; in this less stringent case the general property (2.3) yields (3.2) and (3.3). A slightly more general case is now considered

Definition 3.1. A manifold is nearly conformally symmetric, $(\mathrm{NCS})_n$, (Roter [RO]) if

$$\nabla_a R_{bc} - \nabla_b R_{ac} = \frac{1}{2(n-1)}[g_{bc}\nabla_a R - g_{ac}\nabla_b R], \tag{3.4}$$

where $R = R_a{}^a$ is the curvature scalar.

Since $\nabla_a R_{bc} - \nabla_b R_{ac} = -\nabla_m R_{abc}{}^m$, $(\mathrm{NCS})_n$ are a special case of (2.5) with $\nabla_m K_{bce}{}^m = 0$ (trivial Codazzi tensor and $\varphi = R$). Other particular cases are $K = 0$ (K-flat) and $\nabla_a K_{bcd}{}^e = 0$ (K-symmetric). They yield, for the different choices of K, various types of $K-$flat/symmetric manifolds [SI1]: conformally flat/symmetric ($K = C$) [CH, DZ1], projectively flat/symmetric ($K = P$) [GL], concircular or conharmonic symmetric [AD], and quasi conformally flat/symmetric. Because of Prop. 2.4, the following is true:

Proposition 3.2. *For $(NCS)_n$ manifolds, and for $K-$flat/symmetric manifolds, eqs. (3.2) and (3.3) hold.*

By weakening the defining condition of a locally symmetric space, one introduces a *semisymmetric space*: $[\nabla_a, \nabla_b] R_{cde}{}^f = 0$ [SZ].

Proposition 3.3. *For a semisymmetric space, eqs.(3.1), (3.2) and (3.3) hold.*

Proof. First statement: eq.(2.2) simplifies to $0 = \nabla_{(a}\nabla_b R_{cd)e}{}^f$; by eq.(2.1) the identity (3.1) follows. Second statement: the definition implies a relation for the Ricci tensor: $[\nabla_a, \nabla_b] R_{ce} = 0$. By inserting the identity $\nabla_m R_{abc}{}^m = \nabla_b R_{ac} - \nabla_a R_{bc}$ in the l.h.s. of eqs. (2.3) and (2.7), those sides become sums of respectively three and four commutators of derivatives acting on Ricci tensors, and thus vanish. This implies eqs.(3.2) and (3.3). □

The algebraic property (3.2) holds in presence of even more general differential structures.

Definition 3.4. A manifold is pseudosymmetric (Deszcz [DS1]) if:

$$[\nabla_a, \nabla_b] R_{cdef} = L_R Q(g,R)_{cdefab} \tag{3.5}$$

where L_R is a scalar function and the Tachibana tensor is

$$\begin{aligned} Q(g,R)_{cdefab} = &-g_{cb} R_{adef} + g_{ca} R_{bdef} - g_{db} R_{caef} + g_{da} R_{cbef} \\ &-g_{eb} R_{cdaf} + g_{ea} R_{cdbf} - g_{fb} R_{cdea} + g_{fa} R_{cdeb}. \end{aligned} \tag{3.6}$$

Theorem 3.5. *For pseudosymmetric manifolds, the identities (3.2) and (3.3) hold.*

Proof. The l.h.s. of eq.(3.2) can be written as a sum of commutators acting on Ricci tensors: $[\nabla_a, \nabla_c] R_{be} + [\nabla_b, \nabla_a] R_{ce} + [\nabla_c, \nabla_b] R_{ae}$. A commutator is obtained by contracting two indices in (3.5); for example, contraction of c with f gives

$$[\nabla_a, \nabla_b] R_{de} = L_R(-g_{db} R_{ea} + g_{da} R_{eb} - g_{eb} R_{da} + g_{ea} R_{db}),$$

i.e. the Ricci-pseudosymmetry property[DS2]. Although each commutator is nonzero, their sum vanishes. Veblen's type identity is proven in a similar way. ☐

We now show that (3.1), (3.2) or (3.3) do hold in manifolds with recurrent structure.

Definition 3.6. A Riemannian manifold is a *generalized recurrent manifold* if there exist two vector fields λ_a and μ_a such that

$$\nabla_a R_{bcd}{}^e = \lambda_a R_{bcd}{}^e + \mu_a (\delta_b{}^e g_{cd} - \delta_c{}^e g_{bd}) \tag{3.7}$$

The manifolds were first introduced by Dubey [DU], and studied by several authors [DE1, MA, AR]. In particular, if $\mu_a = 0$ the manifold is a *recurrent space*. Again, we shall prove that the algebraic identities (3.1), (3.2) and (3.3) hold in this case. We need the following lemma, with a content slightly different than the statement by Singh and Khan [SI2].

Lemma 3.7. *In a generalized recurrent manifold with curvature scalar* $R \neq 0$

1. *if the curvature scalar R is a constant then λ is proportional to μ and either λ is closed (i.e. $\nabla_a\lambda_b - \nabla_b\lambda_a = 0$) or the manifold is a space of constant curvature, $R_{abcd} = \frac{R}{n(n-1)}(g_{bd}g_{ac} - g_{ad}g_{bc})$;*

2. *if the curvature scalar is not constant, then λ is closed.*

Proof. We need some relations that easily come from eq.(3.7): a) the contraction $a = e$ gives $\nabla_a R_{bcd}{}^a = \lambda_a R_{bcd}{}^a + \mu_b g_{cd} - \mu_c g_{bd}$. A further divergence ∇^d gives zero in the l.h.s, by eq.(2.8), and the r.h.s. in few steps is evaluated as

$$0 = \frac{1}{2}[(\nabla_d\lambda_a) - (\nabla_a\lambda_d)]R_{bc}{}^{da} - \mu_b\lambda_c + \mu_c\lambda_b + \nabla_c\mu_b - \nabla_b\mu_c; \tag{3.8}$$

b) the contraction of $c = e$ in (3.7) yields $\nabla_a R_{bd} = \lambda_a R_{bd} - (n-1)\mu_a g_{bd}$, and $\nabla_a R = \lambda_a R - n(n-1)\mu_a$; c) the commutator of covariant derivatives on the Riemann tensor of type (3.7) is

$$\begin{aligned}[\nabla_a, \nabla_b]R_{cde}{}^f &= (\nabla_a\lambda_b - \nabla_b\lambda_a)R_{cde}{}^f \\ &+ (\delta_c{}^f g_{de} - \delta_d{}^f g_{ce})(\nabla_a\mu_b - \nabla_b\mu_a - \lambda_a\mu_b + \lambda_b\mu_a)\end{aligned} \tag{3.9}$$

From b) we conclude that, if $\nabla_a R = 0$, λ and μ are collinear (R is a number). Then, eq. (3.8) simplifies to

$$\begin{aligned}0 &= \tfrac{1}{2}[(\nabla_d\lambda_a) - (\nabla_a\lambda_d)]R_{bc}{}^{da} + \tfrac{R}{n(n-1)}(\nabla_c\lambda_b - \nabla_b\lambda_c) \\ &= \tfrac{1}{2}[(\nabla_d\lambda_a) - (\nabla_a\lambda_d)][R_{bc}{}^{da} + \tfrac{R}{n(n-1)}\delta^{da}_{cb}] \equiv \tfrac{1}{2}A_{da}\tilde{C}_{bc}{}^{da}\end{aligned} \tag{3.10}$$

($\tilde{C}$ is the (2,2) concircular tensor and $\delta^{da}_{cb} = \delta^a{}_b\delta^d{}_c - \delta^a{}_c\delta^d{}_b$). Also eq.(3.9) simplifies,

$$[\nabla_a, \nabla_b]R_{cde}{}^f = A_{ab}\tilde{C}_{cde}{}^f. \tag{3.11}$$

Walker's identity (1.2) for the Riemann tensor (3.7) yields the algebraic relation

$$0 = A_{ab}\tilde{C}_{cdef} + A_{cd}\tilde{C}_{abef} + A_{ef}\tilde{C}_{abcd}. \tag{3.12}$$

Now Walker's lemma [WA] is invoked: it implies that either $A_{ab} = 0$ or $\tilde{C}_{abcd} = 0$. We give a proof based on (3.10): 1) Saturate in eq.(3.12) with A^{ef} and use (3.10): one gets $A^{ef}A_{ef}\tilde{C}_{abcd} = 0 \Rightarrow \tilde{C}_{abcd} = 0$; 2) in the same way, by saturation with $\tilde{C}^{cdef}$ one gets $\tilde{C}_{abcd}\tilde{C}^{abcd}A_{ef} = 0 \Rightarrow A_{ef} = 0$. Therefore either λ is closed or the manifold is a space of constant curvature.

We now discuss the case $\nabla_a R \neq 0$. Take the covariant derivative ∇_b of $\nabla_a R = \lambda_a R - n(n-1)\mu_a$, and interchange a and b. Then

$$0 = A_{ab}R + n(n-1)[\lambda_a\mu_b - \lambda_b\mu_a - \nabla_a\mu_b + \nabla_b\mu_a]$$

Enter this in (3.8),(3.9), and get again (3.10),(3.11) where now $\tilde{C} \neq 0$. The same procedure as above gives $A = 0$, i.e. λ is closed. □

Theorem 3.8. *In a generalized recurrent manifold the properties (3.1), (3.2) and (3.3) hold.*

Proof. If $\nabla R \neq 0$ then, by the previous Lemma 3.7, λ is always closed and, by eq.(3.11), the space is semisymmetric. Then eqs.(3.1),(3.2) and (3.3) hold by Prop.3.3.
If $\nabla R = 0$ then λ and μ are collinear (Lemma 3.7) and eq.(3.11) holds again. The Lemma states that either λ is closed or the space has constant curvature. In both cases the manifold is semisymmetric and (3.1),(3.2),(3.3) hold. □

The afore mentioned recurrent structures are special cases of a new one, which we now define. It arises naturally from the invariance stated in eq. (2.6) stemming from Lovelock's identity.

Definition 3.9. A Riemannian manifold with a curvature tensor K such that eq.(2.5) is true, is called a *K-recurrent manifold* (KRM) if $\nabla_a K_{bcd}{}^e = \lambda_a K_{bcd}{}^e$ where λ is a nonzero covector field.

Therefore, KR-manifolds include, as special cases, those which are conformally-recurrent, concircular-recurrent etc. (see [KH] for a compendium).

In general, the Bianchi identity for a tensor K contains a tensor source B: $\nabla_{(a}K_{bc)d}{}^e = B_{abcd}{}^e$ (see appendix for some relevant examples). In a KRM it is $\lambda_{(a}K_{bc)d}{}^e = B_{abcd}{}^e$. When λ is closed, one obtains a remarkable property:

Theorem 3.10. *In a KRM with closed λ*

$$R_{am}R_{bce}{}^m + R_{bm}R_{cae}{}^m + R_{cm}R_{abe}{}^m = -\frac{1}{A}\nabla_m B_{abce}{}^m \tag{3.13}$$

Proof. $\nabla_a\nabla_m K_{bcd}{}^m = (\nabla_a\lambda_m)K_{bcd}{}^m + \lambda_m\lambda_a K_{bcd}{}^m$. Cyclic permutation on (abc) and summation yield $\nabla_a\nabla_m K_{bcd}{}^m + \nabla_b\nabla_m K_{cad}{}^m + \nabla_c\nabla_m K_{abd}{}^m = (\nabla_a\lambda_m)K_{bcd}{}^m + (\nabla_b\lambda_m)K_{cad}{}^m + (\nabla_c\lambda_m)K_{abd}{}^m + \lambda_m\lambda_{(a}K_{bc)d}{}^m$. Evaluate ∇_m of Bianchi identity with $e = m$: $(\nabla_m\lambda_{(a)})K_{bc)d}{}^m + \lambda_m\lambda_{(a}K_{bc)d}{}^m = \nabla_m B_{abcd}{}^m$. Use closure property and Lovelock's identity to conclude. □

Corollary 3.11. *For the tensors $K = C, P, \tilde{C}, N, W$ listed in the appendix, Theorem 3.10 holds with null r.h.s.*

Proof. In the appendix one notes that $\nabla_m B_{abce}{}^m$ is either 0 or a multiple of the l.h.s. (different from A). □

Remark. It is well known that *concircular recurrency* is equivalent to *generalized recurrency* [AR, DU].

4 Weakly Ricci symmetric manifolds $(\mathrm{WRS})_n$

Definition 4.1. A $(\mathrm{WRS})_n$ is a Riemannian manifold with non-zero Ricci tensor such that

$$\nabla_a R_{bc} = A_a R_{bc} + B_b R_{ac} + D_c R_{ab} \tag{4.1}$$

with A, B and D are nonzero covector fields.

These manifolds were introduced by Tamássy and Binh [TA], and include the physically relevant Robertson-Walker space-times [DE3], or the perfect fluid space-time [DE5]. If $B = D = 0$ the manifold is *Ricci-recurrent*. Most of the literature concentrates on the difference $B - D$, and prove that in $(\mathrm{WRS})_n$ that are conformally flat [DE4, DE6] or quasi-conformally flat [JA], $B - D$ is a concircular vector. We here show that Lovelock's identity (2.3) allows to discuss new general properties of A, B, D.

Lemma 4.2. *For $\alpha = A - B$ or $A - D$:*

$$\begin{aligned} &R_{cb}(\nabla_d \alpha_a - \nabla_a \alpha_d) + R_{ca}(\nabla_b \alpha_d - \nabla_d \alpha_b) + R_{cd}(\nabla_a \alpha_b - \nabla_b \alpha_a) \\ &= -R_{dm} R_{bac}{}^m - R_{bm} R_{adc}{}^m - R_{am} R_{dbc}{}^m \end{aligned} \tag{4.2}$$

Proof. From the definition of $(\mathrm{WRS})_n$ and the contracted second Bianchi identity $\nabla_m R_{bac}{}^m = \nabla_a R_{bc} - \nabla_b R_{ac}$ one gets immediately $\nabla_m R_{bac}{}^m = \alpha_a R_{bc} - \alpha_b R_{ac}$, with $\alpha = A - B$. A further covariant derivative gives

$$\nabla_d \nabla_m R_{bac}{}^m = (\nabla_d \alpha_a) R_{bc} - (\nabla_d \alpha_b) R_{ac} + \alpha_a \nabla_d R_{bc} - \alpha_b \nabla_d R_{ac}$$

Summation is done on cyclic permutation of d, b, a:

$$\begin{aligned} &\nabla_d \nabla_m R_{bac}{}^m + \nabla_b \nabla_m R_{adc}{}^m + \nabla_a \nabla_m R_{dbc}{}^m = \\ &(\nabla_d \alpha_a - \nabla_a \alpha_d) R_{bc} + (\nabla_b \alpha_d - \nabla_d \alpha_b) R_{ac} + (\nabla_a \alpha_b - \nabla_b \alpha_a) R_{dc} \\ &+ \alpha_a (\nabla_d R_{bc} - \nabla_b R_{dc}) + \alpha_d (\nabla_b R_{ac} - \nabla_a R_{bc}) + \alpha_b (\nabla_a R_{dc} - \nabla_d R_{ac}) \end{aligned}$$

The terms with derivatives of Ricci tensors vanish because $\nabla_d R_{bc} - \nabla_b R_{dc} = \alpha_d R_{bc} - \alpha_b R_{dc}$. Then, by eq.(2.3), we obtain (4.2). The case $\alpha = A - D$ is proven in the same way starting from the identity $\nabla_m R_{abc}{}^m = \nabla_b R_{ac} - \nabla_a R_{bc}$. □

Theorem 4.3. *If rank* $[R^a{}_b] > 1$ *then* $B = D$.

Proof. Let us assume that $\beta = B - D$ is nonzero. Because the Ricci tensor is symmetric, the antisymmetric part of eq.(4.1) is: $0 = \beta_b R_{ac} - \beta_c R_{ab}$. Left multiplication by g^{ab} and summation on a give: $0 = R^b{}_c \beta_b - R\beta_c$, where R is the nonzero scalar curvature. On the other hand, multiplication by β^b gives: $0 = \beta^b \beta_b R_{ac} - R\beta_a \beta_c$, [DE2], i.e. the Ricci tensor has rank one. □

Remark. The validitity of Lemma 4.2 for both $A - B$ and $A - D$ implies, by subtraction, an equation for β:

$$R_{cb}(\nabla_d \beta_a - \nabla_a \beta_d) + R_{ca}(\nabla_b \beta_d - \nabla_d \beta_b) + R_{cd}(\nabla_a \beta_b - \nabla_b \beta_a) = 0, \tag{4.3}$$

and left multiplication by β^c gives the differential identity

$$\beta_b(\nabla_d \beta_a - \nabla_a \beta_d) + \beta_a(\nabla_b \beta_d - \nabla_d \beta_b) + \beta_d(\nabla_a \beta_b - \nabla_b \beta_a) = 0. \tag{4.4}$$

Theorem 4.4. *In a (WRS)$_n$ manifold with nonsingular Ricci tensor, the covector $A - B$ is closed iff*

$$R_{dm} R_{bac}{}^m + R_{bm} R_{adc}{}^m + R_{am} R_{dbc}{}^m = 0 \tag{4.5}$$

Proof. If $A - B$ (which equals $A - D$ because $\det R \neq 0$) is closed then (4.5) holds because of the Lemma. If the r.h.s. of Lemma vanishes,

$$R_{cb}(\nabla_d \alpha_a - \nabla_a \alpha_d) + R_{ca}(\nabla_b \alpha_d - \nabla_d \alpha_b) + R_{cd}(\nabla_a \alpha_b - \nabla_b \alpha_a) = 0$$

the index c is raised and multiplication is made by $(R^{-1})^s{}_c$:

$$\delta^s{}_b(\nabla_d \alpha_a - \nabla_a \alpha_d) + \delta^s{}_a(\nabla_b \alpha_d - \nabla_d \alpha_b) + \delta^s{}_d(\nabla_a \alpha_b - \nabla_b \alpha_a) = 0$$

Put $s = b$ and sum: $(n-2)(\nabla_d \alpha_a - \nabla_a \alpha_d) = 0$. Then, if $n > 2$, α is closed. □

(WRS)$_n$ manifolds of physical relevance that fulfill the condition (4.5) are the conformally flat WRS-manifolds, i.e. (WRS)$_n$ manifolds whose Weyl tensor (see appendix) vanishes [DE3, DE4].

Corollary 4.5. *If a (WRS)$_n$ manifold is conformally flat and the Ricci matrix is nonsingular, then $A - B$ is closed.*

Proof. The divergence of the Weyl tensor (appendix) takes the form (2.5), where the Codazzi tensor is g_{ab}. Because of the general proposition 2.4 we have

$$\begin{aligned}&\nabla_a \nabla_m C_{bdc}{}^m + \nabla_b \nabla_m C_{dac}{}^m + \nabla_d \nabla_m C_{abc}{}^m \\ &= -\frac{n-3}{n-2}(R_{am} R_{bdc}{}^m + R_{bm} R_{dac}{}^m + R_{dm} R_{abc}{}^m)\end{aligned}$$

If $n > 3$ and if the Weyl tensor itself or its covariant divergence vanish, Theorem 4.4 applies (for $n = 3$ Weyl's tensor is zero). □

5 A wave equation for the Riemann tensor

Proposition 5.1. *For a Levi-Civita connection with* $R_{ab} = 0$*, the contraction in* (2.1) *with* g^{ab} *yields Lichnerowicz's non linear wave equation (1.3).*

Proof. Since $\nabla_a g_{bc} = 0$, indices can be lowered or raised freely under covariant derivation. The Riemann tensor gains the symmetry $R_{abcd} = R_{cdab}$ and the further condition $R_{ab} = 0$ implies that $\nabla_k R_{abc}{}^k = 0$. Eq.(1.3) follows immediately. □

Appendix

We collect the useful formulae for the K-curvature tensors in a $n-$dimensional Riemannian manifold: a) definition, b) divergence, c) cyclic sum of derivatives (unlike the II Bianchi identity, we get a nonzero tensor B), d) divergence of B (the r.h.s. of c).

Projective tensor

a) $$P_{bcd}{}^e = R_{bcd}{}^e + \frac{1}{n-1}(\delta^e{}_b R_{cd} - \delta^e{}_c R_{bd})$$

b) $$\nabla_m P_{bcd}{}^m = \frac{n-2}{n-1} \nabla_m R_{bcd}{}^m$$

c) $$\nabla_a P_{bcd}{}^e + \nabla_b P_{cad}{}^e + \nabla_c P_{abd}{}^e = \frac{1}{n-1}(\delta^e{}_a \nabla_p R_{bcd}{}^p + \delta^e{}_b \nabla_p R_{cad}{}^p + \delta^e{}_c \nabla_p R_{abd}{}^p)$$

d) $$\nabla_m B_{abcd}{}^m = \frac{1}{n-1}(\nabla_a \nabla_p R_{bcd}{}^p + \nabla_b \nabla_p R_{cad}{}^p + \nabla_c \nabla_p R_{abd}{}^p)$$

Conformal (Weyl) tensor

$$a)\qquad C_{abc}{}^{d} = R_{abc}{}^{d} + \frac{\delta_a{}^d R_{bc} - \delta_b{}^d R_{ac} + R_a{}^d g_{bc} - R_b{}^d g_{ac}}{n-2} - R\frac{\delta_a{}^d g_{bc} - \delta_b{}^d g_{ac}}{(n-1)(n-2)}$$

$$b)\qquad \nabla_m C_{abc}{}^{m} = \frac{n-3}{n-2}\left[\nabla_m R_{abc}{}^{m} + \frac{1}{2(n-1)}(g_{bc}\nabla_a R - g_{ac}\nabla_b R)\right]$$

$$c)\qquad \nabla_a C_{bcd}{}^{e} + \nabla_b C_{cad}{}^{e} + \nabla_c C_{abd}{}^{e} = \frac{1}{n-2}[\delta^e{}_a\nabla_p R_{bcd}{}^{p} + \delta^e{}_b\nabla_p R_{cad}{}^{p} + \delta^e{}_c\nabla_p R_{abd}{}^{p}$$

$$+g_{cd}(\nabla_a R_b{}^e - \nabla_b R_a{}^e) + g_{ad}(\nabla_b R_c{}^e - \nabla_c R_b{}^e) + g_{bd}(\nabla_c R_a{}^e - \nabla_a R_c{}^e)] - \frac{1}{(n-1)(n-2)}$$

$$[\delta^e{}_a(g_{bd}\nabla_c R - g_{cd}\nabla_b R) + \delta^e{}_b(g_{cd}\nabla_a R - g_{ad}\nabla_c R) + \delta^e{}_c(g_{ad}\nabla_b R - g_{bd}\nabla_a R)]$$

$$d)\qquad \nabla_m B_{abcd}{}^{m} = \frac{1}{n-2}(\nabla_a\nabla_p R_{bcd}{}^{p} + \nabla_b\nabla_p R_{cad}{}^{p} + \nabla_c\nabla_p R_{abd}{}^{p})$$

Concircular tensor

$$a)\qquad \tilde{C}_{bcd}{}^{e} = R_{bcd}{}^{e} + \frac{R}{n(n-1)}(\delta^e{}_b g_{cd} - \delta^e{}_c g_{bd})$$

$$b)\qquad \nabla_m \tilde{C}_{bcd}{}^{m} = \nabla_m R_{bcd}{}^{m} + \frac{1}{n(n-1)}(\nabla_b R g_{cd} - \nabla_c R g_{bd})$$

$$c)\qquad \nabla_a \tilde{C}_{bcd}{}^{e} + \nabla_b \tilde{C}_{cad}{}^{e} + \nabla_c \tilde{C}_{abd}{}^{e} = \frac{1}{n(n-1)}[\delta^e{}_a(\nabla_c R g_{bd} - \nabla_b R g_{cd})$$

$$+\delta^e{}_b(\nabla_a R g_{cd} - \nabla_c R g_{ad}) + \delta^e{}_c(\nabla_b R g_{ad} - \nabla_a R g_{bd})]$$

$$d)\qquad \nabla_m B_{abcd}{}^{m} = 0$$

Conharmonic tensor

$$a)\qquad N_{bcd}{}^{e} = R_{bcd}{}^{e} + \frac{1}{n-2}[\delta_b{}^e R_{cd} - \delta_c{}^e R_{bd} + R_b{}^e g_{cd} - R_c{}^e g_{bd}]$$

$$b)\qquad \nabla_m N_{bcd}{}^{m} = \frac{n-3}{n-2}\nabla_m R_{bcd}{}^{m} + \frac{1}{2(n-2)}(\nabla_b R g_{cd} - \nabla_c R g_{bd})$$

$$c)\qquad \nabla_a N_{bcd}{}^{e} + \nabla_b N_{cad}{}^{e} + \nabla_c N_{abd}{}^{e} = \frac{1}{n-2}[\delta_a{}^e\nabla_p R_{bcd}{}^{p} + \delta_b{}^e\nabla_p R_{cad}{}^{p} + \delta_c{}^e\nabla_p R_{abd}{}^{p}$$

$$+g_{cd}(\nabla_a R_b{}^e - \nabla R^e) + g_{ad}(\nabla_b R_c{}^e - \nabla_c R_b{}^e) + g_{cd}(\nabla_c R_a{}^e - \nabla_a R_c{}^e)]$$

$$d)\qquad \nabla_m B_{abcd}{}^{m} = \frac{1}{n-2}(\nabla_a\nabla_p R_{bcd}{}^{p} + \nabla_b\nabla_p R_{cad}{}^{p} + \nabla_c\nabla_p R_{abd}{}^{p})$$

Quasi-conformal tensor

$$a)\qquad W_{bcd}{}^{e} = a\tilde{C}_{bcd}{}^{e} + b(n-2)[C_{bcd}{}^{e} - \tilde{C}_{bcd}{}^{e}]$$

$$b)\qquad \nabla_m W_{bcd}{}^{m} = (a+b)\nabla_m R_{bcd}{}^{m} + \frac{2a - b(n-1)(n-4)}{2n(n-1)}(\nabla_b R g_{cd} - \nabla_c R g_{bd})$$

$$c)\qquad \nabla_a W_{bcd}{}^{e} + \nabla_b W_{cad}{}^{e} + \nabla_c W_{abd}{}^{e} = -b(n-2)[\nabla_a C_{bcd}{}^{e} + \nabla_b C_{cad}{}^{e} + \nabla_c C_{abd}{}^{e}]$$

$$+[a + b(n-2)][\nabla_a \tilde{C}_{bcd}{}^{e} + \nabla_b \tilde{C}_{cad}{}^{e} + \nabla_c \tilde{C}_{abd}{}^{e}]$$

$$d)\qquad \nabla_m B_{abcd}{}^{m} = -b(\nabla_a\nabla_p R_{bcd}{}^{p} + \nabla_b\nabla_p R_{cad}{}^{p} + \nabla_c\nabla_p R_{abd}{}^{p})$$

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References

[AD] T. Adati and T. Miyazawa, *On conformally symmetric spaces*, Tensor (N.S.) 18 (1967), 335–342.

[AR] K. Arslan, U. C. De C. Murathan and A. Yildiz, *On generalized recurrent Riemannian manifolds*, Acta Math. Hungar. 123 (2009), 27–39.

[BE] A. L. Besse, *Einstein Manifolds*, Springer (1987).

[BOU] J. P. Bourguignon, *Les variétés de dimension 4 à signature non nulle dont la courbure est harmonique sont d'Einstein*, Invent. Math. 63 (1981), no. 2, 263-286.

[CH] M. C. Chaki and B. Gupta, *On conformally symmetric spaces*, Indian J. Math. 5 (1963), 113-122.

[DE1] U. C. De and N. Guha, *On generalized recurrent manifolds*, J. National Acad. of Math. India (1991), 85–92.

[DE2] U. C. De and B. K. De, *On conformally flat generalized pseudo Ricci symmetric manifolds*, Soochow Journal of Mathematics 23 (1997), 381–389.

[DE3] U. C. De and S. K. Ghosh, *On weakly symmetric spaces*, Publ. Math. Debrecen 60 (2002), 201–8.

[DE4] U. C. De, *On weakly symmetric structures on a Riemannian manifold*, Facta Universitatis, Series; Mechanics, Automatic Control and Robotics vol. 3 n.14, (2003), 805–819.

[DE5] U. C. De and G. C. Ghosh, *On weakly Ricci symmetric spacetime manifolds*, Radovi Matematicki 13 (2004), 93-101.

[DE6] U. C. De and G. C. Ghosh, *Some global properties of weakly Ricci symmetric manifolds*, Soochow Journal of Mathematics 31 n.1 (2005), 83-93.

[DS1] R. Deszcz and W. Grycak, *On some classes of warped product manifolds*, Bull. Inst. Math. Acad. Sinica 15 (1987), 311-322.

[DS2] R. Deszcz, *On Ricci pseudosymmetric warped products*, Demonstratio Mathematica 22 (1989), 1053-1055.

[DZ1] A. Derdzinski and W. Roter, *Some theorems on conformally symmetric manifolds*, Tensor (N.S.) 32 (1978), 11–23.

[DZ2] A. Derdzinski and C.L. Shen, *Codazzi tensor fields, curvature and Pontryagin forms*, Proc. London Math. Soc. 47, n.3 (1983), 15–26.

[DU] R. S. D. Dubey, *Generalized recurrent spaces*, Indian J. Pure Appl. Math. 10 n.12 (1979), 1508–13.

[EI] L. P. Eisenhart, *Non-Riemannian Geometry*, reprint Dover Ed. 2005.

[HU] L. P. Hughston and K. P. Tod, *An introduction to General Relativity*, London Math. Soc. Student Texts 5, Cambridge University Press, 1990.

[KH] Q. Khan, *On recurrent Riemannian manifolds*, Kyungpook Math. J. 44 (2004) 269-266.

[JA] S. K. Jana and A. A. Shaikh, *On quasi-conformally flat weakly Ricci symmetric manifolds*, Acta Math. Hungar. 115 (3) (2007), 197-214.

[GL] E. Glodek, *Some remarks on conformally symmetric Riemannian spaces*, Colloq. Math. 23 (1971), 121–123.

[KO] S. Kobayashi and K. Nomizu, *Foundations of differential geometry*, Vol.1, Interscience, 1963, New York.

[LO] D. Lovelock and H. Rund, *Tensors, differential forms and variational principles*, reprint Dover Ed. 1988.

[MA] Y. B. Maralabhavi and M. Rathnamma, *On generalized recurrent manifolds*, Pure Appl. Math. 30 (1999), 1167–71.

[MT] C. W. Misner, K. S. Thorne, J. A. Wheeler, *Gravitation*, W. H. Freeman and Company, San Francisco, 1973.

[MI] R. S. Mishra, *Structures on a Differentiable Manifold and their applications*, Chandrama Prakashan, Allahabad (1984).

[PO] M. M. Postnikov, *Geometry VI, Riemannian geometry*, Encyclopaedia of Mathematical Sciences, Vol. 91, 2001, Springer.

[RO] W. Roter, *On a generalization of conformally symmetric metrics*, Tensor (N.S.) 46 (1987), 278-286

[SH] J. A. Schouten, *Ricci-calculus*, Springer Verlag, 2nd Ed., 1954.

[SI1] H. Singh and Q. Khan, *On symmetric manifolds*, Novi Sad J. Math. 29 n.3 (1999), 301–308.

[SI2] H. Singh and Q. Khan, *On generalized recurrent Riemannian manifolds*, Publ. Math. Debrecen 56 (2000), 87-95.

[SZ] Z. Szabò, *Structure theorems on Riemannian spaces satisfying* $R(X,Y)\cdot R=0$*, I. The local version*, J. Diff. Geom. 17 (1982), 531–582.

[TA] L. Tamássy and T. Q. Binh, *On weakly symmetric and weakly projective symmetric Riemannian manifolds*, Coll. Math. Soc. J. Boljai 50 (1989), 663-670.

[WA] A. G. Walker, *On Ruse's spaces of recurrent curvature*, Proc. London Math. Soc. 52 (1951), 36-64.

[YA1] K. Yano, *Concircular geometry I, concircular transformations*, Proc. Imp. Acad. Tokyo 16 (1940), 195–200.

[YA2] K. Yano and S. Sawaki, *Riemannian manifolds admitting a conformal transformation group*, J. Diff. Geom. 2 (1968), 161–184.