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Gradient Estimates and Liouville Theorems for Diffusion-type Operators on complete Riemannian Manifolds

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Ai miei genitori

People think it's an obsession. A compulsion. As if there were an irresistible impulse to act. It's never been like that. I CHOSE this life. I KNOW what I'm doing. And on any given day, I could stop doing it. Today, however, isn't that day. And tomorrow won't be either.

Batman in *Identity Crisis* (2004), DC Comics

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Introduction

The classical Liouville theorem in complex analysis says that *an entire function* (i.e., a function $f : \mathbb{C} \to \mathbb{C}$ holomorphic on the whole complex plane) *which is also bounded is necessarily constant*; this result can be proved directly, applying Cauchy's integral formula (see [Hil59], [Kra04], or any text on basic complex analysis), or using the property that, in the aforementioned case, $\operatorname{Re} f$ and $\operatorname{Im} f$ are bounded harmonic function on $\mathbb{R}^2$. Indeed, the following holds:

Theorem 0.1. *A bounded, global C^2-solution of $\Delta u = 0$ on $\mathbb{R}^m$, $m \geq 1$, is constant.*

For a proof see, for example, [Eva98] or the elegant [Nel61]. Theorem 0.1 can be generalized in a number of directions: one can consider Riemannian manifolds instead of $\mathbb{R}^m$, or operators more general than the Laplacian. For an interesting overview on the subject, we refer to the survey [Far07].

The aim of this work is essentially twofold. Our first main concern is analytical. We study, using the method of *gradient estimates*, various Liouville-type theorems which are extensions of Theorem 0.1. We generalize the setting - from $\mathbb{R}^m$ to complete Riemannian manifolds - and the relevant operator - from Δ to a general diffusion operator - and we also consider "relaxed" boundedness conditions (such as non-negativity, controlled growth and so on).

The second main concern is geometrical, and is deeply related to the first: we prove some triviality results for Einstein warped products and quasi-Einstein manifolds studying a specific Poisson equation for a particular, and geometrically relevant, diffusion operator (see below for details).

Let us first depict the analytical framework; the geometrical one will be analyzed later, when we describe the content of Chapter 4.

Let $(M, \langle\ ,\ \rangle)$ be a complete, non-compact, connected Riemannian manifold of dimension $m \geq 2$. We want to determine Liouville theorems and *a priori* estimates for the gradient of global solutions (i.e., solutions defined on the whole M) of equations of the type

$$\boxed{\Delta u + \langle \nabla a, \nabla u \rangle = bf(u)} \tag{0.1}$$

for some sufficiently regular functions a and b on M, with b positive, $f \in C^1(\mathbb{R})$ and where Δ and ∇ are, respectively, the Laplace-Beltrami operator and the gradient in the metric $\langle\ ,\ \rangle$ of M. Of course, our results depend on a, b and f as well as on the geometry of $(M, \langle\ ,\ \rangle)$. The key point in the analysis of (0.1) is the following observation: let us set $A = e^a$, $B = be^a$ and consider the diffusion-type operator

$$L = \frac{1}{B} \operatorname{div}(A\nabla\ ); \tag{0.2}$$

a simple computation shows that (0.1) rewrites as the Poisson equation

$$\boxed{Lu = f(u),} \tag{0.3}$$

so every result concerning (0.1) can be interpreted from this point of view. In order to relate L with the geometry of M we consider the *modified Bakry-Emery Ricci tensor* that we now introduce. First, we fix an origin $o \in M$ and we set $r(x) := \operatorname{dist}_{(M,\langle\ ,\rangle)}(x, o)$; we shall denote by $B_R(o)$ the geodesic ball centered in o with radius $R > 0$. Following Z. Qian, [Qia97], for $n \in \mathbb{R}$, $n > m$ and $L_A = \frac{B}{A} L = \frac{1}{A} \operatorname{div}(A\nabla\ )$ we set

$$\begin{aligned} \operatorname{Ricc}_{n,m}(L_A) &= \operatorname{Ricc}_M - \frac{1}{A}\operatorname{Hess}(A) + \frac{n-m-1}{n-m}\frac{1}{A^2}\mathrm{d}A \otimes \mathrm{d}A \\ &= \operatorname{Ricc}(L_A) - \frac{1}{n-m}\frac{1}{A^2}\mathrm{d}A \otimes \mathrm{d}A, \end{aligned} \tag{0.4}$$

where $\operatorname{Hess}(A)$ is the Hessian of A and $\operatorname{Ricc}(L_A)$ is the usual Bakry-Emery

Ricci tensor. As shown in great generality in [MRS10], Proposition 2.3, estimates from below on $\operatorname{Ricc}_{n,m}(L_A)$ yield estimates from above on $L_A r$; indeed, in the simplest case

$$\operatorname{Ricc}_{n,m}(L_A) \geq -(n-1)Z^2 \tag{0.5}$$

for some constant $Z \geq 0$ on the geodesic ball $B_T(o)$, one has

$$L_A r \leq (n-1)Z\coth(Zr), \tag{0.6}$$

pointwise on $B_T(o) \setminus \{\operatorname{cut}(o) \cup \{o\}\}$ ($\operatorname{cut}(o)$ denotes the cut-locus of o) and weakly on all of $B_T(o)$. Thus, for instance, an assumption like

$$\operatorname{Ricc} + (n-1)H^2 \langle\ ,\ \rangle \geq \ \operatorname{Hess}(a) + \frac{1}{n-m}\, \mathrm{d}a \otimes \mathrm{d}a, \qquad H \geq 0,$$

which appears in some of our results, can be geometrically interpreted in the light of the generalized Bakry-Emery Ricci tensor defined in (0.4), and as such it can be considered a curvature condition. Our Liouville theorems rely essentially on careful gradient estimates for solutions of equation (0.3). We focus only on "classical" solutions (usually C^3) because the starting point of our computations is a pointwise application of a generalized version of the Bochner-Weitzenböck formula (see below for details), and also to avoid some technicalities due to weak formulation. Nevertheless, we point out that some of our result can be generalized lowering some regularity assumptions.

Gradient estimates in the solution of geometrical problems have a long history (see, for instance, the pioneering work of Yau, [Yau75]); for a symmetric diffusion operator (i.e., when $A = B$) they seem to have first appeared in the work of A. G. Setti ([Set92], [Set98]); see also the recent [Li05], where X.-D. Li derives various Liouville theorems for $\mathcal{L}$-harmonic functions, with $\mathcal{L} = \Delta - \langle \nabla f, \nabla\ \rangle$ for $f \in C^\infty(M)$ (note that $\mathcal{L} = L_A$ with $A = e^{-f}$). Some of the results we obtain in the thesis extend previous work of A. Ratto and M. Rigoli [RR95], where the authors consider the classical Poisson equation $\Delta u = f(u)$ on complete manifolds, the case of $\mathbb{R}^m$ having been previously treated, for instance, by Serrin [Ser72] and Modica [Mod85].

The work is organized as follows:

1. Gradient estimates;

2. Liouville-type theorems;

3. More Liouville theorems (and beyond);

4. Geometric applications;

5. Appendix.

In the first Chapter, the most technical, we derive some gradient estimates for solutions u of equation (0.3). Towards this aim we use a method inspired by the old work of Ahlfors [Ahl38], studying the inequality $LG \leq 0$ which holds at any relative maximum of G, where G is a suitable function of u, $|\nabla u|^2$ and ρ, the distance function from a fixed point; the key tool to obtain the fundamental inequalities (1.11) and (1.12) in Lemma 1.2 (which has to be compared to Lemma 12 in [RR95]) is a generalized version of the Bochner - Weitzenböck formula (equation (1.4)), which expresses $\frac{1}{2}L|\nabla u|^2$ (with $u \in C^3(M)$) in terms of geometrically relevant quantities (in particular, $\mathrm{Ricc}_{n,m}(L_A)(\nabla u, \nabla u)$). As a consequence we obtain (see Theorem 1.4)

Theorem 0.2. *Let $(M, \langle\ ,\ \rangle)$ be a complete manifold of dimension m. Fix $o \in M$ and let $r(x) = \mathrm{dist}_{(M,\langle\ ,\ \rangle)}(x, o)$. Suppose that $A, B \in C^1(M)$, $A, B > 0$, and that*

$$\mathrm{Ricc}_{n,m}(L_A) \geq -(n-1)H^2\big(1 + r(x)^2\big)^{\delta/2}$$

for some $n > m$, $H \geq 0$ and $\delta \in \mathbb{R}$. Let $f \in C^1(\mathbb{R})$ and $u \in C^3(M)$ a global solution of (0.3) *satisfying*

$$|u(x)| \leq D(1 + r(x))^{\nu},$$

$$\left| f(u(x))\nabla\left(\frac{B}{A}\right)(x)\right| \leq \Theta(1 + r(x))^{\theta}$$

and

$$\frac{B}{A}(x)f'(u(x)) \geq -K[1 + r(x)]^{\gamma}$$

on M for some constants $\nu \geq 0$, $\gamma, \theta \in \mathbb{R}$ and constants $D, K, \Theta > 0$. Then

$$|\nabla u|^2(x) \leq \widetilde{C}_1 r(x)^{2\nu} \max\Big\{ r(x)^{-2} + r(x)^{\frac{\delta}{2}-1} + r(x)^{\delta} + r(x)^{\gamma},$$
$$r(x)^{\theta-\frac{\delta}{2}} \frac{1+\widetilde{C} r(x)^{\frac{\gamma-\delta}{2}}}{1+\widetilde{C} r(x)^{\gamma-\delta}} \Big\}$$

for $r(x) \gg 1$ and constants $\widetilde{C}, \widetilde{C}_1 \geq 0$.

The other main result we prove (see Theorem 1.7) is an extension of Theorem B in [RR95]:

Theorem 0.3. *Let $(M, \langle\ ,\rangle)$ be a complete manifold of dimension m. Suppose that $A, B \in C^2(M)$, $A, B > 0$ and $h \in C^2(M)$, $h \geq 0$ satisfy*

$$\begin{cases} (i) \frac{B}{A} < C; \\ (ii) h < C; \\ (iii) |\nabla h| < C; \\ (iv) |Lh| < C \end{cases}$$

on M for some $C > 0$. Furthermore, suppose that, for some $n > m$, $H \geq 0$

$$\mathrm{Ricc}_{n,m}(L_A) \geq -(n-1)H^2$$

on M. For $f \in C^1(\mathbb{R})$ let $u \in C^3(M)$ be a global solution of (0.3) such that

$$\begin{cases} (i)\ |\nabla u| < C \qquad \text{on } M; \\ (ii)\ \inf_M |\nabla u| = 0. \end{cases}$$

Assume the existence of a function $Q \in C^2(\mathbb{R})$ with the following properties:

$$\begin{cases} (i)\, Q(u), |Q'(u)| < C; \\ (ii)\, \inf_M Q(u) = 0; \\ (iii)\, \big[Q'(u)h - 2\frac{B}{A} f(u)\big] Q'(u) \geq 0; \\ (iv)\, 2\frac{B}{A} f'(u) - 2(n-1)H^2 - hQ''(u) \geq 0; \\ (v)\, \big|Q'(u)\nabla h - f(u)\nabla\big(\frac{B}{A}\big)\big| < C \end{cases}$$

on M. Then

$$|\nabla u|^2 \leq h(x)Q(u) \qquad \text{on } M.$$

Of course, the applicability of Theorem 0.3 depends on the possible choices for h and Q, as we point out in the proofs of some related results.

In the second Chapter we use the previous estimates to derive our Liouville theorems for Poisson equations associated to L and L_A (and, consequently, for equation (0.1)) under geometric conditions on the manifold $(M, \langle\ ,\ \rangle)$ and appropriate growth conditions on both the solution and the non-linearity f. For instance we prove the following two results (see Theorem 2.1 and Corollary 2.4):

Theorem 0.4. *Let $(M, \langle\ ,\ \rangle)$ be a complete manifold of dimension m. Suppose that $A, B \in C^1(M)$, $A, B > 0$ and that, for some $n > m$, $H \geq 0$, $\delta \in \mathbb{R}$*

$$\mathrm{Ricc}_{n,m}(L_A) \geq -(n-1)H^2\big(1 + r(x)^2\big)^{\delta/2}$$

on M. Let $f \in C^1(\mathbb{R})$ and $u \in C^3(M)$ be a global solution of (0.3) *Assume*

$$|u(x)| \leq D(1 + r(x))^{\nu},$$

$$\frac{B}{A}(x) f'(u(x)) \geq (n-1)H^2\big(1 + r(x)^2\big)^{\delta/2},$$

$$\left| f(u(x))\nabla\left(\frac{B}{A}\right)(x)\right| \leq \Theta(1 + r(x))^{\theta}$$

on M, for some constants $\Theta, D > 0, \nu, \theta \in \mathbb{R}$. Then u is constant provided

$$0 \leq \nu < \min \left\{ 1, 1 - \frac{\delta}{4}, -\frac{\theta}{2} \right\}.$$

Corollary 0.5. *Let $(M, \langle\ ,\rangle)$ be a complete manifold of dimension m, $a, b \in C^2(M)$ and suppose that, for some $n > m$, we have the validity of*

$$\mathrm{Ricc}_M \geq \mathrm{Hess}(a) + \frac{1}{n-m}\, \mathrm{d}a \otimes \mathrm{d}a.$$

Assume

$$0 < b < C, \qquad |\nabla b| < C$$

on M for some constant $C > 0$. Let $f \in C^1(\mathbb{R})$ with $f' \geq 0$, $f \geq 0$ on $[0, +\infty)$ and let u be a non-negative bounded global solution of (0.1)*. Then u is constant.*

In the third Chapter we consider the notion of *stability* for solutions of equation (0.3). First we compute the *first* and *second variation* of the *generalized energy functional* associated to equation (0.3) and we define the notion of *L-stability* of global solutions, which generalize the concept of stability for global solutions of $\Delta u = f(u)$ (see, for example, [FCS80], [MP78], [DF09], [FSV08]); we then relate the L-stability to the non-negativity of the first eigenvalue of an appropriate linear operator, and we exploit this relation to derive a version of a theorem of Fisher-Colbrie and Schoen ([FCS80]). Successively, we prove the analogue of Theorem 4.5 in [PRS08] for global stable solutions under a particular condition on f and f': namely we have (see Theorem 3.5 and also Corollary 3.6)

Theorem 0.6. *Let $(M, \langle\ ,\rangle)$ be a complete manifold, $A, B \in C^2(M)$, $f \in C^1(\mathbb{R})$. Let $u \in C^3(M), u \geq 0$ be a global solution of* (0.3)*. Suppose that*

$$Hf(t) - f'(t)t \geq 0$$

for $t \geq 0$ and some $H \geq 1$. If $\varphi \in C^2(M)$ is a positive solution of

$$-\mathcal{L}_u\varphi = L_A\varphi - \frac{B}{A}f'(u)\varphi \leq 0 \qquad \text{on } M,$$

then there exists a constant $C \geq 0$ such that

$$C\varphi = u^H,$$

provided

$$\left(\int_{\partial B_r} |u|^{2\widetilde{\beta}}\, d\mu\right)^{-1} \notin L^1(+\infty)$$

for some $1 \leq \widetilde{\beta} \leq H$.

Finally we deduce a Liouville theorem for L_A-harmonic functions under an L^p condition on their gradient (Theorem 3.8) and a uniqueness result for equation (0.1) (Theorem 3.10), based on a particular form of the weak maximum principle valid for (symmetric) diffusion operators.

In the last Chapter the geometry becomes the main character. Our purpose is to prove triviality results for complete Einstein warped products $N^{m+k} = M^m \times_u F^k$, exploiting the relations between these latter and the *quasi-Einstein manifolds*, a generalization of the Ricci solitons. After a detailed introduction, where we present the relevant geometrical objects (in particular, the f-Laplacian Δ_f and the k-Bakry-Emery Ricci tensor Ricc_f^k), we enlight their connection with our previous investigations and discuss the recent literature on the subject (see references below), in the second Section we adapt to this new scenario Theorem 2.1 and Corollary 2.5 from Chapter 2. For instance we prove (see Theorem 4.12)

Theorem 0.7. *Let $N = M^m \times_u F^k$ be a complete Einstein warped product with Einstein constant $\lambda < 0$, warping function $u = e^{-f/k}$ and Einstein fibre F^k with Einstein constant $\mu < 0$. Suppose that*

$$f \geq \frac{k}{2}\log\left(\frac{\lambda}{2\mu}\right) \qquad \text{for all } x \in M$$

and that

$$|f| \leq D(1 + r(x))^{\nu}$$

for some $D \geq 0$, $\nu \in \mathbb{R}$. Then N is a Riemannian product, provided

$$0 \leq \nu < 1.$$

In the third Section we prove a weighted version of Theorem 1.31 in [PRS05a] and a sufficient condition for the validity of the full Omori-Yau maximum principle for the f-Laplacian; then we deduce a triviality result (Corollary 4.16) for complete Einstein warped products which is a Corollary of Theorem 1 in [Rim10]:

Corollary 0.8. *Let $N^{m+k} = M^m \times_u F^k$ be a complete Einstein warped product with non-positive scalar curvature $(m+k)\lambda = {}^N S \leq 0$, warping function $u(x) = e^{-\frac{f(x)}{k}}$ satisfying $\inf_M f = f_* > -\infty$ and complete Einstein fibre F. Suppose also that ${}^F S < 0$. Then N is simply a Riemannian product if either one of the following further conditions is satisfied:*

(i) the base manifold M is complete and non-compact, the warping function satisfies $f \in L^p(M)$, for some $1 < p < +\infty$, and $f(x_0) \leq 0$ for some point $x_0 \in M$;

(ii) the base manifold M is complete and non-compact, the warping function satisfies $f \in L^p(M)$, for some $1 < p < +\infty$, and the scalar curvatures of M and N satisfy

$$^M S \geq \frac{m}{m+k} {}^N S + \varepsilon,$$

for some $\varepsilon > 0$.

In the last Section we prove, again applying the Ahlfors technique, a further gradient estimate (Theorem 4.17), which extends the one in [Cas10]:

Theorem 0.9. *Let $\left(M^m, g, e^{-f} d\mu_0\right)$ be a weighted manifold (not necessarily complete); suppose that, for some $k < +\infty, Z \geq 0$*

$$\mathrm{Ricc}_f^k \geq \lambda = -(m+k-1)Z^2,$$

and that

$$\Delta_f f = \psi(f),$$

where $\psi : \mathbb{R} \to \mathbb{R}$ *satisfies*

$$\psi'(t) + \frac{2}{m}\psi(t) - (m+k-1)Z^2 \geq 0$$

for all $t \in \mathbb{R}$. *Then for all* $q \in M$ *and* $T > 0$ *such that* $B_T(q)$ *is geodesically connected in* M *and the closure* $\overline{B_T(q)}$ *is compact,*

$$|\nabla f|^2(q) \leq \frac{1}{G(m \parallel k)}\left[\frac{2(m+k+6)}{T^2} - \frac{4\sqrt{3}}{9}\frac{\lambda}{Z}\frac{1}{T}\right],$$

having defined

$$G(m \parallel k) := \frac{1}{m} + \frac{1}{k}.$$

Finally, we obtain another triviality result (Theorem 4.18) when the function f (related to the warping function u by $u = e^{-f/k}$) is bounded below by a constant depending on $m = \dim M, k$ and on the Einstein constants λ and μ, respectively of the warped product and of the fibre:

Theorem 0.10. *Let* $N = M^m \times_u F^k$ *be a complete Einstein warped product with Einstein constant* $\lambda < 0$, *warping function* $u = e^{-f/k}$ *and Einstein fibre* F^k *with Einstein constant* $\mu < 0$. *Suppose that*

$$f \geq \frac{k}{2}\log\left(\frac{\lambda}{2\mu}\frac{m+2k}{m+k}\right) \qquad \text{for all } x \in M.$$

Then N *is a Riemannian product.*

In the Appendix, for the convenience of the reader, we prove some results and some relations not so easily available in the literature. In the first Section we deduce the generalized Bochner-Weitzenböck formula for the operator L_A; in the second Section we derive a consequence of the L_A-comparison theorem and a particular Newton inequality. The third (and last) Section is devoted to a volume estimate due to Calabi and Yau.

The material presented in the first two Chapters and the uniqueness result of the third Chapter have already appeared, in a slightly different form, in [MR10a]; most of Chapter 4 and part of Chapter 3 appear in a paper with M. Rimoldi (submitted for publication, see [MR10b]).

Chapter 1

Gradient Estimates

> One thing I've learned. You can know anything. It's all there. You just have to find it.
>
> J. Constantine in *Sandman*, Season of Mists, DC Comics-Vertigo

The aim of this Chapter is to establish some gradient estimates for solution of the diffusion Poisson equation

$$\boxed{Lu = f(u)} \tag{1.1}$$

on M. Here L is the operator $L = \frac{1}{B}\operatorname{div}(A\nabla\ )$ defined for some sufficiently regular positive functions A and B on M. Towards this end we use (here, and in Chapter 4 below) a method inspired by the old work of Ahlfors, [Ahl38]: we basically obtain estimates by studying the inequality $LG \leq 0$ which holds at any relative maximum of G, where G is a suitable function of u, $|\nabla u|^2$ and ρ, the distance function from a fixed base point; see also [Yau75] and [SY94] for other applications of this technique. From now on, $(M, \langle\ ,\ \rangle)$ will be a complete, non-compact, connected Riemannian manifold of dimension $\dim M = m \geq 2$.

1.1 The main technical Lemma

In what follows we shall repeatedly use the following elementary facts: for $u, v \in C^2(M)$, $f \in C^2(\mathbb{R})$,

$$L(uv) = uLv + 2\frac{A}{B}\langle \nabla u, \nabla v\rangle + vLu, \tag{1.2}$$

$$Lf(u) = f'(u)Lu + \frac{A}{B}f''(u)|\nabla u|^2, \tag{1.3}$$

and the generalized Bochner-Weitzenböck formula contained in the next

Lemma 1.1. *Let $u \in C^3(M)$. Then*

$$\begin{aligned}\frac{1}{2}L|\nabla u|^2 = \frac{A}{B}|\mathrm{Hess}(u)|^2 + \frac{A}{B}\,\mathrm{Ricc}_{n,m}(L_A)(\nabla u, \nabla u) + \langle \nabla Lu, \nabla u\rangle + \\ + \frac{A}{B}Lu\left\langle \nabla\left(\frac{B}{A}\right), \nabla u\right\rangle + \frac{A}{B}\frac{1}{n-m}\frac{\langle \nabla A, \nabla u\rangle^2}{A^2}.\end{aligned} \tag{1.4}$$

Proof. Since $L = \frac{A}{B}L_A$, (1.4) follows immediately from the Bochner-Weitzenböck formula for the operator L_A

$$\frac{1}{2}L_A|\nabla u|^2 = |\mathrm{Hess}(u)|^2 + \mathrm{Ricc}(L_A)(\nabla u, \nabla u) + \langle \nabla L_A u, \nabla u\rangle \tag{1.5}$$

and the definition (0.4) of the modified Bakry-Emery Ricci tensor. To prove (1.5) one starts with the classical Bochner formula

$$\frac{1}{2}\Delta|\nabla u|^2 = |\mathrm{Hess}(u)|^2 + \mathrm{Ricc}_M(\nabla u, \nabla u) + \langle \nabla \Delta u, \nabla u\rangle \tag{1.6}$$

and uses the two computational identities

$$\left\langle \nabla|\nabla u|^2, \nabla A\right\rangle = 2\,\mathrm{Hess}(u)(\nabla A, \nabla u), \tag{1.7}$$

$$\langle \nabla\langle \nabla A, \nabla u\rangle, \nabla u\rangle = \mathrm{Hess}(u)(\nabla A, \nabla u) + \mathrm{Hess}(A)(\nabla u, \nabla u). \tag{1.8}$$

We refer to the Appendix for a proof of these latter facts using the moving frame formalism. □

We now come to the main technical point of this section.

Lemma 1.2. *Let $B_T(q)$ denote the geodesic ball of radius $T>0$ centered at q and $\rho(x)=\mathrm{dist}_{(M,\langle\,,\rangle)}(x,q)$. Assume that, for some $n>m$, $Z\geq 0$,*

$$\mathrm{Ricc}_{n,m}(L_A)\geq -(n-1)Z^2 \tag{1.9}$$

on $B_T(q)$. On the same ball consider the non-negative function

$$G(x)=\left[T^2-\rho^2(x)\right]^2|\nabla u|^2(x)g(u(x)) \tag{1.10}$$

where u is a $C^3(M)$ solution of (1.1) on M for some $f\in C^1(\mathbb{R})$ and $g\in C^2(\mathbb{R})$ with $g(u)>0$ on $B_T(q)$. If $\bar{x}\in B_T(q)$ is a positive maximum of G on $B_T(q)$, then at $\bar{x}$ we have

$$\begin{aligned}0\geq&\left\{\frac{2g(u)g''(u)-3g'(u)^2}{2g(u)^2}\right\}|\nabla u|^2-\left\{\frac{4T|g'(u)|}{(T^2-\rho^2)g(u)}\right\}|\nabla u|- \\ &-\left\{\frac{4[n+(n-1)ZT]}{T^2-\rho^2}+\frac{16T^2}{(T^2-\rho^2)^2}-2\frac{B}{A}f'(u)+2(n-1)Z^2-\frac{B}{A}\frac{g'(u)}{g(u)}f(u)\right\} \\ &-2\frac{|f(u)|}{|\nabla u|}\left|\nabla\left(\frac{B}{A}\right)\right|,\end{aligned} \tag{1.11}$$

$$\begin{aligned}0\geq&\left\{\frac{8g(u)g''(u)-(16+n)g'(u)^2}{8g(u)^2}\right\}|\nabla u|^2-\left\{\frac{8|g'(u)|}{g(u)}\frac{\rho}{T^2-\rho^2}\right\}|\nabla u|- \\ &-\left\{\frac{4[n+(n-1)Z\rho]}{T^2-\rho^2}+\frac{24\rho^2}{(T^2-\rho^2)^2}-2\frac{B}{A}f'(u)+2(n-1)Z^2\right\}- \\ &-2\frac{|f(u)|}{|\nabla u|}\left|\nabla\left(\frac{B}{A}\right)\right|.\end{aligned} \tag{1.12}$$

Proof. Since $\bar{x}$ is a positive maximum, at $\bar{x}$ we must have

$$i)\ \nabla\log G=\frac{\nabla G}{G}=0;\qquad ii)\ L\log G=\frac{LG}{G}-\frac{A}{B}\frac{|\nabla G|^2}{G^2}\leq 0. \tag{1.13}$$

From the definition of G, a computation shows that (1.13) i) is equivalent to

$$\frac{g'(u)}{g(u)}\nabla u + \frac{\nabla|\nabla u|^2}{|\nabla u|^2} = \frac{2\nabla\rho^2}{T^2-\rho^2} \qquad \text{at } \bar{x}. \tag{1.14}$$

Using formulas (1.2) and (1.3) a tedious calculation yields the equivalence of (1.13) ii) with

$$\begin{aligned} 0 \geq &-2\frac{L\rho^2}{T^2-\rho^2} - 2\frac{A}{B}\frac{\left|\nabla\rho^2\right|^2}{\left(T^2-\rho^2\right)^2} + \frac{g'(u)}{g(u)}Lu + \frac{A}{B}\frac{g(u)g''(u)-g'(u)^2}{g(u)^2}|\nabla u|^2 \\ &+ \frac{L|\nabla u|^2}{|\nabla u|^2} - \frac{A}{B}\frac{\left|\nabla|\nabla u|^2\right|^2}{|\nabla u|^4} \end{aligned} \tag{1.15}$$

at $\bar{x}$. From the generalized Bochner-Weitzenböck formula (1.4) of Lemma 1.1 we deduce the inequality

$$\begin{aligned} L|\nabla u|^2 \geq\ & 2\frac{A}{B}|\mathrm{Hess}(u)|^2 + 2\frac{A}{B}\,\mathrm{Ricc}_{n,m}(L_A)(\nabla u,\nabla u) + 2\left\langle \nabla Lu, \nabla u\right\rangle + \\ &+ 2\frac{A}{B}Lu\left\langle \nabla\left(\frac{B}{A}\right), \nabla u\right\rangle. \end{aligned} \tag{1.16}$$

Thus, combining with the elementary inequality (see the Appendix)

$$\left|\nabla|\nabla u|^2\right|^2 \leq 4|\nabla u|^2|\mathrm{Hess}(u)|^2 \tag{1.17}$$

and (1.1), (1.9) we obtain

$$\frac{L|\nabla u|^2}{|\nabla u|^2} \geq \frac{1}{2}\frac{A}{B}\frac{\left|\nabla|\nabla u|^2\right|^2}{|\nabla u|^4} - 2\frac{A}{B}(n-1)Z^2 + 2f'(u) + 2\frac{A}{B}\frac{f(u)}{|\nabla u|^2}\left\langle \nabla\left(\frac{B}{A}\right), \nabla u\right\rangle \tag{1.18}$$

at $\bar{x}$. As observed in the Introduction, assumption (1.9) implies

$$L\rho^2 \leq 2\frac{A}{B}[n+(n-1)Z\rho] \tag{1.19}$$

on $B_T(q)$ (see also the Appendix). We now use (1.18) and (1.19) into (1.15) together with (1.1) to obtain

$$0 \geq \frac{A}{B}\frac{g(u)g''(u)-g'(u)^2}{g(u)^2}|\nabla u|^2 - \frac{A}{B}\frac{4[n+(n-1)Z\rho]}{T^2-\rho^2} - \frac{A}{B}\frac{8T^2}{(T^2-\rho^2)^2} + \tag{1.20}$$
$$+ \frac{g'(u)}{g(u)}f(u) - \frac{1}{2}\frac{A}{B}\frac{\left|\nabla|\nabla u|^2\right|^2}{|\nabla u|^4} - 2\frac{A}{B}(n-1)Z^2 + 2f'(u) +$$
$$+ 2\frac{A}{B}\frac{f(u)}{|\nabla u|^2}\left\langle \nabla\left(\frac{B}{A}\right), \nabla u\right\rangle$$

at $\bar{x}$. Finally we observe that (1.14) implies

$$\frac{1}{2}\frac{\left|\nabla|\nabla u|^2\right|^2}{|\nabla u|^4} \leq \frac{1}{2}\left(\frac{g'(u)}{g(u)}\right)^2|\nabla u|^2 + \frac{8T^2}{(T^2-\rho^2)^2} + 4\frac{T|g'(u)|}{(T^2-\rho^2)g(u)}|\nabla u| \tag{1.21}$$

at $\bar{x}$. Inserting (1.21) into (1.20) we have

$$0 \geq \frac{2g(u)g''(u)-3g'(u)^2}{2g(u)^2}|\nabla u|^2 - 4\frac{T|g'(u)|}{(T^2-\rho^2)g(u)}|\nabla u| - \frac{4[n+(n-1)ZT]}{T^2-\rho^2} + \tag{1.22}$$
$$- \frac{16T^2}{(T^2-\rho^2)^2} - 2(n-1)Z^2 + \frac{B}{A}\frac{g'(u)}{g(u)}f(u) + 2\frac{B}{A}f'(u) +$$
$$+ 2\frac{f(u)}{|\nabla u|^2}\left\langle \nabla\left(\frac{B}{A}\right), \nabla u\right\rangle$$

at $\bar{x}$, from which (1.11) follows immediately.

To derive (1.12) we use again the Bochner-Weitzenböck formula (1.4) and (1.9), (1.1) to have

$$\frac{L|\nabla u|^2}{|\nabla u|^2} \geq 2\frac{A}{B}\frac{|\mathrm{Hess}(u)|^2}{|\nabla u|^2} - 2\frac{A}{B}(n-1)Z^2 + 2f'(u) + 2\frac{A}{B}\frac{f(u)}{|\nabla u|^2}\left\langle \nabla\left(\frac{B}{A}\right), \nabla u\right\rangle + \tag{1.23}$$
$$+ 2\frac{A}{B}\frac{1}{n-m}\frac{\langle \nabla A, \nabla u\rangle^2}{A^2}\frac{1}{|\nabla u|^2}$$

on $B_T(q)$. On the other hand, by Newton inequalities (see the Appendix)

$$\frac{|\mathrm{Hess}(u)|^2}{|\nabla u|^2} \geq \frac{1}{m}\frac{(\Delta u)^2}{|\nabla u|^2} = \frac{1}{m}\frac{1}{|\nabla u|^2}\left\{\frac{B}{A}Lu - \frac{\langle \nabla A, \nabla u\rangle}{A}\right\}^2, \tag{1.24}$$

and using the elementary inequality

$$(a-b)^2 \geq \frac{a^2}{1+\gamma} - \frac{b^2}{\gamma}$$

valid for $a, b \in \mathbb{R}$ and $\gamma > 0$ we obtain

$$2\frac{A}{B}\frac{|\mathrm{Hess}(u)|^2}{|\nabla u|^2} \geq 2\frac{B}{A}\frac{1}{m}\frac{1}{|\nabla u|^2}\frac{(Lu)^2}{1+\gamma} - 2\frac{A}{B}\frac{1}{m}\frac{1}{|\nabla u|^2}\frac{1}{\gamma}\frac{\langle \nabla A, \nabla u\rangle^2}{A^2}.$$

Inserting this latter into (1.23) yields

$$\begin{aligned}\frac{L|\nabla u|^2}{|\nabla u|^2} \geq & -2\frac{A}{B}(n-1)Z^2 + 2f'(u) + 2\frac{B}{A}\frac{1}{m|\nabla u|^2}\frac{(Lu)^2}{1+\gamma} + \\ & + 2\frac{A}{B}\left(\frac{1}{n-m} - \frac{1}{\gamma m}\right)\frac{\langle \nabla A, \nabla u\rangle^2}{|\nabla u|^2 A^2} + 2\frac{A}{B}\frac{f(u)}{|\nabla u|^2}\left\langle \nabla\left(\frac{B}{A}\right), \nabla u\right\rangle.\end{aligned} \tag{1.25}$$

From the modified Young inequality

$$-2ab \leq \varepsilon a^2 + \frac{b^2}{\varepsilon}$$

(valid for $a, b \in \mathbb{R}$ and $\varepsilon > 0$) we obtain

$$\frac{B}{A}\frac{\varepsilon}{2}\frac{(Lu)^2}{|\nabla u|^2} \geq -\frac{g'(u)Lu}{g(u)} - \frac{1}{2\varepsilon}\frac{A}{B}\left(\frac{g'(u)}{g(u)}\right)^2 |\nabla u|^2.$$

Choosing $\varepsilon = \frac{4}{m(1+\gamma)}$ and inserting into (1.25) gives

$$\begin{aligned}\frac{L|\nabla u|^2}{|\nabla u|^2} \geq & \, 2f'(u) - 2\frac{A}{B}(n-1)Z^2 - \frac{g'(u)}{g(u)}Lu - \frac{A}{B}\frac{m(1+\gamma)}{8}\left(\frac{g'(u)}{g(u)}\right)^2 |\nabla u|^2 + \\ & + 2\frac{A}{B}\left(\frac{1}{n-m} - \frac{1}{\gamma m}\right)\frac{\langle \nabla A, \nabla u\rangle^2}{|\nabla u|^2 A^2} + 2\frac{A}{B}\frac{f(u)}{|\nabla u|^2}\left\langle \nabla\left(\frac{B}{A}\right), \nabla u\right\rangle.\end{aligned}$$

Thus, the choice $\gamma = \frac{n-m}{m} > 0$ yields

$$\frac{L|\nabla u|^2}{|\nabla u|^2} \geq 2f'(u) - 2\frac{A}{B}(n-1)Z^2 - \frac{g'(u)}{g(u)}Lu - \frac{A}{B}\frac{n}{8}\left(\frac{g'(u)}{g(u)}\right)^2|\nabla u|^2 + \\ + 2\frac{A}{B}\frac{f(u)}{|\nabla u|^2}\left\langle \nabla\left(\frac{B}{A}\right), \nabla u\right\rangle. \tag{1.26}$$

We now use (1.19), (1.21) and (1.26) into (1.15). Then, at $\bar{x}$, we obtain

$$0 \geq \frac{g(u)g''(u) - \left(2+\frac{n}{8}\right)g'(u)^2}{g(u)^2}|\nabla u|^2 - \frac{8\rho}{T^2-\rho^2}\frac{|g'(u)|}{g(u)}|\nabla u| - \frac{24\rho^2}{(T^2-\rho^2)^2} + \\ - \frac{4[n+(n-1)Z\rho]}{T^2-\rho^2} - 2(n-1)Z^2 + 2\frac{B}{A}f'(u) + 2\frac{f(u)}{|\nabla u|^2}\left\langle \nabla\left(\frac{B}{A}\right), \nabla u\right\rangle$$

from which (1.12) follows immediately.

□

1.2 Consequences of Lemma 1.2

In the next result we elaborate on Lemma 1.2.

Lemma 1.3. *Let $B_T(q), \rho(x)$ be as in Lemma 1.2 and assume* (1.9). *Consider the function $G(x)$ on $B_T(q)$ given in* (1.10) *and set*

$$S = \min\left\{\inf_{B_T(q)} \frac{B}{A}f'(u), 0\right\}, \tag{1.27}$$

$$E = \sup_{B_T(q)} 2|f(u)|\left|\nabla\left(\frac{B}{A}\right)\right|. \tag{1.28}$$

If $\bar{x} \in B_T(q)$ is a positive maximum of G on $B_T(q)$, assume that

$$a = \frac{g(u)g''(u) - \left(2+\frac{n}{8}\right)g'(u)^2}{g(u)^2}(\bar{x}) > 0. \tag{1.29}$$

Then

$$|\nabla u|^2(q)g(u(q)) \leq 2\alpha^2 \max\left\{ \frac{32}{T^2a^2}\frac{|g'(u(\bar{x}))|^2}{g(u(\bar{x}))}, \right. \tag{1.30}$$

$$\frac{4g(u(\bar{x}))}{a}\left[\frac{2(n+6)}{T^2}+\frac{2(n-1)Z}{T}+(n-1)Z^2-S\right],$$

$$\left. \frac{E}{\alpha\sqrt{a}}\frac{g(u(\bar{x}))}{\sqrt{2(n-1)Z^2-2S}}\right\},$$

with $\alpha = \frac{3+2\sqrt{2}}{4}$.

Proof. From Lemma 1.2, (1.12) holds at $\bar{x}$. We set

$$z = |\nabla u|(\bar{x}) > 0,$$

$$b = \frac{8|g'(u)|(\bar{x})}{g(u(\bar{x}))}\frac{\rho(\bar{x})}{T^2-\rho^2(\bar{x})} \geq 0,$$

$$c = \frac{24\rho^2(\bar{x})}{[T^2-\rho^2(\bar{x})]^2}+\frac{4[n+(n-1)Z\rho(\bar{x})]}{T^2-\rho^2(\bar{x})}+2(n-1)Z^2-2S > 0.$$

Thus, (1.12) becomes

$$az^2 - bz - c - \frac{E}{z} \leq 0. \tag{1.31}$$

(1.31) implies $|\nabla u|^2(\bar{x}) = z^2 \leq z_0^2$, where z_0 is the (unique) real positive root of the third degree equation $az^3 - bz^2 - cz - E = 0$. Let z_1 be the positive root of the quadratic polynomial $az^2 - bz - c$ and l the tangent straight line to the parabola $\Gamma : y = az^2 - bz - c$ passing through z_1 (i.e. the line $y = \sqrt{b^2+4ac}(z-z_1)$). If we denote with z_A the (positive) absciss of the intersection between l and the hyperbola $y = \frac{E}{z}$, a computation then shows that

$$
\begin{aligned}
|\nabla u|^2(\bar{x}) \leq z_0^2 \leq z_A^2 \leq \alpha \max &\left\{ \frac{b^2 + 2ac + b\sqrt{b^2+4ac}}{2a^2}, \frac{4E}{\sqrt{b^2+4ac}} \right\} \\
&\leq \alpha \max \left\{ \alpha \max \left\{ \frac{b^2}{a^2}, 4\frac{c}{a} \right\}, \frac{4E}{\sqrt{b^2+4ac}} \right\} \\
&= \alpha^2 \max \left\{ \frac{b^2}{a^2}, 4\frac{c}{a}, \frac{4E}{\sqrt{b^2+4ac}} \right\}.
\end{aligned}
$$

Computing and recalling the definition of $G(x)$ in (1.10), after some manipulation we obtain

$$
\begin{aligned}
G(\bar{x}) &= |\nabla u|^2(\bar{x}) \left[T^2 - \rho^2(\bar{x})\right]^2 g(u(\bar{x})) \\
&\leq \alpha^2 \max \left\{ \frac{64}{T^2 a^2} \frac{|g'(u(\bar{x}))|^2}{g(u(\bar{x}))}, \right. \\
&\quad \frac{4g(u(\bar{x}))}{a} \left[\frac{4(n+6)}{T^2} + \frac{4(n-1)Z}{T} + 2(n-1)Z^2 - 2S \right], \\
&\quad \left. \frac{4E}{\alpha} \frac{g(u(\bar{x}))}{\sqrt{64 \frac{|g'(u(\bar{x}))|^2}{g(u(\bar{x}))} \frac{\rho^2(\bar{x})}{[T^2-\rho^2(\bar{x})]^2} + 4a\left[\frac{24\rho^2(\bar{x})}{[T^2-\rho^2(\bar{x})]^2} + \frac{4[n+(n-1)Z\rho(\bar{x})]}{T^2-\rho^2(\bar{x})} + 2(n-1)Z^2 - 2S \right]}} \right\} \\
&\leq \alpha^2 \max \left\{ \frac{64}{T^2 a^2} \frac{|g'(u(\bar{x}))|^2}{g(u(\bar{x}))}, \right. \\
&\quad \frac{4g(u(\bar{x}))}{a} \left[\frac{4(n+6)}{T^2} + \frac{4(n-1)Z}{T} + 2(n-1)Z^2 - 2S \right], \\
&\quad \left. \frac{2E}{\alpha\sqrt{a}} \frac{g(u(\bar{x}))}{\sqrt{2(n-1)Z^2 - 2S}} \right\}.
\end{aligned}
$$

Hence, since

$$
G(q) = T^4 |\nabla u|^2(q) g(u(q)) \leq G(\bar{x}),
$$

we deduce the validity of (1.30). □

We are now ready to prove

Theorem 1.4. *Let $(M, \langle\ ,\ \rangle)$ be a complete manifold of dimension m. Fix*

$o \in M$ and let $r(x) = \mathrm{dist}_{(M,\langle\ ,\rangle)}(x,o)$. Suppose that $A, B \in C^1(M)$, $A, B > 0$, and that

$$\mathrm{Ricc}_{n,m}(L_A) \geq -(n-1)H^2\big(1+r(x)^2\big)^{\delta/2} \tag{1.32}$$

for some $n > m$, $H \geq 0$ and $\delta \in \mathbb{R}$. Let $f \in C^1(\mathbb{R})$ and $u \in C^3(M)$ a global solution of (1.1) satisfying

$$|u(x)| \leq D(1+r(x))^{\nu}, \tag{1.33}$$

$$\left| f(u(x))\nabla\left(\frac{B}{A}\right)(x)\right| \leq \Theta(1+r(x))^{\theta} \tag{1.34}$$

and

$$\frac{B}{A}(x)f'(u(x)) \geq -K[1+r(x)]^{\gamma} \tag{1.35}$$

on M for some constants $\nu \geq 0$, $\gamma, \theta \in \mathbb{R}$ and constants $D, K, \Theta > 0$. Then

$$|\nabla u|^2(x) \leq \widetilde{C}_1 r(x)^{2\nu} \max\Big\{ r(x)^{-2} + r(x)^{\frac{\delta}{2}-1} + r(x)^{\delta} + r(x)^{\gamma}, \tag{1.36}$$
$$r(x)^{\theta-\frac{\delta}{2}} \frac{1+\widetilde{C}r(x)^{\frac{\gamma-\delta}{2}}}{1+\widetilde{C}r(x)^{\gamma-\delta}} \Big\}$$

for $r(x) \gg 1$ and constants $\widetilde{C}, \widetilde{C}_1 \geq 0$.

Proof. Fix a geodesic ball $B_T(q)$ and let

$$N \geq \left(\sup_{B_T(q)} u\right) + 1. \tag{1.37}$$

Define

$$g(t) = (N-t)^d \tag{1.38}$$

with $d \in \mathbb{R}$, $t < N$; note that $N - u \geq 1$ on $B_T(q)$, so $g(u) > 0$ on $B_T(q)$. In this case, if we consider a defined in (1.29), we have that $a > 0$ if

$$d\Big[\Big(1+\frac{n}{8}\Big)d+1\Big](N-u)^{2(d-1)} < 0$$

on $B_T(q)$. This forces d to be chosen such that

$$-\frac{8}{n+8} < d < 0.$$

Next, (1.30) and a simple computation yield, with Z as in (1.9),

$$\begin{aligned}|\nabla u|^2(q)g(u(q)) \leq 2\alpha^2[N-u(\bar{x})]^{d+2}\cdot \qquad\qquad (1.39)\\ \cdot \max\Bigg\{ &\frac{32}{\big[\big(1+\frac{n}{8}\big)d+1\big]^2T^2}, \\ &-\frac{4}{d\big[\big(1+\frac{n}{8}\big)d+1\big]}\left[\frac{2(n+6)}{T^2}+\frac{2(n-1)Z}{T}+(n-1)Z^2-S\right], \\ &\frac{E}{\alpha}\frac{1}{\sqrt{-d\big[\big(1+\frac{n}{8}\big)d+1\big]}}\frac{1}{\sqrt{2(n-1)Z^2-2S}}\frac{1}{N-u(\bar{x})}\Bigg\}.\end{aligned}$$

For the geodesic ball $B_T(q)$, $q \neq o \in M$, we choose $T = \frac{1}{2}r(q) > 0$. Then, there exists a constant $\Lambda = \Lambda(\delta) > 0$, depending only on the sign of δ, such that we can choose $Z = H\big(1+\Lambda r(q)^2\big)^{\delta/4}$. Furthermore, using (2.2), (2.4), (1.35) we choose

$$N = D\left[1+\frac{3}{2}r(q)\right]^{\nu}+1 > 0, \qquad (1.40)$$

$$0 \geq S \geq -K(1+\Lambda_1 r(q))^{\gamma}, \qquad (1.41)$$

$$0 \leq E \leq 2\Theta(1+\Lambda_2 r(q))^{\theta} \qquad (1.42)$$

on $B_T(q)$, with Λ_1, Λ_2 positive constants depending respectively only on the signs of γ and θ. Inserting into (1.39) and computing yields

$$
\begin{aligned}
&|\nabla u|^2(q) \le 2\alpha^2\big[2D\big(1+\tfrac{3}{2}r(q)\big)^{\nu}+1\big]^2\cdot \\
&\cdot \max\Bigg\{\frac{32}{\big[\big(1+\frac{n}{8}\big)d+1\big]^2}\cdot 4\cdot r(q)^{-2}, \\
&-\frac{4}{d\big[\big(1+\frac{n}{8}\big)d+1\big]}\Bigg[8(n+6)r(q)^{-2}+\frac{4(n-1)H\big(1+\Lambda r(q)^2\big)^{\delta/4}}{r(q)}+ \\
&+(n-1)H^2\big(1+\Lambda r(q)^2\big)^{\delta/2}+K(1+\Lambda_1 r(q))^{\gamma}\Bigg], \\
&\tfrac{2\Theta(1+\Lambda_2 r(q))^{\theta}}{\alpha}\tfrac{1}{\sqrt{-d\left[\left(1+\frac{n}{8}\right)d+1\right]}}\tfrac{1}{\sqrt{2(n-1)H^2(1+\Lambda r(q)^2)^{\delta/2}+2K(1+\Lambda_1 r(q))^{\gamma}}}\Bigg\} \\
&\le C_1 r(q)^{2\nu}\max\Big\{C_2 r(q)^{-2}, C_3 r(q)^{-2}+C_4 r(q)^{\frac{\delta}{2}-1}+C_5 r(q)^{\delta}+C_6 r(q)^{\gamma}, \\
&C_7 r(q)^{\theta}\frac{C_8 r(q)^{\delta/2}+C_9 r(q)^{\gamma/2}}{C_8 r(q)^{\delta}+C_9 r(q)^{\gamma}}\Big\}
\end{aligned}
$$

for $r(q) \gg 1$ and constants $C_i \ge 0$, $i = 1, \dots, 9$. This easily implies (1.36).

$\square$

Corollary 1.5. *In the assumptions of Theorem 1.4, if*

$$\frac{B}{A} < C, \qquad \left|\nabla\left(\frac{B}{A}\right)\right| < C$$

on M for some constant $C > 0$ and u is a bounded global solution of (1.1), *then $|\nabla u|$ is bounded provided $\delta \le 0$.*

Proof. Just observe that (1.33), (1.34) and (1.35) are verified choosing $\nu = \theta = \gamma = 0$. $\square$

1.3 More gradient estimates

In this section we prove more gradient estimates for global bounded solutions of $Lu = f(u)$. Towards this end we shall need the next

Lemma 1.6. *Let* $N, T, \Lambda, \varepsilon > 0$*,* $H \geq 0$ *be fixed constants. Then there exists a function* $\eta = \eta_{\varepsilon,T} : [T, +\infty) \to [0, 1]$ *with the following properties:*

$$\eta \in C^2((T, +\infty)); \tag{1.43}$$

$$\eta(T) = 1,\ \eta > 0,\ \eta' < 0 \ \textit{on}\ (T, +\infty);\ \lim_{t \to +\infty} \eta(t) = 0; \tag{1.44}$$

$$\lim_{\varepsilon \to 0^+} \eta_{\varepsilon,T}(t) = 1 \qquad \forall \quad \textit{fixed}\ t \in [T, +\infty); \tag{1.45}$$

$$\left(\frac{\eta}{\eta'}\right)^2 \left\{ \frac{2(\eta')^2}{\eta} - \left(\frac{N}{\varepsilon} + \frac{n-1}{t} + (n-1)H\right)\eta' - \eta'' \right\} = \frac{\varepsilon}{\Lambda} \ \textit{on}\ [T, +\infty); \tag{1.46}$$

$$\left|\frac{\eta}{\eta'}\right| \leq \frac{\varepsilon^2}{\eta\Lambda(N + (n-1)H\varepsilon)} \ \textit{on}\ [T, +\infty). \tag{1.47}$$

Proof. The idea of the proof is taken from Modica, [Mod85]. We define the decreasing function

$$g_\varepsilon : \left[0, \frac{\Lambda}{\varepsilon} e^{-\frac{\varepsilon}{\Lambda}}\right] \to [0, 1]$$

by setting

$$g_\varepsilon(t) = -\frac{\varepsilon}{\Lambda} \frac{1}{\log\left(e^{-\frac{\varepsilon}{\Lambda}} - \frac{\varepsilon}{\Lambda} t\right)}$$

on $\left[0, \frac{\Lambda}{\varepsilon} e^{-\frac{\varepsilon}{\Lambda}}\right)$ and extending it by continuity on the right end side of the interval. Furthermore, we let

$$h_{\varepsilon,T} : [T, +\infty) \to \mathbb{R}$$

be given by

$$h_{\varepsilon,T}(t) = \int_T^t s^{1-n} e^{-\left[\frac{N}{\varepsilon} + (n-1)H\right]s}\, ds. \tag{1.48}$$

We note that $h_{\varepsilon,T}$ is increasing, $h_{\varepsilon,T}(T) = 0$,

$$0 < I = h_{\varepsilon,T}(+\infty) = \int_T^{+\infty} s^{1-n} e^{-\left[\frac{N}{\varepsilon} + (n-1)H\right]s}\, ds < +\infty.$$

We set

$$\eta(t) = g_\varepsilon\left(\frac{\Lambda}{\varepsilon}\frac{e^{-\frac{\varepsilon}{\Lambda}}}{I}h_{\varepsilon,T}(t)\right) \tag{1.49}$$

on $[T, +\infty)$. Now, properties (1.43), (1.44) and (1.45) follows easily. As for (1.46), from (1.49) we have

$$g_\varepsilon^{-1}(\eta(t)) = \frac{\Lambda}{\varepsilon}\frac{e^{-\frac{\varepsilon}{\Lambda}}}{I}h_{\varepsilon,T}(t). \tag{1.50}$$

Noting that

$$g_\varepsilon^{-1}(t) = \int_t^1 s^{-2}e^{-\frac{\varepsilon}{\Lambda s}}\,ds,$$

differentiating (1.50) and using (1.48) we get

$$-\frac{e^{-\frac{\varepsilon}{\Lambda\eta(t)}}}{\eta(t)^2}\eta'(t) = \frac{\Lambda}{\varepsilon}\frac{1}{I}\frac{e^{-\left[\frac{N}{\varepsilon}+(n-1)H\right]t-\frac{\varepsilon}{\Lambda}}}{t^{n-1}}. \tag{1.51}$$

Taking the logarithm of both members of (1.51) and differentiating once more we obtain (1.46). It remains to prove (1.47). From (1.50), differentiating we get

$$\frac{\eta(t)}{\eta'(t)} = \eta(t)\frac{\varepsilon I}{\Lambda}e^{\frac{\varepsilon}{\Lambda}}\frac{\left(g_\varepsilon^{-1}\right)'(\eta(t))}{h'_{\varepsilon,T}(t)}. \tag{1.52}$$

A simple computation shows that

$$\left(g_\varepsilon^{-1}\right)'(\eta(t)) = -\frac{e^{-\frac{\varepsilon}{\Lambda\eta(t)}}}{\eta(t)^2}. \tag{1.53}$$

On the other hand (1.49) gives

$$\eta(t) = -\frac{\varepsilon}{\Lambda}\frac{1}{\log\left[e^{-\frac{\varepsilon}{\Lambda}}\left(1-\frac{h_{\varepsilon,T}(t)}{I}\right)\right]}.$$

Inserting into (1.53) yields

$$\left(g_\varepsilon^{-1}\right)'(\eta(t)) = -\frac{e^{-\frac{\varepsilon}{\Lambda}}}{\eta(t)^2}\left(1-\frac{h_{\varepsilon,T}(t)}{I}\right). \tag{1.54}$$

Substituting (1.54) into (1.52) and using the definition (1.48) of $h_{\varepsilon,T}$ to compute its derivative, we obtain

$$\frac{\eta(t)}{\eta'(t)} = -\frac{1}{\eta(t)}\frac{\varepsilon}{\Lambda}(I - h_{\varepsilon,T}(t))t^{n-1}e^{\left[\frac{N}{\varepsilon}+(n-1)H\right]t}.$$

(1.48) and the definition of I allow us to rewrite the above as

$$\frac{\eta(t)}{\eta'(t)} = -\frac{1}{\eta(t)}\frac{\varepsilon}{\Lambda}t^{n-1}e^{\left[\frac{N}{\varepsilon}+(n-1)H\right]t}\int_t^{+\infty} s^{1-n}e^{-\left[\frac{N}{\varepsilon}+(n-1)H\right]s}\,ds.$$

It follows that

$$\left|\frac{\eta(t)}{\eta'(t)}\right| \leq \frac{1}{\eta(t)}\frac{\varepsilon}{\Lambda}e^{\left[\frac{N}{\varepsilon}+(n-1)H\right]t}\int_t^{+\infty} e^{-\left[\frac{N}{\varepsilon}+(n-1)H\right]s}\,ds$$

and (1.47) follows at once explicitating the integral. □

We are now ready to prove the following general theorem, which is the main result of this Chapter and compares directly to Theorem B in [RR95]:

Theorem 1.7. *Let $(M, \langle\ ,\ \rangle)$ be a complete manifold of dimension m. Suppose that $A, B \in C^2(M)$, $A, B > 0$ and $h \in C^2(M)$, $h \geq 0$ satisfy*

$$\begin{cases} i)\ \ \frac{B}{A} < C; \\ ii)\ \ h < C; \\ iii)\ |\nabla h| < C; \\ iv)\ |Lh| < C \end{cases} \tag{1.55}$$

on M, with $L = \frac{1}{B}\operatorname{div}(A\nabla\ )$. Furthermore, suppose that, for some $n > m$, $H \geq 0$

$$\mathrm{Ricc}_{n,m}(L_A) \geq -(n-1)H^2 \tag{1.56}$$

on M. For $f \in C^1(\mathbb{R})$ let $u \in C^3(M)$ be a global solution of

$$Lu = f(u) \tag{1.1}$$

such that

$$\begin{cases} i) \;\; |\nabla u| < C \;\; on \; M; \\ ii) \;\; \inf_M |\nabla u| = 0. \end{cases} \tag{1.57}$$

Assume the existence of a function $Q \in C^2(\mathbb{R})$ with the following properties:

$$\begin{cases} i) \;\; Q(u), |Q'(u)| < C; \\ ii) \;\; \inf_M Q(u) = 0; \\ iii) \; \left[Q'(u)h - 2\frac{B}{A}f(u)\right]Q'(u) \geq 0; \\ iv) \;\; 2\frac{B}{A}f'(u) - 2(n-1)H^2 - hQ''(u) \geq 0; \\ v) \;\; \left|Q'(u)\nabla h - f(u)\nabla\left(\frac{B}{A}\right)\right| < C \end{cases} \tag{Q}$$

on M. Then

$$|\nabla u|^2 \leq h(x)Q(u) \quad on \;\; M. \tag{1.58}$$

Remark. Of course the applicability of Theorem 1.7 depends on the possible choices for h and Q: this is the case in the proofs of Theorem 1.8, 1.9 and Corollary 2.4 below. Note also that, if $A = B = 1$ and $h = 1$, we recover Theorem B in [RR95].

Proof. (of Theorem 1.7) Let u be a global solution of (1.1) and define the function

$$P = |\nabla u|^2 - h(x)Q(u) \tag{1.59}$$

on M. Let d a positive constant. Because of (1.57) ii) there exists $q \in M$ such that

$$|\nabla u|^2(q) < d. \tag{1.60}$$

Fix $\varepsilon, T > 0$ and define a function $v : M \setminus B_T(q) \to \mathbb{R}$ by setting

$$v(x) = \eta_{\varepsilon,T}(\rho(x))P(x), \tag{1.61}$$

where $\rho(x) = \mathrm{dist}_{(M,\langle\;,\rangle)}(x,q)$ and $\eta(t) = \eta_{\varepsilon,T}(t)$ is the function defined in Lemma 1.6 with $N, \Lambda > 0$ to be chosen later. We may assume that $v > 0$ somewhere, for otherwise, since $\eta > 0$, $P \leq 0$ on $M \setminus B_T(q)$. Because of assumptions (1.57) i), (1.55) i) and (Q) i) $P(x)$ is bounded; thus, (1.44) of

Lemma 1.6 implies that

$$v(x) \to 0 \qquad \text{as} \quad \rho(x) \to +\infty. \tag{1.62}$$

Next, we prove that, given $\delta > 0$, we have

$$v(x) \le \max\left\{\delta, \max_{\partial B_T(q)} v\right\} \tag{1.63}$$

on $M \setminus B_T(q)$. Towards this aim it is enough to show that $v(\bar{x}) \le \delta$ at any maximum point $\bar{x} \in M \setminus \overline{B_T(q)}$, if any. At $\bar{x}$ we must have $\nabla v = 0$ and $Lv \le 0$: these are respectively equivalent to

$$\nabla P = -\frac{\eta'(\rho)}{\eta(\rho)} P \nabla \rho \tag{1.64}$$

and

$$0 \ge \eta'(\rho) P L\rho + \eta''(\rho)\frac{A}{B} P + \eta(\rho) LP - 2\frac{A}{B}\frac{\eta'(\rho)^2}{\eta(\rho)} P \tag{1.65}$$

at $\bar{x}$, where in (1.65) we have used (1.64), (1.2) and (1.3). We need now to estimate LP. Using (1.59), (1.1) and the generalized Bochner-Weitzenböck formula (1.4) together with the curvature restriction (1.56) we obtain

$$\begin{aligned}
LP &= L|\nabla u|^2 - L(h(x)Q(u)) \\
&= L|\nabla u|^2 - \left[h(x)LQ(u) + 2\frac{A}{B}\langle \nabla h, \nabla Q(u)\rangle + Q(u)Lh\right] \\
&= L|\nabla u|^2 - hQ'(u)Lu - h\frac{A}{B}Q''(u)|\nabla u|^2 - 2\frac{A}{B}Q'(u)\langle \nabla h, \nabla u\rangle - Q(u)Lh \\
&\ge 2\frac{A}{B}|\mathrm{Hess}(u)|^2 - 2\frac{A}{B}Q'(u)\langle \nabla h, \nabla u\rangle - Q(u)Lh + 2\frac{A}{B}f(u)\left\langle \nabla\left(\frac{B}{A}\right), \nabla u\right\rangle - \\
&\quad - hQ'(u)f(u) - \left[h\frac{A}{B}Q''(u) + 2\frac{A}{B}(n-1)H^2 - 2f'(u)\right]|\nabla u|^2.
\end{aligned}$$

Multiplying both sides by $|\nabla u|^2$ and using inequality (1.17) we have

$$|\nabla u|^2 LP \geq \frac{1}{2}\frac{A}{B}\left|\nabla|\nabla u|^2\right|^2 + 2\frac{A}{B}f(u)|\nabla u|^2\left\langle \nabla\left(\frac{B}{A}\right), \nabla u\right\rangle - hf(u)Q'(u)|\nabla u|^2 - \tag{1.66}$$

$$- 2\frac{A}{B}Q'(u)|\nabla u|^2\langle \nabla h, \nabla u\rangle - Q(u)|\nabla u|^2 Lh.$$

Next, since $\eta' < 0$, using Schwarz inequality, $Q(u) \geq 0$ and (1.64) we have, at $\bar{x}$,

$$\begin{aligned}
\left|\nabla|\nabla u|^2\right|^2 &= \left|\nabla P + hQ'(u)\nabla u + Q(u)\nabla h\right|^2 \\
&= \left|-\frac{\eta'(\rho)}{\eta(\rho)}P\nabla\rho + hQ'(u)\nabla u + Q(u)\nabla h\right|^2 \\
&= \left(\frac{\eta'(\rho)}{\eta(\rho)}\right)^2 P^2 + h^2 Q'(u)^2|\nabla u|^2 + Q(u)^2|\nabla h|^2 - \\
&- 2\frac{\eta'(\rho)}{\eta(\rho)}PhQ'(u)\langle \nabla u, \nabla\rho\rangle - 2\frac{\eta'(\rho)}{\eta(\rho)}PQ(u)\langle \nabla\rho, \nabla h\rangle + \\
&+ 2hQ(u)Q'(u)\langle \nabla u, \nabla h\rangle + \\
&\geq \left(\frac{\eta'(\rho)}{\eta(\rho)}\right)^2 P^2 + h^2 Q'(u)^2|\nabla u|^2 + Q(u)^2|\nabla h|^2 + \\
&+ 2\frac{\eta'(\rho)}{\eta(\rho)}P\left[h\left|Q'(u)\right||\nabla u| + Q(u)|\nabla h|\right] - \\
&- 2hQ(u)\left|Q'(u)\right||\nabla u||\nabla h|.
\end{aligned}$$

Inserting this inequality into (1.66) we obtain

$$\begin{aligned}|\nabla u|^2 LP \geq & \frac{1}{2}\frac{A}{B}\left(\frac{\eta'(\rho)}{\eta(\rho)}\right)^2 P^2 + \frac{1}{2}\frac{A}{B}h^2 Q'(u)^2|\nabla u|^2 + \frac{1}{2}\frac{A}{B}Q(u)^2|\nabla h|^2 + \\ & + \frac{A}{B}\frac{\eta'(\rho)}{\eta(\rho)}P\big[h\big|Q'(u)\big||\nabla u| + Q(u)|\nabla h|\big] - \frac{A}{B}hQ(u)\big|Q'(u)\big||\nabla h||\nabla u| + \\ & + 2\frac{A}{B}f(u)|\nabla u|^2\left\langle\nabla\left(\frac{B}{A}\right),\nabla u\right\rangle - hf(u)Q'(u)|\nabla u|^2 - \\ & - 2\frac{A}{B}Q'(u)|\nabla u|^2\left\langle\nabla h,\nabla u\right\rangle - Q(u)|\nabla u|^2 Lh.\end{aligned} \tag{1.67}$$

From (1.56) we deduce

$$|\nabla u|^2\eta P\left\{\frac{A}{B}\left[\frac{2(\eta')^2}{\eta} - \eta''\right] - \eta' L\rho\right\} - \eta^2|\nabla u|^2 LP \geq 0 \tag{1.68}$$

at $\bar{x}$. Observe that (1.56) implies

$$L\rho \leq (n-1)\frac{A}{B}\left[\frac{1}{\rho} + H\right] \tag{1.69}$$

on $M \setminus B_T(q)$. Using (1.67) and (1.69) into (1.68) and then (Q) iii), $\eta \leq 1$ and Schwarz inequality, after some tedious computations we get

$$\begin{aligned}\frac{1}{2}\left(\frac{\eta'}{\eta}\right)^2 v^2 \leq & |\nabla u|^2 v\left\{2\frac{(\eta')^2}{\eta} - \eta'' - \left[\frac{n-1}{\rho} + (n-1)H + h\frac{|Q'(u)|}{|\nabla u|} + \frac{Q(u)}{|\nabla u|^2}|\nabla h|\right]\eta'\right\} + \\ & + hQ(u)\big|Q'(u)\big||\nabla u||\nabla h| + 2|\nabla u|^3\left|Q'(u)\nabla h - f(u)\nabla\left(\frac{B}{A}\right)\right| + \\ & + \frac{B}{A}Q(u)|\nabla u|^2|Lh|.\end{aligned} \tag{1.70}$$

If $|\nabla u|^2(\bar{x}) \leq \delta$, then $v(\bar{x}) \leq P(\bar{x})$, $h(\bar{x}) \geq 0$ and $Q(u(\bar{x})) \geq 0$ by (Q) ii) immediately imply $v(\bar{x}) \leq \delta$: thus, it remains to consider the case $|\nabla u|^2(\bar{x}) >$

δ. Set

$$L = \sup_M 2|\nabla u|^2, \ N_1 = \sup_M |Q'(u)||\nabla u|, \ N_2 = \sup_M \left| Q'(u)\nabla h - f(u)\nabla\left(\frac{B}{A}\right)\right|,$$

$$K_1 = \sup_M Q(u).$$

By (1.57) i) and (Q) i), v), $L, N_1, N_2, K \in [0, +\infty)$. Using (1.55) and $\eta' < 0$ from (1.70) we deduce that, at $\bar{x}$,

$$\frac{1}{2}\left(\frac{\eta'}{\eta}\right)^2 v^2 \le |\nabla u|^2 v\left\{2\frac{(\eta')^2}{\eta} - \eta'' - \left[\frac{n-1}{\rho} + (n-1)H + \frac{C(N_1+K_1)}{\delta}\right]\eta'\right\} + \\ + CN_1K + L\sqrt{L}N_2 + CK_1L;$$

in other words, for some appropriate constants $N, K > 0$,

$$v^2 \le vL\left(\frac{\eta}{\eta'}\right)^2\left\{2\frac{(\eta')^2}{\eta} - \eta'' - \left[\frac{n-1}{\rho} + (n-1)H + \frac{N}{\delta}\right]\eta'\right\} + 2\left(\frac{\eta}{\eta'}\right)^2 K. \tag{1.71}$$

Choosing in Lemma 1.6 $\varepsilon = \delta, \Lambda = L$ and T, N as above, from (1.71) we obtain at $\bar{x}$

$$v^2 \le \delta\left[v + \frac{2\delta^3 K}{\eta^2 L^2 (N + (n-1)H\delta)^2}\right]. \tag{1.72}$$

Letting $\delta \downarrow 0^+$ in (1.72) and using (1.45) we deduce $v(\bar{x}) = P(\bar{x}) = 0$. This implies

$$v(x) \le \max\left\{\delta, \max_{\partial B_T(q)} v\right\} = \max\left\{\delta, \max_{\partial B_T(q)} P\right\}$$

on $M \setminus B_T(q)$. Therefore, letting $\delta \downarrow 0^+$ again, we arrive at

$$P(x) \le \max\left\{0, \max_{\partial B_T(q)} P\right\}$$

on $M \setminus B_T(q)$. Letting $T \downarrow 0^+$ in this last inequality we obtain

$$P(x) \le \max\{0, P(q)\}$$

on M. From (1.59), (1.60) and (Q) ii) we get

$$P(x) \leq |\nabla u|^2(q) < d$$

on M. Since $d > 0$ was arbitrary, we conclude $P(x) \leq 0$ on M, that is, (1.58). □

Next result is a consequence of Theorem 1.7 and Corollary 1.5; the case $H \equiv 0$ will be treated separately.

Theorem 1.8. *Let $(M, \langle\ ,\ \rangle)$ be a complete manifold of dimension m, $A, B \in C^2(M)$, $A, B > 0$ and such that*

$$\begin{cases} i)\ \frac{B}{A} < C; \\ ii)\ \left|\nabla\left(\frac{B}{A}\right)\right| < C; \\ iii)\ \left|L\left(\frac{B}{A}\right)\right| < C \end{cases} \tag{1.73}$$

on M for some constant $C > 0$. Suppose that for some $n > m$, $H > 0$,

$$\mathrm{Ricc}_{n,m}(L_A) \geq -(n-1)H^2. \tag{1.56}$$

Let $F \in C^2(\mathbb{R})$ and set $f = F'$. Let $u \in C^2(M)$ be a global bounded solution of

$$Lu = f(u) \tag{1.1}$$

and let

$$d \geq \frac{2}{(n-1)H^2} \sup_M |f(u)| + 2 \sup_M |u|. \tag{1.74}$$

Assume

$$\inf_M \left[2F(u) - (n-1)H^2(u^2 + du)\right] = 0. \tag{1.75}$$

Then,

$$|\nabla u|^2 \leq \frac{B}{A}(x)\left[2F(u) - (n-1)H^2(u^2 + du)\right] \tag{1.76}$$

on M.

Proof. First of all we use Corollary 1.5 and boundedness of u (see also the proof of Corollary 2.4 below) to deduce the validity of (1.57). In Theorem 1.7 we now choose $h = \frac{B}{A}$ so that (1.55) are guaranteed by (1.73). We let

$$Q(t) = 2F(t) - (n-1)H^2\big(t^2 + dt\big)$$

where d is chosen to satisfy (1.74). These choices of d, Q and (1.73) together with u bounded imply the validity of (Q). Applying (1.58) of Theorem 1.7 we obtain the desired estimate (1.76). □

In the special case $H = 0$, we choose $h = \frac{B}{A}$, $Q(t) = 2F(t)$ and we apply Theorem 1.7 without requiring u bounded, which is necessary in (1.74). We have then

Theorem 1.9. *Let $(M, \langle\ ,\ \rangle)$ be a complete manifold of dimension m, $A, B \in C^2(M)$ with $A, B > 0$ and such that*

$$\begin{cases} i)\ \ \frac{B}{A} < C; \\ ii)\ \left|\nabla\big(\frac{B}{A}\big)\right| < C; \\ iii)\ \left|L\big(\frac{B}{A}\big)\right| < C \end{cases} \tag{1.77}$$

on M for some constant $C > 0$. Suppose that for some $n > m$

$$\mathrm{Ricc}_{n,m}(L_A) \geq 0. \tag{1.78}$$

Let $F \in C^2(\mathbb{R})$, and set $f = F'$. Let $u \in C^3(M)$ be a global solution of

$$Lu = f(u) \tag{1.1}$$

such that

$$\begin{cases} F(u), |f(u)| < C; \\ \inf_M F(u) = 0; \\ \inf_M |\nabla u| = 0; \\ |\nabla u| < C \end{cases} \tag{1.79}$$

on M for some $C > 0$. Then

$$|\nabla u|^2(x) \leq 2\frac{B}{A}(x)F(u) \tag{1.80}$$

on M.

If we interpret Theorem 1.9 for equation (0.1) and require u bounded, we obtain the following

Corollary 1.10. *Let $(M, \langle\ ,\rangle)$ be a complete manifold of dimension m, $a, b \in C^2(M)$ and suppose that, for some $n > m$,*

$$\mathrm{Ricc}_M \geq \mathrm{Hess}(a) + \frac{1}{n-m}\,\mathrm{d}a \otimes \mathrm{d}a.$$

Assume

$$i)\, 0 < b < C; \quad ii)\, |\nabla b| < C; \quad iii)\, |\Delta b + \langle \nabla a, \nabla b\rangle| \leq Cb \tag{1.81}$$

on M for some constant $C > 0$. Let $F \in C^2(\mathbb{R})$, $f = F'$ and $u \in C^3(M)$ a bounded global solution of (0.1) such that

$$\inf_M F(u) = 0. \tag{1.82}$$

Then

$$|\nabla u|^2 \leq 2b(x)F(u) \qquad \textit{on } M. \tag{1.83}$$

Chapter 2

Liouville Theorems

The paths fork and divide. With each step you take through Destiny's garden you make a choice, and every choice determines future paths. However, at the end of a lifetime of walking you might look back, and see only one path stretching out behind you; or look ahead, and see only darkness.

Sandman, Season of Mists, DC Comics-Vertigo

The aim of this Chapter is to derive some Liouville-type theorems for the diffusion Poisson equation (1.1) and the related equation (0.1). This is obtained under geometric condition on the manifold $(M, \langle\ ,\ \rangle)$ and appropriate growth conditions on both the solution and the non-linearity f. The key tools, *ça va sans dire*, will be the gradient estimates developed in Chapter 1.

2.1 Liouville theorems for solutions of sublinear growth

As an application of Lemma 1.2 we first prove

Theorem 2.1. *Let $(M, \langle\ ,\rangle)$ be a complete manifold of dimension m. Fix $o \in M$ and let $r(x) = \mathrm{dist}_{(M,\langle\ ,\rangle)}(x, o)$. Suppose that $A, B \in C^1(M), A, B > 0$ and that for some $n > m, H \geq 0, \delta \in \mathbb{R}$,*

$$\mathrm{Ricc}_{n,m}(L_A) \geq -(n-1)H^2\big(1 + r(x)^2\big)^{\delta/2} \tag{2.1}$$

on M. Let $f \in C^1(\mathbb{R})$ and $u \in C^3(M)$ be a global solution of

$$Lu = f(u). \tag{1.1}$$

Assume

$$|u(x)| \leq D(1 + r(x))^{\nu}, \tag{2.2}$$

$$\frac{B}{A}(x) f'(u(x)) \geq (n-1)H^2\big(1 + r(x)^2\big)^{\delta/2}, \tag{2.3}$$

$$\left| f(u(x))\nabla\left(\frac{B}{A}\right)(x)\right| \leq \Theta(1 + r(x))^{\theta} \tag{2.4}$$

on M, for some constants $\Theta, D > 0$, $\nu, \theta \in \mathbb{R}$. Then u is constant provided

$$0 \leq \nu < \min\left\{1, 1 - \frac{\delta}{4}, -\frac{\theta}{2}\right\}. \tag{2.5}$$

Proof. For $q \in M, q \neq o$, let $B_T(q)$ the geodesic ball centered at q with radius $T > 0$. Set $\rho(x) = \mathrm{dist}_{(M,\langle\ ,\rangle)}(x, q)$ and choose

$$N(T) \geq \sup_{B_T(q)} |u|. \tag{2.6}$$

We use Lemma 1.2 with the choice $g(u) = [3N(T) - u]^{-d}$ with $d > 0$ to be chosen later. We note that, since $\bar{x}$ is a positive maximum for G in $B_T(q)$, we have

$$\begin{aligned} |\nabla u|^2(\bar{x}) &\geq \frac{G(q)}{\big(T^2 - \rho^2(\bar{x})\big)^2 g(u(\bar{x}))} = \frac{T^4|\nabla u|^2(q)}{(T^2 - \rho^2(\bar{x}))^2}\frac{g(u(q))}{g(u(\bar{x}))} \geq \\ &\geq \frac{T^4}{2^d\big(T^2 - \rho^2(\bar{x})\big)^2}|\nabla u|^2(q). \end{aligned} \tag{2.7}$$

Assume, by contradiction, that u is non constant. Then, without loss of generality we can suppose that, for some $\varepsilon > 0$, $|\nabla u|^2(q) \geq \varepsilon^2$. Thus (2.7) gives

$$|\nabla u|^2(\bar{x}) \geq \frac{T^4\varepsilon^2}{2^d\left(T^2-\rho^2(\bar{x})\right)^2}. \tag{2.8}$$

We now substitute the expression of $g(u)$ into (1.12) obtaining

$$\begin{aligned} 0 \geq\, & \frac{d\left[1-\left(1+\frac{n}{8}\right)d\right]}{\left[3N(T)-u(\bar{x})\right]^2}|\nabla u|^2(\bar{x}) - \frac{8d}{3N(T)-u(\bar{x})}\frac{\rho(\bar{x})}{T^2-\rho^2(\bar{x})}|\nabla u|(\bar{x}) + \\ & -2\frac{\left|f(u)\nabla\left(\frac{B}{A}\right)\right|}{|\nabla u|}(\bar{x}) - \frac{24\rho^2(\bar{x})}{\left(T^2-\rho^2(\bar{x})\right)^2} - \frac{4[n+(n-1)Z\rho(\bar{x})]}{T^2-\rho^2(\bar{x})} - 2(n-1)Z^2 + \\ & +2\frac{B}{A}(\bar{x})f'(u(\bar{x})) \end{aligned} \tag{2.9}$$

where $-(n-1)Z^2$ is a lower bound for $\mathrm{Ricc}_{n,m}(L_A)$ on $B_T(q)$. Suppose now that

$$\frac{B}{A}f'(u) \geq (n-1)Z^2 \tag{2.10}$$

on $B_T(q)$ and divide (2.9) by $|\nabla u|^2(\bar{x})$. We then have

$$\begin{aligned} 0 \geq\, & \frac{d\left[1-\left(1+\frac{n}{8}\right)d\right]}{\left[3N(T)-u(\bar{x})\right]^2} - \frac{8d}{3N(T)-u(\bar{x})}\frac{\rho(\bar{x})}{T^2-\rho^2(\bar{x})}\frac{1}{|\nabla u|(\bar{x})} + \\ & -\left\{\frac{24\rho^2(\bar{x})}{\left(T^2-\rho^2(\bar{x})\right)^2} + \frac{4[n+(n-1)Z\rho(\bar{x})]}{T^2-\rho^2(\bar{x})}\right\}\frac{1}{|\nabla u|^2(\bar{x})} + \\ & -2\frac{\left|f(u)\nabla\left(\frac{B}{A}\right)\right|(\bar{x})}{|\nabla u|^3(\bar{x})}. \end{aligned}$$

We now use (2.8) to deduce

$$\begin{aligned} 0 \geq\, & \frac{d\left[1-\left(1+\frac{n}{8}\right)d\right]}{4N(T)^2} - \frac{d\cdot 2^{\frac{d}{2}+2}}{N(T)T\varepsilon} - \frac{3\cdot 2^{d+3}}{T^2\varepsilon^2} - \frac{[n+(n-1)ZT]2^{d+2}}{T^2\varepsilon^2} + \\ & -\left|f(u)\nabla\left(\frac{B}{A}\right)\right|(\bar{x})\frac{2^{\frac{3}{2}d+1}}{\varepsilon^2}. \end{aligned}$$

Multiplying times $N(T)^2$,

$$0 \geq \frac{d}{4}\Big[1-\Big(1+\frac{n}{8}\Big)d\Big] - \frac{d\cdot 2^{\frac{d}{2}+2}}{T\varepsilon}N(T) - [6+n+(n-1)ZT]2^{d+2}\frac{N(T)^2}{T^2\varepsilon^2} + \tag{2.11}$$
$$- \left|f(u)\nabla\Big(\frac{B}{A}\Big)\right|(\bar{x})\frac{2^{\frac{3}{2}d+1}}{\varepsilon^2}N(T)^2.$$

Next we fix $T=\frac{1}{2}r(q)$; assumption (2.1) allows us to choose, as in the proof of Theorem 1.4,

$$Z = H\big(1+\Lambda r(q)^2\big)^{\delta/4}$$

on $B_T(q)$ where $\Lambda=\Lambda(\delta)>0$ is a constant depending only on the sign of δ. Note that, with this choice, (2.3) implies the validity of (2.10). Furthermore (2.2) enables us to choose

$$N(T) = D(1+\Gamma r(q))^{\nu},$$

where $\Gamma=\Gamma(\nu)>0$ is a constant depending only on the sign of ν. With these choices, from (2.11) and (2.4) we deduce

$$\begin{aligned} 0 \geq\ & \frac{d}{4}\Big[1-\Big(1+\frac{n}{8}\Big)d\Big] - 2^{\Gamma d+1}(1+\Lambda r(q))^{\theta}\frac{\Theta D^2}{\varepsilon^2}(1+\Gamma r(q))^{2\nu} + \\ & - \frac{3\cdot 2^{d+5}}{\varepsilon^2}\frac{D^2}{r(q)^2}(1+\Gamma r(q))^{2\nu} - \frac{d\cdot 2^{\frac{d}{2}+3}}{\varepsilon}\frac{D}{r(q)}(1+\Gamma r(q))^{\nu} + \\ & - \Big[n+(n-1)H\big(1+\Lambda r(q)^2\big)^{\delta/4}\Big]\frac{2^{d+4}}{\varepsilon^2}\frac{D^2}{r(q)^2}(1+\Gamma r(q))^{2\nu}. \end{aligned} \tag{2.12}$$

We fix $d>0$ sufficiently small that

$$\frac{d}{4}\Big[1-\Big(1+\frac{n}{8}\Big)d\Big] > 0;$$

having made this choice, from inequality (2.12) we deduce that there exists a constant $\omega=\omega(d,\Lambda,\Theta,\Gamma,D,H,n)>0$ such that

$$0 \geq \frac{d}{4}\Big[1-\Big(1+\frac{n}{8}\Big)d\Big] - \frac{\omega}{\varepsilon^2}\Big[r(q)^{2\nu+\theta} + r(q)^{2\nu-2} + r(q)^{2\nu+\frac{\delta}{2}-2} + \varepsilon r(q)^{\nu-1}\Big].$$

Under assumption (2.5) we reach a contradiction by letting $r(q) \to +\infty$. □

In the special case $A = B$, Theorem 2.1 becomes

Corollary 2.2. *Let $(M, \langle\ ,\rangle)$ be a complete manifold of dimension m. Fix $o \in M$ and let $r(x) = \mathrm{dist}_{(M,\langle\ ,\rangle)}(x, o)$. Suppose that $A \in C^2(M)$, $A > 0$ and that, for some $n > m, H \geq 0, \delta \in \mathbb{R}$*

$$\mathrm{Ricc}_{n,m}(L_A) \geq -(n-1)H^2\big(1 + r^2\big)^{\delta/2} \tag{2.1}$$

on M. Let $f \in C^1(\mathbb{R})$ and $u \in C^3(M)$ be a global solution of

$$L_A u = f(u). \tag{2.13}$$

Assume the validity of

$$f'(u(x)) \geq (n-1)H^2\big(1 + r(x)^2\big)^{\delta/2} \tag{2.14}$$

and of (2.2) on M for some constants $D > 0$, $\nu \in \mathbb{R}$. Then u is constant provided

$$0 \leq \nu < \min\left\{1, 1 - \frac{\delta}{4}\right\}.$$

Remark. The above Corollary compares to Theorem 2.2 of [Li05] which holds for $\delta = 0, f \equiv 0$.

Interpreting Theorem 2.1 for equation (0.1) we immediately obtain the following

Corollary 2.3. *Let $(M, \langle\ ,\rangle)$ be a complete manifold of dimension m, $a, b \in C^2(M)$, $b > 0$, and suppose that, for some $n > m$, $H \geq 0$,*

$$\mathrm{Ricc}_M + (n-1)H^2 \langle\ ,\ \rangle \geq \mathrm{Hess}(a) + \frac{1}{n-m}\,\mathrm{d}a \otimes \mathrm{d}a. \tag{2.15}$$

Let $f \in C^1(\mathbb{R})$ and $u \in C^3(M)$ be a global solution of (0.1); suppose

$$|u(x)| \leq D(1 + r(x))^{\nu},$$

$$b(x)f'(u(x)) \geq (n-1)H^2,$$

$$|f(u(x))\nabla b(x)| \leq \Theta(1+r(x))^{\theta}$$

on M, for some constants $\Theta, D > 0$, $\nu, \theta \in \mathbb{R}$. Then u is constant provided

$$0 \leq \nu < \min\left\{1, -\frac{\theta}{2}\right\}.$$

Note that the above Corollary in particular implies that under assumption

$$\mathrm{Ricc}_M \geq \mathrm{Hess}(a) + \frac{1}{n-m}\,\mathrm{d}a \otimes \mathrm{d}a.$$

a global solution u of

$$\Delta u + \langle \nabla a, \nabla u \rangle = 0$$

with absolute value of sublinear growth on M has to be constant. In this way, when a is constant we recover a well known result on harmonic functions on a complete manifold first due to Yau, [Yau75]; see also the work of S. Y. Cheng, [Che80], where he proves the analogous result for harmonic maps between Riemannian manifolds. Moreover, as noted by the referee of [MR10a], the above consequence has been previously proved in a non-submitted paper of the Habilitation Thesis of X.-D. Li, which was defended at the Université Paul Sabatier in December 2007. For a family of results in this direction, for example when u is in class L^p, see [PRS05a].

2.2 Consequences of the main Theorem

We now analyze two consequences of Theorem 1.7. First we consider non-negative bounded solutions of (0.1).

Corollary 2.4. *Let $(M, \langle\ ,\ \rangle)$ be a complete manifold of dimension m, $a, b \in C^2(M)$ and suppose that, for some $n > m$, we have the validity of*

$$\mathrm{Ricc}_M \geq \mathrm{Hess}(a) + \frac{1}{n-m}\,\mathrm{d}a \otimes \mathrm{d}a. \tag{2.16}$$

Assume

$$0 < b < C, \qquad |\nabla b| < C \tag{2.17}$$

on M for some constant $C > 0$. Let $f \in C^1(\mathbb{R})$ with $f' \geq 0$, $f \geq 0$ on $[0, +\infty)$ and let u be a non-negative bounded global solution of

$$\Delta u + \langle \nabla a, \nabla u \rangle = bf(u). \tag{0.1}$$

Then u is constant.

Proof. First note that, since u is bounded, by Corollary 1.5 with the choices $A = e^a$, $B = be^a$ and $\delta = 0$ (for instance) we have that u has bounded gradient and also that $\inf_M |\nabla u| = 0$. Thus, assumptions (1.57) i), ii) of Theorem 1.7 are satisfied. With the above choices of A and B, the validity of (0.1) implies that u satisfies (1.1). We now choose $h(x) \equiv 0$ and

$$Q(t) = \int_0^t \left[f(s) - f\left(\sup_M u \right) \right] ds.$$

With these choices one easily verifies the validity of (Q); the remaining assumptions, that is (1.56) and (1.55), follow respectively from (2.16) with the choice $H = 0$ in (1.56) and (2.17). Now conclusion (1.58) of Theorem 1.7 becomes $|\nabla u|^2 \equiv 0$ on M; hence the result. □

As a second consequence (see also Theorem 1.8) we have

Corollary 2.5. *Let $(M, \langle\ ,\ \rangle)$ be a complete manifold of dimension m, $a, b \in C^2(M)$ satisfying*

$$\begin{cases} 0 < b < C; \\ |\nabla b| < C; \\ |\Delta b + \langle \nabla a, \nabla b \rangle| \leq Cb \end{cases} \tag{2.18}$$

on M for some constant $C > 0$. Assume that, for some $n > m$ and $H > 0$,

$$\mathrm{Ricc}_M + (n-1)H^2 \langle , \rangle \geq \mathrm{Hess}(a) + \frac{1}{n-m} \mathrm{d}a \otimes \mathrm{d}a. \tag{2.15}$$

Let $F \in C^2(\mathbb{R})$, $d > 0$ *and set* $f = F'$. *Define*

$$\Phi(t) = 2F(t) - (n-1)H^2\big(t^2 + dt\big)$$

and suppose that $\Phi(t)$ *is non-negative. If* $u \in C^3(M)$ *is a bounded global solution of* (0.1) *with the property that*

$$d \geq \frac{2}{(n-1)H^2} \sup_M |f(u)| + 2 \sup_M |u| \tag{1.74}$$

and for which there exists $x_0 \in M$ *such that*

$$\Phi(u(x_0)) = 0, \tag{2.19}$$

then u *is constant.*

Proof. We transform (0.1) into (1.1) with $A = e^a$, $B = be^a$. Then (1.73), (1.56), (1.74), (1.75) are satisfied and Theorem 1.8 yields the estimate

$$|\nabla u|^2 \leq b(x)\Phi(u). \tag{2.20}$$

We set $t_0 = u(x_0)$ and

$$\Lambda = \{x \in M : u(x) = t_0\}.$$

Λ is a non empty closed set; if we prove that Λ is open, connectedness of M would imply $M = \Lambda$ and thus u is constant. Now, since $\Phi(t) \geq 0$ and $\Phi(t_0) = 0$, t_0 is an (absolute) minimum: it follows that there exist $\delta > 0$ sufficiently small and $C \geq 0$ such that

$$\Phi(t) \leq C(t - t_0)^2$$

on $(t_0 - \delta, t_0 + \delta)$. Consider now the geodesic ball $B_\delta(x_0)$: for $t \in (-\delta, \delta)$ and $w \in T_{x_0}M$, $|w| = 1$ define

$$\varphi(t) = u\big(\exp_{x_0} tw\big) - u(x_0),$$

where $\exp_{x_0} tw = \gamma_w(t)$ is the unit speed geodesic uniquely determined by x_0 and w. Then, using (2.20),

$$\begin{aligned}\left|\varphi'(t)\right|^2 &= |\langle \nabla u(\gamma_w(t)), \dot{\gamma}_w(t)\rangle|^2 \leq |\nabla u(\gamma_w(t))|^2 \\ &\leq b(\gamma_w(t))\Phi(u(\gamma_w(t))) \leq \left(\sup_{B_\delta(x_0)} b\right) C(u(\gamma_w(t)) - u(x_0))^2 \\ &\leq \widetilde{C}\,\varphi(t)^2.\end{aligned}$$

Since $\varphi(0) = 0$, by Gronwall inequality we deduce $\varphi(t) \equiv 0$ on $(-\delta, \delta)$. Hence, u is constant on $B_\delta(x_0)$, proving that Λ is an open set. $\square$

Chapter 3

More Liouville theorems (and beyond)

Adrian Veidt: I did the right thing, didn't I? It all worked out in the end.
Dr. Manhattan: "In the end"? Nothing ends, Adrian. Nothing ever ends.

Watchmen (1986), DC Comics

In this Chapter we consider the notion of *stability* for solutions of equation (1.1). In the first section we compute the *first* and *second variation* of the *generalized energy functional* associated to equation (1.1) and we define the notion of *L-stability* of global solutions, which generalizes the concept of stability for a global solution of $\Delta u = f(u)$ (see, for example, [FCS80], [MP78], [DF09], [FSV08]). We then relate the L-stability to the non-negativity of the first eigenvalue of an appropriate linear operator, and we exploit this relation to derive a useful and more general version of a theorem of Fisher-Colbrie and Schoen ([FCS80]). In the next section we prove the analogue of Theorem 4.5 in [PRS08] for global stable solutions under a particular condition on f and f', then we deduce a Liouville theorem for L_A-harmonic functions under an L^p condition on their gradient. The last section is devoted to a uniqueness result for equation (0.1), based on a particular form of the weak maximum principle valid for (symmetric) diffusion operators.

3.1 Stable Solutions and Liouville-type Theorems under L^p conditions

3.1.1 First and second variation of the generalized energy functional

The aim of this section is to compute the *first* and *second variation* of the *generalized energy functional* associated to equation (1.1) and to define the notion of L-*stability* of a global solution. First observe that

$$Lu = f(u) \Longleftrightarrow \frac{1}{B}\operatorname{div}(A\nabla u) = f(u) \Leftrightarrow L_A u = \frac{B}{A} f(u). \tag{3.1}$$

Next we let $F \in C^2(M)$ be a function such that $F'(u) = f(u)$; define, for a domain Ω of the weighted manifold $(M, \langle\ ,\ \rangle, \mathrm{d}\mu = A\mathrm{d}\mu_0)$ (see also the next Chapter), the (possibly formal) *generalized energy functional* $\mathcal{E}^L(\Omega, \cdot) : C^1(M) \longrightarrow \mathbb{R} \cup \{+\infty\}$ associated to (3.1)

$$\boxed{\mathcal{E}^L(\Omega, u) := \int_\Omega \left[\frac{1}{2}|\nabla u|^2 + \frac{B}{A}F(u)\right] \mathrm{d}\mu.} \tag{3.2}$$

Let $u_t := u + t\xi$ and $\mathcal{E}^L_t := \mathcal{E}^L(\Omega, u_t)$, with $u \in C^3(M), \xi \in C^\infty_0(\Omega)$, $t \in (-\varepsilon, \varepsilon)$ for some $\varepsilon > 0$. By definition we have

$$\begin{aligned}
\mathcal{E}^L_t &= \int_\Omega \left[\frac{1}{2}|\nabla u_t|^2 + \frac{B}{A}F(u_t)\right] \mathrm{d}\mu \\
&= \int_\Omega \left[\frac{1}{2}\Big(|\nabla u|^2 + 2t\langle \nabla u, \nabla \xi\rangle + t^2|\nabla \xi|^2\Big) + \frac{B}{A}F(u + t\xi)\right] \mathrm{d}\mu,
\end{aligned}$$

then

$$\frac{\mathrm{d}\mathcal{E}^L_t}{\mathrm{d}t} = \int_\Omega \left[\frac{1}{2}\Big(2\langle \nabla u, \nabla \xi\rangle + 2t|\nabla \xi|^2\Big) + \frac{B}{A}F'(u + t\xi)\xi\right] \mathrm{d}\mu. \tag{3.3}$$

Now, using the divergence theorem,(3.3) implies

$$\begin{aligned}\left.\frac{\mathrm{d}\mathcal{E}_t^L}{\mathrm{d}t}\right|_{t=0} &= \int_\Omega [\operatorname{div}(\xi A\nabla u) - \xi \operatorname{div}(A\nabla u) + Bf(u)\xi]\,\mathrm{d}\mu_0 \\ &= \int_\Omega \Big(-\frac{1}{A}\operatorname{div}(A\nabla u) + \frac{B}{A}f(u)\Big)\xi\,\mathrm{d}\mu.\end{aligned}$$

Thus the **first variation formula** for the generalized energy functional $\mathcal{E}^L$ is

$$\begin{aligned}\left.\frac{\mathrm{d}\mathcal{E}_t^L}{\mathrm{d}t}\right|_{t=0} &= \int_\Omega \Big(-\frac{1}{A}\operatorname{div}(A\nabla u) + \frac{B}{A}f(u)\Big)\xi\,\mathrm{d}\mu, \\ \left.\frac{\mathrm{d}\mathcal{E}_t^L}{\mathrm{d}t}\right|_{t=0} &= 0\ \forall\,\xi \in C_0^\infty(\Omega) \Leftrightarrow u \text{ is a solution of (3.1) on } \Omega.\end{aligned}$$

From (3.3) we also deduce that

$$\frac{\mathrm{d}^2\mathcal{E}_t^L}{\mathrm{d}t^2} = \int_\Omega \Big[|\nabla\xi|^2 + \frac{B}{A}f'(u+t\xi)\xi^2\Big]\,\mathrm{d}\mu, \tag{3.4}$$

and, accordingly, the **second variation formula** for the generalized energy functional $\mathcal{E}^L$ is

$$\boxed{\left.\frac{\mathrm{d}^2\mathcal{E}_t^L}{\mathrm{d}t^2}\right|_{t=0} = \int_\Omega \Big[|\nabla\xi|^2 + \frac{B}{A}f'(u)\xi^2\Big]\,\mathrm{d}\mu.} \tag{3.5}$$

The second variation formula is the starting point for the following

Definition 3.1. A global solution u of (3.1) is said to be ***L*-stable**[1] if

$$\int_M \Big[|\nabla\xi|^2 + \frac{B}{A}f'(u)\xi^2\Big]\,\mathrm{d}\mu \geq 0 \qquad \forall\,\xi \in C_0^\infty(M). \tag{3.6}$$

If we now define the linear operator $\mathcal{L}_u$ associated to a global solution u of (3.1) by

$$\mathcal{L}_u = -L_A + \frac{B}{A}f'(u), \tag{3.7}$$

and we consider the usual variational characterization of its *first eigenvalue*

[1] L_A-*stable* if $A = B$.

$\lambda_1^{\mathcal{L}_u}(M)$, that is

$$\lambda_1^{\mathcal{L}_u}(M) = \inf_{\varphi\in C_0^\infty,\,\varphi\not\equiv 0} \frac{(\mathcal{L}_u\varphi,\varphi)_{L^2(M,\mathrm{d}\mu)}}{\|\varphi\|^2_{L^2(M,\mathrm{d}\mu)}} = \inf_{\varphi\in C_0^\infty,\,\varphi\not\equiv 0} \frac{\int_M \left[|\nabla\varphi|^2 + \frac{B}{A}f'(u)\varphi^2\right]\mathrm{d}\mu}{\int_M \varphi^2\,\mathrm{d}\mu}, \tag{3.8}$$

we immediately deduce that

$$\boxed{\lambda_1^{\mathcal{L}_u}(M) \geq 0 \qquad \Longleftrightarrow \qquad u \text{ is a global } L-\text{stable solution of (3.1).}}$$

3.1.2 A "Fisher-Colbrie - Schoen type" result

In this section we prove a more general version of a theorem of Fisher-Colbrie and Schoen (see [FCS80], and also [MP78]).

Theorem 3.2. *Let $(M,\langle\ ,\rangle)$ be a Riemannian manifold and $\Omega\subseteq M$ a domain; assume $A\in C^2(M), A>0$, $q\in L^\infty_{loc}(M)$ and let*

$$\mathfrak{L} = -L_A + q(x).$$

The following facts are equivalent:

(i) There exists $w\in C^1(M), w>0$, weak solution on Ω of

$$L_A w - q(x)w = 0; \tag{3.9}$$

(ii) There exists $\varphi\in H^1_{loc}(M), \varphi>0$, weak solution on Ω of

$$L_A\varphi - q(x)\varphi \leq 0; \tag{3.10}$$

(iii) $\lambda_1^{\mathfrak{L}}(M)\geq 0$.

Proof. The proof of the theorem follows from a slight modification of the arguments in [MP78] and [FCS80]. An alternative approach (see [Li05] and [Vol]) starts from the following observation: if we consider the multiplication map $M_{\sqrt{A}} : (L^2(M),\mathrm{d}\mu)\to(L^2(M),\mathrm{d}\mu_0)$, $M_{\sqrt{A}}(u) = \sqrt{A}u$, then we have that $\mathfrak{L}$ is unitarily equivalent to the Schrödinger operator $\mathfrak{H} = -\Delta + p + q$, with $p = \frac{\Delta A}{2A} - \frac{|\nabla A|^2}{4A^2}$, i.e.

$$\left(M_{\sqrt{A}} \circ \mathfrak{L} \circ M_{\sqrt{A}}{}^{-1} \right)(v) = \mathfrak{H}(v).$$

The result then follows applying Lemma 3.10 in [PRS08]. □

From Theorem 3.2 and classical regularity results we deduce the following

Corollary 3.3. *Let $(M, \langle\ ,\rangle)$ be a Riemannian manifold and $\Omega \subseteq M$ a domain; let $A, B \in C^2(M), A, B > 0$ and $u \in C^3(M)$ be a global solution of*

$$Lu = f(u) \Longleftrightarrow L_A u = \frac{B}{A} f(u). \tag{3.1}$$

Then, the following facts are equivalent:

(i) There exists $w \in C^2(M), w > 0$, solution on $(M, \langle\ ,\rangle, d\mu)$ of

$$-\mathcal{L}_u w = L_A w - \frac{B}{A} f'(u) w = 0; \tag{3.11}$$

(ii) There exists $\varphi \in C^2(M), \varphi > 0$, solution on $(M, \langle\ ,\rangle, d\mu)$ of

$$-\mathcal{L}_u \varphi = L_A \varphi - \frac{B}{A} f'(u) \varphi \le 0; \tag{3.12}$$

(iii) u is a global L-stable solution of (3.1) (equivalently: $\lambda_1^{\mathcal{L}_u}(M) \ge 0$).

3.1.3 Liouville theorems for stable solutions under conditions on f and f'

In this section, adapting techniques of S. Pigola, M. Rigoli and A. G. Setti (see [PRS08] and also [PRS05b]), we prove the analogue of Theorem 4.5 in [PRS08] for global stable solutions of the diffusion Poisson equation under a particular condition on f and f'. First we recall

Theorem 3.4. *([PRS08], Theorem 4.1) Let $(M, \langle\ ,\ \rangle)$ be a complete manifold. Assume that $0 < \varphi \in L^\infty_{loc}(M)$ and $\psi \in L^\infty_{loc}(M) \cap W^{1,2}_{loc}(M)$ satisfy*

$$\psi \operatorname{div}(\varphi \nabla \psi) \geq 0, \qquad \text{weakly on } M. \tag{3.13}$$

If, for some $p > 1$,

$$\left(\int_{\partial B_r} |\psi|^p \varphi \, d\mu_0 \right)^{-1} \notin L^1(+\infty) \tag{3.14}$$

then ψ is constant.

Next we consider the two non-negative functions

$$\widetilde{\varphi} := A \varphi^{2\widetilde{\beta}/H}, \tag{3.15}$$

$$\psi := \varphi^{-\widetilde{\beta}/H} v^{\widetilde{\beta}}, \tag{3.16}$$

for some constants $H > 0, \widetilde{\beta} > 1$ and functions $\varphi, v \in C^2(M), \varphi > 0, v \geq 0$. A long but straightforward calculation shows that, on the set $\{v \neq 0\} \subseteq M$,

$$\begin{aligned} \operatorname{div}(\widetilde{\varphi} \nabla \psi) =& \varphi^{\widetilde{\beta}/H} \widetilde{\beta} v^{\widetilde{\beta}-2} A \cdot \\ & \cdot \left[v L_A v + (\widetilde{\beta} - 1) |\nabla v|^2 - \frac{v^2}{H} \varphi^{-1} L_A \varphi + \frac{v^2}{H} \left(1 - \frac{\widetilde{\beta}}{H} \right) \frac{|\nabla \varphi|^2}{\varphi^2} \right]. \end{aligned} \tag{3.17}$$

We are now ready to prove the analogue, in the present setting, of Theorem 4.5 in [PRS08].

Theorem 3.5. *Let $(M, \langle\ ,\ \rangle)$ be a complete manifold, $A, B \in C^2(M)$, $f \in C^1(\mathbb{R})$. Let $u \in C^3(M), u \geq 0$ be a global solution of*

$$Lu = f(u). \tag{3.1}$$

Suppose that

$$Hf(t) - f'(t)t \geq 0 \tag{3.18}$$

for $t \geq 0$ *and some* $H \geq 1$. *If* $\varphi \in C^2(M)$ *is a positive solution of*

$$-\mathcal{L}_u \varphi = L_A \varphi - \frac{B}{A} f'(u)\varphi \leq 0 \qquad \textit{on } M, \tag{3.12}$$

then there exists a constant $C \geq 0$ *such that*

$$C\varphi = u^H, \tag{3.19}$$

provided

$$\left(\int_{\partial B_r} |u|^{2\widetilde{\beta}}\, d\mu \right)^{-1} \notin L^1(+\infty) \tag{3.20}$$

for $1 \leq \widetilde{\beta} \leq H$.

Remark. Conditions similar to (3.18) are not new in the study of Poisson-type PDEs; see for instance the work of Tertikas, [Ter92], [Ter95].

Proof. From (3.17) with $v = u$ we deduce, using (3.12) and (3.18),

$$\begin{aligned}
\operatorname{div}(\widetilde{\varphi}\nabla\psi) = {} & \varphi^{\widetilde{\beta}/H}\widetilde{\beta} v^{\widetilde{\beta}-2} A \left[\frac{B}{A} u f(u) + (\widetilde{\beta}-1)|\nabla u|^2 - \frac{u^2}{H}\frac{B}{A} f'(u) + \right. \\
& \left. + \frac{u^2}{H}\left(1 - \frac{\widetilde{\beta}}{H}\right)\frac{|\nabla\varphi|^2}{\varphi^2} \right] \\
= {} & \varphi^{\widetilde{\beta}/H}\widetilde{\beta} v^{\widetilde{\beta}-2} A \left\{ \frac{1}{H}\frac{B}{A} u \big[H f(u) - u f'(u) \big] + (\widetilde{\beta}-1)|\nabla u|^2 + \right. \\
& \left. + \frac{u^2}{H}\left(1 - \frac{\widetilde{\beta}}{H}\right)\frac{|\nabla\varphi|^2}{\varphi^2} \right\} \geq 0.
\end{aligned}$$

Since

$$|\psi|^p \widetilde{\varphi} = \varphi^{-\frac{\widetilde{\beta}}{H}p} u^{\widetilde{\beta}p} A \varphi^{\frac{2\widetilde{\beta}}{H}} = u^{\widetilde{\beta}p} A \varphi^{\frac{\widetilde{\beta}}{H}(2-p)},$$

(3.20) implies (3.14) with $p = 2$, and ψ is constant by Theorem 3.4. Equality (3.19) now follows at once. $\square$

From Theorem 3.5 we deduce a Liouville theorem for nonnegative global L-stable solutions of equation (3.1).

Corollary 3.6. *Suppose that* (3.18) *and* (3.20) *hold with* $H = \widetilde{\beta} = 1$*. If*

$$f(t) - f'(t)t \not\equiv 0,$$

equation (3.1) *has no nontrivial, non-negative global L-stable solutions.*

Proof. By contradiction, suppose that there exists $u \geq 0, u \not\equiv 0$, global L-stable solution of (3.1). Then, by Theorem 3.3, there exists $\varphi > 0$, solution of (3.12). Theorem 3.5 now implies the existence of a constant $C > 0$ such that $C\varphi = u^H = u$. The last relation forces u to be strictly positive. Without loss of generality, we can choose $C = 1$. From (3.1) and (3.12) we have

$$L_A u = \frac{B}{A} f(u)$$

and

$$L_A u - \frac{B}{A} f'(u)u \leq 0,$$

then

$$\frac{B}{A}\big(f(u) - f'(u)u\big) \leq 0,$$

contradiction. □

3.1.4 A Liouville theorem for L_A-harmonic functions

In this section we deduce a Liouville theorem for L_A-harmonic functions under an appropriate L^p condition on their gradient. First we recall the following formula which can be found, for instance, in Lemma 2.1 of [Li05]:

Lemma 3.7. *([Li05], Lemma 2.1) Let* $u \in C^3(M)$ *be a solution of* $L_A u = 0$ *on* M*, and let* $n > m = \dim M$ *; then*

$$|\nabla u| L_A |\nabla u| \geq \frac{1}{n-1} |\nabla |\nabla u||^2 + \mathrm{Ricc}_{n,m}(L_A)(\nabla u, \nabla u). \tag{3.21}$$

For a proof based on the generalized Bochner-Weitzenböck formula see Li's paper, pp. 1310-1311, and the Appendix.

The previous estimate is the key tool for our next

Theorem 3.8. *Let $(M, \langle\ ,\ \rangle)$ be a complete, non-compact Riemannian manifold, and suppose that*

$$\mathrm{Ricc}_{n,m}(L_A) \geq a(x) \tag{3.22}$$

for some $a(x) \in C^0(M)$. Let $\varphi > 0$, $\varphi \in C^2(M)$ be a solution of

$$L_A\varphi - Ha(x)\varphi \leq 0, \ \ H > \frac{n-2}{n-1}. \tag{3.23}$$

Then every solution $u \in C^3(M)$ of

$$L_A u = 0 \qquad on\ \ M$$

for which

$$|\nabla u| \in L^{2\beta}(M, d\mu) \tag{3.24}$$

is constant provided

$$\frac{n-2}{n-1} \leq \beta \leq H. \tag{3.25}$$

Proof. Set $v := |\nabla u|$; then, using (3.22), (3.21) rewrites as

$$vL_A v - a(x)v^2 \geq \frac{1}{n-1}|\nabla v|^2. \tag{3.26}$$

If we now choose $\widetilde{\beta} = \beta$, $\widetilde{\varphi} := A\varphi^{2\beta/H}$ and $\psi := \varphi^{-\beta/H}v^\beta$, a straightforward calculation shows that the expression (3.17) is nonnegative under condition (3.25), while (3.24) assures the validity of (3.14) with $p = 2$. Theorem 3.4 now implies that ψ is constant, i.e.

$$v^H = C\varphi,$$

for some $C \geq 0$. If $v \equiv 0$, then $|\nabla u| \equiv 0$ and u is constant; suppose then $v \not\equiv 0$. Since $\varphi > 0$, necessarily $v > 0$ and $C > 0$ as well (compare with the proof of Corollary 3.6), so we may choose without loss of generality $C \equiv 1$, i.e. $v^H = \varphi$. Equation (3.23) then implies

$$L_A v^H - Ha(x)v^h \leq 0, \tag{3.27}$$

which can be rewritten, using the diffusion property and (3.26), as

$$\begin{aligned}
0 &\geq Hv^{H-1}L_A v + H(H-1)v^{H-2}|\nabla v|^2 - Ha(x)v^H \\
&= Hv^{H-2}\Big\{vL_A v + (H-1)|\nabla v|^2 - a(x)v^2\Big\} \\
&\geq Hv^{H-2}\Big\{\frac{1}{n-1}|\nabla v|^2 + (H-1)|\nabla v|^2\Big\} \\
&= Hv^{H-2}\Big(H - \frac{n-2}{n-1}\Big)|\nabla v|^2 \geq 0.
\end{aligned}$$

This in turn implies

$$|\nabla v| \equiv 0,$$

so $v \equiv \widetilde{C}$, with $\widetilde{C}$ a positive constant, and $\varphi = \widetilde{C}^H$. From (3.26) we have

$$a(x) \leq 0,$$

while from (3.23) we deduce

$$a(x) \geq 0,$$

so, necessarily, $a(x) \equiv 0$ and

$$\mathrm{Ricc}_{m,n}(L_A) \geq 0. \tag{3.28}$$

By a mild generalization of a result of Calabi and Yau (see the Appendix), (3.28) implies

$$\int_M \mathrm{d}\mu = \mu(M) = +\infty,$$

so (3.24) forces

$$\widetilde{C} \equiv 0,$$

i.e. $|\nabla u| \equiv 0$. □

3.2 A uniqueness result

The aim of this section is to look for a uniqueness result for solutions of the equation

$$\Delta u + \langle \nabla a, \nabla u \rangle = bf(u). \tag{0.1}$$

We observe that, setting $A = e^a$, the above takes the form

$$L_A u = bf(u) \tag{3.29}$$

where, as before, $L_A = \frac{1}{A} \operatorname{div}(A\nabla \ )$. We shall thus concentrate on (3.29). To achieve our goal we first recall the following form of the weak maximum principle valid for (symmetric) diffusion operators. For a proof we refer to Theorem 5.1 of [MRS10].

Theorem 3.9. *Let $(M, \langle \ , \rangle)$ be a complete manifold, $A \in C^2(M)$, $A > 0$ on M. Given $\sigma, \mu \in \mathbb{R}$ suppose*

$$\sigma \geq 0, \qquad \eta = 2 - \sigma - \mu > 0. \tag{3.30}$$

Assume that

$$\liminf_{r \to +\infty} \frac{\log \int_{B_r} A \, \mathrm{d}x}{r^\eta} = d_0 < +\infty. \tag{3.31}$$

Let $u \in C^2(M)$ and suppose that

$$\widehat{u} = \limsup_{r(x) \to +\infty} \frac{u(x)}{r(x)^\sigma} < +\infty. \tag{3.32}$$

Then, given $\gamma \in \mathbb{R}$ such that

$$\Omega_\gamma = \{x \in M : u(x) > \gamma\} \neq \emptyset,$$

we have

$$\inf_{\Omega_\gamma} [1 + r(x)]^\mu L_A u \leq d_0 \max\{\widehat{u}, 0\}\, C(\sigma, \mu), \tag{3.33}$$

with

$$C(\sigma,\mu) = \begin{cases} 0 & \text{if} \quad \sigma = 0, \\ \eta^2 & \text{if} \quad \sigma > 0,\ \mu + 2(\sigma - 1) < 0, \\ \sigma\eta & \text{if} \quad \sigma > 0,\ \mu + 2(\sigma - 1) \geq 0. \end{cases}$$

We now state our uniqueness result:

Theorem 3.10. *Let* $(M, \langle\ ,\ \rangle)$ *be a complete manifold,* $a \in C^2(M), b \in C^0(M)$*, with* $b > 0$ *on* M*, and*

$$\liminf_{r(x)\to+\infty} \frac{b(x)}{r(x)^\beta} > 0 \tag{3.34}$$

for some $\beta \in \mathbb{R}$*. Let* $f \in C^1((0,+\infty)) \cap C^0([0,+\infty))$ *satisfy*

$$\begin{cases} i) \ \ \dfrac{f(t)}{t^\sigma} \text{ is non decreasing on } (0,+\infty); \\ ii) \ \ \liminf\limits_{t\to0^+} \dfrac{f(t)}{t^\sigma} > 0; \\ iii) \ \ \limsup\limits_{t\to+\infty} \dfrac{f'(t)}{t^\sigma} < +\infty \end{cases} \tag{3.35}$$

for some $\sigma > 1$*. Let* $\tau \geq 0$ *and suppose*

$$\liminf_{r\to+\infty} \frac{\log \int_{B_r} e^{a(x)}\,\mathrm{d}x}{r^{2+\beta+\tau(\sigma-1)}} < +\infty. \tag{3.36}$$

Then, the equation (0.1) *has at most one non-negative global solution* $u \in C^2(M)$ *satisfying*

$$C^{-1}r(x)^\tau \leq u(x) \leq Cr(x)^\tau \tag{3.37}$$

for $r(x) \gg 1$ *and some constant* $C > 0$*.*

Note that (3.37) does not assign the asymptotic behaviour of the solution.

Theorem 3.10 is an immediate consequence of the following comparison result:

Theorem 3.11. *Let* $(M, \langle\ ,\ \rangle)$ *be a complete manifold,* $A \in C^2(M)$*,* $b \in C^0(M)$*,* $A, b > 0$ *on* M*. Let* $u, v \in C^2(M)$ *be global, non-negative solutions* *of*

$$L_A u - b(x)f(u) \geq 0 \geq L_A v - b(x)f(v) \tag{3.38}$$

satisfying for some $\tau \geq 0$

$$\begin{cases} i) \;\; \liminf_{r(x)\to+\infty} \frac{v(x)}{r(x)^\tau} > 0; \\ ii) \;\; \limsup_{r(x)\to+\infty} \frac{u(x)}{r(x)^\tau} < +\infty. \end{cases} \tag{3.39}$$

Suppose

$$\liminf_{r(x)\to+\infty} \frac{b(x)}{r(x)^\beta} > 0 \tag{3.40}$$

for some $\beta \in \mathbb{R}$. *Furthermore assume* $f \in C^1(\mathbb{R}^+) \cap C^0(\mathbb{R}^+_0)$,

$$\begin{cases} i) \;\; t^{-\sigma} f(t) \quad \textit{non-decreasing on } \mathbb{R}^+, \\ ii) \;\; \liminf_{t\to 0^+} t^{-\sigma} f(t) > 0, \\ iii) \;\; \limsup_{t\to+\infty} t^{-\sigma} f'(t) < +\infty, \end{cases} \tag{3.41}$$

for some $\sigma > 1$. *If*

$$\liminf_{r\to+\infty} \frac{\log \int_{B_r(o)} A \,\mathrm{d}x}{r^{2+\beta+\tau(\sigma-1)}} < +\infty, \tag{3.42}$$

then $u \leq v$ *on* M.

Proof. The argument, *mutatis mutandis*, follows the same lines of that in [RZ07]. We report it here with the necessary modifications for the convenience of the reader. First of all let $u(x) \not\equiv 0$, otherwise there is nothing to prove. Next, we claim that $v(x) > 0$ on M. Indeed, by contradiction suppose $v(x_0) = 0$ for some $x_0 \in M$; then from (3.38), the strong maximum principle and connectedness of M we deduce $v \equiv 0$ contradicting (3.39) i). This fact, $u \not\equiv 0$ and (3.39) imply that

$$\zeta = \sup_M \frac{u(x)}{v(x)}$$

satisfies

$$0 < \zeta < +\infty.$$

If $\zeta \le 1$, then $u(x) \le v(x)$ on M. Let us assume, by contradiction, $\zeta > 1$ and define

$$\varphi(x) = u(x) - \zeta v(x).$$

Note that $\varphi \le 0$ on M. We claim

$$\sup_M \frac{\varphi(x)}{r(x)^\tau} = 0. \tag{3.43}$$

Indeed, let $\{x_n\} \subset M$ be a sequence realizing ζ. Then

$$\frac{\varphi(x_n)}{r(x_n)^\tau} = \frac{v(x_n)}{r(x_n)^\tau}\left\{\frac{u(x_n)}{v(x_n)} - \zeta\right\}. \tag{3.44}$$

Now observe that $\frac{v(x_n)}{r(x_n)^\tau}$ is bounded, because otherwise (3.39) ii) would imply $\zeta = 0$. From (3.44) it thus follows $\frac{\varphi(x_n)}{r(x_n)^\tau} \to 0$ as $n \to +\infty$, proving (3.43). We now use (3.38) to obtain

$$L_A\varphi \ge b(x)[f(u) - f(\zeta v)] + b(x)[f(\zeta v) - \zeta f(v)]. \tag{3.45}$$

We define

$$h(x) = \begin{cases} f'(u(x)) & \text{if } u(x) = \zeta v(x) \\ \dfrac{1}{u(x) - \zeta v(x)} \displaystyle\int_{\zeta v(x)}^{u(x)} f'(t)\, dt & \text{if } u(x) \ne \zeta v(x) \end{cases}$$

Note that h is continuous and non-negative on M. Furthermore, since as we have already observed $v(x)r(x)^{-\tau}$ is bounded above, (3.39), (3.41) iii) and the mean value theorem imply

$$h(x) \le C(1 + r(x))^{\sigma\tau} \qquad \text{on } M \tag{3.46}$$

for some constant $C > 0$. To simplify the writing let $g(t) = t^{-\sigma} f(t)$. Then, using (3.41),

$$f(\zeta v) - \zeta f(v) = v^\sigma \zeta\left[\zeta^{\sigma-1} g(\zeta v) - g(v)\right] \ge v^\sigma \zeta\left(\zeta^{\sigma-1} - 1\right) g(v) \ge v^\sigma \zeta\left(\zeta^{\sigma-1} - 1\right) g(0^+).$$

Inserting into (3.45), since $b(x) > 0$ we obtain

$$L_A\varphi \geq b(x)C(1+r(x))^{\sigma\tau}\varphi + b(x)g(0^+)v^\sigma\zeta\big(\zeta^{\sigma-1}-1\big).$$

On the other hand, by (3.41) ii), $g(0^+) = C_1 > 0$ and by (3.39)

$$v^\sigma(x) \geq D(1+r(x))^\tau$$

on M for some constant $D > 0$. It follows that

$$(1+r(x))^{-\sigma\tau}\frac{1}{b(x)}L_A\varphi \geq C\varphi + C_2\zeta\big(\zeta^{\sigma-1}-1\big)$$

for some constants $C, C_2 > 0$. We now choose $\varepsilon > 0$ so small that on $\Omega_\varepsilon = \{x \in M : \varphi(x) > -\varepsilon\}$ we have

$$C\varphi(x) > -\frac{1}{2}C_2\zeta\big(\zeta^{\sigma-1}-1\big).$$

Then, on Ω_ε, $L_A\varphi \geq 0$ and using (3.40) we obtain

$$(1+r(x))^{-\sigma\tau-\beta}L_A\varphi \geq \frac{1}{2}C_2\zeta\big(\zeta^{\sigma-1}-1\big) \quad \text{on } \Omega_\varepsilon.$$

It follows that, since $\zeta > 1$ and $\sigma > 1$,

$$\inf_{\Omega_\varepsilon}(1+r(x))^{-\sigma\tau-\beta}L_A\varphi > 0.$$

This fact, together with (3.42), contradicts Theorem 3.9. $\square$

Chapter 4

Geometric Applications

> You know what sticks people to something? The desire to know how it's all going to end.
>
> Loki, *Sandman*, Season of Mists, DC Comics-Vertigo

In this final Chapter we definitely shift from analysis toward geometry. Our main purpose is to prove triviality results for complete Einstein warped products, exploiting the relations between these latter and the *quasi-Enstein manifolds*, a generalization of Ricci solitons (we refer to the next sections for details). After a detailed introduction, where we present the relevant geometrical objects (the f-Laplacian Δ_f, Ricc_f^k, ...) and discuss the recent literature on the subject, in the second Section we adapt to this new scenario two results from Chapter 2. In the third Section we prove a weighted version of Theorem 1.31 in [PRS05a] and a sufficient condition for the validity of the full Omori-Yau maximum principle for the f-Laplacian, and then we deduce a triviality result for complete Einstein warped products which is a Corollary of Theorem 1 in [Rim10]. In the final Section we prove a further gradient estimate, which extend the one in [Cas10], and we obtain another triviality result when the function f (related to the warping function u by $u = e^{-f/k}$) is bounded below by a constant depending on $m = \dim M, k$ and the Einstein constants λ and μ, respectively of the warped product and of the fibre. For other triviality result we refer to the paper [MR10b].

4.1 The Geometry

4.1.1 Einstein warped products

Our reference for this Section is the classical book by O'Neill [O'N83]. Let (M^m, g_M) and $\left(F^k, g_F\right)$ be two Riemannian manifolds. The *Riemannian product* $\left(P^{m+k}, g_P\right)$ is the Riemannian manifold $P^{m+k} = M^m \times F^k$ endowed with the product metric $g_P = \pi^* g_M + \sigma^* g_F$, where π and σ are the canonical projections $\pi : M \times F \to M$, $\pi(x,q) = x$ and $\sigma : M \times F \to F$, $\sigma(x,q) = q$ for all $(x,q) \in M \times F$. We can construct a wide class of metrics on $M \times F$ homothetically warping g_P on each fibre $\{x\} \times F$, $x \in M$: see for instance the seminal [BO69], where the authors study manifolds of negative curvature generalizing the concept of surface of revolution.

Definition 4.1. Let (M^m, g_M) and $\left(F^k, g_F\right)$ be two Riemannian manifold, and let $u \in C^\infty(M), u > 0$. The *warped product* $N^{m+k} = M^m \times_u F^k$ is the product manifold $M \times F$ endowed with the metric

$$g_N = \pi^* g_M + (u \circ \pi)^2 \sigma^* g_F. \tag{4.1}$$

M is called the *base* of N, F the *fibre* and u is the *warping function.*

It can be shown that, for all $x \in M$ and $q \in F$, the *fibres* $\{x\} \times F = \pi^{-1}(x)$ and the *leaves* $M \times \{q\} = \sigma^{-1}(q)$ are Riemannian submanifolds of N, and the warped metric satisfies the following properties:

(i) $\forall\, q \in F$, $\pi_{|_{M \times \{q\}}}$ is an isometry onto M;

(ii) $\forall\, x \in M$, $\sigma_{|_{\{x\} \times F}}$ is a positive homothety onto F;

(iii) $\forall\, (x,q) \in N$, the leaf $M \times \{q\}$ and the fibre $\{x\} \times F$ are orthogonal at (x,q).

If $h \in C^\infty(M)$, the *lift* of h to N is $\widetilde{h} = h \circ \pi \in C^\infty(N)$; if $w \in T_x M$ and $q \in F$, then the *lift* $\widetilde{w}$ of w to (x,q) is the (unique) vector in $T_{(x,q)} N$ such that $\pi_*(\widetilde{w}) = w$, while if $W \in \mathfrak{X}(M)$ (where $\mathfrak{X}(M)$ is the set of smooth vector fields on M) the *lift* of W to N is the (smooth) vector field $\widetilde{W}$ whose value at each (x,q) is the lift of W_x to (x,q). In other words, $\widetilde{W}$ is the unique element of $\mathfrak{X}(N)$ that is π-related to W and σ-related to the zero vector field on F. Functions, tangent vectors and vector fields on F are lifted to N in the same way using the projection σ. A vector X tangent to a leaf (i.e. $\sigma_*(X) = 0$) is called *horizontal*, while a vector V tangent to a fibre (i.e. $\pi_*(V) = 0$) is called *vertical*. We have the following

Lemma 4.2. *([O'N83], Lemma 7.34) If $h \in C^\infty(M)$, then ${}^N\nabla(h \circ \pi)$, the gradient on N of the lift $h \circ \pi$, is the lift to N of $\nabla^M h$, the gradient of h on M.*

Proof. We have to show that ${}^N\nabla(h \circ \pi)$ is horizontal and π-related to $\nabla^M h$. If V is a vertical tangent vector to N, then $g_N\big({}^N\nabla(h\circ\pi), V\big) = V(h \circ \pi) = \pi_*(V)(h) = 0$, since $\pi_*(V) \equiv 0$. Thus ${}^N\nabla(h \circ \pi)$ is horizontal. If X is horizontal,

$$\begin{aligned} g_M\big(\pi_*\big({}^N\nabla(h\circ\pi)\big), \pi_*(X)\big) &= g_N\big({}^N\nabla(h\circ\pi), X\big) = X(h\circ\pi) = \\ &= \pi_*(X)(h\circ\pi) = g_M\big(\nabla^M h, \pi_*(X)\big). \end{aligned}$$

Hence at each point $\pi_*\big({}^N\nabla(h\circ\pi)\big) = \nabla^M h$, i.e. ${}^N\nabla(h\circ\pi)$ is π-related to $\nabla^M h$. □

The previous Lemma allows us to simplify the notation by writing h for $h \circ \pi$ and ∇h for ${}^N\nabla(h \circ \pi)$.

Denote now by ${}^M\operatorname{Ricc}, {}^F\operatorname{Ricc}, {}^N\operatorname{Hess}(u)$ the lifts to N (i.e., the pullbacks *via* π) of the $(0,2)$-tensors $\operatorname{Ricc}_M, \operatorname{Ricc}_F$ and $\operatorname{Hess}(u)$ respectively. We have the

Proposition 4.3. *([O'N83], Corollary 7.43) If Ricc_N is the Ricci curvature of the warped product $N^{m+k} = M^m \times_u F^k$, X, Y horizontal vector fields and U, V vertical vector fields, then:*

1. $\operatorname{Ricc}_N(X,Y) = {}^M\operatorname{Ricc}(X,Y) - \frac{k}{u}\,{}^N\operatorname{Hess}(u)(X,Y)$;

2. $\operatorname{Ricc}_N(X,U) = 0$*;*

3. $\operatorname{Ricc}_N(U,V) = {}^F\operatorname{Ricc}(U,V) - g_N(U,V)u^\sharp$*,*

where $u^\sharp = \frac{\Delta u}{u} + \frac{k-1}{u^2}|\nabla u|^2$ *and* Δ *is the Laplacian on* M*.*

Note that, by Lemma 4.2,

$$|\nabla u|^2 = g_M\big(\nabla^M u, \nabla^M u\big) = g_N\big({}^N\nabla(u\circ h),\, {}^N\nabla(u\circ h)\big).$$

We now recall

Definition 4.4. A Riemannian manifold (M^m, g_M) is called *Einstein* if its Ricci tensor Ricc_M satisfies

$$\operatorname{Ricc}_M = \lambda g_M \tag{4.2}$$

for some $\lambda \in \mathbb{R}$.

Obviously, an *Einstein warped product* is a warped product which is also an Einstein manifold. Proposition 4.3 implies, for $k \geq 3$ (to ensure the validity of Schur's Lemma), the following characterization of Einstein warped products (see [KK03]):

Corollary 4.5. *([KK03], Corollary 3) The warped product* $N^{m+k} = M^m \times_u F^k$ *is Einstein with* $\operatorname{Ricc}_N = \lambda g_N$ *if and only if*

$$\operatorname{Ricc}_M = \lambda g_M + \frac{k}{u}\operatorname{Hess}(u), \tag{4.3}$$

$$(F, g_F) \text{ is Einstein with } \operatorname{Ricc}_F = \mu g_F \text{ for some } \mu \in \mathbb{R} \tag{4.4}$$

and

$$u\Delta u + (k-1)|\nabla u|^2 + \lambda u^2 = \mu. \tag{4.5}$$

In [KK03], Proposition 5, it is also proved that

Proposition 4.6. *If* (M^m, g_M) *is a Riemannian manifold and* $u \in C^\infty(M)$ *satisfies* (4.3) *for* $\lambda \in \mathbb{R}$ *and* $k \in \mathbb{N}$*, then* u *also satisfies* (4.5) *for some constant* $\mu \in \mathbb{R}$*.*

Remark. This result is stated in [KK03] for compact manifolds, but since the proof is local it works also in the general case.

4.1.2 Quasi-Einstein manifolds vs. Einstein warped products

A *weighted manifold*, also known in the literature as a *smooth metric measure space*, is a triple $\left(M^m, g_M, e^{-f}\mathrm{d}\mu_0\right)$, where M^m is a complete m-dimensional Riemannian manifold with metric g_M, $f \in C^\infty(M)$ and $\mathrm{d}\mu_0$, as in the previous Chapters, denotes the canonical Riemannian volume form on M. The Ricci tensor can be naturally extended to weighted manifolds introducing the modified *k-Bakry-Emery Ricci tensor*

$$\mathrm{Ricc}_f^k = \mathrm{Ricc}_M + \mathrm{Hess}(f) - \frac{1}{k}\mathrm{d}f \otimes \mathrm{d}f, \qquad \text{for } 0 < k \leq \infty. \tag{4.6}$$

When f is constant, $\mathrm{Ricc}_f^k \equiv \mathrm{Ricc}_M$, while, if $k = \infty$, $\mathrm{Ricc}_f^k = \mathrm{Ricc}_f$, the usual Bakry-Emery Ricci tensor. For a detailed introduction to weighted manifolds and the k-Bakry-Emery Ricci tensor, we refer to the interesting papers of Wei and Wylie ([WW09], [WW07]) and Li ([Li05]).

In [CSW08] the authors give the following

Definition 4.7. A triple $\left(M^m, g_M, e^{-f}\mathrm{d}\mu_0\right)$ (with M, g_M and f as before) is called a (k)*-quasi-Einstein* or simply a *quasi-Einstein* manifold (and g_M is a *quasi-Einstein metric*) if

$$\boxed{\mathrm{Ricc}_f^k = \lambda g_M} \tag{4.7}$$

for some $\lambda \in \mathbb{R}$.

We note that:

- if $f =$ constant, (4.7) gives the Einstein equation (4.2), and in this case we call the quasi-Einstein metric *trivial*;
- if $k = \infty$, (4.7) is exactly the gradient Ricci soliton equation. In the last years, since the appearance of the seminal works of R. Hamilton [Ham88] and G. Perelman [Per03], the study of Ricci solitons (and of

their generalizations) has become the matter of a rapidly increasing investigation, directed mainly toward problems of *classification* and *triviality*; among the enormous literature on the subject we only quote, as a few examples, the papers [PW09a], [PW09c], [PW09b], [PRS10], [PRRS10], [ENM08].

The case $k \in \mathbb{N}$ is the one we are interested in, because of its relation with Einstein warped product metrics. Indeed, in [CSW08], elaborating on [KK03], it is proved a characterization of quasi-Einstein metrics as base metrics of Einstein warped product metrics. This characterization can be formulated in the following, elegant form (see [Rim10], Theorem 2):

Theorem 4.8. *If $N^{m+k} = M^m \times_u F^k$ is an Einstein warped product with Einstein constant λ, warping function $u = e^{-f/k}$ and Einstein fibre F^k, then the weighted manifold $\left(M^m, g_M, e^{-f}\, d\mu_0\right)$ satisfies the quasi-Einstein equation* (4.7)*; furthermore, the Einstein constant μ of the fibre satisfies the equation*

$$\Delta f - |\nabla f|^2 = k\lambda - k\mu e^{\frac{2}{k}f}. \tag{4.8}$$

Conversely, if the weighted manifold $\left(M^m, g_M, e^{-f}\, d\mu_0\right)$ satisfies (4.7)*, then f satisfies* (4.8) *for some constant $\mu \in \mathbb{R}$. Consider the warped product $N^{m+k} = M^m \times_u F^k$, with $u = e^{-f/k}$, and Einstein fibre F with Einstein constant μ. Then N is Einstein with* $\mathrm{Ricc}_N = \lambda g_N$.

The proof of Theorem 4.8 is a direct consequence of Corollary 4.5 and Proposition 4.6, once we observe that

$$\nabla u = -\frac{1}{k} e^{-\frac{f}{k}} \nabla f,$$

$$\frac{k}{u} \operatorname{Hess}(u) = -\operatorname{Hess}(f) + \frac{1}{k} \mathrm{d}f \otimes \mathrm{d}f,$$

and

$$\frac{k}{u} \Delta u = -\Delta f + \frac{1}{k} |\nabla f|^2.$$

The previous characterization enables us to *study Einstein warped products by focusing only on equation* (4.7) *on the base* (M^m, g_M), which in turn implies (by Proposition 4.6) equation (4.8).

Before we proceed, we need to point out some rather simple, but fundamental, relations that allow us to exploit some of the machinery developed in the previous Chapters also in our new geometrical setting. Following the notation of Petersen and Wylie (see [PW09a], [PW09c], [PW09b]) define, for $f \in C^\infty(M)$ (but C^2 is enough) the *f-Laplacian* Δ_f as

$$\Delta_f u = e^f \operatorname{div}\left(e^{-f}\nabla u\right) = \Delta u - \langle \nabla f, \nabla u\rangle, \qquad u \in C^2(M). \tag{4.9}$$

$\Delta_f f$ is a diffusion-type operator, symmetric on $L^2\big(M, e^{-f}\mathrm{d}\mu_0\big)$, and it coincides with the operator $\mathcal{L}$ defined in [Li05] and quoted in the Introduction. A simple look at equation (4.8) shows that this latter can be rewritten as

$$\boxed{\Delta_f f = k\lambda - k\mu e^{\frac{2}{k}f}.} \tag{4.10}$$

Since, for

$$A = e^{-f} \tag{4.11}$$

and

$$k = n - m, \tag{4.12}$$

Δ_f *is nothing else than the operator* L_A studied before and

$$\boxed{\mathrm{Ricc}_f^k = \mathrm{Ricc}_{n,m}(L_A),} \tag{4.13}$$

we expect that, under appropriate assumptions on f and the geometry of M, some results and techniques of Chapters 1 and 2 can be applied to (4.10). This will be the content of the next Sections.

4.1.3 Examples in the literature

As pointed out in [Cas10], examples of quasi-Einstein manifolds with $\lambda < 0$ and μ of arbitrary sign, or with $\lambda = 0$ and $\mu \geq 0$ are constructed in [Bes08]. Moreover, in the latter case, all non-trivial examples have $\mu > 0$, because the trivial quasi-Einstein metric with $\lambda = 0$ necessarily satisfies $\mu = 0$.

Other non-trivial examples with $\lambda > 0$, $k > 1$ and $\mu > 0$ are constructed in [LPP04]. Since, if $k < \infty$ and $\lambda > 0$, M is necessarily compact (see [Qia97], [WW07]), the maximum principle applied to (4.8) yields that $\mu > 0$ in this situation. Triviality in case $\lambda = 0$ and $\mu \leq 0$ is discussed in [Cas10]; the author proves the following two results[1]:

Theorem 4.9. *([Cas10], Theorem 1.1) Let (M, g) be a complete Riemannian manifold such that $\mathrm{Ricc}_f^k = 0$ for some smooth function f and $0 < k \leq \infty$, and let μ be the constant given by*

$$\Delta_f f = -k\mu e^{\frac{2}{k}f}. \tag{4.14}$$

Then $\mu \geq 0$, and equality holds if and only if (M, g) is Ricci-flat.

Theorem 4.10. *([Cas10], Theorem 1.2) Let (M, g) be a complete Riemannian manifold such that $\mathrm{Ricc}_f^k \geq 0$ for some smooth function f and $0 < k \leq \infty$, and suppose that*

$$\Delta_f f = c_1 e^{c_2 f} \tag{4.15}$$

for constants $c_1, c_2 \geq 0$. Then f is constant.

In [Rim10] M. Rimoldi extends the triviality result of [KK03] for Einstein warped product with nonpositive scalar curvature and compact base to the case of a non-compact base, obtaining the next

Theorem 4.11. *([Rim10], Theorem 1) Let $N^{m+k} = M^m \times_u F^k$ be a complete Einstein warped product with non-positive scalar curvature ${}^N S \leq 0$, warping function $u = e^{-\frac{f}{k}}$ satisfying $\inf_M f = f_* > -\infty$ and complete Einstein fibre F. Then N is a Riemannian product if either one of the following further conditions is satisfied:*

(a) f has a local minimum;

(b) the base manifold M is complete and non-compact, the warping function satisfies $\int_M |f|^p e^{-\frac{f}{k}}\, d\mu_0 < +\infty$ for some $1 < p < +\infty$, and $f(x_0) \leq 0$ for some point $x_0 \in M$.

[1]Note that the second is stronger than the first.

In the next Sections we concentrate on the case (geometrically meaningful, by the previous discussion) $\lambda \leq 0$, $\mu < 0$.

4.2 Two consequences from the first Chapters

In this section we apply some of the results of Chapter 1 and 2 to Einstein warped products. We begin with a consequence of Theorem 2.1.

Theorem 4.12. *Let $N = M^m \times_u F^k$ be a complete Einstein warped product with Einstein constant $\lambda < 0$, warping function $u = e^{-f/k}$ and Einstein fibre F^k with Einstein constant $\mu < 0$. Suppose that*

$$f \geq \frac{k}{2} \log\left(\frac{\lambda}{2\mu}\right) \qquad \text{for all } x \in M \tag{4.16}$$

and that

$$|f| \leq D(1 + r(x))^{\nu} \tag{4.17}$$

for some $D \geq 0$, $\nu \in \mathbb{R}$. Then N is a Riemannian product, provided

$$0 \leq \nu < 1. \tag{4.18}$$

Proof. Since N is an Einstein warped product, from the previous discussions we know that f satisfies (4.10). Now, condition (2.1) is satisfied (with equality sign) for $\delta = 0$ and $\lambda = -(n-1)H^2 = -(m+k-1)H^2$, condition (2.3) is guaranteed by (4.16) and (2.4) is valid for all $\theta \in \mathbb{R}$, since $A = B = e^{-f}$, so we can choose, for instance, $\theta = -2$. Hence f is constant by Theorem 2.1. □

Analogously, as a consequence of Corollary 2.5 we easily deduce

Theorem 4.13. *Let $N = M^m \times_u F^k$ be a complete Einstein warped product with Einstein constant $\lambda < 0$, warping function $u = e^{-f/k}$ and Einstein fibre F^k with Einstein constant $\mu < 0$. Let $\Phi : \mathbb{R}_0^+ \to \mathbb{R}$ be the function*

$$\Phi(t) = 2k\left(\lambda t - \frac{k\mu}{2} e^{\frac{2}{k}t} + C\right) + \lambda\big(t^2 + dt\big) \tag{4.19}$$

for some $C \in \mathbb{R}$, $d > 0$ chosen in such a way that $\Phi \geq 0$. Suppose also that f is non-negative and bounded and that

$$d \geq -\frac{2}{\lambda}\sup_M \left|k\lambda - k\mu e^{\frac{2}{k}f}\right| + 2\sup_M |f|. \tag{4.20}$$

If there exists a point $x_0 \in M$ such that

$$\Phi(f(x_0)) = 0$$

then N is a Riemannian product.

4.3 Triviality under L^p conditions

In the present Section we state a weighted version of Theorem 1.31 in [PRS05a], which can be proved by minor changes to the proof of this latter, and a sufficient condition for the validity of the full Omori-Yau maximum principle for the f-Laplacian; our goal is to deduce a triviality result for complete Einstein warped products, which is a Corollary of Theorem 1 in [Rim10], replacing the integrability assumptions with weight $e^{-\frac{f}{k}}$ in the aforementioned Theorem with more natural conditions. We recall that $(M, \langle\ ,\ \rangle)$ is said to *satisfy the Omori-Yau maximum principle* if for each $u \in C^2(M)$ such that $u^* = \sup_M u < +\infty$ there exists a sequence $\{x_k\} \subset M$ such that

$$(i)\, u(x_k) > u^* - \frac{1}{k}, \quad (ii)\, |\nabla u(x_k)| < \frac{1}{k}, \quad (iii)\, \Delta u(x_k) < \frac{1}{k}$$

for each $k \in \mathbb{N}$. First we have (compare with Theorem 1.31 in [PRS05a]):

Theorem 4.14. *Assume on the complete weighted manifold $(M, g_M, e^{-f}d\mathrm{vol})$ the validity of the Omori-Yau maximum principle for the f-Laplacian. Let $v \in C^2(M)$ be a solution of the differential inequality*

$$\Delta_f v \geq \Phi(v, |\nabla v|),$$

with $\Phi(t,y)$ continuous in t, C^2 in y and such that

$$\frac{\partial^2\Phi}{\partial y^2}(t,y) \geq 0.$$

Set $\varphi(t) = \Phi(t,0)$. Then a sufficient condition to guarantee that

$$v^* = \sup_M v < +\infty$$

is the existence of a continuous function F positive on $[a,+\infty)$ for some $a \in \mathbb{R}$ satisfying

$$\left\{\int_a^t F(s)ds\right\}^{-\frac{1}{2}} \in L^1(+\infty), \tag{4.21}$$

$$\limsup_{t\to+\infty} \frac{\int_a^t F(s)ds}{tF(t)} < +\infty, \tag{4.22}$$

$$\liminf_{t\to+\infty} \frac{\varphi(t)}{F(t)} > 0 \tag{4.23}$$

and

$$\liminf_{t\to+\infty} \frac{\left\{\int_a^t F(s)ds\right\}^{\frac{1}{2}}}{F(t)} \left.\frac{\partial\Phi}{\partial y}\right|_{(t,0)} > -\infty. \tag{4.24}$$

Furthermore in this case

$$\varphi(v^*) \leq 0.$$

Now consider again the equation (4.10) and let $\mu < 0$. If we choose $\varphi(t) = \Phi(t,y) = k\lambda - k\mu e^{\frac{2}{k}t}$ and $F(t) = (t-a)^\sigma$, with $t \in [a,\infty)$ and $\sigma > 1$, then F satisfies the assumptions of Theorem 4.14. However, to use this theorem, we have also to assure on $\left(M^m, g_M, e^{-f}\mathrm{d}\mu_0\right)$ the validity of the Omori-Yau maximum principle. We will use the following result, which is a consequence of Theorem 4.1 in [PRRS10].

Corollary 4.15. *Let $(M^m, g_M, e^{-f}d\mu_0)$ be a complete weighted manifold such that*

$$\mathrm{Ricc}_f^k(\nabla r, \nabla r) \geq -(m+k-1)G(r) \tag{4.25}$$

for a smooth positive function G on $[0,+\infty)$, even at the origin, satisfying

$$\begin{array}{ll} (i)\; G(0) > 0, & (ii)\; G'(t) \geq 0 \;\; on \;\; [0,+\infty), \\ (iii)\, G(t)^{-\frac{1}{2}} \notin L^1(+\infty), & (iv)\; \limsup_{t\to+\infty} \frac{tG\left(t^{\frac{1}{2}}\right)}{G(t)} < +\infty. \end{array} \tag{4.26}$$

Then the Omori-Yau maximum principle for the f-laplacian holds on M.

Proof. Let h be the solution on $\mathbb{R}_0^+$ of the Cauchy problem

$$\begin{cases} h'' - Gh = 0 \\ h(0) = 0; \;\; h'(0) = 1. \end{cases}$$

Then, by Proposition 2.3 in [MRS10], the inequality

$$\Delta_f r \leq -(m+k-1)\frac{h'}{h} \leq C_1 G(r)^{\frac{1}{2}},$$

holds pointwise in $M \setminus (cut(o) \cup \{o\})$ for some constant C_1. Thus

$$\Delta_f r^2 = 2r\Delta_f r + 2 \leq 2 + 2rC_1 G(r)^{\frac{1}{2}} \leq C_2 r G(r)^{\frac{1}{2}}, \tag{4.27}$$

off a compact set, and the hypotheses (4.1), (4.2) and (4.3) of Theorem 4.1 in [PRRS10] are satisfied with $\gamma = r^2$. In that theorem it is also assumed a bound on the gradient of f, but here we don't need this further hypothesis. Indeed by (4.27) we can replace the last part of the proof of Theorem 4.1 in

[PRRS10] with the following computation:

$$
\begin{aligned}
\Delta_f u\left(x_k\right) &= \Delta u\left(x_k\right)-\langle\nabla u, \nabla f\rangle\left(x_k\right) \\
&\leq \frac{\left(u\left(x_k\right)-u(p)+1\right)}{k}\left\{\frac{\varphi'(\gamma(x_k))}{\varphi(\gamma(x_k))}\Delta\gamma(x_k)+\frac{1}{k}\left(\frac{\varphi'(\gamma(x_k))}{\varphi(\gamma(x_k))}\right)^2|\nabla\gamma|^2\left(x_k\right)\right\} \\
&\quad -\frac{\left(u\left(x_k\right)-u(p)+1\right)}{k}\frac{\varphi'(\gamma(x_k))}{\varphi(\gamma(x_k))}\left\langle\nabla\gamma(x_k), \nabla f(x_k)\right\rangle \\
&\leq \frac{\left(u\left(x_k\right)-u(p)+1\right)}{k}\left\{\frac{\varphi'(\gamma(x_k))}{\varphi(\gamma(x_k))}\Delta_f\gamma(x_k)+\frac{1}{k}\left(\frac{\varphi'(\gamma(x_k))}{\varphi(\gamma(x_k))}\right)^2|\nabla\gamma|^2\left(x_k\right)\right\} \\
&\leq \frac{\left(u\left(x_k\right)-u(p)+1\right)}{k}\left\{\frac{c}{\gamma^{1/2}G\left(\gamma^{1/2}\right)^{1/2}}C_2\gamma^{1/2}G\left(\gamma^{1/2}\right)^{1/2}\right. \\
&\quad \left.+\frac{1}{k}\cdot\frac{c^2}{\gamma G\left(\gamma^{1/2}\right)}A^2\gamma\right\},
\end{aligned}
$$

and the RHS tends to zero as $k \to +\infty$. $\square$

Hence, choosing $G(t) = t^2 + \frac{|\lambda|+\varepsilon}{m+k-1}$, for some $\varepsilon > 0$, we obtain the following corollary of Theorem 1 in [Rim10].

Corollary 4.16. *Let $N^{m+k} = M^m \times_u F^k$ be a complete Einstein warped product with non-positive scalar curvature $(m+k)\lambda = {}^N S \leq 0$, warping function $u(x) = e^{-\frac{f(x)}{k}}$ satisfying $\inf_M f = f_* > -\infty$ and complete Einstein fibre F. Suppose also that ${}^F S < 0$. Then N is simply a Riemannian product if either one of the following further conditions is satisfied:*

(i) the base manifold M is complete and non-compact, the warping function satisfies $f \in L^p(M)$, for some $1 < p < +\infty$, and $f\left(x_0\right) \leq 0$ for some point $x_0 \in M$;

(ii) the base manifold M is complete and non-compact, the warping function satisfies $f \in L^p(M)$, for some $1 < p < +\infty$, and the scalar curvatures of M and N satisfy

$$
{}^M S \geq \frac{m}{m+k}{}^N S + \varepsilon,
$$

for some $\varepsilon > 0$.

Proof. Let $\widehat{f} = \frac{f}{k}$. Since $f^* < +\infty$ by Theorem 4.14 and, by assumption, $f_* > -\infty$, the point (i) follows from (b) of Theorem 1 in [Rim10] noting that in this case $\widehat{f}$-weighted volumes are equivalent to Riemannian volumes.
For the same reason, since

$$vol_{\widehat{f}}(M) \leq vol_f(M)e^{\frac{k-1}{k}f^*},$$

we have, from the volume estimates in [Qia97] and by Theorem 9 in [Rim10], that the weak maximum principle at infinity for the $\widehat{f}$-Laplacian holds on M. Hence we can construct a sequence $\{x_n\}$ such that $f(x_n) \to f_*$ and $\Delta_{\widehat{f}} f(x_n) \geq -\frac{1}{n}$. Thus, since tracing (4.7) we have that $\Delta_{\widehat{f}} f = m\lambda - {}^M S$, we obtain that

$$-\frac{1}{n} \leq m\lambda - {}^M S(x_n) \leq m\lambda - {}^M S_* \leq 0,$$

where in the last inequality we have used the estimates of Theorem 3 in [Rim10]. Taking the limit for $n \to +\infty$ we get ${}^M S_* = m\lambda$. Using this, under assumption (ii), since we have that $\inf_M {}^M S > m\lambda$ we conclude the constancy of u. $\square$

4.4 A further gradient estimate and another Liouville-type theorem

In this final Section we prove a further gradient estimate, which extend that in [Cas10], and that allows us to obtain another triviality result when the function f (related to the warping function u by $u = e^{-f/k}$) is bounded below by a constant depending on $m = \dim M, k$ and on the Einstein constants λ and μ, respectively of the warped product and of the fibre.

Theorem 4.17. *Let* $\left(M^m, g, e^{-f}\, d\mu_0\right)$ *be a weighted manifold (not necessarily complete); suppose that, for some* $k < +\infty$, $Z \geq 0$,

$$\mathrm{Ricc}^k_f \geq \lambda = -(m+k-1)Z^2 \tag{4.28}$$

and that

$$\Delta_f f = \psi(f), \tag{4.29}$$

where $\psi : \mathbb{R} \to \mathbb{R}$ *satisfies*

$$\psi'(t) + \frac{2}{m}\psi(t) - (m+k-1)Z^2 \geq 0 \tag{4.30}$$

for all $t \in \mathbb{R}$*. Then for all* $q \in M$ *and* $T > 0$ *such that* $B_T(q)$ *is geodesically connected in* M *and the closure* $\overline{B_T(q)}$ *is compact,*

$$|\nabla f|^2(q) \leq \frac{1}{G(m \parallel k)}\left[\frac{2(m+k+6)}{T^2} - \frac{4\sqrt{3}}{9}\frac{\lambda}{Z}\frac{1}{T}\right], \tag{4.31}$$

having defined

$$G(m \parallel k) := \frac{1}{m} + \frac{1}{k}.$$

Proof. Let $\lambda := -(m+k-1)Z^2$, so that (4.28) and (4.30) become, respectively,

$$\mathrm{Ricc}_f^k \geq \lambda \tag{4.32}$$

and

$$\psi'(t) + \frac{2}{m}\psi(t) + \lambda \geq 0. \tag{4.33}$$

The Bockner-Weitzenböck formula for L_A (see the Appendix) tells us that, if $u \in C^3(M)$, then

$$\frac{1}{2}\Delta_f|\nabla u|^2 = |\mathrm{Hess}(u)|^2 + \langle \nabla u, \nabla \Delta_f u\rangle + \mathrm{Ricc}_f^k\left(\nabla u, \nabla u\right) + \frac{1}{k}\left\langle \nabla f, \nabla u\right\rangle^2.$$

Applying the previous formula to f and using (4.32), (4.33), Newton inequalities and $\Delta f = \Delta_f f + |\nabla f|^2$ we obtain

$$\begin{aligned}
\frac{1}{2}\Delta_f|\nabla f|^2 &= |\mathrm{Hess}(f)|^2 + \langle \nabla f, \nabla \Delta_f f\rangle + \mathrm{Ricc}_f^k\,(\nabla f, \nabla f) + \frac{1}{k}|\nabla f|^2 \\
&\geq |\mathrm{Hess}(f)|^2 + \psi'(f)|\nabla f|^2 + \lambda|\nabla f|^2 + \frac{1}{k}|\nabla f|^4 \\
&\geq \frac{1}{m}(\Delta f)^2 + \psi'(f)|\nabla f|^2 + \lambda|\nabla f|^2 + \frac{1}{k}|\nabla f|^4 \\
&= \frac{1}{m}\psi^2(f) + \left(\frac{2}{m}\psi(f) + \psi'(f) + \lambda\right)|\nabla f|^2 + \left(\frac{1}{m} + \frac{1}{k}\right)|\nabla f|^4 \\
&\geq \left(\frac{1}{m} + \frac{1}{k}\right)|\nabla f|^4,
\end{aligned}$$

and then we deduce

$$\Delta_f f|\nabla f|^2 \geq 2G(m \parallel k)|\nabla f|^4. \tag{4.34}$$

Let now $\rho(x) := \mathrm{dist}\,(q, x)$ (using a trick of Calabi, [Cal57], we can suppose that ρ is smooth) and consider on $B_T(q)$ the function

$$F(x) = \left[T^2 - \rho^2(x)\right]^2|\nabla f|^2. \tag{4.35}$$

If $|\nabla f| \equiv 0$ we have nothing to prove; if $|\nabla f| \not\equiv 0$, since $F \geq 0$ and $F|_{\partial B_T(q)} \equiv 0$, there exists a point $x_0 \in B_T(q)$ such that $F(x_0) = \max\limits_{B_T(q)} F(x) > 0$. At x_0 we then have

$$\frac{\nabla F}{F}(x_0) = 0, \tag{4.36}$$

$$\frac{\Delta_f F}{F}(x_0) \leq 0. \tag{4.37}$$

A long but straightforward calculation (analogous to the one carried out in the proof of Lemma 1.2) shows that (4.36) is equivalent to

$$\frac{\nabla|\nabla f|^2}{|\nabla f|^2} = \frac{2\nabla\rho^2}{T^2 - \rho^2} \qquad \text{at } x_0, \tag{4.38}$$

while (4.37), using (4.38) and the Gauss lemma, is equivalent to

$$0 \geq -2\frac{\Delta_f \rho^2}{T^2-\rho^2} + \frac{\Delta_f |\nabla f|^2}{|\nabla f|^2} - 24\frac{\rho^2}{(T^2-\rho^2)^2} \quad \text{at } x_0. \tag{4.39}$$

As a consequence of the f-Laplacian comparison theorem (see [MRS10] and the Appendix) we have

$$\Delta_f \rho^2 \leq 2[(m+k)+(m+k-1)Z\rho]; \tag{4.40}$$

combining (4.34), (4.39) and (4.40) we find, at x_0,

$$0 \geq -4\frac{[(m+k)+(m+k-1)Z\rho]}{T^2-\rho^2} + 2G(m \parallel k)|\nabla f|^2 - 24\frac{\rho^2}{(T^2-\rho^2)^2},$$

which implies, multiplying through by $\left(T^2-\rho^2\right)^2$, that at x_0 we have

$$0 \geq -4[(m+k)+(m+k-1)Z\rho]\left(T^2-\rho^2\right) + 2G(m \parallel k)F - 24\rho^2. \tag{4.41}$$

The previous relation can be rewritten as

$$0 \geq -4(m+k)\left(T^2-\rho^2\right) + 2G(m \parallel k)F - 24\rho^2 + H_3(\rho), \tag{4.42}$$

where $H_3 : [0,T] \to \mathbb{R}$ is defined by $H_3(\rho) = 4(m+k-1)Z\left(\rho^3 - T^2\rho\right)$. Since H_3 assumes its minimum value $-\frac{8\sqrt{3}}{9}(m+k-1)ZT^3 =$ (for $Z \neq 0$) $\frac{8\sqrt{3}\lambda}{9Z}T^3$ for $\bar{t} = \frac{T}{\sqrt{3}}$, equation (4.42) implies

$$0 \geq -4(m+k)T^2 + 2G(m \parallel k)\left[T^2-\rho^2(x)\right]^2|\nabla f|^2 + \frac{8\sqrt{3}\lambda}{9Z}T^3 - 24\rho^2,$$

and so

$$2G(m \parallel k)\left[T^2-\rho^2(x)\right]^2|\nabla f|^2 \leq 4(m+k+6)T^2 - \frac{8\sqrt{3}\lambda}{9Z}T^3,$$

which easily implies the thesis taking the sup on $B_T(q)$. □

Remark. The previous estimates should be compared with the one in [Cas10], valid for $\lambda = 0$.

Theorem 4.17 implies the following Liouville-type result.

Theorem 4.18. *Let $N = M^m \times_u F^k$ be a complete Einstein warped product with Einstein constant $\lambda < 0$, warping function $u = e^{-f/k}$ and Einstein fibre F^k with Einstein constant $\mu < 0$. Suppose that*

$$f \geq \frac{k}{2} \log\left(\frac{\lambda}{2\mu}\frac{m+2k}{m+k}\right) \qquad \textit{for all } x \in M. \tag{4.43}$$

Then N is a Riemannian product.

Proof. Since N is an Einstein warped product f satisfies (4.10), so, with the notation used above, we have that $\psi(t) = k\lambda - k\mu e^{\frac{2}{k}t}$. Equation (4.43) implies (4.30), so we can apply Theorem 4.17. Since M is complete, letting $T \to +\infty$ we obtain the thesis. $\square$

Appendix A

Some useful results

In this Appendix we prove a couple of results and some relations not so easily available in literature. In the first Section we deduce, using the moving frame method, a generalized Bochner-Weitzenböck formula for the operator L_A; in the second Section we derive a useful version Cauchy-Schwarz inequality, a consequence of the L_A-comparison theorem and a particular Newton inequality. The third and last Section is devoted to the proof of the Calabi-Yau volume estimate.

A.1 The generalized Bochner-Weitzenböck formula

In this section we prove a Bochner-Weitzenböck-type formula for the operator L_A, i.e.

$$\boxed{\frac{1}{2}L_A|\nabla u|^2 = |\mathrm{Hess}(u)|^2 + \mathrm{Ricc}(L_A)(\nabla u, \nabla u) + \langle \nabla L_A u, \nabla u\rangle\,,} \tag{1.5}$$

with

$$\mathrm{Ricc}(L_A) = \mathrm{Ricc} - \frac{1}{A}\,\mathrm{Hess}(A) + \frac{1}{A^2}\mathrm{d}A \otimes \mathrm{d}A$$

and $u \in C^3(M)$.

Proof. We begin with

Lemma A.1. *(The classical Bochner-Weitzenböck formula) For all $u \in C^3(M)$ we have*

$$\frac{1}{2}\Delta|\nabla u|^2 = |\mathrm{Hess}(u)|^2 + \mathrm{Ricc}_M(\nabla u, \nabla u) + \langle \nabla\Delta u, \nabla u\rangle . \tag{1.6}$$

Proof. (of the Lemma) We use the method of the moving frame referring to a local orthonormal coframe $\{\theta^i\}$ for the metric and corresponding Levi-Civita and curvature forms, respectively indicated with $\left\{\theta^i_j\right\}$ and $\left\{\Theta^i_j\right\}$, $1 \le i, j, \ldots \le m = \dim M$. By definition of covariant derivative, if $v \in C^2(M)$ we have

$$v_{ik}\theta^k = \mathrm{d}v_i - v_t\theta^t_i,$$

so $\Delta v = v_{kk}$. Set now $v = |\nabla u|^2 = \sum_{k=1}^m (u_k)^2$; using the skew-symmetry of the connection forms,

$$\mathrm{d}v = v_i\theta^i = 2u_k\mathrm{d}u_k = 2u_k\big(u_{kt}\theta^t + u_t\theta^t_k\big) = 2u_ku_{ki}\theta^i,$$

so

$$v_i = 2u_ku_{ki}.$$

Now we compute v_{ik}:

$$\begin{aligned} v_{ik}\theta^k &= \mathrm{d}v_i - v_t\theta^t_i = 2\mathrm{d}(u_ku_{ki}) - 2u_ku_{kt}\theta^t_i = \\ &= 2u_{ki}\mathrm{d}u_k + 2u_k\mathrm{d}u_{ki} - 2u_ku_{kt}\theta^t_i = \\ &= 2u_{ki}\big(u_{kt}\theta^t + u_t\theta^t_k\big) + 2u_k\big(u_{kit}\theta^t + u_{ti}\theta^t_k + u_{kt}\theta^t_i\big) - 2u_ku_{kt}\theta^t_i = \\ &= (2u_{ki}u_{kt} + 2u_ku_{kit})\theta^t + 2u_{ki}u_t\theta^t_k - 2u_ku_{kt}\theta^t_i + 2u_ku_{ti}\theta^t_k + 2u_ku_{kt}\theta^t_i = \\ &= (2u_{ki}u_{kt} + 2u_ku_{kit})\theta^t, \end{aligned}$$

and then

$$\frac{1}{2}\Delta|\nabla u|^2 = u_{kt}u_{kt} + u_ku_{ktt}. \tag{A.1}$$

The conclusion is now achieved using the commutation relations

$$u_{ijk} = u_{jik} = u_{jki} + u_tR_{tjik},$$

from which we deduce

$$u_{ktt} = u_{ttk} + u_s R_{stkt} = u_{ttk} + u_s R_{sk}.$$

□

From the definition of L_A and the Bochner-Weitzenböck formula (1.6) we have

$$\begin{aligned} L_A|\nabla u|^2 &= \Delta|\nabla u|^2 + \frac{1}{A}\left\langle \nabla A, \nabla|\nabla u|^2 \right\rangle = \\ &= 2|\mathrm{Hess}(u)|^2 + 2\,\mathrm{Ricc}_M\left(\nabla u, \nabla u\right) + 2\left\langle \nabla\Delta u, \nabla u\right\rangle + \\ &+ \frac{1}{A}\left\langle \nabla A, \nabla|\nabla u|^2 \right\rangle = \\ &= 2|\mathrm{Hess}(u)|^2 + 2\,\mathrm{Ricc}_M\left(\nabla u, \nabla u\right) + 2\left\langle \nabla L_A u, \nabla u\right\rangle + \\ &+ \frac{1}{A}\left\langle \nabla A, \nabla|\nabla u|^2 \right\rangle - 2\left\langle \nabla\left(\frac{1}{A}\left\langle \nabla A, \nabla u\right\rangle\right), \nabla u\right\rangle. \end{aligned} \tag{A.2}$$

Since $\mathrm{d}\big(\sum_{i=1}^m u_i^2\big) = 2u_i\mathrm{d}u_i = 2u_i\big(u_{it}\theta^t + u_t\theta_i^t\big) = 2u_iu_{it}\theta^t$, we deduce that

$$\left\langle \nabla A, \nabla|\nabla u|^2 \right\rangle = 2\,\mathrm{Hess}(u)(\nabla u, \nabla A). \tag{A.3}$$

Finally, since $\mathrm{d}(\langle \nabla A, \nabla u\rangle) = \mathrm{d}(A_iu_i) = u_i\mathrm{d}A_i + A_i\mathrm{d}u_i = (u_iA_{ik} + A_iu_{ik})\theta^k$, we get

$$\left\langle \nabla\left\langle \nabla A, \nabla u\right\rangle, \nabla u\right\rangle = \mathrm{Hess}(A)(\nabla u, \nabla u) + \mathrm{Hess}(u)(\nabla u, \nabla A). \tag{A.4}$$

Inserting (A.3) and (A.4) into (A.2) we obtain the thesis. □

A.2 Some inequalities

In this section we prove in detail some relations exploited in the previous Chapters.

A.2.1 Cauchy-Schwarz revisited

$$\boxed{\left|\nabla|\nabla u|^2\right|^2 \leq 4|\nabla u|^2|\mathrm{Hess}(u)|^2.} \tag{1.17}$$

Proof. Inequality (1.17) follows from the more general relation

$$\left|\nabla|X|^2\right|^2 \leq 4|X|^2|\nabla X|^2 \tag{A.5}$$

valid for a general vector field X on M. Equation (A.5) is a direct consequence of the Cauchy-Schwarz inequality:

$$\left|\nabla|X|^2\right|^2 = |\nabla \langle X, X\rangle|^2 = 4|\langle X, \nabla X\rangle|^2 \leq 4|X|^2|\nabla X|^2.$$

□

A.2.2 A consequence of the L_A-comparison theorem

As observed in the Introduction, defining $r(x) = \mathrm{dist}\,(x, o)$ for $x \in B_T(o)$, assumption

$$\mathrm{Ricc}_{n,m}(L_A) \geq -(n-1)Z^2 \tag{0.5}$$

for some constant $Z \geq 0$ on the geodesic ball $B_T(o)$ implies

$$L_A r \leq (n-1)Z\coth(Zr), \tag{0.6}$$

pointwise on $B_T(o) \setminus \{\mathrm{cut}(o) \cup \{o\}\}$ and weakly on all of $B_T(o)$ (see [MRS10] for details). We want to show that (0.6) also implies

$$\boxed{Lr^2 \leq 2\frac{A}{B}[n + (n-1)Zr]} \tag{1.19}$$

on $B_T(q)$.

Proof. From the L-diffusion property and Gauss lemma we have

$$Lr^2 = 2rLr + 2\frac{A}{B}|\nabla r|^2 = 2\frac{A}{B}(rL_Ar + 1).$$

Using (0.6) we then deduce

$$Lr^2 \leq 2\frac{A}{B}(1 + (n-1)Zr\coth(Zr)),$$

so the thesis will be proved once we show that

$$1 + (n-1)Zr\coth(Zr) \leq n + (n-1)Zr \tag{A.6}$$

on $B_T(o)$. Set $y = Zr$; a simple computation yields that (A.6) is equivalent to

$$y\coth(y) \leq y + 1, \qquad 0 \leq y \leq ZT. \tag{A.7}$$

Inequality (A.7) is true in a limit sense for $y = 0$ (just remember that $\coth(y) \sim \frac{1}{y}$ for $y \to 0^+$), while for $y > 0$ is implied (taking the inverses of both sides) by

$$e^{2y} \geq 2y - 1,$$

which is valid for all $y \geq 0$.

□

A.2.3 A Newton inequality

$$\boxed{|\mathrm{Hess}(u)|^2 \geq \frac{1}{m}(\Delta u)^2.} \tag{1.24}$$

Proof. The previous relation is a consequence of the more general inequality

$$\|A\|^2 \geq \frac{(\operatorname{tr} A)^2}{m}, \tag{A.8}$$

where $A \in \mathrm{Mat}(m, \mathbb{R})$, $\|A\|$ is the norm of A and tr stands for trace. To prove (A.8) we consider the m-dimensional vectors $a = (a_{11}, a_{22}, \ldots, a_{mm})$ and $v = (1, 1, \ldots, 1)$ and we apply Cauchy-Schwarz inequality to deduce

$$|a \cdot v|^2 = \left(\sum_{i=1}^m a_{ii}\right)^2 \leq m \sum_{i=1}^m (a_{ii})^2 \leq m \sum_{i,j=1}^m (a_{ij})^2,$$

which easily implies (1.24). □

A.3 The Calabi-Yau volume estimate

Proposition A.2. *Let $(M, \langle\ ,\ \rangle)$ be complete, non-compact Riemannian manifold, $A \in C^2(M), A > 0$. Suppose that*

$$\mathrm{Ricc}_{n,m}(L_A) \geq 0 \tag{A.9}$$

for some $n > m = \dim M$. Then

$$\mu(B_R(o)) \geq CR, \qquad C > 0, \ \ R \gg 1. \tag{A.10}$$

Proof. Define the vector field

$$Z = A\nabla r^2,$$

so that, using the L_A-comparison Theorem (see [MRS10]) and Gauss lemma, we deduce

$$\operatorname{div} Z = AL_A r^2 = A\Big(2rL_A r + r|\nabla r|^2\Big) \leq A2r\frac{n-1}{r} + 2A = 2mA,$$

weakly on M and pointwise on $M \setminus \{\mathrm{cut}(o) \cup \{o\}\}$. Fix now a geodesic ball of radius R, $B_R(o)$, a point $x_0 \in \partial B_R(o)$ and set $\rho(x) = \mathrm{dist}(x, x_0)$. Again we deduce

$$\operatorname{div}\left(A\nabla\rho^2\right) \leq 2mA \qquad \text{(weakly)},$$

thus, $\forall\, \psi \in Lip_0(M)$, $\psi : M \to \mathbb{R}_0^+$ with $\operatorname{supp}\psi \subseteq B_{R+\varepsilon}(x_0)$,

$$-\int_{B_{R+\varepsilon}(x_0)} \left\langle \nabla\psi, \nabla\rho^2 \right\rangle \mathrm{d}\mu \leq 2m\int_{B_{R+\varepsilon}(x_0)} \mathrm{d}\mu, \tag{A.11}$$

where $\varepsilon > 0$ and, as in the previous Chapters, $\mathrm{d}\mu = A\mathrm{d}\mu_0$. We choose now $\psi(x) = u(\rho(x))$, with

$$u = \begin{cases} 1, & 0 \leq \rho \leq R - \varepsilon; \\ -\frac{1}{2\varepsilon}\rho + \frac{R+\varepsilon}{2\varepsilon}, & R - \varepsilon \leq \rho \leq R + \varepsilon; \\ 0, & \rho \geq R + \varepsilon. \end{cases}$$

Thus

$$-\int_{B_{R+\varepsilon}(x_0)} \left\langle \nabla\psi, \nabla\rho^2 \right\rangle \mathrm{d}\mu = -2\int_{B_{R+\varepsilon}(x_0)\setminus B_{R-\varepsilon}(x_0)} u'(\rho) \left\langle \nabla\rho, \nabla\rho \right\rangle A\rho =$$
$$= \frac{1}{\varepsilon}\int_{B_{R+\varepsilon}(x_0)\setminus B_{R-\varepsilon}(x_0)} \rho A \mathrm{d}\mu_0 \geq \frac{R-\varepsilon}{\varepsilon}\int_{B_{R+\varepsilon}(x_0)\setminus B_{R-\varepsilon}(x_0)} A\mathrm{d}\mu_0,$$

which implies, substituting in (A.11),

$$\frac{R-\varepsilon}{\varepsilon}\int_{B_{R+\varepsilon}(x_0)\setminus B_{R-\varepsilon}(x_0)} A\mathrm{d}\mu_0 \leq 2m\int_{B_{R+\varepsilon}(x_0)} A\mathrm{d}\mu_0. \tag{A.12}$$

Since $\int_{B_t(x_0)} A\mathrm{d}\mu_0 = \mu(B_t(x_0))$ by definition, equation (A.12) can be written as

$$2m\mu(B_{R+\varepsilon}(x_0)) \geq \frac{R-\varepsilon}{\varepsilon}[\mu(B_{R+\varepsilon}(x_0)) - \mu(B_{R-\varepsilon}(x_0))];$$

moreover, $B_{R+\varepsilon}(x_0) \setminus B_{R-\varepsilon}(x_0) \supseteq B_\varepsilon(o)$, so that

$$2m\mu(B_{R+\varepsilon}(x_0)) \geq \frac{R-\varepsilon}{\varepsilon}\mu(B_\varepsilon(o)).$$

Since $B_{R+\varepsilon}(x_0) \subseteq B_{2R+\varepsilon}(o)$, we finally deduce

$$\mu(B_{2R+\varepsilon}(o)) \geq \frac{R-\varepsilon}{2m\varepsilon}\mu(B_\varepsilon(o)), \tag{A.13}$$

which implies the thesis for $R \gg 1$. □

Corollary A.3. *Under the hypothesis of Proposition A.2,*

$$\mu(M) = \int_M Ad\mu_0 = +\infty.$$

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