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**VECTOR LATTICES,  
POLYHEDRAL GEOMETRY,  
AND VALUATIONS**

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# Introduction

The present thesis explores the connections between Hadwiger's Characterization Theorem for valuations, vector lattices, and MV-algebras.

The study of valuations can be seen as a precursor to the measure theory of modern probability. For this reason, valuations are one of the most important topics of geometric probability. From a general point of view, geometric probability studies sets of geometric objects bearing a common feature, and invariant measures over them. This is motivated by the belief that the mathematically natural probability models are those that are invariant under certain transformation groups, representable in a geometric way (cf. [17]).

In such a context, one of the topics that turns out to be central is the study of measures on polyconvex sets (i.e., finite unions of compact convex sets) in Euclidean spaces of arbitrary finite dimension, that are invariant under the group of Euclidean motions. One may conjecture that there is only one such measure, namely, the volume. This, however, is actually disproved by one of the most important results in the field: Hadwiger's Characterization Theorem. This fundamental theorem states that the linear space of such invariant measures is of dimension  $n + 1$ , if the ambient has dimension  $n$ . Moreover, its proof shows that another basic invariant measure besides the volume is the Euler-Poincaré characteristic. The Euler-Poincaré characteristic, indeed, is the unique such measure that assigns value one to each compact and convex set.

There is a tight connection between the number of vertices  $v$ , the number of edges  $e$ , and the number of faces  $f$  of a compact and convex polyhedron in the Euclidean space of dimension 3. As proved by Euler,  $v - e + f = 2$ . This equality, called the Euler formula, is a well-known result of elementary geometry (Lakatos, for example, chose it as the topic of his imaginary dialogues in [18]). The left-hand side  $v - e + f$  of the Euler formula can be extended to any polyhedron in any arbitrary Euclidean space, considering the alternate sum of the numbers of faces of dimension, respectively, 0, 1, 2, and so on. The resulting value is the Euler-Poincaré characteristic of the polyhedron itself. Moreover, the Euler-Poincaré characteristic can be further extended to any topological space, by a homological definition. One of the topics of the present work is to investigate the Euler-Poincaré characteristic, in Hadwiger's style, in the algebraic context of vector lattices.

A partially ordered real vector space  $L$  is a vector space which is at the same time a partially ordered set such that its vector space structure and its order structure are compatible. More precisely, for any two elements  $x$  and  $y$  in  $L$ , if  $x \leq y$ , then  $x + t \leq y + t$ , for all  $t \in L$ . Vector lattices (also known as Riesz spaces) are partially ordered real vector spaces such that their order structure

is, in addition, a lattice structure. The theory of vector lattices was founded, independently, by Riesz, Freudenthal, and Kantorovitch, in the Thirties of the last century.

Vector lattices are important in the study of measure theory, where some fundamental results are special cases of results for Riesz spaces. For example, the well-known Radon-Nikodym Theorem and the Spectral Theorem for Hermitian operators in Hilbert spaces are both corollaries of the Freudenthal Spectral Theorem for vector lattices (cf. [19]).

Moreover, in the special case of finitely presented unital vector lattices, the Baker-Beynon duality provides a very useful representation of the elements of a vector lattice in terms of piecewise linear and continuous real-valued functions on a suitable polyhedron in some Euclidean space. This powerful tool, that acts as a bridge between the algebra of vector lattices and the geometry of polyhedra, inspires the present thesis in its entirety. On one hand, we use the Baker-Beynon duality to associate vector lattices to polyhedra, and then to define our own notion of the Euler-Poincaré characteristic for vector lattices, using the standard geometric one. On the other hand, we explore two different ways to associate continuous and piecewise linear functions (and hence, by the Baker-Beynon duality, elements of vector lattices) to geometric objects. The first one involves the notion of support function. The second one makes use of gauge functions. Both support functions and gauge functions are well-studied in the literature, within the theory of convex bodies (cf. [13, 30, 35, 34]).

We use here the notion of support function to import into the algebraic context of vector lattices some extension results about additive valuations, typically used in convex geometry. First we establish a correspondence between additive valuations on a suitable set of the free vector lattice  $FVL_n$  and additive valuation on the set of polytopes in  $\mathbb{R}^n$ . Then we apply to our context the Volland-Groemer Extension Theorem (see [17] and [35]) to extend additive valuations on polytopes to additive valuations on polyconvex sets. This allows us to prove a one-to-one correspondence between additive valuations on  $FVL_n$  and additive valuations on polyconvex sets of  $\mathbb{R}^n$ .

In the association via gauge functions, instead, we define and study a new class of subsets of the Euclidean space, that we call star-shaped objects. Our aim, in this case, is to import into the geometric setting some well-known results obtained in the algebraic context. In particular, we translate in the language of star-shaped objects a fundamental result proved by Mundici for MV-algebras.

MV-algebras are algebraic structures introduced by Chang in [8] to prove the completeness theorem for the Łukasiewicz calculus. They turn out to be the equivalent algebraic semantics for Łukasiewicz infinite-valued logic.

An MV-algebra is a commutative monoid  $(A, \oplus, 0)$  equipped with an involutive negation  $\neg$ , such that  $a \oplus \neg 0 = \neg 0$  and  $\neg(\neg a \oplus b) \oplus b = \neg(\neg b \oplus a) \oplus a$ , for all  $a, b \in A$ . MV-algebras form a variety that contains all Boolean algebras. They can also be equivalently defined (cf. [15]) as residuated lattices  $(A, \wedge, \vee, \otimes, \rightarrow, 0, 1)$  which satisfy the conditions  $a \wedge b = a \otimes (a \rightarrow b)$ ,  $(a \rightarrow b) \vee (b \rightarrow a) = 1$ , and  $a = ((a \rightarrow 0) \rightarrow 0)$ , for all  $a, b \in A$ .

MV-algebras are tightly related to lattice-ordered abelian groups, that are abelian groups equipped with a lattice structure compatible with the group operations. Specifically, Mundici's  $\Gamma$ -functor Theorem states that the category of MV-algebras is equivalent to the category of lattice-ordered abelian groups with



distinguished unit (see [28] and [11]). The proof of this fundamental theorem uses the notion of good sequence, introduced by Mundici himself in [28]. Good sequences are special sequences of elements of MV-algebras, and it can be shown that each  $x \geq 0$  element of a lattice-ordered group with distinguished unit can be associated to a good sequence of elements of the corresponding MV-algebra, in a unique way. This is precisely the main lemma that we will import in our geometric context of star-shaped objects.

Let us now summarize the contents of the thesis.

In Chapter 1 we give the necessary geometric and algebraic background. In particular, we define the notions of polyhedron, triangulation, vector lattice, lattice-ordered group, MV-algebra, good sequence, and valuation. Moreover, we collect some basic results about them, that will be in the following chapters.

The main topic of Chapter 2 is the characterization of the Euler-Poincaré characteristic as a valuation on finitely presented unital vector lattices. By the Baker-Baynon duality, we represent each finitely presented unital vector lattice as the lattice of continuous and piecewise linear real-valued functions on a suitable polyhedron in the Euclidean space. Then we define vl-Schauder hats, that are special elements of the vector lattice with a “pyramidal shape”, and that can be used to generate the vector lattice, via addition and products by real scalars. On the positive cone  $V^+$  of every finitely presented vector lattice  $V$  we define a *pc-valuation* as a valuation (in the usual classical sense) that is insensitive to addition. The (Euler-Poincaré) number  $\chi(f)$  of any function  $f \in V^+$  is next defined as the Euler-Poincaré characteristic of the support  $f^{-1}(\mathbb{R}_{>0})$  of  $f$ . We then prove that pc-valuations uniquely extend to a suitable kind of valuations over  $V$ , called *vl-valuations*. In Theorem 2.3.6 we prove a Hadwiger-like theorem, to the effect that our  $\chi$  is the only vl-valuation assigning 1 to each vl-Schauder hat of  $V$ .

In Chapter 3 we use the notion of support function to establish a correspondence between a suitable subset of the free vector lattice on  $n$  generators and the set of polytopes of  $\mathbb{R}^n$  (see Theorem 3.3.4). This special set of algebraic objects generates the whole free vector lattice via finite meets. We call it the set of support elements. Then we consider valuations on the free vector lattice that are also additive on the set of support elements. By the Volland-Groemer Extension Theorem, we prove that such valuations are in a one-to-one correspondence with the valuations on the lattice of polyconvex sets that are additive on the subset of convex objects (see Theorems 3.4.8 and 3.4.10).

In Chapter 4 we proceed in a similar way, using gauge functions. In this case, the first correspondence that we prove is between the positive cone of the vector lattice of continuous and positively homogeneous real-valued functions of  $\mathbb{R}^n$  and a lattice, equipped with appropriate vector space operations, of a new kind of geometric objects (see Theorem 4.2.5). We call these sets in  $\mathbb{R}^n$  star-shaped objects. Then we define a geometric notion of good sequence, and we prove an analogue of Mundici’s main lemma for MV-algebras (see Theorem 4.3.14). Imposing a polyhedral condition on our star-shaped objects, we obtain a correspondence between them and the elements of the positive cone of the free vector lattice on  $n$  generators. Finally, we specialize the result obtained for good sequences to these polyhedral star-shaped objects (see Theorem 4.4.10).

The final Chapter 5 contains remarks on further research.



# Chapter 1

## Background

The aim of this thesis is to study the connections between a special class of algebraic structures, named vector lattices, and some suitable geometrical objects in the Euclidean space (quite often, polyhedra). Some of these connections are new, even if they use concepts and techniques already known in the literature. This applies, for example, to the identification between support functions and support elements of the free vector lattice on  $n$  generators given in Chapter 3, or to the one involving gauge functions and star-shaped objects presented in Chapter 4. Other such connections, notably the Baker-Beynon duality presented later in this chapter, are well-known, and are used as a dictionary to translate concepts and ideas from algebra to geometry, and vice versa. These translations are fundamental to investigate the behaviour of the third main ingredient of the present thesis: valuations. The idea is to use the properties of the geometric counterparts of vector lattices to define and characterize some suitable subclasses of algebraic valuations, such as the vl-valuations in Chapter 2, or to import results about geometric valuations in the algebraic context, as in Chapter 3.

For these reasons, the concepts of polyhedra, vector lattices and valuations are the kernel of the next chapters. Hence, in the following we will collect the principal algebraic and geometric definitions, concerning the three aforementioned topics.

### 1.1 Polyhedra

Throughout the thesis, we write  $\mathbb{R}$  for the set of real numbers,  $\mathbb{Q}$  for the set of rational numbers,  $\mathbb{Z}$  for the set of integers, and  $\mathbb{N} = \{0, 1, 2, \dots\}$  for the set of natural numbers.

In the following, we present the main definitions and results concerning polyhedra that form the common geometrical background of our work. Polyhedra are central to geometry, and especially to the study of piecewise linear topology. They can be defined in many different ways. The one we have chosen here is taken from [25]. For more details and proofs, see [16] and [25].

### 1.1.1 Simplicial complexes and polyhedra

In the Euclidean space  $\mathbb{R}^n$ , the  $m + 1$  points  $x_0, \dots, x_m$  are called *affinely independent* if the vectors  $x_1 - x_0, x_2 - x_0, \dots, x_m - x_0$  are linearly independent. It can be proved that this definition does not depend on the order of the points  $x_0, \dots, x_m$ . Hence, also the following definitions depend only on the points themselves, and not on their order.

Given  $m + 1$  affinely independent points  $x_0, \dots, x_m$  in some  $\mathbb{R}^n$ , we say that the  $m$ -*simplex*  $\sigma_m = (x_0, \dots, x_m)$  is the set of all the convex combinations of  $x_0, \dots, x_m$ , that is, the set of points  $\sum_{i=0}^m \lambda_i x_i$ , where the  $\lambda_i$  are real numbers such that  $\lambda_i \geq 0$  for all  $i$  and  $\sum_{i=0}^m \lambda_i = 1$ . (Note that the set of convex combinations of  $\emptyset$  is  $\emptyset$ ). The points  $x_0, \dots, x_m$  are called the *vertices* of  $\sigma_m$ , and the number  $m$  is the *dimension* of  $\sigma_m$ . (The dimension of  $\emptyset$  is  $-1$ ). The (*relative*) *interior* of  $\sigma_m$  is the subset of  $\sigma_m$  of those points  $\sum_{i=0}^m \lambda_i x_i$  such that  $\lambda_i > 0$  for all  $i$ .

**Remark 1.1.1.** Throughout the thesis we write *interior* to mean *relative interior* (of a simplex). To avoid possible confusion, we always write *topological interior* when we mean *interior in the topological sense*.

The *barycentre* of  $\sigma_m$  is the point

$$\hat{\sigma}_m = \left( \frac{1}{m+1} \right) (x_0 + \dots + x_m).$$

A *face* of  $\sigma_n$  is the collection of all the convex combinations of a subset of its vertices. (Hence  $\emptyset$  is a face of any simplex). The *boundary*  $\partial \sigma_n$  of  $\sigma_n$  is the set of all faces of  $\sigma_n$  other than  $\sigma_n$  itself.

A *simplicial complex*  $K$  is a finite set of simplices such that

1. if  $\sigma_m \in K$  and  $\tau_p$  is a face of  $\sigma_m$ , then  $\tau_p \in K$ ,
2. if  $\sigma_m, \tau_p \in K$ , then  $\sigma_m \cap \tau_p$  is a (possibly empty) common face of  $\sigma_m$  and  $\tau_p$ .

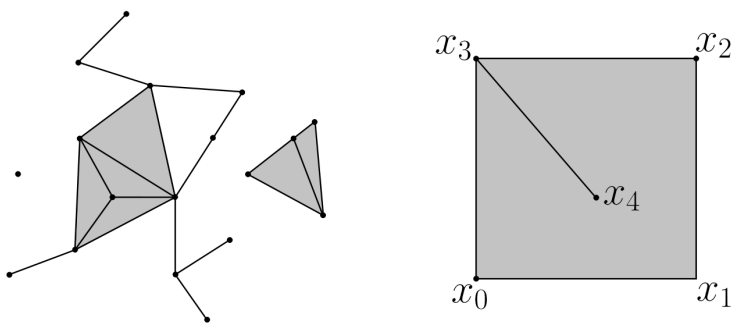


Figure 1.1: The picture on the left is a simplicial complex. The picture on the right is not, for three different reasons: 1) the square  $(x_0, x_1, x_2, x_3)$  is not a simplex; 2) the vertex  $x_1$  is not in the complex, whence the property of closure under faces is not satisfied; 3) the intersection between  $(x_0, x_1, x_2, x_3)$  and  $(x_3, x_4)$  is  $(x_3, x_4)$ , that is not a face of  $(x_0, x_1, x_2, x_3)$ .

The *dimension* of a simplicial complex  $K$  is the maximum of the dimensions of its simplices. A *subcomplex*  $L$  of  $K$  is a subset of simplices of  $K$  that is itself a simplicial complex. For each  $r \geq 0$ , the  $r$ -*skeleton*  $K^r$  of  $K$  is the subset of its simplices of dimension at most  $r$ : clearly, it is a subcomplex of  $K$ . The (*underlying*) *polyhedron*  $|K|$  of  $K$  is the set of points of  $\mathbb{R}^n$  that lie in at least one of the simplices of  $K$ , topologized as a subspace of  $\mathbb{R}^n$ .

**Definition 1.1.2** (Polyhedron). A *polyhedron* is a subset  $P$  of  $\mathbb{R}^n$  such that there exists a simplicial complex  $K$  with  $|K| = P$ .

Each simplicial complex  $K$  with polyhedron  $P$  is called a *triangulation* of  $P$ . It can be proved that any two finite triangulations of  $P$  have the same dimension. Hence the *dimension* of  $P$  is defined to be the dimension of any one of its triangulations.

Given a triangulation  $K$  of the polyhedron  $P$ , a *refinement* of  $K$  is a triangulation  $K^*$  of  $P$  such that for all simplices  $\sigma$  of  $K^*$  there is a simplex  $\tau$  of  $K$  such that  $|\sigma| \subseteq |\tau|$ . Moreover, given any two triangulations  $K_1$  and  $K_2$  of the same polyhedron  $P$ , there always exists a common refinement  $K^*$  of  $K_1$  and  $K_2$ .

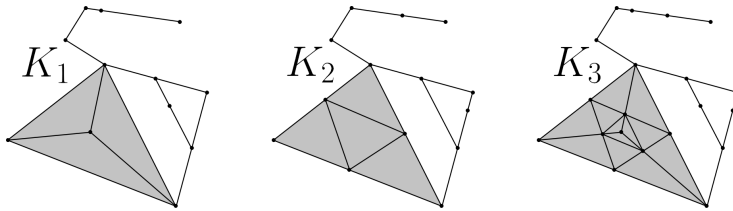


Figure 1.2: Two different triangulations  $K_1$  and  $K_2$  of the same polyhedron, and a common refinement  $K_3$ .

**Proposition 1.1.3** ([25, Proposition 2.3.6]). *Let  $K$  be a simplicial complex. Then each point  $x$  of  $|K|$  is in the interior of exactly one simplex  $\sigma^x$  of  $K$ .*

The simplex  $\sigma^x$  of Proposition 1.1.3 is the inclusion-smallest simplex containing  $x$ , and it is called the *carrier* of  $x$ .

**Remark 1.1.4.** We recall that a *closed half-space* in  $\mathbb{R}^n$  is a subset  $H \subseteq \mathbb{R}^n$  of the form

$$H = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n \mid a \cdot x + b = a_1x_1 + \dots + a_nx_n + b \geq 0\}, \quad (1.1)$$

where  $0 \neq a = (a_1, \dots, a_n) \in \mathbb{R}^n$  and  $b$  is a fixed real number. It is a standard result that we can use the concept of closed half-space to give a characterization of polyhedra. A subset  $P$  of  $\mathbb{R}^n$  is a polyhedron if and only if it is compact and if it can be written as a finite union of finite intersections of closed half-spaces of  $\mathbb{R}^n$ . Convex polyhedra turn out to be precisely the compact finite intersections of closed half-spaces; they are called *polytopes*. Equivalently, polytopes are the convex hulls of finite (possibly empty) sets of points of  $\mathbb{R}^n$ . Here, the convex hull of a set of points  $J$  in  $\mathbb{R}^n$  is defined as the set of all finite convex combinations of them:

$$\text{conv}(J) = \left\{ \sum_{i=1}^l \lambda_i x_i \mid l < \infty, x_i \in J, \sum_{i=1}^l \lambda_i = 1, 0 \leq \lambda_i \leq 1 \right\}.$$

We will use the fact that, if  $K$  and  $H$  are two convex subsets of  $\mathbb{R}^n$ , the convex hull of their union is the set

$$\text{conv}(K \cup H) = \{\lambda k + (1 - \lambda)h \mid k \in K, h \in H, 0 \leq \lambda \leq 1\}.$$

See [35, Section 2.4] for more details about the equivalence between the given definitions of polytopes and polyhedra.  $\square$

**Definition 1.1.5.** We denote by  $\mathcal{P}^n$  the set of all the polytopes of  $\mathbb{R}^n$ , and with  $\mathcal{P}_*^n$  the set of non-empty polytopes in  $\mathbb{R}^n$ . The set of all polyhedra of  $\mathbb{R}^n$  is denoted  $\mathcal{K}^n$ .

Note that  $\mathcal{K}^n$  is the closure of  $\mathcal{P}^n$  under the set-theoretic operation of union. Moreover, it can be shown that  $\mathcal{P}^n$  is an *intersectional* family of sets, that is, a collection of sets closed under finite intersections.

### 1.1.2 The supplement

The concept of supplement is introduced in [25, Definition 2.5.18], in the treatment of simplicial approximations of continuous functions between polyhedra. We will use supplements in Chapter 2, as inner approximations of the supports of piecewise linear and continuous real-valued functions.

The following technical definition is needed, although it is not transparent.

**Definition 1.1.6** (Supplement). Let  $L$  be a subcomplex of a simplicial complex  $K$  and let  $n$  be the dimension of  $K$ . Let  $M^m = K^m \cup L$  for all  $m \leq n$  and define  $(M^0, L)' = M^0$ . Inductively, we define

$$(M^m, L)' = (M^{m-1}, L)' \cup \{\hat{\sigma}\tau\} \cup \{(\hat{\sigma})\},$$

where  $\sigma$  runs through all  $m$ -simplices of  $K - L$  and  $\tau$  through all simplices in each  $(\hat{\sigma})'$ , that is a subcomplex (it always exists) of  $(M^{m-1}, L)'$  such that  $|\hat{\sigma}| = |(\hat{\sigma})'|$ . In this notation,  $\hat{\sigma}\tau$  is the simplex  $(\hat{\sigma}, y_0, \dots, y_r)$ , where  $(y_0, \dots, y_r) = \tau$  and  $\hat{\sigma}$  is the barycentre of  $\sigma$ . The *derived complex of  $K$  relative to  $L$*  is  $(K, L)' = (M^n, L)'$  and we have  $|(K, L)'| = |K|$ . The *derived complex of  $K$*  is  $K' = (K, \emptyset)'$ . The *supplement of  $L$  in  $K$*  is the set  $\bar{L}$  of simplices of  $(K, L)'$  that have no vertex in  $L$ .

As suggested in [25], we can give an equivalent characterization of the supplement that is more expensive in terms of calculation, but also much more understandable. The derived complex  $K'$  of the complex  $K$  is nothing else but the first barycentric subdivision of  $K$ , obtained introducing a new vertex at the barycentre of each simplex of  $K$ , and then joining up the vertices. Hence the supplement  $\bar{L}$  of  $L$  in  $K$  is precisely the subcomplex of  $K'$  consisting of those simplices having no vertex in  $L'$ .

## 1.2 Vector lattices

Vector lattices are algebraic structures also known as *Riesz spaces*; standard references are [7] and [19].

**Definition 1.2.1** (Vector lattice). A (*real*) *vector lattice* is an algebra  $\mathbf{V} = (V, +, \wedge, \vee, \{\lambda\}_{\lambda \in \mathbb{R}}, \mathbf{0})$  such that

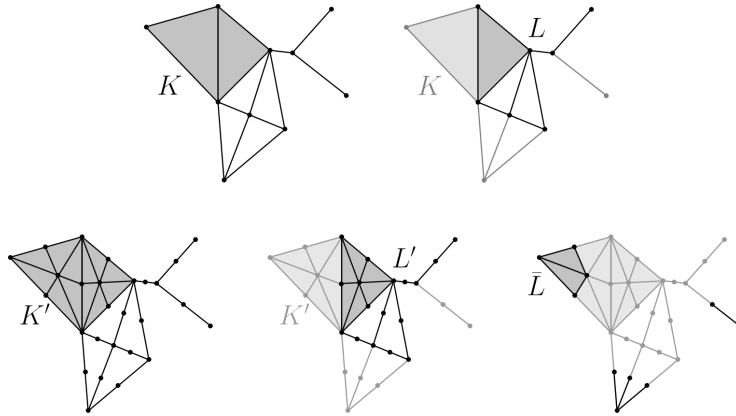


Figure 1.3: An example of supplement:  $K$  is a simplicial complex,  $L$  is a subcomplex of  $K$ .  $K'$  and  $L'$  are the derived complexes of  $K$  and  $L$ , and  $\bar{L}$  is the supplement of  $L$  in  $K$ .

VL1)  $(V, +, \{\lambda\}_{\lambda \in \mathbb{R}}, \mathbf{0})$  is a (real) vector space;

VL2)  $(V, \wedge, \vee)$  is a lattice;

VL3)  $t + (v \wedge w) = (t + v) \wedge (t + w)$ , for all  $t, v, w \in V$ ;

VL4) if  $\lambda \geq 0$  then  $\lambda(v \wedge w) = \lambda v \wedge \lambda w$  for all  $v, w \in V$  and for all  $\lambda \in \mathbb{R}$ .

The lattice structure given in VL2) induces on  $V$  a partial order  $\leq$  defined as usual:

$$\text{for all } v, w \in V \quad v \leq w \text{ if and only if } v \wedge w = v.$$

Vector lattices form a variety of algebras (with continuum-many operations) by their very definition. It is well known that the underlying lattice of  $\mathbf{V}$  is necessarily distributive (cf. Proposition 1.2.2 below). Morphisms of vector lattices are homomorphisms in the variety, that is, linear maps that also preserve the lattice structure. From now on we shall follow common practice and blur the distinction between  $\mathbf{V}$  and its underlying set  $V$ . Moreover, we will denote both the element  $\mathbf{0}$  of  $V$  and the real number zero by the same symbol  $0$ : the meaning will be clear from the context.

The following properties are standard results in vector lattice theory (see [7] for more details).

**Proposition 1.2.2** (Some properties of vector lattices). *In any vector lattice  $V$ , the following properties are satisfied.*

1.  $t + (v \vee w) = (t + v) \vee (t + w)$ , for all  $t, v, w \in V$ ;
2. if  $\lambda \geq 0$  then  $\lambda(v \vee w) = \lambda v \vee \lambda w$ , for all  $v, w \in V$ ;
3. if  $\lambda < 0$  then  $\lambda(v \wedge w) = \lambda v \vee \lambda w$ , for all  $v, w \in V$ ;
4. if  $\lambda < 0$  then  $\lambda(v \vee w) = \lambda v \wedge \lambda w$ , for all  $v, w \in V$ ;
5.  $t \wedge (v \vee w) = (t \wedge v) \vee (t \wedge w)$ , for all  $t, v, w \in V$ ;

6.  $t \vee (v \wedge w) = (t \vee v) \wedge (t \vee w)$ , for all  $t, v, w \in V$ ;

7.  $(v \vee w) + (v \wedge w) = v + w$ , for all  $v, w \in V$ .

We say that a subset  $L$  of  $V$  is *generating* if the intersection of all linear subspaces that are also sublattices of  $V$  containing  $L$  is  $V$  itself. When  $L$  generates  $V$ , then each element  $v \in V$  can be written as a finite combination of elements of  $L$ , using the vector-lattice operations of  $V$ .

The next theorem contains two parts. The first part is a standard result on vector lattices which provides a weak normal form for elements. The second part provides a different such normal form, which to the best of our knowledge is new here.

**Theorem 1.2.3.** *Let  $L$  be a generating subset of the vector lattice  $V$ . Let  $S(L)$  be the vector subspace of  $V$  generated by  $L$ . Let  $J(L)$  be the closure of  $S(L)$  under the join operation of  $V$ .*

1. *Let  $M(L)$  be the closure of  $J(L)$  under the meet operation of  $V$ . Then  $V = M(L)$ .*

2. *Let  $D(L)$  be the vector subspace of  $V$  generated by  $J(L)$ . Then  $V = D(L)$ .*

*Proof.* By definition,  $S(L)$  is the set of all the finite linear combinations of elements of  $L$ , and, by Proposition 1.2.2,  $J(L)$  is closed under the operations of addition and products by scalars  $0 \leq \lambda \in \mathbb{R}$ .

1. This is an immediate consequence of the distributivity properties in Proposition 1.2.2.

2. We will show that the closure of  $J(L)$  under the vector space operations of  $V$  coincides with the set of all the possible differences between any two elements of  $J(L)$ :

$$D(L) = \{a - b \mid a, b \in J(L)\}.$$

To check this, we can proceed by induction. Obviously, each element of  $J(L)$  is a difference of elements of  $J(L)$  itself. If  $f \in D(L)$  is of the form  $g + h$ , with  $g, h \in D(L)$ , then, by the induction hypothesis,  $g = a_1 - b_1$  and  $h = a_2 - b_2$ , with  $a_1, a_2, b_1, b_2 \in J(L)$ . Then  $f = (a_1 + a_2) - (b_1 + b_2)$ , with  $a_1 + a_2, b_1 + b_2 \in J(L)$ . If  $f = \lambda g$ , with  $\lambda \geq 0$ ,  $g = a - b$  and  $a, b \in J(L)$ , then  $f = \lambda a - \lambda b$ , with  $\lambda a, \lambda b \in J(L)$ . If  $f = \lambda g$ , with  $\lambda < 0$ ,  $g = a - b$  and  $a, b \in J(L)$ , then  $f = (-\lambda)b - (-\lambda)a$ , with  $(-\lambda)b, (-\lambda)a \in J(L)$ . Then  $D(L)$  is contained in the set of all the differences between two elements of  $J(L)$ . The other inclusion is trivial.

Now we prove that  $D(L)$  is closed under the vector lattice operations of  $V$ . Since  $L \subseteq D(L)$ , and since  $L$  generates  $V$ , this will show  $D(L) = V$ . The closure under addition and products by real scalars is trivial, and the closure under the meet operation is a consequence of the closure under the join operation, because of the last equality of Proposition 1.2.2. To show that  $D(L)$  is closed under the join operation we proceed as in [1, Proposition I.1.1]. Let  $f, g \in D(L)$ . Then  $f = a_1 - b_1$  and  $g = a_2 - b_2$ , with  $a_1 + a_2, b_1 + b_2 \in J(L)$ . Then the three elements

$$\begin{aligned} f' &= f + (b_1 + b_2) = a_1 + b_2, \\ g' &= g + (b_1 + b_2) = a_2 + b_1 \\ &f' \vee g' \end{aligned}$$



are elements of  $J(L)$ . Then, by Proposition 1.2.2,

$$f \vee g = (f' - (b_1 + b_2)) \vee (g' - (b_1 + b_2)) = (f' \vee g') - (b_1 + b_2).$$

This completes the proof.  $\square$

**Corollary 1.2.4.** *Under the hypotheses of Theorem 1.2.3, each element of  $V$  can be written as the difference of two elements in  $J(L)$ .*

**Definition 1.2.5** (Piecewise linear function). Given a set  $S$  in  $\mathbb{R}^n$ , a function  $f : S \rightarrow \mathbb{R}$  is said to be *piecewise linear* if there is a finite set  $f_1, \dots, f_m$  of affine linear functions  $\mathbb{R}^n \rightarrow \mathbb{R}$  such that for all  $s \in S$  there exists an index  $i \leq m$  for which  $f(s) = f_i(s)$ .

**Example 1.2.6** (The vector lattice  $\nabla(P)$ ). We consider a polyhedron  $P$  in  $\mathbb{R}^n$  and the set of all functions  $f : P \rightarrow \mathbb{R}$  that are continuous with respect to the Euclidean metric and piecewise linear, equipped with pointwise defined addition, supremum, infimum, products by real scalars and the zero function. It is easy to show that this set is a vector lattice under the mentioned operations; we will denote it  $\nabla(P)$ .  $\square$

**Definition 1.2.7** (Positively homogeneous function). A function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  is *positively homogeneous* if for each  $x \in \mathbb{R}^n$  and for all  $0 \leq \lambda$ ,  $f(\lambda x) = \lambda f(x)$ .

**Example 1.2.8.** The set of all functions  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  that are continuous, positively homogeneous and piecewise linear, equipped with pointwise defined addition, supremum, infimum, products by real scalars and the zero function, is a vector lattice under the mentioned operations.  $\square$

**Definition 1.2.9** (Unital vector lattice). Given a vector lattice  $V$ , an element  $u \in V$  is a *strong order unit*, or just a *unit* for short, if for all  $0 \leq v \in V$  there exists a  $0 \leq \lambda \in \mathbb{R}$  such that  $v \leq \lambda u$ . A *unital vector lattice* is a pair  $(V, u)$ , where  $V$  is a vector lattice and  $u$  is a unit of  $V$ .

It turns out that every finitely generated vector lattice admits a unit. For if  $v_1, \dots, v_u$  is a finite set of generators for  $V$ , then it is easily checked that  $|v_1| + \dots + |v_u|$  is a unit of  $V$ , where  $|v_i| = v_i \vee (-v_i)$  is the *absolute value* of  $v_i$ . Morphisms of unital vector lattices are the vector-lattice homomorphisms that carry units to units. Such homomorphisms are called *unital*. Note that unital vector lattices do not form a variety of algebras, because the Archimedean property of the unit is not even definable by first-order formulæ, as is shown via a standard compactness argument.

In Example 1.2.6, we can consider the function  $\mathbf{1} : P \rightarrow \mathbb{R}$ , identically equal to 1 on  $P$ . It is a unit of  $\nabla(P)$ , and hence the pair  $(\nabla(P), \mathbf{1})$  is a unital vector lattice.

### 1.2.1 Baker-Beynon duality

The unital vector lattices of the form  $(\nabla(P), \mathbf{1})$  presented above play a central role in the characterization of a special class of vector lattices.

We denote the free vector lattice on  $n$  generators by  $\text{FVL}_n$ . We also notice that free vector lattices actually exist, because vector lattices form a variety of algebras. From universal algebra and the fact that  $\mathbb{R}$  generates the variety of

vector lattices (see [7]), we can describe  $\text{FVL}_n$  in the following way (see [3]). Consider the set of all real-valued functions on  $\mathbb{R}^n$ , equipped with the same pointwise defined operations of  $\nabla(P)$ . This set is a vector lattice, and  $\text{FVL}_n$  may be identified with the Riesz subspace (sublattice and linear subspace) generated by the *coordinate projections*  $\pi_1, \dots, \pi_n : \mathbb{R}^n \rightarrow \mathbb{R}$ ,  $\pi_i : (x_1, \dots, x_n) \mapsto x_i$ . It can be proved that, under this identification, the elements of  $\text{FVL}_n$  are precisely the continuous positively homogeneous piecewise linear functions from  $\mathbb{R}^n$  to  $\mathbb{R}$ :  $\text{FVL}_n$  can be represented as the vector lattice of Example 1.2.8.

We say that a *cone* of  $\mathbb{R}^n$  is a subset  $C \subseteq \mathbb{R}^n$  which is invariant under multiplication scalars  $0 < \lambda \in \mathbb{R}$ . A *(closed) polyhedral cone* is a cone which is actually obtainable as a finite union of finite intersections of closed half-spaces having the origin in their topological boundary. A closed half-space containing 0 in its topological boundary is of the form (1.1) with  $b = 0$ .

A subset  $I$  of a vector lattice  $V$  is an *ideal* if it is a Riesz subspace that is *(order-)convex*:  $x \in I$ ,  $z \in V$  and  $x \leq z$  imply  $z \in I$ . Ideals are precisely kernels of homomorphisms between vector lattices, and the quotient vector lattice  $V/I$  is defined in the obvious manner. Arbitrary intersections of ideals are again ideals. The ideal generated by a subset  $S \subseteq V$  is the intersection of all ideals containing  $S$ ; it is *finitely generated* if  $S$  can be chosen finite.

Let  $I \subseteq \text{FVL}_n$  be an ideal, and consider the set

$$Z(I) = \{x \in \mathbb{R}^n \mid f(x) = 0 \text{ for all } f \in I\}.$$

We say that a vector lattice  $V$  is *finitely presented* if there exists a finitely generated ideal  $I$  such that  $V$  is isomorphic to the quotient  $\text{FVL}_n/I$ . In this case, as shown in [3] and [4],  $Z(I)$  is a polyhedral cone, and  $V$  is isomorphic to the vector lattice of all continuous, positively homogeneous and piecewise linear functions on  $Z(I)$ . The well-known *Baker-Beynon duality* (see [5]) states that the category of finitely presented vector lattices with vector lattice morphisms is dually equivalent to (that is, equivalent to the opposite category of) the category of polyhedral cones in some Euclidean space, with piecewise homogeneous linear continuous maps as morphisms. Moreover, there is an induced duality between the category of finitely presented vector lattices with a distinguished unit and unital morphisms and the category of polyhedra and piecewise linear continuous maps. This duality entails that finitely presented unital vector lattices are exactly the ones representable as  $(\nabla(P), \mathbf{1})$  to within a unital isomorphism, for some polyhedron  $P$  in some Euclidean space  $\mathbb{R}^n$ . The polyhedron  $P$  is called the *support* of the vector lattice itself, and  $(\nabla(P), \mathbf{1})$  is its *coordinate vector lattice*.

In light of the foregoing, we will identify finitely presented unital vector lattices with their functional representation. The elements of a finitely presented unital vector lattice will be treated as continuous piecewise linear real-valued functions on some suitable polyhedron  $P$ . On the other hand, the elements of  $\text{FVL}_n$  will be represented as continuous, positively homogeneous and piecewise linear real-valued functions on  $\mathbb{R}^n$ .

## 1.2.2 Positive cone and triangulations

Here we present some standard results that will be useful in the following chapters, in order to investigate the behaviour of the elements of vector lattices, considered as real-valued functions.

The *positive cone* of a vector lattice  $V$  is the set

$$V^+ = \{v \in V \mid v \geq 0\}.$$

**Remark 1.2.10.** Given a vector lattice  $V$ , each element  $x \in V$  can be written in a unique way as the difference between two disjoint elements of the positive cone  $V^+$ . This means that there exists exactly two elements  $x^+, x^- \in V^+$  such that  $x^+ \wedge x^- = 0$  and  $x = x^+ - x^-$ . It turns out that these two elements are the *positive part*  $x^+ = x \vee 0$  and the *negative part*  $x^- = (-x) \vee 0$  of  $x$ . Moreover, the absolute value of  $x$  satisfies  $|x| = x^+ + x^-$ .

If we consider a vector lattice of functions (either on  $\mathbb{R}^n$  or on some polyhedron  $P \subseteq \mathbb{R}^n$ ), we obtain that the positive and the negative part of a function  $f \in V$  are precisely the functions

$$f^+(x) = \begin{cases} f(x) & \text{if } f(x) \geq 0 \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad f^-(x) = \begin{cases} -f(x) & \text{if } f(x) \leq 0 \\ 0 & \text{otherwise,} \end{cases}$$

for each  $x$  in the domain of  $f$ . □

For the rest of this section,  $P \subseteq \mathbb{R}^n$  denotes an element of  $\mathcal{P}^n$ , that is a polyhedron.

**Definition 1.2.11** (Linearizing triangulation). Given a function  $f \in \nabla(P)$ , a *linearizing triangulation for  $f$*  is a triangulation  $K_f$  of  $P$  such that  $f$  is linear on each simplex of  $K_f$ .

**Remark 1.2.12.** Because  $f \in \nabla(P)$  is piecewise linear, a standard argument (c.f. [11, p. 183], [31, Corollary 2.3]) shows that there always exists a linearizing triangulation  $K_f$  of  $P$ . For any two functions  $f, g \in \nabla(P)$  there exists a triangulation of  $P$  that is linearizing for both  $f$  and  $g$ . Moreover, if  $K^*$  is a refinement of a linearizing triangulation  $K$  for  $f$ , then  $K^*$  is linearizing for  $f$ . Henceforth,  $K_f$  will denote a linearizing triangulation of  $P$  for  $f$ . □

We observe the following.

**Claim 1.2.13.** *Let  $f \in \nabla(P)^+$  and let  $Z_{K_f, f}$  be the subcomplex of  $K_f$  of those simplices where  $f$  is identically zero. Then  $|Z_{K_f, f}|$  is the zero-set  $f^{-1}(0)$  of  $f$ , and does not depend on the particular choice of the triangulation  $K_f$  that linearizes  $f$ .*

*Proof.* We have to prove that, given a linearizing triangulation  $K_f$ ,  $|Z_{K_f, f}|$  is the zero-set of  $f$ , that is, the set  $\{y \in P \mid f(y) = 0\}$ . The inclusion of  $|Z_{K_f, f}|$  in the zero-set of  $f$  is trivial: if  $y \in |Z_{K_f, f}|$ , then  $y$  is a point of  $P$  that lies in at least one simplex of  $Z_{K_f, f}$  whence  $f(y) = 0$ . For the inverse inclusion, if  $y$  is a point of  $P = |K_f|$  such that  $f(y) = 0$ , whence, by Proposition 1.1.3, there is a simplex  $\sigma$  of  $K_f$  such that  $y$  is a point of the interior of  $\sigma$ . Recalling that  $f \geq 0$ , the linearity of  $f$  on the simplices of  $K_f$ , and in particular on  $\sigma$ , ensures that  $f$  is identically zero on the whole simplex  $\sigma$ . So  $\sigma$  is a simplex of  $Z_{K_f, f}$  and  $y \in |Z_{K_f, f}|$ . □

### 1.3 $\ell$ -groups and MV-algebras

In 1986, Mundici introduced the notion of *good sequences* (cf. [28]) to prove his  $\Gamma$ -*functor Theorem*. This fundamental result asserts the equivalence between two types of algebraic structures, namely, MV-algebras and unital  $\ell$ -groups. In Chapter 4, we will translate the concept of good sequence in a suitable geometric language, and then we will give an analogue, in our context, of one of the main lemmas that Mundici used to prove his theorem.

Introduced by Chang in [8], MV-algebras are the algebras of Łukasiewicz logic, just as Boolean algebras are the algebras of Boolean logic. The following definition, essentially due to Mangani (see [21], and [11] for more details), is equivalent to Chang's original one.

**Definition 1.3.1** (MV-algebra). An *MV-algebra* is an algebra  $\mathbf{A} = (A, \oplus, \neg, \mathbf{0})$  such that

*MV1)*  $(A, \oplus, \mathbf{0})$  is an abelian monoid;

*MV2)*  $\neg\neg x = x$ , for all  $x \in A$ ;

*MV3)*  $x \oplus \neg\mathbf{0} = \neg\mathbf{0}$ , for all  $x \in A$ ;

*MV4)*  $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$ , for all  $x, y \in A$ .

**Example 1.3.2.** The algebra  $[0, 1] = ([0, 1], \oplus, \neg, \mathbf{0})$ , where  $[0, 1] = \{x \in \mathbb{R} \mid 0 \leq x \leq 1\}$  is the real unit interval equipped with the operations

$$x \oplus y = \min(1, x + y) \quad \text{and} \quad \neg x = 1 - x,$$

is an MV-algebra.

For all integers  $n > 1$ , the sets

$$L_n = \left\{ 0, \frac{1}{n-1}, \dots, \frac{n-2}{n-1}, 1 \right\}$$

equipped with the restrictions of the operations defined above are MV-algebras, too. Each  $L_n$  is a subalgebra of  $[0, 1]$ .

The algebra  $[0, 1]$  plays a central role in the theory of MV-algebras. As stated by *Chang's Completeness Theorem* an equation holds in  $[0, 1]$  if and only if it holds in every MV-algebra. This, intuitively, means that  $[0, 1]$  is the analogue of the two-element Boolean algebra  $\{0, 1\}$ . A proof of Chang's Completeness Theorem that makes use of good sequences can be found in [11].  $\square$

**Example 1.3.3.** We say that  $f: [0, 1]^n \rightarrow \mathbb{R}$  is a *McNaughton function* if it is continuous and there exist finitely many polynomials  $p_1, \dots, p_k$ ,

$$p_i(x) = p_i(x_1, \dots, x_n) = a_{i0} + a_{i1}x_1 + \dots + a_{in}x_n,$$

with integer coefficients  $a_{ij} \in \mathbb{Z}$ , such that for each point  $y \in [0, 1]^n$  there exists an index  $j \in \{1, \dots, k\}$  such that  $f(y) = p_j(y)$ . The set of all McNaughton functions  $f: [0, 1]^n \rightarrow [0, 1]$ , equipped with the operations

$$(f \oplus g)(x) = \min(1, f(x) + g(x)) \quad \text{and} \quad (\neg f)(x) = 1 - f(x)$$

and with the function identically equal to 0, is an MV-algebra. As stated by *McNaughton Theorem*, it can be shown that, up to isomorphisms, this MV-algebra is the free MV-algebra over  $n$  generators (cf. [11]).  $\square$

In the following we just give the definition of lattice-ordered abelian group, along with a couple of examples. Standard references on the subject are [6] and [12].

**Definition 1.3.4** ( $\ell$ -group). A *lattice-ordered abelian group* (or  $\ell$ -group, for short) is an algebra  $\mathbf{G} = (G, +, -, \wedge, \vee, \mathbf{0})$  such that

VL1)  $(G, +, -, \mathbf{0})$  is an abelian group;

VL2)  $(G, \wedge, \vee)$  is a lattice;

VL3)  $t + (v \wedge w) = (t + v) \wedge (t + w)$ , for all  $t, v, w \in V$ .

In a way very similar to the one used for vector lattice (see Definition 1.2.9), we can define the concept of unit for an  $\ell$ -group.

**Definition 1.3.5** (Unit (for  $\ell$ -groups)). Given an  $\ell$ -group  $G$ , an element  $u \in G$  is a *strong order unit*, or just a *unit* for short, if for all  $0 \leq v \in G$  there exists  $n \in \mathbb{N}$  such that  $v \leq nu$ . A *unital  $\ell$ -group* is a pair  $(G, u)$ , where  $G$  is an  $\ell$ -group and  $u$  is a unit of  $G$ .

**Example 1.3.6.** The additive groups  $\mathbb{R}$ ,  $\mathbb{Q}$ , and  $\mathbb{Z}$ , equipped with their natural order, are examples of  $\ell$ -groups. In this examples, each element  $x > 0$  is a unit.  $\square$

**Example 1.3.7.** The set of all McNaughton functions  $f: [0, 1]^n \rightarrow \mathbb{R}$ , equipped with the pointwise defined operations of addition, difference, minimum, maximum and with the function identically zero, is an  $\ell$ -group. An example of a unit in this case is the function identically equal to 1 on  $[0, 1]^n$ .  $\square$

As we have done for vector lattices, in the following we will blur the distinction between an MV-algebra  $\mathbf{A}$  (or an  $\ell$ -group  $\mathbf{G}$ ) and its underlying set  $A$  ( $G$ , respectively). Moreover, we will denote both the element  $\mathbf{0}$  of  $A$  (or  $G$ ) and the real number zero by the same symbol  $0$ : the meaning will be clear from the context.

The importance of a categorical equivalence between MV-algebras and unital  $\ell$ -groups lies, for example, in the fact that the definition of a unit for an  $\ell$ -group cannot be expressed in an equational way. Actually, the notion of unit is not even elementary (i.e., definable by first-order formulæ), by a standard compactness argument. However, up to categorical equivalence, unital  $\ell$ -groups can be defined by equations: we can use the equations of MV-algebras.

### 1.3.1 The $\Gamma$ -functor Theorem

In the following we will consider the two categories of MV-algebras and unital  $\ell$ -groups. The objects of these categories are clear. For the morphisms, we consider the maps that preserve the MV-algebraic operations (*homomorphisms of MV-algebras*), and the maps that preserve both the  $\ell$ -group operations and the fixed units (*unital  $\ell$ -homomorphisms*).

**Theorem 1.3.8** ( $\Gamma$ -functor Theorem,[28, Theorem 3.9]). *There is a natural equivalence between the category of unital  $\ell$ -groups, and the category of MV-algebras.*

The name “ $\Gamma$ -functor Theorem” is due to the fact that Mundici denoted by  $\Gamma$  the functor that gives the equivalence in the previous theorem.

If we consider a unital  $\ell$ -group  $(G, u)$ , we can “truncate  $G$  to  $u$ ” and obtain an MV-algebra. More precisely, we consider the *unit interval* of  $G$

$$\Gamma(G, u) = [0, u] = \{x \in G \mid 0 \leq x \leq u\},$$

equipped with the operations

$$x \oplus_u y = (x + y) \wedge u \quad \text{and} \quad \neg_u x = u - x.$$

It can be shown that, proceeding in this way,  $\Gamma(G, u)$  is an MV-algebra. Then we define  $\Gamma$  as the functor that maps each unital  $\ell$ -group  $(G, u)$  into the MV-algebra  $\Gamma(G, u)$ , and each unital  $\ell$ -homomorphism with domain  $G$  into its restriction to the unit interval  $[0, u]$ .

To obtain a categorical equivalence, we have to invert the  $\Gamma$  functor. Roughly speaking, we need to rebuild the  $\ell$ -group  $G$  from its associated MV-algebra  $\Gamma(G, u)$ . The details of this construction, together with the complete proof of the  $\Gamma$ -functor Theorem, can be found in [28] and [11]. Here we will just report one of the crucial lemmas. The idea is that each element of  $G$  can be split into a uniquely determined sequence of elements of  $\Gamma(G, u)$ . Such a sequence enjoys special properties, and is defined as follows.

**Definition 1.3.9** (Good sequence). Given an MV-algebra  $(A, \oplus, \neg, 0)$ , a *good sequence* is a sequence  $(a_i)_{i \in \mathbb{N}}$  of elements  $a_i \in A$  such that

*GS1)* there exists an index  $j \in \mathbb{N}$  such that, for all  $i \geq j$ ,  $a_i = 0$ ;

*GS2)*  $a_i \oplus a_{i+1} = a_i$ , for all  $i \in \mathbb{N}$ .

We observe that the previous definition is purely MV-algebraic. This fact, together with the uniqueness of the representation of the elements of  $G$  stated in the following lemma, assures the possibility of recovering  $G$  from  $\Gamma(G, u)$ .

**Lemma 1.3.10** ([11, Lemma 7.1.3]). *Let  $(G, u)$  be a unital  $\ell$ -group, and let  $A = \Gamma(G, u)$ . Then, for each  $0 \leq a \in G$ , there exists a unique good sequence  $(a_i)_{i \in \mathbb{N}}$  in  $A$  such that  $a = \sum_{i \in \mathbb{N}} a_i$ .*

(Note that the sum in Lemma 1.3.10 is finite because of condition 1 in Definition 1.3.9.)

The lemma thus states that when we cut  $G$  to the unit  $u$  we do not lose information. We will try to mimic this fundamental Lemma 1.3.10 in geometric terms in Chapter 4.

## 1.4 Valuations

Valuations are a central topic of geometric probability. They are functionals over lattices that can be seen as generalizations of measures.

**Definition 1.4.1** (Valuation). Given a lattice  $L$ , a *valuation* on  $L$  is a function  $\nu: L \rightarrow \mathbb{R}$  such that, for all  $x, y \in L$ , the following valuation property is satisfied:

$$\nu(x \vee y) + \nu(x \wedge y) = \nu(x) + \nu(y).$$

**Example 1.4.2.** If we consider the Boolean algebra of Borel subsets of  $\mathbb{R}^n$ , equipped with the operations of intersection and union, the function  $\text{vol}$ , that assigns to each element of the Boolean algebra its  $n$ -dimensional Lebesgue measure, is a valuation that assigns to the bottom  $\emptyset$  of the algebra the value 0.  $\square$

**Example 1.4.3.** Using the same Boolean algebra as in the previous example, we can find infinitely many valuations that assign the value 0 to the set  $\emptyset$ , in the following way. We fix a point  $x \in \mathbb{R}^n$  and then define, for each Borel subset  $A$  of  $\mathbb{R}^n$ ,

$$\delta_x(A) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A. \end{cases}$$

The functional  $\delta_x$  is called a *Dirac valuation*.  $\square$

As presented in [17], the study of valuations on lattices can be motivated by the main method used to solve one of the best-known problems in geometric probability, the *Buffon needle problem*. Consider a needle of fixed length  $l$ , and drop it at random on the plane  $\mathbb{R}^2$ , where parallel straight lines at a fixed distance  $d$  from each other are drawn. We would like to find the probability that the needle shall meet at least one of the lines.

The standard solution of the Buffon needle problem is given via the characterization of an additive functional on a suitable collection of sets in the plane. In this special case, the functional is also required to be monotonically increasing and invariant under the group of Euclidean motions of sets in the plane.

One of the most important results of geometric probability, *Hadwiger's Characterization Theorem*, is actually a characterization of all continuous and invariant valuations on the lattice generated by compact convex subsets in the Euclidean space.





## Chapter 2

# The Euler-Poincaré characteristic

A *polyconvex set* is a finite union of compact convex subsets of  $\mathbb{R}^n$ . In particular, every polyhedron is a polyconvex set. Moreover, the family of polyconvex sets of  $\mathbb{R}^n$  forms a distributive lattice, with respect to the set-theoretic operations of intersection and union. In Hadwiger's terminology, this lattice is known as the *Konvexring*. The term "polyconvex" was introduced in [17] after a suggestion by Ennio De Giorgi.

Hadwiger's Theorem states that we can characterize the valuations on the Konvexring that are continuous, with respect to a suitable metric, and invariant with respect to the rigid motions in  $\mathbb{R}^n$ . (Rigid motions are the elements of the group of Euclidean transformations generated by translations and rotations.) Specifically, such valuations form a linear space under pointwise operations. Moreover, *there exists a finite basis*  $\mu_0, \dots, \mu_n$  *of these valuations on polyconvex sets of*  $\mathbb{R}^n$ , and the elements of the basis are precisely identified: they are the so-called *intrinsic volumes*. The first element  $\mu_0$  is the Euler-Poincaré characteristic (that will be treated in detail later in this chapter), and the last one,  $\mu_n$ , is the volume ( $n$ -dimensional Lebesgue measure) in  $\mathbb{R}^n$ . Further, Hadwiger proved that *the Euler-Poincaré characteristic is the unique continuous invariant valuation on the Konvexring, that takes the value 1 on each non-empty compact convex set, and 0 on the empty one*. For a proof, see [9] or [17, Theorem 5.2.1].

In this chapter we will obtain an analogue of this last part of Hadwiger's characterization. We will define a suitable class of valuations on unital vector lattices, and then we will give an appropriate notion of Euler-Poincaré characteristic of the elements of a fixed finitely presented unital vector lattice. Then we will prove that our Euler-Poincaré characteristic is the unique such valuation that assigns the value 1 to each vl-Schauder hat of the vector lattice. Here, the vl-Schauder hats, to be defined below, can be seen as building blocks of the vector lattice itself, just as the compact convex sets are the building blocks for Hadwiger's Konvexring.

## 2.1 The Euler-Poincaré characteristic

The Euler-Poincaré characteristic is a topological invariant, studied in algebraic topology and in polyhedral combinatorics. It was originally defined for polyhedra, and it was used to prove many different theorems about them, as, for example, the classification of Platonic solids. Classically, it was defined just for the surfaces of polytopes in  $\mathbb{R}^3$  by the formula

$$\chi = v - e + f,$$

where  $v$  is the number of “vertices” (0-simplices),  $e$  the number of “edges” (1-simplices), and  $f$  the number of “faces” (2-simplices) of a polytope. *Euler’s polyhedron formula* states that the characteristic of the surface of a polytope in  $\mathbb{R}^3$  is equal to 2.

In modern mathematics, the concept of Euler-Poincaré characteristic has been extended to polyhedra in any dimension, and then to topological spaces.

**Definition 2.1.1.** Given a polyhedron  $P$  triangulated by the simplicial complex  $K$  with dimension  $n$ , the *Euler-Poincaré characteristic* of  $P$  is the number

$$\chi(P) = \sum_{m=0}^n (-1)^m \alpha_m, \quad (2.1)$$

where  $\alpha_m$  is the number of faces of  $K$  that have dimension  $m$ .

More generally, we have the following definition.

**Definition 2.1.2.** Given a topological space  $T$  and an integer  $m \geq 0$ , write  $\beta_m$  for its  $m$ th Betti number, that is, the rank of its  $m$ th singular homology group. Assume that  $T$  is such that  $\beta_m = 0$  for each sufficiently large  $m$ . Then its *Euler-Poincaré characteristic* is the number

$$\chi(T) = \sum_{m=0}^{\infty} (-1)^m \beta_m. \quad (2.2)$$

Note that if a space  $T$  embeds into  $\mathbb{R}^n$ , then its Euler-Poincaré characteristic is well-defined, because  $T$  cannot have a nontrivial homology in dimension  $> n$ .

One of the most important results in homotopy theory about the Euler-Poincaré characteristic is that it is a homotopy-type invariant (see [25, Lemma 4.5.17] and the remarks following it). Moreover, it can be shown that, for polyhedra, the two definitions given above coincide. Hence the Euler-Poincaré characteristic of a polyhedron  $P$  given in Definition 2.1.1 does not depend on the choice of the triangulation  $K$ . For more details, see e.g. [25].

The aim of this chapter is to find a way to define the Euler-Poincaré characteristic for the elements of a fixed finitely presented unital vector lattice, represented as  $(\nabla(P), \mathbb{1})$  for some polyhedron  $P$ , and then to characterize it in terms of valuations on vector lattices.

From now on, we fix a polyhedron  $P$  in some Euclidean space  $\mathbb{R}^n$ , and we consider the unital finitely presented vector lattice  $(\nabla(P), \mathbb{1})$ .

## 2.2 Hats

Here, we isolate a special class of elements of  $(\nabla(P), \mathbf{1})$  that we call *vl-Schauder hats* (or just *hats*, for short). Hats form a generating set of the underlying vector space of the vector lattice: each element of  $\nabla(P)$  is a linear combination of a suitable finite set of hats. Schauder hats originate in Banach space theory (see [33], and references therein). In the present context, vl-Schauder hats play the same role as Schauder hats in MV-algebras (see [27], [11, 9.2.1]) and lattice-ordered Abelian groups ([20, and references therein]). Pursuing the analogy with above-mentioned Hadwiger's theorem, we will see in due course that our version of the Euler-Poincaré characteristic assigns value one to each hat.

Formally, we define hats as follows.

**Definition 2.2.1** (vl-Schauder hats). A *vl-Schauder hat* is an element  $h \in \nabla(P)$  for which there are a triangulation  $K_h$  of  $P$  linearizing  $h$  and a vertex  $\bar{x}$  of  $K_h$  such that  $h(\bar{x}) = 1$  and  $h(x) = 0$  for any other vertex  $x$  of  $K_h$ .

We remark that it is possible to characterize vl-Schauder hats abstractly in the language of vector lattices, transposing to our context the results obtained for MV-algebras and  $\ell$ -groups (see [22]). Abstract Schauder hats are today known as elements of a basis. Bases of MV-algebras and unital  $\ell$ -groups appear, e.g., in [24]. The existence of a basis is a necessary and sufficient condition for an MV-algebra or an  $\ell$ -group to be finitely presented (see [26]). Therefore, the results of this chapter, stated below, may be regarded as theorems about unital vector lattices that do not depend on any geometric representation.

Defining vl-Schauder hats as in Definition 2.2.1, given a triangulation  $K$  of  $P$  with vertices  $\{x_0, \dots, x_m\}$ , the *vl-Schauder hats of  $K$*  are those vl-Schauder hats  $\{h_i\}$  such that, for all  $i, j \in \{0, \dots, m\}$ ,  $h_i(x_i) = 1$  and  $h_i(x_j) = 0$ . The uniquely determined  $x_i$  such that  $h_i(x_i) = 1$  is called the *vertex* of  $h_i$ . Each  $f \in \nabla(P)$  can be written as a sum  $\sum_{i=0}^m a_i h_i$  (where  $a_i \in \mathbb{R}$ ) of distinct vl-Schauder hats  $h_0, \dots, h_m$  of a common linearizing triangulation  $K_f$  for  $f$ . If  $f \in \nabla(P)^+$ , then necessarily  $0 \leq a_i$  for all  $i = 0, \dots, m$ .

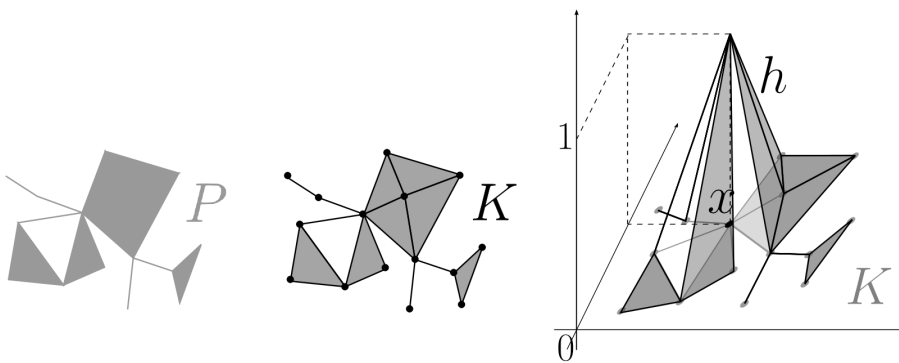


Figure 2.1: The function  $h$  is an example of vl-Schauder hat. Consider the polyhedron  $P$  given in the picture on the left, and the triangulation  $K$  of  $P$  shown in the central picture. The function  $h$  (in the picture on the right) is the vl-Schauder hat of  $K$  with vertex  $x$ .

**Remark 2.2.2.** Given any two vl-Schauder hats  $h_i$  and  $h_j$  of a triangulation  $H$ , with vertices  $x_i$  and  $x_j$ , respectively, the element  $h_{ij} = 2(h_i \wedge h_j)$  is either zero or a vl-Schauder hat with vertex  $x_{ij} = \widehat{(x_i, x_j)}$ . (Recall that  $\widehat{(x_i, x_j)} = \frac{1}{2}(x_i + x_j)$  is the barycentre of the simplex  $(x_i, x_j)$ .) Hence  $h_{ij}$  is a vl-Schauder hat of the triangulation  $H^*$  obtained from  $H$  by performing the barycentric subdivision of the simplex  $(x_i, x_j)$ . Explicitly, by adding the 0-simplex  $(x_{ij})$  to  $H$ , and replacing each  $n$ -simplex  $\sigma = (x_{k_0}, \dots, x_i, \dots, x_j, \dots, x_{k_n})$  with the  $n$ -simplices  $\tau = (x_{k_0}, \dots, x_i, \dots, x_{ij}, \dots, x_{k_n})$  and  $\rho = (x_{k_0}, \dots, x_{ij}, \dots, x_j, \dots, x_{k_n})$ . The vl-Schauder hats associated with  $x_i$  and  $x_j$  in the new triangulation  $H^*$  are, respectively,  $h'_i = h_i - (h_i \wedge h_j)$  and  $h'_j = h_j - (h_i \wedge h_j)$ .  $\square$

**Remark 2.2.3.** The construction in Remark 2.2.2 provides an algebraic encoding of Alexander's stellar subdivision in the language of  $\ell$ -groups. See [26] for further background and references.

## 2.3 A characterization theorem

### 2.3.1 Vl-valuations and pc-valuations

As Hadwiger's Theorem involves just continuous and invariant valuations on the Konvexring, our characterization result considers only a subclass of all the valuations that we can define on a fixed vector lattice. We will call these special valuations *vl-valuations*. We define them in the following way.

**Definition 2.3.1** (Vl-valuations). Let  $V$  be a vector lattice, and let  $V^+$  be its positive cone. A *vl-valuation* on  $V$  is a function  $\nu: V \rightarrow \mathbb{R}$  such that:

- V1)  $\nu(0) = 0$ ,
- V2) for all  $x, y \in V$ ,  $\nu(x) + \nu(y) = \nu(x \vee y) + \nu(x \wedge y)$ ,
- V3) for all  $x, y \in V^+$ ,  $\nu(x + y) = \nu(x \vee y)$ ,
- V4) for all  $x, y \in V^+$ , if  $x \wedge y = 0$  then  $\nu(x - y) = \nu(x) - \nu(y)$ .

We presently show that a vl-valuation is uniquely determined by its values at the positive cone. To do that we define a new kind of valuations that are actually restrictions of vl-valuations to  $V^+$ .

**Definition 2.3.2** (Pc-valuation). Let  $V$  be a vector lattice. A *pc-valuation* on the positive cone is a function  $\nu^+: V^+ \rightarrow \mathbb{R}$  such that:

- P1)  $\nu^+(0) = 0$ ,
- P2) for all  $x, y \in V^+$ ,  $\nu^+(x) + \nu^+(y) = \nu^+(x \vee y) + \nu^+(x \wedge y)$ ,
- P3) for all  $x, y \in V^+$ ,  $\nu^+(x + y) = \nu^+(x \vee y)$ .

**Lemma 2.3.3.** *The operation of restriction of a vl-valuation to the positive cone is a bijection between the set of all vl-valuations on  $V$  and the set of all pc-valuations on  $V^+$ . The inverse bijection is the operation that extends a pc-valuation  $\nu^+$  to the vl-valuation*

$$\nu^\pm: x \mapsto \nu^+(x^+) - \nu^+(x^-), \quad (2.3)$$

where  $x^+$  and  $x^-$  are, respectively, the positive and the negative part of  $x$ .

*Proof.* Trivially, if  $\nu$  is a vl-valuation on  $V$ , then its restriction  $\nu|_{V^+}$  is a pc-valuation. On the other hand, if we consider a pc-valuation  $\nu^+$  defined on  $V^+$ , then its extension  $\nu^\pm$  given in (2.3) is a vl-valuation:

$$\text{V1) } \nu^\pm(0) = \nu^+(0^+) - \nu^+(0^-) = \nu^+(0) - \nu^+(0) = 0;$$

$$\text{V2) for all } x, y \in V, \text{ we have } (x \vee y)^+ = (x^+ \vee y^+), (x \wedge y)^+ = (x^+ \wedge y^+), \\ (x \vee y)^- = (x^- \wedge y^-), (x \wedge y)^- = (x^- \vee y^-), \text{ and hence}$$

$$\begin{aligned} \nu^\pm(x \vee y) &= \nu^+((x \vee y)^+) - \nu^+((x \vee y)^-) = \\ &= \nu^+(x^+ \vee y^+) - \nu^+(x^- \wedge y^-) = \\ &= \nu^+(x^+) + \nu^+(y^+) - \nu^+(x^+ \wedge y^+) \\ &\quad + \nu^+(x^- \vee y^-) - \nu^+(x^-) - \nu^+(y^-) = \\ &= \nu^+(x^+) - \nu^+(x^-) + \nu^+(y^+) - \nu^+(y^-) \\ &\quad - (\nu^+((x \wedge y)^+) - \nu^+((x \wedge y)^-)) = \\ &= \nu^\pm(x) + \nu^\pm(y) - \nu^\pm(x \wedge y); \end{aligned}$$

$$\text{V3) for all } x, y \in V^+, \text{ we have } x+y \in V^+, (x+y)^+ = x+y \text{ and } (x+y)^- = 0, \\ \text{and so}$$

$$\begin{aligned} \nu^\pm(x+y) &= \nu^+((x+y)^+) - \nu^+((x+y)^-) = \nu^+(x+y) - \nu^+(0) = \\ &= \nu^+(x \vee y) - 0 = \nu^+(x^+ \vee y^+) = \nu^+((x \vee y)^+) - 0 = \\ &= \nu^+((x \vee y)^+) - \nu^+((x \vee y)^-) = \nu^\pm(x \vee y); \end{aligned}$$

$$\text{V4) for all } x, y \in V^+, \text{ if } x \wedge y = 0, \text{ then we have } x^- = y^- = 0, (x-y)^+ = x^+ \\ \text{and } (x-y)^- = y^+, \text{ and so}$$

$$\begin{aligned} \nu^\pm(x-y) &= \nu^+((x-y)^+) - \nu^+((x-y)^-) = \\ &= \nu^+(x^+) - \nu^+(y^+) = \\ &= \nu^+(x^+) - \nu^+(x^-) - (\nu^+(y^+) - \nu^+(y^-)) = \\ &= \nu^\pm(x) - \nu^\pm(y). \end{aligned}$$

Moreover, if  $\nu$  is a vl-valuation, then, for all  $x \in V$ ,

$$(\nu|_{V^+})^\pm(x) = \nu|_{V^+}(x^+) - \nu|_{V^+}(x^-) = \nu(x^+) - \nu(x^-) = \nu(x^+ - x^-) = \nu(x).$$

On the other hand, if  $\nu^+$  is a pc-valuation with extension  $\nu^\pm$ , then, for all  $x \in V^+$ ,

$$\nu^\pm|_{V^+}(x) = \nu^\pm(x) = \nu^+(x^+) - \nu^+(x^-) = \nu^+(x^+) = \nu^+(x).$$

□

So, from now on, without loss of generality, we can consider only pc-valuations and positive cones.

**Lemma 2.3.4.** *Let  $\nu^+$  be a pc-valuation, and  $x, y$  be elements of  $V^+$ . Then for all  $0 < a \in \mathbb{R}$*

$$\nu^+(x + ay) = \nu^+(x + y).$$

*Proof.* Let  $0 < b \in \mathbb{R}$ . Then for all  $\frac{b}{2} \leq c \leq b$  the inequality  $0 \leq b - c \leq \frac{b}{2}$  holds. It follows that both  $x + cy$  and  $(b - c)y$  are in  $V^+$ , and  $x + (b - c)y \leq x + cy$ . Therefore,

$$\begin{aligned} \nu^+(x + by) &= \nu^+((x + cy) + (b - c)y) = \nu^+((x + cy) \vee (b - c)y) = \\ &= \nu^+(x + cy) + \nu^+((b - c)y) - \nu^+((x + cy) \wedge (b - c)y) = \\ &= \nu^+(x + cy) + \nu^+((b - c)y) - \nu^+((b - c)y) = \nu^+(x + cy). \end{aligned}$$

By induction,  $\nu^+(x + by) = \nu^+(x + cy)$  for all  $b \geq c \geq \frac{b}{2^n}$  (for all  $n \in \mathbb{N} \setminus \{0\}$ ), whence for all  $b \geq c > 0$ . Then we can choose  $b = \max\{a, 1\}$  to have

$$\nu^+(x + ay) = \nu^+(x + y).$$

□

More generally:

**Corollary 2.3.5.** *Let  $\nu^+$  be a pc-valuation on  $V^+$ , and let  $x = \sum_{i=0}^m a_i x_i$  be such that  $0 < a_i \in \mathbb{R}$  and  $x_0, \dots, x_m \in V^+$ . Then*

$$\nu^+(x) = \nu^+\left(\sum_{i=0}^m x_i\right).$$

### 2.3.2 The main result

Our characterization theorem is the following.

**Theorem 2.3.6.** *Let  $P$  be a polyhedron in  $\mathbb{R}^n$ , for some integer  $n \geq 1$ , and let  $(\nabla(P), \mathbf{1})$  be the finitely presented unital vector lattice of real-valued piecewise linear functions on  $P$ . Then there is a unique vl-valuation*

$$\alpha: \nabla(P) \rightarrow \mathbb{R}$$

*assigning value 1 to each vl-Schauder hat of  $\nabla(P)$ . Further, for each  $0 < f \in \nabla(P)$ ,  $\alpha(f)$  coincides with the Euler-Poincaré characteristic  $\chi(f^{-1}(\mathbb{R}_{>0}))$ , given in (2.2). In particular,  $\alpha(\mathbf{1}) = \chi(P)$ .*

In the preceding statement,  $f^{-1}(\mathbb{R}_{>0})$  is the *support* of  $f$ , that is, the complement of the zero-set  $f^{-1}(0)$ . Since the support is an open set, it is not in general compact and therefore cannot be triangulated by a finite simplicial complex. Thus the classical combinatorial formula (2.1) cannot be used to define the characteristic of the support of  $f$ . Nonetheless, we can use the *supplement* of the support of  $f$ . As we said in Chapter 1, the supplement is a standard construction in algebraic topology: it is a simplicial complex  $\bar{L}$  that approximates the set-theoretic difference between the underlying polyhedra  $|K|$  and  $|L|$  of a simplicial complex  $K$  and its subcomplex  $L$ . It can be shown (see [25, Proposition 5.3.9]) that  $|\bar{L}|$  is homotopically equivalent to the set-theoretic difference  $|K| \setminus |L|$ . So the Euler-Poincaré characteristic of  $|\bar{L}|$  given by (2.1) is exactly the Euler-Poincaré characteristic of  $|K| \setminus |L|$ , defined by (2.2).

To prove Theorem 2.3.6, we will characterize exactly the vl-valuation that assigns value one to each vl-Schauder hat. In light of Lemma 2.3.3, we can restrict attention to pc-valuations on the positive cone.

First of all we observe that, as suggested by Lemma 2.3.4, a pc-valuation forgets the height of functions, so the only information it retains is concerned with supports and zero-sets. We therefore try to use the Euler-Poincaré characteristic of the support  $\text{supp}(f)$  of the functions in  $\nabla(P)^+$  to construct our pc-valuation. To this aim, we use the supplement. For each  $f \in \nabla(P)^+$ , we choose a linearizing triangulation  $K_f$  and isolate the zero-set  $Z_{K_f, f}$  of  $f$ . Then we build its supplement  $\overline{Z_{K_f, f}}$  in  $K_f$ . Now we have a simplicial complex (and so an associated polyhedron) which approximates the support of  $f$ , and we can compute its Euler-Poincaré characteristic. Recalling that  $|Z_{K_f, f}|$  is the zero-set of  $f$ , and that by [25, Proposition 5.3.9] there is a homotopy equivalence between  $|\overline{Z_{K_f, f}}|$  and  $\text{supp}(f) = P \setminus f^{-1}(0) = |K_f| \setminus |Z_{K_f, f}|$ , the Euler-Poincaré characteristic of  $|\overline{Z_{K_f, f}}|$  does not depend on the choice of the linearizing triangulation  $K_f$ , but just on the homotopy type of  $|\overline{Z_{K_f, f}}|$ , and so only on the homotopy type of  $\text{supp}(f)$ .

Because of this, the following is well defined.

**Definition 2.3.7.** We define  $\alpha^+ : \nabla(P)^+ \rightarrow \mathbb{R}$  as

$$\alpha^+(f) = \chi(\text{supp}(f)) = \chi(|\overline{Z_{K_f, f}}|),$$

where  $K_f$  is a linearizing triangulation for  $f \in \nabla(P)^+$ , and  $\chi$  is the Euler-Poincaré characteristic defined in (2.1), and in (2.2).

**Lemma 2.3.8.** *Let  $f \in \nabla(P)^+$  and  $K_f$  be a triangulation of  $P$  linearizing  $f$ . Then*

$$\overline{Z_{K_f, f}} = \{\sigma \in K'_f \mid \sigma \subseteq \text{supp}(f)\}.$$

*Proof.* First, we observe that, by the linearity of  $f$  on  $K_f$ , on the barycentres of the simplices of  $Z_{K_f, f}$  the function  $f$  takes value 0. Hence  $f$  is identically 0 on each vertex of  $Z'_{K_f, f}$ . Furthermore, again by linearity, the vertices of  $Z'_{K_f, f}$  are exactly all vertices of  $K'_f$  where  $f$  is 0. Therefore, if we compute  $\overline{Z_{K_f, f}}$  as the set of simplices of  $K'_f$  with no vertices in  $Z'_{K_f, f}$ , we have that it is the subset of  $K'_f$  of all those simplices whose vertices are in the support of  $f$ . The linearity of  $f$  on  $K'_f$  completes the proof.  $\square$

We now prove that  $\alpha^+$  is a pc-valuation that assigns 1 to each vl-Schauder hat.

**Lemma 2.3.9.** *The following hold.*

1.  $\alpha^+(0) = 0$ ;
2. if  $h \in \nabla(P)^+$  is a vl-Schauder hat, then  $\alpha^+(h) = 1$ ;
3. for all  $f, g \in \nabla(P)^+$ ,

$$\alpha^+(f + g) = \alpha^+(f \vee g) = \alpha^+(f) + \alpha^+(g) - \alpha^+(f \wedge g).$$

*Proof.* (1)  $\alpha^+(0) = \chi(\text{supp}(0)) = \chi(\emptyset) = 0$ .

(2) We can choose a triangulation  $K_h$  that linearizes  $h$  and such that the vertex  $\tilde{x}$  is the only one on which  $h > 0$ . Then we observe that  $\overline{Z_{K_h, h}}$  is the *simplicial neighbourhood* of  $\tilde{x}$  in  $K'_h$  ( $K'_h$  being the first barycentric subdivision of  $K_h$ ):  $\overline{Z_{K_h, h}}$  is the smallest subcomplex of  $K'_h$  containing each simplex of  $K'_h$  which contains  $\tilde{x}$ . It can be shown (using, for example, [25, Proposition 2.4.4]) that  $|\overline{Z_{K_h, h}}|$  is *contractible* (homotopically equivalent to the point  $\tilde{x}$ ). It follows that the Euler-Poincaré characteristic of  $|\overline{Z_{K_h, h}}|$  is the same as the one of the single point  $\tilde{x}$ . This proves that  $\alpha^+(h) = 1$ .

(3) By Remark 1.2.12, we can always choose a triangulation  $K$  of  $P$  that simultaneously linearizes  $f + g$ ,  $f \vee g$ ,  $f$ ,  $g$  and  $f \wedge g$ . Let us compute the Euler-Poincaré characteristic using this common linearizing triangulation.

Applying Lemma 2.3.8 to  $f$ ,  $g$ ,  $f \wedge g$ ,  $f \vee g$  and  $f + g$ , and observing that  $\text{supp}(f \wedge g) = \text{supp}(f) \cap \text{supp}(g)$  and  $\text{supp}(f + g) = \text{supp}(f \vee g) = \text{supp}(f) \cup \text{supp}(g)$ , we have:

$$\overline{Z_{K, f \wedge g}} = \{\sigma \in K' \mid \sigma \subseteq \text{supp}(f) \cap \text{supp}(g)\},$$

$$\overline{Z_{K, f \vee g}} = \{\sigma \in K' \mid \sigma \subseteq \text{supp}(f) \cup \text{supp}(g)\},$$

$$\overline{Z_{K, f + g}} = \{\sigma \in K' \mid \sigma \subseteq \text{supp}(f) \cup \text{supp}(g)\}.$$

Since  $\overline{Z_{K, f \vee g}} = \overline{Z_{K, f + g}}$ , the first equality  $\alpha^+(f + g) = \alpha^+(f \vee g)$  is trivial. For the second one, the  $m$ -simplex  $\sigma_m$  is in  $\overline{Z_{K, f \wedge g}}$  if, and only if, it is in both  $\overline{Z_{K, f}}$  and  $\overline{Z_{K, g}}$ , and, in this case, it also lies in  $\overline{Z_{K, f \vee g}}$ . Therefore, if  $\alpha_{m, \star}$  is the number of  $m$ -simplices in  $\overline{Z_{K, \star}}$ , then  $\alpha_{m, f \vee g} = \alpha_{m, f} + \alpha_{m, g} - \alpha_{m, f \wedge g}$ . Summing over  $m$  completes the proof.  $\square$

**Remark 2.3.10.** Observe that  $\alpha(\mathbb{1}) = \chi(P)$ . In fact, each triangulation  $K$  of  $P$  is a linearizing triangulation for  $\mathbb{1}$  and  $Z_{K, \mathbb{1}} = \emptyset$ ; then  $\overline{Z_{K, \mathbb{1}}} = K'$  and so  $|\overline{Z_{K, \mathbb{1}}}| = |K'| = P$ .  $\square$

The following technical result is crucial. It allows us to reduce the meet between vl-Schauder hats on the left-hand side of (2.4) to its right-hand side, where only sums occur.

**Lemma 2.3.11.** *Let  $h_0, \dots, h_n$  be distinct vl-Schauder hats of the same triangulation  $H$  of  $P$ . Let  $h_n^0 = h_n$ ,  $k_0 = h_0 \wedge h_n$ , and, for all  $i = 1, \dots, n$ , consider the elements  $k_i$  and  $h_n^i$ , recursively defined in the following way:*

$$\begin{aligned} k_i &= h_i \wedge h_n^i, \\ h_n^i &= h_n^{i-1} - (h_{i-1} \wedge h_n^{i-1}) = h_n^{i-1} - k_{i-1}. \end{aligned}$$

Then

$$\left( \sum_{i=0}^{n-1} h_i \right) \wedge h_n = \sum_{i=0}^{n-1} k_i. \quad (2.4)$$

Moreover, there is a unique triangulation  $K$  of  $P$  such that the non-zero elements of the set  $\{2k_0, \dots, 2k_{n-1}\}$  are distinct vl-Schauder hats of  $K$ .

*Proof.* First we notice that:

1.  $h_n^i \leq h_n$ ,



2.  $k_i \leq h_i$  and  $k_i \leq h_n$ ,
3. if  $h_n^i = 0$ , then  $\forall j \geq i$   $h_n^j = 0$  and  $k_j = 0$ ,
4. if  $2k_i \neq 0$  and  $2k_j \neq 0$  (with  $i \neq j$ ), then they are distinct: in fact, recalling Remark 2.2.2,  $h_n^i$  is always a vl-Schauder hat associated with the point  $x_n$ , whence  $2k_i$  attains its maximum at  $x_{in}$ , but the maximum of  $2k_j$  is attained at  $x_{jn}$ , and  $x_{in} \neq x_{jn}$  because  $h_i \neq h_j$ .

The proof proceeds by induction on  $n$ . If  $n = 1$ , there is nothing to prove. The only thing we need to observe is that  $2k_0 = 2(h_0 \wedge h_1)$  is either zero or a vl-Schauder hat of the triangulation  $H^* = K$  given in Remark 2.2.2.

Assume the thesis to be true for all  $m < n$ . In particular,

$$\left( \sum_{i=0}^{n-2} h_i \right) \wedge h_n = \sum_{i=0}^{n-2} k_i,$$

where  $2k_0, \dots, 2k_{n-2}$  are either the zero function or vl-Schauder hats of the single triangulation  $\bar{K}$ , together with the hats  $h'_i = h_i - k_i$  for  $i = 0, \dots, n-2$  and  $h_n^{n-1}$ . By Remark 1.2.12, we can now take a new triangulation  $L$  that linearizes all the functions involved in the proof and then consider the restrictions of these functions on each single simplex of  $L$ . There are three cases:

*Case 1.*  $h_n \leq \sum_{i=0}^{n-2} h_i \leq \sum_{i=0}^{n-1} h_i$ . In this case

$$\left( \sum_{i=0}^{n-1} h_i \right) \wedge h_n = h_n = \left( \sum_{i=0}^{n-2} h_i \right) \wedge h_n = \sum_{i=0}^{n-2} k_i.$$

Therefore, the only thing to prove is  $k_{n-1} = 0$ . If  $\exists j \in \{0, \dots, n-2\}$  such that  $h_n^j \leq h_j$ , then  $k_j = h_n^j$  and  $\forall i > j$   $h_n^i = k_i = 0$ ; in particular  $k_{n-1} = 0$ . Else, if  $\forall i \in \{0, \dots, n-2\}$   $h_i < h_n^i \leq h_n$ , then  $k_i = h_i$  and  $h_n^i = h_n - \sum_{j=1}^{i-1} h_j$ . Then

$$h_n^{n-1} = h_n - \sum_{i=0}^{n-2} h_i \leq \sum_{i=0}^{n-2} h_i - \sum_{i=0}^{n-2} h_i = 0,$$

whence  $k_{n-1} = 0$ .

*Case 2.*  $\sum_{i=0}^{n-2} h_i \leq \sum_{i=0}^{n-1} h_i < h_n$ . In this case  $\forall i \in \{0, \dots, n-1\}$  we have  $h_i < h_n$  and  $\forall j \in \{0, \dots, n-1\}$  we have  $\sum_{i=0}^j h_i < h_n$ . Moreover,  $\forall i \in \{0, \dots, n-1\}$  we have  $h_i < h_n^i$ . To prove the latter inequality, suppose (absurdum hypothesis) that there is a first index  $j$  (necessarily strictly greater than 0) such that  $h_n^j \not\leq h_j$ . Then  $h_n^j \leq h_j$ , because the triangulation  $H$  is supposed to be fine enough to linearize  $k_j = h_n^j \wedge h_j$ , and also  $h_i < h_n^i$  for all  $i < j$ . Therefore,  $k_i = h_i$ ,  $h_n^{i+1} = h_n - \sum_{l=0}^i h_l$ . It follows that  $h_n^j = h_n - \sum_{i=0}^{j-1} h_i$ , and hence  $0 \leq h_n^j - h_j = h_n - \sum_{i=0}^j h_i$ . As a consequence, the contradiction  $h_n \leq \sum_{i=0}^j h_i$ . It follows that  $k_{n-1} = h_{n-1} \wedge h_n^{n-1} = h_{n-1}$ , whence

$$\begin{aligned} \left( \sum_{i=0}^{n-1} h_i \right) \wedge h_n &= \sum_{i=0}^{n-1} h_i = \sum_{i=0}^{n-2} h_i + h_{n-1} = \\ &= \left( \left( \sum_{i=0}^{n-2} h_i \right) \wedge h_n \right) + h_{n-1} = \sum_{i=0}^{n-2} k_i + k_{n-1} = \sum_{i=0}^{n-1} k_i. \end{aligned}$$

Case 3.  $\sum_{i=0}^{n-2} h_i < h_n \leq \sum_{i=0}^{n-1} h_i$ . As in the previous case,  $\forall i \in \{0, \dots, n-2\}$  we have  $h_i < h_n^i$ , whence  $h_n^i = h_n - \sum_{j=0}^{i-1} h_j$  and  $k_i = h_i$ . Further,  $h_n^{n-1} = h_n - \sum_{i=0}^{n-2} h_i$  and

$$k_{n-1} = h_{n-1} \wedge \left( h_n - \sum_{i=0}^{n-2} h_i \right).$$

Therefore:

$$\begin{aligned} \left( \sum_{i=0}^{n-1} h_i \right) \wedge h_n &= \left( \sum_{i=0}^{n-1} h_i \right) \wedge \left( \sum_{i=0}^{n-2} h_i + h_n - \sum_{i=0}^{n-2} h_i \right) = \\ &= \left( \sum_{i=0}^{n-2} h_i + h_{n-1} \right) \wedge \left( \sum_{i=0}^{n-2} h_i + \left( h_n - \sum_{i=0}^{n-2} h_i \right) \right) = \\ &= \sum_{i=0}^{n-2} h_i + \left( h_{n-1} \wedge \left( h_n - \sum_{i=0}^{n-2} h_i \right) \right) = \sum_{i=0}^{n-2} k_i + k_{n-1} = \sum_{i=0}^{n-1} k_i. \end{aligned}$$

This proves (2.4). Finally, we show that  $\{2k_0, \dots, 2k_{n-1}\}$  (when non-zero) are distinct vl-Schauder hats of the same triangulation. Take  $\tilde{K}$ , and construct the triangulation  $K = (\tilde{K})^*$  as in Remark 2.2.2. Adopting the notation in that remark, if  $k_{n-1} = 0$ , then  $K = \tilde{K}$ ; else, we add the 0-simplex  $(x_{(n-1)n})$  and replace each  $m$ -simplex of the form  $\sigma = (x_{u_0}, \dots, x_{n-1}, x_n)$  with the  $m$ -simplices  $\tau = (x_{u_0}, \dots, x_{n-1}, x_{(n-1)n})$  and  $\rho = (x_{u_0}, \dots, x_{(n-1)n}, x_n)$ . The vl-Schauder hats of this new triangulation  $K$  are  $2k_0, \dots, 2k_{n-1}$ , together with the hats  $h'_i = h_i - k_i$  for  $i = 0, \dots, n-1$  and  $h_n^n$ . Clearly, for all  $i \neq j$ ,  $2k_i \neq 2k_j$  unless  $k_i = k_j = 0$ : trivially,  $2k_i$  attains its maximum at  $x_{in}$ ,  $2k_j$  attains its maximum at  $x_{jn}$ , and these two points are distinct because  $x_i \neq x_j$ .  $\square$

**Lemma 2.3.12.** *Let  $\nu^+$  be a pc-valuation on  $\nabla(P)^+$  that assigns 1 to each vl-Schauder hat. Then  $\nu^+(f) = \alpha^+(f)$  for all  $f \in \nabla(P)^+$ .*

*Proof.* We can write each  $0 \neq f \in \nabla(P)^+$  as a sum  $\sum_{i=0}^m a_i h_i$  (where  $0 < a_i \in \mathbb{R}$ ) of distinct vl-Schauder hats  $h_0, \dots, h_m$  of a common triangulation  $K$  that linearizes  $f$ . By Corollary 2.3.5, we also have

$$\nu^+(f) = \nu^+\left(\sum_{i=0}^m h_i\right) \quad \text{and} \quad \alpha^+(f) = \alpha^+\left(\sum_{i=0}^m h_i\right).$$

We proceed by induction on  $m$ . If  $m = 0$ , then, by Lemma 2.3.9,

$$\nu^+(f) = \nu^+(h_1) = 1 = \alpha^+(h_1) = \alpha^+(f).$$

If  $m > 0$ , by the induction hypothesis, for all  $n < m$ ,  $\nu^+\left(\sum_{j=0}^n l_j\right) = \alpha^+\left(\sum_{j=0}^n l_j\right)$  for distinct vl-Schauder hats  $l_0, \dots, l_n$  of the same triangulation

$H_n$  of  $P$ . Then, by Lemma 2.3.11 and Corollary 2.3.5,

$$\begin{aligned}
 \nu^+(f) &= \nu^+\left(\sum_{i=0}^{m-1} h_i + h_m\right) = \\
 &= \nu^+\left(\sum_{i=0}^{m-1} h_i\right) + \nu^+(h_m) - \nu^+\left(\left(\sum_{i=0}^{m-1} h_i\right) \wedge h_m\right) = \\
 &= \nu^+\left(\sum_{i=0}^{m-1} h_i\right) + \nu^+(h_m) - \nu^+\left(\sum_{i=0}^{m-1} k_i\right) = \\
 &= \nu^+\left(\sum_{i=0}^{m-1} h_i\right) + \nu^+(h_m) - \nu^+\left(\sum_{i=0}^{m-1} 2k_i\right) = \\
 &= \alpha^+\left(\sum_{i=0}^{m-1} h_i\right) + \alpha^+(h_m) - \alpha^+\left(\sum_{i=0}^{m-1} 2k_i\right) = \\
 &= \alpha^+\left(\sum_{i=0}^{m-1} h_i\right) + \alpha^+(h_m) - \alpha^+\left(\sum_{i=0}^{m-1} k_i\right) = \\
 &= \alpha^+\left(\sum_{i=0}^{m-1} h_i\right) + \alpha^+(h_m) - \alpha^+\left(\left(\sum_{i=0}^{m-1} h_i\right) \wedge h_m\right) = \\
 &= \alpha^+\left(\sum_{i=0}^{m-1} h_i + h_m\right) = \alpha^+(f).
 \end{aligned}$$

□

Now we can extend the pc-valuation  $\alpha^+$  to a vl-valuation  $\alpha$ . To complete the proof, we define  $\alpha: \nabla(P) \rightarrow \mathbb{R}$  as the map such that, for all  $f \in \nabla(P)$ ,

$$\alpha(f) = \alpha^+(f^+) - \alpha^+(f^-).$$

By the uniqueness of the extension of a pc-valuation granted by Lemma 2.3.3,  $\alpha$  is the unique vl-valuation that assigns 1 to each vl-Schauder hat of  $\nabla(P)$ , and Theorem 2.3.6 is proved. □

**Example 2.3.13.** Consider the function  $f \in \nabla([0, 1])$  of Figure 2.2. Using the linearizing triangulation in the picture, one easily computes  $\alpha(f) = 2 - 1 = 1$ .

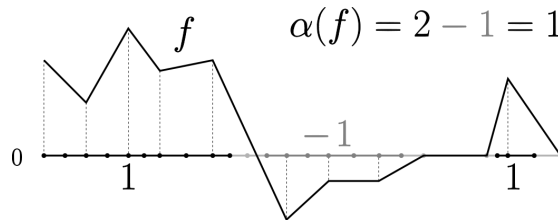


Figure 2.2: The function  $\alpha$ .



## Chapter 3

# Support functions

In Chapter 2, we have seen how dualities between algebra and geometry can be useful to investigate the behaviour of valuations on vector lattices. Specifically, the Baker-Beynon duality allows us to represent the elements of a finitely presented unital vector lattice as continuous and piecewise linear functions on some suitable polyhedron in some Euclidean space. Hence, we can use the properties of these functions, polyhedra and triangulations to define and characterize the Euler-Poincaré characteristic among the algebraically defined class of vl-valuations.

In this chapter we explore another way to connect vector lattices and geometric objects. We fix our attention only on the free vector lattice on  $n$  generators  $\text{FVL}_n$ . Then we set up a correspondence between some special elements of  $\text{FVL}_n$ , that we call *support elements*, and polytopes in  $\mathbb{R}^n$ . This allows us to prove the main results of this chapter (Theorem 3.4.8 and Theorem 3.4.10): they state a relationship between valuations on  $\text{FVL}_n$  and valuations on the lattice of polyhedra  $\mathcal{K}^n$ , under appropriate conditions of additivity. Geometrically, additivity is ensured by equipping  $\mathcal{K}^n$  with *Minkowski addition*.

The main tool used here to make the correspondence between elements of  $\text{FVL}_n$  and polytopes is the so-called *support function*. This is a tool of fundamental importance in Brunn-Minkowski theory (see [35] for a detailed treatment), in convex analysis, in functional analysis, etc.

### 3.1 Minkowski addition

We recall that we denote by  $\mathcal{P}^n$  the set of all polytopes in the Euclidean space  $\mathbb{R}^n$ , and by  $\mathcal{P}_*^n$  the set  $\mathcal{P}^n \setminus \{\emptyset\}$ .

We equip the lattice  $\mathcal{K}^n$  of polyhedra in  $\mathbb{R}^n$  with a sum operation, that will be used as a geometric counterpart of the addition operation of  $\text{FVL}_n$ .

**Definition 3.1.1** (Minkowski addition). Given any two subsets  $A$  and  $B$  of  $\mathbb{R}^n$ , their *Minkowski addition* is the set

$$A + B = \{a + b \mid a \in A, b \in B\}. \quad (3.1)$$

**Definition 3.1.2** (Product by real scalars). Given a subset  $A$  of  $\mathbb{R}^n$  and a real number  $\lambda$  we define the *product* of  $A$  by  $\lambda$  the set

$$\lambda A = \{\lambda a \mid a \in A\}.$$

The meaning of Minkowski addition is quite intuitive. The set  $A + B$  is nothing else than the union of all the possible translations of  $A$  by the vectors of  $B$ . Equation (3.1) can actually be rewritten in the following form:

$$A + B = \bigcup_{b \in B} (A + b).$$

On the other hand, the result of the product of  $A$  by  $\lambda$  is the image of  $A$  under the homothety with center at the origin of  $\mathbb{R}^n$  and ratio  $\lambda$ .

It is a standard result that the Minkowski addition of two convex sets, compact sets, polytopes or polyhedra is, respectively, a convex set, a compact set, a polytope or a polyhedron. Further, the Minkowski addition of two convex polytopes is the convex hull of the sum of their vertices, as Figure 3.1 suggests. The same results hold for the product by real scalars. Therefore, the sets  $\mathcal{K}^n$  and  $\mathcal{P}^n$  are closed under both Minkowski addition and products by real scalars.

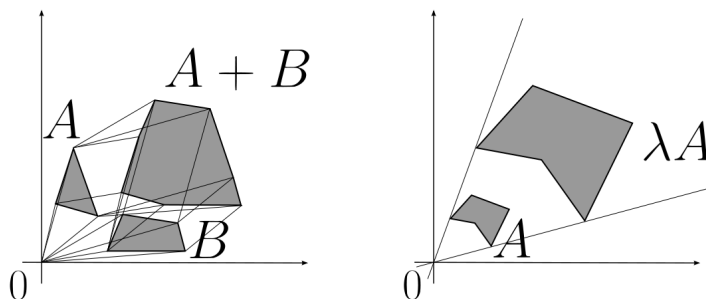


Figure 3.1: Examples of Minkowski addition (on the left) and product by scalars (on the right).

Moreover, we have the following properties:

**Proposition 3.1.3** ([35, p. 127]). *For any  $A, B, C \subseteq \mathbb{R}^n$  and for all  $0 \leq \lambda, \mu \in \mathbb{R}$ , the following hold:*

1.  $(A \cup B) + C = (A + C) \cup (B + C)$ ;
2.  $(A \cap B) + C \subseteq (A + C) \cap (B + C)$ ;
3.  $\lambda A + \lambda B = \lambda(A + B)$ ;
4.  $(\lambda + \mu)A \subseteq \lambda A + \mu A$ .

## 3.2 Support functions

Generally speaking, the definition of support function can be given for any nonempty convex set of  $\mathbb{R}^n$ . Here we only consider compact convex sets. Then, the support function, defined as in (3.2), turns out to take on real values everywhere.

**Definition 3.2.1** (Support function). For any compact convex set  $\emptyset \neq K \subseteq \mathbb{R}^n$  the *support function*  $f_K : \mathbb{R}^n \rightarrow \mathbb{R}$  is defined by stipulating that, for any  $x \in \mathbb{R}^n$ ,

$$f_K(x) = \sup\{x \cdot k \mid k \in K\}, \quad (3.2)$$

where  $\cdot$  denotes the scalar product in  $\mathbb{R}^n$ .

The intuitive meaning of the support function is closely related to the concept of support plane. The *support plane* (or *support hyperplane*) of a subset  $A$  of  $\mathbb{R}^n$  is a hyperplane  $H$  of  $\mathbb{R}^n$  such that  $A \cap H$  is nonempty, and  $A$  is contained in one of the two closed half-spaces bounded by  $H$ . Focusing our attention on a nonempty compact convex set  $K$ , we can describe its support planes via its support function. As is well known, the support planes of  $K$  have the form

$$H_K(x) = \{y \in \mathbb{R}^n \mid x \cdot y = f_K(x)\},$$

letting  $x$  range over  $\mathbb{R}^n$ . If we now consider a point  $u$  in the unit sphere  $\mathcal{S}^{n-1}$  of  $\mathbb{R}^n$ , then the value  $f_K(u)$  is the signed distance from the origin of the support plane of  $K$  with exterior normal vector  $u$ . The distance is negative if, and only if,  $u$  points into the half-space containing the origin. Then the value  $\frac{f_K(x)}{|x|}$  is the (signed) distance between the origin and the support plane  $H_K(x)$ .

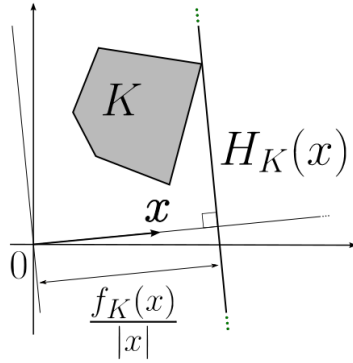


Figure 3.2: The meaning of the support function of  $K$  at the point  $x$ .

The aim of the next section will be to find a characterization of those elements of the free vector lattice  $\text{FVL}_n$  that can satisfactorily represent the support functions of polytopes, the convex objects in the lattice of polyhedra of  $\mathbb{R}^n$ . To figure out the right way to proceed with this task, we study here some basic properties of support functions.

Recalling Definition 1.2.7, a positively homogeneous function  $f$  satisfies the identity  $f(\lambda x) = \lambda f(x)$ , for all  $0 \leq \lambda \in \mathbb{R}$  and  $x \in \mathbb{R}^n$ .

**Definition 3.2.2** (Subadditive function). A function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  is *subadditive* if, for each  $x, y \in \mathbb{R}^n$ ,  $f(x + y) \leq f(x) + f(y)$ .

**Definition 3.2.3** (Sublinear function). A function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  is *sublinear* if it is both positively homogeneous and subadditive.

The following two propositions state the existence of a bijection between the sublinear real-valued functions on  $\mathbb{R}^n$  and the convex compact sets of  $\mathbb{R}^n$ .

**Proposition 3.2.4.** *The support function  $f_K$  of a convex compact set  $K \subseteq \mathbb{R}^n$  is sublinear.*

*Proof.* We first notice that, for  $K$  compact, the supremum in (3.2) is not only finite, but actually a maximum. For each real  $\lambda \geq 0$  and for each  $x \in \mathbb{R}^n$

$$f_K(\lambda x) = \max\{\lambda x \cdot k \mid k \in K\} = \lambda \max\{x \cdot k \mid k \in K\} = \lambda f_K(x).$$

Since scalar product is bilinear, for each  $x, y \in \mathbb{R}^n$ , we have

$$\begin{aligned} f_K(x) + f_K(y) &= \max\{x \cdot k \mid k \in K\} + \max\{y \cdot h \mid h \in K\} \geq \\ &\geq \max\{x \cdot k + y \cdot h \mid k, h \in K\} \geq \\ &\geq \max\{x \cdot l + y \cdot l \mid l \in K\} = \\ &= \max\{(x + y) \cdot l \mid l \in K\} = f_K(x + y). \end{aligned}$$

□

**Proposition 3.2.5** ([35, Theorem 1.7.1]). *Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  be a sublinear function. Then there is a unique convex compact subset  $K$  in  $\mathbb{R}^n$  with support function  $f$ .*

Moreover, we can prove that each support function of a convex compact set is convex and continuous. We recall here the definition of convexity for a real-valued function.

**Definition 3.2.6** (Convex function). A function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  is *convex* if, for each  $x, y \in \mathbb{R}^n$  and for all  $0 \leq \lambda \leq 1$ ,

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y).$$

**Proposition 3.2.7.** *The support function  $f_K$  of a convex set  $K \subseteq \mathbb{R}^n$  is both convex and continuous.*

*Proof.* Convexity is a direct consequence of sublinearity: for any  $x, y \in \mathbb{R}^n$  and  $\lambda \in [0, 1]$ ,

$$f(\lambda x + (1 - \lambda)y) \leq f(\lambda x) + f((1 - \lambda)y) = \lambda f(x) + (1 - \lambda)f(y).$$

Continuity is a consequence of convexity, as proved in [35, Theorem 1.5.1]. □

**Example 3.2.8.** Let  $a = (a_1, \dots, a_n)$  be a point in the space  $\mathbb{R}^n$ . Obviously,  $\{a\}$  is a polytope of  $\mathbb{R}^n$ , and its support function is

$$f_{\{a\}}(x) = a \cdot x. \tag{3.3}$$

By definition of scalar product, we can rewrite (3.3) in terms of the projection functions  $\pi_1, \dots, \pi_n : \mathbb{R}^n \rightarrow \mathbb{R}$ :

$$f_{\{a\}}(x) = \sum_{i=1}^n a_i \pi_i(x).$$

□

**Proposition 3.2.9.** *For any two compact convex sets  $K, H \subseteq \mathbb{R}^n$  and for each  $x \in \mathbb{R}^n$ ,*

$$f_{\text{conv}(K \cup H)}(x) = f_K(x) \vee f_H(x),$$

where  $\vee$  denotes pointwise maximum.



*Proof.* Without loss of generality, let  $f_K(x) \vee f_H(x) = f_K(x)$ .

Then:

$$\begin{aligned}
f_{\text{conv}(K \cup H)}(x) &= \sup\{x \cdot l \mid l \in \text{conv}(K \cup H)\} = \\
&= \sup\{x \cdot (tk + (1-t)h) \mid k \in K, h \in H, t \in [0, 1]\} = \\
&= \sup_{t \in [0, 1]} \sup\{x \cdot tk + x \cdot (1-t)h \mid k \in K, h \in H\} \leq \\
&\leq \sup_{t \in [0, 1]} (t \sup\{x \cdot k \mid k \in K\} + (1-t) \sup\{x \cdot h \mid h \in H\}) = \\
&= \sup_{t \in [0, 1]} (tf_K(x) + (1-t)f_H(x)) \leq \\
&\leq \sup_{t \in [0, 1]} (tf_K(x) + (1-t)f_K(x)) = f_K(x) = f_K(x) \vee f_H(x).
\end{aligned}$$

On the other hand,  $K \subseteq \text{conv}(K \cup H)$ , whence

$$f_{\text{conv}(K \cup H)}(x) \geq f_K(x) = f_K(x) \vee f_H(x).$$

□

We are now ready to give an explicit representation of support functions (of polytopes), which involves projection functions and the join operation of maximum between real-valued functions. In the next section we will use this representation to identify the elements of  $\text{FVL}_n$  which are the best candidates to correspond to the support functions of polytopes.

**Lemma 3.2.10.** *The support function of a polytope  $P$  of  $\mathbb{R}^n$  can always be written in the form*

$$f_K = \bigvee_{j \in J} \sum_{i=1}^n \lambda_{ij} \pi_i,$$

with  $J$  a finite set of indices.

*Proof.* Let  $\{t_1, \dots, t_m\}$  be a set of points of  $\mathbb{R}^n$  whose convex hull is  $P$ . By Example 3.2.8, their support functions are of the form

$$f_{\{t_l\}} = \sum_{i=1}^n \lambda_{li} \pi_i, \text{ for each } l = 1, \dots, m.$$

Therefore, Proposition 3.2.9 yields the representation

$$f_K = f_{\text{conv}(\{t_1, \dots, t_m\})} = \bigvee_{j=1}^m \sum_{i=1}^n \lambda_{ij} \pi_i.$$

□

The following proposition will be useful in the next section to establish a correspondence between the operation of  $\text{FVL}_n$  and the operations of  $\mathcal{K}^n$ .

**Proposition 3.2.11.** *For any two compact convex subsets  $K, H \subseteq \mathbb{R}^n$  and for each  $x \in \mathbb{R}^n$ ,*

$$f_{K+H}(x) = f_K(x) + f_H(x).$$

*Further, for any  $K \subseteq H$  we have  $f_K(x) \leq f_H(x)$ , for all  $x \in \mathbb{R}^n$ .*

*Proof.*

$$\begin{aligned} f_{K+H}(x) &= \sup\{x \cdot l \mid l \in K + H\} = \sup\{x \cdot (k + h) \mid k \in K, h \in H\} = \\ &= \sup\{x \cdot k + x \cdot h \mid k \in K, h \in H\} = \\ &= \sup\{x \cdot k \mid k \in K\} + \sup\{x \cdot h \mid h \in H\} = f_K(x) + f_H(x). \end{aligned}$$

The second part of the statement follows immediately from the definition of support function.  $\square$

### 3.3 Support elements

We recall that the free vector lattice over  $n$  generators  $\text{FVL}_n$ , in its functional representation, is generated by the projection functions  $\pi_1, \dots, \pi_n$  of  $\mathbb{R}^n$ . Moreover, we have the following property.

**Proposition 3.3.1.** *Each element  $f \in \text{FVL}_n$  can be written in a normal form as*

$$f = \bigwedge_{k \in K} \bigvee_{j \in J} \sum_{i=1}^n \lambda_{ijk} \pi_i,$$

where  $K$  and  $J$  are finite sets of indices and the  $\lambda_{ijk}$ 's are real coefficients.

*Proof.*  $\text{FVL}_n$  is generated by the projections, using the vector lattice operations. Thus, the result follows immediately from the first statement in Theorem 1.2.3.  $\square$

The representation given in Proposition 3.3.1 and the result about support functions of Lemma 3.2.10, suggest the following definition.

**Definition 3.3.2** (Support element). A *linear word* is an element of  $\text{FVL}_n$  that can be represented as a sum  $\sum_{i=1}^n \lambda_i \pi_i$ . A *support element* is an element of  $\text{FVL}_n$  that can be written in the form:  $\bigvee_{j \in J} \sum_{i=1}^n \lambda_{ij} \pi_i$ . We let  $\mathcal{S}$  denote the set of support elements of  $\text{FVL}_n$ . Note that  $\mathcal{S}$  is closed under the join operation of  $\text{FVL}_n$ ; by Proposition 1.2.2 it is also closed by addition.

**Theorem 3.3.3.** *A function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  is the support function of some nonempty polytope  $P$  in  $\mathcal{P}_*^n$  if and only if  $f \in \mathcal{S}$ .*

*Proof.* We have already proved one direction of the correspondence in Lemma 3.2.10. For the other direction, we consider a function  $f \in \mathcal{S}$ , and its normal form

$$\bigvee_{j \in J} \sum_{i=1}^n \lambda_{ij} \pi_i,$$

with  $J$  finite. As observed in Example 3.2.8, each linear word  $\sum_{i=1}^n \lambda_{ij} \pi_i$  is the support function  $f_{\{x_j\}}$  of the point  $x_j = (\lambda_{1j}, \dots, \lambda_{nj}) \in \mathbb{R}^n$ . Then we can use Proposition 3.2.9 to obtain the equality

$$f = \bigvee_{j \in J} f_{\{x_j\}} = f_{\text{conv}(\{x_j\}_{j \in J})}.$$

The fact that the convex hull of a finite set of points in  $\mathbb{R}^n$  is always a polytope completes the proof.  $\square$

As promised, Theorem 3.3.3 gives a one-to-one correspondence between the support elements of  $\text{FVL}_n$  and the support functions of polytopes of  $\mathbb{R}^n$ . Moreover, we can extend this correspondence to the operations induced by those of  $\text{FVL}_n$  and  $\mathcal{K}^n$  on, respectively,  $\mathcal{S}$  and  $\mathcal{P}^n$ . Let us consider the two structures  $(\mathcal{S}, \tilde{\wedge}, \vee, +)$  and  $(\mathcal{P}^n, \cap, \tilde{\cup}, +)$ , where  $\tilde{\wedge}$  is the meet operation on  $\mathcal{S}$  induced by the order of  $\text{FVL}_n$ , and  $\tilde{\cup}$  is the join operation induced on  $\mathcal{P}^n$  by the order of  $\mathcal{K}^n$ . The last one turns out to be the “convexification of the union”: for each  $K, H \in \mathcal{P}^n$

$$K \tilde{\cup} H = \text{conv}(K \cup H).$$

Then, by Proposition 3.2.9 and Proposition 3.2.11, we have the following result.

**Theorem 3.3.4.** *The map  $\iota : \mathcal{P}^n \rightarrow \mathcal{S}$  defined as  $\iota(P) = f_P$  is an isomorphism between  $(\mathcal{P}^n, \cap, \tilde{\cup}, +)$  and  $(\mathcal{S}, \tilde{\wedge}, \vee, +)$ . More precisely, for any two polytopes  $P, Q \in \mathcal{P}^n$  we have the following equalities:*

$$\iota(P \cap Q) = f_P \tilde{\wedge} f_Q, \quad \iota(P \tilde{\cup} Q) = f_P \vee f_Q \quad \text{and} \quad \iota(P + Q) = f_P + f_Q.$$

We conclude this section with a result about support elements, which will be useful in the remainder of this chapter.

**Lemma 3.3.5.** *Each element  $f \in \text{FVL}_n$  is the difference  $s_1 - s_2$  of some elements  $s_1, s_2 \in \mathcal{S}$ .*

*Proof.* If we set  $V = \text{FVL}_n$ ,  $L = \{\pi_i \mid i = 1, \dots, n\}$ , then, using the notation of Theorem 1.2.3, we have  $J(L) = \mathcal{S}$ . Thus, we can apply Corollary 1.2.4.  $\square$

## 3.4 Geometric and algebraic valuations

In this section we use Theorem 3.3.4 to give a correspondence between valuations on  $\text{FVL}_n$  and valuations on  $\mathcal{K}^n$ , under suitable conditions of additivity.

We specialize the notion of lattice-theoretic valuation given in Definition 1.4.1 to the two structures considered in this chapter.

**Definition 3.4.1** (Valuations on  $\text{FVL}_n$  and  $\mathcal{K}^n$ ). *A valuation on  $\text{FVL}_n$  is a map  $\nu : \text{FVL}_n \rightarrow \mathbb{R}$  such that, for each  $f, g \in \text{FVL}_n$ ,  $\nu(f \wedge g) + \nu(f \vee g) = \nu(f) + \nu(g)$ . A valuation on  $\mathcal{K}^n$  is a map  $\nu : \mathcal{K}^n \rightarrow \mathbb{R}$  such that, for each  $H, K \in \mathcal{K}^n$ ,  $\nu(H \cap K) + \nu(H \cup K) = \nu(H) + \nu(K)$ .*

We remark that the vl-valuations used in Chapter 2 to axiomatize the Euler-Poincaré characteristic are a special case of the notion of valuation used in Definition 3.4.1.

We now turn to additive valuations. The interest of additive maps in convex geometry is due to the fact that Minkowski addition is a basic operation on the lattice of convex objects in the Euclidean space. Thus, maps defined on convex bodies that are compatible with this addition play a central role in the study of the properties of Hadwiger’s Konvexring.

**Definition 3.4.2** (Additivity). *A valuation  $\nu : \text{FVL}_n \rightarrow \mathbb{R}$  is additive if, for each  $f, g \in \text{FVL}_n$ ,  $\nu(f + g) = \nu(f) + \nu(g)$ . We say that a valuation  $\nu : \text{FVL}_n \rightarrow \mathbb{R}$  is additive on  $\mathcal{S}$  if, for each  $f, g \in \mathcal{S}$ ,  $\nu(f + g) = \nu(f) + \nu(g)$ . A valuation  $\nu : \mathcal{K}^n \rightarrow \mathbb{R}$  is  $\mathcal{M}$ -additive if, for each  $H, K \in \mathcal{K}^n$ ,  $\nu(H + K) = \nu(H) + \nu(K)$ . We say that a valuation  $\nu : \mathcal{K}^n \rightarrow \mathbb{R}$  is  $\mathcal{M}$ -additive on  $\mathcal{P}_\star^n$  if, for each  $H, K \in \mathcal{P}_\star^n$ ,  $\nu(H + K) = \nu(H) + \nu(K)$ .*

**Definition 3.4.3** (Linearity). A valuation  $\nu: \text{FVL}_n \rightarrow \mathbb{R}$  is *linear* if it is additive and, for each  $f \in \text{FVL}_n$  and for each  $0 < \lambda \in \mathbb{R}$ ,  $\nu(\lambda f) = \lambda\nu(f)$ .

A valuation  $\nu: \mathcal{K}^n \rightarrow \mathbb{R}$  is  *$\mathcal{M}$ -linear* if it is  $\mathcal{M}$ -additive and, for each  $H \in \mathcal{K}^n$  and  $0 < \lambda \in \mathbb{R}$ , we have the identity  $\nu(\lambda H) = \lambda\nu(H)$ .

The following property, known as the *inclusion-exclusion principle*, has a crucial role in the proof that every valuation on polytopes has an  $\mathcal{M}$ -additive extension to  $\mathcal{K}^n$ .

**Definition 3.4.4** (Full additivity). A map  $\nu: \mathcal{S} \rightarrow \mathbb{R}$  is *fully additive* if, whenever  $f_1, \dots, f_n$  and  $\bigwedge_{i=1}^n f_i$  are in  $\mathcal{S}$ , the following equation holds:

$$\nu\left(\bigwedge_{i=1}^n f_i\right) = \sum_{\emptyset \neq V \subseteq \{1, \dots, n\}} (-1)^{|V|-1} \nu\left(\bigvee_{j \in V} f_j\right).$$

A map  $\nu: \mathcal{P}^n \rightarrow \mathbb{R}$  is *fully additive* if, whenever  $K_1, \dots, K_n$  and  $\bigcup_{i=1}^n K_i$  are in  $\mathcal{P}^n$ , the following equation holds:

$$\nu\left(\bigcup_{i=1}^n K_i\right) = \sum_{\emptyset \neq V \subseteq \{1, \dots, n\}} (-1)^{|V|-1} \nu\left(\bigcap_{j \in V} K_j\right).$$

We recall that an *intersectional family* of sets  $\mathcal{F}$  is a collection of sets closed by finite intersections. We denote by  $U(\mathcal{F})$  its closure under finite unions.  $\mathcal{P}^n$  is an intersectional family of sets in  $\mathcal{K}^n$ , and  $\mathcal{K}^n = U(\mathcal{P}^n)$ .

The next result is known as the *Volland-Groemer Extension Theorem*:

**Theorem 3.4.5** ([35, Theorem 3.4.11]). *Let  $\nu$  be a valuation with  $\nu(\emptyset) = 0$  on an intersectional family  $\mathcal{F}$  of sets. Then  $\nu$  has an additive extension to the lattice  $U(\mathcal{F})$  if and only if  $\nu$  is fully additive. The extension is unique.*

*Idea of proof.* The general idea is to write an element  $K \in U(\mathcal{F})$  as  $K_1 \cup \dots \cup K_m$ , for  $K_i \in \mathcal{F}$ . Using full additivity, one then obtains  $\nu(K)$  from the  $\nu(K_i)$  along with the values  $\nu(J)$  where  $J$  ranges over arbitrary intersections of the  $K_i$ : note that any such  $J$  belongs to  $\mathcal{F}$ .  $\square$

**Corollary 3.4.6.** *Let  $\nu$  be a valuation with  $\nu(\emptyset) = 0$  on  $\mathcal{P}^n$ . Then  $\nu$  has an  $\mathcal{M}$ -additive extension to  $\mathcal{K}^n$  if and only if  $\nu$  is fully additive on  $\mathcal{P}^n$ . The extension is unique.*

The following Theorem yields a useful condition for a valuation to be fully additive.

**Theorem 3.4.7** ([35, Theorem 3.4.13]). *Every valuation  $\nu$  on  $\mathcal{P}^n$  that is  $\mathcal{M}$ -additive on  $\mathcal{P}_*^n$  and such that  $\nu(\emptyset) = 0$  is fully additive on  $\mathcal{P}^n$ .*

The next results (Theorems 3.4.8 and 3.4.10) yield a correspondence between valuations on  $\mathcal{K}^n$  and valuations on  $\text{FVL}_n$ . To this purpose we will use the isomorphism between  $\mathcal{P}^n$  and  $\mathcal{S}$ , and then we extend every additive valuation to  $\mathcal{K}^n$  and  $\text{FVL}_n$ , using the inclusion-exclusion principle and together with full additivity.

**Theorem 3.4.8.** *Let  $\nu: \mathcal{K}^n \rightarrow \mathbb{R}$  be an  $\mathcal{M}$ -additive valuation on  $\mathcal{P}_\star^n$  such that  $\nu(\emptyset) = 0$ . Then there exists a unique valuation  $\nu^\star: \text{FVL}_n \rightarrow \mathbb{R}$  such that  $\nu^\star(f_P) = \nu(P)$ , for each  $f_P \in \mathcal{S}$ . Moreover  $\nu^\star$  is additive.*

*Proof.* We can use Lemma 3.3.5 to rewrite  $f$  as the sum  $g - h$ , with  $g, h \in \mathcal{S}$ . By Theorem 3.3.3, there are uniquely determined polytopes  $G, H \in \mathbb{R}^n$  such that  $f_G = g$  and  $f_H = h$ . Then we define  $\nu^\star: \text{FVL}_n \rightarrow \mathbb{R}$  as

$$\nu^\star(f) = \nu(G) - \nu(H).$$

First of all we observe that  $\nu^\star$  is well defined. To show this, consider two different decompositions  $g_1 - h_1$  and  $g_2 - h_2$  of  $f$ . Writing  $G_1, G_2, H_1, H_2$  for the polytopes whose support functions are  $g_1, g_2, h_1, h_2$ , respectively, we obtain

$$\begin{aligned} g_1 + h_2 &= g_2 + h_1 \\ G_1 + H_2 &= G_2 + H_1 \\ \nu(G_1 + H_2) &= \nu(G_2 + H_1) \\ \nu(G_1) + \nu(H_2) &= \nu(G_2) + \nu(H_1), \end{aligned}$$

whence  $\nu(G_1) - \nu(H_1) = \nu(G_2) - \nu(H_2)$ .

Further,  $\nu^\star$  is additive:

$$\begin{aligned} \nu^\star(f_1 + f_2) &= \nu^\star((g_1 - h_1) + (g_2 - h_2)) = \\ &= \nu^\star((g_1 + g_2) - (h_1 + h_2)) = \\ &= \nu(G_1 + G_2) - \nu(H_1 + H_2) = \\ &= \nu(G_1) + \nu(G_2) - \nu(H_1) - \nu(H_2) = \\ &= \nu^\star(g_1 - h_1) + \nu^\star(g_2 - h_2) = \nu^\star(f_1) + \nu^\star(f_2). \end{aligned}$$

Finally,  $\nu^\star$  is a valuation:

$$\nu^\star(f_1 + f_2) = \nu^\star((f_1 \wedge f_2) + (f_1 \vee f_2)) = \nu^\star(f_1 \wedge f_2) + \nu^\star(f_1 \vee f_2).$$

□

**Corollary 3.4.9.** *Let the valuation  $\nu$  in Theorem 3.4.8 be  $\mathcal{M}$ -linear on  $\mathcal{P}_\star^n$ . Then  $\nu^\star$  is also linear.*

*Proof.* Let  $0 < \lambda \in \mathbb{R}$  and let  $f$  be an element of  $\text{FVL}_n$ . Write  $f = g - h$ , with  $g, h \in \mathcal{S}$ , using Lemma 3.3.5. Then  $\lambda g$  and  $\lambda h$  are elements of  $\mathcal{S}$  and their associated polytopes are  $\lambda G$  and  $\lambda H$ . Therefore,

$$\begin{aligned} \nu^\star(\lambda f) &= \nu^\star(\lambda g - \lambda h) = \nu(\lambda G) - \nu(\lambda H) = \\ &= \lambda(\nu(G) - \nu(H)) = \lambda\nu^\star(g - h) = \lambda\nu^\star(f). \end{aligned}$$

□

**Theorem 3.4.10.** *Let  $\nu^\star: \text{FVL}_n \rightarrow \mathbb{R}$  be an additive valuation on  $\mathcal{S}$  such that  $\nu^\star(0) = 0$ . Then there exists a unique valuation  $\nu: \mathcal{K}^n \rightarrow \mathbb{R}$  such that  $\nu(\emptyset) = 0$  and  $\nu(P) = \nu^\star(f_P)$  for each  $P \in \mathcal{P}_\star^n$ . Moreover,  $\nu$  is  $\mathcal{M}$ -additive.*

*Proof.* The additivity of  $\nu^\star$  ensures that the valuation  $\nu_{\mathcal{P}^n}: \mathcal{P}_\star^n \rightarrow \mathbb{R}$  defined as  $\nu_{\mathcal{P}^n}(P) = \nu^\star(f_P)$  for all  $P \in \mathcal{P}_\star^n$  is  $\mathcal{M}$ -additive. Defining  $\nu_{\mathcal{P}^n}(\emptyset) = 0$ , we can use Theorem 3.4.7 to get the full additivity of  $\nu_{\mathcal{P}^n}$ . Thus, by Corollary 3.4.6, there exists a unique extension  $\nu$  of  $\nu_{\mathcal{P}^n}$  to the whole  $\mathcal{K}^n$ . □



## Chapter 4

# Gauge functions

In Chapter 3, we used support functions to establish a correspondence between polytopes in  $\mathbb{R}^n$  and the set of support elements of  $\text{FVL}_n$ . In this chapter, each element of the positive cone of  $\text{FVL}_n$  will be associated with a star-shaped polyhedral set containing the origin of the Euclidean space in its topological interior. The association is set up via the so-called *gauge functions*. These were introduced by Minkowski, in his study of convex bodies. For convex bodies, indeed, gauge functions are tightly related to support functions, via the construction of the polar body. In the following, we will give a more general definition of gauge function, considering all the subsets of  $\mathbb{R}^n$ . Then we will prove a correspondence between our star-shaped objects and an enlargement of  $\text{FVL}_n^+$  consisting of certain gauge functions, namely, the set of continuous positively homogeneous functions on  $f: \mathbb{R}^n \rightarrow \mathbb{R}$  such that  $f(x) \geq 0$ , for all  $x \in \mathbb{R}^n$ . Finally, we recover the piecewise linearity condition, forcing the polyhedrality of the geometric objects under consideration.

Our main results of this chapter (Theorem 4.2.5) will be used to give a geometric counterpart of *good sequences* (see Definition 4.3.6). The latter have a pivotal role in Mundici's construction of the categorical equivalence  $\Gamma$  between MV-algebras and unital  $\ell$ -groups, [28]. As a consequence of our geometrization of good sequences, we obtain the following generalization of the fact that the structure of any unital  $\ell$ -groups is unambiguously recovered from its unit interval, via the adjoint of the  $\Gamma$ -functor: the information contained in the totality of star-shaped objects is already contained in the much smaller subset given by  $[U, \mathbb{R}^n]$ , the unit interval of Definition 4.3.2.

### 4.1 Gauge functions and star-shaped objects

Let  $g$  belong to the positive cone of  $\text{FVL}_n$ , the latter being canonically identified with a set of real valued function as in Chapter 1 (see Section 1.2.1). The  $g$  determines the set of points  $C_g = \{x \in \mathbb{R}^n \mid g(x) \leq 1\}$ .  $C_g$  inherits various properties from the continuity and positive homogeneity of  $g$ . For instance,  $C_g$  is closed with polyhedral boundary  $\partial C_g$ , and the origin  $0$  belongs to the topological interior of  $C_g$ . Further, every ray  $\rho$  having  $0$  as a vertex intersects  $\partial C_g$  in at most one point  $x_\rho$ , and the segment  $\text{conv}(0, x_\rho)$  is contained in  $C_g$ .

In the literature there are several inequivalent notions of star-shaped sets.

The one we provide in Definition 4.1.1 is essentially the same of [10, Definition 19.13]. Then we define *star-shaped objects*: they are star-shaped sets with two more closure properties (see Definition 4.1.5). This latter notion seems to be new here.

For  $x \neq 0 \in \mathbb{R}^n$ , we will henceforth denote by  $\sigma_x$  the ray (i.e., the half-line) with vertex 0 passing through  $x$ , in symbols,:

$$\sigma_x = \{\lambda x \mid 0 \leq \lambda \in \mathbb{R}\}.$$

Moreover, for  $x \in \mathbb{R}^n$ , we denote by  $[0, x]$  the closed line segment with extremes 0 and  $x$ :

$$[0, x] = \{\lambda x \mid 0 \leq \lambda \leq 1\}.$$

We say that  $[0, x]$  is *nondegenerate* if  $x \neq 0$ .

**Definition 4.1.1** (Star-shaped sets). A set  $A \subseteq \mathbb{R}^n$  is *star-shaped* if for each  $a \in A$ , the line segment  $[0, a]$  is contained in  $A$ . We denote the set of all star-shaped subsets of  $\mathbb{R}^n$  by  $\mathbb{S}^n$ . The set of all star-shaped subsets of  $\mathbb{R}^n$  which contain 0 in their topological interior is  $\mathbb{S}_0^n$ .

**Lemma 4.1.2.** *Let  $A$  be a closed set of  $\mathbb{S}_0^n$ . Then each ray  $\sigma_x$ , with  $x \neq 0$ , is either contained in  $A$ , or its intersection with  $A$  is a closed nondegenerate line segment  $[0, w]$ .*

*Proof.* If  $\sigma_x \setminus A \neq \emptyset$ , there exists a unique point  $w \in \sigma_x$  with  $|w| = \inf\{|y| \mid y \in \sigma_x \setminus A\}$ . Since 0 is in the topological interior of  $A$ , we have  $|w| > 0$ , and, by definition of  $w$ , each  $y \in \sigma_x$  with  $|y| < |w|$  is in  $A$ . Since  $A$  is closed, then  $w \in A$ . Since  $A$  is star-shaped, then there is no point  $y \in \sigma_x$  with  $|y| > |w|$  lying in  $A$ . Therefore,  $\sigma_x \cap A = [0, w]$ .  $\square$

Lemma 4.1.2 motivates the following definition.

**Definition 4.1.3** (Formal boundary). If  $A$  is in  $\mathbb{S}_0^n$ , then its *formal boundary*  $\text{bd}(A)$  is the set of all points  $w$  such that there exists a ray  $\sigma_x$  departing from 0 with  $\sigma_x \cap A = [0, w]$ .

We recall that the *topological boundary* of a set  $A \subseteq \mathbb{R}^n$  is the set  $\partial A$  of points  $x \in \mathbb{R}^n$  such that each neighborhood of  $x$  contains at least one point of  $A$  and at least one point that is not in  $A$ .

**Proposition 4.1.4.** *Let  $A \in \mathbb{S}_0^n$  be closed with closed formal boundary. Then its formal boundary  $\text{bd}(A)$  and its topological boundary  $\partial A$  coincide.*

*Proof.* If  $x \in \text{bd}(A)$ , then, by definition of formal boundary,  $\sigma_x \cap A = [0, x]$ . Hence, the sequence  $((1+1/(n+1))x)_{n \in \mathbb{N}} \subseteq \sigma_x$  converges to  $x$  and it is contained in  $\mathbb{R}^n \setminus A$ . Then  $x \in \partial A$ . On the other hand, if  $x \in \partial A$ , then we can find a sequence  $(x_n)_{n \in \mathbb{N}} \subseteq \mathbb{R}^n \setminus A$  that converges to  $x$ . Since  $A$  is closed,  $\partial A \subseteq A$ , and, therefore,  $x \in A$ . Since  $A$  is star-shaped, we have  $[0, x] \subseteq \sigma_x \cap A$ . Consider now the sequence  $(b_n \mid \sigma_{x_n} \cap A = [0, b_n])_{n \in \mathbb{N}}$ . The  $b_n$ 's always exist, because 0 is in the topological interior of  $A$  and the  $x_n$ 's are not in  $A$ . Since  $x_n \rightarrow x$  and  $b_n \in \sigma_{x_n}$ , the angles  $\alpha_n$  between  $\sigma_x$  and  $\sigma_{b_n}$  converge to 0. If there is just a finite number of distinct  $b_n$ 's, then they are eventually equal to some point  $b \in \sigma_x$  (because  $\alpha_n \rightarrow 0$ ). If the distinct  $b_n$ 's are infinitely many, then, since  $|b_n| < |x_n|$  and  $|x_n| \rightarrow |x| < +\infty$ , they form an infinite set of some compact



subset of  $\mathbb{R}^n$ . It follows that they have an accumulation point. This, together with  $\alpha_n \rightarrow 0$ , ensures that they admit a subsequence converging to some point  $b \in \sigma_x$ . Writing without loss of generality  $(b_n)_{n \in \mathbb{N}}$  for such a subsequence, in both cases we have  $|b_n| \rightarrow |b|$ ,  $|b_n| < |x_n|$  and  $|x_n| \rightarrow |x|$ . Then  $b \in [0, x]$ . Since  $\text{bd}(A)$  is closed,  $b \in \text{bd}(A)$ , whence  $x = b$ . This completes the proof.  $\square$

The converse of Proposition 4.1.4 does not hold in general: the formal boundary of a closed star-shaped subset of  $\mathbb{R}^n$  (whence having a closed topological boundary) need not be closed. See Figure 4.1 for an example.

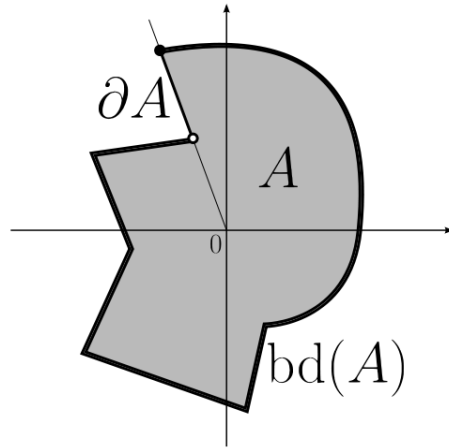


Figure 4.1: The star-shaped set  $A$  is closed, but its formal boundary (the thick line) is not closed. Therefore, the formal boundary  $\text{bd}(A)$  does not coincide with the topological boundary  $\partial A$ . In this example,  $A$  is a star-shaped set, but it is not a star-shaped object (in the sense of Definition 4.1.5).

**Definition 4.1.5** (Star-shaped object). We say that  $A \subseteq \mathbb{R}^n$  is a *star-shaped object* if  $A$  is star-shaped and closed, with closed formal boundary. We denote by  $\Sigma_0^n$  the subset of  $\mathbb{S}_0^n$  of star-shaped objects.

**Definition 4.1.6.** We let  $\mathcal{G}^n$  denote the set of continuous positively homogeneous functions  $f: \mathbb{R}^n \rightarrow \mathbb{R}$  such that  $f(x) \geq 0$  for all  $x \in \mathbb{R}^n$ .

In Theorem 4.1.10 below we will establish a one-one correspondence between  $\mathcal{G}^n$  and the set  $\Sigma_0^n$  of star-shaped objects. To this purpose we prepare:

**Definition 4.1.7** (1-cut). The *1-cut* of a function  $f: \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$  is the set

$$C_f = \{x \in \mathbb{R}^n \mid f(x) \leq 1\},$$

where we always assume  $r \leq +\infty$ , for each  $r \in \mathbb{R}$ .

**Definition 4.1.8** (Gauge function). The *gauge function* of a subset  $A$  of  $\mathbb{R}^n$  is the function  $g_A: \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$  such that, for all  $x \in \mathbb{R}^n$

$$g_A(x) = \inf\{\lambda \geq 0 \mid x \in \lambda A\},$$

where  $\lambda A$  is as in Definition 3.1.2.

In the previous definitions we allow the functions involved to take also the value  $+\infty$ . This is because we do not want, in general, to impose any special condition on the associated sets  $A$  or  $C_f$ . The gauge function of a subset  $A \subseteq \mathbb{R}^n$  that does not contain 0 in its topological interior can take on  $+\infty$  as value. This happens when there exists some  $x \in \mathbb{R}^n$  such that the ray  $\sigma_x$  is disjoint from  $A$ . In this case, the set of  $0 \leq \lambda \in \mathbb{R}$  such that  $x \in \lambda A$  is empty, and  $g_A(x) = +\infty$ . We will meet an example of this situation in the proof of Theorem 4.4.4. However, if  $A$  contains the origin in its topological interior, its gauge function can only take finite real values.

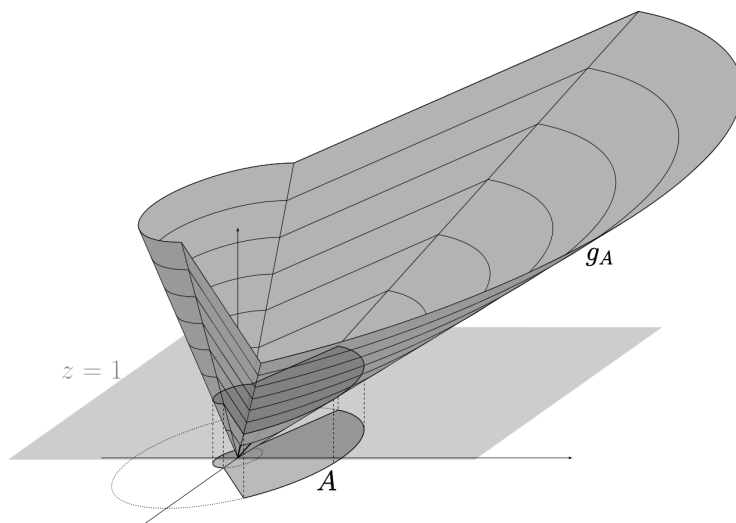


Figure 4.2: The star-shaped object  $A$  is the 1-cut of its gauge function  $g_A$ .

**Lemma 4.1.9.** *Let  $A \in \Sigma_0^n$ . Then*

- i)  $g_A(0) = 0$ ;
- ii)  $g_A(x) = 0$  for each  $x$  such that  $\sigma_x \subseteq A$ ;
- iii) for all  $x \in \mathbb{R}^n$ ,  $g_A(x) = 1$  if and only if  $x \in \text{bd}(A)$ .

*Proof.* Since  $0 \in A$ , for each  $\lambda \geq 0$ , we have  $0 \in \lambda A$ , whence  $g_A(0) = 0$ . Further, if  $x$  is a point of the ray  $\sigma_x \subseteq A$ , then, for each  $\lambda > 0$ ,  $\frac{x}{\lambda} \in A$ , whence  $g_A(x) = 0$ . Lastly, if  $g_A(x) = 1$ , then  $x \neq 0$  and the ray  $\sigma_x$  is not contained in  $A$ . For every  $0 < \lambda < 1$  we have  $x \notin \lambda A$ , and hence  $\frac{x}{\lambda} \notin A$ : each  $y \in \sigma_x$  with  $|y| > |x|$  is not in  $A$ . Moreover, we can find a sequence of numbers  $\lambda_i > 0$  that converges to 1 and such that  $x \in \lambda_i A$ , whence  $\frac{x}{\lambda_i} \in A$  with  $\frac{x}{\lambda_i} \rightarrow x$ . Since  $A$  is closed,  $x \in A$ . Since  $A$  is star-shaped, the segment  $[0, x]$  is contained in  $A$ , whence  $x \in \text{bd}(A)$ . Conversely, if  $x \in \text{bd}(A)$  then  $x \in A$ , while for each  $0 < \lambda < 1$  the point  $\lambda^{-1}x$  does not belong to  $A$ . We conclude that 1 is the smallest  $\lambda \geq 0$  such that  $x \in \lambda A$ , whence  $g_A(x) = 1$ .  $\square$

We are now ready to prove a first correspondence theorem.

**Theorem 4.1.10.** *The maps*

$$\begin{aligned} \omega: \mathcal{G}^n &\rightarrow \Sigma_0^n & \text{and} & & \gamma: \Sigma_0^n &\rightarrow \mathcal{G}^n \\ f &\mapsto C_f & & & A &\mapsto g_A \end{aligned}$$

are inverses to each other, whence define a bijection between  $\mathcal{G}^n$  and  $\Sigma_0^n$ .

*Proof.* First of all we show  $\omega(f) = C_f \in \Sigma_0^n$ , for all  $f \in \mathcal{G}^n$ , and  $\gamma(A) = g_A \in \mathcal{G}^n$ , for all  $A \in \Sigma_0^n$ . Let  $f \in \mathcal{G}^n$ . Then  $f(0) = 0$  and, by continuity, there exists a neighbourhood of 0 where  $f < 1$ , whence 0 is in the topological interior of  $C_f$ . The continuity of  $f$  also ensures that both  $C_f$  and  $\text{bd}(C_f)$  are closed. Lastly, the positive homogeneity of  $f$  makes  $C_f$  star-shaped: if  $a \in C_f$ , then  $f(a) \leq 1$  and hence, for all  $\lambda \in [0, 1]$ ,  $f(\lambda a) = \lambda f(a) \leq 1$ . Thus,  $[0, a] \subseteq C_f$ . On the other hand, if  $A \in \Sigma_0^n$ , 0 is in the topological interior of  $A$ . It follows that, for all  $x \in \mathbb{R}^n$ , the set  $\Lambda_x = \{\lambda \geq 0 \mid x \in \lambda A\}$  is nonempty; hence  $g_A(x) \geq 0$ , and  $g_A(0) = 0$ . If  $g_A(x) = l \geq 0$ , then, for each  $t > 0$ , if  $tx \in \lambda A$ , there exists  $y \in A$  such that  $tx = \lambda y$ , whence  $x = \frac{\lambda}{t}y$  and  $x \in \frac{\lambda}{t}A$ . By definition of infimum, we have  $l \leq \frac{\lambda}{t}$ , and hence  $tl \leq \lambda$ . Moreover, if  $r \geq 0$  is such that  $r \leq \lambda$  for all  $\lambda \in \Lambda_{tx}$ , then  $\frac{r}{t} \leq \mu$  for all  $\mu \in \Lambda_x = \frac{1}{t}\Lambda_{tx}$  ( $tx = \lambda y$  if and only if  $x = \frac{1}{t}\lambda y$ , with  $y \in A$ ). As a consequence  $\frac{r}{t} \leq l$  and  $r \leq tl$ . Therefore,  $\inf_{\lambda \geq 0} \Lambda_{tx} = tl$  and  $g_A$  is positively homogeneous.

To prove the continuity of  $g_A$ , we proceed by way of contradiction. If  $x$  is a point of discontinuity of  $g_A$ , then there exists a sequence  $(x_i)_{i \in \mathbb{N}}$  of points in  $\mathbb{R}^n$  which converges to  $x$  but such that  $g_A(x_i) \not\rightarrow g_A(x)$ . The idea is that we can always shift this discontinuity to the (formal) boundary of  $A$ . First of all we notice that  $x \neq 0$ . The point 0, actually, is in the topological interior of  $A$ , whence there exists  $\varepsilon > 0$  such that  $B_\varepsilon(0) = \{x \in \mathbb{R}^n \mid |x| \leq \varepsilon\} \subseteq A$  and, by Lemma 4.1.9,  $g_A(0) = 0$ . Since  $g_A(x_i) \not\rightarrow 0$ , there exists  $\delta > 0$  such that the sequence  $(g_A(x_i))_{i \in \mathbb{N}}$  is eventually  $> \delta$ . Therefore, we can extract a subsequence (whose elements, for simplicity, will be denoted  $x_i$ , too) such that  $x_i \rightarrow 0$ ,  $g_A(x_i) > \delta$  and  $x_i \in B_\varepsilon(0)$ . Thus  $\delta < g_A(x_i) \leq 1$  for each  $i \in \mathbb{N}$ , and hence  $\frac{x_i}{\delta} > \frac{x_i}{g_A(x_i)} \geq x_i$ . It follows that  $\frac{x_i}{g_A(x_i)} \rightarrow 0$  and, by positive homogeneity,  $\frac{x_i}{g_A(x_i)} \in \text{bd}(A)$ . The fact that  $\text{bd}(A)$  is closed entails the absurdum  $0 \in \text{bd}(A)$ . Thus, we may assume  $x \neq 0$ . Consider the ray  $\sigma_x$  for 0 and through  $x$ . If  $\sigma_x$  is contained in  $A$ , then, by Lemma 4.1.9,  $g_A(x) = 0$  and  $\sigma_x \cap \text{bd}(A) = \emptyset$ . Therefore, we can consider a subsequence  $(x_i)_{i \in \mathbb{N}}$  that converges to  $x$  and such that  $g_A(x_i) > \varepsilon$ , for some  $\varepsilon > 0$  and for all  $i \in \mathbb{N}$ . Now we shift this sequence to the (formal) boundary of  $A$ , defining  $b_i = \frac{x_i}{g_A(x_i)}$ . Since  $g_A$  is positively homogeneous, for each  $i \in \mathbb{N}$ , we have  $b_i \in \text{bd}(A)$ , and  $|b_i| < \frac{|x_i|}{\varepsilon}$ , with  $|x_i| \rightarrow |x|$ . Thus the  $b_i$ 's (that are infinitely many) eventually lie in a compact set, whence they have an accumulation point that must be on the ray  $\sigma_x$ : calling  $\alpha_i$  the angles between the ray  $\sigma_x$  and the rays  $\sigma_{x_i}$  the convergence  $x_i \rightarrow x$  ensures the convergence  $\alpha_i \rightarrow 0$ . As a consequence, the  $b_i$ 's converge to a point  $b \in \sigma_x$  and this is a contradiction: since  $\text{bd}(A)$  is closed,  $b$  must be on the (formal) boundary, but, because no point of  $\sigma_x$  is in  $\text{bd}(A)$ , it cannot be. On the other hand, if  $\sigma_x$  is not contained in  $A$  then  $\sigma_x \cap A = [0, w]$ , with  $g_A(w) = 1$ ,  $g_A(x) > 0$  and  $w = \frac{x}{g_A(x)}$ . Moreover, the sequence  $(g_A(x_i))_{i \in \mathbb{N}}$  cannot be eventually 0: if it were, by positive homogeneity, the whole rays  $\sigma_{x_i}$  would eventually be contained in  $A$ , whence the sequence  $(2w_i)_{i \in \mathbb{N}}$  with  $w_i \in \sigma_{x_i}$  and  $|w_i| = |w|$  would eventually be in  $A$  and would converge to  $2w \notin A$  ( $|2w_i| = |2w|$  and  $\alpha_i \rightarrow 0$ ), contradicting

the fact that  $A$  is closed. Thus, we can consider a subsequence  $(x_i)_{i \in \mathbb{N}}$  such that  $x_i \rightarrow x$ ,  $0 < g_A(x_i) \not\rightarrow g_A(x)$ , and  $b_i = \frac{x_i}{g_A(x_i)} \in \text{bd}(A)$  for all  $i \in \mathbb{N}$ . If the sequence  $(|b_i|)_{i \in \mathbb{N}}$  diverges, we get a contradiction, using again the sequence  $(2w_i)_{i \in \mathbb{N}}$ , with  $w_i \in \sigma_{x_i}$  and  $|w_i| = |w|$ . The elements of  $(2w_i)_{i \in \mathbb{N}}$  eventually lie in  $A$ , but the sequence converges to the point  $2w \notin A$ , contradicting the fact that  $A$  is closed. If, instead, the sequence does not diverge, then it is bounded. It follows that the infinitely many  $b_i$ 's are contained in a compact set of  $\mathbb{R}^n$ . Thus they have an accumulation point  $b$  that must be on the ray  $\sigma_x$ . We finally show  $b \neq w$ . If  $b = w$  then  $\frac{x_i}{g_A(x_i)} \rightarrow w = \frac{x}{g_A(x)}$ , whence  $\frac{|x_i|}{g_A(x_i)} \rightarrow \frac{|x|}{g_A(x)}$ , with  $|x| \neq 0$ ,  $|x_i| \neq 0$ ,  $|x_i| \rightarrow |x|$ ,  $g_A(x_i) \neq 0$  and  $g_A(x) \neq 0$ . Thus

$$\frac{1}{g_A(x_i)} = \frac{\frac{|x_i|}{g_A(x_i)}}{|x_i|} \rightarrow \frac{\frac{|x|}{g_A(x)}}{|x|} = \frac{1}{g_A(x)},$$

and hence  $g_A(x_i) \rightarrow g_A(x)$ , contradicting the hypothesis of discontinuity.

Now, we have to prove that  $\omega$  and  $\gamma$  are mutual inverses. It follows that, for all  $f \in \mathcal{G}^n$  and for all  $A \in \Sigma_0^n$ ,  $\gamma(\omega(f)) = f$  and  $\omega(\gamma(A)) = A$ . Let  $f \in \mathcal{G}^n$ , then  $\omega(f) = \{x \in \mathbb{R}^n \mid f(x) \leq 1\}$ , whence, for all  $y \in \mathbb{R}^n$ ,

$$\begin{aligned} \gamma(\omega(f))(y) &= \inf\{\lambda \geq 0 \mid y \in \lambda\{x \in \mathbb{R}^n \mid f(x) \leq 1\}\} = \\ &= \inf\{\lambda \geq 0 \mid y \in \{\lambda x \in \mathbb{R}^n \mid f(x) \leq 1\}\} = \\ &= \inf\{\lambda \geq 0 \mid y \in \{x \in \mathbb{R}^n \mid f(x) \leq \lambda\}\} = \\ &= \inf\{\lambda \geq 0 \mid f(y) \leq \lambda\} = f(y). \end{aligned}$$

On the other hand, if  $A \in \Sigma_0^n$ , then  $\gamma(A)(x) = \inf\{\lambda \geq 0 \mid x \in \lambda A\}$ , for all  $x \in \mathbb{R}^n$ , and hence

$$\omega(\gamma(A)) = \{x \in \mathbb{R}^n \mid \inf\{\lambda \geq 0 \mid x \in \lambda A\} \leq 1\}.$$

If  $x \in A$ , then  $\inf\{\lambda \geq 0 \mid x \in \lambda A\} \leq 1$ , whence  $A \subseteq \omega(\gamma(A))$ . If  $x \in \omega(\gamma(A))$ , then the star-shaped property ensures that if  $x \in \lambda A$ , then  $x \in \mu A$  for all  $\mu \geq \lambda$ . Thus,  $x \in \lambda A$  for all  $\lambda > 1$ . It follows that  $\frac{x}{\lambda} \in A$ , for all  $0 < \lambda < 1$ . Hence, the half-open segment  $[0, x)$  is contained in  $A$ . The closure of  $A$  entails  $x \in A$ , whence  $\omega(\gamma(A)) \subseteq A$ . This completes the proof.  $\square$

Next we show that  $\omega$  and  $\gamma$  preserve the vector lattice operations. We recall that the main goal of our discussion is to establish a correspondence between the positive cone of the vector lattice  $\text{FVL}_n$  and the set of polyhedral star-shaped object, equipped with appropriate operations. Hence we equip  $\mathcal{G}^n$  with the same operations of  $\text{FVL}_n$  (in its canonical functional representation). We also equip  $\Sigma_0^n$  with the lattice operations of intersection and union. The zero element of  $\mathcal{G}^n$  is the constant function 0 on  $\mathbb{R}^n$ . The zero element of  $\Sigma_0^n$  will be given by  $\mathbb{R}^n$ . In the next section we will introduce the remaining vector lattice operations on  $\Sigma_0^n$ .

**Proposition 4.1.11.**  $\Sigma_0^n$  is closed under set-theoretic union and intersection.

*Proof.* If  $A, B \in \Sigma_0^n$ , then 0 is contained in both their topological interiors, whence there exist  $a > 0$  and  $b > 0$  such that  $B_a(0) \subseteq A$  and  $B_b(0) \subseteq B$ . Then  $B_{\min(a,b)} \subseteq A \cap B$  and  $B_{\max(a,b)} \subseteq A \cup B$ : 0 is in the topological interior of both  $A \cap B$  and  $A \cup B$ . Moreover, for every  $x \in \mathbb{R}^n$ , if  $x \in A \cap B$ , then  $[0, x] \subseteq A$

and  $[0, x] \subseteq B$ , and hence  $[0, x] \subseteq A \cap B$ . If  $x \in A \cup B$ , then  $x \in A$  or  $x \in B$ ; it follows that  $[0, x] \subseteq A$  or  $[0, x] \subseteq B$ , whence  $[0, x] \subseteq A \cup B$ . This proves that  $A \cap B$  and  $A \cup B$  are in  $\mathbb{S}_0^n$ . Furthermore,  $A \cap B$  and  $A \cup B$  are trivially closed. There remains to be proved that the formal boundaries of  $A \cap B$  and  $A \cup B$  are closed.

By way of contradiction, suppose  $\text{bd}(A \cap B)$  is not closed. Then  $\text{bd}(A \cap B)$  contains a sequence  $(x_i)_{i \in \mathbb{N}}$  converging to some point  $x \notin \text{bd}(A \cap B)$ . Since  $A$  and  $B$  are star-shaped, by definition of (formal) boundary, for each  $i \in \mathbb{N}$  we have  $[0, x_i] = \sigma_{x_i} \cap A \subseteq \sigma_{x_i} \cap B$  or  $[0, x_i] = \sigma_{x_i} \cap B \subseteq \sigma_{x_i} \cap A$ . It follows that either  $x_i \in \text{bd}(A)$  or  $x_i \in \text{bd}(B)$ , for each  $i \in \mathbb{N}$ . There is a subsequence of  $(x_i)_{i \in \mathbb{N}}$  converging to  $x$  and included either in  $\text{bd}(A) \cap B$  or in  $\text{bd}(B) \cap A$ . In the first case, from  $\text{bd}(A)$  and  $B$  both being closed we get  $x \in \text{bd}(A) \cap B$ , and hence  $[0, x] = (\sigma_x \cap A) \cap B = \sigma_x \cap (A \cap B)$ . Similarly, in the second case, from  $\text{bd}(B)$  and  $A$  both being closed we get  $x \in \text{bd}(B) \cap A$ , whence  $[0, x] = (\sigma_x \cap B) \cap A = \sigma_x \cap (B \cap A)$ . We have thus reached the contradiction  $x \in \text{bd}(A \cap B)$  in both cases.

Passing now to consider  $A \cup B$ , if  $\text{bd}(A \cup B)$  is not closed (absurdum hypothesis) then there is a sequence  $(x_i)_{i \in \mathbb{N}}$  of points in  $\text{bd}(A \cup B)$  converging to a point  $x \notin \text{bd}(A \cup B)$ . Therefore, for each  $i \in \mathbb{N}$ , we either have  $\sigma_{x_i} \cap A \subseteq \sigma_{x_i} \cap B = [0, x_i]$  or else  $\sigma_{x_i} \cap B \subseteq \sigma_{x_i} \cap A = [0, x_i]$ . Arguing as in the first case, we obtain either  $\sigma_x \cap A \subseteq \sigma_x \cap B = [0, x]$  or  $\sigma_x \cap B \subseteq \sigma_x \cap A = [0, x]$ . Whence  $x \in \text{bd}(A \cup B)$ , which is a contradiction.  $\square$

**Proposition 4.1.12.** *The maps  $\omega$  and  $\gamma$  of Theorem 4.1.10 are order reversing, and they satisfy the identities:*

$$\begin{aligned} \omega(f \wedge g) &= \omega(f) \cup \omega(g), & \gamma(A \cup B) &= \gamma(A) \wedge \gamma(B), \\ \omega(f \vee g) &= \omega(f) \cap \omega(g), & \gamma(A \cap B) &= \gamma(A) \vee \gamma(B), \\ \omega(0) &= \mathbb{R}^n, & \gamma(\mathbb{R}^n) &= 0. \end{aligned}$$

*Proof.* It suffices to prove the second statement, because  $\omega$  and  $\gamma$  are obviously order-reversing. We have

$$\begin{aligned} \omega(f) \cup \omega(g) &= \{x \in \mathbb{R}^n \mid f(x) \leq 1\} \cup \{x \in \mathbb{R}^n \mid g(x) \leq 1\} = \\ &= \{x \in \mathbb{R}^n \mid \min(f(x), g(x)) \leq 1\} = \omega(f \wedge g), \end{aligned}$$

and

$$\begin{aligned} \omega(f) \cap \omega(g) &= \{x \in \mathbb{R}^n \mid f(x) \leq 1\} \cap \{x \in \mathbb{R}^n \mid g(x) \leq 1\} = \\ &= \{x \in \mathbb{R}^n \mid \max(f(x), g(x)) \leq 1\} = \omega(f \vee g). \end{aligned}$$

On the other hand, if  $A, B \in \Sigma_0^n$  then for each  $x \in \mathbb{R}^n$

$$\begin{aligned} (\gamma(A) \wedge \gamma(B))(x) &= \min(\inf\{\lambda \geq 0 \mid x \in \lambda A\}, \inf\{\lambda \geq 0 \mid x \in \lambda B\}) = \\ &= \inf\{\lambda \geq 0 \mid x \in \lambda A \cup \lambda B = \lambda(A \cup B)\} = \gamma(A \cup B)(x), \end{aligned}$$

and

$$\begin{aligned} (\gamma(A) \vee \gamma(B))(x) &= \max(\inf\{\lambda \geq 0 \mid x \in \lambda A\}, \inf\{\lambda \geq 0 \mid x \in \lambda B\}) = \\ &= \inf\{\lambda \geq 0 \mid x \in \lambda A \cap \lambda B = \lambda(A \cap B)\} = \gamma(A \cap B)(x). \end{aligned}$$

The correspondence between 0 and  $\mathbb{R}^n$  follows immediately from the definitions of 1-cut and gauge function.  $\square$

## 4.2 Vector space operations

The isomorphism between the lattice structures of  $\mathcal{G}^n$  and  $\Sigma_0^n$  is the first step in the proof of the existence of a correspondence between them, as structures equipped with vector lattice operations. We now equip  $\Sigma_0^n$  with suitable operations of addition and multiplication by scalars, making it a vector lattice.

### 4.2.1 Gauge sum

As we have seen in Chapter 3, Minkowski addition corresponds to the addition of the support functions of convex bodies in  $\mathbb{R}^n$ . In the same spirit, we want here to define a sum operation that translates the pointwise addition of gauge functions into the geometric context of star-shaped objects. Then we will give a geometric description of the addition between star-shaped objects, specifying its behaviour on each single ray of the Euclidean space.

**Definition 4.2.1** (Gauge sum). Given two star-shaped objects  $A, B \in \Sigma_0^n$ , their *gauge sum*  $A +_g B$  is the element  $C \in \Sigma_0^n$  such that

$$g_C(x) = g_A(x) + g_B(x),$$

for all  $x \in \mathbb{R}^n$ .

We notice that the gauge sum is well-defined. The set  $\mathcal{G}^n$  is closed under addition, whence, by Theorem 4.1.10, there exists exactly one element  $C \in \Sigma_0^n$  with gauge function  $g_A + g_B$ , for any two star-shaped objects  $A$  and  $B$ .

**Lemma 4.2.2.** *Each  $A \in \Sigma_0^n$  is completely described by its intersections with the rays  $\sigma_x$  in  $\mathbb{R}^n$ . Specifically, we have the identity*

$$A = \bigsqcup_{|x|=1} \sigma_x \cap A,$$

where  $\bigsqcup$  denotes disjoint union.

*Proof.* The proof immediately follows from the fact that the disjoint union of the rays from 0 and through the points of the unit sphere  $S^{n-1}$  completely covers the entire space  $\mathbb{R}^n$ .  $\square$

Then we can give a geometric representation of the gauge sum and the other operations on  $\Sigma_0^n$ , simply describing their behaviour on the rays  $\sigma_x$ . We can easily check that, given any two elements  $A, B \in \Sigma_0^n$  and for each  $x \in \mathbb{R}^n$  with  $|x| = 1$ ,

$$\sigma_x \cap (A +_g B) = \begin{cases} \sigma_x \cap A & \text{if } \sigma_x \cap B = \sigma_x, \\ \sigma_x \cap B & \text{if } \sigma_x \cap A = \sigma_x, \\ \left[0, \frac{ab}{a+b}\right] & \text{if } \sigma_x \cap A = [0, a], \sigma_x \cap B = [0, b]. \end{cases}$$

### 4.2.2 Products by scalars

In this section we endow  $\Sigma_0^n$  with an operation of multiplication by scalars  $\geq 0$ , in such a way that the bijection between  $\mathcal{G}^n$  and  $\Sigma_0^n$  given by Theorem 4.1.10 preserves all operations. The definition of multiplication by scalars for star-shaped objects will be crucial for the definition of units in  $\Sigma_0^n$ .

The correctness of the following definition is ensured by Theorem 4.1.10.

**Definition 4.2.3** (Gauge products by real scalars). Given a star-shaped object  $A \in \Sigma_0^n$  and a scalar  $0 \leq \lambda \in \mathbb{R}$ , the *gauge product* of  $A$  by  $\lambda$  is the set  $\lambda.A \in \Sigma_0^n$  such that  $g_{\lambda.A}(x) = \lambda g_A(x)$ , for all  $x \in \mathbb{R}^n$ .

Also in this case, by Lemma 4.2.2, we can provide a geometric description of the set  $\lambda.A$  just defined. We have that for  $\lambda > 0$  and for each  $x \in \mathbb{R}^n$  with  $|x| = 1$ ,

$$\sigma_x \cap (\lambda.A) = \begin{cases} \sigma_x & \text{if } \sigma_x \cap A = \sigma_x, \\ \left[0, \frac{1}{\lambda}a\right] & \text{if } \sigma_x \cap A = [0, a]. \end{cases} \quad (4.1)$$

Moreover, by Proposition 4.1.12,  $0.A = \mathbb{R}^n$ .

**Remark 4.2.4.** The above definition does not coincide with the usual definition of multiplication by scalars as given, e.g., in Definition 3.1.2. Hence, in this latter case, the representation by rays differs from the one in (4.1):

$$\sigma_x \cap (\lambda A) = \begin{cases} \sigma_x & \text{if } \sigma_x \cap A = \sigma_x, \\ [0, \lambda a] & \text{if } \sigma_x \cap A = [0, a]. \end{cases}$$

Accordingly, we have introduced the new notation “ $\lambda.$ ” to denote gauge products. Nevertheless, there is a simple relation between these two multiplications: as a matter of fact, for each  $A \in \Sigma_0^n$  and  $0 < \lambda \in \mathbb{R}$  we have  $\lambda.A = \frac{1}{\lambda}A$ . Further, note that  $\lim_{\lambda \rightarrow 0^+} \lambda.A = \lim_{\lambda \rightarrow +\infty} \lambda A$ .  $\square$

The set  $\Sigma_0^n$  is now equipped with the gauge sum and the gauge product operations, beyond the set-theoretic lattice operations of union and intersection. Then we can finally state the correspondence between  $\mathcal{G}^n$  and  $\Sigma_0^n$ , equipped with their vector lattice operations.

**Theorem 4.2.5.** *There is an isomorphism between the structures*

$$(\mathcal{G}^n, \min, \max, +, \{\lambda\}_{\lambda \in \mathbb{R}^+}, 0) \quad \text{and} \quad (\Sigma_0^n, \cup, \cap, +_g, \{\lambda.\}_{\lambda \in \mathbb{R}^+}, \mathbb{R}^n).$$

*The isomorphism is provided by the maps  $\omega$  and  $\gamma$  of Theorem 4.1.10.*

*Proof.* From Theorem 4.1.10, Proposition 4.1.12, and from the definitions of gauge sum and gauge products.  $\square$

## 4.3 Unit interval and good sequences

In this section we develop a geometric counterpart of *good sequences*. The latter were introduced by Mundici as a key ingredient for his construction of the categorical equivalence between MV-algebras and unital  $\ell$ -groups, [28]. We

will preliminarily introduce geometric counterparts of unit, unit interval, and truncated sum. In Theorem 4.3.14 we will associate to each element of  $\Sigma_0^n$  a unique “good sequence” of star-shaped objects.

To this purpose, we prepare:

**Definition 4.3.1** (Unit). A (*strong order*) *unit* of  $\Sigma_0^n$  is any element  $U \in \Sigma_0^n$  such that for any element  $A \in \Sigma_0^n$  there is  $0 < \lambda \in \mathbb{R}$  such that  $\lambda U \subseteq A$ .

**Definition 4.3.2** (Unit interval). Given a unit  $U \in \Sigma_0^n$ , its associated *unit interval* is the set

$$[U, \mathbb{R}^n] = \{A \in \Sigma_0^n \mid U \subseteq A\}.$$

An element  $g \in \mathcal{G}^n$  is a *unit* if for any  $f \in \mathcal{G}^n$  there is  $0 < \lambda \in \mathbb{R}$  such that  $f \leq \lambda g$ . As an immediate consequence of Theorem 4.2.5, a star-shaped object  $A$  is a unit of  $\Sigma_0^n$  if, and only if, its gauge function  $g_A$  is a unit in  $\mathcal{G}^n$ . The maps  $\omega$  and  $\gamma$  of Theorems 4.1.10 and 4.2.5 are unit-preserving. Further, these maps also *preserve unit intervals*, in the sense that the image under  $\omega$  of a unit interval on  $\mathcal{G}^n$  is a unit interval of  $\Sigma_0^n$ , and the image under  $\gamma$  of a unit interval of  $\Sigma_0^n$  is a unit interval of  $\mathcal{G}^n$ .

Units of  $\Sigma_0^n$  can be given the following purely geometric characterization:

**Proposition 4.3.3.** *The units of  $\Sigma_0^n$  coincide with its compact elements.*

*Proof.* Trivially, an element  $g$  of  $\mathcal{G}^n$  is a unit if and only if  $g$  vanishes only at the origin. An element  $U \in \Sigma_0^n$  is a unit if and only if its gauge function  $g_U$  is a unit for  $\mathcal{G}^n$ . In particular,  $g_U$  is continuous and such that  $g_U(x) > 0$  on the unit (compact) sphere  $\mathcal{S}^{n-1}$ . Hence, on  $\mathcal{S}^{n-1}$ ,  $g_U$  attains its minimum value  $r$ , with  $r > 0$ . Let  $g: \mathbb{R}^n \rightarrow \mathbb{R}$  be the unique continuous and positively homogeneous function that is identically equal to  $r$  on the sphere  $\mathcal{S}^{n-1}$ . Then its 1-cut is the closed ball  $B_{\frac{1}{r}}(0)$ . Because  $g \leq g_U$ , then  $U \subseteq B_{\frac{1}{r}}(0)$ . Hence,  $U$  is bounded. Moreover,  $U$  is closed. Hence  $U$  is compact. Conversely, if  $U$  is compact then it does not contain any ray  $\sigma_x$  of  $\mathbb{R}^n$ . Then,  $g_U$  is zero only at 0. Hence,  $g_U$  is a unit of  $\mathcal{G}^n$ , whence  $U$  is a unit of  $\Sigma_0^n$ .  $\square$

### 4.3.1 Truncated gauge sum and good sequences of star-shaped objects

Upon fixing a unit  $u$  of  $\mathcal{G}^n$  we equip  $\mathcal{G}^n$  with a “truncated addition” operation  $\oplus$ , where

$$f \oplus g = (f + g) \wedge u.$$

Given a unit  $U$  of  $\Sigma_0^n$ , we can also truncate the  $+_g$  operation and define a new sum operation  $\oplus_g$ , in such a way that the corresponding operation in  $\mathcal{G}^n$  via the map  $\gamma$  is precisely the truncated sum  $\oplus$ :

**Definition 4.3.4** (Truncated gauge sum). Given two star-shaped objects  $A, B$  in  $\Sigma_0^n$ , their *truncated gauge sum*  $A \oplus_g B$  is the star-shaped object

$$C = (A +_g B) \cup U.$$

The gauge function  $g_C$  trivially satisfies the identity  $g_C(x) = g_A(x) \oplus g_B(x)$ , for all  $x \in \mathbb{R}^n$ .



By Lemma 4.2.2, we can describe the truncated gauge sum operation as follows: for any two elements  $A, B \in \Sigma_0^n$  and for each  $x \in \mathbb{R}^n$  with  $|x| = 1$ ,

$$\sigma_x \cap (A \oplus_g B) = \begin{cases} \sigma_x \cap A & \text{if } \sigma_x \cap B = \sigma_x, \sigma_x \cap U \subseteq \sigma_x \cap A, \\ \sigma_x \cap B & \text{if } \sigma_x \cap A = \sigma_x, \sigma_x \cap U \subseteq \sigma_x \cap B, \\ \sigma_x \cap U & \text{if } \sigma_x \cap B = \sigma_x, \sigma_x \cap A \subseteq \sigma_x \cap U, \\ \sigma_x \cap U & \text{if } \sigma_x \cap A = \sigma_x, \sigma_x \cap B \subseteq \sigma_x \cap U, \\ (\sigma_x \cap U) \cup \left[0, \frac{ab}{a+b}\right] & \text{if } \sigma_x \cap A = [0, a], \sigma_x \cap B = [0, b]. \end{cases}$$

**Corollary 4.3.5.** *The isomorphisms  $\omega$  and  $\gamma$  defined in Theorem 4.1.10 preserve both units and truncated sums.*

We have thus prepared all necessary ingredients for our geometric good sequences:

**Definition 4.3.6** (Good sequences of star-shaped objects). Given a fixed unit  $U \in \Sigma_0^n$ , a *good sequence of star-shaped objects* is a sequence  $(A_i)_{i \in \mathbb{N}}$  of elements  $A_i \in \Sigma_0^n$  such that

SS1) there exists an index  $j \in \mathbb{N}$  such that, for all  $i \geq j$ ,  $A_i = \mathbb{R}^n$ ;

SS2)  $U \subseteq A_i$ , for all  $i \in \mathbb{N}$ ;

SS3)  $A_i \oplus_g A_{i+1} = A_i$ , for all  $i \in \mathbb{N}$ .

### 4.3.2 Good sequences of real intervals

Lemma 4.2.2 allows us to give a representation by rays also for good sequences. Any good sequence of star-shaped objects  $(A_i)_{i \in \mathbb{N}}$  restricted to a fixed ray  $\sigma_x$  is a sequence of real intervals of the form  $([0, a_i])_{i \in \mathbb{N}}$ , with  $0 < a_i \in A_i$ . This suggests the idea of working first on rays, defining a suitable concept of good sequence for real intervals. Then, given a star-shaped object  $A$ , we will obtain the good sequence of star-shaped objects that represents it just by gluing together the good sequences of real intervals that represent each  $\sigma_x \cap A$ , with  $|x| = 1$ .

In the following, we will consider only real intervals of the form  $[0, a]$  with  $0 < a \in \mathbb{R}$  together with the set  $\mathbb{R}^+$ . To simplify the treatment, we will refer to this latter set also with the notation  $[0, \infty]$  (instead of the more precise  $[0, \infty)$ ).

**Definition 4.3.7** (Gauge sum of real intervals). The *gauge sum* of any two real intervals  $[0, a]$  and  $[0, b]$  with  $0 < a, b \in \mathbb{R}$  is the real interval

$$[0, a] +_g [0, b] = \begin{cases} [0, a] & \text{if } [0, b] = [0, \infty], \\ [0, b] & \text{if } [0, a] = [0, \infty], \\ \left[0, \frac{ab}{a+b}\right] & \text{if } a, b < \infty. \end{cases}$$

Trivially, the gauge sum of any two real intervals is commutative, and satisfies the inclusion  $[0, a] +_g [0, b] \subseteq [0, a], [0, b]$ , which immediately follows from the inequality  $\frac{ab}{a+b} < a, b$ , for all  $0 < a, b < \infty$ . Further, we notice that the previous definition is not a special case of Definition 4.2.1: real intervals are not in  $\Sigma_0^n$ .

Nevertheless, the gauge sum of real intervals agrees with a gauge sum on  $\mathcal{G}^n$  restricted to a single ray. The same happens for the truncated gauge sum of real intervals:

**Definition 4.3.8** (Truncated gauge sum of real intervals). For any  $0 < u \in \mathbb{R}$ , the *truncated gauge sum with respect to  $u$*  of any two real intervals  $[0, a]$  and  $[0, b]$  with  $0 < a, b \in \mathbb{R}$  is the real interval

$$[0, a] \oplus_{\mathbf{g}}^u [0, b] = [0, u] \cup ([0, a] +_{\mathbf{g}} [0, b]).$$

With these preliminary definitions we are ready to introduce:

**Definition 4.3.9** (Good sequences of real intervals). For any real number  $u > 0$ , a *good sequence of real intervals with respect to  $u$*  is a sequence  $([0, a_i])_{i \in \mathbb{N}}$  of intervals  $[0, a_i] \subseteq \mathbb{R}$  such that

*RI1)* there exists an index  $j \in \mathbb{N}$  such that, for all  $i \geq j$ ,  $[0, a_i] = [0, \infty]$ ;

*RI2)*  $u \leq a_i$ , for all  $i \in \mathbb{N}$ ;

*RI3)*  $[0, a_i] \oplus_{\mathbf{g}}^u [0, a_{i+1}] = [0, a_i]$ , for all  $i \in \mathbb{N}$ .

Conditions SS3 and RI3 above can be given more amenable equivalent reformulations as follows:

*SS3')* for all  $i \in \mathbb{N}$  and for each  $|x| = 1$ ,  $\sigma_x \cap A_i = \sigma_x \cap U$  or  $\sigma_x \cap A_{i+1} = \sigma_x$ ;

*RI3')*  $a_i = u$  or  $a_{i+1} = \infty$ .

Both equivalences can be checked by an easy computation.

Given a fixed unit  $U \in \Sigma_0^n$  and an element  $A \in \Sigma_0^n$ , we consider separately each ray  $\sigma_x$  with  $|x| = 1$ , restricting  $U$  and  $A$  to it. Since both  $U$  and  $A$  are star-shaped, we obtain  $\sigma_x \cap U = [0, u(x)]$  and  $\sigma_x \cap A = [0, a(x)]$ , where  $0 < u(x) < \infty$  and  $0 < a(x)$ . Now we look for good sequences  $(a_i(x))_{i \in \mathbb{N}}$  such that, for each  $|x| = 1$ ,  $[0, a(x)] = [0, a_1(x)] +_{\mathbf{g}} [0, a_2(x)] +_{\mathbf{g}} \dots$ , and then glue them together to obtain the good sequence  $(A_i = \bigsqcup_{|x|=1} [0, a_i(x)])_{i \in \mathbb{N}}$ . To ensure that the sequence  $(A_i)_{i \in \mathbb{N}}$  is good some additional assumptions must be made about the good sequences  $([0, a_i(x)])_{i \in \mathbb{N}}$ . Indeed, some properties of good sequences of star-shaped objects, such as SS2 or SS3 or the star-shaped property, are local to each ray: they are equivalent to a property required to hold at each ray of a star-shaped set that only depend on the ray itself. For example SS3 is equivalent to SS3', and the latter is explicitly a property on each ray of the star-shaped set. Other properties, such as SS1 or the closure property of the  $A_i$ 's and their formal boundaries, are not local in this sense.

In order to present Theorem 4.3.11, we prepare:

**Definition 4.3.10.** The sequence of real intervals  $([0, a_i])_{i \in \mathbb{N}}$  *converges to the real interval  $[0, a]$*  if the following conditions hold:

1. if  $a = \infty$ , then the sequence of real numbers  $(a_i)_{i \in \mathbb{N}}$  either diverges or else there is an index  $j \in \mathbb{N}$  such that  $a_i = \infty$  for all  $i \geq j$ ;
2. if  $a_i(x) < \infty$ , then the sequence of real numbers  $(a_i)_{i \in \mathbb{N}}$  converges to  $a$ .

**Theorem 4.3.11.** *Let  $(A_i)_{i \in \mathbb{N}}$  be a good sequence of star-shaped objects with respect to a fixed unit  $U \in \Sigma_0^n$ . Then, for each  $x \in \mathbb{R}^n$  with  $|x| = 1$ , the sequences  $([0, a_i] = \sigma_x \cap A_i)_{i \in \mathbb{N}}$  are good sequences of real intervals with respect to the real numbers  $u(x) > 0$  such that  $\sigma_x \cap U = [0, u(x)]$ . Further:*

RI4) *there exists an index  $j \in \mathbb{N}$  such that  $a_i(x) = \infty$ , for each  $x \in \mathbb{R}^n$  with  $|x| = 1$  and for all  $i \geq j$ ;*

RI5) *let  $(x_l)_{l \in \mathbb{N}}$  be a sequence converging to the point  $x$ , where  $|x_l| = 1$ , whence automatically  $|x| = 1$ . Then for all  $i \in \mathbb{N}$  the sequence  $([0, a_i(x_l)])_{l \in \mathbb{N}}$  converges to the interval  $[0, a_i(x)]$ .*

*Conversely: let, for each  $x \in \mathbb{R}^n$  with  $|x| = 1$ ,  $([0, a_i(x)])_{i \in \mathbb{N}}$  be good sequences of real intervals with respect to real numbers  $u(x) > 0$  such that  $\sigma_x \cap U = [0, u(x)]$  for some unit  $U \in \Sigma_0^n$ , and let them satisfy conditions RI4 and RI5. Then their gluing  $(A_i = \bigsqcup_{|x|=1} [0, a_i(x)])_{i \in \mathbb{N}}$  is a good sequence of star-shaped objects, with respect to the unit  $U$ .*

*Proof.* It is easy to see that  $([0, a_i] = \sigma_x \cap A_i)_{i \in \mathbb{N}}$  is a good sequence of real intervals: the star-shaped condition on the  $A_i$ 's and the fact that 0 is in their topological interiors ensure that, for all  $i \in \mathbb{N}$  and for each  $|x| = 1$ ,  $\sigma_x \cap A_i$  is actually an interval  $[0, a_i(x)]$ , with  $0 < a_i(x)$ . Moreover, conditions RI1 and RI2 immediately follow from SS1, and SS2, respectively. Condition RI3 is clear once we notice that RI3' is equivalent to RI3, SS3' is equivalent to SS3', and RI3' immediately follows from SS3'.

Condition RI4 immediately follows from SS1. There remains to be proved that RI5 holds. To this purpose, we first fix  $i \in \mathbb{N}$  and recall that the gauge function  $g_{A_i}$  is continuous and that, for each  $|y| = 1$ ,  $g_{A_i}(y) = 0$  if and only if  $\sigma_y \cap A_i = \sigma_y$  (see Theorem 4.1.10). Thus, if  $a_i(x) = \infty$ , then  $g_{A_i}(x) = 0$  whence  $g_{A_i}(x_l) \rightarrow 0$ . We have the following alternatives: either  $(g_{A_i}(x_l))_{l \in \mathbb{N}}$  is equal to 0 for all sufficiently large  $l$ , whence  $(a_i(x_l))_{l \in \mathbb{N}}$  is equal to  $\infty$  for sufficiently large  $l$ , or for each  $\varepsilon > 0$  there exists an index  $j_\varepsilon \in \mathbb{N}$  such that, for all  $l \geq j_\varepsilon$ ,  $g_{A_i}(x_l) < \varepsilon$ . In this latter case, for all  $l \geq j_\varepsilon$ ,  $\sigma_{x_l} \cap A_i = [0, a_i(x_l)]$  with  $a_i(x_l) > \frac{1}{\varepsilon}$  (see Lemma 4.1.9 and recall that  $|x_l| = 1$ ). As a consequence, for each  $M > 0$ , there exists an index  $j_{\frac{1}{M}}$  such that  $a_i(x_l) > M$  for each  $l \geq j_{\frac{1}{M}}$ , whence the sequence  $(a_i(x_l))_{l \in \mathbb{N}}$  diverges. If, instead,  $a_i(x) < \infty$  then  $g_{A_i}(x_l) \rightarrow g_{A_i}(x) = \frac{1}{a_i(x)} > 0$ . It follows that there exists an index  $j$  such that, for each  $l \geq j$ ,  $g_{A_i}(x_l) > 0$  and hence  $a_i(x_l) = \frac{1}{g_{A_i}(x_l)} \rightarrow \frac{1}{g_{A_i}(x)} = a_i(x)$ .

Conversely, conditions RI4, RI2, and RI3' imply conditions SS1, SS2, and SS3'. The point is to prove that each  $A_i \bigsqcup_{|x|=1} [0, a_i(x)]$  actually is in  $\Sigma_0^n$ . We fix  $i \in \mathbb{N}$ . The star-shaped condition is easy to show. If  $a \in A_i$ , then there exists a point  $x$  with  $|x| = 1$  such that  $a \in [0, a_i(x)]$ , whence  $[0, a] \subseteq [0, a_i(x)] \subseteq A_i$ . Trivially  $0 \in A_i$ , and to prove that 0 is also in the topological interior of  $A_i$ , we proceed by way of contradiction. If 0 does not belong to the topological interior of  $A_i$  then the star-shaped condition ensures that for all  $\varepsilon > 0$  there exists a point  $y_\varepsilon$  with  $|y_\varepsilon| = \varepsilon$  which is not in  $A_i$ . Thus we consider the sequence

$$\left( x_l = \frac{y_l}{|y_l|} \right)_{l \in \mathbb{N}}.$$

For all  $l$ ,  $|x_l| = 1$ , and  $a_i(x_j) < |y_{\frac{1}{j}}| = \frac{1}{j}$ . Moreover, the  $x_l$ 's are all contained in a compact subset of  $\mathbb{R}^n$  (the unit sphere  $\mathcal{S}^{n-1}$ ), and hence they have an accumulation point. Therefore, there is a subsequence  $(x_{l_j})_{j \in \mathbb{N}}$  that converges to some  $x \in \mathbb{R}^n$  with  $|x| = 1$ . By RI5, the sequence  $(a_i(x_{l_j}))_{j \in \mathbb{N}}$  converges to  $a_i(x)$ , but we also have  $a_i(x_{l_j}) < \frac{1}{l_j} \rightarrow 0$ , whence  $a_i(x) = 0$ , against RI2.

Next we prove that  $A_i$  is closed. We consider a sequence  $(y_l)_{l \in \mathbb{N}}$  of points of  $A_i$  which converges to the point  $y$ . If  $y = 0$ , then  $y \in A_i$ , whence we are done. Otherwise, replace  $(y_l)_{l \in \mathbb{N}}$  with the subsequence such that  $y_l \neq 0$  for each  $l \in \mathbb{N}$ . Then we can take  $x_l = \frac{y_l}{|y_l|}$  and  $x = \frac{y}{|y|}$ . Since  $0 \neq |y_l| \rightarrow |y| \neq 0$  and  $y_l \rightarrow y$ , then  $|x_l| \rightarrow |x|$  and  $x_l \rightarrow x$ . By RI5, the  $a_i(x_l)$ 's converge to  $a_i(x)$  (in the sense of RI5). Trivially, if  $a_i(x) = \infty$ , then  $y \in A_i$ . Otherwise, the  $a_i(x_l)$ 's eventually belong to  $\mathbb{R}$ , and  $0 < |y_l| \leq a_i(x_l) \rightarrow a_i(x)$  and  $|y_l| \rightarrow |y|$ . Thus  $|y| \leq a_i(x)$ , and hence  $y \in A_i$ .

Finally, we show that the formal boundary of  $A_i$  is closed. So we take a sequence  $(y_l)_{l \in \mathbb{N}}$  of points  $y_l \in \text{bd}(A_i)$  which converges to a point  $y$ . Since  $0$  is in the topological interior of  $A_i$ ,  $y_l \neq 0$  for all  $l \in \mathbb{N}$ , whence we can consider the points  $x_l = \frac{y_l}{|y_l|}$  and their associated  $a_i(x_l)$ , that are exactly the values  $|y_l|$ . Moreover,  $y \neq 0$ : if  $y = 0$ , for all  $\varepsilon > 0$  we can find an index  $j_\varepsilon$  such that  $|y_{j_\varepsilon}| < \varepsilon$ . Then, the points  $\varepsilon x_l$  are not in  $A_i$  and converge to  $0$ , contradicting the fact that  $0$  is in the topological interior of  $A_i$ . Therefore,  $0 \neq y_l \rightarrow y \neq 0$ , and  $x_l \rightarrow x$  on  $\mathcal{S}^{n-1}$ . Since  $y_l$  lies on the (formal) boundary, we have  $a_i(x_l) = |y_l| < \infty$  and, by RI5,  $a_i(x_l) \rightarrow a_i(x)$ . This, together with  $|y_l| \rightarrow |y|$ , ensures that  $a_i(x) = |y|$ , whence  $y \in \text{bd}(A_i)$ . This completes the proof.  $\square$

We are now in a position to represent each element of  $\Sigma_0^n$  as a good sequence of star-shaped objects.

**Lemma 4.3.12.** *Let  $0 < u \in \mathbb{R}$ . Then for each  $0 < a \leq \infty$  there exists a unique good sequence (with respect to  $u$ ) of real intervals  $([0, a_i])_{i \in \mathbb{N}}$  such that*

$$[0, a] = [0, a_1] +_{\text{g}} [0, a_2] +_{\text{g}} \cdots .$$

*Proof.* For all  $n \in \mathbb{N}$  and  $0 < b \leq \infty$  we have the identity  $n \cdot [0, b] = [0, \frac{1}{n}b]$ . If  $a = \infty$ , then we can take  $a_i = \infty$  for all  $i \in \mathbb{N}$ . Moreover, condition RI3' implies that a good sequence of real intervals is always of the form

$$\underbrace{([0, u], [0, u], \dots, [0, u])}_{m \text{ times}}, [0, c], [0, \infty], [0, \infty], \dots,$$

with  $0 \leq m < \infty$  and  $u < c \leq \infty$ . The gauge sum of the intervals of this sequence is given by

$$\begin{cases} [0, c] & \text{if } m = 0, \\ \left[0, \frac{u}{m}\right] & \text{if } m \neq 0 \text{ and } c = \infty, \\ \left[0, \frac{\frac{u}{m}c}{\frac{u}{m} + c}\right] & \text{otherwise.} \end{cases}$$

In any case, the result of the sum must be  $[0, a]$ . If  $a > u$ , the only possible case is  $m = 0$  and  $c = a$ , because  $\frac{u}{m} \leq u < a$  for  $m \neq 0$  and  $\frac{\frac{u}{m}c}{\frac{u}{m} + c} \leq \frac{u}{m}$ , for

$m \neq 0$  and  $c \neq \infty$ . If  $a \leq u$ , then there exists a unique  $k \in \mathbb{N}/\{0\}$  such that  $\frac{1}{k+1}u < a \leq \frac{1}{k}u$ . If  $a = \frac{1}{k}u$ , then the only possible choice is  $m = k$  and  $c = \infty$ , because  $\frac{1}{k}u \leq u$ . If  $\frac{1}{k+1}u < a < \frac{1}{k}u$ , then we have to choose  $m \neq 0$  and  $c \neq \infty$ , whence  $c = \frac{au}{u-ma}$ . Furthermore, the condition  $c > u$  implies  $au > u^2 - mau$ , and hence  $\frac{1}{k}u > a > \frac{1}{1-m}u$ . Then  $m > k - 1$ . On the other hand, since the gauge sum is not increasing,  $\frac{1}{k+1}u < a = \frac{\frac{u}{m}c}{\frac{u}{m}+c} < \frac{u}{m}$  implies that  $m$  must be strictly less than  $k + 1$ . In conclusion,  $m = k$  and  $c = \frac{au}{u-ka}$ .  $\square$

**Lemma 4.3.13.** *Let  $U$  be a fixed unit in  $\Sigma_0^n$  and  $A \in \Sigma_0^n$ . Suppose that for all points  $x \in \mathbb{R}^n$  with  $|x| = 1$  the sequence  $([0, a_i(x)])_{i \in \mathbb{N}}$  is the good sequence (with respect to  $u(x)$  such that  $\sigma_x \cap U = [0, u(x)]$ ) of real intervals such that*

$$\sigma_x \cap A = [0, a(x)] = [0, a_1(x)] +_{\mathbf{g}} [0, a_2(x)] +_{\mathbf{g}} \cdots .$$

*Then conditions RI4 and RI5 are satisfied.*

*Proof.* Since 0 is in the topological interior of  $A$ , there exists  $\varepsilon > 0$  such that the ball  $B_\varepsilon(0)$  is contained in  $A$ . Moreover,  $U$  is compact, whence there exists  $\infty > M > 0$  such that, for all  $y \in U$ ,  $|y| < M$ . Since  $0 < M \in \mathbb{R}$  and  $0 < \varepsilon \in \mathbb{R}$ , then there exists  $0 < k \in \mathbb{N}$  such that  $\frac{M}{k} < \varepsilon$ . Thus,  $\{\frac{y}{k} \mid y \in U\} = k.U \subseteq A$ . Therefore, for all  $|x| = 1$ ,  $[0, \frac{1}{k}u(x)] = \sigma_x \cap k.U \subseteq \sigma_x \cap A = [0, a(x)]$ . By the construction of  $([0, a_i(x)])_{i \in \mathbb{N}}$  given in the proof of Lemma 4.3.12, for each  $l \geq k + 1$ ,  $a_l(x) = \infty$ . This proves RI4.

To show RI5, we take a sequence  $(x_l)_{l \in \mathbb{N}}$ , with  $|x_l| = 1$ , that converges to  $x$  with  $|x| = 1$ . Then we have the associated good sequences  $([0, a_i(x_l)])_{i \in \mathbb{N}}$  and  $([0, a_i(x)])_{i \in \mathbb{N}}$ . By continuity of the gauge function,  $g_A(x_l) \rightarrow g_A(x)$ . If  $g_A(x) \neq 0$ , then  $(g_A(x_l))_{l \in \mathbb{N}}$  is eventually  $> 0$ , whence

$$a(x_l) = \frac{1}{g_A(x_l)} \rightarrow \frac{1}{g_A(x)} = a(x).$$

Otherwise,  $g_A(x_l) \rightarrow 0$  and  $a(x) = \infty$ . There are two cases: if  $(g_A(x_l))_{l \in \mathbb{N}}$  is eventually 0 then  $(a(x_l))_{l \in \mathbb{N}}$  is eventually  $\infty$ . Else, in the second case, Replacing  $(g_A(x_l))_{l \in \mathbb{N}}$  with a subsequence, we can assume  $g_A(x_l) \neq 0$  for each  $l \in \mathbb{N}$ , and hence  $a(x_l) = \frac{1}{g_A(x_l)} \rightarrow \infty$ . In both cases,  $a(x_l) \rightarrow a(x)$ . Thus, by Lemma 4.3.12, we have

$$\begin{aligned} ([0, a_i(x_l)])_{i \in \mathbb{N}} &= \\ &= \left( \underbrace{[0, u(x_l)], \dots, [0, u(x_l)]}_{k_l \text{ times}}, \left[0, \frac{a(x_l)u(x_l)}{u(x_l) - k_l a(x_l)}\right], [0, \infty], [0, \infty], \dots \right), \\ ([0, a_i(x)])_{i \in \mathbb{N}} &= \\ &= \left( \underbrace{[0, u(x)], \dots, [0, u(x)]}_k, \left[0, \frac{a(x)u(x)}{u(x) - ka(x)}\right], [0, \infty], [0, \infty], \dots \right), \end{aligned}$$

with  $\frac{a(x_l)u(x_l)}{u(x_l) - k_l a(x_l)} = \infty$  if  $a(x_l) = \frac{u(x_l)}{k_l}$ , and  $\frac{a(x)u(x)}{u(x) - ka(x)} = \infty$  if  $a(x) = \frac{u(x)}{k}$ . If  $a(x) = \infty$ , then  $k = 0$ ,  $\frac{a(x)u(x)}{u(x) - ka(x)} = a(x) = \infty$  and  $(a(x_l))_{l \in \mathbb{N}}$  diverges. Since  $U$

is compact, there exists  $M > 0$  such that, for all  $y$  with  $|y| = 1$ ,  $u(y) < M$ , and there exists an index  $j$  such that, for all  $l \geq j$ ,  $a(x_l) > u(x_l)$ . Then, for all  $l \geq j$ ,  $k_l = 0$  and  $\frac{a(x_l)u(x_l)}{u(x_l) - k_l a(x_l)} = a(x_l)$ . Thus, for all  $i \in \mathbb{N}$ ,  $[0, a_i(x_l)] \rightarrow [0, a_i(x)]$ . If  $u(x) < a(x) < \infty$ , then  $k = 0$  and  $\frac{a(x)u(x)}{u(x) - ka(x)} = a(x)$ . Since the (formal) boundary of  $U$  is closed,  $U$  is compact, and  $x_l \rightarrow x$ , then  $u(x_l) \rightarrow u(x)$ . Since  $a(x_l) \rightarrow a(x)$ ,  $u(x_l) \rightarrow u(x)$  and  $u(x) < a(x) < \infty$ , there exists an index  $j \in \mathbb{N}$ , such that, for all  $l \geq j$ ,  $u(x_l) < a(x_l) < \infty$ . It follows that, for  $l \geq j$ ,  $k_l = 0$  and  $\frac{a(x_l)u(x_l)}{u(x_l) - k_l a(x_l)} = a(x_l) \rightarrow a(x)$ . If  $\frac{u(x)}{k+1} < a(x) < \frac{u(x)}{k}$  for some natural number  $k > 0$ , the argument is very similar: there exists an index  $j \in \mathbb{N}$  such that, for  $l \geq j$ ,  $\frac{u(x_l)}{k+1} < a(x_l) < \frac{u(x_l)}{k}$ ; then  $k_l = k$  and  $\frac{a(x_l)u(x_l)}{u(x_l) - k_l a(x_l)} = a(x_l) \rightarrow a(x)$ . For  $a(x) = \frac{u}{k}$  (with  $k > 0$ ), we have  $\frac{a(x)u(x)}{u(x) - ka(x)} = \infty$ , and we obtain, in the same way as before,  $\frac{u(x_l)}{k+1} < a(x_l) < \frac{u(x_l)}{k-1}$ , for  $l \geq j$ , for some index  $j \in \mathbb{N}$ , whence  $k_l = k$  or  $k_l = k - 1$ . Then, for  $l \geq j$  and  $i < k$ ,  $[0, a_i(x_l)] = [0, u(x_l)]$  and  $u(x_l) \rightarrow u(x) = a_i(x)$ . Further for  $i > k + 1$  the identity  $[0, a_i(x_l)] = [0, \infty] = [0, a_i(x)]$  holds. In both cases,  $[0, a_i(x_l)] \rightarrow [0, a_i(x)]$ . We have to check the convergence just for  $i = k$  and  $i = k + 1$ . Since  $k_l = k - 1$  or  $k_l = k$ , we have

$$\begin{aligned} [0, a_k(x_l)] &= \left[ 0, \frac{a(x_l)u(x_l)}{u(x_l) - (k-1)a(x_l)} \right] \rightarrow \left[ 0, \frac{a(x)u(x)}{u(x) - (k-1)a(x)} \right] = \\ &= \left[ 0, \frac{\frac{u(x)}{k}u(x)}{u(x) - (k-1)\frac{u(x)}{k}} \right] = [0, u(x)] = [0, a_k(x)] \end{aligned}$$

or

$$[0, a_k(x_l)] = [0, u(x_l)] \rightarrow [0, u(x)] = [0, a_k(x)].$$

Thus  $[0, a_k(x_l)] \rightarrow [0, a_k(x)]$ , in the sense of condition RI5. On the other hand, we have

$$[0, a_{k+1}(x_l)] = [0, \infty] = [0, a(x)]$$

or

$$\begin{aligned} [0, a_{k+1}(x_l)] &= \left[ 0, \frac{a(x_l)u(x_l)}{u(x_l) - ka(x_l)} \right] \rightarrow \left[ 0, \frac{a(x)u(x)}{u(x) - ka(x)} \right] = \\ &= \left[ 0, \frac{\frac{u(x)}{k}u(x)}{u(x) - k\frac{u(x)}{k}} \right] = [0, \infty] = [0, a_{k+1}(x)]. \end{aligned}$$

So  $[0, a_{k+1}(x_l)] \rightarrow [0, a_{k+1}(x)]$ . This completes the proof.  $\square$

Combining Theorem 4.3.11, Lemma 4.3.12, and Lemma 4.3.13, we get the following theorem.

**Theorem 4.3.14** (Good sequences representation). *Fix a unit  $U \in \Sigma_0^n$  and an element  $A \in \Sigma_0^n$ . Then there exists a unique good sequence  $(A_i)_{i \in \mathbb{N}}$  in the unit interval  $[U, \mathbb{R}^n]$ , such that*

$$A = A_1 +_{\mathfrak{g}} A_2 +_{\mathfrak{g}} A_3 +_{\mathfrak{g}} \cdots$$

By the previous theorem, we can conclude that, as in the case of unital  $\ell$ -groups, the information carried by the set of star-shaped objects equipped with its vector lattice operations, is actually contained in any unit interval  $[U, \mathbb{R}^n]$ .

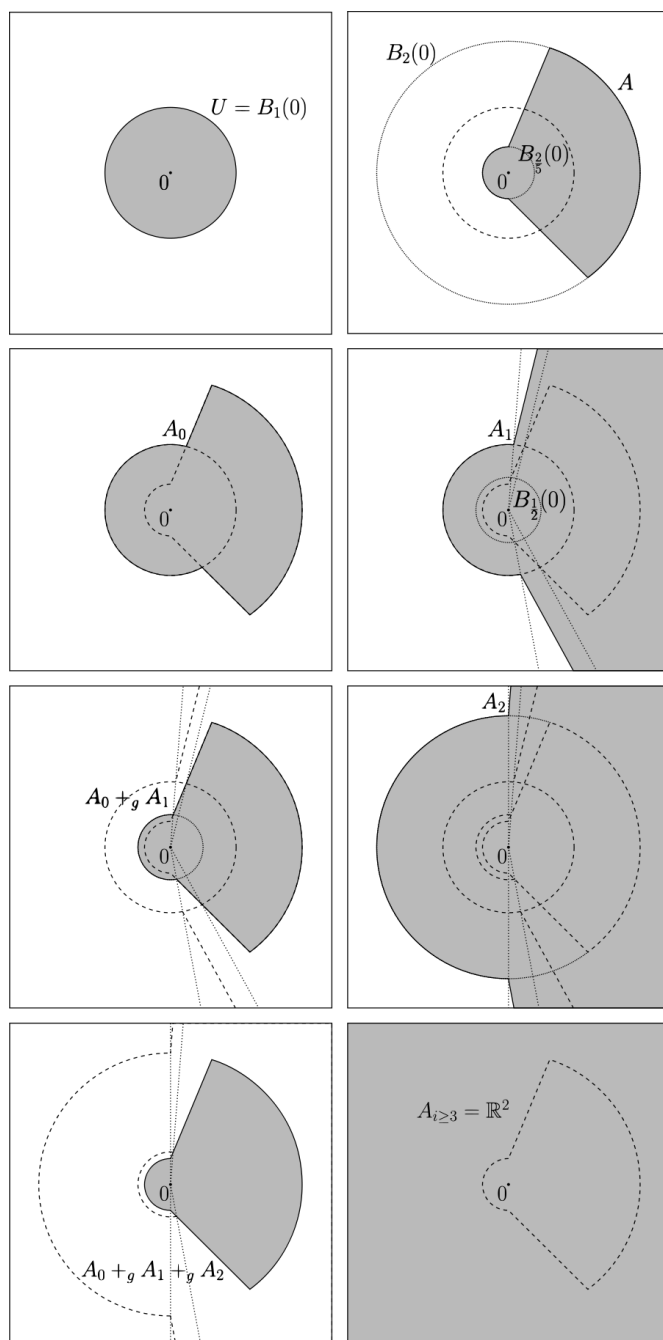


Figure 4.3: Good sequences of star-shaped objects.

## 4.4 Piecewise linearity and polyhedrality

In the previous sections we have obtained our main representation results (Theorems 4.2.5 and 4.3.14) for  $\mathcal{G}^n$ . In this section we show that the subset of  $\Sigma_0^n$  which corresponds to  $\text{FVL}_n^+ \subseteq \mathcal{G}^n$  via the isomorphisms  $\omega$  and  $\gamma$  consists of the “polyhedral” elements, in the sense of the following definition. (Recall from Remark 1.1.4 that polyhedra can be written as finite unions of finite intersections of closed half-spaces.)

**Definition 4.4.1** (Polyhedral star-shaped object). A star-shaped subset  $A$  of  $\mathbb{R}^n$  is *polyhedral* if it is a finite union of finite intersections of half-spaces, that is, if there exists a finite number of half-spaces  $H_{ij}$ , such that  $A = \bigcup_i \bigcap_j H_{ij}$ . In the following we will denote by  $\Pi_0^n$  the set of polyhedral star-shaped objects of  $\Sigma_0^n$ .

The idea used in associating a polyhedral star-shaped object to a function of  $\text{FVL}_n^+$  is to decompose both of them into simpler elements. In more detail, we will prove a correspondence between closed half-spaces (with the origin in their topological interiors) and linear words (joined with 0), and then we will put them together using the lattice operations to extend the correspondence.

In order to do that, we need:

**Lemma 4.4.2.**  $\text{FVL}_n^+$  is the subset of  $\text{FVL}_n$  of those elements that can be written as finitely many meets of finitely many joins of linear words joined with 0:

$$\text{FVL}_n^+ = \left\{ f \in \text{FVL}_n \mid f = \bigwedge_{k \in K} \bigvee_{j \in J} \left( \sum_{i=1}^{n_{jk}} \lambda_{ijk} \pi_i \vee 0 \right) \right\},$$

where  $J$  and  $K$  are finite sets of indices, and  $\pi_i: \mathbb{R}^n \rightarrow \mathbb{R}$  denotes the projection on the  $i$ th coordinate.

*Proof.* The desired conclusion easily follows from Proposition 3.3.1, the fact that, for all  $f \in \text{FVL}_n^+$ ,  $f = f \vee 0$ , and the distributivity of the lattice structure of  $\text{FVL}_n$ .  $\square$

**Lemma 4.4.3.**  $H$  is a closed half-space of  $\mathbb{R}^n$  which contains 0 in its topological interior if and only if its gauge function  $g_H$  is of the form  $l \vee 0$ , where  $l \neq 0$  is a linear word of  $\text{FVL}_n$ .

*Proof.*  $H$  can be written as

$$H = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n \mid a \cdot x + b = a_1 x_1 + \dots + a_n x_n + b \geq 0\},$$

with  $0 \neq a \in \mathbb{R}^n$  and  $b \in \mathbb{R}$ . Since 0 lies in the topological interior of  $H$ , we can always choose  $a$  and  $b$  such that  $b > 0$ . We recall that for all  $\lambda > 0$  the point  $x$  belongs to  $\lambda H$  if and only if  $\frac{1}{\lambda} x \in H$ . If  $x \in \mathbb{R}^n$  is such that  $a \cdot x \geq 0$ , then for each  $\lambda > 0$  we have  $a \cdot \frac{1}{\lambda} x + b \geq a \cdot \frac{1}{\lambda} x = \frac{1}{\lambda} (a \cdot x) \geq 0$ , whence  $g_H(x) = 0$ . Otherwise,  $a \cdot x = c < 0$ . Therefore, for all  $\lambda > 0$  the inequality  $a \cdot \frac{1}{\lambda} x + b \geq 0$  holds if and only if  $\lambda \geq -\frac{c}{b}$ . In this case,  $g_H(x) = -\frac{1}{b} a \cdot x = -\frac{a_1}{b} x_1 - \dots - \frac{a_n}{b} x_n = \sum_{i=1}^n \left(-\frac{a_i}{b}\right) \pi_i(x)$ . Finally, we observe that  $-\frac{1}{b} a \cdot x \geq 0$  if and only if  $a \cdot x \leq 0$ . We have thus proved the first statement of the lemma:

$$g_H(x) = \left( \sum_{i=1}^n \left(-\frac{a_i}{b}\right) \pi_i(x) \right) \vee 0.$$



On the other hand, given any

$$f(x) = \left( \sum_{i=1}^n \lambda_i \pi_i(x) \right) \vee 0 \in \text{FVL}_n$$

we get

$$C_f = \{x \in \mathbb{R}^n \mid f(x) \leq 1\} = \left\{ x \in \mathbb{R}^n \mid \sum_{i=1}^n -\lambda_i x_i + 1 \geq 0 \right\}.$$

Thus, if  $f \neq 0$  then  $\sum_{i=1}^n \lambda_i \pi_i \neq 0$  and  $C_f$  is a closed half-space containing 0 in its topological interior.  $\square$

The following result provides a characterization of polyhedral star-shaped objects in terms of closed half-spaces. This is the kernel of the proof of Theorem 4.4.5. Moreover it allows us to forget, in the polyhedral case, the definition of a star-shaped object, and just to use the more familiar one of half-space.

**Theorem 4.4.4** (Characterization of  $\Pi_0^n$ ). *The elements of  $\Pi_0^n$  are exactly those subsets of  $\mathbb{R}^n$  that can be written as finite unions of finite intersections of closed half-spaces whose topological interiors contain the point 0:*

$$\begin{aligned} \Pi_0^n &= \\ &= \left\{ A \subseteq \mathbb{R}^n \mid A = \bigcup_{i=1}^p \bigcap_{j=1}^{q_i} H_{ij}, H_{ij} \subseteq \mathbb{R}^n \text{ closed half-space, } 0 \in \text{int}(H_{ij}) \right\}. \end{aligned}$$

*Proof.* If  $A$  is a finite union of finite intersections of half-spaces containing the origin in their topological interior, then, by Lemma 4.4.3, Theorem 4.1.10 and Proposition 4.1.12, the gauge function of  $A$  is a finite meet of finite joins of linear words joined with 0, whence  $g_A$  is an element of  $\text{FVL}_n^+$ . This ensures that  $A$  is in  $\Sigma_0^n$  and, of course, in  $\Pi_0^n$ .

For the converse, we have to split the proof into two parts. First of all, we show that if  $A \in \Pi_0^n$  is such that  $A = \bigcup_{i=1}^p \bigcap_{j=1}^{q_i} H_{ij}$ , with the  $H_{ij}$ 's closed half-spaces, then  $A$  is also a finite union of finite intersections of closed half-spaces  $H'_{ij}$  which contain the point 0 (not necessarily in their topological interiors). To do that, for all  $i = 1, \dots, p$ , we set  $K_i = \bigcap_{j=1}^{q_i} H_{ij}$  and we prove that there exists  $K'_i = \bigcap_{j=1}^{q_i} H'_{ij}$ , with  $H'_{ij}$  closed half-spaces containing the origin, such that  $K_i \subseteq K'_i \subseteq A$ . If  $0 \in K_i$  it suffices to take  $K'_i = K_i$ . Otherwise, we can take the set

$$K'_i = \text{Conv}(K_i, 0) = \{\lambda x \in \mathbb{R}^n \mid 0 \leq \lambda \leq 1, x \in K_i\}.$$

Since  $A$  is star-shaped, then  $K'_i \subseteq A$ , whence  $A = \bigcup_{i=1}^p K'_i$ , because  $K'_i$  contains  $K_i$ . Since  $K_i$  is a convex polyhedron (not necessarily compact), also  $K'_i$  is a convex polyhedron, and hence it is a finite intersection of closed half-spaces  $H'_{ij}$ . Moreover, since  $0 \in K'_i$ , then  $0 \in H'_{ij}$  for each  $j$ . This completes the first step of our proof.

In the second part, we can consider only the closed half-spaces  $H'_{ij}$  containing 0 in their topological interior. Let  $A = \bigcup_{i=1}^p \bigcap_{j=1}^{q_i} H_{ij}$ , with the  $H_{ij}$ 's closed

half-spaces containing 0. Then, we can proceed in the same way as in the proof of Proposition 4.1.12 to show that  $g_A = \bigwedge_{i=1}^p \bigvee_{j=1}^{q_i} g_{H_{ij}}$ . In the present situation,  $g_{H_{ij}}$  may take on value  $\infty$ , and we are assuming that  $r \leq \infty$ , for each  $r \in \mathbb{R}$ ; with these conventions the same proof goes true. Moreover, by Theorem 4.1.10,  $g_A \in \mathcal{G}^n$  and, by Lemma 4.4.3,  $g_{H_{ij}} = l_{ij} \vee 0$ , for each  $H_{ij}$  with 0 in its topological interior and for some linear word  $l_{ij}$  of  $\text{FVL}_n$ . The other  $H_{ij}$ 's, the ones which do not have 0 in their topological interiors, are of the form

$$H_{ij} = \{x \in \mathbb{R}^n \mid a \cdot x = 0\},$$

for some  $a \in \mathbb{R}^n$ . Hence, their gauge function is quite simple to describe:

$$g_{H_{ij}}(x) = \begin{cases} 0 & \text{if } x \in H_{ij} \\ \infty & \text{otherwise.} \end{cases}$$

Because  $g_A$  is in  $\mathcal{G}^n$ , the gauge function  $g_A$  is finite everywhere, and hence, it may coincide with  $g_{H_{ij}}$  for some closed half-space  $H_{ij}$  not containing 0 in its topological interior, only if both  $g_A$  and  $g_{H_{ij}}$  are equal to 0. This allows us to restrict our attention to linear words  $l_{ij}$  together with the function 0. Let  $W = \{w_1, \dots, w_s\}$  denote this set. For each permutation  $\tau \in \text{Perm}(s)$  of the index set  $\{1, \dots, s\}$  let the set  $T_\tau$  be defined by

$$T_\tau = \{x \in \mathbb{R}^n \mid w_{\tau(1)}(x) < \dots < w_{\tau(s)}(x)\}.$$

Each nonempty  $T_\tau$  is full-dimensional open cone of  $\mathbb{R}^n$ . Moreover their set-theoretic union

$$T = \bigcup_{\tau \in \text{Perm}(s)} T_\tau$$

is dense in  $\mathbb{R}^n$ .

Now we focus our attention on a fixed  $x$  in a fixed nonempty  $T_\tau$ . Here  $g_A$  is finite and, because it is a finite meet of finite joins of  $g_{H_{ij}}$ 's, it must be equal to some  $w_\tau \in W$ . Since  $T_\tau$  is connected,  $g_A$  coincides with a uniquely determined  $w_\tau$  on  $T_\tau$ . Then, in  $T_\tau$ , we have  $g_A = \bigvee \{w_i \mid w_i(x) \leq w_\tau(x), \forall x \in T_\tau\}$ , whence

$$g_A = \bigwedge_{T_\tau \neq \emptyset} \bigvee \{w_i \mid w_i(x) \leq w_\tau(x), \forall x \in T_\tau\}$$

on  $T$ . The continuity of  $g_A$ , the fact that  $g_A(x) \geq 0$  for all  $x \neq 0$ , and the density of  $T$  in  $\mathbb{R}^n$  ensure that

$$\begin{aligned} g_A &= g_A \vee 0 = \left( \bigwedge_{T_\tau \neq \emptyset} \bigvee \{w_i \mid w_i(x) \leq w_\tau(x), \forall x \in T_\tau\} \right) \vee 0 = \\ &= \bigwedge_{T_\tau \neq \emptyset} \bigvee \{w_i \vee 0 \mid w_i(x) \leq w_\tau(x), \forall x \in T_\tau\}. \end{aligned}$$

We can now use the map  $\omega$  of Theorem 4.1.10 to translate the last identity into the geometric language of half-spaces:

$$A = \bigcup_{T_\tau \neq \emptyset} \bigcap \{\omega(w_i \vee 0) \mid w_i(x) \leq w_\tau(x), \forall x \in T_\tau\}.$$

By Lemma 4.4.3, in the last equation  $\omega(w_i \vee 0)$  is not a half-space if and only if  $w_i = 0$ . In this case  $\omega(w_i \vee 0) = \mathbb{R}^n$ . This, however, is not a problem:  $\mathbb{R}^n$  can be simply deleted in any intersection where it does not appear alone. If there is some intersection which involves  $\mathbb{R}^n$  only, then  $A = \mathbb{R}^n$ , and hence it is easy to write it as a finite union of finite intersections of half-spaces with 0 in their topological interior. For example, we can take the union of the following two half-spaces:

$$H_1 = \{x \in \mathbb{R}^n \mid x_1 + 1 \geq 0\} \quad \text{and} \quad H_2 = \{x \in \mathbb{R}^n \mid -x_1 + 1 \geq 0\}.$$

This completes the proof.  $\square$

**Theorem 4.4.5.** *The maps  $\omega$  and  $\gamma$  are isomorphisms between*

$$(\text{FVL}_n^+, \vee, \wedge, +, \{\lambda\}_{\lambda \in \mathbb{R}^+}, 0) \quad \text{and} \quad (\Pi_0^n, \cap, \cup, +_g, \{\lambda \cdot\}_{\lambda \in \mathbb{R}^+}, \mathbb{R}^n).$$

*Proof.* The bijection between the elements of  $\text{FVL}_n^+$  and  $\Pi_0^n$  is given by Proposition 4.1.12 and Lemma 4.4.3 applied to Lemma 4.4.2 and Theorem 4.4.4. The fact that  $\omega$  and  $\gamma$  are order-reversing isomorphism between the two structures follows immediately from Theorem 4.2.5.  $\square$

#### 4.4.1 Polyhedral good sequences

To prove the polyhedral version of Theorem 4.3.14, we make essential use of the isomorphism result of Theorem 4.4.5. We begin by specializing the definition of good sequences for star-shaped objects to the polyhedral case.

**Definition 4.4.6** (Polyhedral unit). A polyhedral star-shaped object  $U$  is a *polyhedral unit* if it is a unit of  $\Sigma_0^n$ . Equivalently,  $U \in \Pi_0^n$  is a polyhedral unit if for any  $P \in \Pi_0^n$  there exists  $0 \leq \lambda \in \mathbb{R}$  such that  $\lambda \cdot U \subseteq P$ .

**Definition 4.4.7** (Polyhedral good sequence). Fix a polyhedral unit  $U \in \Pi_0^n$ . The sequence  $(P_i)_{i \in \mathbb{N}}$  of elements of  $\Pi_0^n$  is a *polyhedral good sequence* if it is a good sequence of star-shaped objects, that is, if

*PS1)* there exists an index  $j \in \mathbb{N}$  such that, for all  $i \geq j$ ,  $P_i = \mathbb{R}^n$ ;

*PS2)*  $U \subseteq P_i$ , for all  $i \in \mathbb{N}$ ;

*PS3)*  $P_i \oplus_g P_{i+1} = P_i$ , for all  $i \in \mathbb{N}$ .

Now, we translate Definition 4.4.7 into algebraic language. The maps  $\omega$  and  $\gamma$  then preserve good sequences, meaning that the element-wise image of a polyhedral good sequence is a good sequence of  $\text{FVL}_n$ , and vice versa. We repeat here for clarity the definition of good sequence even though it is an instance of Definition 1.3.9.

**Definition 4.4.8** (Good sequence in  $\text{FVL}_n$ ). Fix a unit  $u$  of  $\text{FVL}_n$ . A *good sequence* is a sequence  $(f_i)_{i \in \mathbb{N}}$  of elements in  $\text{FVL}_n^+$  such that

*FS1)* there exists an index  $j \in \mathbb{N}$  such that, for all  $i \geq j$ ,  $f_i = 0$ ;

*FS2)*  $f_i \leq u$ , for all  $i \in \mathbb{N}$ ;

*FS3)*  $f_i \oplus f_{i+1} = f_i$ , for all  $i \in \mathbb{N}$ .

Here,  $a \oplus b$  is defined as the element  $(a + b) \wedge u$ , for all  $a, b \in \text{FVL}_n$ .

The following lemma is a special case of Lemma 1.3.10 in [11].

**Lemma 4.4.9.** *Let  $u$  be a unit of  $\text{FVL}_n$ . Then for each  $f \in \text{FVL}_n^+$  there exists a unique good sequence  $(f_i)_{i \in \mathbb{N}}$  of elements of the unit interval  $[0, u]$  such that  $f = f_1 + f_2 + \dots$*

Now, coming back to geometry, we obtain the desired result:

**Theorem 4.4.10** (Polyhedral good sequences representation). *Fix a unit  $U \in \Pi_0^n$  and an element  $P \in \Pi_0^n$ . There exists a unique polyhedral good sequence  $(P_i)_{i \in \mathbb{N}}$  in the unit interval  $[U, \mathbb{R}^n]$  of  $\Pi_0^n$ , such that*

$$P = P_1 +_g P_2 +_g P_3 +_g \dots$$

*Proof.* From Lemma 4.4.9 and Theorem 4.4.5. □

# Chapter 5

## Conclusions

In this final chapter we indicate directions for possible further research.

### 5.1 A Riesz Representation Theorem for star-shaped objects

Let  $X$  be a metrizable Hausdorff compact space, and let  $\mathcal{C}(X)$  be the set of all the real-valued continuous functions on  $X$ . Then  $\mathcal{C}(X)$  is a vector lattice with respect to the pointwise operations of minimum, maximum, addition and products by real scalars. Moreover, by the compactness of  $X$ , the function identically equal to 1, denoted by  $\mathbf{1}$ , is a unit of  $\mathcal{C}(X)$ , whence the pair  $(\mathcal{C}(X), \mathbf{1})$  is a unital vector lattice.

**Definition 5.1.1** (State of a unital vector lattice). A *state* of a unital vector lattice  $(V, u)$  is a map  $s: V \rightarrow \mathbb{R}$  that is

*S1)* linear:  $s(\alpha f + \beta g) = \alpha s(f) + \beta s(g)$ , for all  $f, g \in V$  and  $\alpha, \beta \in \mathbb{R}$ ;

*S2)* order-preserving: if  $f \geq 0$  then  $s(f) \geq 0$ , for all  $f \in V$ ;

*S3)* normalized:  $s(u) = 1$ .

We denote by  $\mathcal{S}(X)$  the set of all states of  $(\mathcal{C}(X), \mathbf{1})$ .

**Remark 5.1.2.** We immediately notice that, by linearity, a state of  $(V, u)$  is completely determined by the values it takes on the positive cone  $V^+$ .

**Definition 5.1.3** (Probability measure on  $X$ ). Let  $\mathcal{B}(X)$  be the Borel  $\sigma$ -algebra of  $X$ . A *probability measure* on the space  $X$  is a map  $\mu: \mathcal{B}(X) \rightarrow [0, 1]$  that is

*PM1)* a measure on  $\mathcal{B}(X)$ ;

*PM2)* normalized:  $\mu(X) = 1$ .

We denote by  $\mathcal{M}(X)$  the set of all probability measures on  $X$ . Note that each element of  $\mathcal{M}(X)$  is regular.

**Theorem 5.1.4** (Riesz Representation Theorem, [32, Theorem 18.4.1]). *The map*

$$\begin{aligned} \mathcal{R}: \mathcal{M}(X) &\rightarrow \mathcal{S}(X) \\ \mu &\mapsto \left( s_\mu(f): f \mapsto \int_X f d\mu \right) \end{aligned}$$

*is a bijection.*

The idea is to translate the Riesz Representation Theorem in the language of star-shaped objects.

Let  $\mathcal{CP}(\mathbb{R}^n)$  be the set of all the continuous and positively homogeneous real-valued functions on  $\mathbb{R}^n$ . This set (equipped with the usual pointwise operations of minimum, maximum, addition and products by real scalars) is a vector lattice, and its positive cone is  $\mathcal{G}^n$  (see Chapter 4). Moreover, if we fix a unit  $f$  on  $\mathcal{CP}(\mathbb{R}^n)$ , it can be shown that the pair  $(\mathcal{CP}(\mathbb{R}^n), f)$  is unitaly isomorphic to  $(\mathcal{C}(X), \mathbf{1})$ , where  $X = f^{-1}(1) = \{x \in \mathbb{R}^n \mid f(x) = 1\}$ .

Using the vector-lattice version of the Stone-Weierstrass Theorem ([2, Theorem 11.3, p.88]), we obtain the following:

**Corollary 5.1.5** (Riesz Representation Theorem for  $\text{FVL}_n$ ). *Let  $f$  be a unit of  $\text{FVL}_n$ . Then there is a bijection between the set  $\mathcal{M}(f^{-1}(1))$  of probability measures on  $f^{-1}(1) = \{x \in \mathbb{R}^n \mid f(x) = 1\}$  and the set  $\mathcal{S}(f^{-1}(1))$  of states of the unital vector lattice  $(\text{FVL}_n, f)$ . This bijection is given by the map*

$$\mathcal{R}: \mu \mapsto \left( s_\mu: g \mapsto \int_{f^{-1}(1)} g d\mu \right).$$

We notice that the set  $f^{-1}(1)$  is precisely the (formal) boundary of the compact polyhedral star-shaped object  $C_f = \{x \in \mathbb{R}^n \mid f(x) \leq 1\}$ , that is, the set associated to  $f$  by the maps  $\omega$  and  $\gamma$  of Theorem 4.1.10. This, together with Remark 5.1.2, indicates that there is a theory of Borel measures on boundaries of compact polyhedral star-shaped objects to be developed.

## 5.2 Integral polyhedral star-shaped objects

As we have seen in Chapter 1, the free vector lattice  $\text{FVL}_n$  is generated by the projections

$$\begin{aligned} \pi_i: \quad \mathbb{R}^n &\rightarrow \mathbb{R} \\ (x_1, \dots, x_n) &\mapsto x_i, \end{aligned}$$

for  $i = 1, \dots, n$ .

If we now consider the set generated by the projections  $\pi_i$ 's with the  $\ell$ -group operations of pointwise minimum, maximum, addition, and additive inverse, we obtain the set of all continuous, positively homogeneous and piecewise linear real-valued functions on  $\mathbb{R}^n$ , such that each linear piece (see Definition 1.2.5) has integer coefficients. By the Baker-Beynon duality for  $\ell$ -groups, this set, equipped with the aforesaid operations, is actually the free  $\ell$ -group over  $n$  generators  $\text{FG}_n$ .

Hence, we can specialize the correspondence of Theorem 4.1.10 to  $\text{FG}_n$ . The resulting set is a subset of the set of polyhedral star-shaped objects  $\Pi_0^n$  (see Definition 4.4.1). More precisely, the positive cone of  $\text{FG}_n$  corresponds, via the

maps  $\omega$  and  $\gamma$ , to the set of *integral polyhedral star-shaped objects*. They are the subsets of  $\mathbb{R}^n$  that can be written as finite unions of finite intersections of *rational closed half-spaces* with 0 in their topological interior, that is, half-spaces of the form

$$H = \{x \in \mathbb{R}^n \mid a \cdot x + b \geq 0\},$$

with  $0 \neq a \in \mathbb{Z}^n$ , and  $0 < b \in \mathbb{Z}$ .

Using an analogue of Definition 5.1.1, we can consider the notion of state on the unital  $\ell$ -group  $(\text{FG}_n, u)$ . The Riesz Representation Theorem (and hence Corollary 5.1.5) can be adapted to this context by taking the divisible hull of  $\text{FG}_n$ , and then applying the Stone-Weierstrass Approximation Theorem to the resulting lattice-ordered rational vector space.

Let us fix, for example, the unit  $u = \sum_{i=1}^n |\pi_i|$  in  $\text{FG}_n$ . We observe that  $u^{-1}(1)$  is the (topological) boundary  $\partial O$  of the octahedron

$$O = C_u = \text{conv}\{\pm e_i \mid i = 1, \dots, n\},$$

where  $\{e_i \mid i = 1, \dots, n\}$  is the standard basis of  $\mathbb{R}^n$ . There is a natural measure  $\eta$  on  $\partial O$  induced by the  $(n-1)$ -dimensional Lebesgue measure. Results in the literature on the characterization of the Lebesgue state of the free MV-algebra on  $n$  generators (cf. [29, 24, 23]) can be adapted to give a characterization of the state of  $\text{FG}_n$  induced by  $\eta$ . A key ingredient here is the invariance of  $\eta$  under the unital automorphisms of  $(\text{FG}_n, u)$ . The passage from vector spaces to groups is essential, because no such characterization is available on vector lattices:  $\eta$  is not invariant under the unital vector lattice automorphisms of  $(\text{FVL}_n, u)$ .

This suggests that there should be a way to characterize a natural correspondent of the Lebesgue measure on integral polyhedral star-shaped objects in terms of invariance with respect to the appropriate group of automorphisms. We hope to be able to obtain such a result in the future.

### 5.3 Integrals of support functions and states of gauge functions

We notice that the states on  $\text{FVL}_n$  with the fixed unit  $u$  are examples of additive valuations on  $\text{FVL}_n$ . This fact, together with the correspondence given by Theorems 3.4.8 and 3.4.10, suggests that each state  $s$  of  $(\text{FVL}_n, u)$  induces a unique  $\mathcal{M}$ -additive valuation  $\nu_s$  on  $\mathcal{P}_\star^n$  (see Definitions 3.4.2 and 1.1.5). In particular, for each  $P \in \mathcal{P}_\star^n$ ,

$$\nu_s(P) = \int_{\partial u^{-1}(1)} f_P d\mu_s, \quad (5.1)$$

where  $f_P$  is the support function of  $P$ , and  $\mu_s$  is the unique measure associated to  $s$  by the Riesz Representation Theorem.

Let us recall that there is a well-known connection between support functions and gauge functions of convex bodies. Let  $C$  be a closed convex subset of  $\mathbb{R}^n$  that contains the origin 0 in its topological interior. If we consider its *polar set*

$$C^\star = \{x \in \mathbb{R}^n \mid x \cdot y \leq 1 \text{ for all } y \in C\},$$

then  $C^\star$  is a closed convex subset of  $\mathbb{R}^n$  which contains 0 in its topological interior, too. Moreover, the gauge function  $g_{C^\star}$  of  $C^\star$  is precisely the support function  $f_C$  of  $C$  (see [13] for more details).

In conclusion, the integral formula (5.1) suggests an interesting four-way connection between gauge functions, support functions, measures on compact convex star-shaped objects, and states of vector lattices. This, we believe, is well-worth exploring in future research.



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