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Solutions of a Nonlinear Dirac Equation with External Fields

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Abstract

We study the stationary Dirac equation:

$$-ic\hbar\sum_{k=1}^{3}\alpha_k\partial_k u + mc^2\beta u + M(x)u = R_u(x,u),$$

where $M(x)$ is a matrix potential describing the external field, and $R(x,u)$ stands for an asymptotically quadratic nonlinearity modeling various types of interaction without any periodicity assumption. For $\hbar$ fixed our discussion includes the Coulomb potential as a special case, and for the semiclassical situation ($\hbar \to 0$), we handle the scalar fields. We obtain existence and multiplicity results of stationary solutions via critical point theory.

1. Introduction

Nonlinear Dirac equations occur in the attempt to model extended relativistic particles in external fields (see [9,18,25]). In a general form, such equations are given by

$$-i\hbar\partial_t\psi = ic\hbar\sum_{k=1}^{3}\alpha_k\partial_k\psi - mc^2\beta\psi - M(x)\psi + G_\psi(x,\psi)\,; \qquad (1.1)$$

here $x=(x_1,x_2,x_3)\in\mathbb{R}^3$, $\partial_k=\frac{\partial}{\partial x_k}$, c denotes the speed of light, $m>0$ the mass of the electron, and $\hbar$ denotes Planck's constant. Furthermore, $\alpha_1,\alpha_2,\alpha_3$ and β are 4×4 complex matrices whose standard form (in 2×2 blocks) is

$$\beta=\begin{pmatrix} I & 0\\ 0 & -I\end{pmatrix},\quad \alpha_k=\begin{pmatrix} 0 & \sigma_k\\ \sigma_k & 0\end{pmatrix},\quad k=1,2,3$$

with

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

One verifies that $\beta = \beta^*$, $\alpha_k = \alpha_k^*$, $\alpha_k\alpha_l + \alpha_l\alpha_k = 2\delta_{kl}$ and $\alpha_k\beta + \beta\alpha_k = 0$; due to these relations, the linear operator $\mathscr{H}_0 = -ic\hbar\sum_{k=1}^{3}\alpha_k\partial_k\psi + mc^2\beta\psi$ is a symmetric operator, such that

$$\mathscr{H}_0^2 = -c^2\hbar^2\Delta + m^2c^4 \ .$$

A solution $\psi : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{C}^4$ of (1.1), with $\Psi(t, \cdot) \in L^2(\mathbb{R}^3, \mathbb{C}^4)$, is a *wave function* which represents the state of a relativistic electron.

The external fields are given by the real matrix potential $M(x)$, and the nonlinearity $G : \mathbb{R}^3 \times \mathbb{C}^4 \to \mathbb{R}$ represents a nonlinear self-coupling. We assume throughout the paper that G satisfies $G(x, e^{i\theta}\psi) = G(x, \psi)$, for all $\theta \in [0, 2\pi]$. We are looking for stationary solutions of (1.1) which may be regarded as “particle-like solutions” (see [25]): they propagate without changing their shape and thus have a soliton-like behavior.

The stationary solutions of Equation (1.1) are found by the Ansatz

$$\psi(t, x) = e^{\frac{i\theta t}{\hbar}} u(x) \ ;$$

then $u : \mathbb{R}^3 \to \mathbb{C}^4$ satisfies the equation

$$-ic\hbar\sum_{k=1}^{3}\alpha_k\partial_k u + mc^2\beta\, u + M(x)u = G_u(x, u) - \theta u \ . \tag{1.2}$$

Dividing Equation (1.2) by $\hbar c$, we are led to study equations of the form

$$-i\sum_{k=1}^{3}\alpha_k\,\partial_k u + a\,\beta\, u + \omega\, u + M(x)u = G_u(x, u) \ , \tag{1.3}$$

where $a > 0$ and $\omega \in \mathbb{R}$. We look for weak solutions which are localized in space; more precisely, the solutions we find satisfy $u \in \bigcap_{2 \leqq q < \infty} W^{1,q}(\mathbb{R}^3, \mathbb{C}^4)$.

Nonlinear problems of the form (1.3) have been studied in recent years by several authors.

In [3,4,11,24], the so-called Soler problem was considered, in which $M \equiv 0$ and G has the form

$$G(u) = \tfrac{1}{2}\, H(\tilde{u}u), \quad H \in \mathcal{C}^2(\mathbb{R}, \mathbb{R}), \quad H(0) = 0; \text{ here } \ \tilde{u}u := (\beta u, u)_{\mathbb{C}^4} \tag{1.4}$$

Note that G does not depend explicitly on x and that no external field is present. In this case, using a suitable Ansatz for the solution u, Equation (1.3) can be reduced to a system of ODE’s. In [3,4,11,24] shooting methods were used to prove the existence of a solution provided that $\omega \in (-a, 0)$ and under suitable assumptions on H.

In [17], M. Esteban and E. Séré treated the above mentioned system of ODEs variationally, obtaining the existence of infinitely many solutions, under the main additional assumption that $H'(s)\, s \geqq \theta\, H(s)$ for some $\theta > 1$, and all $s \in \mathbb{R}$.

In another model presented by Finkelstein et al. [19], the nonlinearity G has the form

$$G(u) = \tfrac{1}{2}|\tilde{u}u|^2 + b|\tilde{u}\alpha u|^2, \quad \tilde{u}\alpha u := (\beta u, \alpha u)_{\mathbb{C}^4}, \quad \alpha := \alpha_1\alpha_2\alpha_3 \tag{1.5}$$

with $b > 0$. For such nonlinearities, the above mentioned Ansatz cannot be applied. In [17], Esteban and Séré considered nonlinearities of type (1.5), however with a weaker growth

$$G(u) = \mu|\tilde{u}u|^\tau + b|\tilde{u}\alpha u|^\sigma, \quad 1 < \tau,\ \sigma < \tfrac{3}{2},\ \mu, b > 0$$

This growth restriction is due to the variational approach, in which the natural space for the associated functional is given by $H^{1/2}(\mathbb{R}^3, \mathbb{C}^4)$.

In the paper [7], Th. Bartsch and Y. Ding investigated the Dirac equation by using some recently developed critical point theorems from [6] for strongly indefinite functionals. They mainly treated functions $G(x, u)$ which depend periodically on x and which may be superquadratic or asymptotically quadratic in u as $|u| \to \infty$; they obtained infinitely many solutions if G is even, for superquadratic G as well as in the asymptotically quadratic case. They also considered the case where the nonlinearity has a non-vanishing quadratic part in the origin, so that the linearized equation has a potential.

In the present paper, we consider equations of the form (1.3) with symmetric real matrix potentials $M(x)$ (that is $M(x) := \big(m_{j,k}(x)\big)$ is a symmetric real 4×4-matrix). In the following, for convenience, any real function $U(x)$ will be regarded as the symmetric matrix $U(x)I_4$ where I_4 denotes the 4×4 identity matrix. For two given symmetric 4×4 real matrix functions $L_1(x)$ and $L_2(x)$, we write that $L_1(x) \leqq L_2(x)$ if and only if

$$\max_{\xi \in \mathbb{C}^4, |\xi|=1} (L_1(x) - L_2(x))\, \xi \cdot \bar{\xi} \leqq 0.$$

We are interested in

a. *Vector potentials* $M(x)$ (see Thaller [29]) which either
   i) are of *Coulomb-type* that is, tend to 0 as $|x| \to \infty$ and are singular at the origin (for example the Coulomb potential $\frac{\kappa}{|x|}$), or
   ii) have the property that for some $b > 0$ the measure of the sublevel set Ω_b of $\beta M(x)$ is finite (i.e $|\Omega_b| = |\{x \in \mathbb{R}^3 : \beta M(x) < b\}| < \infty$).

   Vector potentials serve, for example, to take into account external electromagnetic fields.

b. *Scalar potentials* of the form $M(x) = \beta V(x)$, where $V : \mathbb{R}^3 \to \mathbb{R}$; such potentials can be used as a model for quark confinement (see [29]). A Dirac operator with this type of potential is also referred to as a *Dirac operator with supersymmetry*. We will assume that V is non-positive in some point x_0, and that for some $b > 0$ the sublevel set Ω_b of V has finite measure (that is $|\Omega_b| = |\{x \in \mathbb{R}^3 : V(x) < b\}| < \infty$ for some $b > 0$).

To treat the nonlinear problem, it is crucial to have information about the spectrum of the linearized operator $\mathscr{A} := -i\sum_{k=1}^{3}\alpha_k\partial_k + a\beta + \omega + M$ in the origin. Our assumptions on M will guarantee that $\mathscr{A}$ is selfadjoint acting in $L^2(\mathbb{R}^3, \mathbb{C}^4)$ with $\mathscr{D}(\mathscr{A}) \subset H^{1/2}(\mathbb{R}^3, \mathbb{C}^4)$ and has a spectral gap around the origin, and that there exist a finite number (or infinitely many) eigenvalues in the spectral gap.

We will consider nonlinearities $G_u(x, u)$ which are asymptotically linear, that is, $G_u(x, u) = Q(x)u + o(|u|)$ for $|u| \to \infty$, where $Q(x)$ is a continuous and symmetric real 4×4-matrix-function. We assume $q_0 := \inf_x Q_{\min}(x) > 0$ where $Q_{\min}(x)$ denotes the minimal eigenvalue of $Q(x)$. Furthermore, we assume that $G_u(x, u) = o(|u|)$ for u near 0, that $q_\infty := \limsup_{|x|\to\infty} Q_{\max}(x)$ lies in the spectral gap where $Q_{\max}(x)$ denotes the maximal eigenvalue of $Q(x)$, and that between 0 and q_0 lie some eigenvalues of $\mathscr{A}$. We recall that nonlinearities of this type have been introduced by Amann–Zehnder [1] in other contexts (see also [28]).

Roughly speaking, the results we prove are:

Theorem A. *Suppose that M is a vector potential having either the form*

a.i)*, and $q_\infty < a$, where $a > 0$ is the upper bound of the spectral gap; or the form*
a.ii)*, and $q_\infty < a + b_{\max}$, where $b_{\max} := \sup\{b : |\Omega_b| < \infty\}$.*

Then (provided G satisfies some additional technical conditions) problem (1.3) has at least one solution. If in addition $G_u(x, u)$ is odd in u, then (1.3) has at least ℓ pairs of solutions, where ℓ is the number of eigenvalues of $\mathscr{A}$ between 0 and q_0.

We also consider the so-called semi-classical limit, that is, when (formally) Planck's constant $\hbar$ tends to zero.

Theorem B. *Suppose that M is a scalar potential satisfying the above condition* (**b**). *If $G(x, u)$ is as in Theorem A, and if $q_\infty < a + b_{\max}$, then there exists a $\varepsilon_0 > 0$ such that problem (1.2) has at least one solution for $\varepsilon^2 := \hbar < \varepsilon_0$. If $G_u(x, u)$ is odd in u, then for each $m \in \mathbb{N}$ there exists $\varepsilon_m > 0$ such that Equation (1.2) has at least m solutions for $\varepsilon^2 < \varepsilon_m$.*

The paper is organized as follows. In Section 2 we state the precise hypotheses and our main results. In Section 3 we formulate the variational setting and we discuss the required critical point theory. We prove our theorems for fixed $\hbar$ in Section 4, and finally, in Section 5 for the singularly perturbed equation (semi-classical solutions).

2. Main results

Specifically, we are interested in the Dirac equation with external fields of the form

$$-\varepsilon^2\sum_{k=1}^{3} i\alpha_k\partial_k u + a\beta u + M(x)u = R_u(x, u) \tag{2.1}$$

where $\varepsilon^2 = \hbar$, $a = mc > 0$, $M(x) = \big(m_{jk}(x)\big)$ is a 4×4 symmetric real matrix function defined almost everywhere on $\mathbb{R}^3$, that is, $m_{jk}(x) = m_{kj}(x) \in \mathbb{R}$ for $j, k = 1, 2, 3, 4$ and almost every $x \in \mathbb{R}^3$, such that

$$A := H_0 + M \ \text{ with } \ H_0 := -i \sum_{k=1}^{3} \alpha_k \partial_k + a\beta$$

is a selfadjoint operator in $L^2(\mathbb{R}^3, \mathbb{C}^4)$ with domain $\mathscr{D}(A) \subset H^1(\mathbb{R}^3, \mathbb{C}^4)$, and $R(x, u)$ satisfies

- (R_1) $R(x, u) \geqq 0$ and $R_u(x, u) = o(|u|)$ as $u \to 0$ uniformly in x;
- (R_2) $R_u(x, u) - Q(x)u = o(|u|)$ uniformly in x as $|u| \to \infty$, where Q is a continuous symmetric 4×4 real matrix function;
- (R_3) Either (i) $0 \notin \sigma(A - Q)$, or (ii) $\tilde{R}(x, u) \geqq 0$ and there exist δ_0, $\nu_0 > 0$ such that $\tilde{R}(x, u) \geqq \delta_0$ if $|u| \geqq \nu_0$;
- (R_4) $q_0 := \inf_x Q_{\min}(x) > \inf \sigma(A) \cap (0, \infty)$.

Here (and below) we denote by $\sigma(B)$ the spectrum of an operator B, and we write

$$\tilde{R}(x, u) := \frac{1}{2} R_u(x, u) \cdot u - R(x, u)$$

($u \cdot v$ or uv denotes the scalar product of $\mathbb{C}^4$). Set

$$q_\infty := \limsup_{|x| \to \infty} \left(\sup_u \frac{|R_u(x, u)|}{|u|} \right).$$

First we consider the case that $\varepsilon = 1$:

$$-i \sum_{k=1}^{3} \alpha_k \partial_k u + a\beta u + M(x)u = R_u(x, u) \tag{$\mathcal{P}$}$$

with the Coulomb type potential

- (M_1) M is a symmetric continuous real 4×4-matrix function on $\mathbb{R}^3 \setminus \{0\}$ with $0 \geqq M(x) \geqq -\frac{\kappa}{|x|}$ where $\kappa < \frac{\sqrt{3}}{2}$.

It is known that the corresponding operator A is selfadjoint with domain $\mathscr{D}(A) = H^1(\mathbb{R}^3, \mathbb{C}^4)$, $\sigma_e(A) = \mathbb{R} \setminus (-a, a)$ and $\sigma_d(A) \cap (0, a) \neq \emptyset$ where $\sigma_e(A)$ denotes the essential spectrum and $\sigma_d(A)$ the eigenvalues of finite multiplicity (cf. [21,29]). We assume, in addition to $(R_1) - (R_4)$, that

- (R_5) $q_\infty < a$.

Involving (R_4), let ℓ be the number of $(0, q_0) \cap \sigma(A)$. We are going to prove the following result:

Theorem 2.1. *Assume that (M_1) and (R_1)–(R_5) hold. Then $(\mathcal{P})$ has at least one solution. If, additionally, $R_u(x, u)$ is odd in $u \in \mathbb{C}^4$, then $(\mathcal{P})$ has ℓ pairs of solutions.*

Next we consider again the problem $(\mathcal{P})$ with the matrix potential $M(x)$ satisfying

(M_2) $M \in L^\infty(\mathbb{R}^3, \mathbb{R}^{4\times 4})$, and there is $b > 0$ such that $|\Omega_b| < \infty$ where $\Omega_b := \{x \in \mathbb{R}^3 : \beta M(x) < b\}$.

Here we write $|S|$ for the Lebesgue measure of $S \subset \mathbb{R}^3$. We define the number $b_{\max} := \sup\{b : |\Omega_b| < \infty\}$. Assume instead of (R_5) that

$(\hat{R}_5)$ $q_\infty < a + b_{\max}$.

Note that, since $M \in L^3_{loc}$, it is known from [22] (see also [29], p. 306) that A is selfadjoint. Under assumption (R_4), let ℓ be the number of points in $(0, q_0) \cap \sigma(A)$.

Theorem 2.2. *Assume that (M_2), (R_1)–(R_4) and $(\hat{R}_5)$ hold. Then $(\mathcal{P})$ has at least one solution. If, additionally, $R_u(x, u)$ is odd in $u \in \mathbb{C}^4$, then $(\mathcal{P})$ has ℓ pairs of solutions.*

Finally we consider the semi-classical solutions of the Dirac equation with the *scalar potential* $M(x) = V(x)\beta$ (cf. [29]):

$$-\varepsilon^2 \sum_{k=1}^{3} i\alpha_k \partial_k u + (a + V(x))\beta u = R_u(x, u) \qquad (\mathcal{P}_\varepsilon)$$

where V is a real function satisfying

(V) $V \in L^\infty(\mathbb{R}^3, \mathbb{R})$, and there are $x_0 \in \mathbb{R}^3$ and $b > 0$ such that $V(x_0) \leqq 0$ and $|\Omega_b| < \infty$ where $\Omega_b := \{x \in \mathbb{R}^3 : V(x) < b\}$.

This type of matrix is also referred to the *Dirac operator with supersymmetry* (cf. [29]). The semiclassical point of view is important for studying Dirac operators and semiclassical methods are employed in treating Dirac equation problems (see [29, pp. 308] and the references therein).

Theorem 2.3. *Let (V), (R_1)–(R_3) and $(\hat{R}_5)$ be satisfied. Assume $q_0 > a$. Then there is $\mathcal{E}_0 > 0$ such that $(\mathcal{P}_\varepsilon)$ has at least one solution for each $\varepsilon \in (0, \mathcal{E}_0)$. If, additionally, $R_u(x, u)$ is odd in $u \in \mathbb{C}^4$, then for each $m \in \mathbb{N}$ there is $\mathcal{E}_m > 0$ such that $(\mathcal{P}_\varepsilon)$ has m solutions for each $\varepsilon \in (0, \mathcal{E}_m)$.*

Note that in this theorem we assume only that $q_0 > a$, which is weaker than (R_4).

Remark 2.4. The assumption $M \in L^\infty$ in (M_2) can be weakened. It is sufficient to require that each $m_{jk}(x)$ is measurable such that A is selfadjoint acting in L^2 with $\mathscr{D}(A) \subset H^1$. We assume $M \in L^\infty$ only for simplicity. Similarly, the assumption $V \in L^\infty$ in (V) can be weakened. See Section 3.

Here are some examples where the assumptions apply.

Example 2.5. (a) $R(x, u) = \frac{1}{2} Q(x) u \cdot u \left(1 - \frac{1}{\ln(e+|u|)}\right)$.

(b) $R(x, u) = Q(x)\varphi(\frac{1}{2}|u|^2)$ where $\varphi : [0, \infty) \to [0, \infty)$ is of class $\mathcal{C}^2$ with $\varphi(0) = \varphi'(0) = 0$, and $\varphi'(s) \to 1$ as $s \to \infty$, $\varphi''(s) \geqq 0$.

(c) $R_u(x, u) = f(x, |u|)u$, where $f(x, s)$ is even in s; $f(x, s) \to 0$ as $s \to 0$ uniformly in x; $f(x, s)$ is non-decreasing for $s \in [0, \infty)$; and $f(x, s) \to q(x)$ as $s \to \infty$.

3. Variational arguments

In what follows, by $|\cdot|_q$ we denote the usual L^q-norm, and by $(\cdot,\cdot)_2$ the usual L^2-inner product. Throughout the section we always assume that the matrix $M(x)$ is such that $A = H_0 + M$ is a selfadjoint operator on $L^2(\mathbb{R}^3, \mathbb{C}^4)$ with domain $\mathscr{D}(A) \subset H^1(\mathbb{R}^3, \mathbb{C}^4)$, and consider the equation $(\mathcal{P})$ with $R(x,u)$ satisfying (R_1)-(R_4).

Let

$$\mu_e^- := \sup\left(\sigma_e(A) \cap (-\infty, 0)\right), \quad \mu_e^+ := \inf\left(\sigma_e(A) \cap (0, \infty)\right),$$

and $\mu_e := \min\{-\mu_e^-,\ \mu_e^+\}$. We assume

(A_0) $\mu_e^- < 0 < \mu_e^+$;
(R_0) $q_\infty < \mu_e$.

We are going to prove the following result:

Theorem 3.1. *Assume that (R_1)–(R_4), (A_0) and (R_0) hold. Then $(\mathcal{P})$ has at least one solution. If, additionally, $R_u(x,u)$ is odd in $u \in \mathbb{C}^4$, then $(\mathcal{P})$ has ℓ pairs of solutions.*

3.1. A variational setting

Observe that, since we have assumed M is such that A is a selfadjoint operator with $\mathscr{D}(A) \subset H^1(\mathbb{R}^3, \mathbb{C}^4)$, $\mathscr{D}(A)$ is a Hilbert space with the graph inner product

$$(u, v)_A := (Au, Av)_2 + (u, v)_2$$

and the induced norm $|u|_A := (u,u)_A^{1/2}$. Let $\{F_\lambda : \lambda \in \mathbb{R}\}$ denote the spectral family and $|A|$ the absolute value of A. A has the polar decomposition $A = U|A|$ with $U = 1 - F_0 - F_{-0}$ (see [16]). The assumption (A_0) induces an orthogonal decomposition of $L^2(\mathbb{R}^3, \mathbb{C}^4)$:

$$L^2 = L^- \oplus L^0 \oplus L^+, \quad u = u^- + u^0 + u^+$$

so that A is negative definite (resp. positive definite) in L^- (resp. L^+) and $L^0 = \ker A$. In fact, $L^\pm = \{u \in L^2 : Uu = \pm u\}$ and $L^0 = \{u \in L^2 : Uu = 0\}$ (see [16, Theorem IV, 3.3]). Let $P^0 : L^2 \to L^0$ denote the projector. Then P^0 commutes with A and $|A|$. Note that (A_0) implies also $\dim(L^0) < \infty$. On $\mathscr{D}(A)$ we introduce the inner product

$$\begin{aligned}\langle u, v\rangle_A :=& (Au, Av)_2 + (P^0u, P^0v)_2 \\ =& (|A|u, |A|v)_2 + (P^0u, v)_2\end{aligned}$$

whose induced norm will be denoted by $\|u\|_A$. Since (A_0) implies that 0 is at most an isolated eigenvalue of finite multiplicity of A, it is clear that $|\cdot|_A$ and $\|\cdot\|_A$ are equivalent norms on $\mathscr{D}(A)$: $d_1\,|u|_A \leqq \|u\|_A \leqq d_2\,|u|_A$ for all $u \in \mathscr{D}(A)$. Define

$$\tilde{A} := |A| + P^0.$$

Then $\mathscr{D}(\tilde{A}) = \mathscr{D}(A)$. Noting that $P^0|A| = |A|P^0 = 0$ we have for u and $v \in \mathscr{D}(A)$,

$$\begin{aligned}(\tilde{A}u, \tilde{A}v)_2 &= (|A|u, |A|v)_2 + (|A|u, P^0v)_2 + (P^0u, |A|v)_2 + (P^0u, P^0v)_2 \\ &= (|A|u, |A|v)_2 + (P^0u, P^0v)_2 = \langle u, v\rangle_A ,\end{aligned}$$

hence,

$$d_1|u|_A \leqq \|u\|_A = |\tilde{A}u|_2 \leqq d_2|u|_A \quad \text{for all } u \in \mathscr{D}(A). \tag{3.1}$$

Let $E := \mathscr{D}(|A|^{1/2})$ be the domain of the selfadjoint operator $|A|^{1/2}$, which is a Hilbert space equipped with the inner product

$$(u, v) = (|A|^{1/2}u, |A|^{1/2}v)_2 + (P^0u, P^0v)_2$$

and the induced norm $\|u\| = (u, u)^{1/2}$. E possesses the following decomposition

$$E = E^- \oplus E^0 \oplus E^+ \quad \text{with} \quad E^\pm = E \cap L^\pm \text{ and } E^0 = L^0,$$

orthogonal with respect to both $(\cdot, \cdot)_2$ and $(\cdot, \cdot)$ inner products. In fact, the $(\cdot, \cdot)_2$ orthogonality follows from the decomposition of L^2. To show the $(\cdot, \cdot)$ orthogonality, observe that, for $u^\pm \in L^\pm \cap \mathscr{D}(A)$,

$$\begin{aligned}(u^+, u^-) &= (|A|^{1/2}u^+, |A|^{1/2}u^-)_2 = (|A|u^+, u^-)_2 = (|A|Uu^+, u^-)_2 \\ &= (Au^+, u^-)_2 = (u^+, Au^-)_2 = (u^+, |A|Uu^-)_2 = -(u^+, |A|u^-)_2 \\ &= -(|A|^{1/2}u^+, |A|^{1/2}u^-)_2 \\ &= -(u^+, u^-),\end{aligned}$$

hence $(u^+, u^-) = 0$. Since $\mathscr{D}(A)$ is dense in E, one sees that E^+ and E^- are orthogonal in $(\cdot, \cdot)$. Similarly, one checks that $E^\pm$ are orthogonal to E^0 in $(\cdot, \cdot)$. Observe that for all $u \in \mathscr{D}(A)$ and $v \in \mathscr{D}(|A|^{1/2})$

$$\begin{aligned}(\tilde{A}^{1/2}u, \tilde{A}^{1/2}v)_2 &= (\tilde{A}u, v)_2 = ((|A| + P^0)u, v)_2 = (|A|u, v)_2 + (P^0u, v)_2 \\ &= (|A|^{1/2}u, |A|^{1/2}v)_2 + (P^0u, P^0v)_2 = (u, v).\end{aligned}$$

Consequently, since $\mathscr{D}(A) = \mathscr{D}(\tilde{A})$ is a core of $\tilde{A}^{1/2}$, we have

$$(u, v) = (\tilde{A}^{1/2}u, \tilde{A}^{1/2}v)_2 \quad \text{for all } u, v \in \mathscr{D}(|A|^{1/2})$$

which induces in particular that

$$\|u\| = |\tilde{A}^{1/2}u|_2 \quad \text{for all } u \in E. \tag{3.2}$$

In order to study certain embedding properties of E we set

$$\tilde{H}_0 := -i\sum_{k=1}^{3}\alpha_k\partial_k + \beta$$

(selfadjoint with $\mathscr{D}(\tilde{H}_0) = H^1(\mathbb{R}^3, \mathbb{C}^4)$). Then $\tilde{H}_0^2 = -\Delta + 1$ and, letting $|\tilde{H}_0|$ denote the absolute value of $\tilde{H}_0$,

$$||\tilde{H}_0|u|_2^2 = |\tilde{H}_0 u|_2^2 = (\tilde{H}_0 u, \tilde{H}_0 u)_2 = (\tilde{H}_0^2 u, u)_2$$
$$= ((-\Delta + 1)u, u)_2 = |\nabla u|_2^2 + |u|_2^2$$

which implies that

$$\|u\|_{H^1} = ||\tilde{H}_0|u|_2 \tag{3.3}$$

for all $u \in \mathscr{D}(\tilde{H}_0^2) = H^2(\mathbb{R}^3, \mathbb{C}^4)$, hence for all $u \in H^1(\mathbb{R}^3, \mathbb{C}^4)$ because of the density of H^2 in H^1.

Lemma 3.2. *The assumption $\mathscr{D}(A) \subset H^1(\mathbb{R}^3, \mathbb{C}^4)$ implies that*

$$\|u\|_{H^1} = ||\tilde{H}_0|u|_2 \leqq d_3|\tilde{A}u|_2 \quad \textit{for all } u \in \mathscr{D}(A). \tag{3.4}$$

Proof. Let $\tilde{H}_A$ be the restriction of $\tilde{H}_0$ to $\mathscr{D}(A)$. $\tilde{H}_A$ is a linear map from $\mathscr{D}(A)$ to L^2. We claim that $\tilde{H}_A$ is closed. Indeed, let $u_n \xrightarrow{|\cdot|_A} u$ and $\tilde{H}_A u_n \xrightarrow{|\cdot|_2} v$. Then $u \in \mathscr{D}(A)$, and since $\tilde{H}_0$ is closed, $\tilde{H}_A u_n = \tilde{H}_0 u_n \to \tilde{H}_0 u = \tilde{H}_A u$, hence the claim. Now the Closed-Graph theorem implies that $\tilde{H}_A \in \mathscr{B}(\mathscr{D}(A), L^2)$ (the Banach space of bounded linear maps), so $|\tilde{H}_0 u|_2 = |\tilde{H}_A u|_2 \leqq d_4|u|_A$ for all $u \in \mathscr{D}(A)$. This, together with (3.1) and (3.3), implies (3.4). □

By interpolation theory we have that $H^{1/2} = [L^2, H^1]_{1/2}$ (see [30, Theorem 2.4.1]). Since $\mathscr{D}(|\tilde{H}_0|^0) = L^2$ and $\|u\|_{H^1} = ||\tilde{H}_0|u|_2$, one has

$$H^{1/2} = [\mathscr{D}(|\tilde{H}_0|^0),\ \mathscr{D}(|\tilde{H}_0|)]_{1/2}$$

with equivalent norms. It then follows from [30, Theorem 1.18.10] that

$$H^{1/2} = [\mathscr{D}(|\tilde{H}_0|^0),\ \mathscr{D}(|\tilde{H}_0|)]_{1/2} = \mathscr{D}(|\tilde{H}_0|^{1/2}),$$

hence $\|u\|_{H^{1/2}}$ and $||\tilde{H}_0|^{1/2}u|_2$ are equivalent norms in $H^{1/2}$:

$$d_5\|u\|_{H^{1/2}} \leqq ||\tilde{H}_0|^{1/2}u|_2 \leqq d_6\|u\|_{H^{1/2}} \quad \text{for all } u \in H^{1/2}(\mathbb{R}^3, \mathbb{C}^4). \tag{3.5}$$

Lemma 3.3. *E embeds continuously into $H^{1/2}(\mathbb{R}^3, \mathbb{C}^4)$, hence, E embeds continuously into L^p for all $p \in [2, 3]$ and compactly into L^p_{loc} for all $p \in [1, 3)$.*

Proof. By (3.4),

$$||\tilde{H}_0|u|_2 \leqq d_3|\tilde{A}u|_2 = |(d_3\tilde{A})u|_2$$

for all $u \in \mathscr{D}(A)$. Thus $(|\tilde{H}_0|u, u)_2 \leqq (d_3\tilde{A}u, u)_2$ for all $u \in \mathscr{D}(A)$ (see [16, Proposition III 8.11]). This implies

$$||\tilde{H}_0|^{1/2}u|_2^2 = (|\tilde{H}_0|u, u)_2 \leqq (d_3\tilde{A}u, u)_2 = d_3|\tilde{A}^{1/2}u|_2^2$$

for all $u \in \mathscr{D}(A)$ (see, [16, Proposition III 8.12]). Since $\mathscr{D}(A)$ is a core of $\tilde{A}^{1/2}$ we obtain that $||\tilde{H}_0|^{1/2}u|_2^2 \leqq d_3|\tilde{A}^{1/2}u|_2^2$ for all $u \in E$. This, jointly with (3.2), shows that

$$||\tilde{H}_0|^{1/2}u|_2^2 \leqq d_3\|u\|^2 \quad \text{for all } u \in E$$

which, together with (3.5), implies that

$$\|u\|_{H^{1/2}} \leqq d_7\|u\| \quad \text{for all } u \in E$$

ending the proof. □

For further requirements, we arbitrarily fix a positive number γ with

$$q_\infty < \gamma < \mu_e. \tag{3.6}$$

Let n be the number of eigenvalues in the interval $[-\gamma, \gamma]$. We write η_j and f_j $(1 \leqq i \leqq n)$ for the eigenvalues and eigenfunctions. Setting

$$L^d := \text{span}\{f_1, \cdots, f_n\},$$

we have another orthogonal decomposition

$$L^2 = L^d \oplus L^e, \quad u = u^d + u^e.$$

Correspondingly, E has the decomposition

$$E = E^d \oplus E^e \text{ with } E^d = L^d \text{ and } E^e = E \cap L^e,$$

orthogonal with respect to both the inner products $(\cdot,\cdot)_2$ and $(\cdot,\cdot)$.

We define on E the following functional

$$\Phi(u) := \frac{1}{2}\left(\|u^+\|^2 - \|u^-\|^2\right) - \Psi(u) \text{ with } \Psi(u) := \int_{\mathbb{R}^3} R(x, u).$$

Note that by assumptions (R_1)-(R_2) and (R_0), given $p \in (2, 3]$, for any $\varepsilon > 0$, there is $C_\varepsilon > 0$ such that

$$|R_u(x, u)| \leqq \varepsilon|u| + C_\varepsilon|u|^{p-1} \tag{3.7}$$

and

$$R(x, u) \leqq \varepsilon|u|^2 + C_\varepsilon|u|^p \tag{3.8}$$

for all (x, u). Thus $\Phi \in \mathcal{C}^1(E, \mathbb{R})$ and a standard argument shows that critical points of Φ are weak solutions of $(\mathcal{P})$. Moreover, by [17], such solutions are in $W^{1,s}(\mathbb{R}^3, \mathbb{C}^4)$ for all $s \geqq 2$ (see also [7]).

In order to find critical points of Φ we will use the following abstract theorems.

3.2. Critical point theorems

The following two critical point theorems are quoted from [6] (see also [5] and [23] for earlier versions of Theorem 3.4).

Let E be a Banach space with direct sum decomposition $E = X \oplus Y$ and corresponding projections P_X, P_Y onto X, Y, respectively. For a functional $\Phi \in \mathcal{C}^1(E, \mathbb{R})$ we write $\Phi_a = \{u \in E : \Phi(u) \geqq a\}$. Recall that a sequence $(u_n) \subset E$ is said to be a $(C)_c$-sequence if $\Phi(u_n) \to c$ and $(1 + \|u_n\|)\Phi'(u_n) \to 0$. Φ is said to satisfy the $(C)_c$-condition if any $(C)_c$-sequence has a convergent subsequence.

From now on, we assume that X is separable and reflexive, and we fix a countable dense subset $\mathcal{S} \subset X^*$. For each $s \in \mathcal{S}$ there is a semi-norm on E defined by

$$p_s : E \to \mathbb{R}, \quad p_s(u) = |s(x)| + \|y\| \quad \text{for } u = x + y \in X \oplus Y.$$

We denote by $\mathcal{T}_{\mathcal{S}}$ the induced topology. Let w^* denote the weak*-topology on E^*.

Suppose:

- (Φ_0) For any $c \in \mathbb{R}$, Φ_c is $\mathcal{T}_{\mathcal{S}}$-closed, and $\Phi' : (\Phi_c, \mathcal{T}_{\mathcal{S}}) \to (E^*, w^*)$ is continuous.
- (Φ_1) For any $c > 0$, there exists $\zeta > 0$ such that $\|u\| < \zeta \|P_Y u\|$ for all $u \in \Phi_c$.
- (Φ_2) There exists $\rho > 0$ with $\nu := \inf \Phi(S_\rho Y) > 0$ where $S_\rho Y := \{u \in Y : \|u\| = \rho\}$.

The following theorem is a special case of the Theorem 3.4 of [6]; see also [23].

Theorem 3.4. *Let* (Φ_0)–(Φ_2) *be satisfied and suppose there are* $R > \rho > 0$ *and* $e \in Y$ *with* $\|e\| = 1$ *such that* $\sup \Phi(\partial Q) \leqq \nu$ *where* $Q = \{u = x + te : x \in X, t \geqq 0, \|u\| < R\}$. *Then* Φ *has a* $(C)_c$*-sequence with* $\nu \leqq c \leqq \sup \Phi(Q)$.

For our next result on multiple critical points we assume:

- (Φ_3) There is a finite-dimensional subspace $Y_0 \subset Y$ and $R > \rho$ such that we have for $E_0 := X \oplus Y_0$ and $B_0 := \{u \in E_0 : \|u\| \leqq R\} : b := \sup \Phi(E_0) < \infty$ and $\sup \Phi(E_0 \setminus B_0) < \inf \Phi(B_\rho \cap Y)$.

A special case of Theorem 4.6 of [6] is

Theorem 3.5. *If* Φ *is even, satisfies* (Φ_0), (Φ_2), (Φ_3) *and the* $(C)_c$ *condition for all* $c \in [\kappa, b]$, *then it has at least* $n := \dim Y_0$ *pairs of critical points.*

3.3. Weakly sequential continuity and linking structure

Lemma 3.6. *Let* (R_1)–(R_2), (A_0) *and* (R_0) *be satisfied. Then* Ψ *is weakly sequentially lower semicontinuous and* Φ' *is weakly sequentially continuous. Moreover, there is* $\zeta > 0$ *such that for any* $c > 0$*:*

$$\|u\| < \zeta \|u^+\| \quad \textit{for all} \quad u \in \Phi_c. \tag{3.9}$$

Proof. The first conclusion follows easily because $E \hookrightarrow H^{1/2}(\mathbb{R}^3, \mathbb{C}^4)$, so E embeds continuously into $L^q(\mathbb{R}^3, \mathbb{C}^4)$ for $q \in [2, 3]$ and compactly into $L^q_{loc}(\mathbb{R}^3, \mathbb{C}^4)$ for $q \in [1, 3)$. For showing (3.9) we adopt an argument of [14]. Arguing indirectly, assume by contradiction that for some $c > 0$, there is a sequence $u_n \in \Phi_c$ and $\|u_n\|^2 \geqq n\|u^+\|^2$. This, jointly with the form of Φ, yields that

$$\|u_n^- + u_n^0\|^2 \geqq (n-1)\|u^+\|^2 \geqq (n-1)\left(2c + \|u_n^-\|^2 + 2\int_{\mathbb{R}^3} R(x, u_n)\right),$$

or

$$\|u_n^0\|^2 \geqq (n-1)2c + (n-2)\|u_n^-\|^2 + 2(n-1)\int_{\mathbb{R}^3} R(x, u_n).$$

Since $c > 0$ and $R(x, u) \geqq 0$, it follows that $\|u_n^0\| \to \infty$, hence $\|u_n\| \to \infty$. Set $w_n = u_n/\|u_n\|$. We have $\|w_n^+\|^2 \leqq 1/n \to 0$. By

$$1 \geqq \|w_n^0\|^2 \geqq \frac{(n-1)2c}{\|u_n\|^2} + (n-2)\|w_n^-\|^2 + 2(n-1)\int_{\mathbb{R}^3} \frac{R(x, u_n)}{\|u_n\|^2},$$

we also have $\|w_n^-\|^2 \leqq 1/(n-2) \to 0$. Therefore, $w_n \to w = w^0$ in E and $\|w^0\| = 1$. By (R_2) we set

$$r(x, u) := R(x, u) - \frac{1}{2}Q(x)u \cdot u. \tag{3.10}$$

Then $|r(x, u)|/|u|^2 \to 0$ as $|u| \to \infty$ uniformly in x. In particular $|r(x, u)| \leqq c_1|u|^2$. Observe that $|u_n(x)| \to \infty$ for $w(x) \neq 0$. Therefore,

$$\begin{aligned}\int_{\mathbb{R}^3} \frac{r(x, u_n)}{\|u_n\|^2} &= \int_{w(x)\neq 0} \frac{r(x, u_n)}{|u_n|^2}|w_n|^2 + \int_{w(x)=0} \frac{r(x, u_n)}{|u_n|^2}|w_n - w|^2 \\ &\leqq 2\int_{w(x)\neq 0} \frac{|r(x, u_n)|}{|u_n|^2}|w|^2 + 2c_1|w_n - w|_2^2 \to 0.\end{aligned}$$

This implies

$$\begin{aligned}\frac{1}{2(n-1)} &\geqq \int_{\mathbb{R}^3} \frac{R(x, u_n)}{\|u_n\|^2} = \frac{1}{2}\int_{\mathbb{R}^3} Q(x)w_n \cdot w_n + \int_{\mathbb{R}^3} \frac{r(x, u_n)}{\|u_n\|^2} \\ &\geqq \frac{q_0}{2}|w_n|_2^2 + o(1),\end{aligned}$$

consequently, $w^0 = 0$, a contradiction. □

Lemma 3.7. *Under the assumptions of Lemma (3.6), there is $\rho > 0$ such that* $\nu := \inf \Phi(\partial B_\rho \cap E^+) > 0$.

Proof. Choosing $p \in (2, 3)$, it follows from (3.8),

$$\Psi(u) \leqq \varepsilon|u|_2^2 + C_\varepsilon|u|_p^p \leqq C(\varepsilon\|u\|^2 + C_\varepsilon\|u\|^p)$$

for all $u \in E$. The desired conclusion now follows easily. □

In the following, we arrange all the eigenvalues (counted in multiplicity) of A in $(0, q_0)$ by $0 < \mu_1 \leqq \mu_2 \leqq ... \leqq \mu_\ell < q_0$ and let e_j denote the corresponding eigenfunctions: $Ae_j = \mu_j e_j$ for $j = 1, ..., \ell$. Set $Y_0 := \text{span}\{e_1, ..., e_\ell\}$. Note that

$$\mu_1 |w|_2^2 \leqq \|w\|^2 \leqq \mu_\ell |w|_2^2 \quad \text{for all } w \in Y_0. \tag{3.11}$$

For any subspace F of Y_0 set $E_F = E^- \oplus E^0 \oplus F$.

Lemma 3.8. *Let (R_1), (R_2), (R_4), (A_0) and (R_0) be satisfied. Then for any subspace F of Y_0, $\sup \Phi(E_F) < \infty$, and there is $R_F > 0$ such that $\Phi(u) < \inf \Phi(B_\rho)$ for all $u \in E_F$ with $\|u\| \geqq R_F$.*

Proof. See [14]. For the reader's convenience, we repeat here with apparent modifications. Clearly, it is sufficient to check that $\Phi(u) \to -\infty$ as $u \in E_F$, $\|u\| \to \infty$. Arguing indirectly, one can assume that for some sequence $u_j \in E_F$ with $\|u_j\| \to \infty$, there is $c > 0$ such that $\Phi(u_j) \geqq -c$ for all j. Then, setting $w_j = u_j/\|u_j\|$, we have $\|w_j\| = 1$, $w_j \rightharpoonup w$, $w_j^- \rightharpoonup w^-$, $w_j^0 \to w^0$, $w_j^+ \to w^+ \in Y$ and

$$-\frac{c}{\|u_j\|^2} \leqq \frac{\Phi(u_j)}{\|u_j\|^2} = \frac{1}{2}\|w_j^+\|^2 - \frac{1}{2}\|w_j^-\|^2 - \int_{\mathbb{R}^3} \frac{R(x, u_j)}{\|u_j\|^2}. \tag{3.12}$$

Note that $w^+ \neq 0$. Indeed, if not, then it follows from (3.12) that

$$0 \leqq \frac{1}{2}\|w_j^-\|^2 + \int_{\mathbb{R}^3} \frac{R(x, u_j)}{\|u_j\|^2} \leqq \frac{1}{2}\|w_j^+\|^2 + \frac{c}{\|u_j\|^2} \to 0,$$

in particular, $\|w_j^-\| \to 0$, hence $w_j \to w = w^0$. Since $r(x, u)/|u|^2 \to 0$ uniformly in x as $|u| \to \infty$ and $|u_j(x)| \to \infty$ if $w(x) \neq 0$, we have

$$\begin{aligned}
\int_{\mathbb{R}^3} \frac{r(x, u_j)}{\|u_j\|^2} &= \int_{\mathbb{R}^3} \frac{r(x, u_j)}{|u_j|^2} |w_j|^2 \\
&\leqq 2\int_{\mathbb{R}^3} \frac{|r(x, u_j)|}{|u_j|^2} |w_j - w|^2 + 2\int_{\mathbb{R}^3} \frac{|r(x, u_j)|}{|u_j|^2} |w|^2 \\
&= o(1) + 2\int_{w(x)\neq 0} \frac{|r(x, u_j)|}{|u_j|^2} |w|^2 = o(1)
\end{aligned}$$

and

$$\frac{1}{2}\int_{\mathbb{R}^3} \frac{Q(x)u_j \cdot u_j}{\|u_j\|^2} = \frac{1}{2}\int_{\mathbb{R}^3} \frac{Q(x)u_j \cdot u_j}{|u_j|^2} |w_j|^2 \geqq \frac{q_0}{2}|w_j|_2^2.$$

It then follows from $\int_{\mathbb{R}^3} \frac{R(x,u_j)}{\|u_j\|^2} \to 0$ that $|w_j|_2 \to 0$, consequently $1 = \|w_j\| \to 0$, a contradiction. Now since

$$\begin{aligned}
\|w^+\|^2 - \|w^-\|^2 - \int_{\mathbb{R}^3} Q(x) w \cdot w &\leqq \|w^+\|^2 - \|w^-\|^2 - q_0 |w|_2^2 \\
&\leqq -\Big((q_0 - \mu_\ell)|w^+|_2^2 + \|w^-\|^2 + q_0 |w^0|_2^2\Big) < 0,
\end{aligned}$$

there is $d > 0$ such that

$$\|w^+\|^2 - \|w^-\|^2 - \int_{B_d} Q(x)w \cdot w \; < \; 0. \tag{3.13}$$

Since $|r(x,u)| \leqq c_1|u|^2$ it follows from the fact $|w_j - w|_{L^2(B_d)} \to 0$ that

$$\lim_{j\to\infty} \int_{B_d} \frac{r(x,u_j)}{\|u_j\|^2} = \lim_{j\to\infty} \int_{B_d} \frac{r(x,u_j)|w_j|^2}{|u_j|^2} = 0.$$

Thus (3.12) and (3.13) imply that

$$\begin{aligned} 0 &\leqq \lim_{j\to\infty} \left(\frac{1}{2}\|w_j^+\|^2 - \frac{1}{2}\|w_j^-\|^2 - \int_{B_d} \frac{R(x,u_j)}{\|u_j\|^2} \right) \\ &\leqq \frac{1}{2} \left(\|w^+\|^2 - \|w^-\|^2 - \int_{B_d} Q(x)w \cdot w \right) \; < \; 0, \end{aligned}$$

a contradiction. □

As a special case we have

Lemma 3.9. *Under the conditions of Lemma (3.8), letting $e \in Y_0$ with $\|e\| = 1$, there is $r_0 > 0$ such that* $\sup \Phi(\partial Q) = 0$ *where* $Q := \{u = u^- + u^0 + se : u^- + u^0 \in E^- \oplus E^0, s \geqq 0, \|u\| \leqq r_0\}$.

3.4. The Cerami condition

We now discuss the Cerami condition. We adapt an argument of [14] (see also [13,15]). Observe that by (R_0) and (3.10), given $\gamma_0 \in (q_\infty, \ \gamma)$, there exists $t_0 > 0$ large so that

$$\sup_u \frac{|R_u(x,u)|}{|u|} < \gamma_0 \quad \text{if } |x| \geqq t_0. \tag{3.14}$$

Set

$$I_0 := \{x \in \mathbb{R}^3 : \, |x| < t_0\} \;\; \text{and} \;\; I_0^c := \mathbb{R}^3 \setminus I_0.$$

Lemma 3.10. *Let (R_1)–(R_4), (R_0) and (A_0) be satisfied. Then any $(C)_c$-sequence is bounded.*

Proof. Let $(u_j) \subset E$ be such that

$$\Phi(u_j) \to c \quad \text{and} \quad (1 + \|u_j\|)\Phi'(u_j) \to 0. \tag{3.15}$$

Then

$$C_0 \geqq \Phi(u_j) - \frac{1}{2}\Phi'(u_j)u_j = \int_{\mathbb{R}^3} \tilde{R}(x,u_j)\,. \tag{3.16}$$

Arguing indirectly we assume that, up to a subsequence, $\|u_j\| \to \infty$ and set $v_j = u_j/\|u_j\|$. Then $\|v_j\| = 1$, $|v_j|_s \leqq C_s\|v_j\| = C_s$ for all $s \in [2, 3]$, and passing to a subsequence if necessary, $v_j \rightharpoonup v$ in E, $v_j \to v$ in L^s_{loc} for all $s \in [1,3)$, $v_j(x) \to v(x)$ for almost every $x \in \mathbb{R}^3$. Since, by (R_2), $|R_u(x,u)| \leqq c_1|u|$ and $|u_j(x)| \to \infty$ if $v(x) \neq 0$, it is easy to see that

$$\int_{\mathbb{R}^3} \frac{R_u(x, u_j(x))v_j\varphi(x)}{|u_j(x)|} \to \int_{\mathbb{R}^3} Q(x)v\varphi$$

for all $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^3, \mathbb{C}^4)$, hence

$$Av = Q(x)v. \tag{3.17}$$

We claim that $v \neq 0$. Arguing by contradiction, one can assume $v = 0$. Then $v_j^d \to 0$ in E and $v_j \to 0$ in L^s_{loc}. Observe that

$$\frac{\Phi'(u_j)(u_j^{e+} - u_j^{e-})}{\|u_j\|^2} = \|v_j^e\|^2 - \int_{\mathbb{R}^3} \frac{R_u(x,u_j)}{|u_j|}(v_j^{e+} - v_j^{e-})|v_j|. \tag{3.18}$$

It follows from (3.18) and (3.14) that

$$\begin{aligned}
\|v_j^e\|^2 &= \int_{I_0} \frac{R_u(x,u_j)}{|u_j|}(v_j^{e+} - v_j^{e-})|v_j| \\
&\quad + \int_{I_0^c} \frac{R_u(x,u_j)}{|u_j|}(v_j^{e+} - v_j^{e-})|v_j| + o(1) \\
&\leqq c_1 \int_{I_0} |v_j||v_j^{e+} - v_j^{e-}| + \gamma_0 \int_{I^c} |v_j||v_j^{e+} - v_j^{e-}| + o(1) \\
&\leqq o(1) + \gamma_0 |v_j^e|_2^2 \\
&\leqq o(1) + \frac{\gamma_0}{\gamma}\|v_j^e\|^2
\end{aligned}$$

hence $\left(1 - \frac{\gamma_0}{\gamma}\right)\|v_j^e\|^2 \to 0$, which implies that $1 = \|v_j\|^2 = \|v_j^d\|^2 + \|v_j^e\|^2 \to 0$, a contradiction.

Therefore, $v \neq 0$. This is a contradiction if (i) of (R_4) is satisfied.

Assume (ii) of (R_4) is satisfied. Motivated by [15], set $\Omega_j(r,\infty) := \{x \in \mathbb{R}^3 : |u_j(x)| \geqq r\}$ for $r \geqq 0$. By assumption $\tilde{R}(x,u) \geqq \delta_0$ if $|u| \geqq \nu_0$, hence, $|\Omega_j(\nu_0,\infty)| \leqq C_0/\delta_0$ by (3.16). Note that v is a solution of (3.17). Set $\Omega := \{x : v(x) \neq 0\}$. By the weak unique continuation property for Dirac operators one has $|\Omega| = \infty$ (cf. [8,10]). There exist $\varepsilon > 0$ and $\omega \subset \Omega$ such that $|v(x)| \geqq 2\varepsilon$ for $x \in \omega$ and $2C_0/\delta_0 \leqq |\omega| < \infty$. By Egoroff's theorem we can find a set $\omega' \subset \omega$ with $|\omega'| > C_0/\delta_0$ such that $v_j \to v$ uniformly on ω'. So for almost all j, $|v_j(x)| \geqq \varepsilon$ and $|u_j(x)| \geqq \nu_0$ in ω'. Then

$$\frac{C_0}{\nu_0} < |\omega'| \leqq |\Omega_j(\nu_0,\infty)| \leqq \frac{C_0}{\nu_0},$$

a contradiction. The proof hereby is completed. □

In the following lemma we discuss further the $(C)_c$-sequence $(u_j) \subset E$. By Lemma 3.9 it is bounded, hence, we may assume without loss of generality that $u_j \rightharpoonup u$ in E, $u_j \to u$ in L^q_{loc} for $q \in [1, 3)$ and $u_j(x) \to u(x)$ almost everywhere in x. Plainly u is a critical point of Φ.

Choose $p \in (2, 3)$ such that $|R_u(t, u)| \leqq |u| + C_1|u|^{p-1}$ for all (x, u), and let q stand for either 2 or p. Set $I_d := \{x \in \mathbb{R}^3 : |x| \leqq d\}$ for $d > 0$.

Lemma 3.11. *Let* $2 \leqq q < 3$. *Under the conditions of Lemma* (3.10), *along a subsequence, for any* $\varepsilon > 0$, *there exists* $r_\varepsilon > 0$ *such that*

$$\limsup_{n\to\infty} \int_{I_n \setminus I_r} |u_{j_n}|^q \leqq \varepsilon \tag{3.19}$$

for all $r \geqq r_\varepsilon$.

Proof. Note that, for each $n \in \mathbb{N}$, $\int_{I_n} |u_j|^q \to \int_{I_n} |u|^q$ as $j \to \infty$. There exists $i_n \in \mathbb{N}$ such that

$$\int_{I_n} \left(|u_j|^q - |u|^q\right) < \frac{1}{n} \quad \text{for all } j = i_n + m,\ m = 1, 2, 3, \ldots.$$

Without loss of generality, we can assume $i_{n+1} \geqq i_n$. In particular, for $j_n = i_n + n$ we have

$$\int_{I_n} \left(|u_{j_n}|^q - |u|^q\right) < \frac{1}{n}.$$

Observe that there is r_ε satisfying

$$\int_{\mathbb{R}^3 \setminus I_r} |u|^q < \varepsilon \tag{3.20}$$

for all $r \geqq r_\varepsilon$. Since

$$\begin{aligned} \int_{I_n \setminus I_r} |u_{j_n}|^q &= \int_{I_n} \left(|u_{j_n}|^q - |u|^q\right) + \int_{I_n \setminus I_r} |u|^q + \int_{I_r} \left(|u|^q - |u_{j_n}|^q\right) \\ &\leqq \frac{1}{n} + \int_{\mathbb{R}^3 \setminus I_r} |u|^q + \int_{I_r} \left(|u|^q - |u_{j_n}|^q\right), \end{aligned}$$

the lemma now follows easily. □

Let $\eta : [0, \infty) \to [0, 1]$ be a smooth function satisfying $\eta(s) = 1$ if $s \leqq 1$, $\eta(s) = 0$ if $s \geqq 2$. Define $\tilde{u}_n(x) = \eta(2|x|/n)u(x)$ and set $h_n := u - \tilde{u}_n$. Since u solves $(\mathcal{P})$, we have, by definition, that $h_n \in H^1$ and

$$\|h_n\| \to 0 \ \text{ and } \ |h_n|_p \to 0 \ \text{ as } n \to \infty \tag{3.21}$$

for $p \in [2, 3]$.

Lemma 3.12. *Under the conditions of Lemma* (3.10), *we have*

$$\lim_{n\to\infty}\left|\int_{\mathbb{R}^3}\big(R_u(t,u_{j_n})-R_u(t,u_{j_n}-\tilde u_n)-R_u(t,\tilde u_n)\big)\,\varphi\right|=0$$

uniformly in $\varphi\in E$ *with* $\|\varphi\|\leqq 1$.

Proof. Note that (3.19), (3.21) and the compactness of Sobolev embeddings imply that, for any $r>0$,

$$\lim_{n\to\infty}\left|\int_{I_r}\big(R_u(t,u_{j_n})-R_u(t,u_{j_n}-\tilde u_n)-R_u(t,\tilde u_n)\big)\,\varphi\right|=0$$

uniformly in $\|\varphi\|\leqq 1$. For any $\varepsilon>0$ let $r_\varepsilon>0$ so large that (3.19) and (3.20) hold. Then

$$\limsup_{n\to\infty}\int_{I_n\setminus I_r}|\tilde u_n|^q\;\leqq\;\int_{\mathbb{R}^3\setminus I_r}|u|^q\leqq\varepsilon$$

for all $r\geqq r_\varepsilon$. Using (3.19) for $q=2,\,p$ we have

$$\begin{aligned}
&\limsup_{n\to\infty}\left|\int_{\mathbb{R}^3}\big(R_u(x,u_{j_n})-R_u(x,u_{j_n}-\tilde u_n)-R_u(x,\tilde u_n)\big)\,\varphi\right|\\
&\quad=\limsup_{n\to\infty}\left|\int_{I_n\setminus I_r}\big(R_u(x,u_{j_n})-R_u(x,u_{j_n}-\tilde u_n)-R_u(x,\tilde u_n)\big)\,\varphi\right|\\
&\quad\leqq c_1\limsup_{n\to\infty}\int_{I_n\setminus I_r}\big(|u_{j_n}|+|\tilde u_n|\big)\,|\varphi|\\
&\qquad+c_2\limsup_{n\to\infty}\int_{I_n\setminus I_r}\Big(|u_{j_n}|^{p-1}+|\tilde u_n|^{p-1}\Big)\,|\varphi|\\
&\quad\leqq c_1\limsup_{n\to\infty}\big(|u_{j_n}|_{L^2(I_n\setminus I_r)}+|\tilde u_n|_{L^2(I_n\setminus I_r)}\big)\,|\varphi|_2\\
&\qquad+c_2\limsup_{n\to\infty}\Big(|u_{j_n}|^{p-1}_{L^p(I_n\setminus I_r)}+|\tilde u_n|^{p-1}_{L^p(I_n\setminus I_r)}\Big)\,|\varphi|_p\\
&\quad\leqq c_3\varepsilon^{1/2}+c_4\varepsilon^{(p-1)/p},
\end{aligned}$$

which implies the conclusion as required. □

Lemma 3.13. *Under the conditions of Lemma* (3.10), *one has along a subsequence:*

1) $\Phi(u_{j_n}-\tilde u_n)\to c-\Phi(u)$;
2) $\Phi'(u_{j_n}-\tilde u_n)\to 0$.

Proof. One has

$$\begin{aligned}
\Phi(u_{j_n}-\tilde u_n)&=\Phi(u_{j_n})-\Phi(\tilde u_n)\\
&\qquad+\int_{\mathbb{R}^3}\big(R(x,u_{j_n})-R(x,u_{j_n}-\tilde u_n)-R(x,\tilde u_n)\big).
\end{aligned}$$

Using (3.20), one can easily check that

$$\int_{\mathbb{R}^3} \big(R(x,u_{j_n}) - R(x,u_{j_n}-\tilde{u}_n) - R(x,\tilde{u}_n)\big) \to 0.$$

This, together with the facts $\Phi(u_{j_n}) \to c$ and $\Phi(\tilde{u}_n) \to \Phi(u)$, gives 1).

To verify 2), observe that, for any $\varphi \in E$,

$$\begin{aligned}\Phi'(u_{j_n}-\tilde{u}_n)\varphi &= \Phi'(u_{j_n})\varphi - \Phi'(\tilde{u}_n)\varphi \\ &\quad + \int_{\mathbb{R}^3} \big(R_u(x,u_{j_n}) - R_u(x,u_{j_n}-\tilde{u}_n) - R_u(x,\tilde{u}_n)\big)\,\varphi.\end{aligned}$$

By Lemma 3.12 we get

$$\lim_{n\to\infty}\int_{\mathbb{R}^3} \big(R_u(x,u_{j_n}) - R_u(x,u_{j_n}-\tilde{u}_n) - R_u(x,\tilde{u}_n)\big)\,\varphi = 0$$

uniformly in $\|\varphi\| \leqq 1$, proving 2). □

Lemma 3.14. *Under the conditions of Lemma (3.10), Φ satisfies the $(C)_c$ condition.*

Proof. In the following we will utilize the decomposition $E = E^d \oplus E^e$. Recall that $\dim(E^d) < \infty$. Write

$$y_n := u_{j_n} - \tilde{u}_n = y_n^d + y_n^e.$$

Then $y_n^d = (u_{j_n}^d - u^d) + (u^d - \tilde{u}_n^d) \to 0$ and, by Lemma 3.13, $\Phi(y_n) \to c - \Phi(u)$, $\Phi'(y_n) \to 0$. Set $\bar{y}_n^e = y_n^{e+} - y^{e-}$. Observe that

$$o(1) = \Phi'(y_n)\bar{y}_n^e = \|y_n^e\|^2 - \int_{\mathbb{R}^3} R_u(x,y_n)\bar{y}_n^e. \tag{3.22}$$

It follows from (3.22) that

$$\begin{aligned}\|y_n^e\|^2 &\leqq o(1) + \int_{I_0} \frac{|R_u(x,y_n)|}{|y_n|}|y_n||\bar{y}_n^e| + \int_{I_0^c} \frac{|R_u(x,y_n)|}{|y_n|}|y_n||\bar{y}_n^e| \\ &\leqq o(1) + c_1\int_{I_0} |y_n||\bar{y}_n^e| + \gamma_0\int_{I_0^c} |y_n||\bar{y}_n^e| \\ &\leqq o(1) + \gamma_0|y_n^e|_2^2 \leqq o(1) + \frac{\gamma_0}{\gamma}\|y_n^e\|^2,\end{aligned}$$

hence $(1-\gamma_0/\gamma)\|y_n\| \leqq o(1)$ that is, $y_n \to 0$, finishing the proof. □

3.5. Proof of Theorem 3.1

In order to prove Theorem 3.1 we set $X = E^- \oplus E^0$ and $Y = E^+$ with $u = x + y$, $x = u^- + u^0$, $y = u^+$ for $u \in E$. Then X is separable and reflexive and so is X^*. We may assume $\mathcal{S}$ is countable and dense in X^*. Therefore, $\mathcal{T_S}$ is metrizable so its convergence is equivalent to sequential convergence.

Proof of Theorem 3.1 (Existence). Observe that if $c > 0$ and $u_n \in \Phi_c$ with $u_n = x_n + y_n \to u = x + y$ in $\mathcal{T_S}$ then $y_n \to y$ in norm. (3.9) then implies $\|u_n\|$ is bounded, consequently, $u_n \rightharpoonup u$. Thus by Lemma 3.6

$$c \leqq \lim_{n\to\infty} \Phi(u_n) \leqq \frac{1}{2}\|y\|^2 - \frac{1}{2}\|u^-\|^2 - \Psi(u) = \Phi(u)$$

which proves that Φ_c is $\mathcal{T_S}$ closed. Lemma 3.6 implies also that $\Phi'(u_n)v \to \Phi'(u)v$ for all $v \in E$ that is, $\Phi' : (\Phi_c, \mathcal{T_S}) \to (E^*, w^*)$ is continuous. Thus Φ verifies (Φ_0). Remark that (3.4) is nothing but the condition (Φ_1). Lemma 3.7 implies (Φ_2). Lemma 3.9 shows that Φ possesses the linking structure of Theorem 3.4. Finally, Φ satisfies the $(C)_c$-condition by virtue of Lemma 3.14. Therefore, Φ has at least one critical point u with $\Phi(u) \geqq \nu > 0$.

(Multiplicity). Assume, moreover, that $R(x, u)$ is even in u. Then Φ is even. Lemma 3.8 says that Φ satisfies (Φ_3) with $\dim Y = \ell$. Therefore, Φ has at least ℓ pairs of nontrivial critical points by Theorem 3.5. □

4. Proofs of Theorems 2.1 and 2.2

Proof of Theorem 2.1. Assume (M_1) holds. Then one has $\mu_e^- = -a$ and $\mu_e^+ = a$. Now Theorem 3.1 applies. □

Remark 4.1. Similarly, one can get existence and multiplicity results of solutions to $(\mathcal{P})$ if the Coulomb potential is replaced by the electrostatic potential $M(x) = \gamma\phi_{el} I_4$, where γ is a positive constant and ϕ_{el} is a real function satisfying, for example,

$(\hat{M}_1)$ $\phi_{el} \in L^3(\mathbb{R}^3) \cap L^{3/2}(\mathbb{R}^3)$, $\phi_{el}(x) \leqq 0$

(see [29]). Another typical example is

$$H = H_0 + \frac{\gamma}{1 + |x|^2}$$

which has finitely many eigenvalues in $(-mc^2, mc^2)$ if $\gamma < 1/8m$, and infinitely many eigenvalues for $\gamma > 1/8m$.

For proving Theorem 2.2, we recall that the operator A is selfadjoint in L^2 ([12]). Additionally, we have the following result:

Lemma 4.2. *Assume that* (M_2) *is satisfied. Then* $\mathscr{D}(A) \subset H^1$ *and*

$$\sigma_e(A) \subset \mathbb{R} \setminus (-(a + b_{\max}),\ (a + b_{\max})),$$

that is, $\mu_e^- \leqq -(a + b_{\max})$ *and* $\mu_e^+ \geqq (a + b_{\max})$.

Proof. Note that $|H_0 u|_2^2 = \int_{\mathbb{R}^3}(|\nabla u|^2 + a^2|u|^2)$, hence the norms $\|u\|_{H^1}$, $||\tilde{H}_0|u|_2$ and $|H_0 u|_2$ are equivalent on H^1. Since $M \in L^\infty$,

$$|H_0 u|_2 = |Au - Mu|_2 \leqq |Au|_2 + |M|_\infty |u|_2 \leqq (1 + |M|_\infty)|u|_A, \qquad (4.1)$$

one sees $\|u\|_{H^1} \leqq c_1|u|_A$ for all $u \in \mathscr{D}(A)$, hence $\mathscr{D}(A) \subset H^1$.

Let $b > 0$ be such that $|\Omega_b| < \infty$. Set

$$(\beta M(x) - b)^+ := \begin{cases} \beta M(x) - b & \text{if } \beta M(x) - b \geqq 0 \\ 0 & \text{if } \beta M(x) - b < 0 \end{cases}$$

and $(\beta M(x) - b)^- := (\beta M(x) - b) - (\beta M(x) - b)^+$. We have $A = A_1 + \beta(\beta M(x) - b)^-$ where

$$A_1 = -i \sum_{k=1}^{3} \alpha_k \partial_k + (a + b)\beta + \beta(\beta M(x) - b)^+.$$

Since $\beta^2 = I$ and $\beta\alpha_j = -\alpha_j\beta$, we have, for $u \in \mathscr{D}(A)$,

$$\begin{aligned}
&(A_1 u, A_1 u)_2 \\
&\quad = \left|\left(-i \sum \alpha_k \partial_k + \beta(\beta M - b)^+ + (a + b)\beta\right) u\right|_2^2 \\
&\quad = \left|\left(-i \sum \alpha_k \partial_k + \beta(\beta M - b)^+\right) u\right|_2^2 + (a + b)^2|u|_2^2 \\
&\qquad + \left(-i \sum \alpha_k \partial_k u,\ (a + b)\beta u\right)_2 + \left((a + b)\beta u,\ -i \sum \alpha_k \partial_k u\right)_2 \\
&\qquad + \left(\beta(\beta M - b)^+ u,\ (a + b)\beta u\right)_2 + \left((a + b)\beta u,\ \beta(\beta M - b)^+ u\right)_2 \\
&\quad = \left|\left(-i \sum \alpha_k \partial_k + \beta(\beta M - b)^+\right) u\right|_2^2 + (a + b)^2|u|_2^2 \\
&\qquad + 2(a + b)\left((\beta M - b)^+ u,\ u\right)_2 \\
&\quad \geqq\ (a + b)^2|u|_2^2 .
\end{aligned}$$

Thus $\sigma(A_1) \subset \mathbb{R} \setminus (-(a + b), (a + b))$.

We claim that $\sigma_e(A) \cap (-(a + b), (a + b)) = \emptyset$. Assume by contradiction that there is $\mu \in \sigma_e(A)$ with $|\mu| < a + b$. Let $u_n \in \mathscr{D}(A)$ with $|u_n|_2 = 1$, $u_n \rightharpoonup 0$ in L^2 and $|(A - \mu)u_n|_2 \to 0$. It follows from (4.1) that $\|u_n\|_{H^1} \leqq c_1|u_n|_A = c_1(|Au_n|_2^2 + |u_n|_2^2)^{1/2} \leqq c_2(|(A - \mu)u_n|_2^2 + \mu^2 + 1)^{1/2} \leqq c_3$. Thus $|\beta(\beta M - b)^- u_n|_2 \to 0$. We get

$$\begin{aligned}
o(1) = |(A - \mu)u_n|_2 &= |A_1 u_n - \mu u_n + \beta(\beta M - b)^- u_n|_2 \\
&\geqq |A_1 u_n|_2 - |\mu| - o(1) \\
&\geqq (a + b) - |\mu| - o(1)
\end{aligned}$$

which implies that $0 < (a + b) - |\mu| \leqq 0$, a contradiction.

Since the claim holds true for any $b > 0$ with $|\Omega^b| < \infty$, one sees that $\sigma_e(A) \subset \mathbb{R} \setminus (-(a + b_{\max}),\ (a + b_{\max}))$. $\square$

Remark 4.3. From the proof of Lemma 4.2, one sees that if (M_2) is replaced by the stronger one

$(\hat{M}_2)$ $|\Omega_b| < \infty$ for any $b > 0$,

then $\sigma(A) = \sigma_d(A)$, that is, the Dirac operator A has only eigenvalues of finite multiplicity.

Proof of Theorem 2.2. Lemma 4.2 implies (A_0), hence Theorem 3.1 applies and yields the desired conclusions. □

5. Semi-classical solutions

Observe that by dividing by ε^2 and setting $\lambda = 1/\varepsilon^2$ in the equation $(\mathcal{P}_\varepsilon)$, we have the following equivalent problem:

$$-i\sum_{k=1}^{3}\alpha_k\partial_k u + \lambda(a + V(x))\beta u = \lambda R_u(x, u). \qquad (\mathcal{P}_\lambda)$$

We are led to study the existence and multiplicity of solutions of $(\mathcal{P}_\lambda)$ for $\lambda \to \infty$. For distinguishability we will write $A_\lambda = -i\sum_{k=1}^{3}\alpha_k\partial_k + \lambda(a + V)\beta$ instead of A, $\|\cdot\|_\lambda$ instead of $\|\cdot\|$, $E_\lambda^\pm$ instead of $E^\pm$, etc. Note that the assumption (V) implies that the matrix $\lambda\beta V$ satisfies (M_2). Therefore $\mathscr{D}(A_\lambda) \subset H^1$ and we have the following result by Lemma 4.2.

Lemma 5.1. *Assume that* (V) *holds. Then*

$$\sigma_e(A_\lambda) \subset \mathbb{R} \setminus (-\lambda(a + b_{\max}),\ \lambda(a + b_{\max}))\,.$$

Next, we prove

Lemma 5.2. *Assume that* (V) *holds. Then for any* $m \in \mathbb{N}$ *there is* $\Lambda_m > 0$ *such that* A_λ *has at least* m *eigenvalues (counted with multiplicity) lying in* $(0, \lambda q_0)$ *for each* $\lambda \in [\Lambda_m, \infty)$.

We will establish this lemma constructively. Observe that since $\sigma_e(A_\lambda) \subset \mathbb{R} \setminus (-\lambda(a + b_{\max}),\ \lambda(a + b_{\max}))$, it is sufficient to show that there are m linearly independent elements $\varphi \in E_\lambda^+$ with $|\varphi|_2 = 1$ and $\|\varphi\|_\lambda < \lambda q_0$. By assumption, $q_0 > a$. Given

$$0 < \theta < \min\left\{\frac{q_0 - a}{2q_0},\ \frac{1}{2}\right\}$$

set

$$D_\theta := \{x \in \mathbb{R}^3 : \frac{\theta q_0}{2} \leqq V(x) \leqq \theta q_0\} \quad \text{and} \quad \Omega_\theta := \operatorname{int} D_\theta\,.$$

For each $m \in \mathbb{N}$, we choose m real functions $\omega^j \in \mathcal{C}_0^\infty(\Omega_\theta, \mathbb{R})$, $j = 1, \ldots, m$, satisfying

$$|\omega^j|_2 = 1 \quad \text{and} \quad \operatorname{supp}\omega^j \cap \operatorname{supp}\omega^k = \emptyset \text{ if } j \neq k.$$

Set

$$\varphi_j = (\omega^j, 0, 0, 0) \in \mathcal{C}_0^\infty(\Omega_\theta, \mathbb{C}^4) \quad \text{for } j = 1, \ldots, m.$$

Clearly $\varphi_1, \ldots, \varphi_m$ are linearly independent,

$$\begin{aligned} A_\lambda \varphi_j &= (0, 0, -i\partial_3\omega^j, -i\partial_1\omega^j + \partial_2\omega^j) + (\lambda(a+V)\omega^j, 0, 0, 0) \\ &= (\lambda(a+V)\omega^j, 0, -i\partial_3\omega^j, -i\partial_1\omega^j + \partial_2\omega^j), \end{aligned}$$

$$\left(-i \sum_{k=1}^{3} \alpha_k \partial_k \varphi_j,\ \varphi_j \right)_2 = 0,$$

and

$$\begin{aligned} \lambda \left(a + \frac{\theta q_0}{2} \right) &\leqq (A_\lambda \varphi_j, \varphi_j)_2 = \lambda \int_{\mathbb{R}^3} (a+V)|\omega^j|^2 \leqq \lambda \left(a + \theta q_0 \right), \\ |A_\lambda \varphi_j|_2^2 &= (A_\lambda^2 \varphi_j, \varphi_j)_2 = |\nabla \omega^j|_2^2 + \lambda^2 \int_{\mathbb{R}^3} (a+V)^2 |\omega^j|^2 \end{aligned}$$

so

$$|\nabla \omega^j|_2^2 + \lambda^2 \left(a + \frac{\theta q_0}{2} \right)^2 \leqq |A_\lambda \varphi_j|_2^2 \leqq |\nabla \omega^j|_2^2 + \lambda^2 \left(a + \theta q_0 \right)^2.$$

For each $\lambda > 0$ we have the representation $\varphi_j = \varphi_{\lambda j}^- + \varphi_{\lambda j}^0 + \varphi_{\lambda j}^+$ $(j = 1, \ldots, m)$. Set

$$Z_m := \operatorname{span}\{\varphi_1, \ldots, \varphi_m\}, \quad Z_{\lambda m} := \operatorname{span}\{\varphi_{\lambda 1}^+, \ldots, \varphi_{\lambda m}^+\}.$$

Lemma 5.3. *For each $\lambda > 0$ and $m \in \mathbb{N}$, $\dim(Z_{\lambda m}) = m$.*

Proof. See [7, Lemma 4.7]. □

In the following, we set

$$\alpha := \max \left\{ |\nabla \omega^j|_2^2 :\ j = 1, \ldots, m \right\}$$

which depends on m and the choice of ω^j, but is independent of λ. Denote

$$\hat{u} := \sum_{j=1}^{m} c_j \varphi_j \in Z_m \quad \text{for} \quad u = \sum_{j=1}^{m} c_j \varphi_{\lambda j}^+ \in Z_{\lambda m}.$$

It is clear that

$$\hat{u}^+ = u \quad \text{and} \quad |\hat{u}|_2^2 = \sum_{j=1}^{m} c_j^2.$$

Lemma 5.4. *We have:*

(i) *for each* $\lambda \geqq 1$, $\zeta|\hat{u}|_2 \leqq |u|_2 \leqq |\hat{u}|_2$ *for all* $u \in Z_{\lambda m}$, *where* $\zeta > 0$ *is independent of* λ;

(ii) *for each* $\lambda \geqq 1$ *and all* $u \in Z_{\lambda m}$,

$$\lambda\left(a + \frac{\theta q_0}{2}\right)|\hat{u}|_2^2 \leqq \|u\|_\lambda^2 \leqq \lambda\left(\frac{\alpha}{\lambda^2} + (a+\theta q_0)^2\right)^{1/2}|\hat{u}|_2|u|_2\,;$$

(iii) *there is* $\Lambda_m > 0$ *such that for each* $\lambda \geqq \Lambda_m$ *and all* $u \in Z_{\lambda m}$,

$$\|u\|_\lambda^2 - \lambda q_0|u|_2^2 \leqq -\lambda q_0 \xi_\theta |\hat{u}|_2|u|_2$$

where

$$\xi_\theta = \frac{2a\,(q_0 - a - 2\theta q_0) + (1-2\theta)\theta q_0^2}{4q_0(a+\theta q_0)}$$

Proof. Let $u \in Z_{\lambda m}$. Observe that

$$\begin{aligned}
\|u\|_\lambda^2 - \|\hat{u}^-\|_\lambda^2 &= (A_\lambda \hat{u}, \hat{u})_2 = \sum_{j=1}^m |c_j|^2 (A_\lambda \varphi_j, \varphi_j)_2 \\
&\geqq \lambda\left(a + \frac{\theta q_0}{2}\right)|\hat{u}|_2^2, \\
|A_\lambda \hat{u}|_2^2 &= \sum_{j=1}^m |c_j|^2 |A_\lambda \varphi_j|_2^2 \leqq \sum_{j=1}^m |c_j|^2 \left(|\nabla\omega^j|_2^2 + \lambda^2 (a+\theta q_0)^2\right) \\
&\leqq \left(\alpha + \lambda^2 (a+\theta q_0)^2\right)|\hat{u}|_2^2, \\
\|u\|_\lambda^2 &= (A_\lambda \hat{u}, u)_2 \leqq |A_\lambda \hat{u}|_2 |u|_2 \leqq \left(\alpha + \lambda^2(a+\theta q_0)^2\right)^{1/2}|\hat{u}|_2|u|_2.
\end{aligned}$$

Hence

$$\lambda\left(a + \frac{\theta q_0}{2}\right)|\hat{u}|_2^2 \leqq \|u\|_\lambda^2 \leqq \lambda\left(\frac{\alpha}{\lambda^2} + (a+\theta q_0)^2\right)^{1/2}|\hat{u}|_2|u|_2 \tag{5.1}$$

which proves ii).

Obviously, $|u|_2 \leqq |\hat{u}|_2$. In order to check the first inequality of (i), we note that by (5.1)

$$|u|_2 \geqq f(\lambda)|\hat{u}|_2 \quad \text{where} \quad f(\lambda) := \frac{\lambda(2a+\theta q_0)}{2\left(\lambda^2(a+\theta q_0)^2 + \alpha\right)^{1/2}}. \tag{5.2}$$

It is clear that $f(\lambda)$ is strictly increasing and

$$\lim_{\lambda\to\infty} f(\lambda) = \frac{2a+\theta q_0}{2(a+\theta q_0)}\,.$$

Hence

$$\frac{2a+\theta q_0}{2\left(\alpha+(a+\theta q_0)^2\right)^{1/2}} \leqq f(\lambda) < \frac{2a+\theta q_0}{2(a+\theta q_0)} \quad \text{for all } \lambda \geqq 1$$

and (i) follows.

Using (5.1) and (5.2) one sees

$$\begin{aligned}
&\|u\|_\lambda^2 - \lambda q_0 |u|_2^2 \\
&\quad = (A_\lambda \hat{u}, u)_2 - \lambda q_0 |u|_2^2 \\
&\quad \leqq \left(|A_\lambda \hat{u}|_2 - \lambda q_0 |u|_2\right) |u|_2 \\
&\quad \leqq \left(\left(\alpha + \lambda^2 (a+\theta q_0)^2\right)^{1/2} - \lambda q_0 \frac{\lambda(2a+\theta q_0)}{2\left(\lambda^2 (a+\theta q_0)^2 + \alpha\right)^{1/2}} \right) |\hat{u}|_2 |u|_2 \\
&\quad = -\lambda q_0 h(\lambda) |\hat{u}|_2 |u|_2
\end{aligned} \tag{5.3}$$

where

$$h(\lambda) = \frac{2a+\theta q_0}{2\left(\frac{\alpha}{\lambda^2} + (a+\theta q_0)^2\right)^{1/2}} - \frac{\left(\frac{\alpha}{\lambda^2} + (a+\theta q_0)^2\right)^{1/2}}{q_0}.$$

Note that

$$\begin{aligned}
\lim_{\lambda\to\infty} h(\lambda) &= \frac{2a+\theta q_0}{2(a+\theta q_0)} - \frac{a+\theta q_0}{q_0} \\
&= \frac{2a\,(q_0 - a - 2\theta q_0) + (1-2\theta)\theta q_0^2}{2q_0(a+\theta q_0)} \\
&= 2\xi_\theta.
\end{aligned} \tag{5.4}$$

Now (iii) follows from (5.3) and (5.4). □

Proof of Lemma 5.2. From (iii) of Lemma 5.4 we obtain for $\lambda \geqq \Lambda_m$

$$\begin{aligned}
\mu_m\left(A_\lambda|_{L_\lambda^+}\right) := &\inf_{\substack{F\subset E_\lambda^+ \\ \dim(F)=m}} \sup_{\substack{\varphi\in E_\lambda^-\oplus F \\ |\varphi|_2=1}} (A_\lambda\varphi,\varphi)_2 \\
&\leqq \sup_{\substack{u\in Z_{\lambda m} \\ |u|_2=1}} (A_\lambda u, u)_2 \\
&\leqq \sup_{\substack{u\in Z_{\lambda m} \\ |u|_2=1}} \lambda q_0 \left(1 - \xi_\theta |\hat{u}|_2\right) \\
&< \lambda q_0
\end{aligned}$$

as required. □

Proof of Theorem 2.3. By Lemma 5.1, we see that (A_0) is satisfied, and we have, additionally, $\mu_e \geqq \lambda(a + b_{\max})$ which, jointly with $(\hat{R}_5)$, implies $\lambda q_\infty < \mu_e$, that is, (R_0) holds. By Lemma 5.2, for any $m \in \mathbb{N}$, there is $\Lambda_m > 0$ such that the number $\#[(0, \lambda q_0) \cap \sigma(A_\lambda)] \geqq m$ for all $\lambda \geqq \Lambda_m$. This implies, in particular, that (R_4) holds, therefore, Theorem 3.1 applies. □

Remark 5.5. Let $\gamma > 0$ be a parameter and consider the supersymmetric Dirac operator $H_\gamma := H_0 + \gamma V\beta$, where H_0 is the free Dirac operator and the scalar field $\gamma V(x)\beta$ satisfies the condition (V). Checking the proof of Lemma 5.2, we have, as a by-product, the following asymptotic estimate on the number of eigenvalues of H_γ.

Lemma 5.6. *Let* (V) *be satisfied. Then*

$$\sigma_e(H_\gamma) \subset \mathbb{R} \setminus (-(a + \gamma b_{\max}),\ a + \gamma b_{\max})$$

and the number $\mathcal{N}(\gamma) := \#[(0,\ a + \gamma b_{\max}) \cap \sigma_d(H_\gamma)] \to \infty$ *as* $\gamma \to \infty$.

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