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Parameter Identification, Population and Economic Growth in an Extended Lucas and Uzawa-type Two Sector Model[♦]

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Abstract

The aim of this paper is twofold. First of all we re-examine the long-run relationship between population and economic growth. To do this we extend the Lucas-Uzawa model along two different directions: we introduce the growth of the physical capital stock into the human capital supply equation and include in the intertemporal maximization problem of the representative household a preference parameter controlling for the degree of agents' altruism towards future generations. These two extensions allow us to capture eventual complementarity/substitutability links between physical and human capital in the production of new human capital and to study how such links, along with agents' altruism, may impact on the interplay between economic and demographic growth along the balanced growth path equilibrium.

In the second part of this paper we develop the inverse problem for this extended Lucas-Uzawa model. The method we are going to use is based on fractals and has been developed by two of the authors in recent papers. Through the solution of the inverse problem one can get the estimation of some key-parameters such as the total factor productivity, the productivity of human capital in the production of new skills, the physical capital share in total income, the inverse of the intertemporal elasticity of substitution in consumption, the depreciation rate of (physical and human) capital and the parameter controlling for the degree of altruism towards future generations.

***Key Words*:** Population Growth; Two-Sector Endogenous Growth Models; Human Capital Investment; Physical Capital Accumulation; Fractal-based Methods, Inverse Problems; Collage Theorem.

***JEL Codes*:** O41; J24; J10;

PACS Classification: 02.30.Zz; 02.30.Hq; 02.60.Nm;

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1 INTRODUCTION

The Lucas (1988) and Uzawa (1965) – henceforth simply Lucas-Uzawa – approach to growth theory still represents one of the milestones of endogenous growth literature. The reason of the success of that research line is twofold. From an economic point of view, it succeeds in formalizing the idea that human capital investment, in the form of time spent on intentional education activities and/or on-the-job-training, is an important determinant of long run per capita income growth. From a mathematical point of view, instead, it gives rise to a sophisticated dynamical system – with two control variables (consumption and the fraction of man-hours to be devoted to production and educational activities) and two state variables (human and physical capital) – that displays some really interesting properties. As Boucekkine and Ruiz-Tamarit (2008, p.34) have recently emphasized, exactly because of its mathematical appeal the Lucas-Uzawa model *"…has been studied by many authors, using different approaches, therefore allowing for a stimulating methodological discussion"*.[1]

The main objective of our paper is to build upon the celebrated Lucas-Uzawa model, both economically and mathematically.

From the economic point of view, we extend that model to shed a new light on the debate concerning the long-run relationship between population change and economic growth. In a very influential paper some years ago Kelley (1988) claimed that, depending on the country, population growth might have contributed, deterred or even had no impact on economic development. Later on many other contributions explained this ambiguous result by the fact that the effects of population growth are not stable over time. For example, one more individual in a society has both a short-term negative effect caused by the cost of rearing him/her when child and a long-run positive effect through the larger labour force that the same individual contributes to generate over that horizon (Crenshaw *et al.*, 1997). In order to reconcile within the same framework the ambiguous views (pessimistic, optimistic and neutral) about the consequences of population change on income growth[2] we modify the basic Lucas-Uzawa model along two distinct directions. First of all, we introduce the growth of the physical capital stock into the supply function of skills. In so doing we aim at capturing the potential relationships of *complementarity*/*substitutability* between physical and human capital in the production of new human capital and, hence, at studying the impact of such relationships on the interplay between economic and

[1] For a short, but comprehensive survey of these approaches and methodological discussions see Boucekkine and Ruiz-Tamarit (2008), pp.34-35.
[2] See, among many others, Bloom *et al*. (2003), Ehrlich and Lui (1997), Kelley and Schmidt (2003), Laincz and Peretto (2006) and Tournemaine (2007) for a review of these views.

demographic growth.[3] Secondly, since we are interested in studying the way population change might affect per capita income growth (driven by factor accumulation in the model), we follow Strulik (2005, p.135) and include in the intertemporal maximization problem of the representative household a preference parameter controlling for the degree of agents' *altruism* towards future generations. This consents to analyze whether, and eventually how, altruism shapes the link between demographic and economic growth in the long run.

In the second part of the paper we analyze the *inverse problem* for this extended version of the Lucas-Uzawa model. The method we are going to use is based on the so called "Collage Theorem". This result is an easy consequence of Banach theorem and Barnsley (1985, 1989) has been the first one who showed the importance of this result for solving inverse problems in fractal analysis and image approximation. Recently Kunze and Vrscay (1999, 2003, 2004), Kunze, La Torre and Vrscay (2007a, 2007b, 2007c), La Torre and Mendivil (2008), Capasso, Kunze, La Torre and Vrscay (2008) showed the importance of this result and its generalizations for solving inverse problems for deterministic, random and stochastic differential equations with initial or boundary conditions.

For practical purposes, the main aim of solving an inverse problem is to get parameter estimations of economic models which involve the solution of a differential equation or an optimal control problem. In our case it allows us estimating important parameters such as the total factor productivity, the productivity of human capital in the production of new human capital, the physical capital share in total income, the inverse of the intertemporal elasticity of substitution in consumption, the depreciation rate of (physical and human) capital and the parameter measuring the degree of altruism towards future generations.

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[3] Rebelo (1991) considers an extension of the Lucas-Uzawa model where, likewise human capital, physical capital is employed partly in goods production and partly in human capital accumulation. In his original formulation Rebelo considered a setup with two Cobb-Douglas production functions. Bond *et al.* (1996) and Mino (1996) analyze the same model with more general neoclassical production functions. Unlike these contributions, the presence of the growth rate of physical capital into the law of motion of human capital allows us to introduce in the Lucas-Uzawa model a sort of *endogenous* mechanism of depreciation (or appreciation) of embodied knowledge. Moreover, and unlike Bucci and La Torre (2007) and Bucci (2008b), in this paper we assume that, in maximizing its own intertemporal utility, the representative household takes the growth rate of physical capital as endogenous.

The main conclusions we reach from the economic model are the following. First of all, we find that along the balanced growth path (BGP, hereafter) equilibrium the relationship between population and economic growth crucially depends on how much altruistic agents are towards the future generations: when the degree of altruism is higher than a given threshold level, population growth exerts always a positive effect on economic growth; below the same threshold, instead, the effect of population change on economic growth is negative. Secondly, we see that population growth can have an ambiguous effect on economic growth depending also on whether physical and human capital are complementary or substitutes for each other in the production of new human capital and on the degree of complementarity between these two forms of capital. Finally, we examine the conditions on the parameter values under which our model is able to deliver the same results of Lucas-Uzawa as presented in Barro and Sala-i-Martin (2004).

At the end the application of the "Collage Method" allows us to get parameter estimation for some crucial unknown key-parameter of the model. We show the method through numerical simulations.

The paper is organized as follows: Section 2 presents the model economy and discusses in detail our main extensions to the Lucas-Uzawa setup. Section 3 analyzes the model's balanced growth path properties. …Section… As usual, Section…concludes.

2 THE MODEL

The economy is closed and composed of households (that receive wages and interest income, purchase consumption goods and choose how much to save and how much to invest in human capital) and firms (that produce consumption goods). Population (*L*) coincides with the available number of workers (there exists full employment) and grows at a constant exogenously given rate, g_L. Following Barro and Sala-i-Martin (2004, Chap.5, p.240) the total stock of human capital available at time *t* (H_t) is given by the number of workers at *t* (L_t) times the average level of human capital of each worker (h_t). Consumption goods are produced competitively, with prices being taken as given and each input compensated according to its own marginal product. Human and physical capital represent the two reproducible factors. While physical capital accumulates through foregone consumption, the production of new human capital is postulated to be a human capital intensive economic activity.

2.1 *PRODUCTION*

Consumption goods (*Y*) act as numeraire goods (their price is normalized to one) and are produced competitively using human (H_Y) and physical capital (K). The production function of these goods is given by:

$$Y_t = AK_t^{\alpha} H_{Yt}^{1-\alpha}, \qquad A > 0, \qquad \alpha \in (0;1). \tag{1}$$

In the above equation Y is the output of the (homeogeneous) consumption goods, A is the total factor productivity (taken as constant in the model[4]), K denotes aggregate physical capital, H_Y is the stock of human capital employed in production activities and α is the physical capital share in total income. This production function exhibits constant returns to scale to the two rival and reproducible inputs (*i.e.*, H_Y and K).

In equilibrium each input receives its marginal productivity. Hence:

$$r_t = \frac{\partial Y_t}{\partial K_t} = A\alpha K_t^{\alpha-1}\left(u_t H_t\right)^{1-\alpha}, \qquad u_t H_t \equiv H_{Yt} \tag{2}$$

$$w_t = \frac{\partial Y_t}{\partial\left(H_{Yt}\right)} = AK_t^{\alpha}\left(1-\alpha\right)\left(u_t H_t\right)^{-\alpha} \tag{3}$$

In equations (2) and (3), r_t and w_t are, respectively, the real interest rate and the wage accruing to one unit of productive human capital.

2.2 *HOUSEHOLDS*

The size of the structurally identical households grows over time at the (constant and exogenous) rate of population growth, g_L. A representative infinitely-lived dynasty uses the income it does not consume to accumulate physical capital. Thus:

$$\dot{K}_t = Y_t - \delta K_t - C_t, \qquad K_0 > 0$$

where $\dot{K}_t$ represents net physical capital investment, $\delta > 0$ is the depreciation rate of physical capital and C_t is aggregate consumption. In per capita terms, the law of motion of physical capital reads as:

$$\dot{k}_t = Ak_t^{\alpha}\left(u_t h_t\right)^{1-\alpha} - c_t - \left(g_L + \delta\right)k_t \tag{4}$$

where $y_t \equiv Y_t / L_t = Ak_t^{\alpha}\left(u_t h_t\right)^{1-\alpha}$, $c_t\left(\equiv C_t / L_t\right)$, k_t and $u_t h_t$ $\left(\equiv H_{Yt} / L_t\right)$ represent per capita income, per capita consumption, per capita physical capital and per capita human capital employed in production, respectively. The remaining fraction $1-u_t$ of individual

[4] This means that we omit any technological progress.

skills (h_t) is used to acquire new human capital. At the aggregate level we assume that the law of human capital accumulation is the following:

$$\dot{H}_t = B\left(1-u_t\right)H_t - \left(\phi g_{Kt} + \delta\right)H_t \qquad (5)$$

$$H_0 > 0\,, \qquad u_t \in \left[0;1\right], \qquad B > 0\,, \qquad 1+\phi > 0\,.$$

In (5) B is a positive technological parameter (denoting the productivity of human capital in skill acquisition), δ is the common depreciation rate of (physical and human) capital and ϕ reflects the impact of the growth rate of K (g_{Kt}) –a measure of *learning by using* the new technology embodied in new capital goods[5] (from now on simply *learning*)– on the accumulation of *H*. Under the assumption that $B\left(1-u_t\right) > \delta$ at each *t*, the constraint $\phi > -1$ prevents the growth rate of the model's aggregate variables from either exploding ($\phi = -1$) or being negative ($\phi < -1$) along a BGP equilibrium where H_t / K_t is constant. In a moment we shall give a more formal definition of the model's BGP equilibrium. Simple inspection of equation (5) reveals that when $\phi \neq 0$ the main difference with the Lucas-Uzawa model consists in postulating a technology for human capital formation in which a faster *learning* (higher g_{Kt}) may, *ceteris paribus*, either accelerate ($-1 < \phi < 0$) or slow down ($\phi > 0$) the rate at which human capital accumulates over time, g_{Ht}. However, we show below that, under specific assumptions on some key parameters, our model is general enough to encompass Lucas-Uzawa as a special case. With respect to Alvarez Albelo (1999) equation (5) presents two major differences. The first is that we measure *learning* by $g_{Kt} \equiv \dot{K}_t / K_t$, and not by physical capital investment ($\dot{K}_t$). The reason is technical. In her work, Alvarez Albelo (1999) uses a technology of human capital accumulation that is additive in two components (formal education and learning)[6] and defines the long run equilibrium of the model (what she calls steady state) as a situation where *u* remains constant and the main variables depending on time grow at a constant rate.[7] If we applied her definitions of *learning* ($\dot{K}_t$) and steady state equilibrium

[5] Alvarez Albelo (1999).

[6] Namely, $\dot{H}_t = \beta H_t\left(1-U_t\right) + \gamma \dot{K}_t$, with $\beta, \gamma \in \left(0;1\right)$.

[7] *"…The steady state paths are such that c_t, k_t, h_t grow at a constant rate and u remains constant. …It is easy to check that c_t, k_t, h_t, y_t grow at the same rate"* (p. 359). In her model there is no population growth and variables are in per capita terms.

to our framework, we would write (5) as $\dot{H}_t = B(1-u)H_t - \left(\phi \dot{K}_t + \delta\right)H_t$ and obtain that in the long run growth in physical capital ceases $\left(\lim_{t\to\infty} \dot{K}_t / K_t \to 0\right)$. Instead, using a BGP perspective, equation (5) allows us analyzing the predictions of an endogenous growth model where in the long run *u* can still remain constant but the common and constant growth rate of all (aggregate and per capita) variables is positive (see next section). The second difference is more substantial and has to do with the fact that, as already mentioned, depending on the sign of ϕ, we posit that *learning* can act either as a mechanism of *endogenous* depreciation or as a mechanism of *endogenous* appreciation of H. More precisely, unlike Alvarez Albelo (1999) that analyses only the case where physical and human capital are *complementary* for each other in the production of new human capital,[8] we do not make any *a priori* assumption in this respect. Indeed, it is well recognized (Galor and Moav, 2002) that the time required for learning the latest technology increases with the rate of technical progress. This implies that, especially in those sectors experiencing rapid advancements in technological change, the presence of large time-costs from learning to use the most up-to-date technologies embodied in new capital goods leads to a faster depreciation of the available human capital (*erosion effect*). This is what happens in equation (5) when $\phi > 0$. In this case physical and human capital are *substitutes* for each other in the production of new human capital since in the long period (when g_K and g_H are constant exponential rates) an increase in learning (g_K), and thus in K_t, harms investment in human capital ($\dot{H}_t$) and its growth rate (g_H), ultimately leading to a fall of H_t. On the other hand, when $-1 < \phi < 0$, learning acts as a mechanism of endogenous appreciation of human capital in the equation of skill-supply and our model reproduces the same situation already studied by Alvarez Albelo (1999). In this case physical and human capital are *complementary* for each other in the production of skills since in the long run an increase in g_K, and thus in K_t, stimulates human capital investment ($\dot{H}_t$), its growth rate (g_H), and eventually leads to a rise of H_t.[9]

[8] "*…this new technology implies that both types of capital are complementary for each other in such a way that if U_t were equal to unity we would obtain the AK model*" (p.358).

[9] Bucci (2008a) also uses a technology for human capital accumulation similar to (5) within a model where R&D activity, employing human capital, is the source of technical progress. Recall that in the present model we are leaving technological progress out of the analysis.

Given $\dot{H}_t$ and $g_{K_t} \equiv \dot{K}_t / K_t$, after some algebra the law of motion of per capita human capital becomes:

$$\dot{h}_t = B(1-u_t)h_t - \phi\left[Au_t^{1-\alpha}\left(\frac{h_t}{k_t}\right)^{1-\alpha} - \delta - \frac{c_t}{k_t}\right]h_t - (g_L+\delta)h_t . \qquad (5')$$

With a constant intertemporal elasticity of substitution individual utility function $u(c_t) = \frac{c_t^{1-\theta}-1}{1-\theta}$ the intertemporal maximization household's problem can be recast as:

$$\underset{\{c_t,u_t,k_t,h_t\}_{t=0}^{\infty}}{\text{Max}} \ \text{U} \equiv \int_0^{\infty}\left(\frac{c_t^{1-\theta}-1}{1-\theta}\right)e^{-(\rho - mg_L)t}dt , \qquad (\rho - mg_L) > 0 , \qquad m \in [0;1], \qquad \theta > 1 \quad (6)$$

s.t.:

$$\dot{k}_t = Ak_t^{\alpha}(u_t h_t)^{1-\alpha} - c_t - (g_L+\delta)k_t , \qquad u_t \in [0;1] \quad \forall t \qquad (4)$$

$$\dot{h}_t = B(1-u_t)h_t - \phi\left[Au_t^{1-\alpha}\left(\frac{h_t}{k_t}\right)^{1-\alpha} - \frac{c_t}{k_t} - \delta\right]h_t - (g_L+\delta)h_t \qquad (5')$$

$$\lim_{t\to\infty}\lambda_t k_t = 0 , \qquad \lim_{t\to\infty}\mu_t h_t = 0 \qquad (7)$$

$$k_0 , \ h_0 > 0 .$$

In the problem stated above we normalized to one the size of the representative household at time zero ($L_0 \equiv 1$) and denoted by ρ the pure rate of time preference (or discount rate), by m the parameter controlling for the degree of altruism towards future generations (Strulik, 2005) and by $1/\theta$ the intertemporal elasticity of substitution in consumption. The hypothesis $\theta > 1$ is dictated by available evidence (Growiec, 2006). In the problem stated above the control variables are c_t and u_t, the state variables are k_t and h_t and the co-state variables are μ_t and λ_t.

3 BGP ANALYSIS

In this section we define and characterize the BGP equilibrium of the model.

Definition 3.1: BGP EQUILIBRIUM

A BGP equilibrium is a long-run equilibrium where: (i) *All variables depending on time grow at constant (possibly positive) exponential rates;* (ii) *The shares of human capital* *devoted to production (u) and education (1-u) activities are both constant;* (iii) *The ratio of the two endogenous state variables (h_t / k_t) is constant, as well.*

This definition implies:

PROPOSITION 3.1

Along the BGP equilibrium:

$$\frac{\dot{c}_t}{c_t} = \frac{\dot{k}_t}{k_t} = \frac{\dot{h}_t}{h_t} = \frac{\dot{y}_t}{y_t}.$$

Proof:

Take equations (4) and (5'), solve them for c_t / k_t and apply the definition of BGP equilibrium. Finally, notice that $y_t \equiv Y_t / L_t = Ak_t^{\alpha}\left(u_t h_t\right)^{1-\alpha}$.■

It is possible to show that the following results must hold along the BGP equilibrium (mathematical derivation of such results is in *Appendix A*):

$$u = \frac{(\theta-1)(B-\delta)+(1+\phi)\left[\rho+(1-\theta-m)g_L\right]}{B(\theta+\phi)} \tag{8}$$

$$g_c = g_k = g_h = g_y \equiv g = \frac{\left[B-\rho-\delta-(\phi+1-m)g_L\right]}{(\theta+\phi)} \tag{9}$$

$$\frac{h}{k} = \left\{\frac{B\theta+\phi\left[\delta+\rho+(1-m-\theta)g_L\right]}{A\left[\alpha-\phi(1-\alpha)\right](\theta+\phi)}\right\}^{\frac{1}{1-\alpha}} \cdot \left(\frac{1}{u}\right) \tag{10}$$

$$\frac{c}{k} = \left\{\frac{B\left[\theta-\alpha+\phi(1-\alpha)\right]+\delta\left[\alpha(1-\theta)+\phi(1-\alpha)(\theta+\phi)\right]+\alpha(1+\phi)\left[\rho+(1-m-\theta)g_L\right]}{\left[\alpha-\phi(1-\alpha)\right](\theta+\phi)}\right\} \tag{11}$$

Equation (8) gives the equilibrium fraction of human capital employed in consumption goods production. According to equation (9), per capita consumption (*c*), physical capital (*k*), human capital (*h*) and income (y) grow in the long run at the same constant rate. Equations (10) and (11) provide the equilibrium value for the ratio of the two endogenous state variables of the model (h / k) and the ratio of per capita consumption to per capita physical capital (c / k).

The following two propositions analyze the relationship between population growth (g_L) and economic growth (g) along the BGP equilibrium (see equation 9).

PROPOSITION 3.2

The relationship between population and economic growth along the BGP equilibrium depends crucially on the size of agents' altruism towards the future generations (m). In particular:

- *When* $(1+\phi) < m \le 1$ *population growth exerts a positive effect on economic growth, i.e.* $\frac{\partial g}{\partial g_L} > 0$*;*
- *When* $0 \le m < (1+\phi)$ *population growth exerts a negative effect on economic growth, i.e.* $\frac{\partial g}{\partial g_L} < 0$*;*
- *When* $m = (1+\phi)$ *population growth has no impact on economic growth, i.e.* $\frac{\partial g}{\partial g_L} = 0$.

Proof:

The proof follows immediately from equation (9) and the fact that $(\theta+\phi) > 0$. ■

The intuition behind Proposition 3.2 goes as follows. With given pure time preference rate (ρ) and population growth (g_L), when agents' degree of *altruism* (*m*) is sufficiently high (low), implying that the future size of the family is (not) sufficiently taken into account, households are more (less) patient and, hence, save more (less). Thus, the higher (lower) *m*, the higher (lower) the investment in physical and human capital. The increase of the size of the dynastic family (g_L) has the effect of reinforcing the positive relationship between altruism and factor accumulation. Thus, when *m* is sufficiently high (low) a rise of population size increases (decreases) further agents' investment in reproducible inputs, and hence economic growth.

Proposition 3.3 relates the ambiguous effect of population change on economic growth to whether human and physical capital are complementary or substitutes for each other in the production of new human capital.

PROPOSITION 3.3

- *When human and physical capital are complementary for each other in human capital production* ($-1 < \phi < 0$)*, population growth may have either a positive, or a negative, or else no effect on real per capita income growth:* $\frac{\partial g}{\partial g_L} \gtreqqless 0$.

- *When human and physical capital are substitutes for each other in human capital production (* $\phi > 0$ *), population growth exerts an unambiguously negative effect on real per capita income growth:* $\frac{\partial g}{\partial g_L} < 0$.

Proof:

In proving the proposition we use the fact that $(1+\phi) > 0$ and suppose that $\phi \neq 0$ (otherwise we have the classical Lucas model). According to Proposition 3.2, when $-1 < \phi < 0$ it can happen that either $\frac{\partial g}{\partial g_L} > 0$, $\frac{\partial g}{\partial g_L} < 0$ or $\frac{\partial g}{\partial g_L} = 0$. When $\phi > 0$ then $0 \leq m < (1+\phi)$ and this implies that $\frac{\partial g}{\partial g_L} < 0$. ■

The economic intuition behind Proposition 3.3 is the following. For given per capita physical capital stock, an increase in population size raises the exponential growth rate of aggregate physical capital. If human and physical capital are substitutes for each other in the production of new human capital, the joint increase of population and the physical capital growth rate definitely lowers human capital accumulation at the individual level and, thus, per capita income growth. Instead, if human and physical capital are complements, as long as population and the growth rate of aggregate physical capital increase, the rate of per capita human capital accumulation (and, thus, the rate of per capita income growth) may either go up, down, or else be exactly equal to zero.

The next proposition studies the conditions on the parameter values under which our model is able to replicate the same conclusions of the Lucas-Uzawa model, as presented in Barro and Sala-i-Martin (2004, Ch.5).

PROPOSITION 3.4: *A comparison with Lucas-Uzawa*

Suppose that $g_L = 0$, $m = 1$ *and* $\phi = 0$. *Under these parameter values our model allows to obtain the same results of the Lucas-Uzawa model*

Proof:

Use these parameter values into (8), (9), (10) and (11) and compare results with those provided by Barro and Sala-i-Martin (2004, p. 252, equations 5.28, 5.29 and 5.31).

The following result states the conditions which have to be satisfied in order to guarantee that (8), (9), (10) and (11) are all positive. The proof is very easy and follows from simple calculations.

PROPOSITION 3.5: *The parameters of the model have to satisfy the following conditions:*

- $g_L < \dfrac{\rho}{\theta - 1 + m}$
- $B > \delta + \rho$
- $\phi < \min\left\{\dfrac{\alpha}{1-\alpha}, \dfrac{B-\delta-\rho}{g_L} + m - 1\right\}$
- $\phi > \max\left\{\dfrac{-B\theta}{\delta + \rho + (1-m-\theta)g_L}, \dfrac{-B(\theta-\alpha) - \alpha\delta(1-\theta) - \alpha[\rho + (1-m-\theta)g_L]}{B(1-\alpha) + \alpha[\rho + (1-m-\theta)g_L]}, -1\right\}$

4 Dynamics of the model

The FOCs of the model can be rewritten in terms of the new variables $\psi_t = \dfrac{h_t}{k_t}$, $\Omega_t = \dfrac{c_t}{k_t}$, and u_t, as follows:

$$\frac{\dot{\psi}_t}{\psi_t} = B(1-u_t) - \phi(Au_t^{1-\alpha}\psi_t^{1-\alpha} - \Omega_t - \delta) - Au_t^{1-\alpha}\psi_t^{1-\alpha} + \Omega_t$$

$$\begin{aligned}\frac{\dot{\Omega}_t}{\Omega_t} &= (\psi_t u_t \theta(B(u_t\psi_t)^\alpha + \phi A(1-\alpha)\psi_t))^{-1}((g_L\rho A\alpha + \rho A\alpha + \theta\rho\alpha\Omega_t + \delta\phi A\alpha \\ &+\delta\theta\phi A - \theta\phi A\alpha\Omega_t - \delta\theta\phi A\alpha + mg_L\phi A - g_L\phi A - \rho\phi\alpha - \delta\phi A - mg_L\phi A\alpha \\ &+\phi AB\alpha u_t - AB\theta u_t - g_L\theta\phi A\alpha + \alpha BAu_t + g_L\theta\phi A - \phi ABu_t)u_t\psi_t^2 \\ &+(\Omega_t\theta B + \delta\theta B + g_L\theta B - \rho B + mg_L B - g_L B - \delta B)(u_t\psi_t)^{1+\alpha} \\ &+(-A^2\theta\phi + A^2\theta\alpha - \phi^2A^2 - \alpha^2A^2\phi - \alpha^2\phi^2A^2 + \alpha A^2\phi + 2\alpha\phi^2A^2)(u_t\psi_t)^{2-\alpha}\psi_t)\end{aligned}$$

$$\begin{aligned}\frac{\dot{u}_t}{u_t} &= \alpha(B(u_t\psi_t)^\alpha + (1-\alpha)\phi A\psi_t)^{-1}((1-\alpha)\phi A\alpha\Omega_t + 2B\phi A\alpha - \phi AB\alpha^2 \\ &-\phi^2A\Omega_t + 2\phi^2A\alpha\Omega_t - AB\phi - \phi^2A\alpha^2\Omega_t - \phi^2\delta A + 2\phi^2\delta A\alpha - \phi^2\delta A\alpha^2)\psi_t u_t \\ &-(B\phi A(1-\alpha)\alpha)\psi_t u_t^2 + (-B^2\alpha u_t - (1-\alpha)B\phi\Omega_t + B^2\alpha - (1-\alpha)B\phi\delta \\ &+B\alpha\Omega_t)\psi^\alpha u_t^{1+\alpha}\end{aligned}$$

Of course, the nontrivial equilibrium state of this system is given in equations (8), (10), and (11). The classification of this equilibrium point via linearization is complicated by the fact that there are many parameters in the problem. Of course, once we pick values for the parameters, we can perform this classification.
The key-parameters of our model are the following:

- g_L (the exogenous population growth rate);
- α (the physical capital share in total income);
- δ (the common depreciation rate of physical and human capital);
- ρ (the pure rate of time-preference or agents' subjective discount rate);

- B (the productivity of human capital in the production of new human capital);
- A (Total Factor Productivity);
- θ (the inverse of the intertemporal elasticity of substitution in consumption);
- m (the parameter controlling for the degree of agents' altruism towards future generations);
- ϕ (the parameter controlling for the degree of complementarity/substitutability between human and physical capital in the production of new human capital).

For most of these parameters we have empirical estimates or baseline specifications coming from previous works. We use the following parameter-values:

- $g_L = 0.0144$

This value is suggested by Jones and Williams (2000, Table 1, p. 73) and refers to the average growth rate of the labor force in the U.S. private business sector over the period 1948-1997.

- $\alpha = 0.3$

Mankiw (2000, p. 75) shows that the physical capital share in the United States has been roughly stable at 0.3 since the 1960s if we include depreciation in capital income and exclude proprietors' income from total income.

- $\delta = 0.05$
- $\rho = 0.04$
- $B = 0.12$

These three parameter values are taken from Mulligan and Sala-i-Martin (1993, p.761).

- $A = 1$

Since in our model we omit technological progress and take A as a constant, we normalize this parameter to one.

- $\theta = 1.28$

Empirical evidence supports the assumption that the *intertemporal elasticity of substitution in consumption* ($1/\theta$ in our model) is smaller than one, *i.e.* $\theta > 1$(see Growiec, 2006, pp.17-19). When $\theta = 1.28$, we obtain a value for the intertemporal elasticity of substitution which is close to that (0.78) recently found by Favero (2005).

- $m = 0.13$

Clearly, it is extremely difficult to find a direct and precise estimate of *agents' degree of altruism towards subsequent generations*. However, we have indirect indication that *m* might be rather small and definitely different from one. Indeed, and according to theory, if parents behaved altruistically, a one dollar increase of income redistribution from children to parents would lead to a one dollar increase of transfers from parents to children, as well. Using panel data on *inter-vivos* transfers, Altonji *et al.* (1997) estimated that a one dollar redistribution from children to parents increases parents' transfers to children by less than 13 cents. In other words, they strongly reject the hypothesis that parents behave intergenerationally altruistically. Using panel data on bequests, Laitner and Ohlsson (2001) obtained a similar result. On the basis of such literature, we set *m* equal to 0.13.

- ϕ

As far as we know there exists ni empirical estimate of this parameter. However, we know that, according to equation (5), ϕ must obey the following restriction: $\phi > -1$.

Accordingly, in what follows we consider three possible parameterizations for ϕ:

- $\phi = 0$

- $\phi = -0.5$

- $\phi = 0.2$

Case 1: $\phi = 0$.

We can compute that the nontrivial equilibrium point is

$$(\psi_t, \Omega_t, u_t) = (11.2796, 1.1100, 0.3456).$$

The Jacobian matrix of previous system at this point is computed to be

$$Df = \begin{pmatrix} -0.2800 & 0.7726 & -0.7115 \\ -0.0893 & 0.3220 & -0.1974 \\ 0 & -0.3496 & 0.0420 \end{pmatrix}$$

with eigenvalues -0.2800, 0.0419, 0.3220.

Case 2: $\phi = -0.5$.

We compute that the nontrivial equilibrium point is

$$(\psi_t, \Omega_t, u_t) = (0.2937, 0.1240, 0.3915).$$

The Jacobian matrix of previous system at this point is

$$Df = \begin{pmatrix} -0.0770 & 0.1468 & -0.0930 \\ -0.0320 & 0.1240 & -0.0240 \\ 0 & -0.8483 & 0.0470 \end{pmatrix}$$

with eigenvalues -0.1370, 0.1840, 0.0470

Case 3: $\phi = 0.2$.

In this final case, the nontrivial equilibrium point is

$$(\psi_t, \Omega_t, u_t) = (1.8344, 0.6454, 0.3407) .$$

The Jacobian matrix of previous system at this point is computed to be

$$Df = \begin{pmatrix} -0.6045 & 2.2013 & -3.4746 \\ -0.1551 & 0.6454 & -0.8350 \\ 0 & -0.1817 & 0.0409 \end{pmatrix}$$

with eigenvalues -0.4278, 0.4687, 0.0409.

The nontrivial equilibrium points are a saddle point. To visualize the results we use 2-D and 3-D direction fields.

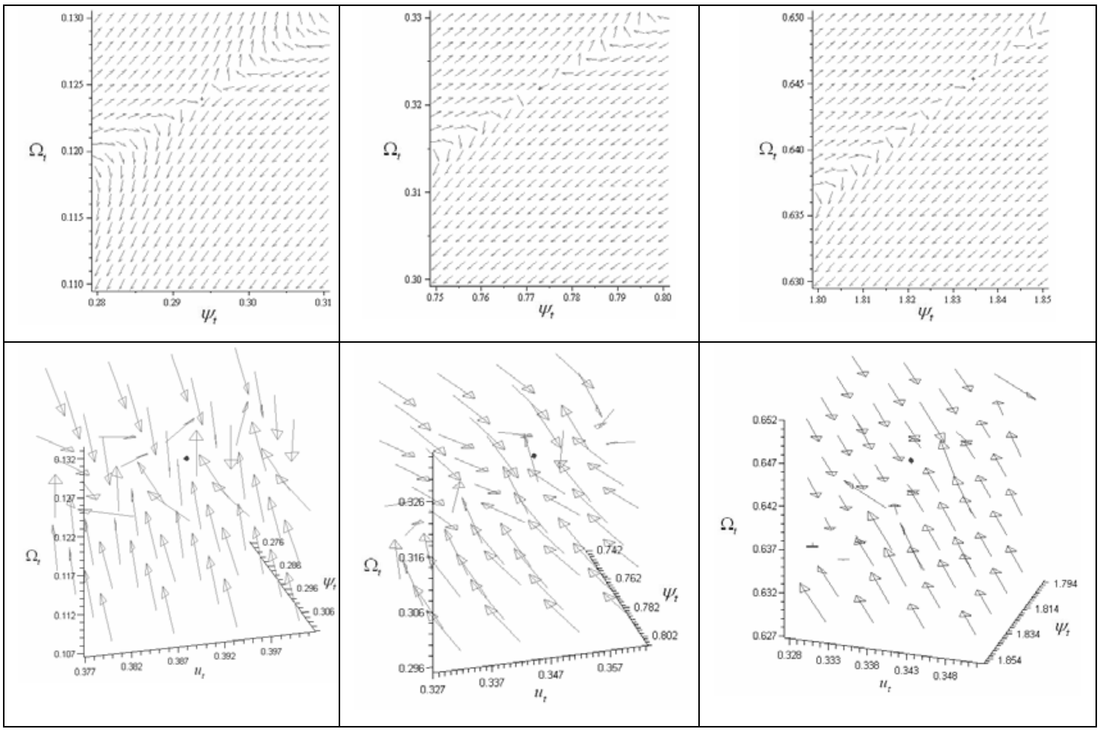

Figure 1: (left to right) 2-D and 3-D direction field plots for cases 1,2 and 3.

5 PARAMETER ESTIMATION THROUGH INVERSE PROBLEM

In Kunze and Vrscay (1999) and subsequent works a collage coding framework to solve inverse problems for systems of ordinary differential equations was developed. For $x \in \mathbb{R}^n$, consider the system

$$\begin{cases} x'(t) = f(t, x(t)) \\ x(0) = x_0 \end{cases}$$

and the associated Picard integral operator

$$(Tx)(t) = x_0 + \int_0^t f(s, x(s))ds$$

Choose $\delta > 0$ and define $I = [-\delta, \delta]$, $X = (C(I))^n$, and

$$d(x, y) = d_\infty(x, y) = \max_{1 \le i \le n} \sup_{t \in I} | x_i(t) - y_i(t) | \text{ for } x, y \in X .$$

With these definitions, (X,d) is complete. When f is Lipschitz, perhaps picking δ quite small, we have that

- $T: X \to X$, and
- T is contractive on (X,d).

With this set up, Banach's fixed point theorem applies, allowing us to conclude that T has a unique fixed point $\bar{x} \in X$. The general inverse problem is:

Given the target observation $x(t)$, $t \in I$, find a system of ordinary differential equations that admits the target as an approximate solution. Equivalently, find a Picard operator T, usually within a chosen class, with fixed point $\bar{x}$ as close as possible to x.

In order to solve the inverse problem, we make use of the following result, a simple consequence of Banach's fixed point theorem.

THEOREM: (Collage theorem) *Let (X,d) be a complete metric space and $T: X \to X$ a contraction map with contraction factor $c \in [0,1)$. Then for any $x \in X$,*

$$d(x,\bar{x}) \le \frac{1}{1-c} d(x,Tx),$$

where $\bar{x}$ is the fixed point of T.

Barnsley in 1985 was the first one who showed the importance of this result for solving inverse problem for fractals and image approximation. The collage theorem says that the approximation error $d(x,\bar{x})$ can be controlled by the collage distance $d(x,Tx)$, provided that the contraction factors of our family of maps T are kept away from one. We desire to work with a computationally convenient metric. In practice, we use d_2, the L^2 metric. For $x, y \in X$, we have

$$d_2(x,y) = \left(\sum_{i=1}^{n} \int_I (x_{i(t)} - y_i(t))^2 dt \right)^{\frac{1}{2}}$$

Working with this metric is alright since $C(I) \subset L^2(I)$. Note that T is contractive in the L^2 metric, as well. Thus, provided that each vector field component is determined by a unique set of parameters, the solution of the inverse problem via collage coding reduces to minimizing each of the squared L^2 collage distances

$$\Delta_i = d_2^2(x_i,(Tx)_i),\ 1 \le i \le n.$$

If parameters are shared across components, we might minimize the sum of those components instead.

In order to simulate an inverse problem for the demographics dynamics model, for a particular choice of parameter values, we first solve the system of four ordinary differential equations numerically for $h(t)$, $k(t)$, $\lambda(t)$, and $\mu(t)$. Next, we sample our solution components at chosen observation times t_i, $1 \le i \le N$, as a way to produce “observational data.” We then fit a polynomial target function of degree M to each set of N data points, possibly adding Gaussian noise with low amplitude ε. At this point, we begin the inverse problem solution process. Beginning with these target functions and forgetting the parameter values that led to them, we seek to find parameter values—

$A, B, \alpha, \phi, \rho, \delta, g_L, m$—so that the system of ordinary differential equations of the correct form admits the target functions as an approximate solution.
In the examples that follow, each component is initially equal to 1, and we choose $N = 15$ data points on the observation intervals lying inside $[0, 0.15]$. Our target functions are degree 4 polynomials. The right-hand sides of the differential equations are rather complicated functions, both of the parameters and of the components. As a result, when we construct the term $f(x(s))$, with the parameters left as variables and our target functions plugged in, we have no hope of integrating directly. Thus, we replace $f_i(x(s))$ by $P_i^{(r)}(s)$, the r^{th} degree Taylor polynomial about $s = 0$. Clearly, the larger we choose r, the better we expect it to approximate the true integral. But the resulting collage distances become ever more complicated functions of the parameters. Putting everything together, we use gradient descent to find the parameter values that minimize

$$\Delta = \sum_{i=1}^{4} \Delta_i = \sum_{i=1}^{4} d_2^2(x_i, (Tx)_i) = \sum_{i=1}^{4} \int_0^1 \left(x_i(t) - x_i(0) - \int_0^t P_i^{(3)}(x(s))ds \right)^2 dt .$$

To generate our target functions, we use the values

$$A = 1, B = 0.12, \alpha = 0.3, \theta = 1.28, \rho = 0.04, \delta = 0.05, g_L = 0.0144, m = 0.13,$$

as well as three different values for ϕ: $-0.5, 0, 0.2$.
The results obtained after 100 gradient descent steps, to four decimal places, are presented in Table 1. We assume that we know the true values of the parameters ρ, g_L and solve the inverse problem for the others.

noise ε	true ϕ	A	B	α	θ	δ	m	ϕ
0	0	1.0000	0.1198	0.3000	1.2800	0.0500	0.1300	0.0000
0.01	0	1.0000	0.1199	0.3001	1.2800	0.0498	0.1300	-0.0004
0.1	0	1.0000	0.1200	0.3000	1.2800	0.0498	0.1300	-0.0006
0	0.2	1.0000	0.1199	0.3000	1.2800	0.0500	0.1300	0.1999
0.01	0.2	1.0001	0.1195	0.2999	1.2800	0.0500	0.1300	0.2000
0.1	0.2	0.9994	0.1248	0.3034	1.2800	0.0490	0.1300	0.1972
0	-0.5	1.0000	0.1199	0.2998	1.2800	0.0500	0.1300	-0.5001
0.01	-0.5	1.0001	0.1193	0.2997	1.2800	0.0500	0.1300	-0.5000
0.1	-0.5	1.0031	0.1029	0.3040	1.2800	0.0486	0.1300	-0.4966

Table 4.1. Results for the Inverse Problem

9 Conclusions

In this paper we have extended the Lucas-Uzawa model along two different directions: we have introduced the growth of the physical capital stock into the human capital supply equation and included in the intertemporal maximization problem of the representative household a preference parameter controlling for the degree of agents' altruism towards future generations. We have found a balanced growth path equilibrium and showed through numerical simulations that this is a saddle point.
The results for the inverse problem listed in Table 4.1 show strong agreement with the true values of the parameters. This example demonstrates how one can solve the parameter

identification problem for such models by reducing them to the analysis of the inverse problem for the system of differential equations induced by the first-order conditions. As shown in other papers in the literature, the approach we have used to solve this inverse problem is based on the collage theorem and has been successfully developed for and applied to problems in many different settings. The results in Table 4.1 confirm the goodness of this method for optimal control problems, as well. Concerning the stability and the robustness of the method, we refer the reader to the papers listed in the references.

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Appendix A

The Hamiltonian function (J_t) associated to the inter-temporal maximization problem of the representative household is:

$$J_t = \frac{c_t^{1-\theta}-1}{1-\theta} e^{-(\rho - mg_L)t} + \mu_t \left[B(1-u_t) h_t - (g_L+\delta) h_t - \phi \left(A k_t^{\alpha-1} u_t^{1-\alpha} h_t^{2-\alpha} - \frac{c_t}{k_t} h_t - \delta h_t \right) \right] +$$
$$\lambda_t \left[A k_t^{\alpha} (u_t h_t)^{1-\alpha} - c_t - (g_L+\delta) k_t \right]$$

The (necessary) first order conditions of the problem read as:

(A1) $$c_t = \left(\lambda_t - \phi \mu_t \frac{h_t}{k_t} \right)^{-\frac{1}{\theta}} e^{-\frac{(\rho - mg_L)t}{\theta}}$$

(A2) $$\frac{\lambda_t}{\mu_t} = \frac{B}{A} \frac{1}{1-\alpha} u_t^{\alpha} \left(\frac{h_t}{k_t} \right)^{\alpha} + \phi \left(\frac{h_t}{k_t} \right)$$

(A3) $$\frac{\dot{\mu}_t}{\mu_t} = -B(1-u_t) + (g_L+\delta) + \phi \left[A(2-\alpha) u_t^{1-\alpha} \left(\frac{h_t}{k_t} \right)^{1-\alpha} - \frac{c_t}{k_t} - \delta \right] - \frac{\lambda_t}{\mu_t} \left[A(1-\alpha) u_t^{1-\alpha} \left(\frac{h_t}{k_t} \right)^{-\alpha} \right]$$

(A4) $$\frac{\dot{\lambda}_t}{\lambda_t} = -A\alpha u_t^{1-\alpha} \left(\frac{h_t}{k_t} \right)^{1-\alpha} + (g_L+\delta) + \frac{\mu_t}{\lambda_t} \left\{ -\phi \left[A(1-\alpha) u_t^{1-\alpha} \left(\frac{h_t}{k_t} \right)^{2-\alpha} - \frac{c_t}{k_t} \frac{h_t}{k_t} \right] \right\}$$

(A5) $$\lim_{t\to\infty} \mu_t h_t = 0 ; \qquad \lim_{t\to\infty} \lambda_t k_t = 0 .$$

The last equation gives the two transversality conditions. Along the BGP equilibrium $\frac{\dot{h}_t}{h_t} = \frac{\dot{k}_t}{k_t}$ (see Proposition 1 in the main text). Using (4) and (5') in the text this equality implies:

(A6) $$\frac{h_t}{k_t} = \left(\frac{1}{A} \right)^{\frac{1}{1-\alpha}} \frac{1}{u} \left[\frac{c_t}{k_t} + \frac{\phi \delta}{1+\phi} + \frac{B(1-u)}{1+\phi} \right]^{\frac{1}{1-\alpha}}$$

Combining (A1) and (A2) yields:

(A7) $$c_t = \left[\frac{B}{A} \frac{1}{1-\alpha} u^{\alpha} \left(\frac{h_t}{k_t} \right)^{\alpha} \mu_t \right]^{-\frac{1}{\theta}} e^{-\frac{(\rho - mg_L)t}{\theta}} .$$

The last equation allows computing the growth rate of consumption ($\dot{c}_t / c_t$) along the BGP equilibrium:

$$\text{(A8)} \quad \frac{\dot{c}_t}{c_t} = -\frac{1}{\theta}\frac{\dot{\mu}_t}{\mu_t} - \frac{(\rho - mg_L)}{\theta}.$$

By using (A3) into (A8) one obtains:

$$\text{(A9)} \quad \begin{aligned} \frac{\dot{c}_t}{c_t} &= \frac{1}{\theta}B(1-u) - \frac{1}{\theta}(g_L + \delta) - \frac{\phi}{\theta}\left[A(2-\alpha)u^{1-\alpha}\left(\frac{h_t}{k_t}\right)^{1-\alpha} - \frac{c_t}{k_t} - \delta\right] + \\ &+ \frac{1}{\theta}\frac{\lambda_t}{\mu_t}\left[A(1-\alpha)u^{1-\alpha}\left(\frac{h_t}{k_t}\right)^{-\alpha}\right] - \frac{(\rho - mg_L)}{\theta} \end{aligned}.$$

Insertion of (A2) into (A9) in the end yields:

$$\text{(A10)} \quad \frac{\dot{c}_t}{c_t} = -\frac{1}{\theta}\left\{-B + \phi\left[Au^{1-\alpha}\left(\frac{h_t}{k_t}\right)^{1-\alpha} - \frac{c_t}{k_t}\right] + \rho - (\phi - 1)\delta + (1-m)g_L\right\}.$$

Along the BGP equilibrium we also have $\frac{\dot{c}_t}{c_t} = \frac{\dot{k}_t}{k_t}$. Thus, using (A10) and (4) this equality leads to:

$$\text{(A11)} \quad \frac{h_t}{k_t} = \left(\frac{1}{A}\right)^{\frac{1}{1-\alpha}} \frac{1}{u}\left[\frac{c_t}{k_t} + \frac{(\theta + \phi - 1)\delta}{(\theta + \phi)} + \frac{(\theta + m - 1)g_L}{(\theta + \phi)} + \frac{B - \rho}{\theta + \phi}\right]^{\frac{1}{1-\alpha}}.$$

We now equate (A6) and (A11) and get the BGP equilibrium value of the share of human capital devoted to goods production (u):

$$\text{(A12)} \quad u = \frac{(\theta - 1)(B - \delta) + (1 + \phi)\left[\rho + (1 - \theta - m)g_L\right]}{B(\theta + \phi)}.$$

From (4) in the main text:

$$g_k \equiv \frac{\dot{k}_t}{k_t} = Ak_t^{\alpha-1}u^{1-\alpha}h_t^{1-\alpha} - \frac{c_t}{k_t} - (g_L + \delta)$$

we obtain an expression for $\frac{c_t}{k_t}$:

$$\text{(A13)} \quad \frac{c_t}{k_t} = Au^{1-\alpha}\left(\frac{h_t}{k_t}\right)^{1-\alpha} - g_k - (g_L + \delta).$$

Inserting (A13) into (A10) yields:

$$\text{(A14)} \quad g_c = \frac{B}{\theta} - \frac{\phi}{\theta}g_k - \frac{\phi}{\theta}(g_L + \delta) - \frac{\rho}{\theta} + \frac{(\phi - 1)}{\theta}\delta - \frac{(1-m)}{\theta}g_L.$$

Along the BGP equilibrium:

$$g_c = g_k = g_h = g_y \equiv g \ ,$$

with g_M denoting the growth rate of variable M.

Solving in g equation (A14) above yields:

(A15) $$g_c = g_k = g_h = g_y \equiv g = \frac{\left[B-\rho-\delta-(\phi+1-m)g_L\right]}{(\theta+\phi)}.$$

Since u and h_t / k_t are constant, it follows that λ_t / μ_t is constant as well along the BGP equilibrium (see A2). In turn, this leads to:

(A16) $$\frac{\dot{\lambda}_t}{\lambda_t} = \frac{\dot{\mu}_t}{\mu_t}.$$

We now combine (A4) and (A2) and obtain:

(A17) $$\frac{\dot{\lambda}_t}{\lambda_t} = \frac{-\left(\frac{\alpha}{1-\alpha}\right)Bu - A\phi u^{1-\alpha}\left(\frac{h_t}{k_t}\right)^{1-\alpha} + (g_L+\delta)\left[\frac{B}{A}\frac{1}{1-\alpha}u^{\alpha}\left(\frac{h_t}{k_t}\right)^{\alpha-1}+\phi\right] + \phi\frac{c_t}{k_t}}{\left[\frac{B}{A}\frac{1}{1-\alpha}u^{\alpha}\left(\frac{h_t}{k_t}\right)^{\alpha-1}+\phi\right]}.$$

Instead, by combining (A3) and (A2) we get:

(A18) $$\frac{\dot{\mu}_t}{\mu_t} = -B + (g_L+\delta) + A\phi u^{1-\alpha}\left(\frac{h_t}{k_t}\right)^{1-\alpha} - \phi\frac{c_t}{k_t} - \phi\delta \ .$$

We can now equate (A17) and (A18). Making use of (A13) we conclude:

(A19) $$\left(\frac{h_t}{k_t}\right)^{\alpha-1} = \left\{\left(\frac{\alpha}{1-\alpha}\right)Bu + \phi\left[-B+\delta+(1+\phi)(g_k+g_L)\right]\right\}\frac{A}{B}(1-\alpha)u^{-\alpha}\left[B-\phi(g_k+g_L)\right]^{-1}.$$

Given equations (A12) and (A15), one can rewrite the first part of (A19) as:

(A20) $$\left\{\left(\frac{\alpha}{1-\alpha}\right)Bu + \phi\left[-B+\delta+(1+\phi)(g_k+g_L)\right]\right\}\frac{A}{B}(1-\alpha) = u\left[\alpha-\phi(1-\alpha)\right]A.$$

Instead, using (A15) yields:

(A21) $$\left[B-\phi(g_k+g_L)\right]^{-1} = \left\{\frac{B\theta+\phi\left[\rho+\delta+(1-m-\theta)g_L\right]}{(\theta+\phi)}\right\}^{-1}.$$

Thus, (A19) becomes:

(A22) $$\left(\frac{h_t}{k_t}\right)^{-(1-\alpha)} = u^{1-\alpha}\left[\alpha-\phi(1-\alpha)\right]A\left\{\frac{B\theta+\phi\left[\rho+\delta+(1-m-\theta)g_L\right]}{(\theta+\phi)}\right\}^{-1},$$

that implies:

(A23) $$\frac{h_t}{k_t} = \left(\frac{1}{u}\right)\left\{\frac{B\theta+\phi\left[\rho+\delta+(1-m-\theta)g_L\right]}{A\left[\alpha-\phi(1-\alpha)\right](\theta+\phi)}\right\}^{\frac{1}{1-\alpha}}.$$

Combination of (A13) and (A23) produces:

(A24) $\frac{c_t}{k_t} = \frac{B\theta + \phi\left[\rho + \delta + (1 - m - \theta) g_L\right]}{\left[\alpha - \phi(1-\alpha)\right](\theta + \phi)} - \left(g_k + g_L + \delta\right)$.

Finally, plug (A15) into the last equation and obtain:

(A25) $\frac{c_t}{k_t} = \left\{\frac{B\left[\theta - \alpha + \phi(1-\alpha)\right] + \delta\left[\alpha(1-\theta) + \phi(1-\alpha)(\theta+\phi)\right] + \alpha(1+\phi)\left[\rho + (1-m-\theta) g_L\right]}{\left[\alpha - \phi(1-\alpha)\right](\theta+\phi)}\right\}$

Finally, we want to prove that along the BGP equilibrium the two transversality conditions:

$$\lim_{t\to\infty} \lambda_t k_t = 0\,, \qquad \lim_{t\to\infty} \mu_t h_t = 0$$

are checked (see equation A5). These conditions can be written as:

$$\lim_{t\to\infty} \lambda_t k_t = \lim_{t\to\infty} \lambda_0 e^{\frac{\dot{\lambda}_t}{\lambda_t} t} k_0 e^{g_k t} = \lambda_0 k_0 \lim_{t\to\infty} e^{\left(\frac{\dot{\lambda}_t}{\lambda_t} + g_k\right)t} = 0$$

$$\lim_{t\to\infty} \mu_t h_t = \lim_{t\to\infty} \mu_0 e^{\frac{\dot{\mu}_t}{\mu_t} t} h_0 e^{g_h t} = \mu_0 h_0 \lim_{t\to\infty} e^{\left(\frac{\dot{\mu}_t}{\mu_t} + g_h\right)t} = 0$$

where k_0, h_0, λ_0 and μ_0 are the (given) initial values (*i.e.*, at t=0) of the two state-variables (*k* and *h*) and their respective shadow prices (λ and μ). Along the BGPE $g_k = g_h$ (equation A15) and $\frac{\dot{\lambda}_t}{\lambda_t} = \frac{\dot{\mu}_t}{\mu_t}$ (equation A16). Therefore, when

$$\left(\frac{\dot{\mu}_t}{\mu_t} + g_h\right) < 0$$

the transversality conditions are both satisfied. Using (A18), (A13) and the fact that $g_k = g_h \equiv g$ the last inequality can be recast as:

(A26) $-B + (1+\phi)(g_L + \delta) + (1+\phi) g - \phi\delta < 0$.

Inserting (A15) into (A26) yields:

(A27) $(\theta - 1)(B - \delta) + (1+\phi)\left[\rho + (1 - \theta - m) g_L\right] > 0$,

or, equivalently:

(A28) $\left[B(\theta + \phi)\right] u > 0$.

With our parameter values (and, more precisely, with $B > 0$, $\theta > 1$ and $1 + \phi > 0$), equation (A28) is always met as long as u is positive. Hence, when $u \in (0;1)$, which we assume in the paper, the two transversality conditions are certainly satisfied along the BGP equilibrium.