Non Hermitian Spectra and Anderson Localization

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Abstract. The spectrum of exponents of the transfer matrix provides the localization lengths of Anderson's model for a particle in a lattice with disordered potential. I show that a duality identity for determinants and Jensen's identity for subharmonic functions, give a formula for the spectrum in terms of eigenvalues of the Hamiltonian with non-Hermitian boundary conditions. The formula is exact; it involves an average over a Bloch phase, rather than disorder. A preliminary investigation of non-Hermitian spectra of Anderson's model in D=1,2 and on the smallest exponent is presented.

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1. Introduction

Several models in physics are described by matrices with banded or block-tridiagonal structure. Examples are the Laplacian matrix, the Anderson Hamiltonian for transport in a lattice with random impurities, band random matrices, tight binding models in condensed matter and chemistry. The matrix structure reproduces that of a system consisting of a chain of units with same number of internal states, with nearest neighbors interaction. Finite size effects are often dealt with by imposing periodicity; the limit of large number of units is eventually taken.

The matrix structure calls for a transfer matrix description of the eigenstates, and the spectrum of exponents of the transfer matrix describes the decay lengths of the eigenstates. For the Anderson model or band random matrices, most of the knowledge on Lyapunov spectra relies on numerical computations.

With great generality, I showed that an analytic tool to access the decay lengths is a *duality relation*, that connects the spectrum of the transfer matrix of a block tridiagonal Hermitian matrix with the spectrum of a non-Hermitian extension of it[1, 2, 3, 4]. The extension arises by mere generalization of boundary conditions for the eigenstates. If $\vec{u}_k$ specifies the state of a unit of the chain ($k = 1, \ldots, n$), the boundary conditions (b.c.) are parametrized by a complex number z:

$$\vec{u}_{n+1} = z^n \vec{u}_1, \quad \vec{u}_0 = \frac{1}{z^n} \vec{u}_n. \tag{1}$$

This destroys Hermiticity of the Hamiltonian matrix, but enlights a nice property of the transfer matrix:

$$T(\epsilon) \begin{bmatrix} \vec{u}_1 \\ \frac{1}{z^n} \vec{u}_n \end{bmatrix} = z^n \begin{bmatrix} \vec{u}_1 \\ \frac{1}{z^n} \vec{u}_n \end{bmatrix}. \tag{2}$$

The ensuing spectral duality and Jensen's identity for subharmonic functions allow to evaluate the counting function of exponents. This paper is intended to introduce the theory and explore its application to the long-studied problem of Anderson's localization. In section II the duality relation is reviewed and the main formula (12) for the exponents is obtained from Jensen's theorem. The theory can be extended to include the spectrum of the (Lyapunov) exponents of the matrix $\mathrm{T}^\dagger\mathrm{T}$, by constructing a corresponding non-Hermitian block tridiagonal matrix, twice the size of the original Hamiltonian matrix. In section III a preliminary study of the eigenvalues of non Hermitian Hamiltonian matrices in D=1 and D=2 is made, with the purpose of illustrating the duality. The spectral formula is used to evaluate the smallest exponent ξ_{min}.

Because of their relevance in mathematics, numerical analysis and physics, block tridiagonal matrices are an active area of research[5, 6, 7]. This work extends in a new perspective the work by Hatano and Nelson[8] which, together with the works by Feinberg and Zee[9], started an interest for non-Hermitian matrix models in physics.

2. Theory

2.1. Transfer Matrix

Consider the following block tridiagonal matrix with corners, of size $nm \times nm$,

$$H = \begin{bmatrix} A_1 & B_1 & & B_n^\dagger \\ B_1^\dagger & \ddots & \ddots & \\ & \ddots & \ddots & B_{n-1} \\ B_n & & B_{n-1}^\dagger & A_n \end{bmatrix}. \tag{3}$$

The blocks have size $m \times m$: B_k are complex matrices with $\det B_k \neq 0$, A_k are Hermitian matrices. To the matrix $\epsilon I_{nm} - H$ there corresponds the *transfer matrix*[1]

$$T(\epsilon) = \begin{bmatrix} B_n^{-1}(\epsilon I_m - A_n) & -B_n^{-1}B_{n-1}^\dagger \\ I_m & 0 \end{bmatrix} \cdots \begin{bmatrix} B_1^{-1}(\epsilon I_m - A_1) & -B_1^{-1}B_n^\dagger \\ I_m & 0 \end{bmatrix}. \tag{4}$$

I_m is the $m \times m$ identity matrix. The transfer matrix is so named because it transforms the eigenvalue equation $Hu = \epsilon u$ into a relation for the end-components of the vector $u = (\vec{u}_1, \dots, \vec{u}_n)^t$:

$$T(\epsilon) \begin{bmatrix} \vec{u}_1 \\ \vec{u}_0 \end{bmatrix} = \begin{bmatrix} \vec{u}_{n+1} \\ \vec{u}_n \end{bmatrix}. \tag{5}$$

The corners in the matrix (3) imply a condition of periodicity $\vec{u}_0 = \vec{u}_n$ and $\vec{u}_{n+1} = \vec{u}_1$, that can be used to obtain the eigenvalue ϵ in alternative to diagonalization of H. By comparing eqs.(2) and (5) one arrives at the main point: *to study the spectrum of $T(\epsilon)$, one must impose the generalized b.c.* (1). We thus introduce an instrumental non-Hermitian matrix depending on a parameter z $(0 \leq \arg z \leq \frac{2\pi}{n})$

$$H(z^n) = \begin{bmatrix} A_1 & B_1 & & \frac{1}{z^n} B_n^\dagger \\ B_1^\dagger & \ddots & \ddots & \\ & \ddots & \ddots & B_{n-1} \\ z^n B_n & & B_{n-1}^\dagger & A_n \end{bmatrix}. \tag{6}$$

The matrix is Hermitian for Bloch b.c. ($|z| = 1$) but, for the purpose of studying the spectrum of $T(\epsilon)$, it will be considered for $z \in C_0$. The matrix can be brought by similarity to the balanced form $H_b(z) = Z^{-1} H(z^n) Z$,

$$H_b(z) = \begin{bmatrix} A_1 & zB_1 & & \frac{1}{z} B_n^\dagger \\ \frac{1}{z} B_1^\dagger & \ddots & \ddots & \\ & \ddots & \ddots & zB_{n-1} \\ zB_n & & \frac{1}{z} B_{n-1}^\dagger & A_n \end{bmatrix}, \tag{7}$$

by means of the block diagonal matrix Z with blocks $\{zI_m, \dots, z^n I_m\}$. Therefore, no site of the chain is privileged. While the matrix $H_b(z)$ does change if arg z is increased by $2\pi/n$, its eigenvalues do not.

2.2. Symplectic properties and exponents

The following relations hold for the transfer matrix:

$$
\begin{aligned}
&T(\epsilon^*)^\dagger \Sigma_n T(\epsilon) = \Sigma_n, \qquad \Sigma_n = \begin{bmatrix} 0 & -B_n^\dagger \\ B_n & 0 \end{bmatrix} \\
&T(\epsilon)\Sigma_n^{-1} T(\epsilon^*)^\dagger = \Sigma_n^{-1}, \qquad \Sigma_n^{-1} = \begin{bmatrix} 0 & B_n^{-1} \\ -B_n^{\dagger\,-1} & 0 \end{bmatrix}
\end{aligned} \tag{8}
$$

Let us denote as $z_1^n \dots z_{2m}^n$ the $2m$ eigenvalues of $T(\epsilon)$. The relations imply that if z_a^n is an eigenvalue of $T(\epsilon)$, then $(z_a^{-n})^*$ is an eigenvalue of $T(\epsilon^*)$. In this study we are concerned with the *exponents*

$$\xi_a(\epsilon) = \ln |z_a|. \tag{9}$$

Since $|\det T(\epsilon)| = 1$, it is always $\sum_a \xi_a(\epsilon) = 0$. For real ϵ the exponents of $T(\epsilon)$ come in pairs $\pm\xi_a$.

2.3. Duality, Jensen, and spectrum of exponents

Since the extremal components $\vec{u}_1$ and $\vec{u}_n$ of the eigenvector $H(z^n)u = \epsilon u$ enter in the eigenvalue equation (2) of $T(\epsilon)$, it follows that the characteristic polynomials of the two matrices are linked by a

Duality relation. *ϵ is an eigenvalue of $H(z^n)$ iff z^n is eigenvalue of $T(\epsilon)$:*

$$\frac{\det[\epsilon I_{nm} - H(z^n)]}{\det[B_1 \cdots B_n]} = \frac{(-1)^m}{z^{nm}} \det[T(\epsilon) - z^n I_{2m}]. \tag{10}$$

A proof of duality that holds also for non-Hermitian matrices, with blocks $B_k^\dagger$ being replaced by blocks C_k, is found in ref.[4].

The spectrum of exponents can be obtained from the spectrum of $H(z^n)$ through the following identity for analytic functions, which is a particular case of a theorem by Poisson and Jensen for subharmonic functions[10]:

Jensen's identity. *Let f be an analytic function in the open disk of radius R, where it has zeros $z_1, \dots, z_k$ that are ordered according to increasing modulus. Then, if $0 < |z_1|$ and for r such that $|z_\ell| \le r \le |z_{\ell+1}|$ we have:*

$$\int_0^{2\pi} \frac{d\varphi}{2\pi} \ln |f(re^{i\varphi})| = \ln \frac{r^\ell |f(0)|}{|z_1 \cdots z_\ell|}. \tag{11}$$

Proposition. *For real ξ and complex ϵ it is*

$$
\begin{aligned}
\frac{1}{m} \sum_{\xi_a < \xi} [\xi - \xi_a(\epsilon)] - \xi \;=\; &-\frac{1}{nm} \sum_{k=1}^{n} \ln |\det B_k| \\
&+ \int_0^{2\pi} \frac{d\varphi}{2\pi} \frac{1}{nm} \ln |\det[\epsilon I_{nm} - H(e^{n\xi + i\varphi})]|.
\end{aligned} \tag{12}
$$

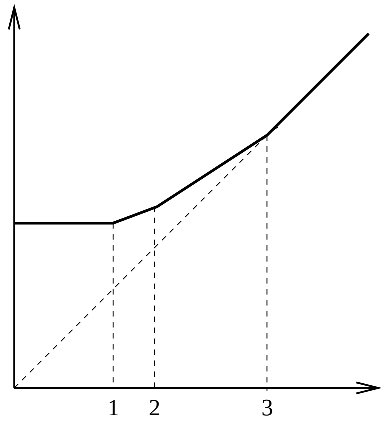

Figure 1. The behaviour of the right hand side of (12) as a function of ξ, for $m = 3$. The constant value for $\xi < \xi_1 = \xi_{min}$ is $\frac{1}{3}(\xi_1 + \xi_2 + \xi_3)$. At $\xi = \xi_1, \xi_2$ and $\xi_3 = \xi_{MAX}$ the slope increases by 1/3.

Proof: Jensen's identity is applied to the polynomial $f(z) = \det[T(\epsilon) - z^n I_{2m}]$, with $|f(0)| = 1$ and $z = \exp(\xi + i\varphi/n)$. The duality relation is then used to obtain the formula.

For $\xi = 0$ a formula for the sum of positive exponents follows. It involves eigenvalue spectra of Hermitian matrices $H(e^{i\varphi})$,

$$\frac{1}{m}\sum_{\xi_a>0}\xi_a(\epsilon) = -\frac{1}{nm}\sum_k \ln|\det B_k| + \int_0^{2\pi}\frac{d\varphi}{2\pi}\frac{1}{nm}\ln|\det[\epsilon I_{nm} - H(e^{i\varphi})]|. \tag{13}$$

Equations (12) and (13) are exact and valid for a single, general transfer matrix. If the parent block tridiagonal matrix is sampled from an ensemble of random matrices, with a probability measure, and the large n limit is taken, it is natural to ask if the exponents given by eq.(12) exist and are sample independent. What is then their relationship with the Lyapunov exponents of random matrix products? Do they scale in the transverse dimension m? These are difficult questions and are not answered here. A preliminary study of the spectra of tridiagonal block matrices with non Hermitian boundary conditions will be presented here, as they are connected by duality to the spectra of exponents.

In the theory of disordered systems, a formula for the sum of exponents is known, where Jensen's angular average is replaced by the ensemble average[11, 12, 13]. No general formula is known for the counting function of the exponents in random multiplicative matrix theory.

The left-hand side of (12) is a non decreasing function of ξ (Fig.1). For all $\xi \geq \xi_{MAX}(\epsilon)$ (the maximum exponent of $T(\epsilon)$), the right side is always equal to ξ. For all positive $\xi < \xi_{min}(\epsilon)$ (the smallest positive exponent), the right hand side is constant and equal to the average value of the exponents (13). For intermediate positive values of ξ the function is piecewise linear, with discontinuities of order $1/m$ in the first derivative, at the values of the exponents.

2.4. The matrix $T^\dagger T$

Let us introduce the matrix $Q(\epsilon) = T(\epsilon^*)^\dagger T(\epsilon)$, with exponents $\gamma_a(\epsilon)$. For real ϵ the matrix is real and positive, and is preferred to T because of better large n behaviour of the exponents. If $B_n B_n^\dagger = I_m$, the matrix Q is symplectic

$$Q(\epsilon)\Sigma_n Q(\epsilon) = \Sigma_n \tag{14}$$

and the exponents come in pairs $\pm\gamma_a$. Hereafter, the matrix B_n will be restricted to be unitary. Under this restriction, it is shown in Appendix A that $(-1)^n Q(\epsilon)$ is unitarily equivalent to the transfer matrix of a block tridiagonal matrix, which is transformed to obtain the duality relation:

$$\frac{\det[K(z^{2n}) - \epsilon I_{2nm}]}{\prod_k |\det B_k|^2} = \frac{(-1)^m}{z^{2nm}} \det[Q(\epsilon) - z^{2n} I_{2m}], \tag{15}$$

$$K(z^{2n}) = \begin{bmatrix} A_1 & B_1 & & & & & & & z^{-2n} \\ B_1^\dagger & \ddots & \ddots & & & & & & \\ & \ddots & \ddots & B_{n-1} & & & & & \\ & & B_{n-1}^\dagger & A_n & -I_m & & & & \\ & & & I_m & A_n & B_{n-1}^\dagger & & & \\ & & & & B_{n-1} & \ddots & \ddots & & \\ & & & & & \ddots & \ddots & B_1^\dagger \\ -z^{2n} & & & & & & B_1 & A_1 \end{bmatrix} \tag{16}$$

Some properties of the matrix K are presented in the appendix. The spectrum of exponents $\{\pm\gamma_a(\epsilon)\}$ (the Lyapunov spectrum) is extracted by means of Jensen's formula:

$$\begin{aligned} \xi + \frac{1}{m}\sum_{\gamma_a>\xi}[\gamma_a(\epsilon) - \xi] = \frac{1}{nm}\sum_{k=1}^{n-1} \ln|\det B_k| \\ + \int_0^{2\pi} \frac{d\varphi}{2\pi}\frac{1}{2nm} \ln|\det[K(e^{2n\xi+i\varphi}) - \epsilon I_{2nm}]|. \end{aligned} \tag{17}$$

3. THE ANDERSON MODEL

The discrete Anderson model describes a particle in a lattice, subject to a random potential. The potential of a sample is specified by a set $\{v_\mathbf{j}\}$ of random numbers chosen independently. Anderson[14] considered a uniform density $p(v) = 1/w$ in the interval $[-w/2, w/2]$. Lloyd[15, 16] studied the Cauchy distribution $p(v) = \frac{\delta}{\pi}(v^2 + \delta^2)^{-1}$, and evaluated the energy distribution exactly in any space dimension. Anderson's choice and the simple hypercubic geometry are here considered. More complex lattices can be studied by transfer matrix[17].

For a given configuration of potential, the eigenvalue equation is

$$\sum_\mathbf{e} u_{\mathbf{j}+\mathbf{e}} + v_\mathbf{j} u_\mathbf{j} = \epsilon u_\mathbf{j}. \tag{18}$$

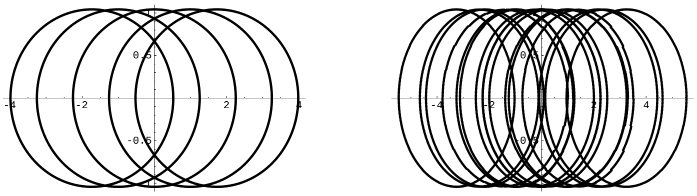

Figure 2. Complex energy spectra for zero disorder, large n, $\xi = 1$: in 2D (left, $m = 5$) and 3D (right, $n_x = n_y = 4$ i.e. $m = 16$).

The sum is on the unit vectors along the $2D$ directions, ϵ is the energy of the particle, the lattice has lengths $n_1, \dots, n_D$. If the D axis is singled out, the sample is viewed as a number $n \equiv n_D$ of sections each containing $m \equiv n_1 \cdots n_{D-1}$ sites. Accordingly, the Hamiltonian matrix is block tridiagonal

$$H = \begin{bmatrix} A_1 & I_m & & I_m \\ I_m & \ddots & \ddots & \\ & \ddots & \ddots & I_m \\ I_m & & I_m & A_n \end{bmatrix} \tag{19}$$

with Hermitian blocks A_i describing sections, and off diagonal blocks describing hopping among sections. The associated transfer matrix is

$$T(\epsilon) = \prod_{j=1}^{n} \begin{bmatrix} \epsilon I_m - A_j & -I_m \\ I_m & 0 \end{bmatrix}. \tag{20}$$

For large n the exponents of $T(\epsilon)$ describe the inverse decay lengths of the eigenstates of Anderson's Hamiltonian. To study them, we introduce b.c. terms $\pm z^n$ in the corner blocks of (19), and choose periodic b.c. in the other $D-1$ directions, that appear in the diagonal blocks.

Remark 0: For zero disorder the eigenvalues of $H(z^n)$ are complex for any nonzero value of the parameter ξ that measures non-Hermiticity ($z = \exp(\xi + i\varphi)$):

$$\mathrm{Re}\,\epsilon = 2\cosh\xi\cos(\varphi + \frac{2\pi}{n}\ell) + \epsilon_r,$$
$$\mathrm{Im}\,\epsilon = 2\sinh\xi\sin(\varphi + \frac{2\pi}{n}\ell),$$

$\ell = 1, \dots, n$ and $r = 1, \dots, m$. There are n eigenvalues on each ellipse centered at $\epsilon_r = 2\sum_{i=1}^{D-1}\cos(2\pi\frac{k_i}{n_i})$, $1 \le k_i \le n_i$. Therefore, the spectrum has support on m identical but shifted ellipses. In $1D$ there is a single ellipse centered in the origin. In $2D$ there are $m = n_x$ distinct ones, while in $3D$ some of the $m = n_x n_y$ ellipses may overlap because centers may be degenerate (Fig.2). In Appendix B it is shown that the exponents of T and $T^\dagger T$ coincide, for large n.

Remark 1: For non-zero disorder the eigenvalues of the Hamiltonian matrix $H(z^n)$ are all contained inside the union of ellipses

$$\frac{(\mathrm{Re}\,\epsilon - \epsilon_0)^2}{4\cosh^2\xi} + \frac{(\mathrm{Im}\,\epsilon)^2}{4\sinh^2\xi} \le 1, \tag{21}$$

where ϵ_0 ranges in the interval $[-2D+2-w/2, 2D-2+w/2]$. Proof: If $H_b(z)u = \epsilon u$, and u is normalized, the inner product $\epsilon = (u|H_b(z)u)$ in C^{nm} is separated into real and imaginary parts:

$$\mathrm{Re}\,\epsilon - (u|Au) = 2\,|(u|Su)|\,\cosh\xi\,\cos(\varphi+\theta),$$
$$\mathrm{Im}\,\epsilon = 2\,|(u|Su)|\,\sinh\xi\,\sin(\varphi+\theta).$$

A is the block diagonal part of H_b, S is the periodic one-block shift matrix ($S\vec{u}_k = \vec{u}_{k+1}$), and θ =arg$(u|Su)$. The real number $(u|Au) = \epsilon_0$ ranges in the spectrum of A. Schwartz's inequality gives the bounds.

Remark 2: Since H is real, under complex conjugation it is $T(\epsilon)^* = T(\epsilon^*)$. Then $\xi_a(\epsilon) = \xi_a(\epsilon^*)$. The symplectic property (8) with $B_n = I_m$ implies that the exponents of $T(\epsilon)$ come in pairs $\pm\xi_a$ for any ϵ.

Remark 3: Since the transposed matrix $H(z^n)^t$ coincides with $H(z^{-n})$, then $\det[\epsilon I_{nm} - H(z^n)]$ is a polynomial of degree m of the variable $(z^n + z^{-n})$.

Remark 4: Since $H_b(ze^{i2\pi/n}) \simeq H_b(z)$ ($\simeq$ means similarity) and $H_b(z)^* = H_b(z^*)$, the following symmetry holds: $H_b(e^{\xi+i(\frac{2\pi}{n}-\varphi)}) \simeq H_b(e^{\xi+i\varphi})^*$.

3.1. The Lyapunov spectrum

The localization properties of Anderson's model are usually derived from the spectrum of positive Lyapunov exponents $\gamma_1 < \ldots < \gamma_m$ of $T(\epsilon)^\dagger T(\epsilon)$, with ϵ real. Oseledec's theorem[18] guarantees that for $n \to \infty$ it *does not depend on the realization of disorder.* The most interesting exponent for physics is γ_1, that controls conductance. It is also the most difficult one to study numerically, because of the larger ones[19, 20]. Thorough investigations of the Lyapunov spectrum, its statistical properties and scaling, have been done in 2D[21] and 3D[22]. The influence of b.c. was studied[23] with the corner parameters $z^{\pm n}$ of the present theory being both replaced by the same parameter $t \in [0,1]$. It was found that the critical values of γ_1 and of the disorder parameter w_c are $t-$dependent, while the critical exponent ν is not.

Analytic results for the Lyapunov spectrum are accessible in perturbation theory for the 2D strip[24, 25, 26], where n is large and m is finite. In such quasi-1D Anderson systems, *the large n limit of the exponent spectrum of T coincides with the Lyapunov spectrum.* For this reason here I concentrate on the spectral features of the matrix $H(z^n)$ to which T is linked by duality. I'll then show in another section that the spectral identity (12) allows to evaluate the smallest exponent ξ_{min}, which converges to γ_1 for large n.

3.2. 1D Anderson model, Hatano and Nelson.

For a chain of n sites the Hamiltonian is a tridiagonal $n \times n$ matrix and the transfer matrix is 2×2. Hatano and Nelson[8] suggested to study 1D Anderson localization through the non-Hermitian extension of the model

$$e^{\xi}\psi_{i+1} + e^{-\xi}\psi_{i-1} + v_i\psi_i = \epsilon\psi_i \tag{22}$$

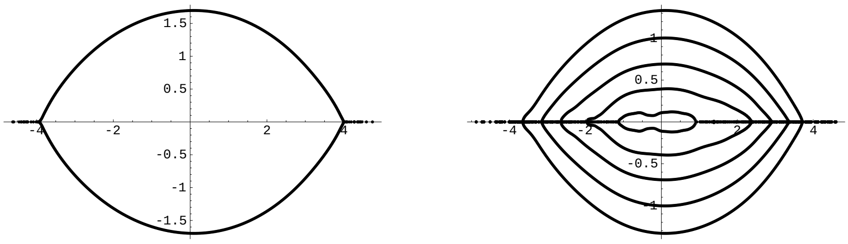

Figure 3. 1D Anderson model: Left: the complex eigenvalues form a closed loop, with two wings of real values ($w = 7$, $n = 600$, $\xi = 1$). Right: superposition of the eigenvalues of five matrices (same w and n) for ξ from 0.5 (inner blob) to 1 (outer).

with periodic b.c. As ξ is increased from zero, the eigenvalues do not distribute randomly in the complex plane but form a loop, Fig.3 (left), whose analytic expression is known for Cauchy disorder in the large n limit[28]. The loop has two outer wings of real eigenvalues, that correspond to enough localized eigenstates, and evolves to a more and more regular shape, while the wings reduce, Fig.3 (right). The value $\xi_c(\epsilon)$ up to which an eigenvalue ϵ persists in the real axis, measures the inverse localization length of the physical eigenvector (the Lyapunov exponent): $\xi_c = \gamma_1(\epsilon)$. The latter is evaluated through Herbert, Jones and Thouless' formula,

$$\gamma_1(\epsilon) = \int d\epsilon' \rho(\epsilon') \ln|\epsilon - \epsilon'|, \tag{23}$$

where $\rho(\epsilon)$ is the disorder-averaged level density of the matrix ensemble in the limit of large n, $\xi = 0$. The model has been studied by several authors[30, 31, 32, 33, 34]; mathematical proofs were established by Goldsheid and Khoruzhenko[35]. Numerical diagonalization of large non-Hermitian matrices is a delicate issue, as approximate eigenvalues may occur which are not close to true ones[36].

The Hatano-Nelson model is a case $m = 1$ of the theory presented in Section II. The duality relation (10) simplifies greatly:

$$\det[\epsilon I_n - H(z^n)] = \mathrm{tr}\, T(\epsilon) - (z^n + z^{-n}) \tag{24}$$

and implies that

$$\mathrm{tr}\, T(\epsilon) = \det[\epsilon I_n - H(i)] \equiv p_n(\epsilon). \tag{25}$$

For a pure Bloch phase (24) describes the energy bands of $H(e^{in\varphi})$ as intersections of the polynomial $y = p_n(\epsilon)$ with the strip $y = 2\cos(n\varphi)$. As the non-Hermitian regime is entered, $y = \pm 2\cosh(n\xi)$, all eigenvalues of $H(e^{n\xi})$ are in the gaps, and approach pairwise for increasing ξ. A pair collides at a zero of p_n' and becomes complex conjugate. This means that $2\cosh(n\xi)$ equals the height $|p_n|$ at an extremum of the polynomial.

The sum of the two eigenvalues $\exp[\pm n(\xi_1 + i\varphi_1)]$ of the transfer matrix is

$$p_n(\epsilon) = 2\cosh(n\xi_1)\cos(n\varphi_1) + 2i\sinh(n\xi_1)\sin(n\varphi_1). \tag{26}$$

Elimination of the phase results in an exact equation for the exponent $\xi_1(\epsilon)$

$$\frac{(\mathrm{Re}\, p_n)^2}{4\cosh^2(n\xi_1)} + \frac{(\mathrm{Im}\, p_n)^2}{4\sinh^2(n\xi_1)} = 1. \tag{27}$$

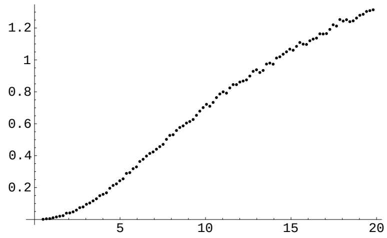

Figure 4. 1D Anderson model: the exponent $\xi_1(0)$ versus disorder parameter w, for $n = 400$, averaged over 15 samples of disorder.

For large n it becomes $\exp[n\xi_1(\epsilon)] = |p_n(\epsilon)|$, and gives a convenient formula to compute the exponent (Fig.4).

$$\xi_1(\epsilon) = \frac{1}{n}\ln|\det[\epsilon I_n - H(i)]|. \tag{28}$$

For the 1D Anderson model, in the limit $n \to \infty$, the exponent ξ_1 becomes independent of disorder sampling and is the Lyapunov exponent. Eq.(28) becomes the Thouless formula for $\gamma_1(\epsilon)$.

Proposition: the eigenvalues of $H(e^{n\xi})$ distribute along the curve $\xi_1(\epsilon) = \xi$. Real eigenvalues (wings) solve $p_n(\epsilon) = 2\cosh(n\xi)$. For large $n\xi$ the eigenvalues form the *Lemniscate* [27] $|p_n(\epsilon)| = \exp(n\xi)$.

3.3. 2D Anderson model

For a rectangular $n \times m$ lattice the Hamiltonian matrix (19) has diagonal blocks

$$A_i = \begin{bmatrix} v_{i,1} & 1 & & 1 \\ 1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ 1 & & 1 & v_{i,m} \end{bmatrix}.$$

The eigenvalue spectrum of 2D non-Hermitian Anderson model is studied, and explained in the light of duality. Fig.5 shows the eigenvalues of two matrices $H(e^{n\xi})$ with same ξ and different m. They are distributed along a number of loops which is precisely given by m, the size of the blocks.

By varying only the phase φ of z, the eigenvalues of the matrix $H_b(z)$ move in the complex plane along arcs which retrace the loops. Fig.6 shows that, as φ goes from 0 to $2\pi/n$, an eigenvalue moves along an arc that terminates where the arc of another eigenvalue starts. The union of such consecutive arcs makes a loop, and there are m closed loops. In the limit of large n, the eigenvalues of one matrix fill the loops. Differently from the $w = 0$ case, loops may contain different numbers of eigenvalues of the matrix. The occurrence of loops is suggested by the duality equation: when a zero of $\det[H_b(z) - \epsilon I_{nm}]$ occurs, it is also a zero of $\det[T(\epsilon) - z^n]$, i.e.

$$\xi_a(\epsilon) = \xi, \quad \varphi_a(\epsilon) = \varphi \mod \frac{2\pi}{n}, \quad (a = 1, \ldots, m). \tag{29}$$

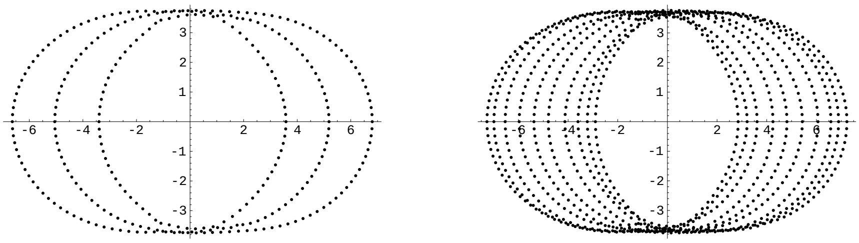

Figure 5. 2D Anderson model: eigenvalues of a single matrix, with parameters w=7, n=100, $\xi = 1.5$, $\varphi = 0$. Size of blocks: $m = 3$ and $m = 10$. The size m of blocks is the number of loops.

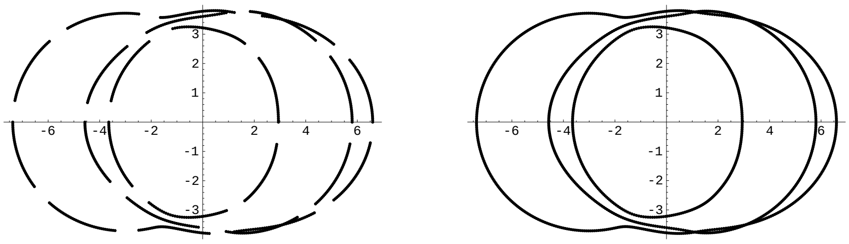

Figure 6. 2D Anderson model: motion of the eigenvalues in the complex plane for fixed disorder w=7 and parameters m=3, n=8, and $\xi = 1.5$ and varying the phase φ. The 24 eigenvalues trace arcs: $0 \le \varphi \le \pi/4 - \delta$ (left) and $0 \le \varphi \le \pi/4$ (right). The arcs join to form three loops. Loops are seen to contain different numbers of eigenvalues.

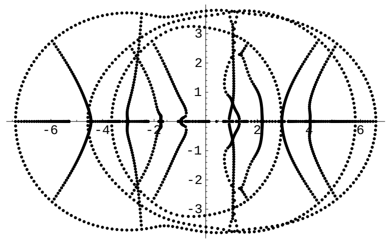

Figure 7. 2D Anderson model: motion of 24 eigenvalues for $0 \le \xi \le 1.5$, w=7, $m = 3$, n=8, $\varphi = 0$. The various wings terminate on $m = 3$ loops (at $\xi = 1.5$ the phase is allowed to vary over 2π.)

Loops are thus *level curves* of the exponents ξ_a, as functions of the complex variable ϵ.

In Fig.7 only ξ is varied: the eigenvalues trace lines that originate on the real axis (at $\xi = 0$ the matrix is Hermitian). For zero disorder the lines would be arcs of hyperbola. In the disordered case, for small ξ the pattern of eigenvalues is complex, but evolves to regular loops. In Fig.8 the parameters ξ and φ are kept fixed, and the eigenvalues are computed for different realizations of disorder, with same strength w. They distribute along m loops, that appear shifted along the real axis for the different samples.

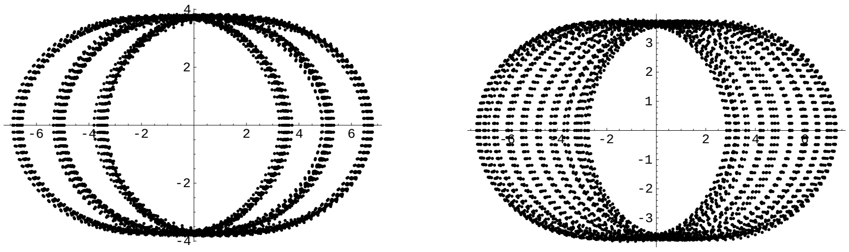

Figure 8. 2D Anderson model: superposition of eigenvalue spectra for various realizations of disorder. For all: w=7, $n = 100$, $\xi = 1.5$, $\varphi = 0$. Left: m=3, 20 realizations of disorder. Right: m=10, 5 realizations. The imaginary part of the eigenvalues is much less sensitive to disorder sampling than the real part.

3.4. The smallest exponent

Kuwae and Taniguchi[37] extended Hatano and Nelson's approach from 1D to 2D, and evaluated numerically the sample-average of the critical value ξ_c where the first pair of eigenvalues of $H(e^{n\xi})$ becomes a complex conjugate pair. They conjectured that the inverse of the localization length coincides with this critical value, i.e. $\gamma_1(0) = \xi_c$.

The spectral formula (12) allows to study the exponents numerically, at least for small m. Let us assume that the exponents $\pm\xi_a$ of $T(\epsilon)$ are isolated, $0 < \xi_{min} < \xi_2 \ldots < \xi_{m-1} < \xi_{MAX}$. By increasing the boundary parameter ξ in the Hamiltonian from the value zero, the r.h.s. of (12) yields a constant value (the value at $\xi = 0$) until the value $\xi = \xi_{min}(\epsilon)$ is reached. Then the function becomes linear with slope $1/m$ until the value $\xi = \xi_2(\epsilon)$ is reached, where a new change of slope occurs. The first change of slope can be used to identify the smallest exponent. Fig.9 illustrates the changes of slope that occur in the numerical evaluation of the r.h.s. of (12); compare it with Fig.1. The numerical data allow to read the values ξ_i and the average value of the positive exponents at a given energy ϵ. For the given example, the eigenvalues of $H(e^{n\xi})$ are evaluated for ξ close to ξ_1 and ξ_3 and the angle φ is varied. Fig.10 (left) shows that most of them are in the complex plane, and at $\xi = \xi_{max}$ the pattern of eigenvalues is regular (right). Then ξ_{min} is not precisely the value at which the first pair of eigenvalues gains imaginary parts. This issue requires further study.

4. Conclusions

Based on a spectral duality relation for block tridiagonal matrices and Jensen's identity, the distribution of exponents of a transfer matrix can be evaluated from the eigenvalue spectrum of the Hamiltonian with non-Hermitian boundary conditions. A preliminary numerical study of the complex energy spectra of Anderson Hamiltonian matrices is made. The spectra have support on loops, that are explained as sections of the exponents at fixed height: $\xi_a(\epsilon) = \xi$. This picture is complementary to the standard direct evaluation of the exponents $\xi_a(\epsilon)$ at fixed energy ϵ, by diagonalization of the transfer matrix. The study of the changes in the topology of the loops as ξ crosses the values

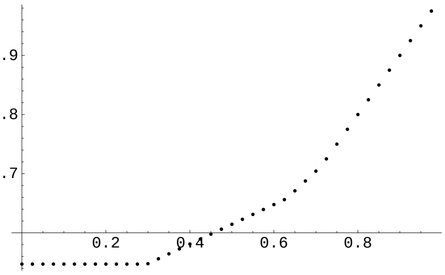

Figure 9. 2D Anderson model: evaluation of the r.h.s. of (12) as a function of ξ for $\epsilon = 0$, $n = 50$, $m = 3$, $w = 7$, average on 40 angles. The constant value is $\frac{1}{3}(\xi_1 + \xi_2 + \xi_3) = 0.547121$. The changes of slope identify $\xi_1 = 0.27$, $\xi_2 = 0.62$ and $\xi_3 = 0.72$.

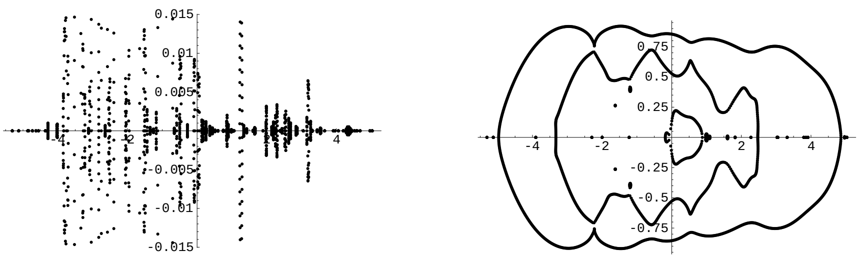

Figure 10. 2D Anderson model: Eigenvalues for $\epsilon = 0$, $w = 7$, $m = 3$, $n = 50$. Left: $\xi = 0.27$ ($\approx \xi_{min}$) Right: $\xi = 0.72$ ($\approx \xi_{max}$). The angle φ is varied to produce the loop structures.

ξ_a, is an interesting subject for further investigation.

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Appendix A

Step 1: $(-1)^n Q(\epsilon)$ is unitarily equivalent to the transfer matrix of a block tridiagonal matrix.
Proof: It is convenient to factor $T(\epsilon)$ as

$$\begin{bmatrix} B_n^{-1} & 0 \\ 0 & I_m \end{bmatrix} t_n \sigma_{n-1} \cdots \sigma_1 t_1 \begin{bmatrix} I_m & 0 \\ 0 & B_n^\dagger \end{bmatrix}. \tag{A.1}$$

Accordingly:

$$Q(\epsilon) = (-1)^n \begin{bmatrix} I_m & 0 \\ 0 & B_n \end{bmatrix} u_1 \sigma_1^\dagger \cdots \sigma_{n-1}^\dagger u_n \begin{bmatrix} (B_n B_n^\dagger)^{-1} & 0 \\ 0 & I_m \end{bmatrix} t_n \sigma_{n-1} \cdots \sigma_1 t_1 \begin{bmatrix} I_m & 0 \\ 0 & B_n^\dagger \end{bmatrix},$$

$$t_k = \begin{bmatrix} \epsilon I_m - A_k & -I_m \\ I_m & 0 \end{bmatrix}, \quad u_k = \begin{bmatrix} A_k - \epsilon I_m & -I_m \\ I_m & 0 \end{bmatrix}, \quad \sigma_k = \begin{bmatrix} B_k^{-1} & 0 \\ 0 & B_k^\dagger \end{bmatrix}.$$

To obtain the structure (A.1) of a transfer matrix, it is necessary that $B_n^\dagger B_n = I_m$. The first and last factors containing B_n are not consistent with (A.1), and only the intermediate product $\theta(\epsilon) = u_1\sigma_1^\dagger \cdots \sigma_1 t_1$ is the transfer matrix of a tridiagonal block matrix. Therefore:

$$(-1)^n Q(\epsilon) = \begin{bmatrix} I_m & 0 \\ 0 & B_n \end{bmatrix} \theta(\epsilon) \begin{bmatrix} I_m & 0 \\ 0 & B_n^\dagger \end{bmatrix}. \tag{A.2}$$

Step 2: $\theta(\epsilon)$ is the transfer matrix of the matrix

$$M(\epsilon) = \begin{bmatrix} A_1-\epsilon & B_1 & & & & & & & I_m \\ B_1^\dagger & \ddots & \ddots & & & & & & \\ & \ddots & \ddots & B_{n-1} & & & & & \\ & & B_{n-1}^\dagger & A_n-\epsilon & I_m & & & & \\ & & & I_m & \epsilon - A_n & B_{n-1}^\dagger & & & \\ & & & & B_{n-1} & \ddots & \ddots & & \\ & & & & & \ddots & \ddots & B_1^\dagger & \\ I_m & & & & & & B_1 & \epsilon - A_1 & \end{bmatrix}.$$

Step 3: Factors $z^{\pm 2n}$ are introduced in the corners as in (6) and a duality relation is obtained,

$$\frac{\det M(\epsilon, z^{2n})}{\prod_k |\det B_k|^2} = \frac{(-1)^m}{z^{2nm}} \det[Q(\epsilon) - (iz)^{2n} I_{2m}]. \tag{A.3}$$

Step 4: Since only the determinant matters, there is freedom to modify M to a form where ϵ enters as a shift. Left and right multiplication of M by the block diagonal matrices $\{I_{nm}, I_m, -I_m, +I_m, \dots\}$ and $\{I_{nm}, -I_m, I_m, -I_m, \dots\}$ give $\det M = (-1)^{nm}\det[K((iz)^{2n}) - \epsilon I_{2nm}]$, where $K(z^{2n})$ is shown in (16).

One can show the following properties of the matrix $K(s)$. J is the matrix with $2n$ blocks I_m along the diagonal from lower left to upper right corners; S_3 is the block diagonal matrix $\{I_{nm}, -I_{nm}\}$. Then:

$$JK(s)J = K(s^*)^\dagger, \tag{A.4}$$

$$S_3 K(s) S_3 = K(1/s^*). \tag{A.5}$$

The two imply that $K(s)$ is similar to $K(1/s)$.

Appendix B

For the Anderson model with no disorder ($w = 0$), real energy ϵ and large n, the exponents of T and $T^\dagger T$ coincide.

Proof: It is

$$T = \begin{bmatrix} \epsilon I_m - A & -I_m \\ I_m & 0 \end{bmatrix}^n \tag{A.6}$$

$$= \begin{bmatrix} U & 0 \\ 0 & U \end{bmatrix} \begin{bmatrix} \epsilon I_m - \Lambda & -I_m \\ I_m & 0 \end{bmatrix}^n \begin{bmatrix} U^\dagger & 0 \\ 0 & U^\dagger \end{bmatrix}$$

where $A = U\Lambda U^\dagger$ and Λ is the diagonal matrix of eigenvalues $\{\lambda_1, \ldots, \lambda_m\}$. The eigenvalues of T are m pairs $z_k^{\pm n}$, where z_k is a root of the equation $z_k^2 - (\epsilon - \lambda_k) z_k + 1 = 0$.

The power n of the matrix can be computed by means of Cayley-Hamilton's formula. Because the blocks are diagonal, only powers zero and one of the matrix are needed:

$$\begin{bmatrix} \epsilon I_m - \Lambda & -I_m \\ I_m & 0 \end{bmatrix}^n = \begin{bmatrix} \alpha & 0 \\ 0 & \alpha \end{bmatrix} + \begin{bmatrix} \beta & 0 \\ 0 & \beta \end{bmatrix} \begin{bmatrix} \epsilon I_m - \Lambda & -I_m \\ I_m & 0 \end{bmatrix}$$

α and β are diagonal matrices with elements constructed with the roots z_k: $z_k^{\pm n} = \alpha_k + \beta_k z_k^{\pm 1}$. Since ϵ is real, α_k and β_k are real. The matrix $T^\dagger T$ is then constructed, and diagonalized. Its eigenvalues are pairs $w_k^{\pm 1}$, with sum

$$w_k + w_k^{-1} = (z_k^{2n} + z_k^{-2n}) \left(\frac{z_k^2 + 1}{z_k^2 - 1} \right)^2 - \frac{8 z_k^2}{(z_k^2 - 1)^2} \tag{A.7}$$

If $|z_k| > 1$ then, for large n, $|w_k| \approx |z_k|^{2n}$ (the spectrum of Lyapunov exponents of $T^\dagger T$ and the spectrum of exponents of T coincide).

References

[1] Molinari L 1997 *J. Phys. A: Math. Gen.* **30** 983–97
[2] Molinari L 1998 *J. Phys. A: Math. Gen.* **31** 8553–62
[3] Molinari L 2003 *J. Phys. A: Math. Gen.* **36** 4081–90
[4] Molinari L G 2008 *Linear Algebra and its Applications* **429** 2221-26, *Preprint* arXiv:0712.0681
[5] Nabben R 1999 *Siam J. Matrix Anal. Appl.* **20** 820–37
[6] Korotyaev E and Kutsenko A 2007 Lyapunov functions for periodic matrix-valued Jacobi operators *Preprint* math/0701448 [math.SP]
[7] Schulz-Baldes H 2008 Geometry of Weyl theory for Jacobi matrices with matrix entries *Preprint* arXiv:0804.3746 [math-ph]
[8] Hatano N and Nelson D R 1996 *Phys. Rev. Lett.* **77** 570
[9] J. Feinberg and A. Zee 1997 *Nucl. Phys.* B **504** 579-608
[10] Markushevich A I 1977 *Theory of Functions of Complex Variable* (New York: Chelsea Publishing Co. 2nd ed.)
[11] Lacroix J 1986 *Computation of the sum of positive Lyapunov exponents for the Lloyd model on a strip* (*Lect. Notes in Mathematics* 1186)
[12] Pastur L and Figotin A 1992 *Spectra of Random and Almost-Periodic Operators* (*Grundlehren der mathematischen Wissenschaften* 297) (Berlin: Springer-Verlag)
[13] Derrida B, Jacobsen J L and Zeitak R 2000 *J. Stat. Phys.* **98** 31–55
[14] Anderson P W 1958 *Phys. Rev.* **109** 1492
[15] Lloyd P 1969 *J. Phys. C: Solid State Phys.* **2** 1717
[16] Mudry C, Brouwer P W, Halperin B I, Gurarie V and Zee A 1998 *Phys. Rev.* B **58** 13539
[17] Eilmes A, Fisher A M and Römer R A 2008 Critical parameters for the disorder-induced metal-insulator transition in FCC and BCC lattices *Preprint* arXiv:0801.3338 [cond-mat.dis-nn]

[18] Crisanti A, Paladin G and Vulpiani A 1993 *Products of random matrices in statistical physics*, (*Springer Series in solid state sciences* 104) (Berlin: Springer-Verlag)
[19] Pichard J L and Sarma G 1981 *J. Phys. C: Solid State Phys.* **14** L127
[20] MacKinnon A and Kramer B 1981 *Phys. Rev. Lett.* **47** 1546
[21] Slevin K, Asada Y and Deych L I 2004 *Phys. Rev.* B **70** 054201
[22] Markoŝ P 1997 *J. Phys. A: Math. Gen.* **30** 3441
[23] Cerovski V Z 2007 *Phys. Rev.* B **75** 113101
[24] Rutman Y, Feingold M, Avishai Y and Piro O 1994, *Europhys. Lett.* **28** 329
[25] Rutman Y, Feingold M and Avishai Y 1996 *Phys. Rev.* B **53** 9634
[26] Römer R A and Schulz-Baldes H 2004 *Europhys. Lett.* **68** 247
[27] Hille E 1982 *Analytic Function Theory* (New York: Chelsea 2nd ed.)
[28] Goldsheid I and Khoruzhenko B 1998 *Phys. Rev. Lett.* **80** 2897
[29] Kolesnikov A V and Efetov K B 2000 *Phys. Rev. Lett.* **84** 5600
[30] Brouwer P W, Silvestrov P G and Beenakker C W J 1997 *Phys. Rev.* B **56** R4333–35
[31] Shnerb N M and Nelson D R 1998 *Phys. Rev. Lett.* **80** 5172
[32] Feinberg J and Zee A 1999 *Phys. Rev.* E **59** 6433
[33] Janik R A , Nowak M A , Papp G and Zahed I 1999 *Acta Phys. Polonica* B **30** 45
[34] Heinrichs J 2001 *Phys. Rev.* B **63** 165108
[35] Goldsheid I Y and Khoruzhenko B 2003 *Comm. Math. Phys.* **238** 505–24
[36] Davies E B 2001 *Proc. R. Soc.* A **457** 191–206
[37] Kuwae T and Taniguchi N 2001 *Phys. Rev.* B **64** 202321(R).