Injectivity and sections

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Abstract

We investigate injectivity in a comma-category $\mathbf{C}/B$ using the notion of the "object of sections" $S(f)$ of a given morphism $f : X \to B$ in $\mathbf{C}$. We first obtain that $f : X \to B$ is injective in $\mathbf{C}/B$ if and only if the morphism $\langle 1_X, f\rangle : X \to X \times B$ is a section in $\mathbf{C}/B$ and the object $S(f)$ of sections of f is injective in $\mathbf{C}$. Using this approach, we study injective objects f with respect to the class of embeddings in the categories $\mathbf{ContL}/B$ ($\mathbf{AlgL}/B$) of continuous (algebraic) lattices over B. As a result, we obtain both topological (every fiber of f has maximum and minimum elements and f is open and closed) and algebraic (f is a complete lattice homomorphism) characterizations.

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Introduction

The relevance in various fields of mathematics of the notion of injectivity is well known and injective objects, with respect to a class $\mathcal{H}$ of morphisms, have been investigated for a long time in different categories. For instance, in the category **Pos** of partial ordered sets and monotone mappings, injective objects, with respect to the class of regular monomorphisms, coincide with the complete lattices, while, in the category **SLat** of (meet) semilattices and semilattice homomorphisms, injective objects are precisely the locales

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(see [5]) and in the category **Boo** of boolean algebras and boolean homomorphisms, they coincide with complete boolean algebras. In the category $\mathbf{Top}_0$ of T_0 topological spaces and continuous functions, injective objects have an algebraic characterization, given by Scott in [11], as continuous lattices (viewed as topological spaces with the so-called Scott topology). Using a result of Wyler in [15], it turns out that, in the category **Top** of topological spaces, injective spaces are exactly those with a T_0-reflection given by a continuous lattice. Since any continuous lattice is then injective in **Top**, every object in the category **ContL** of continuous lattices and Scott-continuous functions (i.e. functions preserving directed sups, see e.g. [8]) is injective in **ContL** with respect to the class $\mathcal{H}$ of topological embeddings between continuous lattices. (The same happens for the category **AlgL** of algebraic lattices (see e.g. [8]) and Scott-continuous functions.)

Recently, new investigations on injective objects have been developed in comma-categories $\mathbf{C}/B$ (whose objects are $\mathcal{C}$-morphisms with fixed codomain B) (see [13], [14], [3], [6], [7]). "Sliced" injectivity is related to weak factorization systems, a concept used in homotopy theory, particularly for model categories. In fact, $\mathcal{H}$-injective objects in $\mathcal{C}/B$, for any B in $\mathcal{C}$, form the right part of a weak factorization system that has morphisms of $\mathcal{H}$ as the left part. So it may be useful to know the nature of $\mathcal{H}$-injectives in $\mathbf{C}/B$ and in this direction there are results in the category **Pos** (by Tholen, Adámek, Herrlich, Rosicky), for $\mathcal{H}$ given by the class of regular monomorphisms. In this case (and in the more general case of the category **Cat** of small categories and functors, where $\mathcal{H}$ is given by the class of full functors) for a morphism to be injective is equivalent to be (if viewed as a functor) topological, a notion introduced in the sixties (for a systematic treatment see [2]).

In this paper, we approach the study of "sliced" injectivity in a category $\mathbf{C}$ with products by using the notion of the "object of sections" $S(f)$ of a given morphism $f : X \to B$ in $\mathbf{C}$, where S is a right adjoint to the functor $\Pi_B : \mathbf{C} \to \mathbf{C}/B$, which assigns to B the second projection $\pi_B^X : X \times B \to B$. (If $\mathbf{C}$ has also equalizers, the existence of S is equivalent to say that B is cartesian in $\mathbf{C}$, that is to the existence of a right adjoint for the functor "product with B" $(-) \times B$.) When such a functor S exists, we find that $f : X \to B$ is injective in $\mathbf{C}/B$ if and only if the morphism $\langle 1_X, f\rangle : X \to X \times B$ is a section in $\mathbf{C}/B$ and the object $S(f)$ of sections of f is injective in $\mathbf{C}$.

Using this result, we first find a new characterization of injective morphisms in the category **Pos** (see Proposition 2.1). In the cartesian closed categories **Dcpo** (ω**Cpo**) of directed complete (ω-complete) posets and continuous maps (see e.g. [8] and [9]), our theorem shows that injective morphisms with respect to the class of regular monomorphisms are necessarily isomorphisms, since injective objects (with respect to the same class) are trivial.

The main result is obtained by the application of our Theorem 1.2 to the category of **ContL** (and to its subcategory **AlgL**). In this way we get characterizations of injective morphisms (with respects to topological embeddings) f between continuous (algebraic) lattices, both topological (every fiber of f has maximum and minimum elements and f is open and closed) and algebraic (f is a complete lattice homomorphism, i.e. f preserves arbitrary sups and arbitrary infs).

1 Injective morphisms via sections

We recall that, given a class $\mathcal{H}$ of morphisms in a category $\mathcal{C}$, an object I is $\mathcal{H}$-*injective* if, for any $h : U \to V$ in $\mathcal{H}$ and any $u : U \to I$, there exists an arrow $s : V \to I$

$$\begin{array}{ccc} U & \xrightarrow{u} & I \\ {\scriptstyle h}\downarrow & \nearrow_{s} & \\ V & & \end{array}$$

such that $sh = u$.

In particular, this means that, in the comma-category $\mathcal{C}/B$ (whose objects are $\mathcal{C}$-morphisms with fixed codomain B), f is $\mathcal{H}$-*injective* if, for any commutative diagram in $\mathcal{C}$

$$\begin{array}{ccc} U & \xrightarrow{u} & X \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle f} \\ V & \xrightarrow[v]{} & B \end{array}$$

with $h \in \mathcal{H}$, there exists an arrow $s : V \to X$

$$\begin{array}{ccc} U & \xrightarrow{u} & X \\ {\scriptstyle h}\downarrow & \nearrow^{s} & \downarrow{\scriptstyle f} \\ V & \xrightarrow[v]{} & B \end{array}$$

such that $sh = u$ and $fs = v$.

Notation. From now on, injective will denote $\mathcal{H}$-injective for $\mathcal{H}$ the class of regular monomorphisms in $\mathbf{C}$.

If a category $\mathbf{C}$ has finite products, we can state that an object B is *cartesian* (or *exponentiable*) in $\mathbf{C}$ if the functor $(-) \times B : \mathbf{C} \to \mathbf{C}$ has a right adjoint $(-)^B$.

Let us now consider the functor $\Pi_B : \mathbf{C} \to \mathbf{C}/B$, which assigns the second projection $\pi_B^X : X \times B \to B$ to any object X and the forgetful functor $\Sigma_B : \mathbf{C}/B \to \mathbf{C}$, which assigns to any f its domain. If $\mathbf{C}$ also has equalizers (that is, $\mathbf{C}$ has all finite limits), by Prop. 1.1 in [10], Π_B has a right adjoint if and only if the functor $\Sigma_B \circ \Pi_B$ has a right adjoint. But $\Sigma_B \circ \Pi_B$ coincides with the functor $(-) \times B$, which by definition has a right adjoint when B is cartesian in $\mathbf{C}$. In conclusion we have that

Proposition 1.1 *B is cartesian in a category $\mathbf{C}$ with finite limits if and only if the functor $\Pi_B : \mathbf{C} \to \mathbf{C}/B$ has a right adjoint $S : \mathbf{C}/B \to \mathbf{C}$.*

Following the proof of Prop. 1.1 in [10] in the case B cartesian, given the morphism $\alpha := \pi_B^{X^B} : X^B \times B \to B$, by adjunction we obtain $\widehat{\alpha} := X^B \to B^B$, which represents a constant morphism of value 1_B. Then, for any $f : X \to B$, the object $S(f)$ is obtained as the equalizer in $\mathbf{C}$ of the two morphisms $\widehat{\alpha}, f^B$, where the latter is the "composition with f". This means that $S(f)$ can be interpreted as the object of sections of f in $\mathbf{C}$. This object turns out to be very useful to obtain a characterization of those f injective in $\mathbf{C}/B$. In fact:

Theorem 1.2 *Let $\Pi_B \dashv S : \mathbf{C}/B \to \mathbf{C}$.*
$f : X \to B$ is injective in $\mathbf{C}/B$ if and only if the following two conditions are satisfied:
1. $\langle 1_X, f\rangle : X \to X \times B$ is a section in $\mathbf{C}/B$;
2. the object $S(f)$ of sections of f is injective in $\mathbf{C}$.

Proof. Let f be injective in $\mathbf{C}/B$. Since $\langle 1, f\rangle$ is a regular monomorphism, corresponding to the commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{1_X} & X \\ {\scriptstyle \langle 1,f\rangle}\downarrow & & \downarrow{\scriptstyle f} \\ X \times B & \xrightarrow[\pi_B^X]{} & B \end{array}$$

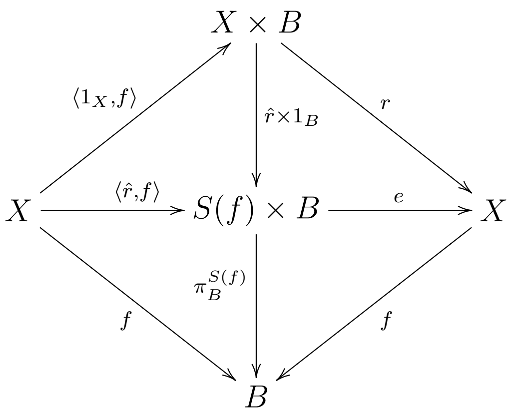
there exists an arrow $r : X \times B \to X$ such that $r\langle 1, f\rangle = 1_x$ and $fr = \pi_B^X$. This means that $\langle 1, f\rangle$ is a section in $\mathbf{C}/B$.

We now have to show that $S(f)$ is injective in $\mathbf{C}$. Given $v : U \to V$ and $u : U \to S(f)$, by adjunction there exists a morphism $\tilde{u} : U \times B \to X$ such that $f\tilde{u} = \pi_B^U$. We can the consider the following commutative diagram:

$$\begin{array}{ccc} U \times B & \xrightarrow{\tilde{u}} & X \\ {\scriptstyle v\times 1_B}\downarrow & & \downarrow{\scriptstyle f} \\ V \times B & \xrightarrow[\pi_B^V]{} & B \end{array}$$

By injectivity of f there exists an arrow $w : V \times B \to X$ such that $w(v \times 1_B) = \tilde{u}$ and $fw = \pi_B^V$. By naturality, $\hat{w}v = u$, where $\hat{w} : V \to S(f)$ is the right adjunct of w.

Now, let f fulfill the conditions 1 and 2. Since $S(f)$ is injective in $\mathbf{C}$, $\pi_B^{S(f)}$ is injective in $\mathbf{C}/B$ (see e.g. Cor. 1.6 in [6], but it follows directly from the definition). Furthermore, since $\langle 1, f\rangle : X \to X \times B$ is a section in $\mathbf{C}/B$, there exists a corresponding retraction r in $\mathbf{C}/B$ with $r\langle 1_X, f\rangle = 1_X$, and its right adjunct $\hat{r} : X \to S(f)$. If $e : S(f) \times B \to X$ denotes the counit of the adjunction, we have the following commutative diagram:



We deduce that $e\langle \hat{r}, f\rangle = r\langle 1_X, f\rangle = 1_X$, then f is a retract (by e) of the injective $\pi_B^{S(f)}$ in $\mathbf{C}/B$. This means that f is injective in $\mathbf{C}/B$.

2 Injective objects in $\mathbf{Pos}/B$

Let **Pos** denote the category of partially ordered sets and monotone mappings. In such a category injective objects, with respect to the class of regular monomorphisms=order embeddings (maps h with $x < y$ iff $h(x) < h(y)$), coincide with the complete lattices (see e.g. [2]). In $\mathbf{Pos}/B$, injectivity (again with respect to order embeddings) has been studied and various characterizations of such injective objects are known (see, for example, [13] and [3]). In this case (and in the more general case of the category **Cat** of small categories and functors, where $\mathcal{H}$ is given by the class of full functors) for a morphism to be injective is equivalent to be (if viewed as a functor) topological, a notion introduced in the sixties (for a systematic treatment see [2]). Since the category **Pos** is cartesian closed, that is every object B is cartesian, we can apply Theorem 1.2 to $\mathbf{Pos}/B$ and find a new characterization:

Theorem 2.1 *$f : X \to B$ is injective in* **Pos**$/B$ *if and only if the following two conditions are satisfied:*

a) $\langle 1_X, f\rangle : X \to X \times B$ *is a section in* **Pos**$/B$*;*

b) *fibers of f are injective in* **Pos***, that is any $f^{-1}(b)$ (as a sub-poset of X) is a complete lattice (in its own right).*

Proof. The necessary conditions are trivial. Viceversa, if condition *a*) holds and $r : X \times B \to X$ is a retraction of π_B^X over f, for any $b \in B$, $r(x, b) \in f^{-1}(b)$, since $f(r(x, b)) = b$, and $r(x, b) = x$, whenever $b = f(x)$. Since any $f^{-1}(b)$ is a complete lattice, we can define a map $s_m : B \to X$ by $s_m(b) =$ minimum of $f^{-1}(b)$. This map is trivially a section of f and is monotone. In fact, if $b_1 \leq b_2$,

$$s_m(b_1) \leq r(s_m(b_2), b_1), \text{ since } \quad r(s_m(b_2), b_1) \in f^{-1}(b_1).$$

By monotony of r, $r(s_m(b_2), b_1) \leq r(s_m(b_2), b_2) = s_m(b_2)$, since $b_2 = f(s_m(b_2))$.

This allows us to say that s_m is the minimum of the pointwise ordered poset $S(f)$, given by the monotone sections of f. In order to prove the injectivity of $S(f)$ in **Pos**, all we need to show is that in $S(f)$ there exists a supremum of any non-empty family $(s_i)_{i \in I}$. For any $b \in B$, we define $(\bigvee_{S(f)} s_i)(b) = \bigvee_{f^{-1}(b)} s_i(b)$. This function is trivially a section and is monotone. In fact,

if $b_1 \leq b_2$, $\forall i \in I$, $s_i(b_1) \leq s_i(b_2) \leq \bigvee_{f^{-1}(b_2)} s_i(b_2) := s_2$

But $s_i(b_1) = r(s_i(b_1), b_1) \leq r(s_2, b_1)$, by monotony of r. Then $s_1 := \bigvee_{f^{-1}(b_1)} s_i(b_1) \leq r(s_2, b_1) \leq r(s_2, b_2) = s_2$.

This suffices to prove that $S(f)$ is a complete lattice, that is an injective object in **Pos**. The conditions 1 and 2 of Theorem 1.2 are therefore satisfied and we can conclude that f is injective in **Pos**$/B$.

3 Injective objects in Dcpo$/B$ and ωCpo$/B$

In this section we are going to consider the categories **Dcpo** (ω**Cpo**) of directed complete (ω-complete) posets and continuous maps (see e.g. [8] and [9]). We shall need some definitions and standard results about them (for which see [8]).

Definition 3.1 A poset B in which every directed subset (ω-chain) has a supremum is called a *directed complete (ω-complete) poset* or *dcpo (ω-cpo)* for short.

Dcpo's (ω-cpo's) are usually considered as topological spaces when endowed with the Scott topology (ω-Scott topology), where C is closed in B if it is a lower set closed under suprema of directed subsets (ω-chains). A map $f : A \to B$ between dcpo's (ω-cpo's) is

1. continuous with respect to the Scott topologies (ω-Scott topologies) if and only if f preserves directed sups (sups of ω-chains);

2. a regular monomorphism if and only if it is a continuous order embedding;

3. a topological homeomorphism if and only if it is an order isomorphism (f and f^{-1} are monotone).

It is known that in **Dcpo** (and in ω**Cpo**) there are regular monomorphisms that are not topological embeddings (see e.g. the example due to Moggi in [12]). This fact enables us to say that the Sierpinski space is not injective with respect to the class of regular monomorphisms. But any topological embedding is a regular monomorphism in **Dcpo** (and in ω**Cpo**), so that any object injective with respect to the class of regular monomorphisms is a continuous lattice, since continuous lattices are the injective objects with respect to the class of topological embeddings in **Dcpo** and in ω**Cpo** (the proof is the same that the one in [11] for the category **Top** of topological spaces). It follows that the injective objects with respect to the class of regular monomorphisms in **Dcpo** and in ω**Cpo** are trivial, since any continuous lattices with at least two elements has the Sierpinski space as a retract.

Both **Dcpo** and ω**Cpo** are cartesian closed, so we can apply Theorem 1.2, obtaining that any injective morphism with respect to the class of regular monomorphisms has exactly one section and then it is necessarily an isomorphism.

4 Injective objects in ContL/B

Now we turn our attention to the category **ContL** of continuous lattices and continuous maps (see e.g. [8] and [9]). We first need to recall some definitions and standard results (for which see [8]).

Definition 4.1 Let B be a dcpo. We recall that, for $a, b \in B$, $a \ll b$ (read: *way below*) if, whenever $b \leq \bigvee D$ for D directed subset, we already have $a \leq d$ for some $d \in D$.

A dcpo is *continuous* if every element is *$\ll$-approximated* , i.e.

$$\forall b \in B, b = \bigvee \twoheaddownarrow b, \quad \text{where } \twoheaddownarrow b = \{b' \in B : b' \ll b\}.$$

If B is a continuous dcpo and a complete lattice, then it is called a *continuous lattice.*

Proposition 4.2 *Let B be a continuous lattice.*

1. *Each point b has a neighborhood basis consisting of the sets $\Uparrow b'$, with $b' \ll b$.*

2. $b = \bigvee\{\bigwedge U, U$ *open in* $B, b \in U\}$

3. $b = \bigvee\{\bigwedge U', U'$ *in a neighborhood basis of* $b\}$

It is well known that **ContL** is a cartesian closed category (see [8]). If we want to apply Theorem 1.2 to **ContL**, we need to know injective objects with respect to the class of regular monomorphisms. But, as in the previous section, these injectives are trivial, since also in **ContL** there are regular monomorphisms that are not topological embeddings, as the following example (suggested by M. Escardó) shows:

Example 4.3 Let $Q = [0,1]^2$ be the square with the componentwise order. Q is a continuous lattice, where $(x,y) \ll (x',y') \Leftrightarrow x < x'$ and $y < y'$. $U \subseteq Q$ is Scott open iff it is an upper set open in the ordinary topology induced by the plane. Let $L = \{(x,y) \in Q \mid y = 1-x\} \cup \{(0,0),(1,1)\}$. The induced order is the discrete one on the diagonal and $(0,0) \leq (x,y) \leq (1,1)$. Then L is trivially a continuous lattice and the sets $(x,y) \cup (1,1)$ are open in the Scott topology on L, while they are not open in the topology induced by Q. This means that the inclusion $i : L \to Q$ is a regular monomorphism that is not a topological embedding.

Consequently, as in **Dcpo**, injective morphisms with respect to the class of regular monomorphisms in **ContL** are only the isomorphisms. We can then consider in **ContL** injectivity with respect the class $\mathcal{H}$ of topological embeddings between continuous lattices. As far as objects are concerned, continuous lattices are the injectives in the category **Top** with respect to $\mathcal{H}$ (see [11]) and this implies that every object in **ContL** is injective in **ContL**.

Now we state some properties of injective objects (with respect to the above class $\mathcal{H}$) in the categories **ContL**$/B$.

Proposition 4.4 *Let $f : X \to B$ be injective in* **ContL**$/B$. *Then*

1. *every fiber of f (as a sub-poset of X) is a continuous lattice (in its own right).*

2. *f has a maximal section and a minimal section.*

Proof. Since f is injective, there exists a retraction $r : X \times B \to X$ in **ContL**$/B$ of π_B^X over f. For any $b \in B$, the restriction r_b of r to $X \times \{b\}$ gives rise to a retraction of X over $f^{-1}(b)$. Therefore, also $f^{-1}(b)$ is a continuous lattice. Furthermore, if we denote with $\max C$ and $\min C$ the maximum and minimum elements of a complete lattice C, $r(\max X, b) = r_b(\max X) = \max f^{-1}(b)$ and $r(\min X, b) = r_b(\min X) = \min f^{-1}(b)$. This means that the restrictions of r respectively to $\{\max X\} \times B$ and to $\{\min X\} \times B$ give rise to a maximal section and to a minimal section of f.

Remark 4.5 The above proposition remains valid for f injective in **AlgL**$/B$, where **AlgL** denotes the full subcategory of **ContL** given by algebraic lattices (a complete lattice is said to be *algebraic* when any element is a directed sup of compact elements, where an element x is compact when $x \ll x$, see e.g. [8]).

In order to apply Theorem 1.2, we use that **ContL** is a full subcategory of **Top** closed under the formation of function spaces, that is: every continuous lattice B is cartesian in **Top** and any space A^B is in **ContL**, when A and B are in **ContL**.

Now we are ready to characterize injective objects in **ContL**$/B$.

Theorem 4.6 *Let $f : X \to B$ be a continuous map between continuous lattices. TFAE:*

1. *$f : X \to B$ is injective in* **ContL**$/B$*;*
2. *$\langle 1_X, f\rangle : X \to X \times B$ is a section in* **ContL**$/B$*;*
3. *every fiber of f has maximum $\max f^{-1}(b)$ and minimum $\min f^{-1}(b)$ elements and the functions $s_M, s_m : B \to X$, defined by $s_M(b) = \max f^{-1}(b)$ and $s_m(b) = \min f^{-1}(b)$ respectively, are sections of f in* **ContL***;*
4. *f is a complete lattice homomorphism, i.e. f preserves arbitrary sups and arbitrary infs.*
5. *every fiber of f has maximum $\max f^{-1}(b)$ and minimum $\min f^{-1}(b)$ elements and the restrictions $f_{|M_f}$ and $f_{|m_f}$ of f respectively to $M_f = \{\max f^{-1}(b) | \, b \in B\}$ and to $m_f = \{\min f^{-1}(b) | \, b \in B\}$ are order isomorphisms, i.e. topological homeomorphisms ;*
6. *every fiber of f has maximum $\max f^{-1}(b)$ and minimum $\min f^{-1}(b)$ elements and f is open and closed.*

Proof. In order to prove the equivalence between conditions 1 and 2, since any B in **ContL** is cartesian in **Top**, we can apply Proposition 1.1 to **Top**, proving the existence of $S(f)$ in **Top**. Under the assumption of condition 2, it is routine to show that $S(f)$ is a retract of the continuous lattice X^B. Hence $S(f)$ is injective in **Top** and then it is in **ContL**. This means that $\Pi_B \dashv S$ holds also in **ContL**. Now we can apply Theorem 1.2 to **ContL**, having the equivalence between 1 and 2.

Let now condition 3 hold. s_M, s_m are monotone functions if and only if, for any $b \in B$, $\max f^{-1}(b) = \max f^{-1}\{b' \mid b' \leq b\}$ and $\min f^{-1}(b) = \min f^{-1}\{b' \mid b' \geq b\}$. But the existence of such maximum and minimum elements for any $b \in B$ is equivalent to say that f has right and left adjoint (see, for example, [1], Prop. 3.1.10). This last condition, being X and B complete lattices, is equivalent (by the Adjoint functor theorem for posets) to say that f is a complete lattice homomorphism, i.e. f preserves arbitrary sups and arbitrary infs. If f preserves arbitrary sups and arbitrary infs, sup and inf in X of any fiber $f^{-1}(b)$ belong to it, then any fiber has maximum and minimum elements. Now let x_1 and x_2 belong to M_f with $f(x_1) < f(x_2)$. Therefore $f(x_1 \vee x_2) = f(x_1) \vee f(x_2) = f(x_2)$, so that $x_1 \vee x_2$ is in the same fiber of which x_2 is the maximum element, then $x_1 \vee x_2 = x_2$. In an analogous way we can prove that also $f_{|m_f}$ is an order isomorphism, so that condition 5 holds. But condition 5 is equivalent to condition 3, since $f_{|M_f}$ and $f_{|m_f}$ are respectively the inverse maps of s_M and s_m, restricted to their images.

Now suppose f injective. By Proposition 4.4, condition 3 holds and $f^{-1}(b)$ is a continuous lattice. Then, if U is an open set of X, $U_b = U \bigcap f^{-1}(b)$ is open in $f^{-1}(b)$, hence an upper set. This means that $b \in f(U)$ if and only if $\max f^{-1}(b) \in U$, so f is open if and only if $f_{|M_f}$ is open. In an analogous way, it can be proved that f is closed if and only if $f_{|m_f}$ is closed. Obviously, if f is open and closed, its bijective restrictions $f_{|M_f}$ and $f_{|m_f}$ are topological homeomorphisms, corresponding to order isomorphisms.

We have then proved till now that

$$② \Leftrightarrow ① \Rightarrow ③ \Leftrightarrow ④ \Leftrightarrow ⑤ \Leftrightarrow ⑥$$

In conclusion, it is sufficient to show that ③ $\Rightarrow$ ②.

We then need to prove the existence of a retraction $r : X \times B \longrightarrow X$ of π_B^X over f in **ContL**$/B$. Let us consider $(x, b) \in X \times B$, the family $\mathcal{V} = \{V \mid V \text{ open in } X, x \in V \text{ and } V \bigcap f^{-1}(b) \neq \emptyset\}$ and the family $\mathcal{U} = \{U \mid U \text{ open in } B,\, b \in U\}$.

In correspondence of any (x, b), we can then define the family $\mathcal{W} = \{W \mid W = \langle id_X, f\rangle^{-1}(V \times U), \text{ for } V \in \mathcal{V} \text{ and } U \in \mathcal{U}\}$.

Any W is an open subset of X, but not empty, since $V \bigcap f^{-1}(b) \neq \emptyset$, for any $V \in \mathcal{V}$. If $\tilde{x} \in W$, $f(\tilde{x}) \in U$ and then $f(W) \subseteq U$. It follows that $\bigwedge f(W) \geq \bigwedge U$ and then, by condition (4) equivalent to (3)

$$f\left(\bigvee \bigwedge W\right) = \bigvee f\left(\bigwedge W\right) = \bigvee \bigwedge f(W) \geq \bigvee \bigwedge U = b,$$

by Proposition 4.2 (2).

On the other hand, $b \in f(W)$, for any $W \in \mathcal{W}$, so that $b \geq f\left(\bigvee \bigwedge W\right)$.

If we then define

$$r(x,b) = \bigvee \bigwedge W,$$

$f(r(x,b)) = b$. We are going to show that such an r is a retraction.

Since any open neighborhood of x has a non-empty intersection with $f^{-1}(f(x))$, $(x, f(x))$ has $\mathcal{V} \times \mathcal{U}$ as a neighborhood basis. Therefore $\mathcal{W}$ is a neighborhood basis for $x \in X$. Hence, by Proposition 4.2 (2),

$$x = \bigvee \bigwedge W = r(x, f(x)),$$

that is $\langle 1_X, f \rangle r = 1_X$.

Let now $(x_1, b_1) \leq (x_2, b_2)$. If V^1 is an open neighborhood of x_1 with $V^1 \bigcap f^{-1}(b_1) \neq \emptyset$, then V^1 is an upper set and so $\max f^{-1}(b_1) \in V^1$. But also $\max f^{-1}(b_2) \in V^1$, since $\max f^{-1}(b_1) \leq \max f^{-1}(b_2)$, because the maximum section $s_{_M}$ is monotone. Consequently $V^1 \bigcap f^{-1}(b_2) \neq \emptyset$. Furthermore, any open neighborhood U^1 of b_1 is an upper set, hence an open neighborhood of b_2. This means that the family $\mathcal{W}^1$ (defined as above for (x_1, b_1)) is contained in the analogous family $\mathcal{W}^2$ (defined for (x_2, b_2)). Consequently

$$r(x_1, b_1) = \bigvee \bigwedge W^1 \leq \bigvee \bigwedge W^2 = r(x_2, b_2),$$

that is, r is monotone.

Let $(x, b) = \bigvee (x_\lambda, b_\lambda)$, where $(x_\lambda, b_\lambda)_\Lambda$ is a direct subset of $X \times B$. Since $r(x_\lambda, b_\lambda) \leq r(x,b)$, $\bigvee r(x_\lambda, b_\lambda) \leq r(x,b)$. Suppose $\bigvee r(x_\lambda, b_\lambda) < r(x,b)$. Then there should exist an open subset O of X with $r(x,b) \in O$ and $\bigvee r(x_\lambda, b_\lambda) \notin O$. By definition of Scott topology, it should exist $W \in \mathcal{W}$ such that $\bigwedge W \in O$. But $W = \langle id_X, f \rangle^{-1}(V \times U)$, with V an open neighborhood of x, such that $V \bigcap f^{-1}(b) \neq \emptyset$ (and then $\max f^{-1}(b)$ is in V) and U an open neighborhood of b. Since $(x,b) = \bigvee (x_\lambda, b_\lambda)$, $\exists \lambda_1$ with $(x_{\lambda_1}, b_{\lambda_1}) \in V \times U$.

But the maximum section $s_{_M} : B \to X$ preserves directed sups. Then

$$V \ni \max f^{-1}(b) = s_{_M}(b) = s_{_M}\left(\bigvee b_\lambda\right) = \bigvee s_{_M}(b_\lambda) = \bigvee \max f^{-1}(b_\lambda),$$

so there exists $\lambda_2 \in \Lambda$ such that $\max f^{-1}(b_{\lambda_2}) \in V$.

Let now $(x_{\lambda_3}, b_{\lambda_3}) \geq (x_{\lambda_1}, b_{\lambda_1}), (x_{\lambda_2}, b_{\lambda_2})$. Then $\max f^{-1}(b_{\lambda_3}) \in V$. Therefore $V \bigcap f^{-1}(b_{\lambda_3}) \neq \emptyset$. Consequently, $V \times U$ is an open neighborhood of $(x_{\lambda_3}, b_{\lambda_3})$ in $X \times B$ and

$$\bigwedge W \leq r(x_{\lambda_3}, b_{\lambda_3}) \leq \bigvee r(x_\lambda, b_\lambda).$$

But this is impossible, since $\bigwedge W \in O, \bigvee r(x_\lambda, b_\lambda) \notin O$, with O upwards closed. This means that $\bigvee r(x_\lambda, b_\lambda) = r(x, b)$. In conclusion, we have proved that r is a retraction of Π_B over f in $\mathbf{ContL}/B$.

Corollary 4.7 *Let $f : X \to B$ be a continuous map between algebraic lattices. TFAE:*

1. *$f : X \to B$ is injective in $\mathbf{AlgL}/B$*
2. *$\langle 1_X, f\rangle : X \to X \times B$ is a section in $\mathbf{AlgL}/B$;*
3. *every fiber of f has maximum $\max f^{-1}(b)$ and minimum $\min f^{-1}(b)$ elements and the functions $s_M, s_m : B \to X$ defined by $s_M(b) = \max f^{-1}(b)$ and $s_m(b) = \min f^{-1}(b)$ respectively, are sections of f in $\mathbf{AlgL}$.*
4. *f is a complete lattice homomorphism, i.e. f preserves arbitrary sups and arbitrary infs.*
5. *every fiber of f has maximum $\max f^{-1}(b)$ and minimum $\min f^{-1}(b)$ elements and the restrictions $f_{|M_f}$ and $f_{|m_f}$ of f respectively to $M_f = \{\max f^{-1}(b) | \, b \in B\}$ and to $m_f = \{\min f^{-1}(b) | \, b \in B\}$ are topological homeomorphisms, i.e. order isomorphisms;*
6. *every fiber of f has maximum $\max f^{-1}(b)$ and minimum $\min f^{-1}(b)$ elements and f is open and closed.*

Proof. Condition 1 implies condition 3, by Remark 4.5. If condition 3 holds, f is an object of $\mathbf{AlgL}/B$ injective in $\mathbf{ContL}/B$, so f is injective in $\mathbf{AlgL}/B$.

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