On the critical Neumann problem with weight in exterior domains

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Abstract

In this paper we investigate the solvability of the Neumann problem (1.1) involving a critical Sobolev exponent in exterior domains. It is assumed that the coefficient Q is a positive and smooth function on Ω^c and $\lambda > 0$ is a parameter. We examine the common effect of the mean curvature of the boundary $\partial\Omega$ and the shape of the graph of the coefficient Q on the existence of least energy solutions.

1 Introduction

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with a smooth boundary $\partial\Omega$ and set $\Omega^c = \mathbb{R}^N - \bar{\Omega}$. In this paper we investigate the nonlinear Neumann problem

$$(1.1) \qquad \begin{cases} -\Delta u + \lambda u &= Q(x)|u|^{2^*-2}u \ \text{ in } \Omega^c, \\ \frac{\partial}{\partial \nu} u(x) &= 0 \ \text{ on } \partial\Omega \ \text{ and } \ u > 0 \ \text{ on } \Omega^c, \end{cases}$$

where the coefficient Q is continuous and positive on $\bar{\Omega}^c$, $\lambda > 0$ is a parameter and $2^* = \frac{2N}{N-2}$, $N \geq 3$, is a critical Sobolev exponent.

If $Q \equiv 1$ on Ω^c some existence results can be found in the paper [19]. For the subcritical case we refer to the paper [22].

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On the other hand the Neumann problem in bounded domains has an extensive literature [1], [2], [3], [5], [17], [21], [23], [24], [25], [26], [27].

To motivate our approach we briefly recall the main results for the Neumann problem in the bounded domain Ω

$$\left\{ \begin{array}{ll} -\Delta u + \lambda u = Q(x)|u|^{2^*-2}u & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega,\ u > 0 \text{ on } \Omega. \end{array} \right. \tag{1.2}$$

First existence results for problem (1.2) with $Q \equiv 1$ are due to Adimurthi - Mancini [1], Adimurthi - Yadava [6] and X.J. Wang [21]. Solutions to problem (1.2) were obtained as the minimizers of the variational problem

$$\begin{aligned} m_\lambda &= \inf_{u \in H^1(\Omega) - \{0\}} \frac{\int_\Omega \left(|\nabla u|^2 + \lambda u^2\right) dx}{\left(\int_\Omega Q(x)|u|^{2^*}\, dx\right)^{\frac{2}{2^*}}} \\ &= \inf_{u \in H^1(\Omega),\, \int_\Omega Q(x)|u|^{2^*}\, dx = 1} \int_\Omega \left(|\nabla u|^2 + \lambda u^2\right) dx. \end{aligned} \tag{1.3}$$

The existence of a minimizer for m_λ is closely related to the best Sobolev constant S. We recall that

$$S = \inf_{u \in D^{1,2}(\mathbb{R}^N) - \{0\}} \frac{\int_{\mathbb{R}^N} |\nabla u|^2\, dx}{\left(\int_{\mathbb{R}^N} |u|^{2^*}\, dx\right)^{\frac{2}{2^*}}},$$

where $D^{1,2}(\mathbb{R}^N)$ is the Sobolev space defined by $D^{1,2}(\mathbb{R}^N) = \{u;\ \nabla u \in L^2(\mathbb{R}^N),\ u \in L^{2^*}(\mathbb{R}^N)\}$. The best Sobolev constant is achieved by

$$U(x) = \frac{c_N}{\left(N(N-2) + |x|^2\right)^{\frac{N-2}{2}}},$$

where $c_N > 0$ is a constant depending on N. The function U, called an instanton, satisfies the equation

$$-\Delta U = U^{2^*-1} \text{ in } \mathbb{R}^N.$$

We have $\int_{\mathbb{R}^N} |\nabla U|^2\, dx = \int_{\mathbb{R}^N} U^{2^*}\, dx = S^{\frac{N}{2}}$. For future use we introduce the notation

$$U_{\epsilon,y}(x) = \epsilon^{-\frac{N-2}{2}} U\left(\frac{x-y}{\epsilon}\right),\ y \in \mathbb{R}^N,\ \epsilon > 0.$$

If $y = 0$ we write $U_\epsilon = U_{\epsilon,0}$.

The main step in establishing the existence of a minimizer for m_λ is to show that

$$m_\lambda < \frac{S}{2^{\frac{2}{N}}}. \tag{1.4}$$

This was established by testing m_λ with $U_{\epsilon,y}$, and $y \in \partial\Omega$ with the mean curvature $H(y) > 0$. Solutions of the minimization problem (1.3) are called the least energy solutions. If $Q \not\equiv$ const, then a new phenomenon occurs, namely, a combined effect of the mean curvature and the graph of Q on the existence and nonexistence of solutions. Let $Q_M = \max_{x\in\bar\Omega} Q(x)$ and $Q_m = \max_{x\in\partial\Omega} Q(x)$. To obtain the least energy solutions one has to distinguish two cases: (i) $Q_M \leq 2^{\frac{2}{N-2}} Q_m$ and (ii) $Q_M > 2^{\frac{2}{N-2}} Q_m$. According to the results in [10] a least energy solution exists if

$$m_\lambda < \min\left(\frac{S}{2^{\frac{2}{N}} Q_m^{\frac{N-2}{N}}}, \frac{S}{Q_M^{\frac{N-2}{N}}} \right) := s_\infty. \tag{1.5}$$

If (i) holds, then $s_\infty = \frac{S}{2^{\frac{2}{N}} Q_m^{\frac{N-2}{N}}}$ and the strict inequality in (1.5) holds for every $\lambda > 0$ if $Q_m = Q(y)$ for some $y \in \partial\Omega$ with $H(y) > 0$ and moreover $|Q(x) - Q(y)| = o(|x - y|)$ for x near y. If (ii) prevails, then $s_\infty = \frac{S}{Q_M^{\frac{N-2}{N}}}$ and there exists $\Lambda > 0$ such that $m_\lambda < \frac{S}{Q_M^{\frac{N-2}{N}}}$ for $\lambda < \Lambda$ and $m_\lambda = \frac{S}{Q_M^{\frac{N-2}{N}}}$ for $\lambda \geq \Lambda$. The least energy solutions exist for $0 < \lambda \leq \Lambda$.

The purpose of this paper is to extend these results to problem (1.1) on exterior domains. Since the domain Ω^c is unbounded the existence of solutions depends also on the behaviour of the coefficient Q at ∞.

2 Preliminaries

We set

$$J_\lambda(u) = \int_{\Omega^c} \left(|\nabla u|^2 + \lambda u^2 \right) dx.$$

As in the case of bounded domains solutions will be sought as minimizers of the variational problem

$$S(\Omega^c, Q, \lambda) = \inf\{ J_\lambda(u);\ u \in H^1(\Omega^c),\ \int_{\Omega^c} Q(x)|u|^{2^*}\, dx = 1 \} \tag{2.1}$$

for $\lambda \geq 0$. Here we make a convention: if $\lambda = 0$, then the space $H^1(\Omega^c)$ in (2.1) is replaced by the Sobolev space

$$D^{1,2}(\Omega^c) = \{u;\ \nabla u \in L^2(\Omega^c) \text{ and } u \in L^{2^*}(\Omega^c)\}.$$

It follows from the results in [19] that

$$0 < S(\Omega^c, 1, \lambda) \leq \frac{S}{2^{\frac{2}{N}}} \tag{2.2}$$

for all $\lambda \geq 0$. Moreover, if $S(\Omega^c, 1, \lambda) < \frac{S}{2^{\frac{2}{N}}}$, then $S(\Omega^c, 1, \lambda)$ is achieved. The strict inequality $S(\Omega^c, 1, \lambda) < \frac{S}{2^{\frac{2}{N}}}$ holds for $\lambda \geq 0$ provided the mean curvature of $\partial\Omega$, viewed from inside of Ω, is negative somewhere (see Theorem 1 and Proposition 2.1 in [19]).

To examine the minimization sequences for (2.1) we need the following version of P.L. Lions' concentration - compactness principle [15].

Let $\{u_n\}$ be a weakly convergent sequence to u in $H^1(\Omega^c)$ (or in $D^{1,2}(\Omega)$). We define two quantities which measure the loss of mass at infinity of $\{u_n\}$:

$$\nu_\infty = \lim_{R\to\infty} \limsup_{n\to\infty} \int_{\Omega^c\cap(|x|\geq R)} |u_n|^{2*}\, dx$$

and

$$\mu_\infty = \lim_{R\to\infty} \limsup_{n\to\infty} \int_{\Omega^c\cap(|x|\geq R)} |\nabla u_n|^2\, dx.$$

It follows from the Sobolev inequality that

$$S\nu_\infty^{\frac{2}{2^*}} \leq \mu_\infty. \tag{2.3}$$

The concentration - compactness principle can be stated in the following way: let $u_n \rightharpoonup u$ in $H^1(\Omega^c)$ (or in $D^{1,2}(\Omega^c)$) be such that

$$|u_n|^{2^*} \overset{*}{\rightharpoonup} \nu \ \text{ and } \ |\nabla u_n|^2 \overset{*}{\rightharpoonup} \mu$$

weakly in the sense of measure. Then there exist numbers $\nu_j > 0$, $\mu_j > 0$ and points $x_j \in \Omega^c \cup \partial\Omega$, $j \in J$, where J is at most a countable set, such that

$$\nu = |u|^{2*} + \sum_{j\in J} \nu_j \delta_{x_j} + \mu_\infty \delta_\infty$$

and

$$\mu = |\nabla u|^2 + \sum_{j\in J} \nu_j \delta_{x_j} + \nu_\infty \delta_\infty.$$

Moreover, if $x_j \in \Omega^c$

$$S\nu_j^{\frac{N-2}{N}} \leq \mu_j\ , \tag{2.4}$$

if $x_j \in \partial\Omega$

$$\frac{S}{2^{\frac{2}{N}}}\nu_j^{\frac{N-2}{N}} \leq \mu_j \tag{2.5}$$

and moreover (2.3) holds.

The following Proposition gives a criterion for the existence of a minimizer for $S(\Omega^c, Q, \lambda)$. We set

$$Q_m = \max_{x\in\partial\Omega} Q(x)\ ,\ Q_M = \sup_{x\in\Omega^c} Q(x)\ ,$$

and we assume throughout the paper that

$$Q_\infty = \lim_{|x|\to\infty} Q(x)$$

exists. Then we have

Proposition 2.1 *Let*

$$S_\infty = \min\left(\frac{S}{2^{\frac{2}{N}} Q_m^{\frac{N-2}{N}}}, \frac{S}{Q_M^{\frac{N-2}{N}}}, \frac{S}{Q_\infty^{\frac{N-2}{N}}}\right).$$

If $S(\Omega^c, Q, \lambda) < S_\infty$ for some $\lambda \geq 0$, then $S(\Omega^c, Q, \lambda)$ is achieved.

Proof Let $\{u_n\} \subset H^1(\Omega^c)$ (or $D^{1,2}(\Omega^c)$ if $\lambda = 0$) be such that

$$J_\lambda(u_n) \to S(\Omega^c, Q, \lambda) \quad \text{and} \quad \int_{\Omega^c} Q(x)|u_n|^{2^*}\, dx = 1.$$

Then

$$|\nabla u_n|^2 \stackrel{*}{\rightharpoonup} \mu = |\nabla u|^2 + \sum_{j\in J} \mu_j \delta_{x_j} + \delta_\infty \mu_\infty$$

and

$$|u_n|^{2^*} \stackrel{*}{\rightharpoonup} \nu = |u|^{2^*} + \sum_{j\in J} \nu_j \delta_{x_j} + \delta_\infty \nu_\infty.$$

Hence

$$\int_{\Omega^c} Q(x)|u|^{2^*}\, dx + \sum_{j\in J} Q(x_j)\nu_j + \nu_\infty Q_\infty = 1 \tag{2.6}$$

and

$$S(\Omega^c, Q, \lambda) = \lim_{n\to\infty} \int_{\Omega^c} \left(|\nabla u_n|^2 + \lambda u_n^2\right) dx \geq \int_{\Omega^c} \left(|\nabla u|^2 + \lambda u^2\right) dx + \sum_{j\in J} \mu_j + \mu_\infty.$$

Using (2.3), (2.4) and (2.5) we derive the estimate from below for $S(\Omega^c, Q, \lambda)$:

$$\begin{aligned}
S(\Omega^c, Q, \lambda) \;&\geq\; S(\Omega^c, Q, \lambda)\left(\int_{\Omega^c} Q|u|^{2^*}\, dx\right)^{\frac{2}{2^*}} + \sum_{x_j\in\Omega^c} \frac{S}{Q(x_j)^{\frac{N-2}{N}}}\left(\nu_j Q(x_j)\right)^{\frac{N-2}{N}} \\
&+ \sum_{x_j\in\partial\Omega} \frac{S}{2^{\frac{2}{N}} Q(x_j)^{\frac{N-2}{N}}}\left(\nu_j Q(x_j)\right)^{\frac{N-2}{N}} + \frac{S}{Q_\infty^{\frac{N-2}{N}}}\left(\nu_\infty Q_\infty\right)^{\frac{N-2}{N}} \\
&\geq\; S(\Omega^c, Q, \lambda)\left(\int_{\Omega^c} Q|u|^{2^*}\, dx\right)^{\frac{2}{2^*}} + \sum_{x_j\in\Omega^c} \frac{S}{Q_M^{\frac{N-2}{N}}}\left(Q(x_j)\nu_j\right)^{\frac{N-2}{N}} \\
&+ \sum_{x_j\in\partial\Omega} \frac{S}{2^{\frac{2}{N}} Q_m^{\frac{N-2}{N}}}\left(\nu_j Q(x_j)\right)^{\frac{N-2}{N}} + \frac{S}{Q_\infty^{\frac{N-2}{N}}}\left(\nu_\infty Q_\infty\right)^{\frac{N-2}{N}}.
\end{aligned}$$

Since $S(\Omega^c, Q, \lambda) < S_\infty$, comparing the above inequality with (2.6) we deduce that $\nu_j = 0$ for all $j \in J \cup \{\infty\}$ and the result follows. □

Lemma 2.2 *For every $\lambda \geq 0$ we have*

$$\frac{S(\Omega^c, 1, \lambda)}{Q_M^{\frac{N-2}{N}}} \leq S(\Omega^c, Q, \lambda) \leq S_\infty.$$

If $Q(x) < Q_\infty$ for every $x \in \Omega^c$, Q_M on the left side should be replaced by Q_∞.

Proof Let $\epsilon > 0$ and choose $R > 0$ so that $\Omega \subset B(0,R)$ and $Q(x) \geq Q_\infty - \epsilon$ for $x \in \mathbb{R}^N - B(0,R)$. We then have

$$\begin{aligned} S(\Omega^c, Q, \lambda) &\leq \frac{\int_{\mathbb{R}^N - B(0,R)} \big(|\nabla u|^2 + \lambda u^2\big)\, dx}{\Big(\int_{\mathbb{R}^N - B(0,R)} Q(x)|u|^{2^*}\, dx\Big)^{\frac{2}{2^*}}} + \frac{\int_{\Omega^c \cap B(0,R)} \big(|\nabla u|^2 + \lambda u^2\big)\, dx}{\Big(\int_{\mathbb{R}^N - B(0,R)} Q(x)|u|^{2^*}\, dx\Big)^{\frac{2}{2^*}}} \\ &\leq \frac{\int_{\mathbb{R}^N - B(0,R)} \big(|\nabla u|^2 + \lambda u^2\big)\, dx}{\big(Q_\infty - \epsilon\big)^{\frac{N-2}{N}} \Big(\int_{\mathbb{R}^N - B(0,R)} |u|^{2^*}\, dx\Big)^{\frac{2}{2^*}}} + \frac{\int_{\Omega^c \cap B(0,R)} \big(|\nabla u|^2 + \lambda u^2\big)\, dx}{\Big(\int_{\mathbb{R}^N - B(0,R)} Q(x)|u|^{2^*}\, dx\Big)^{\frac{2}{2^*}}} \end{aligned}$$

for every $u \in H^1(\Omega^c)$ (or $u \in D^{1,2}(\Omega^c)$ if $\lambda = 0$). Taking inf over $u \in H_0^1(\mathbb{R}^N - B(0,R))$ ($u \in D_0^{1,2}(\mathbb{R}^N - B(0,R))$ if $\lambda = 0$) and since $\epsilon > 0$ is arbitrary we deduce from the above inequality that

$$S(\Omega^c, Q, \lambda) \leq \frac{S}{Q_\infty^{\frac{N-2}{N}}}.$$

If $Q(y) = Q_M$ for some $y \in \Omega^c$, then testing $S(\Omega^c, Q, \lambda)$ by $U_{\epsilon,y}$ we derive the estimate

$$S(\Omega^c, Q, \lambda) \leq \frac{S}{Q_M^{\frac{N-2}{N}}}.$$

Finally, let $y \in \partial\Omega$ and $Q_m = Q(y)$. We need the following estimate
(2.7)

$$\frac{J_\lambda(U_{\epsilon,y})}{\Big(\int_{\Omega^c} U_{\epsilon,y}^{2^*}\, dx\Big)^{\frac{2}{2^*}}} = \frac{S}{2^{\frac{2}{N}}} + \begin{cases} A_N H(y)\epsilon \log\frac{1}{\epsilon} + a_N \lambda\epsilon + O(\epsilon) + o(\lambda\epsilon), & N = 3 \\ A_N H(y)\epsilon + a_N \epsilon^2 \log\frac{1}{\epsilon} + O\big(\epsilon^2 \log\frac{1}{\epsilon}\big) + o\big(\lambda\epsilon^2 \log\frac{1}{\epsilon}\big), & N = 4 \\ A_N H(y)\epsilon + a_N \lambda\epsilon^2 + O\big(\epsilon^2\big) + o(\lambda\epsilon^2), & N \geq 5 \end{cases}$$

where $A_N > 0$ and $a_N > 0$ are constants depending on N. Here $H(y)$ denotes the mean curvature of $\partial\Omega$ at y, when viewed from inside. This estimate can be obtained from the corresponding estimate on bounded domains by truncation. It then follows from (2.7) that

$$S(\Omega^c, Q, \lambda) \leq \frac{S}{2^{\frac{2}{N}} Q_m^{\frac{N-2}{N}}}.$$

The assertion of this lemma follows from the above cases. □

3 Case $Q_M \leq 2^{\frac{2}{N-2}} Q_m$

To apply Proposition 2.1 we need a condition guaranteeing that $S(\Omega^c, Q, \lambda) < S_\infty$ for $\lambda \geq 0$.

In this section we assume that

(A) $Q_M \leq 2^{\frac{2}{N-2}} Q_m$.

Theorem 3.1 *Suppose that (A) is satisfied and that $Q(y) = Q_m$ for some $y \in \partial\Omega$ with $H(y) < 0$, when viewed from inside Ω, and*

$$|Q(x) - Q(y)| = o(|x - y|) \quad \text{for } x \text{ near } y. \tag{3.1}$$

Then for each $\lambda \geq 0$ problem (1.1) has a least energy solution.

Proof It follows from assumption (A) that $S_\infty = \frac{S}{2^{\frac{2}{N}} Q_m^{\frac{N-2}{N}}}$. If $N \geq 5$ we test $S(\Omega^c, Q, \lambda)$ using $U_{\epsilon,y}$. The strict inequality $S(\Omega^c, Q, \lambda) < \frac{S}{2^{\frac{2}{N}} Q_m^{\frac{N-2}{N}}}$ follows from (2.7) and the fact that

$$\int_{\Omega^c} Q(x) U_{\epsilon,y}^{2^*}(x)\, dx = Q_m \int_{\Omega_c} U_{\epsilon,y}^{2^*}\, dx + o(\epsilon).$$

If $N = 3, 4$ instead of $U_{\epsilon,y}$ we use $U_{\epsilon,y}\phi_R$, where $\phi_R \in C^1(\mathbb{R}^N)$, $\phi_R(x) = 1$ for $x \in B(0, R)$, $\phi_R(x) = 0$ for $x \in \mathbb{R}^N - B(0, R+1)$ and $0 \leq \phi_R(x) \leq 1$ on $\mathbb{R}^N$. □

4 Case $Q_M > 2^{\frac{2}{N-2}} Q_m$

In this section we assume

(B) $Q_M > 2^{\frac{2}{N-2}} Q_m$

Moreover, we assume that Q_M is attained at some point of Ω^c.

Note that under assumption (B) we have $S_\infty = \frac{S}{Q_M^{\frac{N-2}{N}}}$.

Theorem 4.1 *Suppose that (B) holds and that $Q_M = Q(y)$ for some $y \in \Omega^c$ with y satisfying*

$$|Q(y) - Q(x)| = o(|x - y|^{N-2}) \quad \text{for } x \text{ near } y. \tag{4.1}$$

Then there exists $\lambda_\circ > 0$ such that problem (1.1) has a least energy solution u_λ for every $0 \leq \lambda < \lambda_\circ$.

Proof According to Proposition 2.1 it is sufficient to show that there exists $\lambda_\circ > 0$ such that

$$S(\Omega^c, Q, \lambda) < \frac{S}{Q_M^{\frac{N-2}{N}}}$$

for all $0 \leq \lambda < \lambda_\circ$. First, we consider the case $N \geq 5$. Let $\delta > 0$. Using (4.1) we write

$$\begin{aligned} \int_{\Omega^c} Q(x) U_{\epsilon,y}^{2^*}\, dx &= \int_{\Omega^c} \big(Q(x) - Q_M\big) U_{\epsilon,y}^{2^*}\, dx + \int_{\Omega^c} Q_M U_{\epsilon,y}^{2^*}\, dx \\ &= Q_M \int_{\mathbb{R}^N} U_{\epsilon,y}^{2^*}\, dx - Q_M \int_{\Omega} U_{\epsilon,y}^{2^*}\, dx + \int_{\Omega^c \cap B(y,\delta)} o(|x-y|^{N-2}) U_{\epsilon,y}^{2^*}\, dx + o(\epsilon^N) \\ &= Q_M K_2 + o\big(\epsilon^{N-2}\big), \end{aligned}$$

where $K_2 = \int_{\mathbb{R}^N} U^{2^*}\,dx$. Since $y \notin \bar\Omega$, there exists a constant $C_1 > 0$ such that

$$\int_{\Omega^c} |\nabla U_{\epsilon,y}|^2\,dx = \int_{\mathbb{R}^N} |\nabla U_{\epsilon,y}|^2\,dx - \int_{\Omega} |\nabla U_{\epsilon,y}|^2\,dx \le K_1 - C_1\epsilon^{N-2},$$

where $K_1 = \int_{\mathbb{R}^N} |\nabla U|^2\,dx$. Using these two asymptotic relations we derive the estimate

$$\begin{aligned}
&\frac{\int_{\Omega^c}\big(|\nabla U_{\epsilon,y}|^2 + \lambda U_{\epsilon,y}^2\big)\,dx}{\big(\int_{\Omega^c} Q(x)U_{\epsilon,y}^{2^*}\,dx\big)^{\frac{2}{2^*}}} \le \frac{K_1 - C_1\epsilon^{N-2} + K_3\lambda\epsilon^2}{\big(Q_M K_2 + o(\epsilon^{N-2})\big)^{\frac{2}{2^*}}} \\
= &\ \big(K_1 - C_1\epsilon^{N-2}K_3 + \lambda\epsilon^2\big)\left(\big(Q_M K_2\big)^{-\frac{N-2}{N}} - \frac{N-2}{N}\big(Q_M K_2\big)^{\frac{-2N+2}{N}} o\big(\epsilon^{N-2}\big)\right) \\
< &\ \frac{S}{Q_M^{\frac{N-2}{N}}}
\end{aligned}$$

for $\epsilon > 0$ and $\lambda \ge 0$ sufficiently small. Here $K_3 > 0$ is a constant independent of ϵ. If $N = 3, 4$, then $U \notin L^2(\mathbb{R}^N)$ and we use a truncation. We replace $U_{\epsilon,y}$ by $U_{\epsilon,y}\phi_R$, where ϕ_R is a C^1-function such that $\phi_R(x) = 1$ if $x \in B(y, R)$, $\phi_R(x) = 0$ for $x \in \mathbb{R}^N - B(y, R+1)$, with $R > 0$ large, and we repeat the above estimates. □

Since $S(\Omega^c, Q, \lambda)$ is continuous and nondecreasing in $\lambda \in (0, \infty)$ we can define

$$\Lambda = \begin{cases} +\infty\ , & \text{if } S(\Omega^c, Q, \lambda) < \frac{S}{Q_M^{\frac{N-2}{N}}}\ \ \forall\ \lambda \in [0, \infty) \\ \min\{\lambda \in [0,\infty), S(\Omega^c, Q, \lambda) = \frac{S}{Q_M^{\frac{N-2}{N}}}\}\ , & \text{otherwise.} \end{cases}$$

It is easy to see that the function $\lambda \to S(\Omega^c, Q, \lambda)$ is strictly increasing on $[0, \Lambda)$. By Proposition 2.1, problem (1.1) has a least energy solution for $\lambda < \Lambda$ and no least energy solution for $\lambda > \Lambda$.

We now show that $\Lambda < \infty$, for $N \ge 5$, and if we assume in addition that $Q_M > Q_\infty$. Arguing by contradiction we assume that for every $\lambda \ge 0$ there exists a least energy solution u_λ of problem (1.1). In Proposition 4.2 we examine the behavior of u_λ as $\lambda \to \infty$.

Proposition 4.2 *Assume the hypothesis of theorem 4.1, and in addition $Q_M > Q_\infty$. Suppose that $\Lambda = \infty$, and denote by u_λ the least energy solution to $\lambda \ge 0$. Then there exist sequences $\lambda_k \to \infty$, $\epsilon_k \to 0$ and a sequence of points $P_k \to P_\circ$, with $P_\circ$ satisfying $Q_M = Q(P_\circ)$, such that*

$$\lim_{k\to\infty} \int_{\Omega^c} |\nabla\big(u_{\lambda_k} - \epsilon_k^{-\frac{N-2}{2}} U\big(Q_M^{\frac{1}{N}} S^{\frac{1}{2}} \frac{\cdot - P_k}{\epsilon_k}\big)\big)|^2\,dx = 0. \tag{4.2}$$

Proof Due to elliptic regularity theory [12] $u_\lambda \in C^2(\Omega^c)$. As in the paper [19] one can show that $\lim_{|x|\to\infty} u_\lambda(x) = 0$. By the concentration - compactness principle we have

$$\lim_{\lambda\to\infty} S(\Omega^c, Q, \lambda) = \frac{S}{Q_M^{\frac{N-2}{N}}}.$$

Let $\lambda_k \to \infty$, $\lambda_k > 0$. We set

$$M_{\lambda_k} = \max_{x\in\Omega^c} u_{\lambda_k}(x) = u_{\lambda_k}(P_{\lambda_k})$$

for some $P_{\lambda_k} \in \Omega^c$. Since

$$0 \le \int_{\Omega^c} |\nabla u_{\lambda_k}|^2\, dx = \int_{\Omega^c} u_{\lambda_k}^2 \left(Qu_{\lambda_k}^{2^*-2} - \lambda_k\right) dx$$

we see that $Q_M M_{\lambda_k}^{2^*-2} \ge \lambda_k$ and $\lim_{\lambda_k\to\infty} M_{\lambda_k} = \infty$. We now use a blow - up technique. We rescale the solutions u_{λ_k} by setting

$$v_k(x) = \epsilon_k^{\frac{N-2}{2}} u_{\lambda_k}(\epsilon_k x + P_{\lambda_k}) \text{ for } x \in \Omega_k = \frac{\Omega^c - P_{\lambda_k}}{\epsilon_k},$$

where $\epsilon_k = M_{\lambda_k}^{-\frac{2}{N-2}}$. The function v_k is a solution of the problem

$$\begin{cases} -\Delta v_k + \lambda_k \epsilon_k^2 v_k &= S_k Q(\epsilon_k x + P_k) v_k^{2^*-1} \text{ in } \Omega_k, \\ \frac{\partial v_k}{\partial \nu} = 0 & \text{on } \partial\Omega_k,\ 0 \le v_k \le 1 \text{ on } \Omega_k,\ v_k(0) = 1, \end{cases}$$

where $S_k = S(\Omega^c, Q, \lambda_k)$ and $P_k = P_{\lambda_k}$. Applying elliptic regularity theory [12] one shows that $v_k \to w$ in $C^2_{\text{loc}}(\Omega_\infty)$, where $\Omega_k \to \Omega_\infty$. We now distinguish two cases:
(i) $\frac{\text{dist}(P_k,\partial\Omega)}{\epsilon_k} \to a < \infty$ and (ii) $\frac{\text{dist}(P_k,\partial\Omega)}{\epsilon_k} \to \infty$. We first show that in both cases $\lambda_k\epsilon_k^2 \to 0$ as $k \to \infty$. Observe that

$$\lim_{k\to\infty} \int_{\Omega^c} \left(|\nabla u_k|^2 + \lambda_k u_k^2\right) dx = \frac{S}{Q_M^{\frac{N-2}{N}}}.$$

Given $\delta > 0$ we can find $\Lambda > 0$ such that

$$S_k = S(\Omega^c, Q, \lambda_k) = \int_{\Omega^c} \left(|\nabla u_k|^2 + \lambda_k u_k^2\right) dx \ge \frac{S}{Q_M^{\frac{N-2}{N}}} - \delta$$

for every $\lambda_k \ge \Lambda$. In particular, we have

$$\int_{\Omega^c} \left(|\nabla u_k|^2 + \Lambda u_k^2\right) dx \ge S(\Omega^c, Q, \Lambda) \ge \frac{S}{Q_M^{\frac{N-2}{N}}} - \delta$$

for every k. Since $u_k \to 0$ in $L^2(\Omega^c)$ and $\delta > 0$ is arbitrary we see that

$$\lim_{k\to\infty} \int_{\Omega^c} |\nabla u_k|^2\, dx = \frac{S}{Q_M^{\frac{N-2}{N}}}.$$

Hence

$$0 = \lim_{k\to\infty} \lambda_k \int_{\Omega^c} u_k^2\, dx = \lim_{k\to\infty} \lambda_k \epsilon_k^2 \int_{\Omega_k} v_k^2\, dx.$$

Since $v_k \to w \not\equiv 0$ in $C^2_{\text{loc}}(\Omega_\infty)$, our claim follows.

We now set $\hat{S} = \frac{S}{Q_M^{\frac{N-2}{N}}}$. If the case (i) holds we can assume that $P_k \to P_\circ \in \partial\Omega$ and w satisfies

$$\begin{cases} -\Delta w &= \hat{S}Q(P_\circ)w^{2^*-1} \text{ in } \Omega_\infty \\ \frac{\partial w}{\partial \nu} = 0 & \text{on } \partial\Omega_\infty,\ 0 \le w \le 1, \text{ and } w(0) = 1. \end{cases}$$

In this case $\Omega_\infty = \mathbb{R}^N_+$. As a consequence of the Pohozaev identity we have $w(x) = U(\beta x)$ with $\beta^2 = \hat{S}Q(P_\circ)$. Thus the Fatou lemma implies that

$$\frac{\beta^{2-N}S^{\frac{N}{2}}}{2} = \int_{\mathbb{R}^N_+} |\nabla w|^2\,dx \le \lim_{k\to\infty} \int_{\Omega_k} |\nabla v_k|^2\,dx = \frac{S}{Q_M^{\frac{N-2}{N}}}.$$

We deduce from this that

$$\left(\frac{Q(P_\circ)^{\frac{1}{2}}S^{\frac{1}{2}}}{Q_M^{\frac{N-2}{2N}}}\right)^{2-N} \frac{S^{\frac{N}{2}}}{2} \le \frac{S}{Q_M^{\frac{N-2}{N}}}.$$

So we must have $Q_M \le 2^{\frac{2}{N-2}}Q(P_\circ) \le 2^{\frac{2}{N-2}}Q_m$, which is impossible. Consequently the case (ii) prevails. We now show that $\{P_k\}$ is bounded. If not we may assume that $|P_k| \to \infty$ and w satisfies

$$\begin{cases} -\Delta w &= \hat{S}Q_\infty w^{2^*-1} \text{ in } \Omega_\infty \\ 0 \le w \le 1 & \text{on } \Omega_\infty \text{ and } w(0) = 1, \end{cases}$$

where $\Omega_\infty = \mathbb{R}^N$. Hence $w(x) = U(\beta_1 x)$ with $\beta_1 = \left(\hat{S}Q_\infty\right)^{\frac{1}{2}}$ and we get

$$\beta_1^{2-N}S^{\frac{N}{2}} = \int_{\mathbb{R}^N} |\nabla w|^2\,dx \le \lim_{k\to\infty} \int_{\Omega_k} |\nabla v_k|^2\,dx = \frac{S}{Q_M^{\frac{N-2}{N}}}.$$

From this we deduce that $Q_M \le Q_\infty$, which contradicts the assumption. Therefore $\{P_k\}$ must be bounded and, up to a subsequence, $P_k \to P_\circ \in \Omega^c$ ($P_\circ$ is an interior point of Ω^c), and the function w satisfies

$$\begin{cases} -\Delta w &= \hat{S}Q(P_\circ)w^{2^*-1} \text{ in } \mathbb{R}^N \\ 0 \le w \le 1 & \text{on } \mathbb{R}^N \text{ and } w(0) = 1. \end{cases}$$

Hence $w(x) = U(\beta_2 x)$ with $\beta_2 = \left(\hat{S}Q(P_\circ)\right)^{\frac{1}{2}}$ and

$$\beta_2^{2-N}S^{\frac{N}{2}} = \int_{\Omega} |\nabla w|^2\,dx \le \lim_{k\to\infty} \int_{\Omega_k} |\nabla v_k|^2\,dx = \frac{S}{Q_M^{\frac{N-2}{N}}}.$$

This yields $Q_M \le Q(P_\circ)$ and then necessarily $Q(P_\circ) = Q_M$. Hence the last inequality becomes equality and formula (4.2) follows. □

Having the asymptotic relation (4.2) we now aim to show that

$$\frac{J_{\lambda_k}(u_k)}{\left(\int_{\Omega^c} Q|u_k|^{2^*}\,dx\right)^{\frac{2}{2^*}}} > \frac{S}{Q_M^{\frac{N-2}{N}}} = S_\infty \tag{4.3}$$

for large k, which will be seen to be impossible. To achieve this we use a modification of a method that has been used in several papers [4], [19], [24], [10] and [28]. This method will be described in the next section.

5 Behaviour of solutions in the case $S_\infty = \frac{S}{Q_M^{\frac{N-2}{N}}}$

In this section we maintain the assumptions from Section 4. We only describe the main steps of the proof of (4.3). We choose $R > 0$ so that $\Omega^c \cap B(0,R) \neq \emptyset$ and set

$$\mathcal{M}_R = \{CU_{\epsilon,y};\ C \in \mathbb{R},\ \epsilon \in (0,1],\ y \in \overline{\Omega^c \cap B(0,R)}\}.$$

Let

$$d(u,\mathcal{M}_R) = \inf\{\|\nabla u - \nabla v\|_2;\ v \in \mathcal{M}_R\}.$$

Lemma 5.1 *Let $\sigma > 0$ and $\{u_n\} \subset H^1(\Omega^c)$ be such that $u_n \rightharpoonup 0$ in $H^1(\Omega^c)$, $d(u_n,\mathcal{M}_R)^2 \leq \|\nabla u_n\|^2 - 2\sigma$. Then there exist $n_\circ \geq 1$ such that for every $n \geq n_\circ$, $d(u_n,\mathcal{M}_R)$ is achieved by some $C_nU_{\epsilon_n,y_n}$. Furthermore, if $y_n \to y_\circ \in \mathrm{Int}(\Omega^c \cap B(0,R))$ and w_n is defined by*

$$u_n = C_nU_{\epsilon_n,y_n} + w_n, \tag{5.1}$$

then up to a subsequence

- *(i)* $\lim_{n\to\infty} \epsilon_n = 0$,
- *(ii)* $\lim_{n\to\infty} C_n = C_\circ \neq 0$,
- *(iii)* $\int_{\Omega^c} U_{\epsilon_n,y_n}^{2^*-1} w_n\, dx = O\big(\epsilon_n^{\frac{N-2}{2}} \|w_n\|\big)$ *and*
- *(iv)* $\int_{\Omega^c} U_{\epsilon_n,y_n}^{2^*-2} w_n \frac{\partial}{\partial x_i} U_{\epsilon_n,y_n}\, dx = O\big(\epsilon_n^{\frac{N-2}{2}} \|w_n\|\big)$.

Proof For each n let $\{C_{kn}U_{\epsilon_{kn},y_{kn}}\}$ be a minimizing sequence for $d(u_n,\mathcal{M}_R)$. Then there exists $k_\circ \geq 1$ such that for every $k \geq k_\circ$, we have

$$\int_{\Omega^c} |\nabla u_n - \nabla\big(C_{kn}U_{\epsilon_{kn},y_{kn}}\big)|^2\, dx \leq \int_{\Omega^c} |\nabla u_n|^2\, dx - \sigma. \tag{5.2}$$

From this we get

$$C_{kn}^2 \int_{\Omega^c} |\nabla U_{\epsilon_{kn},y_{kn}}|^2\, dx - 2C_{kn} \int_{\Omega^c} \nabla u_n \nabla U_{\epsilon_{kn},y_{kn}}\, dx \leq -\sigma. \tag{5.3}$$

First we show that $C_{kn} \to C_n$ as $k \to \infty$, with $C_n \in \mathbb{R} - \{0\}$. Inequality (5.3) can be rewritten as

$$\begin{aligned} &D_N^2 C_{kn}^2 \int_{\frac{\Omega^c - y_{kn}}{\epsilon_{kn}}} \frac{|z|^2}{\big(N(N-2)+|z|^2\big)^N}\, dz \\ &\quad - \ 2D_N C_{kn} \|\nabla u_n\|_2 \left(\int_{\frac{\Omega^c - y_{kn}}{\epsilon_{kn}}} \frac{|z|^2}{\big(N(N-2)+|z|^2\big)^N}\, dz\right)^{\frac{1}{2}} \leq -\sigma, \end{aligned} \tag{5.4}$$

where $D_N>0$ is a constant depending on N. Since $\{y_{kn}\}$ is bounded we may assume that $y_{kn}\to y_n$ and also $\frac{\Omega^c-y_{kn}}{\epsilon_{kn}}\to\Omega_n\neq\emptyset$ as $k\to\infty$. Thus

$$\int_{\frac{\Omega^c-y_{kn}}{\epsilon_{kn}}}\frac{|x|^2}{\big(N(N-2)+|x|^2\big)^N}\,dx\to B_n\in(0,\infty).$$

So letting $k\to\infty$ in (5.4) we obtain

$$D_N^2C_n^2B_n-2C_nD_N\|\nabla u_n\|_2B_n^{\frac{1}{2}}\leq-\sigma.$$

This implies that up to a subsequence $C_n=\lim_{k\to\infty}C_{kn}\in\mathbb{R}-\{0\}$. We now set $\lim_{k\to\infty}\epsilon_{kn}=\epsilon_n$ (up to a subsequence). We claim that $\epsilon_n>0$ for each n. If $\epsilon_n=0$ along a subsequence, then $U_{\epsilon_{kn},y_{kn}}\rightharpoonup 0$ in $H^1(\Omega^c)$. Letting $k\to\infty$ in (5.3) we get

$$C_n^2\int_{\mathbb{R}^N}|\nabla U|^2\,dx\leq-\sigma$$

provided $y_n\in\mathrm{Int}(\Omega^c\cap B(0,R))$, otherwise the integration is on a half-space. In both cases we have a contradiction. Therefore, $d(u_n,\mathcal{M}_R)$ is attained by $C_nU_{\epsilon_n,y_n}$ for $n\geq n_\circ$. We now assume that $y_n\to y_\circ\in\mathrm{Int}(\Omega^c\cap B(0,R))$. We deduce from (5.3) that

$$C_n^2\int_{\Omega^c}|\nabla U_{\epsilon_n,y_n}|^2\,dx-2C_n\int_{\Omega^c}\nabla u_n\nabla U_{\epsilon_n,y_n}\,dx\leq-\sigma. \tag{5.5}$$

Since $u_n\rightharpoonup 0$ in $H^1(\Omega^c)$, we have $C_n\to C_\circ\neq 0$ and $\epsilon_n\to 0$. Thus (i) and (ii) hold.

To establish (iii) we observe that by (5.1) we have

$$\int_{\Omega^c}\nabla U_{\epsilon_n,y_n}\nabla w_n\,dx=0.$$

Since $y_\circ\in\mathrm{Int}(\Omega^c\cap B(0,R))$, we may assume that $\mathrm{dist}(y_n,\partial(\Omega^c\cap B(0,R))\geq\gamma>0$ for all n and some constant γ. We then have

$$\begin{aligned}0&=\int_{\Omega^c\cap B(0,R)}\nabla U_{\epsilon_n,y_n}\nabla w_n\,dx+\int_{\Omega^c\cap(\mathbb{R}^N-B(0,R))}\nabla U_{\epsilon_n,y_n}\nabla w_n\,dx\\&=\int_{\Omega^c\cap B(0,R)}w_nU_{\epsilon_n,y_n}^{2^*-1}\,dx+\int_{\partial(\Omega^c\cap B(0,R))}w_n\frac{\partial U_{\epsilon_n,y_n}}{\partial\nu}\,dS_x+\int_{\Omega^c\cap(\mathbb{R}^N-B(0,R))}\nabla U_{\epsilon_n,y_n}\nabla w_n\,dx.\end{aligned}$$

Since $\mathrm{dist}(y_n,\partial(\Omega^c\cap B(0,R))\geq\gamma>0$ we have

$$\int_{\Omega^c\cap(\mathbb{R}^N-B(0,R))}w_nU_{\epsilon_n,y_n}^{2^*-1}\,dx=O\big(\epsilon_n^{\frac{N-2}{2}}\|w_n\|\big)$$

and

$$\int_{\Omega^c\cap(\mathbb{R}^N-B(0,R))}\nabla U_{\epsilon_n,y_n}\nabla w_n\,dx=O\big(\epsilon_n^{\frac{N-2}{2}}\|w_n\|\big).$$

Using the embedding trace theorem we get

$$\int_{\partial\left(\Omega^c\cap B(0,R)\right)} w_n \frac{\partial U_{\epsilon_n,y_n}}{\partial\nu}\,dS_x = O\left(\epsilon_n^{\frac{N-2}{2}}\|w_n\|\right)$$

and (iii) follows.

Finally, to establish (iv) we observe that by (5.1) we have

$$\begin{aligned}
0 &= \int_{\Omega^c} \nabla w_n \nabla\big(\frac{\partial}{\partial x_i}U_{\epsilon_n,y_n}\big)\,dx = \int_{\Omega^c\cap B(0,R)} \nabla w_n \nabla\big(\frac{\partial}{\partial x_i}U_{\epsilon_n,y_n}\big)\,dx \\
&+ \int_{\Omega^c\cap(\mathbb{R}^N-B(0,R))} \nabla w_n \nabla\big(\frac{\partial}{\partial x_i}U_{\epsilon_n,y_n}\big)\,dx \\
&= -\int_{\Omega^c\cap B(0,R)} w_n \Delta\big(\frac{\partial}{\partial x_i}U_{\epsilon_n,y_n}\big)\,dx + \int_{\partial(\Omega^c\cap B(0,R))} w_n \frac{\partial}{\partial\nu}\big(\frac{\partial}{\partial x_i}U_{\epsilon_n,y_n}\big)\,dS_x \\
&+ \int_{\Omega^c\cap(\mathbb{R}^N-B(0,R))} \nabla w_n \nabla\big(\frac{\partial}{\partial x_i}U_{\epsilon_n,y_n}\big)\,dx \\
&= (2^*-1)\int_{\Omega^c\cap B(0,R)} U_{\epsilon_n,y_n}^{2^*-2} w_n \frac{\partial}{\partial x_i}U_{\epsilon_n,y_n}\,dx + \int_{\partial(\Omega^c\cap B(0,R))} w_n \frac{\partial}{\partial\nu}\big(\frac{\partial}{\partial x_i}U_{\epsilon_n,y_n}\big)\,dS_x \\
&+ \int_{\Omega^c\cap(\mathbb{R}^N-B(0,R))} \nabla w_n \nabla\big(\frac{\partial}{\partial x_i}U_{\epsilon_n,y_n}\big)\,dx.
\end{aligned}$$

Again using the fact that $\mathrm{dist}(y_n,\partial(\Omega\cap B(0,R))\geq\gamma$ we get

$$\int_{\Omega^c\cap(\mathbb{R}^N-B(0,R))} U_{\epsilon_n,y_n}^{2^*-2} w_n \frac{\partial}{\partial x_i}U_{\epsilon_n,y_n}\,dx = O\left(\epsilon^{\frac{N-2}{2}}\|w_n\|\right)$$

and

$$\int_{\Omega^c\cap\left(\mathbb{R}^N-B(0,R)\right)} \nabla w_n \nabla\big(\frac{\partial}{\partial x_i}U_{\epsilon_n,y_n}\big)\,dx = O\left(\epsilon^{\frac{N-2}{2}}\|w_n\|\right).$$

To complete the proof of (iv) we apply the embedding trace theorem. □

In the next step we use the weighted eigenvalue problem

$$\text{(5.6)}\qquad \begin{cases} -\Delta\phi &= \nu U^{2^*-2}\phi \ \text{ in } \mathbb{R}^N \\ \int_{\mathbb{R}^N} U^{2^*-2}\phi^2\,dx &< \infty. \end{cases}$$

Lemma 5.2 *Let $N\geq 5$. The eigenvalue problem (5.6) has a discrete set of eigenvalues $\nu_1<\nu_2\leq\nu_3\leq\ldots$ with orthonormal eigenfunctions $\psi_1,\psi_2,\ldots$ such that $\nu_1=1$, $\nu_i=2^*-1$ for $2\leq i\leq N+1$ and $\nu_{N+2}>2^*-1$. The eigenspaces E_1 and E_{2^*-1} corresponding to 1 and 2^*-1 are given by*

$$\begin{aligned}
E_1 &= \text{span}\,\{U\}, \\
E_{2^*-1} &= \text{span}\,\{\frac{\partial U_{1,y}}{\partial y_i}\Big|_{y=0};\ i=1,\ldots,N\}.
\end{aligned}$$

We now consider the linearized eigenvalue problem corresponding to equation (1.1) on the exterior domain Ω^c:

$$\begin{cases} -\Delta u + \lambda(\epsilon)u &= \mu U_{\epsilon,P_\epsilon}^{2^*-2}u \text{ in } \Omega^c, \\ \frac{\partial u}{\partial \nu} = 0 \text{ on } & \partial\Omega, \end{cases} \tag{5.7}$$

where $\lambda(\epsilon) > 0$, $\epsilon > 0$, $P_\epsilon \to P_\circ \in \Omega^c$ as $\epsilon \to 0$. Here we assume that $P_\circ$ is an interior point of Ω^c. Let $\{\phi_{i\epsilon}\}$, $i = 1, \dots, \infty$, be a complete set of orthonormal eigenfunctions corresponding to the eigenvalues $0 < \mu_{1,\epsilon} \leq \mu_{2,\epsilon} \leq \dots$ of problem (5.7). In the sequel we assume that

$$\int_{\Omega^c} U_{\epsilon,P_\epsilon}^{2^*-2}\phi_{i,\epsilon}\phi_{j,\epsilon}\,dx = \delta_{ij}.$$

Lemma 5.3 *Assume that $\lambda(\epsilon) \to \infty$, $\lambda(\epsilon)\epsilon^2 \to 0$ as $\epsilon \to 0$. We then have up to a subsequence*

(i) $\lim_{\epsilon\to 0}\mu_{i\epsilon} = \mu_i$,

(ii) $\lim_{\epsilon\to 0}\int_{\Omega_\epsilon} U^{2^*-2}\big(\tilde\phi_{i\epsilon} - \phi_i\big)^2\,dx = 0$,

where $\Omega_\epsilon = \frac{\Omega^c - P_\epsilon}{\epsilon}$, $\tilde\phi_{i\epsilon}(x) = \epsilon^{\frac{N-2}{2}}\phi_{i\epsilon}(\epsilon x + P_\epsilon)$ and (μ_i, ϕ_i) satisfies (5.6). In particular, we have $\mu_1 = \nu_1 = 1$, $\phi_1 = \psi_1 = CU$ for some $C > 0$ and there exists $2 \leq k_\circ \leq N+1$ such that $\mu_i = \nu_i = 2^* - 1$ for $2 \leq i \leq k_\circ$ and $\mu_{k_\circ+1} > 2^* - 1$. Moreover $\{\phi_i\}$, $i = 2, \dots, k_\circ$, belong to span $\{\frac{\partial U_{1,y}}{\partial y_i}\Big|_{y=0}, i = 1, \dots, N\}$.

We now state the crucial estimate needed to prove (4.3).

Lemma 5.4 *Let $\epsilon > 0$, $\lambda(\epsilon) > 0$ be such that $\lambda(\epsilon) \to \infty$, $\lambda(\epsilon)\epsilon^2 \to 0$ as $\epsilon \to 0$. Then there exist $0 < \delta < 1$ and $\epsilon_\circ > 0$ such that for all $0 < \epsilon \leq \epsilon_\circ$*

$$\int_{\Omega^c} U_{\epsilon,P_\epsilon}^{2^*-2}w^2\,dx \leq \delta\big(\|\nabla w\|_2^2 + \lambda(\epsilon)\|w\|_2^2\big) \tag{5.8}$$

for each w orthogonal to the tangent space of $\mathcal{M}_R$ at U_{ϵ,P_ϵ}.

The proof is based on Lemma 5.3 (see the proof of Lemma 5.9 in [24]).

Lemma 5.5 *Let $\lambda_n > 0$, $\delta_n > 0$, $y_n \in \overline{\Omega^c \cap B(0,R)}$ and $\{u_n\} \subset H^1(\Omega^c)$ be such that $\lambda_n \to \infty$, $\delta_n \to 0$, $y_n \to P_\circ \in \Omega^c \cap B(0,R)$ (an interior point), $u_n \rightharpoonup 0$ in $H^1(\Omega^c)$ and*

$$\lim_{n\to\infty}\int_{\Omega^c} |\nabla\big(u_n - U_{\delta_n,y_n}\big)|^2\,dx = 0. \tag{5.9}$$

Then up to a subsequence

$$\frac{J_\lambda(u_n)}{\Big(\int_{\Omega^c}|u_n|^{2^*}\,dx\Big)^{\frac{2}{2^*}}} > S. \tag{5.10}$$

Proof Since $d(u_n, \mathcal{M}_R) \to 0$, we can apply Lemma 5.1 to get a decomposition

$$u_n = C_n U_{\epsilon,\bar{y}_n} + w_n, \tag{5.11}$$

with $\|w_n\| \to 0$ and $\{\bar{y}_n\} \subset \overline{\Omega^c \cap B(0,R)}$. By a standard argument (see [4], [24], [28]) we show that $C_n \to 1$, $\frac{\delta_n}{\epsilon_n} \to 1$ and $\bar{y}_n \to P_\circ$. So we can assume that (5.9) holds with $\bar{y}_n = y_n$ and $\delta_n = \epsilon_n$. Having the decomposition (5.11) we can show that (5.10) is valid for a subsequence. In the proof we use the estimates (iii) and (iv) of Lemma 5.1, (5.8) of Lemma 5.4 and the fact that $\int_{\Omega^c} \nabla w_n \nabla U_{\epsilon_n,y_n}\, dx = 0$. For technical details we refer to the proof of Lemma 4.7 in [24]. □

We are now in a position to state the main result of this section.

Theorem 5.6 *Let $N \geq 5$ and $Q_\infty < Q_M$. Suppose that (B) holds. Then $\Lambda < \infty$.*

Proof Arguing by contradiction assume that $\Lambda = \infty$. Let $\lambda_n \to \infty$. Then for each n problem (1.1) with $\lambda = \lambda_n$ has a least energy solution u_n verifying

$$\frac{J_{\lambda_n}(u_n)}{\left(\int_{\Omega^c} Q(x)|u_n|^{2^*}\, dx\right)^{\frac{2}{2^*}}} < \frac{S}{Q_M^{\frac{N-2}{N}}}$$

and moreover (4.2) holds. We set $v_n = Q_M^{\frac{1}{2^*}} u_n$. Then $J_{\lambda_n}(v_n) = Q_M^{\frac{2}{2^*}} J_{\lambda_n}(u_n) < S$ and

$$\lim_{n\to\infty} \int_{\Omega^c} |\nabla (v_n - \sigma_n^{-\frac{N-2}{2}} S^{-\frac{N-2}{4}} U_{\sigma_n,y_n})|^2\, dx = 0,$$

where $\sigma_n = \frac{\epsilon_n}{S^{\frac{1}{2}} Q_M^{\frac{1}{N}}}$. Applying Lemma 5.5 we get a contradiction. □

6 Case $S_\infty = \frac{S}{Q_\infty^{\frac{N-2}{N}}}$

In this case we have $Q(x) < Q_\infty$ for all $x \in \Omega^c$ (otherwise we are in case 4). We assume that there exists some cone $K \subset \mathbb{R}^N$ on which the convergence $Q(x) \to Q_\infty$ is sufficiently fast. More precisely, we assume

(**C**) There exist $\bar{z} \in \partial B(0,1)$, $\delta > 0$ and $c > 0$ such that for $R \to \infty$

$$0 < Q_\infty - Q(R\bar{y}) \leq \frac{c}{R^p}\ , \quad \text{with} \quad p > \frac{N^2}{2}\ ,$$

for every $\bar{y} \in \partial B(0,1) \cap B(\bar{z}, \delta)$. Furthermore, we assume that $Q_\infty > 2^{\frac{2}{N-2}} Q_m$.

Example Let $x = (x_1, \dots, x_N)$ and let $\bar{x}_i$ denote the i^{th} unitary vector in $\mathbb{R}^N$. Consider

$$Q(x) = A\left(1 - \frac{1}{1 + |x_1|^p + |x_2| + \cdots + |x_N|}\right) .$$

Then, for Ω given and a suitable choice of A, assumption (C) is satisfied for $\delta = 1/2$; indeed, $Q_\infty = A$ and for $|\bar{x} - \bar{x}_1| \le \frac{1}{2}$ one has $Q_\infty - Q(R\bar{x}) \le A - A(1 - \frac{1}{(\frac{1}{2}R)^p}) = A\frac{2^p}{R^p}$.

Theorem 6.1 *Suppose (C) holds. Then there exists $\Lambda > 0$ such that for $0 \le \lambda \le \Lambda$ problem (1.1) admits a least energy solution.*

Proof In this case we have $S_\infty = \frac{S}{Q_\infty^{\frac{N-2}{N}}}$. We now show that $S(\Omega^c, Q, \lambda) < \frac{S}{Q_\infty^{\frac{N-2}{N}}}$. Let $\bar{z} \in \partial B(0,1)$ as in (C). Let $R > 0$ be such that $\Omega \subset B(0, \frac{R}{2})$ and let K_δ denote the cone $K_\delta = \{s\bar{y};\ \bar{y} \in \partial B(0,1) \cap B(\bar{z}, \delta), s \ge 0\}$. We then have

$$\begin{aligned}
\int_{\Omega^c} Q(x) U_{\epsilon,R\bar{z}}^{2^*}\,dx &= \int_{B(0,\frac{R}{2})-\Omega} Q(x) U_{\epsilon,R\bar{z}}^{2^*}\,dx + \int_{\mathbb{R}^N - B(0,\frac{R}{2})} \big(Q(x) - Q_\infty\big) U_{\epsilon,R\bar{z}}^{2^*}\,dx \\
&+ \int_{\mathbb{R}^N - B(0,\frac{R}{2})} Q_\infty U_{\epsilon,R\bar{z}}^{2^*}\,dx \\
&\ge \int_{\mathbb{R}^N - \left(B(0,\frac{R}{2}) \cup K_\delta\right)} \big(Q(x) - Q_\infty\big) U_{\epsilon,R\bar{z}}^{2^*}\,dx \\
&+ \int_{K_\delta - B(0,\frac{R}{2})} \big(Q(x) - Q_\infty\big) U_{\epsilon,R\bar{z}}^{2^*}\,dx \\
&+ \int_{\mathbb{R}^N} Q_\infty U_{\epsilon,R\bar{z}}^{2^*}\,dx - \int_{B(0,\frac{R}{2})} Q_\infty U_{\epsilon,R\bar{z}}^{2^*}\,dx = I_1 + I_2 + Q_\infty K_2 + I_3.
\end{aligned}$$

We estimate the integrals as follows:

$$\begin{aligned}
I_1 &= \int_{\mathbb{R}^N - \left(B(0,\frac{R}{2}) \cup K_\delta\right)} \big(Q(x) - Q_\infty\big) U_{\epsilon,R\bar{z}}^{2^*}\,dx \\
&\ge -Q_\infty \int_{\mathbb{R}^N - \left(B(0,\frac{R}{2}) \cup K_\delta\right)} \frac{\epsilon^N c_N^{2^*}}{\big(N(N-2)\epsilon^2 + |x - R\bar{z}|^2\big)^N}\,dx;
\end{aligned}$$

since $|x - R\bar{z}| > \delta R$, for $x \in B(0, 2R) - \big(B(0, \frac{R}{2}) \cup K_\delta\big)$, and $|x - R\bar{z}| \ge \frac{|x|}{2}$, for $x \in \mathbb{R}^N - B(0, 2R)$, we can continue the estimate

$$\ge -Q_\infty \epsilon^N c_N^{2^*} \int_{\frac{R}{2}}^{2R} \frac{r^{N-1}}{(\delta R)^{2N}}\,dr - Q_\infty \epsilon^N c_N^{2^*} \int_{2R}^{\infty} \frac{r^{N-1}}{\left(\frac{r}{2}\right)^{2N}}\,dr = -C_1 \frac{\epsilon^N}{R^N}$$

for some constant $C_1 > 0$. We now estimate the integral I_2, using assumption (C):

$$I_2 = \int_{K_\delta - B(0,\frac{R}{2})} \big(Q(x) - Q_\infty\big) U_{\epsilon,R\bar{z}}^{2^*}\,dx \ge -\frac{c2^p}{R^p} \int_{\mathbb{R}^N} U_{\epsilon,R\bar{z}}^{2^*}\,dx = -\frac{CK_2}{R^p}.$$

Finally, for I_3 we have, using that $|x - R\bar{z}| \geq \frac{R}{2}$ for $x \in B(0, \frac{R}{2})$

$$I_3 = -\int_{B(0,\frac{R}{2})} Q_\infty U_{\epsilon,R\bar{z}}^{2^*}\,dx \geq -\frac{Q_\infty c_N^{2^*} 2^{2N}}{R^{2N}} \int_0^{\frac{R}{2}} r^{N-1}\,dr = -\frac{C_2 \epsilon^N}{R^N}$$

for some constant $C_2 > 0$. Thus we have

$$\int_{\Omega^c} Q(x) |U_{\epsilon,R\bar{z}}|^{2^*}\,dx \geq K_2 Q_\infty - C_3 \frac{\epsilon^N}{R^N} - \frac{CK_2}{R^p},$$

where $C_3 = C_1 + C_2$. On the other hand we have

$$\begin{aligned}
\int_{\Omega^c} |\nabla U_{\epsilon,R\bar{z}}|^2\,dx &= \int_{\mathbb{R}^N} |\nabla U_{\epsilon,R\bar{z}}|^2\,dx - \int_\Omega |\nabla U_{\epsilon,R\bar{z}}|^2\,dx \\
&\leq K_1 - c_N^2 (N-2)^2 \epsilon^{N-2} \int_\Omega \frac{|x - R\bar{z}|^2}{\left(\epsilon^2 N(N-2) + |x - R\bar{z}|^2\right)^N}\,dx \\
&\leq K_1 - \frac{C_4 \epsilon^{N-2}}{R^{2N-2}},
\end{aligned}$$

for some constant $C_4 > 0$. These inequalities lead to the following estimate

$$\frac{J_\lambda(U_{\epsilon,R\bar{z}})}{\left(\int_{\Omega^c} Q(x) U_{\epsilon,R\bar{z}}^{2^*}\,dx\right)^{\frac{2}{2^*}}} \leq \frac{K_1 - C_4 \frac{\epsilon^{N-2}}{R^{2N-2}} + \lambda C_5 \epsilon^2}{\left(K_2 Q_\infty - C_3 \frac{\epsilon^N}{R^N} - \frac{CK_2}{R^p}\right)^{\frac{N-2}{N}}}$$

for some constant $C_5 > 0$. From this we conclude that there exist constants $A > 0$, $B > 0$, $C > 0$ and $D > 0$ such that

$$\frac{J_\lambda(U_{\epsilon,R\bar{z}})}{\left(\int_{\Omega^c} Q(x) U_{\epsilon,R\bar{z}}^{2^*}\,dx\right)^{\frac{2}{2^*}}} \leq \frac{S}{Q_\infty^{\frac{N-2}{N}}} - \frac{A\epsilon^{N-2}}{R^{2N-2}} + \lambda B \epsilon^2 + \frac{C\epsilon^N}{R^N} + \frac{D}{R^p}.$$

We want the negative term to dominate (for $\lambda = 0$); for this we may choose $\epsilon = \epsilon(R)$ such that $\frac{A\epsilon^{N-2}}{2R^{2N-2}} = \frac{C\epsilon^N}{R^N}$, that is, $\frac{1}{\epsilon^2} = \frac{2CR^{N-2}}{A}$. Then we choose $R > 0$ so that

$$\frac{A\epsilon^{N-2}}{2R^{2N-2}} = \frac{A}{2R^{2N-2}} \left(\frac{A}{2CR^{N-2}}\right)^{\frac{N-2}{2}} > \frac{D}{R^p},$$

which is possible if $p > \frac{N^2}{2}$. For this choice of R we have

$$\frac{J_\lambda(U_{\epsilon,R\bar{z}})}{\left(\int_{\Omega^c} Q(x) U_{\epsilon,R\bar{z}}^{2^*}\,dx\right)^{\frac{2}{2^*}}} < \frac{S}{Q_\infty^{\frac{N-2}{N}}}$$

for $\lambda = 0$. Clearly, this inequality remains valid for small $\lambda > 0$. The existence of a least energy solution follows from Proposition 2.1. □

It is worth to mention that if Ω is a ball and $Q(x)$ is radial, i.e. $Q(x) = Q(|x|)$, then assumption (C) can be written as:
$Q(x) = A - f(|x|)$ for $x \in \Omega^c$, where $A > 0$ is a constant and f is a continuous function such that $0 \le f(s) < A$ for $s \ge 0$ and

$$\lim_{s\to\infty} f(s)s^{\frac{N^2}{2}} = 0, \quad A\big(1 - 2^{-\frac{2}{N-2}}\big) < \inf_{x\in\partial\Omega} f(|x|).$$

7 Applications to critical problems on $\mathbb{R}^N$

In the case where $\Omega = B(0,1)$ Theorem 4.1 and Proposition 6.1 give rise to the existence of positive solutions of the problem on $\mathbb{R}^N$ involving the critical Sobolev exponent

$$-\Delta u = \tilde{Q}(x)|u|^{2^*-2}u \ \text{ on } \ \mathbb{R}^N, \tag{7.1}$$

which are not energy minimizing. Here $\tilde{Q}$ is an extension of Q obtained by reflection through $\partial B(0,1)$. We use the approach from the paper [19]. Suppose that (B) and (4.1) are satisfied on $B(0,1)^c$. According to Theorem 4.1 there exists a least energy solution u of (1.1) with $\lambda = 0$. Since u is a minimizer for $S(B(0,1)^c, Q, 0)$, we have

$$\frac{\int_{B(0,1)^c} |\nabla u|^2\,dx - \frac{N-2}{2}\int_{\partial B(0,1)} u^2\,dS_x}{\Big(\int_{B(0,1)^c} Q(x)|u|^{2^*}\,dx\Big)^{\frac{2}{2^*}}} < \frac{S}{Q_M^{\frac{N-2}{N}}}.$$

Therefore the following problem

$$\begin{cases} -\Delta v &= Q(x)|v|^{2^*-2}v \ \text{ in } \ B(0,1)^c, \\ \frac{\partial v}{\partial \nu} + \frac{N-2}{2}v &= 0 \ \text{ on } \ \partial B(0,1) \end{cases} \tag{7.2}$$

has a positive solution v. This can be established with the aid of the concentration-compactness principle. This solution obviously satisfies

$$\frac{\int_{B(0,1)^c} |\nabla v|^2\,dx - \frac{N-2}{2}\int_{\partial B(0,1)} v^2\,dS_x}{\Big(\int_{B(0,1)^c} Q(x)|v|^{2^*}\,dx\Big)^{\frac{2}{2^*}}} < \frac{S}{Q_M^{\frac{N-2}{N}}}. \tag{7.3}$$

We now define a function w on $\mathbb{R}^N$ by

$$w(x) = \begin{cases} |x|^{-(N-2)}v\big(\frac{x}{|x|^2}\big) & \text{for } x \in B(0,1) - \{0\} \\ v(x) & \text{for } x \in B(0,1)^c. \end{cases}$$

Then w satisfies (7.1) with $\tilde{Q}(x)$ given by

$$\tilde{Q}(x) = \begin{cases} Q(x), & x \in B(0,1)^c \\ Q\big(\frac{x}{|x|^2}\big), & x \in B(0,1) \end{cases} \tag{7.4}$$

We now observe that

$$\begin{aligned}
\int_{\mathbb{R}^N} |\nabla w|^2\,dx &= \int_{B(0,1)} |\nabla w|^2\,dx + \int_{B(0,1)^c} |\nabla w|^2\,dx \\
&= \int_{B(0,1)^c} |\nabla v|^2\,dx - (N-2)\int_{\partial B(0,1)} w^2\,dS_x + \int_{B(0,1)^c} |\nabla v|^2\,dx \\
&= 2\left(\int_{B(0,1)^c} |\nabla v|^2\,dx - \frac{N-2}{2}\int_{\partial B(0,1)} v^2\,dS_x\right)
\end{aligned}$$

and similarly

$$\int_{\mathbb{R}^N} \tilde{Q}|w|^{2^*}\,dx = 2\int_{B(0,1)} Q(x)|v|^{2^*}\,dx.$$

Thus

$$\frac{\int_{\mathbb{R}^N} |\nabla w|^2\,dx}{\left(\int_{\mathbb{R}^N} \tilde{Q}|w|^{2^*}\,dx\right)^{\frac{2}{2^*}}} = \frac{2\left(\int_{B(0,1)^c} |\nabla v|^2\,dx - \frac{N-2}{2}\int_{\partial B(0,1)} v^2\,dS_x\right)}{\left(2\int_{B(0,1)^c} |v|^{2^*}\,dx\right)^{\frac{2}{2^*}}} < \frac{2^{\frac{2}{N}}S}{Q_M^{\frac{N-2}{N}}}.$$

Since $\tilde{Q}(x) \le Q_M$ with strict inequality on a set of positive measure we see that

$$\frac{S}{Q_M^{\frac{N-2}{N}}} < \frac{\int_{\mathbb{R}^N} |\nabla w|^2\,dx}{\left(\int_{\mathbb{R}^N} \tilde{Q}(x)|w|^{2^*}\,dx\right)^{\frac{2}{2^*}}} < \frac{2^{\frac{2}{N}}S}{Q_M^{\frac{N-2}{N}}}. \tag{7.5}$$

The above considerations allow us to state the following existence result for (7.1)

Proposition 7.1 *Suppose that assumption (B) and (4.1) hold and let $\tilde{Q}$ be defined by (7.4). Then equation (7.1) has a positive solution satisfying (7.5).*

In a similar manner we can establish the existence result under assumption (C).

Proposition 7.2 *Suppose that assumption (C) holds (we assume in addition that $\lim_{|x|\to\infty} Q(x) = Q_\infty$ exists) and let $\tilde{Q}$ be defined by (7.4). Then equation (7.1) has a positive solution u satisfying*

$$\frac{S}{Q_\infty^{\frac{N-2}{N}}} < \frac{\int_{\mathbb{R}^N} |\nabla u|^2\,dx}{\left(\int_{\mathbb{R}^N} \tilde{Q}(x)|u|^{2^*}\,dx\right)^{\frac{2}{2^*}}} < \frac{2^{\frac{2}{N}}S}{Q_\infty^{\frac{N-2}{N}}}.$$

We close this paper with a nonexistence result for the exterior problem on the ball $B(0,1)$.

Proposition 7.3 *Suppose that $Q(x) < Q_M = Q_m$, for all $x \in \mathbb{R}^N$. Then problem (1.1) with $\Omega = B(0,1)$ and $\lambda = 0$ does not have a least energy solution.*

Proof Assuming that there exists a least energy solution u of problem (1.1) we obtain, as in the proof of Proposition 7.1, a solution v of problem (7.2). This by a reflection leads to a solution w of equation (7.1). This solution w verifies

$$\frac{\int_{\mathbb{R}^N} |\nabla w|^2\,dx}{\left(\int_{\mathbb{R}^N} \tilde{Q}|w|^{2^*}\,dx\right)^{\frac{2}{2^*}}} = \frac{2\left(\int_{B(0,1)^c} |\nabla v|^2\,dx - \frac{N-2}{2}\int_{\partial B(0,1} v^2\,dS_x\right)}{\left(2\int_{B(0,1)^c} Q(x)|v|^{2^*}\,dx\right)^{\frac{2}{2^*}}} \leq 2^{2/N}\frac{S}{2^{2/N}Q_m^{\frac{N-2}{N}}} = \frac{S}{Q_M^{\frac{N-2}{N}}}.$$

On the other hand we always have

$$\frac{\int_{\mathbb{R}^N} |\nabla w|^2\,dx}{\left(\int_{\mathbb{R}^N} \tilde{Q}(x)|w|^{2^*}\,dx\right)^{\frac{2}{2^*}}} > \frac{S}{Q_M^{\frac{N-2}{N}}}$$

which is impossible. □

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