EXTERNAL DERIVATIONS OF INTERNAL GROUPOIDS

S. KASANGIAN, S. MANTOVANI, G. METERE, AND E.M. VITALE

ABSTRACT. If H is a G-crossed module, the set of derivations of G in H is a monoid under Whitehead product of derivations. We interpret Whitehead product using the correspondence between crossed modules and internal groupoids in the category of groups. Working in the general context of internal groupoids in a finitely complete category, we relate derivations to holomorphisms and translations.

1. Introduction

Let G be a group and $\varphi\colon G \to AutH$ a G-group. A derivation of G in H is a map $d\colon G \to H$ such that $d(xy) = d(x) + x \cdot d(y)$. If H is a G-module, i.e. if H is abelian, the set $Der(G,H)$ of derivations is an abelian group w.r.t. the point-wise sum. If H is not abelian, in general $Der(G,H)$ is just a pointed set (the zero-morphism $0\colon G \to H$ is a derivation). Whitehead [8] discovered the following fact.

1.1. THEOREM. *Let* $(H \xrightarrow{\partial} G \xrightarrow{\varphi} AutH)$ *be a crossed module of groups. The set* $Der(G,H)$ *is a monoid w.r.t.* $(d_1 + d_2)(x) = d_1(\partial(d_2(x))x) + d_2(x)$.

The aim of this note is to understand in a more conceptual way Whitehead product of derivations. The idea is to replace crossed modules of groups by the equivalent notion of internal groupoids in the category of groups. Using the language of internal groupoids, Whitehead product becomes clear: it is nothing but the composition in the internal category. The surprise is that, once expressed in terms of internal groupoids, Whitehead theorem, as well as some other basic properties of derivations, has nothing to do with groups, but holds in the very general context of internal groupoids in an arbitrary category $\mathcal{G}$ with finite limits. In this way, these results hold not only for crossed module of groups (when $\mathcal{G}$ is the category of groups), but also for crossed modules of Lie algebras (take for $\mathcal{G}$ the category of Lie algebras), Lie groupoids (take for $\mathcal{G}$ the category of smooth manifolds), étale groupoids (take for $\mathcal{G}$ the category of topological spaces and local homeomorphisms), and of course ordinary groupoids (take for $\mathcal{G}$ the category of sets).

Note that to explain the quoted theorem is also the aim of Gilbert's paper [3]. Gilbert explains Whitehead product of derivations replacing crossed modules by the equivalent notion of groups in the category of groupoids, whereas we use groupoids in the category of groups. Even if the equivalence between groups in groupoids and groupoids in groups is a trivial fact, the advantage of working with groupoids in groups is that this immediately suggests the more general context of internal groupoids in any finitely complete category. This gain of generality allows us, for example, to include in the same theory

---

Financial support by INDAM and FNRS grant 1.5.121.6 are gratefully acknowledged.
2000 Mathematics Subject Classification: ???.
Key words and phrases: ???.

the holomorph of a group: this is possible because a group is a particular groupoid in sets. In fact, it is precisely this easy example the guiding example to describe derivations using holomorphisms and translations as in Sections 4 and 5. Moreover, it is a fact that several definitions, constructions and proofs become more transparent having in mind the set-theoretical case instead of the group-theoretical case. Finally, since internal groupoids are the objects of a 2-category, we can exploit some general 2-categorical facts to define derivations and translations.

2. The monoid of derivations

In this section, we construct the monoid of $\mathbb{C}$-derivations, for $\mathbb{C}$ an internal groupoid.

We fix, once for all, a category $\mathcal{G}$ with finite limits. The notation for an internal groupoid $\mathbb{C}$ in $\mathcal{G}$ is

$$\mathbb{C} = \left(C_0 \underset{cod}{\overset{dom}{\underset{\xrightarrow{\;u\;}}{\overleftarrow{}}\overleftarrow{}}} C_1 \xleftarrow{\;\circ\;} C_1 \times_{C_0} C_1 \;,\; C_1 \xrightarrow{\;(\,)^{-1}\;} C_1 \right)$$

where $C_1 \times_{C_0} C_1$ is the object of "composable pairs", that is

$$\begin{array}{ccc} C_1 \times_{C_0} C_1 & \xrightarrow{\;\pi_2\;} & C_1 \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle dom} \\ C_1 & \xrightarrow[\;cod\;]{} & C_0 \end{array}$$

is a pullback in $\mathcal{G}$. We also write $\circ^2 \colon C_1 \times_{C_0} C_1 \times_{C_0} C_1 \to C_1$ for the diagonal of the commutative square

$$\begin{array}{ccc} C_1 \times_{C_0} C_1 \times_{C_0} C_1 & \xrightarrow{\;\pi_{2,3}\;} & C_1 \times_{C_0} C_1 \\ {\scriptstyle \pi_{1,2}}\downarrow & & \downarrow{\scriptstyle \circ} \\ C_1 \times_{C_0} C_1 & \xrightarrow[\;\circ\;]{} & C_1 \end{array}$$

where

$$\begin{array}{ccccc} C_1 & \xleftarrow{\;\pi_1\;} & C_1 \times_{C_0} C_1 \times_{C_0} C_1 & \xrightarrow{\;\pi_3\;} & C_1 \\ {\scriptstyle cod}\downarrow & & {\scriptstyle \pi_2}\downarrow & & \downarrow{\scriptstyle dom} \\ C_0 & \xleftarrow[\;dom\;]{} & C_1 & \xrightarrow[\;cod\;]{} & C_0 \end{array}$$

is a limit in $\mathcal{G}$.

We denote by $Grpd(\mathcal{G})$ the 2-category of internal groupoids, internal functors and internal natural transformations. For $\mathbb{C}, \mathbb{B}$ internal groupoids, $Grpd(\mathcal{G})(\mathbb{C}, \mathbb{B})$ is the hom-category, and we write

$$[Grpd(\mathcal{G})(\mathbb{C}, \mathbb{B})]_1 \underset{cod}{\overset{dom}{\rightrightarrows}} [Grpd(\mathcal{G})(\mathbb{C}, \mathbb{B})]_0$$

for its sets of arrows and of objects, together with the domain and the codomain maps. In particular, $Grpd(\mathcal{G})(\mathbb{C},\mathbb{C})$ is a strict monoidal category: tensor product is composition of internal functors and horizontal composition of internal natural transformations, the unit object is the identity functor on $\mathbb{C}$. As with any strict monoidal category, the map

$$cod\colon [Grpd(\mathcal{G})(\mathbb{C},\mathbb{C})]_1 \to [Grpd(\mathcal{G})(\mathbb{C},\mathbb{C})]_0$$

is an homomorphism of monoids.

2.1. Definition. *The monoid of $\mathbb{C}$-derivations is the kernel of the codomain map*

$$Der\mathbb{C} = Ker(cod) \to [Grpd(\mathcal{G})(\mathbb{C},\mathbb{C})]_1 \to [Grpd(\mathcal{G})(\mathbb{C},\mathbb{C})]_0$$

Explicitly, a $\mathbb{C}$-derivation is a pair (D,d)

$$\mathbb{C} \underset{Id}{\overset{D}{\rightrightarrows}} \mathbb{C} \qquad \Downarrow d$$

with D an internal functor and d an internal natural transformation.

When $\mathcal{G}$ is the category of sets, to give a $\mathbb{C}$-derivation just means to choose, for each object x of $\mathbb{C}$, an arrow

$$d(x)\colon dom(d(x)) \to x$$

with codomain x. This suggests to describe derivations as sections of the codomain arrow.

2.2. Proposition. *To give a $\mathbb{C}$-derivation amounts to give an arrow $d\colon C_0 \to C_1$ such that the diagram*

$$\begin{array}{ccc} & & C_1 \\ & \overset{d}{\nearrow} & \downarrow{\scriptstyle cod} \\ C_0 & \xrightarrow[1]{} & C_0 \end{array} \qquad (1)$$

commutes.

Proof. Such an arrow d given, we have to construct an internal functor $D\colon \mathbb{C} \to \mathbb{C}$ in such a way that d becomes an internal natural transformation $d\colon D \Rightarrow Id$. The following picture explains the set-theoretical idea behind the construction of D.

$$\begin{array}{ccccc} x & & D_0(x) = dom(d(x)) & \xrightarrow{d(x)} & x \\ {\scriptstyle a}\downarrow & \longmapsto & \vdots\,{\scriptstyle D_1(a)} & & \downarrow{\scriptstyle a} \\ y & & D_0(y) = dom(d(y)) & \xleftarrow[d(y)^{-1}]{} & y \end{array}$$

It suffices now to internalize this idea:

- On objects, the functor $D\colon \mathbb{C} \to \mathbb{C}$ is defined by

$$D_0\colon\ C_0 \xrightarrow{d} C_1 \xrightarrow{dom} C_0$$

- As far as arrows are concerned, we consider the diagram

$$\begin{array}{ccccccccc}
C_0 & \xleftarrow{\quad dom \quad} & & & C_1 & \xrightarrow{cod} & C_0 & \xrightarrow{d} & C_1 \\
{\scriptstyle d}\big\downarrow & & & & {\scriptstyle 1}\big\downarrow & & & & \big\downarrow{\scriptstyle (\ )^{-1}} \\
C_1 & \xrightarrow[cod]{} & C_0 & \xleftarrow[dom]{} & C_1 & \xrightarrow[cod]{} & C_0 & \xleftarrow[dom]{} & C_1
\end{array}$$

By equation (1), this diagram commutes, and we get a unique factorization of the projective cone through the object of composable triples, say

$$\overline{d} = \langle dom \cdot d, 1, cod \cdot d \cdot (\ )^{-1} \rangle \colon C_1 \to C_1 \times_{C_0} C_1 \times_{C_0} C_1$$

Finally, the functor $D\colon \mathbb{C} \to \mathbb{C}$ is defined on arrows by

$$D_1\colon\ C_1 \xrightarrow{\overline{d}} C_1 \times_{C_0} C_1 \times_{C_0} C_1 \xrightarrow{\qquad \circ^2 \qquad} C_1$$

■

We wish now to describe explicitly the operations in $Der\mathbb{C}$ using Proposition 2.2:

- The unit in $Der\mathbb{C}$ is $u\colon C_0 \to C_1$.

- The multiplication in $Der\mathbb{C}$ is the internal version of

$$z \xrightarrow{\qquad d_1(y) \qquad} y = dom(d_2(x)) \xrightarrow{\qquad d_2(x) \qquad} x$$

(In other words, Whitehead product of derivations is just the internal composition in $\mathbb{C}$.) This means that, given two derivations $d_1, d_2\colon C_0 \to C_1$, we start constructing the arrow

$$d_1 \star d_2 = \langle d_2 \cdot dom \cdot d_1, d_2 \rangle \colon C_0 \to C_1 \times_{C_0} C_1$$

and we get the product of d_1 and d_2 by composing internally

$$d_1 \otimes d_2\colon\ C_0 \xrightarrow{\qquad d_1 \star d_2 \qquad} C_1 \times_{C_0} C_1 \xrightarrow{\ \circ\ } C_1$$

2.3. Example. When $\mathcal{G}$ is the category of groups, we recapture the classical notion of derivation. Indeed, it is well-known that to a crossed module of groups

$$(\; H \xrightarrow{\partial} G \xrightarrow{\varphi} AutH \;)$$

we can associate an internal groupoid $\mathbb{C}$, with

$$C_0 = G, \; C_1 = H \rtimes_{\varphi} G, \; m((a,x),(b,y)) = (a+b,y)$$

$$cod(a,x) = x, \; dom(a,x) = \partial(a)x, \; u(x) = (0,x)$$

(see [2, 4]). Moreover, $\mathbb{C}$-derivations in the sense of Definition 2.1 are in bijection with derivations of G in H : a morphism $d\colon C_0 \to C_1$ is a $\mathbb{C}$-derivation precisely when its second component is the identity on C_0 and its first component is a derivation of G in H. (Let us recall here also the converse construction, which is needed later. Given an internal groupoid $\mathbb{C}$ in groups, we get a crossed module with $G = C_0$ and $H = Ker(cod)$; the map ∂ is the restriction of dom to H, and the action of G on H is given by $x{\cdot}a = u(x)+a-u(x)$.)

3. The group of regular derivations

In this section we characterize the invertible (or regular) elements of the monoid $Der\mathbb{C}$.

From Definition 2.1, we get three morphisms of monoids:

- $\mathcal{U}\colon Der\mathbb{C} \to [Grpd(\mathcal{G})(\mathbb{C},\mathbb{C})]_0 \qquad (D,d) \mapsto (D\colon \mathbb{C} \to \mathbb{C})$

- $(\;)_0\colon Der\mathbb{C} \to EndC_0 \qquad (D,d) \mapsto (D_0\colon C_0 \to C_0)$

- $(\;)_1\colon Der\mathbb{C} \to EndC_1 \qquad (D,d) \mapsto (D_1\colon C_1 \to C_1)$

As with any morphism of monoids, these morphisms restrict to the groups of invertible elements:

$$\begin{array}{ccc} Der\mathbb{C} & \xrightarrow{\mathcal{U}} & [Grpd(\mathcal{G})(\mathbb{C},\mathbb{C})]_0 \\ \uparrow & & \uparrow \\ Der^*\mathbb{C} & \dashrightarrow & [Grpd(\mathcal{G})^*(\mathbb{C},\mathbb{C})]_0 \end{array} \qquad \begin{array}{ccc} Der\mathbb{C} & \xrightarrow{(\;)_0} & EndC_0 \\ \uparrow & & \uparrow \\ Der^*\mathbb{C} & \dashrightarrow & Aut\,C_0 \end{array} \qquad \begin{array}{ccc} Der\mathbb{C} & \xrightarrow{(\;)_1} & EndC_1 \\ \uparrow & & \uparrow \\ Der^*\mathbb{C} & \dashrightarrow & Aut\,C_1 \end{array}$$

where $Grpd(\mathcal{G})^*$ is the sub-2-category of $Grpd(\mathcal{G})$ of those internal functors which are isomorphisms. In fact, more is true: the previous diagrams are pullbacks. This is a corollary of the following general fact.

3.1. Lemma. *Let*

$$\mathbb{C} \underset{G}{\overset{F}{\rightrightarrows}} \mathbb{B} \qquad \Downarrow \alpha$$

be a 2-cell in Grpd($\mathcal{G}$). The following conditions are equivalent:

1. *α is invertible with respect to the horizontal composition;*
2. *$F, G \colon \mathbb{C} \to \mathbb{B}$ are in $Grpd(\mathcal{G})^*$;*
3. *$F_1, G_1 \colon C_1 \to B_1$ are isomorphisms in $\mathcal{G}$;*
4. *$F_1 \colon C_1 \to B_1$ and $G_0 \colon C_0 \to B_0$ are isomorphisms in $\mathcal{G}$;*
5. *$G_1 \colon C_1 \to B_1$ and $F_0 \colon C_0 \to B_0$ are isomorphisms in $\mathcal{G}$.*

Proof. QUESTA DIMOSTRAZIONE DOVREBBE ESSERE PIU O MENO UGUALE ALLA DIMOSTRAZIONE DELLA PROPOSIZIONE 3.1 NELLA VERSIONE PRECEDENTE. ∎

3.2. Corollary. *Let (D, d) be a $\mathbb{C}$-derivation. The following conditions are equivalent:*

1. *(D, d) is a regular derivation;*
2. *$D \colon \mathbb{C} \to \mathbb{C}$ is in $Grpd(\mathcal{G})^*$;*
3. *$D_0 \colon C_0 \to C_0$ is an isomorphism (i.e. $C_0 \xrightarrow{d} C_0 \xrightarrow{dom} C_1$ is an isomorphism);*
4. *$D_1 \colon C_1 \to C_1$ is an isomorphism.*

3.3. Example. When $\mathcal{G}$ is the category of groups and $\mathbb{C}$ is the internal groupoid associated with a crossed module $H \to G \to AutH$ as in Example 2.3, the previous corollary extends the following characterization of regular derivations, due to Whitehead [8]:

There are morphisms of monoids $\sigma \colon Der(G, H) \to EndG : \sigma_d(x) = \partial(d(x))x$ and $\theta \colon Der(G, H) \to EndH : \theta_d(a) = d(\partial(a)) + a$. Moreover, a derivation d is invertible iff $\sigma_d \in AutG$ iff $\theta_d \in AutH$.

Our definition of derivation also explains why the group of regular derivations $Der^*(G, H)$ enters in the construction of Norrie's actor of a crossed module (cf. [7], see also Theorem 3.3 in [3]). In fact, for any internal groupoid $\mathbb{C}$ in any finitely complete category $\mathcal{G}$, the data

$$Act\mathbb{C} \colon \begin{cases} Der^*\mathbb{C} \to [Grpd(\mathcal{G})^*(\mathbb{C}, \mathbb{C})]_0 & (D, d) \mapsto (D \colon \mathbb{C} \to \mathbb{C}) \\ [Grpd(\mathcal{G})^*(\mathbb{C}, \mathbb{C})]_0 \times Der^*\mathbb{C} \to Der^*\mathbb{C} & F, (D, d) \mapsto F_0 \cdot d \cdot {F_1}^{-1} \end{cases}$$

define a crossed module of groups: $Act\mathbb{C}$ precisely is the crossed module associated with $Grpd(\mathcal{G})^*(\mathbb{C},\mathbb{C})$, which is an internal groupoid in groups. Recall now that the actor $Act(G,H)$ of a crossed module is a new crossed module intended to recapture, in the category of crossed modules, the idea of "group of automorphisms of a group". If we look at the crossed module $H \to G \to AutH$ as an internal groupoid $\mathbb{C}$ in groups, then the group of automorphisms must be replaced by $Grpd(\mathcal{G})^*(\mathbb{C},\mathbb{C})$, and $Act(G,H)$ is nothing but $Act\mathbb{C}$.

4. The 2-category of holomorphisms

In this section, we give a different description of 2-cells in $Grpd(\mathcal{G})$. For this, we introduce the notion of holomorphism between two groupoids. Our terminology is justified by Example 4.5.

The set-theoretical idea behind the notion of holomorphism is quite easy: given a 2-cell

$$\mathbb{C} \underset{G}{\overset{F}{\rightrightarrows}} \mathbb{B} \quad (\Downarrow \alpha)$$

in $Grpd(\mathcal{G})$, then F, G and α itself are completely determined by the map associating to an internal arrow $(a\colon x \to y) \in C_1$ the diagonal $(Fx \to Gy) \in B_1$ of the commutative square

$$\begin{array}{ccc} Fx & \xrightarrow{\alpha(x)} & Gx \\ {\scriptstyle Fa}\downarrow & & \downarrow{\scriptstyle Ga} \\ Fy & \xrightarrow[\alpha(y)]{} & Gy \end{array}$$

To make this more precise, we need some preliminary work. Consider two internal groupoids $\mathbb{C}, \mathbb{B}$ and let $h\colon C_1 \to B_1$ be an arrow making commutative the following diagrams

$$\begin{array}{ccccc} C_1 & \xrightarrow{h} & B_1 & \xrightarrow{dom} & B_0 \\ {\scriptstyle dom}\downarrow & & (2) & & \uparrow{\scriptstyle dom} \\ C_0 & \xrightarrow[u]{} & C_1 & \xrightarrow[h]{} & B_1 \end{array} \qquad \begin{array}{ccccc} C_1 & \xrightarrow{h} & B_1 & \xrightarrow{cod} & B_0 \\ {\scriptstyle cod}\downarrow & & (3) & & \uparrow{\scriptstyle cod} \\ C_0 & \xrightarrow[u]{} & C_1 & \xrightarrow[h]{} & B_1 \end{array}$$

Thanks to conditions (2) and (3), we get two arrows

$$\widehat{h} = \langle \pi_1 \cdot h, \pi_1 \cdot cod \cdot u \cdot h \cdot (\ )^{-1}, \pi_2 \cdot h \rangle \colon C_1 \times_{C_0} C_1 \to B_1 \times_{B_0} B_1 \times_{B_0} B_1$$

$$\widetilde{h} = \langle \pi_1 \cdot h, \pi_2 \cdot h, \pi_3 \cdot h \rangle \colon P \to Q$$

where P and Q are defined by the following limits in $\mathcal{G}$

$$
\begin{array}{ccccc}
C_1 & \xleftarrow{\pi_1} & P & \xrightarrow{\pi_3} & C_1 \\
{\scriptstyle cod}\downarrow & & \downarrow{\scriptstyle \pi_2} & & \downarrow{\scriptstyle dom} \\
C_0 & \xleftarrow[cod]{} & C_1 & \xrightarrow[dom]{} & C_0
\end{array}
\qquad
\begin{array}{ccccc}
B_1 & \xleftarrow{\pi_1} & Q & \xrightarrow{\pi_3} & B_1 \\
{\scriptstyle cod}\downarrow & & \downarrow{\scriptstyle \pi_2} & & \downarrow{\scriptstyle dom} \\
B_0 & \xleftarrow[cod]{} & B_1 & \xrightarrow[dom]{} & B_0
\end{array}
$$

4.1. Lemma.

1. *Diagram (4) commutes iff diagram (4') commutes*

$$
\begin{array}{ccc}
C_1 \times_{C_0} C_1 & \xrightarrow{\circ} & C_1 \\
{\scriptstyle \widehat{h}}\downarrow & (4) & \downarrow{\scriptstyle h} \\
B_1 \times_{B_0} B_1 \times_{B_0} B_1 & \xrightarrow[\circ^2]{} & B_1
\end{array}
$$

$$
\begin{array}{ccc}
P & \xrightarrow{\widetilde{h}} & Q \\
{\scriptstyle \langle \pi_1, \pi_2 \cdot (\ )^{-1}, \pi_3 \rangle}\downarrow & (4') & \downarrow{\scriptstyle \langle \pi_1, \pi_2 \cdot (\ )^{-1}, \pi_3 \rangle} \\
C_1 \times_{C_0} C_1 \times_{C_0} C_1 & & B_1 \times_{B_0} B_1 \times_{B_0} B_1 \\
{\scriptstyle \circ^2}\downarrow & & \downarrow{\scriptstyle \circ^2} \\
C_1 & \xrightarrow[h]{} & B_1
\end{array}
$$

2. *Diagram (5) commutes iff diagram (5') commutes*

$$
\begin{array}{ccccc}
C_0 & \xrightarrow{u} & C_1 & \xrightarrow{h} & B_1 \\
{\scriptstyle u}\downarrow & & (5) & & \uparrow{\scriptstyle u} \\
C_1 & \xrightarrow[h]{} & B_1 & \xrightarrow[dom]{} & B_0
\end{array}
\qquad
\begin{array}{ccccc}
C_0 & \xrightarrow{u} & C_1 & \xrightarrow{h} & B_1 \\
{\scriptstyle u}\downarrow & & (5') & & \uparrow{\scriptstyle u} \\
C_1 & \xrightarrow[h]{} & B_1 & \xrightarrow[cod]{} & B_0
\end{array}
$$

Proof. L'EQUIVALENZA FRA (5) E (5') DOVREBBE ESSERE FACILE. L'EQUIVALENZA FRA (4) E (4') NON LO SO, MA FORSE SI PUO' PROVARE IMITANDO L'EQUIVALENZA FRA LE DUE DEFINIZIONI DI OLOMORFO NEL CASO DEI GRUPPI, VEDI ESEMPIO 4.5. ∎

4.2. Definition. *Consider two groupoids $\mathbb{C}, \mathbb{B}$ in $\mathcal{G}$.*

1. *An holomorphism $h\colon \mathbb{C} \to \mathbb{B}$ is an arrow $h\colon C_1 \to B_1$ making commutative diagram (2), diagram (3), and diagram (4).*

2. *An holomorphism $h\colon \mathbb{C} \to \mathbb{B}$ is pointed if it makes commutative diagram (5).*

4.3. Remark. If $\mathcal{G}$ is the category of sets and $a\colon x \to y, b\colon z \to y, c\colon z \to w$ are elements of C_1, condition (4') means that $h(a \cdot b^{-1} \cdot c) = h(a) \cdot h(b)^{-1} \cdot h(c)$. Condition (4) expresses the special case of condition (4') where $z = y$ and $b = 1_y$.

4.4. Lemma.

1. *Holomorphisms and pointed holomorphisms are stable under composition in $\mathcal{G}$.*
2. *If $h\colon \mathbb{C} \to \mathbb{B}$ is an holomorphism, then the arrows*

$$\delta_h\colon\ C_1 \xrightarrow{\langle h, cod\cdot u\cdot h\cdot(\ )^{-1}\rangle} B_1 \times_{B_0} B_1 \xrightarrow{\circ} B_1$$

$$\gamma_h\colon\ C_1 \xrightarrow{\langle dom\cdot u\cdot h\cdot(\ )^{-1}, h\rangle} B_1 \times_{B_0} B_1 \xrightarrow{\circ} B_1$$

are pointed holomorphisms from $\mathbb{C}$ to $\mathbb{B}$. We call δ_h the domain of h and γ_h the codomain of h.

Proof. ROUTINE, O ALMENO SPERO ... ∎

We are ready to describe the 2-category $Hol(\mathcal{G})$ of holomorphisms:

- Objects are internal groupoids in $\mathcal{G}$. Morphisms are pointed holomorphisms. 2-cells are holomorphisms.
- Domain, codomain, and horizontal composition of 2-cells are defined in Lemma 4.4. The identity 2-cell on a morphism $f\colon \mathbb{C} \to \mathbb{B}$ is f itself.
- If $h, k\colon \mathbb{C} \to \mathbb{B}$ are holomorphisms with $\gamma_h = \delta_k$, their vertical composition is given by

$$C_1 \xrightarrow{\langle dom\cdot u\cdot h, k\rangle} B_1 \times_{B_0} B_1 \xrightarrow{\circ} B_1$$

or, equivalently, by

$$C_1 \xrightarrow{\langle h, cod\cdot u\cdot k\rangle} B_1 \times_{B_0} B_1 \xrightarrow{\circ} B_1$$

4.5. Example. We can consider a group G as a groupoid (in sets) with just one object, and having the elements of G as arrows. Group homomorphisms correspond then to internal functors. If $f, g\colon G \to H$ are group homomorphisms, a natural transformation $h\colon f \Rightarrow g$ is just an element $h_* \in H$ such that, for all $a \in G$, one has $f(a)+h_* = h_*+g(a)$. We can therefore define a map $h\colon G \to H$ by $h(a) = f(a) + h_*$, so that $h(0) = h_*$. Such a map satisfies the equation $h(a + c) = h(a) - h(0) + h(c)$, which is also equivalent to the equation $h(a - b + c) = h(a) - h(b) + h(c)$ (compare with Lemma 4.1). A map satisfying these equivalent conditions is called a group holomorphism (see, for example, Section IV.1 in [6]). Conversely, an holomorphism $h\colon G \to H$ is an homomorphism precisely when it is pointed, that is when $h(0) = 0$. We can therefore construct two homomorphisms from an holomorphism h :

$$\delta_h\colon G \to H\ ,\ \ \delta_h(a) = h(a) - h(0)\quad ;\quad \gamma_h\colon G \to H\ ,\ \ \gamma_h(a) = -h(0) + h(a)$$

The element $h(0)$ gives then a natural transformation $h(0)\colon \delta_h \Rightarrow \gamma_h$ (compare with Lemma 4.4).

Because of the way holomorphisms compose, we have the following fact.

4.6. Corollary. *An holomorphism $h \colon \mathbb{C} \to \mathbb{B}$ is invertible with respect to horizontal composition iff $h \colon C_1 \to B_1$ is an isomorphism in $\mathcal{G}$.*

As announced at the beginning of this section, $Hol(\mathcal{G})$ provides an equivalent description of $Grpd(\mathcal{G})$. In fact, we have the following result.

4.7. Proposition. *There is a 2-functor $\epsilon \colon Hol(\mathcal{G}) \to Grpd(\mathcal{G})$ which is the identity on objects and an isomorphism on hom-categories. The 2-functor ϵ restricts to the sub-2-categories of isomorphisms $Hol(\mathcal{G})^* \to Grpd(\mathcal{G})^*$.*

Proof. If $f \colon \mathbb{C} \to \mathbb{B}$ is a pointed holomorphism, we get an internal functor $\epsilon(f) = (F_1, F_0) \colon \mathbb{C} \to \mathbb{B}$ by $F_1 = f \colon C_1 \to B_1$ and $F_0 = u \cdot f \cdot dom \colon C_0 \to B_0$.
If $h \colon \mathbb{C} \to \mathbb{B}$ is an holomorphism, we get an internal natural transformation $\epsilon(h) \colon \epsilon(\delta_h) \Rightarrow \epsilon(\gamma_h)$ by $\epsilon(h) = u \cdot h \colon C_0 \to C_1 \to B_1$.
Conversely, if $\alpha \colon F = (F_1, F_0) \Rightarrow G = (G_1, G_0) \colon \mathbb{C} \to \mathbb{B}$ is an internal natural transformation (with $\alpha \colon C_0 \to B_1$), we get an holomorphism $h \colon \mathbb{C} \to \mathbb{B}$ by

$$C_1 \xrightarrow{\langle F_1, cod \cdot \alpha \rangle} B_1 \times_{B_0} B_1 \xrightarrow{\circ} B_1$$

or, equivalently, by

$$C_1 \xrightarrow{\langle dom \cdot \alpha, G_1 \rangle} B_1 \times_{B_0} B_1 \xrightarrow{\circ} B_1$$

■

I DETTAGLI DELLA DIMOSTRAZIONE PRECEDENTE DOVREBBERO ESSERE ESSENZIALMENTE QUELLI DELL'ISOMORFISMO $(End\mathbb{C})_1 \simeq Lue\mathbb{C}$ DEL FAX DI BEPPE.

5. Translations

In this section, we specialize the notion of holomorphism to get a different description of derivations in terms of what we call translations. This name is justified by Example 5.5.

5.1. Definition. *The monoid of $\mathbb{C}$-translations is the kernel of the codomain map*

$$Tr\mathbb{C} = Ker(cod) \to [Hol(\mathcal{G})(\mathbb{C}, \mathbb{C})]_1 \to [Hol(\mathcal{G})(\mathbb{C}, \mathbb{C})]_0$$

As we did in Proposition 2.2 with derivations, we give now a simpler description of translations. Fix an arrow $t \colon C_1 \to C_1$ such that the diagram

$$\begin{array}{ccc} C_1 & \xrightarrow{\quad t \quad} & C_1 \\ & {\scriptstyle cod} \searrow \quad (6) \quad \swarrow {\scriptstyle cod} & \\ & C_1 & \end{array}$$

commutes, and consider the factorizations

$$\widehat{t} = \langle dom \cdot u \cdot t,\, 1 \rangle \colon C_1 \to C_1 \times_{C_0} C_1 \quad , \quad \widetilde{t} = \langle \pi_1 \cdot t,\, \pi_2 \rangle \colon C_1 \times_{C_0} C_1 \to C_1 \times_{C_0} C_1$$

5.2. Lemma. *Diagram (7) commutes iff diagram (7') commutes*

$$
\begin{array}{ccc}
C_1 & \xrightarrow{\quad t \quad} & C_1 \\
 & {\scriptstyle \widehat{t}}\searrow \quad (7) \quad \nearrow{\scriptstyle \circ} & \\
 & C_1 \times_{C_0} C_1 &
\end{array}
\qquad
\begin{array}{ccc}
C_1 \times_{C_0} C_1 & \xrightarrow{\quad \circ \quad} & C_1 \\
{\scriptstyle \widetilde{t}}\downarrow & (7') & \downarrow{\scriptstyle t} \\
C_1 \times_{C_0} C_1 & \xrightarrow[\quad \circ \quad]{} & C_1
\end{array}
$$

Proof. COME NELLA VERSIONE PRECEDENTE. ■

5.3. Remark. If $\mathcal{G}$ is the category of sets and $a\colon x \to y, b\colon y \to z$ are elements of C_1, condition (7') means $t(a \cdot b) = t(a) \cdot b$. Condition (7) expresses the special case of condition (7') where $x = y$ and $a = 1_y$.

5.4. Proposition. *To give a $\mathbb{C}$-translation amounts to give an arrow $t\colon C_1 \to C_1$ such that diagram (6) and diagram (7) commute.*

Proof. COME NELLA VERSIONE PRECEDENTE. ■

5.5. Example. Consider once again a group G as a groupoid $\mathbb{C}$ with a single object. Thanks to Proposition 5.4, a $\mathbb{C}$-translation in the sense of Definition 5.1 is nothing but a map $t\colon G \to G$ such that $t(a) = t(0) + a$ for all $a \in G$. That is, t is the right translation by $t(0)$.Therefore, in this case $Tr\mathbb{C}$ is a group isomorphic to G.

In contrast with the situation described in Example 5.5, the monoid $Tr\mathbb{C}$ in general is not a group.

5.6. Corollary. *The group of regular translation $Tr^*\mathbb{C}$ is given by $Tr\mathbb{C} \cap Aut\, C_1$.*

By Proposition 4.7, we get the following corollary.

5.7. Corollary. *The 2-functor $\epsilon\colon Hol(\mathcal{G}) \to Grpd(\mathcal{G})$ induces two isomorphisms of monoids $Tr\mathbb{C} \simeq Der\mathbb{C}$ and $Tr^*\mathbb{C} \simeq Der^*\mathbb{C}$.*

We can describe the isomorphism $Tr\mathbb{C} \simeq Der\mathbb{C}$ using Propositions 2.2 and 5.4:

- Given $(t\colon C_1 \to C_1) \in Tr\mathbb{C}$, we get $(u \cdot t\colon C_0 \to C_1 \to C_1) \in Der\mathbb{C}$.
- Given $(d\colon C_0 \to C_1) \in Der\mathbb{C}$, we get $C_1 \xrightarrow{\langle dom \cdot d, 1\rangle} C_1 \times_{C_0} C_1 \xrightarrow{\circ} C_1 \ \in Tr\mathbb{C}$.

Let us summarize the situation we have so far with the following picture, where the unlabelled vertical arrows are the inclusion of the kernel.

$$
\begin{array}{ccc}
Tr^*\mathbb{C} & \xrightarrow{\quad \simeq \quad} & Der^*\mathbb{C} \\
\downarrow & & \downarrow \\
[Hol(\mathcal{G})^*(\mathbb{C},\mathbb{C})]_1 & \xrightarrow{\quad \simeq \quad} & [Grpd(\mathcal{G})^*(\mathbb{C},\mathbb{C})]_1 \\
{\scriptstyle dom}\downarrow\downarrow{\scriptstyle cod} & & {\scriptstyle dom}\downarrow\downarrow{\scriptstyle cod} \\
[Hol(\mathcal{G})^*(\mathbb{C},\mathbb{C})]_0 & \xrightarrow{\quad \simeq \quad} & [Grpd(\mathcal{G})^*(\mathbb{C},\mathbb{C})]_0
\end{array}
$$

5.8. Example. Since $Hol(\mathcal{G})^*(\mathbb{C}, \mathbb{C})$ is a groupoid in groups, using the constructions described in Example 2.3 we can pass to a crossed module of groups, and then come back to a groupoid isomorphic to $Hol(\mathcal{G})^*(\mathbb{C}, \mathbb{C})$. Using also the isomorphisms of the previous picture, we get a group isomorphism

$$[Hol(\mathcal{G})^*(\mathbb{C}, \mathbb{C})]_1 \simeq Tr^*\mathbb{C} \rtimes [Grpd(\mathcal{G})^*(\mathbb{C}, \mathbb{C})]_0$$

If we specialize this isomorphism to the case where $\mathcal{G}$ is the category of sets and $\mathbb{C}$ is the one-object groupoid associated to a group G (Exemples 4.5 and 5.5), we get the classical isomorphism

$$HolG \simeq G \rtimes AutG$$

where $HolG$ is the group of bijective holomorphisms from G to G (see [6], Section IV.1).

5.9. Example. QUI CI STAREBBE BENE L'ESEMPIO SULLE AFFINITA'. BEPPE, LET'S GO !

5.10. Example. As in the previous example, we have a group isomorphism

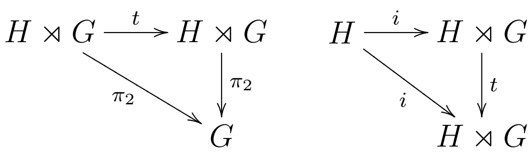
$$[Hol(\mathcal{G})^*(\mathbb{C}, \mathbb{C})]_1 \simeq Der^*\mathbb{C} \rtimes [Grpd(\mathcal{G})^*(\mathbb{C}, \mathbb{C})]_0$$

This generalizes the isomorphism established by Lue in the case where $\mathcal{G}$ is the category of groups, and groupoids are replaced by crossed modules (see [5], Theorem 9). It is interesting to observe that, in that case, only the analogue of our conditions (2) and (3) are used to define the analogue of $[Hol(\mathcal{G})^*(\mathbb{C}, \mathbb{C})]_1$. This is because the category of groups is a Mal'cev category (see [1] for the notion of Mal'cev category). In fact, we have the following result.

5.11. Lemma. *Let $\mathcal{G}$ be a Mal'cev category, and consider two groupoids $\mathbb{C}, \mathbb{B}$ in $\mathcal{G}$. If an arrow $h \colon C_1 \to B_1$ satisfies conditions (2) and (3), then it is an holomorphism.*

Proof. ARGOMENTO DI SANDRA. (E' ANCORA VALIDO IN QUESTA FORMA PIU' GENERALE ?) ∎

5.12. Example. Observe that, by 2.2 and 3.2, if the domain and codomain maps of an internal groupoid $\mathbb{C}$ are equal, then $\mathbb{C}$-derivations are invertible. This is the case for $\mathbb{C}$ the groupoid in groups associated with a crossed module of the form $H \xrightarrow{0} G \xrightarrow{\varphi} AutH$, where $\varphi \colon G \to AutH$ is a G-module and $0 \colon H \to G$ is the zero-morphism. Indeed, in this case, both domain and codomain coincide with the second projection $\pi_2 \colon H \rtimes G \to G$. Moreover, in this case a classical result (see [6], Proposition IV.2.1) asserts that the group $Der^*\mathbb{C}$ is isomorphic to the group of isomorphisms $t \colon H \rtimes G \to H \rtimes G$ making commutative the following duagrams



where $i(a) = (a, 1)$. Since the first diagram is precisely diagram (6), and the commutativity of the second diagram is equivalent to the commutativity of diagram (7), this description of $Der^*\mathbb{C}$ is a specialization of the isomorphism $Der^*\mathbb{C} \simeq Tr^*\mathbb{C}$ established in 5.7. Even for an arbitrary crossed module $H \to G \to AutH$, the group $Der^*(G, H)$ can be described as a suitable subgroup of $Aut(H \rtimes G)$, see Proposition 3.5 in [3]. Once again, this description is a particular case of the isomorphism $Der^*\mathbb{C} \simeq Tr^*\mathbb{C}$.

PROBLEMA : GENERALIZZARE L'ARGOMENTO DI QUESTO ESEMPIO AL CASO IN CUI $\mathcal{G}$ E' UNA CATEGORIA SEMIABELIANA E H E' UN OGGETTO ABELIANO.

6. The embedding category of an internal groupoid

STO ANCORA RIMUGINANDO SULL'ARTICOLO DI MOERDIJK E SUL PREPRINT DI BROWN. CHE IMPRESSIONE NE AVETE ? POSSIAMO TIRARNE FUORI QUALCOSA DI UTILE ?

7. Left exactness of $Der\mathbb{C}$

If ($H \xrightarrow{\partial} G \xrightarrow{\varphi} AutH$) is a crossed module of groups, one of the main properties of the group of regular derivations $Der^*(G, H)$ is that, when it is seen as a functor of the second variable, it preserves kernels. Indeed, this allows one to apply the kernel-cokernel lemma for groups, and obtaining in this way the fundamental exact sequence in nonabelian group cohomology. The aim of this section is to study the main properties of $Der\mathbb{C}$ and $Der^*\mathbb{C}$ as functors.

Consider two internal groupoids $\mathbb{C}$ and $\mathbb{C}'$ in $\mathcal{G}$ having the same object of objects, and an internal functor $F\colon \mathbb{C} \to \mathbb{C}'$ which is the identity on objects

$$\begin{array}{ccc} C_1 & \xrightarrow{F_1} & C'_1 \\ {\scriptstyle dom}\downarrow\downarrow{\scriptstyle cod} & & {\scriptstyle dom'}\downarrow\downarrow{\scriptstyle cod'} \\ C_0 & \xrightarrow[F_0=1]{} & C_0 \end{array}$$

Composing with F_1 gives a morphism of monoids

$$DerF\colon Der\mathbb{C} \to Der\mathbb{C}' \qquad C_0 \xrightarrow{d} C_1 \ \mapsto\ C_0 \xrightarrow{d} C_1 \xrightarrow{F_1} C'_1$$

and its restrictions to the groups of regular derivations $Der^*F\colon Der^*\mathbb{C} \to Der^*\mathbb{C}'$.

In fact, this construction is a functor

$$Der\colon \mathcal{F}_{C_0} \to Mon$$

where Mon is the category of monoids, and $\mathcal{F}_{C_0}$ is the fibre over C_0 of the functor

$$Grpd(\mathcal{G}) \to \mathcal{G}$$

which associate to an internal groupoid $\mathbb{C}$ its object of objects C_0. Moreover, the functor Der factorizes through the comma category

$$Mon/EndC_0$$

because $Der\mathbb{C}$ is equipped with a canonical morphism

$$Der\mathbb{C} \to EndC_0 \qquad C_0 \xrightarrow{d} C_1 \;\mapsto\; C_0 \xrightarrow{d} C_1 \xrightarrow{dom} C_0$$

(cf. Proposition 2.2). In the same way, using regular derivations instead of arbitrary derivations, we obtain two functors

$$Der^* \colon \mathcal{F}_{C_0} \to Grp/Iso\,C_0 \qquad\qquad Der^* \colon \mathcal{F}_{C_0} \to Grp$$

where Grp is the category of groups.

7.1. Proposition.

1. *The functor* $Der \colon \mathcal{F}_{C_0} \to Mon/EndC_0$ *preserves finite limits;*
2. *The functor* $Der \colon \mathcal{F}_{C_0} \to Mon$ *preserves equalizers;*
3. *The functor* $Der^* \colon \mathcal{F}_{C_0} \to Grp/Iso\,C_0$ *preserves finite limits;*
4. *The functor* $Der^* \colon \mathcal{F}_{C_0} \to Grp$ *preserves equalizers.*

Proof. The functor $Mon \to Grp$ which associates to a monoid the group of its invertible elements preserves limits, so that points 3 and 4 follow from points 1 and 2. Moreover, the canonical forgetful functor from a comma category to the base category preserves equalizers, so that point 2 follows from point 1. As far as point 1 is concerned, it is enough to give a glance to finite limits in the fibre $\mathcal{F}_{C_0}$.

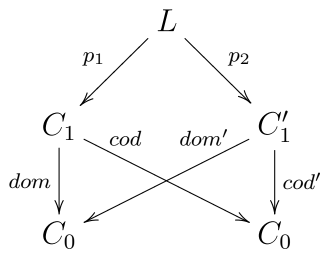
- The object of arrows of the terminal object in $\mathcal{F}_{C_0}$ is the product $C_0 \times C_0$. Domain and codomain are the projections. Composition $C_0 \times C_0 \times C_0 \to C_0 \times C_0$ is the projection on the first and third components. The inverse $C_0 \times C_0 \to C_0 \times C_0$ is the twist.

- The object of arrows of the equalizer in $\mathcal{F}_{C_0}$ of $F, G \colon \mathbb{C} \to \mathbb{C}'$ is the equalizer in $\mathcal{G}$

$$E \xrightarrow{e} C_1 \underset{G_1}{\overset{F_1}{\rightrightarrows}} C'_1$$

  with domain and codomain given by $dom \cdot e, cod \cdot e$. The rest of the structure is inherited from that of $\mathbb{C}$ using the universal property of E.

- The object of arrows of the product in $\mathcal{F}_{C_0}$ of $\mathbb{C}$ and $\mathbb{C}'$ is the limit L as in the following diagram



The domain is $dom \cdot p_1 = dom' \cdot p_2$ and the codomain is $cod \cdot p_1 = cod' \cdot p_2$. The rest of the structure is inherited from those of $\mathbb{C}$ and $\mathbb{C}'$ via the universal property of L.

It is now easy to verify that the functor $Der \colon \mathcal{F}_{C_0} \to Mon/EndC_0$ preserves finite limits. Let us look, for example, at the case of products. Consider a pair

$$(d_1, d_2) \in Der\mathbb{C} \times_{EndC_0} Der\mathbb{C}'$$

(which is the product in the comma category $Mon/EndC_0$). Since $cod \cdot d_1 = 1 = cod' \cdot d_2$ and $dom \cdot d_1 = dom' \cdot d_2$, there is a unique arrow $d \colon C_0 \to L$ such that $p_1 \cdot d = d_1$ and $p_2 \cdot d = d_2$. Moreover, $cod \cdot p_1 \cdot d = cod \cdot d_1 = 1$, so that d is a derivation. Conversely, if $d \colon C_0 \to L$ is a derivation, then $cod \cdot p_1 \cdot d = 1 = cod' \cdot p_2 \cdot d$ and $dom \cdot p_1 \cdot d = dom' \cdot p_2 \cdot d$, so that the pair $(p_1 \cdot d, p_2 \cdot d)$ is an element of the pullback $Der\mathbb{C} \times_{EndC_0} Der\mathbb{C}'$. ■

References

[1] F. Borceux and D. Bourn, Mal'cev, protomodular, homological and semi-abelian categories, *Kluwer Academic Publishers* (2004).

[2] R. Brown and C.B. Spencer, G-groupoids, crossed modules and the fundamental groupoid of a topological group, *Proceedings Nederl. Akad. Wet. Ser. A* **79** (1976) 296-302.

[3] N.D. Gilbert, Derivations, automorphisms and crossed modules, *Communications in Algebra* **18** (1990) 2703-2734.

[4] G. Janelidze, Internal crossed modules, *Georgian Mathematical Journal* **10** (2003) 99-114.

[5] A.S.-T. Lue, Semi-complete crossed modules and holomorphs of groups, *Bulletin of the London Mathematical Society* **11** (1979) 8-16.

[6] S. Mac Lane, Homology, *Springer Verlag* (1975).

[7] K. Norrie, Actions and automorphisms of crossed modules, *Bulletin de la Société Mathématique Française* **118** (1990) 129-146.

[8] J.H.C. Whitehead, On operators in relative homotopy groups, *Annals of Mathematics* **49** (1948) 610-640.

Dipartimento di Matematica
Università di Milano
Via Saldini 50
I 20133 Milano, Italia

Département de Mathématique Pure et Appliquée
Université catholique de Louvain
Chemin du Cyclotron 2
B 1348 Louvain-la-Neuve, Belgique
Email:
Stefano.Kasangian@mat.unimi.it, Sandra.Mantovani@mat.unimi.it,
metere@mat.unimi.it, vitale@math.ucl.ac.be