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Conformal solitons for the mean curvature flow in hyperbolic space

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Abstract
In this paper, we study conformal solitons for the mean curvature flow in hyperbolic space $\mathbb{H}^{n+1}$. Working in the upper half-space model, we focus on horo-expanders, which relate to the conformal field $-\partial_0$. We classify cylindrical and rotationally symmetric examples, finding appropriate analogues of grim-reaper cylinders, bowl and winglike solitons. Moreover, we address the Plateau and the Dirichlet problems at infinity. For the latter, we provide the sharp boundary convexity condition to guarantee its solvability and address the case of non-compact boundaries contained between two parallel hyperplanes of $\partial_\infty \mathbb{H}^{n+1}$. We conclude by proving rigidity results for bowl and grim-reaper cylinders.

Keywords Mean curvature flow · Solitons · Plateau's problem · Dirichlet's problem

Mathematics Subject Classification Primary 53C44 · 53A10 · 53C21 · 53C42

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1 Introduction

Let M and N be complete connected Riemannian manifolds and $f : M \to N$ an isometric immersion. A *mean curvature flow* (MCF for short) issuing from f is a smooth map $F : M \times [0, T) \to N$ such that each $F_t : M \to N$, $F_t(\cdot) = F(\cdot, t)$, is an immersion and satisfies the evolution equation

$$\begin{cases} \dfrac{\partial F_t}{\partial t} = \mathbf{H}(F_t), \\ F_0 = f, \end{cases} \tag{1.1}$$

where $\mathbf{H}(F_t)$ is the unnormalized mean curvature vector field of F_t. If M is compact, then (1.1) admits a smooth and unique (up to reparametrizations) solution, see for example [6, 29, 50]. Although most of the literature regards the MCF in Euclidean space, there are significant applications of the MCF in other ambient manifolds, see [50] for an overview.

In this paper, we will consider a special class of solutions to the MCF. Let X be a smooth vector field on a Riemannian manifold N and $\Phi : \mathscr{D} \subset N \times \mathbb{R} \to N$ its associated flow with maximal domain $\mathscr{D}$. A solution $F : M \times (0, T) \to N$ to the MCF is said to *move along* X if there exists an immersion $f : M \to N$, a reparametrization $s : (0, T) \to \mathbb{R}$ of the flow lines of X and a 1-parameter family of diffeomorphisms $\eta : M \times (0, T) \to M$ such that

$$F(x, t) = \Phi\big(f(\eta(x, t)), s(t)\big), \qquad (x, t) \in M \times (0, T). \tag{1.2}$$

While the definition is meaningful for arbitrary X, we focus on conformal vector fields since in this case the MCF "preserves" the shape of the evolved submanifold. A MCF moving along a conformal field X is said to be a *self-similar solution with respect to* X. Self-similar solutions serve as comparison solutions to investigate the formation of singularities. Differentiating (1.2) with respect to t and estimating at 0, we obtain the equation

$$\mathbf{H} = s'(0) X^{\perp},$$

where $\{\cdot\}^{\perp}$ is the orthogonal projection on the normal bundle of f. This motivates the following definition:

Definition 1.1 An isometric immersion $f : M \to N$ satisfying the elliptic equation

$$\mathbf{H} = cX^{\perp}, \tag{1.3}$$

where $c \in \mathbb{R}$, is called a *soliton with respect to* X with *soliton constant* c. If X is a gradient field (resp. a conformal field or a parallel field), then f is named a gradient (resp. a conformal or a translating) soliton.

Remark 1.2 Let us make some comments regarding solitons in general ambient spaces:

(1) The case where X is conformal and closed was considered in [1, 5, 17, 51]. A soliton with respect to X with constant c is a soliton with respect to cX with constant 1. However, in what follows, it will be more convenient to specify X from the very beginning and keep c as a parameter.

(2) The fact that f solves (1.3) may not imply that the MCF issuing from f moves along X. This is the case, however, when X is a Killing field, since X generates a 1-parameter group of isometries, [31, 32]. Nevertheless, the study of (possibly non-Killing) gradient solitons in more general ambient spaces can also be justified from the parabolic point of view, as shown by Yamamoto [56]. The starting point is the investigation of the MCF in a Ricci flow background; see [40, 41]. More precisely, let $\{g_t\}$ a smooth 1-parameter

family of Riemannian metrics on N and $F_t : M \to N$ a smooth 1-parameter family of immersions for $t \in [0, T)$. We say that $\{(\mathrm{g}_t, F_t)\}$ is a solution to the *Ricci-mean curvature flow* if the following system is satisfied

$$\begin{cases} \dfrac{\partial \mathrm{g}_t}{\partial t} = -2\,\mathrm{Ric}(\mathrm{g}_t), \\ \dfrac{\partial F_t}{\partial t} = \quad \mathbf{H}(F_t), \end{cases} \tag{1.4}$$

where $\mathbf{H}(F_t)$ denotes the mean curvature vector field of $F_t : M \to (N, \mathrm{g}_t)$. One interesting case is that of a MCF in a gradient shrinking Ricci soliton (N, g, u). In this situation, g and u satisfy

$$\mathrm{Ric}(\mathrm{g}) + \mathrm{Hess}_{\mathrm{g}}(u) - (1/2)\,\mathrm{g} = 0$$

and thus $\mathrm{g}_t = (T - t)\Phi_t^* \mathrm{g}$, where $\Phi_t : N \to N$, $t \in (0, T)$ is the flow of the vector field

$$V = \frac{\nabla u}{T - t}.$$

Yamamoto [56] proved the following results:

- Let (N, g, u) be a shrinking gradient Ricci soliton and let $f : M \to N$ be a soliton satisfying

  $$\mathbf{H}(f) = -(\nabla u)^{\perp}.$$

  Then, $F : M \times [0, T) \to N$ given by $F = \Phi_t^{-1} \circ f$ forms, up to tangential reparametrizations, a solution to the Ricci-mean curvature flow (1.4); see [56, Proposition 4.3].
- Let (N, g, u) be a compact shrinking gradient Ricci soliton, M a compact manifold and let $F : M \times [0, T) \to N$, $T < \infty$, be a Ricci-mean curvature flow defined by (1.4). Suppose that the second fundamental forms $\mathbf{A}(F_t)$ of F_t satisfy

  $$\max_M |\mathbf{A}(F_t)| < \frac{C}{\sqrt{T - t}},$$

  where C is a positive constant. For any increasing sequence $\{s_j\}$ of numbers tending to infinity and any sequence of points $\{x_j\}$ in M, the family of the rescaled pointed immersions $G_{s_j} : (M, x_j) \to N$ given by $G_{s_j} = \Phi_{t_j} \circ F_{t_j}$, where t_j is defined by $s_j = -\log(T - t_j)$, subconverges in the Cheeger-Gromov sense to an immersion $f_\infty : M_\infty \to N$ of a complete Riemannian manifold satisfying the equation

  $$\mathbf{H}(f_\infty) = -(\nabla u)^{\perp};$$

  for more details we refer to [56, Theorem 1.5].

Remark 1.3 When X is a gradient field, a further motivation for studying solutions to (1.3) is the tight relation to the theory of minimal submanifolds; see for example [14–16, 28, 30–35, 51].

(1) Isometric immersions $f : M^k \to (N^{n+1}, \mathrm{g})$ satisfying the soliton equation

$$\mathbf{H} = (\nabla u)^{\perp}, \tag{1.5}$$

for $u \in C^\infty(N)$ are precisely the stationary points of the weighted volume functional

$$\Omega \subset M \mapsto \int_\Omega e^{u(f)} \mathrm{d}x,$$

where $\mathrm{d}x$ is the induced Riemannian measure on M. Consequently, solitons are particular examples of u-minimal submanifolds. By considering the conformally related metric $\mathrm{g_I}(k) = e^{2u/k}\,\mathrm{g}$ and endowing M with the induced metric $\mathrm{h}(k) = f^*\mathrm{g_I}(k)$, $e^u \mathrm{d}x$ is the Riemannian volume measure of $\mathrm{h}(k)$ and thus f solves (1.5) if and only if $f : (M, \mathrm{h}(k)) \to (N, \mathrm{g_I}(k))$ is minimal. The metric $\mathrm{g_I}(k)$ is called the *Ilmanen metric*.

(2) According to a result of Smoczyk [51], there exists a 1–1 correspondence between gradient conformal solitons in N and minimal submanifolds in a suitable warped product constructed out of N. Smoczyk proved that if $f : M \to (N, \mathrm{g})$ is a soliton with respect to a conformal gradient field ∇u, then its associated submanifold $\bar{M} = \mathbb{R} \times M$ is minimal in $\bar{N} = \mathbb{R} \times N$ equipped with the warped metric $\bar{\mathrm{g}}_{(s,x)} = e^{2u(x)}\mathrm{d}s^2 + \mathrm{g}_x$. Moreover, he proved that a submanifold M in N converges to a conformal soliton under the MCF if and only if its associated submanifold $\bar{M}$ converges to a minimal submanifold under a rescaled MCF in $\bar{N}$.

In the present paper, we will investigate conformal solitons in the hyperbolic space

$$\mathbb{H}^{n+1} = \big\{(x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1} : x_0 > 0\big\}, \quad \mathrm{g}_{\mathbb{H}} = x_0^{-2} \sum\nolimits_{i=0}^{n} \mathrm{d}x_i^2. \tag{1.6}$$

The class of conformal vector fields of $\mathbb{H}^{n+1}$ is particularly rich. Thus, we shall restrict ourselves to solitons with respect to $-\partial_0$, i.e., solutions to

$$\mathbf{H} = -\partial_0^\perp = (\nabla x_0^{-1})^\perp. \tag{1.7}$$

Such solitons correspond to "limit self-expanders" which we call *horo-expanders*. Horo-expanders share many similarities with translators in Euclidean space. One may suspect that the analogy is a trivial consequence of the fact that the model (1.6) is conformal to the Euclidean upper half-space. However, a direct computation shows that a k-dimensional soliton in $\mathbb{H}^{n+1}$ with respect to $-\partial_0$ is a soliton in the Euclidean half-space $\mathbb{R}^+ \times \mathbb{R}^n$ with respect to $(-x_0^{-1} - kx_0^{-2})\partial_0$, which is not conformal. Hence, a duality between conformal solitons in $\mathbb{H}^{n+1}$ and translators in $\mathbb{R}^{n+1}$ is hardly obtainable via simple transformations.

In view of Remark 1.3(1), we can regard a k-dimensional soliton with respect to $-\partial_0$ as minimal submanifold of $(\mathbb{H}^{n+1}, \mathrm{g_I}(k))$, where

$$\mathrm{g_I}(k) = e^{\frac{2}{kx_0}}\, \mathrm{g}_{\mathbb{H}}\,. \tag{1.8}$$

This allows us to relate the existence of a soliton with prescribed boundary contained in the boundary at infinity $\partial_\infty \mathbb{H}^{n+1}$ to the existence of minimal submanifolds in Riemannian manifolds. It turns out that in codimension one, both the Plateau and the Dirichlet problem at infinity are solvable, the latter under the additional condition that the boundary is mean convex. However, a distinction shall be made between boundary points of $\partial_\infty \mathbb{H}^{n+1}$: in the upper half-space model (1.6) the boundary at infinity can be represented as the union of

$$\partial'_\infty \mathbb{H}^{n+1} = \{x_0 = 0\} \quad \text{and} \quad \mathrm{p}_\infty = \partial_\infty \mathbb{H}^{n+1} \setminus \partial'_\infty \mathbb{H}^{n+1},$$

which behave quite differently for the Ilmanen metric. Hereafter, the space $\partial'_\infty \mathbb{H}^{n+1}$ will be given the Euclidean metric and metric quantities (balls, hyperplanes, neighborhoods, etc..) will be considered with respect to it.

We can also represent the hyperbolic space via the Poincaré model

$$\mathbb{H}^{n+1} = \{x \in \mathbb{R}^{n+1} : \|x\|^2 < 1\}, \quad g_{\mathbb{H}} = 4(1 - \|x\|^2)^{-1} \sum_{i=1}^{n+1} \mathrm{d}x_i^2.$$

In this model, $\partial_\infty \mathbb{H}^{n+1}$ is the n-dimensional unit sphere $\mathbb{S}^n$. Moreover, if $W \subset \mathbb{H}^{n+1}$, then

$$\partial_\infty W \doteq \overline{W} \cap \mathbb{S}^n,$$

where $\overline{W}$ denotes closure of W in the Euclidean topology.

Regarding Plateau's problem, we show its solvability for hypersurfaces. The higher codimensional case remains as an open problem; see Sect. 4. Before stating our result, let us recall some standard notations from geometric measure theory: given a closet subset W with locally finite perimeter in a smooth manifold N, we denote by $[W]$ its associated rectifiable current. If M is a rectifiable current we denote by spt M its support and by ∂M its boundary; for more details, we refer to [49].

Theorem A *[Plateau's problem] Let $\Sigma \subset \partial'_\infty \mathbb{H}^{n+1}$ be the boundary of a relatively compact subset $A \subset \partial'_\infty \mathbb{H}^{n+1}$ with $A = \overline{\mathrm{int}(A)}$. Then, there exists a closed set W of local finite perimeter in $\mathbb{H}^{n+1}$ with $\partial_\infty W = A$ such that $M = \partial[W]$ is a conformal soliton for $-\partial_0$ on the complement of a closed set S of Hausdorff dimension $\dim_{\mathscr{H}}(\mathrm{S}) \leq n - 7$, and that $\partial_\infty \mathrm{spt}(M) = \Sigma$. Furthermore, when $n < 7$, then M is a properly embedded smooth hypersurface of $\mathbb{H}^{n+1}$.*

We next focus on hypersurfaces which are graphs of the form

$$\Gamma(u) = \left\{(u(x); x) \in \mathbb{H}^{n+1} = \mathbb{R}^+ \times \mathbb{R}^n : x \in \Omega \subset \mathbb{R}^n\right\},$$

where $u \in C^\infty(\Omega)$. Given $0 \leq \phi \in C(\partial\Omega)$, it turns out that $\Gamma(u)$ is a soliton with respect to $-\partial_0$ with boundary $\Gamma(\phi)$ if and only if u satisfies

$$\begin{cases} \mathrm{div}\left(\dfrac{Du}{\sqrt{1+|Du|^2}}\right) = -\dfrac{1+nu}{u^2\sqrt{1+|Du|^2}} & \text{on } \Omega, \\ u > 0 & \text{on } \Omega, \\ u = \phi & \text{on } \partial\Omega, \end{cases} \tag{1.9}$$

where D, div, $|\cdot|$ are the gradient, divergence and norm in the Euclidean metric. In particular, for $\phi \equiv 0$, we obtain a complete graphical soliton whose boundary at infinity is $\partial\Omega$. The right-hand side of (1.9) becomes undefined as $u = 0$, which calls for some care. However, we will see that the terms u and $|Du|$ contribute in opposite directions to the size of u. We show the following result, which parallels the seminal one by Jenkins & Serrin [36] for minimal graphs:

Theorem B *[Dirichlet's problem] Let $\Omega \subset \partial'_\infty \mathbb{H}^{n+1}$ be an open, connected subset with C^3-smooth boundary $\partial\Omega$. Assume that Ω is contained between two parallel hyperplanes of $\partial'_\infty \mathbb{H}^{n+1}$, and denote by $H_{\partial\Omega}$ the Euclidean mean curvature of $\partial\Omega$ in the direction pointing toward Ω.*

(1) *If $H_{\partial\Omega} \geq 0$ on $\partial\Omega$, then for each continuous bounded function $\phi : \partial\Omega \to [0, \infty)$ there exists a function $u : \overline{\Omega} \to [0, \infty)$ such that:*

   (a) *$u > 0$ on Ω and the graph $\Gamma(u) \subset \mathbb{H}^{n+1}$ is a conformal soliton with respect to $-\partial_0$.*
   (b) *$u \in C^\infty(\Omega) \cap C(\overline{\Omega}) \cap L^\infty(\Omega)$ and $u \equiv \phi$ on $\partial\Omega$.*

(2) *If* Ω *is bounded and* $H_{\partial\Omega}(y) < 0$ *for some* $y \in \partial\Omega$*, then there exists a continuous boundary value function* $\phi : \partial\Omega \to (0, \infty)$ *such that no* $u : \overline{\Omega} \to [0, \infty)$ *satisfying the properties in* (1) *does exist.*

Remark 1.4 Let us make some comments about the conclusions of Theorem B.

(1) If Ω is bounded, then u realizing (a), (b) is unique, by a direct application of the comparison theorem. It would be interesting to investigate the uniqueness problem for Ω unbounded.
(2) Similar (non-degenerate) Dirichlet problems were considered for prescribed mean curvature graphs in warped product manifolds; see for example [3, 9–11, 19, 21–23, 36, 47]. Among them, the only applicable result to (1.9) is [11, Theorem 1.1], which guarantees the solvability of

$$\begin{cases} \operatorname{div}\left(\dfrac{Du}{\sqrt{1+|Du|^2}}\right) = \dfrac{f(u)}{\sqrt{1+|Du|^2}} & \text{on a smooth } \Omega \subset \mathbb{R}^n, \\ u = \phi & \text{on } \partial\Omega, \end{cases} \tag{1.10}$$

for $C^{2,\alpha}$-smooth positive ϕ provided that $f \in C^1(\mathbb{R})$ and $H_{\partial\Omega}$ satisfy:

$$(n-1)\kappa \doteq \sup_{\mathbb{R}} |f| < \infty \quad \text{and} \quad H_{\partial\Omega} \geq (n-1)\kappa. \tag{1.11}$$

However, application to (1.9) would force a lower bound on $H_{\partial\Omega}$ that diverges as $\min_{\partial\Omega} \phi \to 0$.
(3) Serrin discussed in [47, Chapter IV, pages 477–478] the solvability and the non-solvability of the Dirichlet problem for equations of the form

$$\operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) = \frac{\Lambda}{(1+|Du|^2)^\theta} \quad \text{and} \quad \operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) = \frac{Cu}{(1+|Du|^2)^\theta}, \tag{1.12}$$

where Λ, C and θ are constants and $C > 0$. Note that, for $\Lambda = 1$ and $\theta = 1/2$, the equation (1.12) describes a translating graphical soliton of the MCF in the Euclidean space. Both equations (1.12) appear in an old paper of Bernstein [7]. As a matter of fact, Bernstein studied the 2-dimensional case and showed that the corresponding Dirichlet problems are solvable for arbitrary analytic boundary data in an arbitrary strictly convex analytic domain only for special values of θ. The problem that we treat in (1.9) does not fall in the class of equations defined in (1.12).
(4) Jenkins and Serrin [36] constructed data $(\partial\Omega, \phi)$ for which the Dirichlet problem for the minimal surface equation in the Euclidean space is not solvable. In their work, $\partial\Omega$ has negative mean curvature at a given point and the oscillation of ϕ can be made arbitrarily small. However, this is not the case for the boundary data we provide in Theorem B(2), whose oscillation shall be at least a fixed amount. From the proof of Theorem B(2), we may suspect the impossibility to produce boundary data with arbitrarily small oscillation for which (1.9) is not solvable.
(5) As a direct application of the maximum principle, we can show that there are no solutions to

$$\operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) + \frac{1+nu}{u^2\sqrt{1+|Du|^2}} = 0 \quad \text{on the entire } \mathbb{R}^n.$$

Another goal of our paper is to construct and classify complete codimension one solitons with symmetries. Despite their own interest, such solitons serve as important barriers and will be used in the proofs of Theorems A and B. Up to rotation, all examples are generated by special curves $\gamma : I \to \mathbb{R}^+ \times \mathbb{R}$, $\gamma(t) = (x_0(t), x_1(t))$, in the x_0x_1-plane. The first family to be considered is that of cylindrical solitons:

$$M = \big\{(x_0, x_1, \dots, x_n) \in \mathbb{H}^{n+1} : \ (x_0, x_1) \in \gamma(I)\big\}. \tag{1.13}$$

We shall prove that the only such examples are the *grim-reaper cylinders*, described by the following:

Theorem C *[Grim-reaper cylinders] If M is a complete soliton for $-\partial_0$ of the type (1.13), then it has the following properties:*

(1) *$\partial_\infty M$ is a pair of parallel hyperplanes $\pi_1 \cup \pi_2$ in $\partial'\mathbb{H}^{n+1}$.*
(2) *M is contained between the two totally geodesic hyperplanes Π_1 and Π_2 of $\mathbb{H}^{n+1}$ with $\partial'_\infty \Pi_1 = \pi_1$ and $\partial'_\infty \Pi_2 = \pi_2$.*
(3) *M is symmetric with respect to the reflection sending Π_1 to Π_2, and invariant with respect to translations fixing Π_1 and Π_2.*
(4) *M is (Euclidean) convex with respect to the direction $-\partial_0$.*

We name M a grim-reaper cylinder. Denote by $h = \max x_0(M)$, and let $\mathscr{G}^h$ be the grim-reaper cylinder isometric to M which is symmetric with respect to the hyperplane $\{x_1 = 0\}$. Then, the family $\{\mathscr{G}^h\}_{h\in\mathbb{R}^+}$ foliates $\mathbb{H}^{n+1}$. In particular, given a pair of parallel hyperplanes $\pi_1, \pi_2 \subset \partial'_\infty \mathbb{H}^{n+1}$, the grim-reaper cylinder with $\partial_\infty M = \pi_1 \cup \pi_2$ exists and is unique.

Another way to construct complete conformal solitons is by rotating γ around the x_0-axis. From this procedure, we obtain the hyperbolic winglike and bowl solitons, similar to those existing in Euclidean space; see for more details [13] and [43]. We here report a simplified, slightly informal statement of our main existence and uniqueness result. For the precise one, including further properties of γ, we refer the reader to Lemma 3.6 and Theorem 3.8.

Theorem D *[Rotationally symmetric solitons] There are exactly two families of curves γ giving rise to a complete, smooth rotationally symmetric conformal soliton with respect to $-\partial_0$. They are depicted in Fig. 1, and named γ_B and γ_W. We call a* bowl soliton *the one obtained by rotating γ_B, and a* winglike soliton *that obtained by rotating γ_W. The following holds:*

(1) *γ_B is a strictly concave graph over a domain $(0, h_B)$ of the x_0-axis, and meets the x_1 and x_0 axes orthogonally.*
(2) *γ_W is a bigraph over a domain $(0, h_W)$ of the x_0-axis, and does not touch the x_0-axis. The upper graph (the one from q_1 to p_1) is strictly concave, while the lower graph is the union of a strictly convex branch with a unique minimum (from p_1 to p_2) and a strictly concave branch (from p_2 to q_2). Moreover, γ_W meets the x_1-axis orthogonally at two distinct points ($q_1 \neq q_2$).*

It turns out that bowl solitons foliate $\mathbb{H}^{n+1}$, see Lemma 3.9. As a direct consequence, we deduce the following uniqueness property which may be viewed as an analogue of [43, Theorem A].

Theorem E *[Uniqueness of the bowl soliton] For any Euclidean ball $B_R \subset \partial'_\infty \mathbb{H}^{n+1}$, there exists a bowl soliton M with $\partial_\infty M = \partial B_R$, and it is the unique properly immersed soliton with respect to $-\partial_0$ with boundary at infinity ∂B_R.*

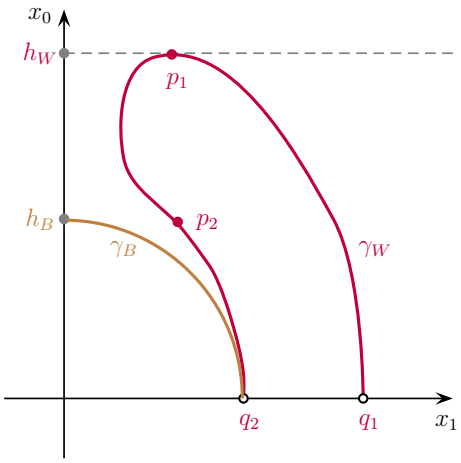

Fig. 1 Curves generating the bowl and winglike solitons

Regarding the uniqueness of grim-reaper cylinders, the problem is more subtle. Quite differently from the Euclidean case, hyperbolic grim-reaper cylinders foliate the hyperbolic space, which is quite helpful. Nevertheless, getting uniqueness under the only assumption that $\partial_\infty M$ is a pair of parallel hyperplanes of $\partial'_\infty \mathbb{H}^{n+1}$ seems difficult. Our last result is that M is a grim-reaper cylinder provided that the height x_0 is bounded and that M is graphical in a small region $\{x_0 < \tau\}$, see Definition 1.5. A similar result was proved in [26, 42] for MCF translators in Euclidean space under the stronger assumption that they are C^1-asymptotic outside a cylinder to a grim-reaper cylinder. In our setting, we will use a calibration argument to get rid of the C^1-bound.

Definition 1.5 A properly embedded hypersurface $M \subset \mathbb{H}^{n+1}$ is said to satisfy the GR-property (see Fig. 2) if the following conditions are satisfied:

(1) $\partial'_\infty M = \pi_1 \cup \pi_2$, where π_1 and π_2 are parallel hyperplanes of $\partial'_\infty \mathbb{H}^{n+1}$.
(2) The x_0-component of M is bounded.
(3) There exist $\tau > 0$, a pair of (Euclidean) hyperplanes $\mathcal{H}_j \subset \mathbb{H}^{n+1}$ and a pair of functions $\varphi_j : \mathcal{H}_j^\tau = \mathcal{H}_j \cap \{x_0 < \tau\} \to \mathbb{R}$, $j \in \{1, 2\}$, such that:
   (a) $\partial'_\infty \mathcal{H}_j = \pi_j$.
   (b) The wings $\mathcal{W}_j = \{x + \varphi_j(x)\nu_j : x \in \mathcal{H}_j^\tau\}$ are contained in M, where ν_j is a fixed (Euclidean) unit normal to H_j.
   (c) $M \cap \{x_0 < \tau\}$ is the portion of $\mathcal{W}_1 \cup \mathcal{W}_2$ inside $\{x_0 < \tau\}$.

Observe that condition (a) in Definition 1.5 implies that

$$\forall\, y \in \pi_j, \qquad \lim_{x \to y} \varphi_j(x) = 0.$$

However, a priori the limit may not be uniform in y, an assumption which was required in [26, 42].

Theorem F *[Uniqueness of the hyperbolic grim-reaper cylinder] Let $M \subset \mathbb{H}^{n+1}$ be a properly embedded conformal soliton with respect to $-\partial_0$ satisfying the GR-property. Then M coincides with a grim-reaper cylinder.*

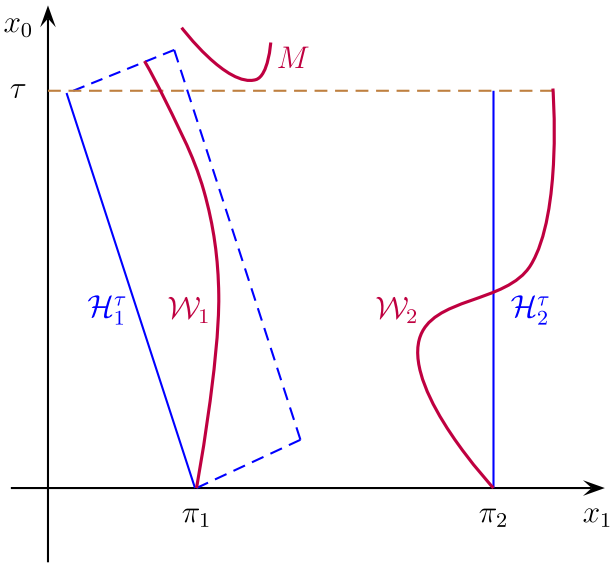

Fig. 2 GR-property

The structure of the paper is as follows. Section 2 we set up the notation and derive basic properties for conformal solitons in the hyperbolic space. In Sect. 3, we examine symmetric horo-expanders and prove Theorems C, D and E. Section 4 is devoted to the Plateau problem at infinity, while in Sect. 5 we consider the Dirichlet problem at infinity and prove Theorem B. Finally, in Sect. 6, we prove Theorem F.

2 Preliminaries

In this section we fix the notation and review some basic formulas.

2.1 Generalities about soliton solutions

A vector field X in an n-dimensional Riemannian manifold (N, g) is called *conformal* if the Lie derivative of g in direction X satisfies

$$\mathcal{L}_X\, \mathrm{g} = 2\psi\, \mathrm{g} \qquad \text{for some } \psi \in C^\infty(N).$$

Taking traces, $\psi = \mathrm{div}_{\mathrm{g}} X/n$. By relating the Lie derivative to the Levi-Civita connection ∇, a conformal vector field is characterized by the identity

$$\mathrm{g}(\nabla_Y X, Z) + \mathrm{g}(\nabla_Z X, Y) = \frac{2}{n}\big(\mathrm{div}_{\mathrm{g}} X\big)\, \mathrm{g}(Y, Z)$$

for $Y, Z \in \mathfrak{X}(N)$. If the field X is conformal and divergence free, it is called *Killing*. If the vector field X satisfies

$$\nabla_Y X = \frac{1}{n}\big(\mathrm{div}_{\mathrm{g}} X\big) Y,$$

namely, the dual form $X_\flat$ is closed, then X is called *closed conformal*. It is a well-known fact that a Riemannian manifold possessing a non-trivial closed conformal vector field is locally isometric to a warped product with a 1-dimensional factor; see for instance [44, page 721]. The hyperbolic space possesses various conformal vector fields X and the study of the corresponding solitons is interesting. Let us see some explicit examples here:

Example 2.1 Denote with $\mathrm{g}_{\mathbb{S}}$, $\mathrm{g}_{\mathbb{R}}$, $\mathrm{g}_{\mathbb{H}}$ the Riemannian metrics of the n-dimensional unit sphere $\mathbb{S}^n$, the Euclidean space $\mathbb{R}^n$ and of the hyperbolic space $\mathbb{H}^n$.

(1) Consider for the hyperbolic space the model $\mathbb{H}^{n+1}\backslash\{0\} = \mathbb{R}^+ \times \mathbb{S}^n$ equipped with the metric $\mathrm{d}r^2 + \sinh^2(r)\,\mathrm{g}_{\mathbb{S}}$. Then the vector field $X = \sinh(r)\partial_r$ is conformal. We call solitons for cX *expanders* if $c > 0$, and *shrinkers* if $c < 0$. Also, rotation vector fields in the $\mathbb{S}^n$ factor extend to Killing fields and give rise to solitons that we call *rotators*.

(2) Consider for the hyperbolic space the model $\mathbb{H}^{n+1} = \mathbb{R} \times \mathbb{R}^n$ endowed with the warped metric $\mathrm{d}r^2 + e^{2r}\,\mathrm{g}_{\mathbb{R}}$. Then, the vector field $X = e^r\,\partial_r$ is conformal. The change of coordinates $x_0 = e^{-r}$ gives rise to an isometry with the upper half-space model sending X to the field $-\partial_0$. Solitons with respect to cX are called *horo-expanders* if $c > 0$ and *horo-shrinkers* if $c < 0$.

(3) Consider for the hyperbolic space the model $\mathbb{H}^{n+1} = \mathbb{R} \times \mathbb{H}^n$ equipped with the Riemannian metric $\mathrm{d}r^2 + \cosh^2(r)\,\mathrm{g}_{\mathbb{H}}$. Then $X = \cosh(r)\partial_r$ is conformal. Given that $\cosh(r)$ is even, we restrict to $c > 0$. A soliton with respect to cX *shrinks* where $r < 0$ and *expands* where $r > 0$.

(4) Consider for the hyperbolic space the model $\mathbb{H}^{n+1} = \mathbb{R} \times \mathbb{H}^n$ with the metric $\cosh^2(\rho)\mathrm{d}s^2 + \mathrm{g}_{\mathbb{H}}$, with ρ is the distance in $\mathbb{H}^n$ to a fixed point. Then, the vector field $X = \partial_s$ is Killing. In the upper half-space model, it corresponds to the position vector field

$$X = \sum_{j=0}^{n} x_j \partial_j.$$

(5) Consider for the hyperbolic space the model $\mathbb{H}^{n+1} = \mathbb{R} \times \mathbb{H}^n$ with the metric $e^{2\rho}\mathrm{d}s^2 + \mathrm{g}_{\mathbb{H}}$, with ρ a Busemann function in $\mathbb{H}^n$. The vector field $X = \partial_s$ is Killing. In the upper half-space model, it corresponds to the vector field ∂_j, for $j \in \{1, \ldots, n\}$.

2.2 Conformal solitons and the Ilmanen metric

Suppose that M and N are connected manifolds with dimensions k and $n+1$, respectively, with $k \leq n$. Let h_1 and h_2 be metrics on N which are conformally related:

$$\mathrm{h}_2 = \lambda^2 \mathrm{h}_1 \qquad \text{for some } 0 < \lambda \in C^\infty(N).$$

Assume that $f : M \to N$ is an immersion and denote by $\mathrm{g}_j = f^*\mathrm{h}_j$ the corresponding induced metrics. Then, the second fundamental forms $\mathbf{A}_j$ of $f_j = f : (M, \mathrm{g}_j) \to (N, \mathrm{h}_j)$ are related by

$$\mathbf{A}_2(X, Y) = \mathbf{A}_1(X, Y) - \mathrm{g}_1(X, Y)\big(\nabla^1 \log\lambda\big)^\perp, \tag{2.1}$$

for any $X, Y \in \mathfrak{X}(M)$. Here, ∇^1 stands for the Levi-Civita connection of h_1 and $\{\cdot\}^\perp$ denotes the orthogonal projection with respect to h_1 on the normal bundle of f_1. Taking traces, we see that the corresponding mean curvature vectors $\mathbf{H}_1$ and $\mathbf{H}_2$ are related by

$$\mathbf{H}_2 = \lambda^{-2}\big\{\mathbf{H}_1 - k(\nabla^1 \log\lambda)^\perp\big\}. \tag{2.2}$$

As immediate consequence of the above formulas we obtain the following:

Lemma 2.2 *Let $M \subset (\mathbb{H}^{n+1}, \mathrm{g}_{\mathbb{H}})$ be a k-dimensional soliton of the MCF with respect to $-\partial_0$. Then, the following hold:*

(1) *M is a minimal submanifold of the Ilmanen space $(\mathbb{H}^{n+1}, \mathrm{g}_{\mathrm{I}}(k))$ given in (1.8).*
(2) *M is a soliton of the Euclidean half-space $\mathbb{R}^+ \times \mathbb{R}^n$ with respect to the field $(-x_0^{-1} - kx_0^{-2})\partial_0$, which is not conformal.*

3 Symmetric conformal solitons

In this section, we examine special conformal solitons and will prove Theorems C, D and E.

3.1 The convex hull property

Recall that a k-dimensional conformal soliton in $\mathbb{H}^{n+1}$ with respect to $-\partial_0$ can be regarded as a minimal submanifold when $\mathbb{H}^{n+1}$ is equipped with the Ilmanen metric

$$\mathrm{g}_{\mathrm{I}}(k) = e^{\frac{2}{kx_0}}\, \mathrm{g}_{\mathbb{H}}\,.$$

Hence, solitons are real analytic submanifolds. According to the strong maximum principle, two different conformal solitons cannot "touch" each other at an interior or boundary point; see [25].

Lemma 3.1 *Let $\mathcal{S} \subset (\mathbb{H}^{n+1}, \mathrm{g}_{\mathbb{H}})$ be a (Euclidean) spherical cap centered at a point of $\partial'_\infty \mathbb{H}^{n+1}$ and $2 \le k \le n$ a natural number. Then, $\mathcal{S} \subset (\mathbb{H}^{n+1}, \mathrm{g}_{\mathrm{I}}(k))$ is strictly convex with respect to the upward pointing normal direction. Moreover, there is no k-dimensional soliton with respect to $-\partial_0$ touching $\mathcal{S}$ from above.*

Proof If ν is a unit normal vector field along $\mathcal{S} \subset (\mathbb{H}^{n+1}, \mathrm{g}_{\mathbb{H}})$, then $\tilde{\nu} = \lambda^{-1}\nu$ is a unit normal along $\mathcal{S} \subset (\mathbb{H}^{n+1}, \mathrm{g}_{\mathrm{I}}(k))$. Denoting with $\mathrm{II}_{\mathrm{g}_{\mathbb{H}}}$ the scalar second fundamental form of $\mathcal{S} \subset (\mathbb{H}^{n+1}, \mathrm{g}_{\mathbb{H}})$ in the direction of ν, and with $\mathrm{II}_{\mathrm{g}_{\mathrm{I}}}$ that of $\mathcal{S} \subset (\mathbb{H}^{n+1}, \mathrm{g}_{\mathrm{I}}(k))$ in the direction of $\tilde{\nu}$, from (2.1) it follows that

$$\mathrm{II}_{\mathrm{g}_{\mathrm{I}}} = e^{\frac{1}{kx_0}} \left\{ \mathrm{II}_{\mathrm{g}_{\mathbb{H}}} - \nu \left(\frac{1}{kx_0} \right) \mathrm{g}_{\mathbb{H}} \right\}. \tag{3.1}$$

Let us choose as ν the upward pointing unit normal along $\mathcal{S}$. Since $\mathcal{S}$ is a totally geodesic hypersurface of $(\mathbb{H}^{n+1}, \mathrm{g}_{\mathbb{H}})$, from the last identity we obtain that

$$\mathrm{II}_{\mathrm{g}_{\mathrm{I}}} = \frac{e^{\frac{1}{kx_0}}}{k}\, \mathrm{g}_{\mathbb{H}}(\nu, \partial_0)\, \mathrm{g}_{\mathbb{H}} > 0.$$

Hence $\mathcal{S}$ is convex when the ambient space is equipped with the Ilmanen metric. The last claim of the lemma follows by the strong maximum principle of Jorge & Tomi [37]. □

Now we show that, similarly to minimal submanifolds, solitons satisfy the convex hull property.

Lemma 3.2 *Let $M \subset \mathbb{H}^{n+1}$ be a k-dimensional, connected and properly immersed soliton with respect to $-\partial_0$. Then, $\partial'_\infty M \neq \emptyset$ and M is contained in the cylinder $\mathbb{R}^+ \times \mathrm{conv}(\partial'_\infty M)$, where* conv *is the (Euclidean) convex hull in $\partial'_\infty \mathbb{H}^{n+1}$.*

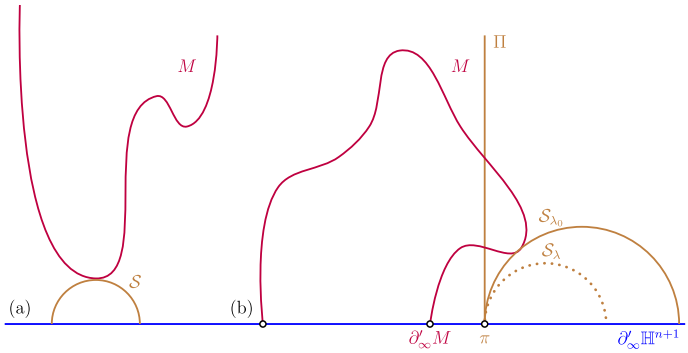

Fig. 3 Contact with spherical cap

Proof Observe at first that $\partial'_\infty M \neq \emptyset$ since otherwise we can touch M from below by a spherical cap $\mathcal{S}$, something which contradicts Lemma 3.1; see Fig. 3a.

Assume now that $\mathrm{conv}(\partial'_\infty M) \not\equiv \partial'_\infty \mathbb{H}^{n+1}$, because otherwise we have nothing to prove. Fix a hyperplane $\pi \subset \partial'_\infty \mathbb{H}^{n+1}$ not intersecting $\partial'_\infty M$, and let $\Pi \subset \mathbb{H}^{n+1}$ be the totally geodesic hyperplane with $\partial'_\infty \Pi = \pi$. Denote with U_π the open half of $\mathbb{H}^{n+1}$ such that

$$\partial U_\pi = \Pi \quad \text{and} \quad \partial'_\infty U_\pi \cap \partial'_\infty M = \emptyset.$$

By the properness of M, we can pick a sufficiently small spherical barrier $\mathcal{S} \subset U_\pi$ centered at a point of $\partial'_\infty U_\pi$ and lying outside of M. Due to Lemma 3.1, we can slide $\mathcal{S}$ toward Π without intersecting M, until its boundary at infinity touches π at a point q. Denote by p the center of $\mathcal{S}$ and consider the family of spherical barriers $\{\mathcal{S}_\lambda\}$ of radius $\lambda > 0$, passing through q and centered in the half-line emanating from q in the direction of p. The family foliates U and, again by Lemma 3.1, M does not intersect any spherical cap $\mathcal{S}_\lambda$; see Fig. 3b. Hence, $M \subset \mathbb{H}^{n+1} \setminus U_\pi$. The conclusion follows by the arbitrariness of π. □

3.2 Graphical solitons

Let us consider solitons of the form

$$\Gamma(u) = \left\{(u(x); x) \in \mathbb{H}^{n+1} = \mathbb{R}^+ \times \mathbb{R}^n : x \in \Omega \subset \mathbb{R}^n\right\},$$

where $u \in C^\infty(\Omega)$.

Lemma 3.3 *The graph $\Gamma(u) \subset \mathbb{H}^{n+1}$ is a soliton with respect to $-\partial_0$, if and only if it satisfies the equation*

$$\mathrm{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) = \frac{-nu-1}{u^2\sqrt{1+|Du|^2}}, \tag{SE}$$

where div *is the Euclidean divergence, D denotes the Euclidean gradient and $|\cdot|$ the Euclidean norm. In particular, $\Gamma(u)$ has nowhere zero mean curvature.*

Proof The graph is the image of $\psi : \Omega \to \mathbb{H}^{n+1}$ given by

$$\psi(x) = (u(x); x),$$

for each point $x \in \Omega$. As usual, denote by $g_{\mathbb{H}}$ the metric of $\mathbb{H}^{n+1}$ and by ∇ its Levi-Civita connection. The components of the induced metric g on the graph in the basis $\{\partial_j\}$ are

$$g_{ij} = \frac{u_i u_j + \delta_{ij}}{u^2}, \tag{3.2}$$

where $i, j \in \{1, \dots, n\}$. Moreover, the components g^{ij} of the inverse of g are given by

$$g^{ij} = u^2 \left(\delta^{ij} - \frac{u^i u^j}{1 + |Du|^2} \right), \tag{3.3}$$

where

$$u^i = \delta^{ij} u_j \quad \text{and} \quad Du = u^j \partial_j.$$

The unit normal ν along the graph is

$$\nu = \frac{u\,\partial_0 - u Du}{\sqrt{1 + |Du|^2}} = \frac{u\,\partial_0 - u u^j \partial_j}{\sqrt{1 + |Du|^2}}. \tag{3.4}$$

Making use of the Koszul formula and (3.4), the components of the second fundamental form are

$$b_{ij} = g_{\mathbb{H}} \left(\nabla_{\psi_i} \psi_j, \nu \right) = g_{\mathbb{H}}(\psi_{ij}, \nu) + u^{-1} \langle \psi_i, \psi_j \rangle\, g_{\mathbb{H}}(\partial_0, \nu) = \frac{u u_{ij} + \delta_{ij} + u_i u_j}{u^2 \sqrt{1 + |Du|^2}}$$

for each $i, j \in \{1, \dots, n\}$, where $\langle \cdot, \cdot \rangle$ stands for the Euclidean standard inner product. Using (3.2) and raising one index by means of the graph metric, the shape operator satisfies

$$b^k_j = g^{ki} b_{ij} = \frac{1}{\sqrt{1 + |Du|^2}} \left[u \left(u^k_j - \frac{u^k u^i u_{ij}}{1 + |Du|^2} \right) + \delta^k_j \right]. \tag{3.5}$$

From (3.5), the unnormalized scalar mean curvature is

$$H = g^{ij} b_{ij} = u \operatorname{div} \left(\frac{Du}{\sqrt{1 + |Du|^2}} \right) + \frac{n}{\sqrt{1 + |Du|^2}}. \tag{3.6}$$

One the other hand, $\Gamma(u)$ is a soliton with respect to $X = -\partial_0$ if and only if

$$H = -\, g_{\mathbb{H}}(\partial_0, \nu) = \frac{-1}{u \sqrt{1 + |Du|^2}}. \tag{3.7}$$

Combining (3.6) with (3.7) we obtain the desired result. □

3.3 Sub- and supersolutions

Let us describe here special sub- and supersolutions
to the quasilinear differential equation (SE) that we will use often in the rest of the paper.

Definition 3.4 Let Ω be a domain of the Euclidean space $\mathbb{R}^n$. A C^2-smooth function $u : \Omega \to (0, \infty)$ is called *subsolution* (resp. *supersolution*) to (SE) if it satisfies

$$\operatorname{div} \left(\frac{Du}{\sqrt{1 + |Du|^2}} \right) \geq \frac{-nu - 1}{u^2 \sqrt{1 + |Du|^2}} \qquad (\text{resp. } \leq).$$

In this case we say that the graph of f is a subsolution (resp. *supersolution*) to (SE).

Remark 3.5 Euclidean half-spheres in $\mathbb{H}^{n+1}$ whose centers are at $\partial'_\infty \mathbb{H}^{n+1}$ and Euclidean half-hyperplanes whose boundaries are at $\partial'_\infty \mathbb{H}^{n+1}$ are subsolutions to the equation (SE). If u is a solution to (SE) and $\varepsilon > 0$, then $u + \varepsilon$ is a supersolution and $u - \varepsilon$ is a subsolution to (SE).

3.4 Proof of Theorem C

We examine here complete solitons of the form $\Gamma \times \mathbb{R}^{n-1} \subset \mathbb{H}^{n+1}$, where Γ is a curve in the x_0x_1-plane. In regions where Γ can be represented as the image of

$$\gamma(t) = (u(t), t)$$

for some function u, equation (SE) becomes:

$$\frac{u''}{1 + (u')^2} = \frac{-nu - 1}{u^2}. \tag{3.8}$$

Notice that u is strictly concave. Hence, up to possibly one point (the maximum of u, if any), Γ can also be rewritten as the union of graphs of the type

$$\gamma(z) = (z, \phi(z))$$

for some functions ϕ. In this case, (3.8) rewrites as

$$\frac{\phi_{zz}}{1 + \phi_z^2} = \frac{nz + 1}{z^2}\phi_z. \tag{3.9}$$

By (3.9), if γ' is parallel to ∂_0 at some point (z_0, ϕ_0) then the unique solution is $\gamma(z) = (z, \phi_0)$ and the corresponding soliton is a vertical hyperplane. Let us treat now the case where γ' is nowhere parallel to ∂_0. In this case, γ can be globally written as a graph of the type $(u(t), t)$. In particular, the derivative ϕ_z does not change sign on each maximal interval where γ can be written in the form $(z, \phi(z))$. Since $\phi_c \doteq \phi + c$ and $\phi^* = -\phi$ still solve (3.9), without loss of generality we can consider a maximal interval where

$$\gamma(z) = (z, \phi(z))$$

and $\phi_z < 0$. Equation (3.9) guarantees that ϕ is concave therein, so the interval of definition of z is of the form $(0, h)$. Rewrite (3.9) as

$$\frac{\mathrm{d}}{\mathrm{d}z} \int_{-\phi_z}^{\infty} \frac{\mathrm{d}t}{t(1 + t^2)} = -\frac{n}{z} - \frac{1}{z^2},$$

and note that the integral can be explicitly computed. Integrating on $[z, b]$ leads to

$$\log \sqrt{1 + \phi_z(b)^{-2}} - \log \sqrt{1 + \phi_z(z)^{-2}} = -n \log\left(\frac{b}{z}\right) + \frac{1}{b} - \frac{1}{z}.$$

If $h = \infty$, letting $b \to \infty$ and recalling that ϕ_z is negative and decreasing we easily get a contradiction. Hence, $h < \infty$, and by concavity and maximality of h we shall have $\phi_z(h^-) = -\infty$. Letting $b \to h$ we get

$$\log \sqrt{1 + \phi_z(z)^{-2}} = n \log\left(\frac{h}{z}\right) + \frac{1}{z} - \frac{1}{h}.$$

In particular, $\phi_z(0^+) = 0$, i.e., γ meets the x_1-axis orthogonally. Extracting ϕ_z and integrating once more on $[z, b]$, we get

$$\phi(z) - \phi(b) = \int_z^b \left\{ \left(\frac{h}{t}\right)^{2n} e^{\frac{2}{t}-\frac{2}{h}} - 1 \right\}^{-\frac{1}{2}} \mathrm{d}t.$$

We deduce that $\phi(h)$ is finite, and up to translation we can assume that $\phi(h) = 0$. Writing γ as a graph of type $(u(t), t)$, noting that $u^*(t) \doteq u(-t)$ still solves (3.8) and $u'(0) = 0$, we get that u is even and $u(t) = \phi^{-1}(t)$ for $t > 0$ has the behavior described in Theorem C. Letting $b \to h$ we obtain

$$\phi(z) = \int_z^h \left\{ \left(\frac{h}{t}\right)^{2n} e^{\frac{2}{t}-\frac{2}{h}} - 1 \right\}^{-\frac{1}{2}} \mathrm{d}t = h \int_{z/h}^1 \left\{ s^{-2n} e^{\frac{2-2s}{hs}} - 1 \right\}^{-\frac{1}{2}} \mathrm{d}s. \tag{3.10}$$

Hence, for h varying in $\mathbb{R}^+$, the functions ϕ in (3.10) provide a foliation of $\{x_1 > 0\}$. This concludes the proof of Theorem C. □

3.5 Rotationally symmetric solitons

We deal now with solitons for $-\partial_0$ which are rotationally symmetric with respect to the x_0-axis. Consider a curve $\gamma(t) = (x_0(t), x_1(t))$ in the x_0x_1-plane, defined for $t \in I \subset \mathbb{R}$ and contained in the set $\{(x_0, x_1) \in \mathbb{R}^2 : x_0 > 0, x_1 > 0\}$, and let us rotate it around the x_0 axis. Locally, γ can be written either as a graph of a function u over the x_1-axis or as a graph of a function ϕ over the x_0-axis. Let us describe the corresponding soliton equations for u and ϕ.

Case 1: The curve γ is written as a graph of the form $\rho \mapsto (u(\rho), \rho)$ defined on an interval (r_1, r_2). In this case, the obtained hypersurface is a graph over the annulus

$$A = \left\{(x_1, \dots, x_n) \in \partial'_\infty \mathbb{H}^{n+1} : r_1 < |x| = (x_1^2 + \cdots + x_n^2)^{1/2} < r_2\right\},$$

and it can be parametrized by the embedding $R_u : A \to \mathbb{R}^+ \times \mathbb{R}^n$ given by

$$R_u(x_1, \dots, x_n) = (u(|x|); x_1, \dots, x_n).$$

Then, R_u is a conformal soliton with respect to $-\partial_0$ if and only if u solves

$$\frac{u''}{1 + (u')^2} + \frac{n-1}{\rho} u' = -\frac{1 + nu}{u^2}, \qquad \rho = |x| \in (r_1, r_2). \tag{3.11}$$

Case 2: The curve γ is written as a graph of the form $z \to (z, \phi(z))$ defined on an interval (z_1, z_2). In this case the hypersurface can be parametrized as a graph over the cylinder $\mathbb{R}^+ \times \mathbb{S}^{n-1} \subset \mathbb{R}^+ \times \mathbb{R}^m$. To state the equation for ϕ, we first locally identify $\mathbb{H}^{n+1} = \mathbb{R}^+ \times \mathbb{R}^n$ with $\mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{S}^{n-1}$ via the local diffeomorphism

$$(x_0; x_1, \dots, x_n) \to (z, \rho, \omega) = \left(x_0; \sqrt{x_1^2 + \cdots + x_n^2}; \frac{(x_1, \dots, x_n)}{\sqrt{x_1^2 + \cdots + x_n^2}}\right).$$

The hyperbolic metric in the new coordinate system has the form

$$\mathrm{g}_{\mathbb{H}} = z^{-2}\left(\mathrm{d}z^2 + \mathrm{d}\rho^2 + \rho^2\, \mathrm{g}_{\mathbb{S}}\right),$$

where $\mathrm{g}_{\mathbb{S}}$ is the standard metric of $\mathbb{S}^{n-1}$. We can parametrize the hypersurface via the map $C_\phi : (z_1, z_2) \times \mathbb{S}^{n-1} \to \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{S}^{n-1}$ given by $C_\phi(z, \omega) = (z, \phi(z), \omega)$. Then C_ϕ is a

soliton if and only if

$$\frac{z\phi''}{1+(\phi')^2} - \frac{1+nz}{z}\phi' - \frac{(n-1)z}{\phi} = 0, \qquad z \in (z_1, z_2), \tag{3.12}$$

where we use ϕ' instead of ϕ_z for notational convenience.

In the next lemma, we examine the qualitative behavior of solutions to (3.12).

Lemma 3.6 *Let $z_0 \in (z_1, z_2)$ with $\phi(z_0) > 0$. The following facts hold:*

(1) **(Concave branch)** *If $\phi'(z_0) < 0$ and $\phi''(z_0) \leq 0$, then ϕ can be extended to the interval $[0, z_0]$. Moreover, $\phi' < 0$, $\phi'' < 0$ on $(0, z_0)$ and*

$$\phi(z) = \phi(0) - \frac{n-1}{3\phi(0)}z^3 + o(z^3) \quad \text{as } z \to 0.$$

(2) **(Convex branch)** *If either $\phi'(z_0) \geq 0$ or $\phi''(z_0) > 0$, then there exists an interval $[\lambda_0, h) \subset (0, \infty)$ containing z_0 where ϕ can be defined, is positive and satisfies*

$$(i): \ \phi'' > 0 \ \text{ on } (\lambda_0, h), \quad \phi'(\lambda_0) < 0 \ \text{ and } \ \phi''(\lambda_0) = 0.$$
$$(ii): \lim_{z\to h}\phi(z) \in \mathbb{R}^+ \ \text{ and } \ \lim_{z\to h}\phi'(z) = \infty.$$

In particular, ϕ has a unique minimum on (λ_0, h).

Proof (1) Let $(z_1, z_0]$ be the maximal interval on the left of z_0 where ϕ is defined. By (3.12), if $\phi'(z) = 0$ for some $z \in (z_1, z_0)$ then $\phi''(z) > 0$. Hence, from $\phi'(z_0) < 0$ we deduce that $\phi' < 0$ on $(z_1, z_0]$. We claim that $\phi'' \leq 0$ on $(z_1, z_0]$. This fact is true since otherwise, from $\phi''(z_0) \leq 0$ we deduce that there exist an interval $(a, b] \subset (z_1, z_0]$ such that $\phi'' > 0$ on (a, b) and $\phi''(b) = 0$. Then, from (3.12) we have

$$-\phi'(a)\left(n + \frac{1}{a}\right) \leq (n-1)\frac{a}{\phi(a)} \quad \text{and} \quad -\phi'(b)\left(n + \frac{1}{b}\right) = (n-1)\frac{b}{\phi(b)}.$$

Now from $0 < \phi(b) < \phi(a)$, we obtain that $0 > \phi'(a) > \phi'(b)$, contradicting that $\phi'' > 0$ on (a, b). To show the strict inequality $\phi'' < 0$ on $(z_1, z_0]$, observe that any point $z \in (z_1, z_0]$ for which $\phi''(z) = 0$ must be a local maximum for φ'', hence a point for which $\varphi'''(z) = 0$. Differentiating (3.12), and evaluating at z, we have

$$\begin{aligned} 0 &= \frac{z\phi'''(z) + \phi''(z)}{1+(\phi'(z))^2} - \frac{2z\phi'(z)(\phi''(z))^2}{(1+(\phi'(z))^2)^2} - \frac{1+nz}{z}\phi''(z) \\ &\quad + \frac{\phi'(z)}{z^2} - \frac{(n-1)}{\phi(z)} + \frac{(n-1)z\phi'(z)}{\phi^2(z)} \\ &= \frac{\phi'(z)}{z^2} - \frac{(n-1)}{\phi(z)} + \frac{(n-1)z\phi'(z)}{\phi^2(z)} < 0, \end{aligned}$$

which gives a contradiction. From $\phi' < 0$ and $\phi'' < 0$, we deduce that the limit $\lim_{z\to z_1}\phi'(z)$ exists and is finite. If $z_1 > 0$, then the function ϕ can be extended beyond z_1, which contradicts the maximality of the interval. Hence, $z_1 = 0$ and so ϕ has a C^1-extension up to $z = 0$. By using spherical barriers and the strong maximum principle, we deduce that $\lim_{z\to 0}\phi' = 0$. To find the asymptotic behavior of ϕ, consider the function

$$F(z) = \frac{z^{-n}\phi'(z)}{\sqrt{1+(\phi'(z))^2}}, \tag{3.13}$$

defined in a neighborhood $(0, a]$. Differentiating and using (3.12), we deduce that

$$F' - \frac{F}{z^2} = \frac{n-1}{z^n \phi \sqrt{1+(\phi')^2}},$$

that is,

$$\left(F e^{\int_z^a r^{-2} \mathrm{d}r}\right)' = \frac{(n-1) e^{\int_z^a r^{-2} \mathrm{d}r}}{z^n \phi \sqrt{1+(\phi')^2}}.$$

Integrating on $[z, a]$, we get

$$F(a) - F(z) e^{-\frac{1}{a}+\frac{1}{z}} = \int_z^a \frac{(n-1) e^{-\frac{1}{a}+\frac{1}{r}}}{r^n \phi(r) \sqrt{1+\phi'(r)^2}} \mathrm{d}r, \tag{3.14}$$

or, equivalently,

$$F(z) = F(a) e^{\frac{1}{a}-\frac{1}{z}} - e^{-\frac{1}{z}} \int_z^a \frac{(n-1) e^{\frac{1}{r}}}{r^n \phi(r) \sqrt{1+\phi'(r)^2}} \mathrm{d}r.$$

Computing the asymptotic behavior of the right hand side as $z \to 0$ and using the facts

$$\lim_{z \to 0} \phi(z) = \phi(0) > 0, \quad \lim_{z \to 0} \phi'(z) = 0 \quad \text{and} \quad F(z) \sim z^{-n} \phi'(z)$$

we infer

$$\phi'(z) = -\frac{n-1}{\phi(0)} z^2 + o(z^2), \quad \text{as } z \to 0.$$

By integration we obtain the desired asymptotic behavior of ϕ close to $z = 0$.

(2) Notice that if $\phi'(z_0) \geq 0$ then (3.12) implies $\phi''(z_0) > 0$. Let (λ_0, h) be the maximal interval containing z_0 where $\phi'' > 0$. By convexity, the limits of ϕ and ϕ' as z tends to h exist and

$$\lim_{z \to h} \phi(z) = \phi_h \in [0, \infty] \quad \text{and} \quad \lim_{z \to h} \phi'(z) = \phi'_h \in (-\infty, \infty]. \tag{3.15}$$

Claim 1: *It is not possible that h, ϕ'_h are finite and $\phi_h > 0$.*

Proof of the claim. Indeed, if this is possible then ϕ can be extended beyond h and therefore, by the maximality of h, $\phi''(h) = 0$. Equation (3.12) then gives $\phi'(h) < 0$ and by part (1), the function ϕ would be a concave branch before h, which is a contradiction. ⊛

Observe that $\phi'' > 0$ implies that ϕ' does not change sign in some interval (λ, h).

Claim 2: *ϕ attains a minimum on (λ_0, h) and the minimum is unique.*

Proof of the claim. Uniqueness is immediate from $\phi'' > 0$. Because of the convexity, for the existence we only have to exclude the possibility that $|\phi'| > 0$ on (λ_0, h).

(a) First, we rule out the possibility that $\phi' < 0$ on (λ_0, h). Suppose to the contrary that this case occurs. If $h = \infty$, then the convexity and the positivity of ϕ imply that $\lim_{z \to \infty} \phi'(z) = 0$. From (3.12) we deduce that the existence of a positive constant C such that $\phi''(z) > C$ for large values of z. Integrating, $\phi'(z) \to \infty$, a contradiction. Assume now that $h < \infty$ and the limit ϕ_h in (3.15) vanishes. By convexity, fixing $\lambda \in (\lambda_0, h)$ there exists a constant $C > 0$ such that $\phi(z) \leq C(h - z)$, on $[\lambda, h)$. From (3.12) and the fact that ϕ' is bounded on $[\lambda, h)$, it follows that there exist positive constants C_1 and C_2 such that

$$\phi'' \geq \frac{C_1}{h - z} - C_2 \quad \text{on } [\lambda, h).$$

Integrating on $[\lambda, z]$ we get

$$\phi'(z) - \phi'(\lambda) \geq -C_1 \log\left(\frac{h-z}{h-\lambda}\right) - C_2(z-\lambda) \to \infty \quad \text{as} \quad z \to h,$$

contradicting $\phi'(z) < 0$. Consequently, h is finite and $\phi_h \in \mathbb{R}^+$. From (3.15) and $\phi' < 0$ we deduce that ϕ_h' is finite as well. By Claim 1, this leads to a contradiction.

(b) We also rule out the possibility that $\phi' > 0$ on (λ_0, h). Arguing again by contradiction, let us suppose that this happens. We may represent the graph of ϕ on (λ_0, h) as the graph of a function u over the x_1-axis defined on an interval $(r_1, r_2) \subset (0, \infty)$, where $r_1 = \phi(\lambda_0^+)$ and $r_2 = \phi(h^-)$. Consider $\Phi : (r_1, r_2) \to \mathbb{R}$ given by

$$\Phi(\rho) = \frac{u'(\rho)\rho^{n-1}}{\sqrt{1 + (u'(\rho))^2}}. \tag{3.16}$$

From (3.11) we deduce that

$$\Phi'(\rho) = -\frac{(1 + nu(\rho))\rho^{n-1}}{u^2(\rho)\sqrt{1 + (u'(\rho))^2}} < 0 \qquad \text{for} \quad r \in (r_1, r_2).$$

Hence Φ is strictly decreasing. Since u is increasing, Φ is positive on (r_1, r_2). If $r_1 = 0$, then letting $\rho \to 0$, we get that $\Phi(0^+) = 0$ which contradicts the aforementioned properties of Φ. Consequently $r_1 > 0$. Moreover, λ_0 must be positive. Indeed, otherwise ϕ is defined, convex and increasing on $(0, h)$. Consider the functional F given by (3.13) on the interval $(0, a] \subset (0, h)$. From the properties of ϕ, it follows that ϕ' is bounded on $(0, a]$. Hence,

$$\lim_{z \to 0} \int_z^a \frac{(n-1)e^{-\frac{1}{a}+\frac{1}{r}}}{r^n \phi(r)\sqrt{1 + (\phi'(r))^2}} \mathrm{d}r = \infty.$$

From (3.14) we now deduce that

$$\lim_{z \to 0} F(z) e^{-\frac{1}{a}+\frac{1}{z}} = -\infty.$$

Consequently $F < 0$ in a neighborhood of $z = 0$. Since F and ϕ' have the same sign, this leads to a contradiction. Thus, λ_0 must be positive. Because $\phi(\lambda_0) > 0$ and $\phi'(\lambda_0) \geq 0$, we can extend the solution below λ_0. The minimality of λ_0 implies that $\phi''(\lambda_0) = 0$ and from (3.12) we obtain that $\phi'(\lambda_0) < 0$, contradiction. ⊛

Claim 3: $\lambda_0 > 0$ *and* (i) *holds.*

Proof of the claim. Suppose that this is not true. Recalling that $\phi'' > 0$ on (λ_0, h), one of the following holds:

(a) $\lambda_0 = 0$;
(b) $\lambda_0 > 0$ and $\lim_{z \to \lambda_0} \phi(z) = \infty$;
(c) $\lambda_0 > 0$ and $\lim_{z \to \lambda_0} \phi(z) < \infty$ and $\lim_{z \to \lambda_0} \phi'(z) = -\infty$.

Cases (a) and (b) are excluded by considering the soliton M obtained by rotating the graph of ϕ and sliding-enlarging a small spherical barrier below M up to a touching point. As for (c), writing the graph in terms of the parametrization $\rho \to (u(\rho), \rho)$, we get that u can be extended near $\phi(\lambda_0)$ with zero derivative and nonnegative second derivative. This contradicts (3.11) and concludes the proof of the claim. ⊛

Claim 4: *h is finite and* (ii) *holds.*

Proof of the claim. By Claim 2, ϕ' and ϕ'' are positive in an interval (λ, h). Then, we represent the graph of ϕ therein as a graph of the form $\rho \mapsto (u(\rho), \rho)$ satisfying $u' > 0$ and $u'' < 0$ for $\rho \in (r_1, r_2)$, where $r_2 = \phi(h^-)$. First we show that $r_2 < \infty$. Suppose that this is not the case and denote by u_∞ the limit of u as ρ tends to ∞. If u_∞ is finite, then $u' \to 0$ as $\rho \to \infty$ and there exists a sequence $\{\rho_j\}_{j\in\mathbb{N}}$ tending to infinity such that $u''(\rho_j) \to 0$. Evaluating (3.11) at ρ_j and passing to the limit we easily get a contradiction. If $u_\infty = \infty$, consider Φ given in (3.16) for $\rho \in (r_1, \infty)$. Fixing $\rho_0 > 1$, from the monotonicity of Φ there exists a constant $C > 0$ such that

$$\frac{u'(\rho)}{\sqrt{1 + (u'(\rho))^2}} \leq C\rho^{1-n} \qquad \text{for all} \quad \rho \geq \rho_0,$$

from where we deduce that

$$u'(\rho) \leq \frac{C\rho^{1-n}}{\sqrt{1 - C^2\rho^{2-2n}}}.$$

If $n \geq 3$, then by integration we get that $u_\infty < \infty$, contradiction. If $n = 2$, integration gives

$$u'(\rho) \leq C_1\rho^{-1} \text{ and } u(\rho) \leq C_1 \log\rho \qquad \text{for some constant } C_1 > 0.$$

Inserting into (3.11), we conclude that there exists a constant $C_2 > 0$ such that

$$u'' \leq -\frac{C_2}{\log\rho} \qquad \text{for large enough } \rho.$$

By integrating we see that $u'(\rho) \to -\infty$ as $\rho \to \infty$, which gives a contradiction. Hence, $r_2 < \infty$, which implies that h is finite by the concavity of u. Moreover, $u' \to \ell \geq 0$ as $\rho \to r_2$, whence u can be extended beyond r_2. From (3.11) we get $u''(r_2) < 0$. If $\ell > 0$, then rephrasing the curve in terms of ϕ, it follows that ϕ can be extended beyond h to a convex function, contradicting the maximality of h. Thus, $\ell = 0$ and (ii) follows. ⊛

This completes the proof of the lemma. □

Lemma 3.7 *Let u_1 and u_2 be solutions to (3.11) on an interval (r_1, r_2). Then, either $u_1 \equiv u_2$ or $u_1 - u_2$ does not have a nonnegative local maximum on (r_1, r_2).*

Proof Assume to the contrary that $u_1 - u_2$ attains a local maximum $c_0 \geq 0$ at a point r_0. Then, $u_1 \leq u_2 + c_0$ near r_0 with equality attained at the point r_0. From $c_0 \geq 0$, it follows that $u_2 + c_0$ is a supersolution to (3.11) and this contradicts the strong maximum principle. □

Now we are ready to prove the main theorem characterizing all rotationally symmetric solitons.

Theorem 3.8 *There are exactly two families of complete solitons with respect to $-\partial_0$ which are rotationally symmetric around the x_0-axis. They are properly embedded and, denoting with*

$$\gamma \subset \{(x_0, x_1) \in \mathbb{R}^2 : x_0 > 0, x_1 \geq 0\}$$

the rotated curve, one of the following cases occurs:

(1) **(Winglike catenoids)** *Suppose that $(x_0 \circ \gamma)'(t_0) = 0$ at some interior point t_0 and let $\gamma(t_0) = (h, R)$, $R > 0$. Then the curve γ can be written as the bi-graph over the x_0 axis of $\phi_1, \phi_2 : (0, h] \to (0, \infty)$ satisfying the following properties:*

(a) *It holds* $\phi_1 < \phi_2$ *on* $(0, h)$ *and* $\phi_1(h) = \phi_2(h) = R$. *Furthermore,* $\phi_1(0^+) < \phi_2(0^+)$, *namely,* γ *cannot have the same end-points;*
(b) *the graph of* ϕ_2 *is a concave branch on* $(0, h)$;
(c) *there exists* $\lambda_0 \in (0, h)$ *such that* ϕ_1 *is the union of a concave branch on* $(0, \lambda_0)$ *and a convex branch on* (λ_0, h);

(2) **(Bowl solitons)** *If* $x_0 \circ \gamma$ *does not have interior stationary points, then* γ *is the graph of* $\phi : (0, h] \to [0, \infty)$ *satisfying the following properties:*

(a) *The graph of* ϕ *is a concave branch on* $(0, h)$;
(b) *it holds* $\phi(h) = 0$, $\phi'(h^-) = \infty$.

Proof (1) Writing γ near (h, R) as a graph of the form $\rho \to (u(\rho), \rho)$, the equation (3.11) and $u'(R) = 0$ gives $u'' < 0$ near R. In particular, u is decreasing after R and increasing before. Hence, by Lemma 3.6, for $\rho > R$ the graph of u extends to a concave branch of the form $z \to (z, \phi_2(z))$ for $\phi_2 : (0, h) \to \mathbb{R}^+$, while for $\rho < R$ it extends to a convex branch of a function $\phi_1 : (\lambda_0, h) \to \mathbb{R}^+$. At the point $(\lambda_0, \phi_1(\lambda_0))$ we can apply Lemma 3.6 again to deduce that ϕ_1 extends to a concave branch on $(0, \lambda_0)$. Set $c = \min \phi_1$. It remains to prove that $\phi_1 < \phi_2$ on $(0, h)$ and $\phi_1(0^+) < \phi_2(0^+)$. Arguing by contradiction, if any of the properties fails then, according to what we already proved, there exists $R_1 \in (c, \phi_2(0^+)]$ such that the curve γ can be written as a bigraph of functions $u_i : (c, R_1) \to \mathbb{R}^+$ over the x_1-axis satisfying $u_1 > u_2$ on (c, R_1), $u_1 - u_2 \to 0$ as $\rho \to c$ and as $\rho \to R_1$. This contradicts Lemma 3.7.

(2) If $x_0 \circ \gamma$ has no stationary points, then γ can be globally written as a graph of the form $z \to (z, \phi(z))$. By Lemma 3.6, the graph of ϕ is the concave branch defined in a maximal domain $(0, h)$. Representing γ as a graph of the form $\rho \to (u(\rho), \rho)$, concavity implies that $u'(0^+)$ exists and is non-positive. If this value is negative, then by inspecting (3.11) for small enough ρ we arrive at a contradiction. Hence, $u'(0^+) = 0$. It remains to prove that a curve of type (2) actually exists. This is addressed in the next lemma. □

Lemma 3.9 *Fix* $R \geq 0$. *For* $h > 0$, *the solution* u_h *to the following problem exists and is unique:*

$$\begin{cases} \dfrac{u''}{1 + (u')^2} + \dfrac{n-1}{\rho} u' = -\dfrac{1 + nu}{u^2} \quad \text{for} \quad \rho > R, \\ u(R^+) = h, \\ u'(R^+) = 0. \end{cases} \tag{3.17}$$

Moreover, u_h *is concave and strictly decreasing. Let* $[R, r_2(h))$ *be the maximal interval where* u_h *is defined. Then, for* $h \in \mathbb{R}^+$ *the graphs of* $\{u_h\}$ *foliate the region* $\{\rho > R\}$, *and* $r_2 : (0, \infty) \to (R, \infty)$ *is a strictly increasing bijection.*

Proof Uniqueness for u_h is a consequence of Lemma 3.7 and the strong maximum principle (or the Hopf Lemma, if $R > 0$) for the solitons M_h obtained by rotating the graphs $\{x_0 = u_h(|x|)\}$ around the x_0-axis. Note that, if $R = 0$, M_h is C^1 near the origin and $\mathrm{g_I}(n)$-minimal, hence it is smooth therein. Existence for (3.17) is standard if $R > 0$. By adapting the proof of Lemma 3.6, one easily sees that u_h is strictly decreasing for $R > 0$ and concave for $\rho > R$. To prove existence for $R = 0$ one may proceed by adapting the techniques in [8, Chapter 5, Theorem 5.10]. However, we give a geometric and simpler proof which exploits the results

we showed so far. Consider a decreasing sequence $\varepsilon_i \downarrow 0$, and for each i let u_i solve

$$\begin{cases} \dfrac{u_i''}{1+(u_i')^2} + \dfrac{n-1}{\rho} u_i' = -\dfrac{1+nu_i}{u_i^2}, \\ u_i(\varepsilon_i) = h, \\ u_i'(\varepsilon_i) = 0, \end{cases}$$

on its maximal interval $[\varepsilon_i, R_i)$. Since

$$u_i''(\varepsilon_i) = -(1+nh)h^{-2} < 0,$$

from Lemma 3.6 we get that the graph of u_i is a concave branch in (ε_i, R_i) and in particular R_i is finite. We claim that R_i is bounded from below away from zero. Suppose to the contrary that, along a subsequence, $R_i \to 0$. According to Lemma 3.6, the graph of u_i is part of a winglike catenoid M_i of height h. Take a grim-reaper cylinder $\mathscr{G}_s$ of height $s > h$ and passing through $(s, 0, \dots, 0)$. Then, for fixed R_i small enough, M_i lies below $\mathscr{G}_s$. Reducing s up to a first touching point with M_i we reach a contradiction. Let now $0 < R^* = \inf R_i$. By (3.11) and since each u_i is decreasing, the sequence $\{u_i\}$ has uniformly bounded C^2-norm on any fixed compact set of $(0, R^*)$. Therefore, up to a subsequence, $u_i \to u$ in $C^2_{\mathrm{loc}}((0, R^*))$, where u solves (3.11) on $(0, R^*)$. Since each u_i is decreasing and concave, u is concave, non-increasing and $u(0^+) = h$. As a matter of fact, $u' < 0$ by (3.11) and so u is a concave branch. In particular, by the first part of the proof of Theorem 3.8(2), $u'(0^+) = 0$.

We next address the properties of u_h for varying h.

(i) First we show that r_2 is strictly increasing and that $\{u_h\}$ is increasing in h. Suppose to the contrary that for $h_1 > h_2$ we have $r_2(h_1) \le r_2(h_2)$. Then, $c \doteq \max(u_1 - u_2)$ is positive and attained at some $r \in [R, r_2(h_1))$. Consider the functions $v_j : B_{r_2(h_1)} \setminus B_R \to \mathbb{R}$ obtained by rotating u_{h_j} along the x_0-axis, and notice that u_1 is a soliton for $-\partial_0$ while $v_2 + c$ is a supersolution for (1.9), equivalently, it is mean convex for $\mathrm{g_I}(n)$ in the downward direction. The interior strong maximum principle (if $r > R$ or $r = R = 0$) or the boundary maximum principle (if $r = R > 0$) in [25] imply $v_1 \equiv v_2 + c$ wherever both are defined, contradiction. The very same reasoning also proves that $\{u_h\}$ is an increasing family in h.

(ii) Fix $h_0 > 0$ and suppose to the contrary that $r_* = \inf\{r_2(h) : h > h_0\} > r_2(h_0) = r_0$. Then a spherical barrier $\mathcal{S}$ with radius $(r_* - r_0)/2$ centered at the point $(0, (r_* + r_0)/2, 0, \dots, 0)$ fits between the sequence of solitons $\{M_h\}$ that are generated by the sequence $\{u_h\}$ for $h > h_0$ and u_{h_0}. Consider now the sequence $\{\overline{u}_h = u_h|_{[R, r_0)}\}$ for $h > h_0$. Observe that $\{\overline{u}_h\}$ uniformly converges to u_{h_0} as h tends to h_0. Hence, for h_1 sufficiently close to h_0 and ρ_1 sufficiently close to r_0, we have that

$$u_{h_1}(\rho_1) < \frac{r_* - r_0}{2}.$$

Hence M_{h_1} intersects $\mathcal{S}$, contradiction. This proves that r_2 is continuous.

(iii) We show that $r_2(h) \to \infty$ as $h \to \infty$. By contradiction, if $r_2(h) \le \ell$ for some $\ell \in \mathbb{R}^+$ and all $h > 0$, consider a grim-reaper cylinder $\mathscr{G}$ of width 4ℓ and symmetric with respect to $\{x_1 = 0\}$, and fix $h < \sup_{\mathscr{G}} x_0$. Let M be the hypersurface obtained by rotating u_h with respect to the x_0-axis. Then, by construction and since $u_h'(R^+) = 0$, we can find a grim-reaper cylinder $\mathscr{G}'$ in the foliation determined by $\mathscr{G}$ that touches M from above at some interior point p, contradiction.

(iv) To show that the graphs of u_h foliate $\{\rho > R\}$, let $\rho_0 > R$ and by (iii), let h_0 satisfy $r_2(h_0) = \rho_0$. It is enough to prove that the map $\eta : h \in (h_0, \infty) \mapsto u_h(\rho_0) \in (0, \infty)$ is

a continuous bijection. First, by (i) η is injective and strictly monotone. Next, if $h_j \to h > h_0$ then $\{u_{h_j}\}$ has uniformly bounded C^0 (hence, C^2) norm on any fixed compact of $(0, \rho_0]$. Up to a subsequence and by the uniqueness of solutions to (3.17), $u_{h_j} \to u_h$ in $C^2_{\text{loc}}((0, \rho_0])$, thus η is continuous. Whence $\eta((h_0, \infty))$ is an interval. By concavity and (iii), $\eta(h) \to \infty$ as $h \to \infty$. On the other hand, if $\delta = \inf \eta((h_0, \infty)) > 0$, then for $h_j \downarrow h_0$ we would have $u_{h_j}(\rho_0) > \delta$ for each j. Let $\rho_1 < \rho_0$ be such that $u_{h_0}(\rho_1) < \delta/2$. From $u_{h_j}(\rho_1) \to u_{h_0}(\rho_1)$, for j large we would have $u_{h_j}(\rho_1) < u_{h_j}(\rho_0)$, contradiction.

This completes the proof of the lemma. □

As an immediate consequence of Theorem 3.8 and Lemma 3.9, we are ready for the:

3.6 Proof of theorem E

Hereafter, all spheres and balls we use are meant to be centered at the center of B_R. By Theorem 3.8, there exists a unique graphical rotationally symmetric bowl soliton $\mathscr{B}_R$ having as boundary at infinity ∂B_R. Suppose now that M is a properly immersed soliton with respect to $-\partial_0$ such that $\partial'_\infty M = \partial B_R$. We take two bowl solitons $\mathscr{B}_{r_1}$ and $\mathscr{B}_{r_2}$, with $\partial'_\infty \mathscr{B}_{r_j} = \partial B_{r_j} \subset \partial'_\infty \mathbb{H}^{n+1}$ and lying above and below M, respectively. Shrinking $\mathscr{B}_{r_2}$ and enlarging $\mathscr{B}_{r_1}$, by the maximum principle we deduce that the soliton M must coincide with $\mathscr{B}_R$.

4 The Plateau problem at infinity

In this section we will show the existence of solitons $M \subset \mathbb{H}^{n+1}$ with respect to $-\partial_0$ that have prescribed asymptotic boundary values on $\partial_\infty \mathbb{H}^{n+1}$.

4.1 Plateau's problem at infinity

Consider an embedded $(k-1)$-dimensional (topological) submanifold $\Sigma \subset \partial_\infty \mathbb{H}^{n+1}$, where $2 \le k \le n$. Our aim is to find a k-dimensional conformal soliton M with respect to the direction $-\partial_0$ whose boundary at infinity satisfies $\partial_\infty M = \Sigma$. Viewing solitons as minimal submanifolds with respect to the Ilmanen metric, our problem can be rephrased as the classical Plateau's problem at infinity for area minimizing submanifolds. Thanks to various works, the solvability theory for Plateau's problem is well understood on Cartan–Hadamard manifolds, that is, on complete, simply connected manifolds N with non-positive sectional curvature; see, for example, [2–4, 9, 10, 12, 53].

It is known that every Cartan–Hadamard manifold N can be compactified by adding a sphere at infinity $\partial_\infty N$; see for details [24]. Given a compact $(k-1)$-dimensional submanifold $\Sigma \subset \partial_\infty N$, Plateau's problem at infinity asks whether exists a k-dimensional area minimizing submanifold $M \subset N$ such that $\partial_\infty M = \Sigma$. It is well-known that Plateau's problem is solvable if $\partial_\infty N$ satisfies certain convexity conditions. Let us recall here a convenient one proposed by Ripoll & Telichevesky in [46].

Definition 4.1 [SC condition] A Cartan–Hadamard manifold N satisfies the strict convexity condition (*SC* condition for short) at a point $x \in \partial_\infty N$ if, for each relatively open subset $W \subset \partial_\infty N$ containing x, there exists a C^2-open subset $\Omega \subset N$ such that $x \in \text{int}(\partial_\infty \Omega) \subset W$ and $N \setminus \Omega$ has convex boundary in the inward pointing direction. We say that N satisfies the *SC* condition if this holds at every $x \in \partial_\infty N$.

For a reason to be explained below, Theorem A is restricted to hypersurfaces. In this case, building on a previous result of Lang [39], Castéras, Holopainen & Ripoll [12, Theorem 1.6] obtained the following.

Theorem 4.2 *Let N^{n+1}, $n \geq 2$, be a Cartan–Hadamard manifold satisfying the* SC *condition and let $\Sigma \in \partial_\infty N$ satisfy $\Sigma = \partial A$ for some open subset $A \subset \partial_\infty N$ with $A = \overline{\mathrm{int}(A)}$. Then, there exists a closed set $W \subset N$ of locally finite perimeter in N such that $M \doteq \partial[W]$ is a locally rectifiable, minimizing n-current in N, $\partial_\infty W = A$ and $\partial_\infty \mathrm{spt} M = \Sigma$.*

Remark 4.3 An n-dimensional area minimizing n-rectifiable current M in a smooth complete manifold N^{n+1} is a smooth, embedded manifold on the complement of a singular set of Hausdorff dimension at most $n-7$. In particular, if $n < 7$, then the singular set is empty, while if $n = 7$ it consists of isolated points. In higher codimensions, the singular set has Hausdorff dimension at most $n-2$; for more details see [20, Section 3, Theorems 3.3, 3.4 and 3.5].

4.2 Geometry of the Ilmanen metric

Let us examine here the geometry of the Ilmanen metric in more detail. As a matter of fact, we will compute its curvatures and geodesics.

Lemma 4.4 *The Riemannian manifold $N = (\mathbb{H}^{n+1}, \mathrm{g_I}(n))$ is Cartan–Hadamard, and its sectional curvatures satisfy:*

(1) $\sec_{\mathrm{g_I}}(\partial_i \wedge \partial_0) = -e^{-\frac{2}{nx_0}} \frac{2+n}{nx_0}$,
(2) $\sec_{\mathrm{g_I}}(\partial_i \wedge \partial_j) = -e^{-\frac{2}{nx_0}} \frac{1+nx_0}{n}$,
(3) $\sec_{\mathrm{g_I}}((\sin\theta\,\partial_0 + \cos\theta\,\partial_i) \wedge \partial_j) = \sin^2\theta \sec_{\mathrm{g_I}}(\partial_0 \wedge \partial_j) + \cos^2\theta \sec_{\mathrm{g_I}}(\partial_i \wedge \partial_j)$,

for any $i, j \in \{1, \ldots, n\}$ and $\theta \in (0, 2\pi)$.

Proof Completeness immediately follows from $\mathrm{g_I}(n) > \mathrm{g}_{\mathbb{H}}$, and the fact that $(\mathbb{H}^{n+1}, \mathrm{g}_{\mathbb{H}})$ is complete. Direct computations gives the sectional curvatures in (1), (2) and (3). Eventually, let $\pi \subset T_p\mathbb{H}^{n+1}$ be a fixed 2-plane. If $\pi \subset \partial_0^\perp$, then up to a rotation, π is generated by $\partial_i \wedge \partial_j$ and (2) gives $\sec_{\mathrm{g_I}}(\pi) \leq 0$. Otherwise, let e_1 be a unit vector generating $\pi \cap \partial_0^\perp$, which up to rotation we can assume to be ∂_1. Complete e_1 to an orthonormal basis $\{e_1, e_2\}$ of π. Again up to rotation, we have that $e_2 = \sin\theta\partial_0 + \cos\theta\partial_2$, for some $\theta \in (0, 2\pi)$. Item (3) gives now $\sec_{\mathrm{g_I}}(\pi) \leq 0$. □

Let $\gamma : (-\varepsilon, \varepsilon) \to \mathbb{H}^{n+1}$ be a geodesic with respect to the Ilmanen metric $\mathrm{g_I}(n)$. Observe that isometries of the hyperbolic space which preserves the direction $-\partial_0$ are also isometries of the Ilmanen metric. Because of this fact, it suffices to examine only geodesics in the x_0x_1-plane. Let us suppose that the geodesic has the form $\gamma = (x_0, x_1)$. By straightforward computations we see that γ is a geodesic if the functions x_0 and x_1 satisfy

$$x_0'' - \frac{1+nx_0}{nx_0^2}\big((x_0')^2 - (x_1')^2\big) = 0 \quad \text{and} \quad x_1'' - 2\frac{1+nx_0}{nx_0^2}x_0'x_1' = 0.$$

Following the same methods as in Sect. 3, we can show the following:

Lemma 4.5 *Let $\gamma : \mathbb{R} \to (\mathbb{H}^{n+1}, \mathrm{g_I}(n))$ be a geodesic lying in the x_0x_1-plane. Then, either γ is a vertical line or:*

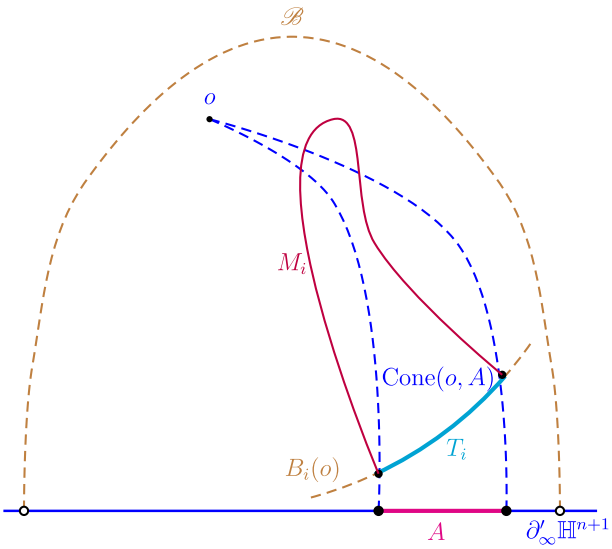

Fig. 4 The cone Cone(o, A)

(1) *There exist* $t_n \in \mathbb{R}$ *such that* $\max x_0 = x_0(t_n)$*, and* γ *is symmetric with respect to the line* $x_1 = t_n$ *in the* x_0x_1*-plane;*
(2) γ *is concave, when regarded as a curve lying the Euclidean space;*
(3) *The two ends of* γ *approach* $\partial'_\infty \mathbb{H}^{n+1}$ *orthogonally.*

4.3 Proof of theorem A

According to Lemma 4.4, the Ilmanen space $N = (\mathbb{H}^{n+1}, \mathrm{g_I}(n))$ is a Cartan–Hadamard manifold. Regarding the SC condition, if $x \in \partial'_\infty \mathbb{H}^{n+1}$ one can consider a spherical barrier $\mathcal{S}$, which by Lemma 3.1 is convex with respect to the upward pointing normal direction. The SC condition therefore holds at any point $x \in \partial'_\infty \mathbb{H}^{n+1}$ by choosing as Ω the half-ball below $\mathcal{S}$. However, at the point p_∞, the SC condition may fail and for this reason we have to slightly complement the strategy in [12, 39], which we now recall. Fix an origin $o \in N$ and consider the cone Cone(o, A) generated by geodesics issuing from o to points in A. Moreover, denote by $B_r(o)$ the geodesic ball in N of radius r centered at o. For each $i \in \mathbb{N}$, consider the set

$$T_i = \partial B_i(o) \cap \mathrm{Cone}(o, A)$$

with orientation pointing outside of $B_i(o)$ and denote by $[T_i]$ its associated n-rectifiable current. Note that the boundary $\partial[T_i]$ is supported in Cone(o, Σ). Meanwhile, according to Lemma 4.5, the geodesics in N either are vertical lines or behave like grim-reaper type curves. Since A is relatively compact in $\partial'_\infty \mathbb{H}^{n+1}$, we can therefore take a large enough bowl soliton $\mathscr{B}$ such that $\overline{\mathrm{Cone}(o, A)}$ (in particular, $\overline{T}_i$) lies in the open subgraph of $\mathscr{B}$, which we call U. According to a result of Lang [39], for each $i \in \mathbb{N}$, there exists a set $W_i \subset \overline{B_i(o)}$ of finite perimeter such that $M_i \doteq \partial[W_i] - [T_i]$, is area minimizing in $\overline{B_i(o)}$. Notice that $\partial M_i = -\partial[T_i]$ is supported in U. Moreover, since $B_i(o)$ is strictly convex, by the strong maximum principle of White [55] we deduce that

$$\mathrm{spt}\, M_i \cap \partial B_i(o) = \mathrm{spt}\, \partial M_i, \qquad i \in \mathbb{N}.$$

We claim now that $\mathrm{spt} M_i \subset U$. Suppose to the contrary that this is not true and consider the foliation of $\mathbb{H}^{n+1}$ determined by $\mathscr{B}$ (Fig. 4). Then, we could find a large bowl soliton $\mathscr{B}'$ lying above $\mathscr{B}$ and touching $\mathrm{spt}\, M_i$ from above at some point $p \notin \mathrm{spt}\, \partial M_i$. Let U' be the open set below $\mathscr{B}'$, and consider the manifold with boundary

$$N' = \overline{U'} \cap B_i(o).$$

Let $v(M_i')$ be the stationary integral varifold obtained, by forgetting orientations, from the connected component of M_i whose support contains p; see [49, Section 27]. The strong maximum principle of White [55, Theorem 4], guarantees that $\mathrm{spt}\, v(M_i') \cap N'$ contains a connected component of $\mathscr{B}' \cap B_i(o)$. In particular, $\mathrm{spt}\, \partial M_i'$ contains a piece of $\mathscr{B}' \cap \partial B_i(o)$. This however contradicts $\partial M_i' \subset \mathrm{spt}\, \partial M_i \subset U$. Having observed that each W_i is contained in U and is therefore separated from p_∞, the rest of the argument follows verbatim as in [12, 39]. □

Remark 4.6 A similar argument (see [12, Theorem 1.5]) would allow to solve Plateau's problem for k-dimensional submanifolds provided that the point p_∞ can be separated from each $\partial[T_i]$ by a k-convex barrier. We have been unable to produce such objects, e.g., by rotating special curves around the x_0-axis. It might be possible that such barriers do not exist.

5 The Dirichlet problem at infinity

In this section, we investigate the Dirichlet problem (1.9). Set for simplicity

$$f(u) = -\frac{1+nu}{u^2} \quad \text{and} \quad W(u) = \sqrt{1+|Du|^2}$$

and consider the operator $\mathcal{Q}$ given by

$$\mathcal{Q}[u] = \mathrm{div}\left(\frac{Du}{W(u)}\right) - \frac{f(u)}{W(u)},$$

acting on positive C^2-functions on Ω.

5.1 Proof of theorem B. Part (1) - existence

We shall first solve the problem for data that do not meet the boundary at infinity of the hyperbolic space, and then we will exploit Perron's method to establish the existence of solutions to the Dirichlet problem at infinity.

We distinguish three cases:

Case A: Assume at first that $\partial\Omega$ is compact, ϕ is positive on $\partial\Omega$ and C^3-smooth. To solve the problem use the continuity method. Since the operator $\mathcal{Q}$ is quasilinear, the method will be applicable once we provide global a priori C^1-estimates on a solution u of (1.9); see for example [27, Chapter 11] or [47, page 417]. Height and gradient estimates will be proved by constructing suitable subsolutions and supersolutions for (1.9) and applying classical comparison theorems, for which we refer to [45, Theorem 2.1.3 & 2.1.4]. Notice that comparison holds because f defined above is increasing.

Claim 1 (*Height estimate*)**:** *There exist positive constants B_1 and B_2 which only depend on ϕ and Ω such that $B_1 \le u \le B_2$.*

Proof of the claim. Observe at first that the constant $u_1 = B_1 = \min_{\partial\Omega} \phi$ is a subsolution to (1.9), namely $\mathcal{Q}[u_1] \ge 0$. By the comparison principle, a solution u to (1.9) satisfies

$u_1 \le u$. Consider now a bowl soliton $\mathscr{B}$ lying above the graph of ϕ on $\partial\Omega$. Then, if $\mathscr{B}$ is generated by the graph of u_2, by the comparison principle we get $u \le u_2$. The claim follows. ⊛

Claim 2 (*Boundary gradient estimate*)**:** *There exists a constant B_3 which only depends on ϕ and Ω such that* $\sup_{\partial\Omega}|Du| \le B_3$.

Proof of the claim. Let ν be the inward pointing Euclidean unit normal on $\partial\Omega$, and fix a tubular neighborhood

$$\Omega_\rho = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) < \rho\}$$

around $\partial\Omega$ for which the Fermi chart $[0, \rho) \times \partial\Omega \to \Omega_\rho$ given by $(r, y) \mapsto y + r\nu(y)$ is well defined and smooth. Define $\hat{\phi}$ on Ω_ρ by $\hat{\phi}(r, y) = \phi(y)$. The goal is to find $l \in (0, \rho)$ and an increasing C^2-smooth function $\psi : [0, l) \to [0, \infty)$ with bounded gradient and satisfying

$$\psi(0) = 0 \quad \text{and} \quad |u(r, y) - \phi(y)| \le \psi(r) \qquad \text{for all} \quad (r, y) \in \Omega_l.$$

To achieve this goal we seek for $l \in (0, \rho)$ and ψ such that $\mathcal{Q}[\psi + \hat{\phi}] \le 0 \le \mathcal{Q}[-\psi + \hat{\phi}]$ on Ω_l. Set $v = \psi + \hat{\phi}$. Then, by a straightforward computation we get that

$$\begin{aligned}\mathcal{Q}[v] &= \frac{1}{W(v)}\left[\Delta v - D^2 v\left(\frac{Dv}{W(v)}, \frac{Dv}{W(v)}\right) - f(v)\right] \\ &= \frac{1}{W(v)}\left[\psi'' + \psi'\Delta r + \Delta\hat{\phi} - \psi''\frac{\langle Dr, Dv\rangle^2}{W^2(v)} - \psi' D^2 r\left(\frac{Dv}{W(v)}, \frac{Dv}{W(v)}\right)\right. \\ &\qquad \left. - D^2\hat{\phi}\left(\frac{Dv}{W(v)}, \frac{Dv}{W(v)}\right) - f(v)\right], \end{aligned} \tag{5.1}$$

where here Δ is the Euclidean Laplacian. Let us examine each term of (5.1) carefully:

- Because f is increasing and negative, we deduce that

$$0 < -f(t) \le C_\phi \doteq -f(B_1) \quad \text{for each} \quad t \ge \inf_{\partial\Omega}\phi > 0. \tag{5.2}$$

- Since ϕ is assumed to be smooth, there exists a constant C_1 such that

$$1 + \|D\hat{\phi}|^2_\infty + \|D^2\hat{\phi}|^2_\infty \le C_1 \quad \text{on} \quad \Omega_\rho. \tag{5.3}$$

- Using Gauss' Lemma we see that

$$Dv = \psi' Dr + D\hat{\phi} \quad \text{and} \quad \langle D\hat{\phi}, Dr\rangle = 0. \tag{5.4}$$

- Because $\mathcal{Y} = Dv/W(v)$ is of length at most 1, there exists a constant C_2 such that

$$\langle Dr, \mathcal{Y}\rangle = \psi' W^{-1}(v) \quad \text{and} \quad \left|\Delta\hat{\phi}\right| + \left|D^2\hat{\phi}(\mathcal{Y}, \mathcal{Y})\right| \le C_2. \tag{5.5}$$

- Since Dr belongs to the kernel of $D^2 r$, we have that

$$\left|D^2 r(\mathcal{Y}, \mathcal{Y})\right| = W^{-2}(v)\left|D^2 r(D\hat{\phi}, D\hat{\phi})\right| \le C_1\|D^2 r\|_\infty \doteq C_3. \tag{5.6}$$

- Since $H_{\partial\Omega} \ge 0$, the Laplacian comparison theorem implies

$$\Delta r \le -\frac{H_{\partial\Omega}}{1 - rH_{\partial\Omega}} \le 0 \quad \text{on} \quad \Omega_\rho. \tag{5.7}$$

Taking into account (5.2), (5.3), (5.4), (5.5), (5.6), (5.7), inequality (5.1) can be estimated by

$$W^3(v)\mathcal{Q}[v] \le W^2(v)\psi'' - \psi''(\psi')^2 + C_3\psi' + CW^2(v), \tag{5.8}$$

where $C = C_2 + C_\phi$. Consider now ψ to be a function of the form $\psi(r) = \mu \log(1 + kr)$, where $\mu > 0$ and $k > 0$ are constants to be chosen later. Then,

$$\psi(0) = 0, \qquad \psi'(r) = \frac{\mu k}{1 + kr} \quad \text{and} \quad \psi''(r) = -\frac{(\psi')^2}{\mu}.$$

For this choice of ψ, inequality (5.8) becomes

$$W^3(v)\mathcal{Q}[v] \leq -\frac{(\psi')^2}{\mu}\big(1 + |D\hat{\phi}|^2\big) + C_3\psi' + CW(v)^2. \tag{5.9}$$

From (5.3) and (5.4), we have

$$W^2(v) = W^2(\psi + \hat{\phi}) = 1 + |D\hat{\phi}|^2 + (\psi')^2 \leq C_1 + (\psi')^2.$$

Hence,

$$W^3(v)\mathcal{Q}[v] \leq -\frac{(\psi')^2}{\mu} + C_3\psi' + C(C_1 + (\psi')^2) \leq \Big(-\frac{1}{\mu} + C\Big)(\psi')^2 + C_3\psi' + CC_1. \tag{5.10}$$

By height estimates, $u \leq B_2$ on Ω. Define

$$\mu = \frac{B_2}{\log(1 + \sqrt{k})},$$

and observe that $\mu \to 0$ as $k \to \infty$. Choose $k > \rho^{-2}$ sufficiently large and set $l_1 = k^{-1/2}$. Then, from (5.10) we deduce that $\mathcal{Q}[v] < 0$ on Ω_{l_1}. Since

$$v(0, y) = \phi(y) = u(0, y) \quad \text{and} \quad v(l_1, y) > B_2 \geq u(l_1, y) \quad \text{for each} \quad y \in \partial\Omega,$$

from the comparison principle, it follows that $u \leq v = \psi + \phi$ on Ω_{l_1}. To prove the existence of $l_2 \in (0, \rho)$ such that $u \geq -\psi + \phi$ on Ω_{l_2}, we may proceed with the same technique as above and making use the fact that $-f(t) \geq -f(B_2)$ for $t \in [B_1, B_2]$. Alternatively, observe that since $f \leq 0$ a standard lower barrier w for the minimal surface equation on Ω_ρ also satisfies $\mathcal{Q}[w] \geq 0$ on $\{w > 0\}$. Take $l = \min\{l_1, l_2\}$ and let us restrict ourselves in Ω_l. For each $y \in \partial\Omega$, we have that

$$\left|\frac{\partial u}{\partial \nu}(y)\right| = \lim_{r \to 0} \frac{|u(r, y) - u(0, y)|}{r} \leq \lim_{r \to 0} \frac{\psi(r)}{r} = \mu k.$$

Let now X is a unit tangent vector field along $\partial\Omega$. Since

$$u|_{\partial\Omega} \equiv \phi|_{\partial\Omega},$$

the derivative of u in the direction X obeys

$$\big|\langle Du, X\rangle\big| \leq \sup_{\partial\Omega} |D\phi|.$$

Combining all these we complete the proof of the claim. ⊛

Claim 3 (*Interior gradient estimate*)**:** *There exists a constant B_4 which depends only on ϕ and Ω such that* $\sup_\Omega |Du| \leq B_4$.

Proof of the claim: From Lemma 3.3, the unit normal ν and the mean curvature of the soliton M are given by

$$\nu = \frac{u\,\partial_0 - u Du}{\sqrt{1 + |Du|^2}} \quad \text{and} \quad H = \frac{-1}{u\sqrt{1 + |Du|^2}} < 0.$$

Let us compute the Laplacian of H with respect to the induced metric g, following the same lines as in [43, Lemma 2.1]. For simplicity, let us denote the metrics of $\mathbb{H}^{n+1}$ and M by the same letter g. As usual let us denote by $\overline{\nabla}$ the Levi-Civita connection of $\mathbb{H}^{n+1}$ and by ∇ the Levi-Civita connection of the induced metric on M. Let $\{e_1, \dots, e_n\}$ be a local orthonormal tangent frame, which is normal at a fixed point $p \in M$, and denote by b_{ij} the coefficients of the second fundamental form $\mathrm{I\!I}$ of M with respect to ν. Differentiating with respect to e_i, we get that

$$e_i H = -e_i\, \mathrm{g}_{\mathbb{H}}(\partial_0, \nu) = -\,\mathrm{g}_{\mathbb{H}}(\overline{\nabla}_{e_i}\partial_0, \nu) - \mathrm{g}_{\mathbb{H}}(\partial_0, \overline{\nabla}_{e_i}\nu), \qquad i \in \{1, \dots, n\}.$$

From the Koszul formula, we have that at p it holds

$$\overline{\nabla}_{e_i}\partial_0 = -u^{-1}e_i, \qquad i \in \{1, \dots, n\}.$$

Consequently,

$$e_i H = -\,\mathrm{g}_{\mathbb{H}}(\partial_0^{\top}, \overline{\nabla}_{e_i}\nu) = \mathrm{I\!I}(\partial_0^{\top}, e_i), \qquad i \in \{1, \dots, n\}.$$

Differentiating once more, using the Codazzi-Mainardi equation, and then estimating at $p \in M$, we obtain

$$\begin{aligned}\Delta_{\mathrm{g}} H &= e_i e_i H = e_i\big(b_{ij}\, \mathrm{g}_{\mathbb{H}}(\partial_0, e_j)\big) = b_{iji}\, \mathrm{g}_{\mathbb{H}}(\partial_0, e_j) + b_{ij}\, \mathrm{g}_{\mathbb{H}}(\overline{\nabla}_{e_i}\partial_0, e_j) + b_{ij}\, \mathrm{g}_{\mathbb{H}}(\partial_0, \overline{\nabla}_{e_i} e_j)\\ &= \mathrm{g}_{\mathbb{H}}(\partial_0^{\top}, b_{iij}e_j) - u^{-1}b_{ij}\delta_{ij} + b_{ij}b_{ij}\, \mathrm{g}_{\mathbb{H}}(\partial_0, \nu)\\ &= \mathrm{g}_{\mathbb{H}}(\partial_0^{\top}, \nabla H) - u^{-1}H - H|\mathrm{I\!I}|^2,\end{aligned}$$

where Δ_{g} is the Laplacian with respect to the induced metric g of the soliton. Since $-H > 0$, according to the maximum principle, we obtain that $\sup_{\Omega} H = \max_{\partial\Omega} H$. Thus, there exists $y_0 \in \partial\Omega$ such that

$$\frac{-1}{u(x)\sqrt{1 + |Du|^2(x)}} \le \frac{-1}{u(y_0)\sqrt{1 + |Du|^2(y_0)}} \qquad \text{for each } x \in \Omega.$$

Hence,

$$|Du|^2(x) \le \frac{u^2(y_0)}{u^2(x)}\Big(1 + |Du|^2(y_0)\Big) - 1 \qquad \text{for each } x \in \Omega.$$

Combining with the estimates we showed in Claims 1 and 2, we deduce the desired estimate on the gradient of u. This completes the proof of the claim. ⊛

Case B: Assume that Ω has C^3-smooth compact mean convex boundary $\partial\Omega$. Furthermore, assume that $\phi : \partial\Omega \to (0, \infty)$ is continuous. Choose a decreasing sequence $\{\phi_j\}$ and an increasing sequence $\{\theta_j\}$ of positive smooth functions uniformly converging to ϕ. For each $j \in \mathbb{N}$ denote by u_j, v_j the solutions given by Case A for boundary data ϕ_j and θ_j, respectively. By the comparison maximum principle, the sequence $\{u_j\}$ is decreasing, $\{v_j\}$ is increasing and $v_j \le u_l$ for each $j, l \in \mathbb{N}$. Moreover, by the height estimates obtained in Case A, there exist positive constants B_1, B_2 such that

$$B_1 \le v_j \le u_l \le B_2 \qquad \text{for all } j, l \in \mathbb{N}.$$

According to a result of Simon [48, Corollary 1, p. 257], the sequences $\{u_j\}$ and $\{v_j\}$ have uniformly bounded gradients on compact subsets $K \subset \Omega$. Then local $C^{1,\alpha}$-estimates follow by Ladyzhenskaya & Ural'tseva [38], and Schauder estimates imply that the sequences $\{u_j\}$ and $\{v_j\}$ are bounded on $C^{2,\alpha}(K)$; for more details see also [27, Section 11.3, Chapter 13

and Theorem 13.6]. Passing to the limit, using the same idea as in Lemma 3.7, we deduce that $u_j \downarrow u$ and $v_j \uparrow u$ locally in $C^{2,\alpha}$ to a unique solution u to $\mathcal{Q}[u] = 0$ which satisfies $u \equiv \phi$ on $\partial\Omega$.

Case C: We conclude the proof by considering the case where Ω, not necessarily compact but satisfying $H_{\partial\Omega} \geq 0$, is contained between two parallel hyperplanes of $\partial'_\infty \mathbb{H}^{n+1}$. We employ Perron's method, the main novelty being the treatment of boundary barriers to force $u = \phi$ on $\partial\Omega$. Recall at first that a function v is said to satisfy $\mathcal{Q}[v] \geq 0$ in the viscosity sense if, for each $x \in \Omega$ and each C^2-smooth test function φ touching v from above at the point x (i.e., $\varphi \geq v$ near x and $\varphi(x) = v(x)$) it holds $\mathcal{Q}[\varphi] \geq 0$; see for details [18]. Consider now a large grim-reaper cylinder $\mathscr{G}$ such that the graph of ϕ over $\partial\Omega$ lies in the region below $\mathscr{G}$. Without loss of generality, we may denote the graph function generating $\mathscr{G}$ with the same name. Define Perron's class

$$\mathscr{F} = \left\{ v \in C(\overline{\Omega}) : \begin{array}{ll} 0 < v \leq \mathscr{G} & \text{on } \Omega, \\ \mathcal{Q}[v] \geq 0 & \text{on } \Omega \text{ in the viscosity sense,} \\ 0 \leq v \leq \phi & \text{on } \partial\Omega. \end{array} \right\}.$$

Claim 4: *The set $\mathscr{F}$ is non-empty.*

Proof of the claim: For each $x \in \Omega$ we can consider the maximal spherical barrier $\mathcal{S}_x$ centered at x whose boundary at infinity is contained in $\overline{\Omega}$. The spherical barrier can be expressed as the graph of a function $s_x : B_x \to \mathbb{R}$ which solves $\mathcal{Q}[s_x] \geq 0$ in the interior of its domain of definition B_x. Note that s_x is zero on ∂B_x and $s_x < \mathscr{G}$ on B_x by comparison. Extend s_x as being zero on $\Omega \setminus B_x$ and define $s : \Omega \to \mathbb{R}$ by $s = \sup_{x\in\Omega} s_x$. Then, $0 < s \leq \mathscr{G}$ on Ω and moreover s is locally Lipschitz. By elementary properties of viscosity solutions, we deduce that $\mathcal{Q}[s] \geq 0$ on the entire Ω. Since Ω is contained between two parallel hyperplanes, the radius of $\mathcal{S}_x$ is uniformly bounded from above by some $R > 0$. Hence, for each $x \in \Omega$, by considering a nearest point $x_0 \in \partial\Omega$ to x and a spherical cap of radius R and center $x_0 + R\nu(x_0)$ we deduce

$$s_y(x) \leq R\sqrt{1 - \frac{R - \operatorname{dist}(x, \partial\Omega)}{R}} \qquad \forall\, y \in \Omega.$$

Consequently, $s \in C(\overline{\Omega})$ and $s \equiv 0$ on $\partial\Omega$. This shows that $\mathscr{F} \neq \emptyset$. ⊛

Define now Perron's envelope

$$u(x) = \sup\{v(x) : v \in \mathscr{F}\}. \tag{5.11}$$

Then, u is lower-semicontinuous on $\overline{\Omega}$, $0 < u \leq \mathscr{G}$ on Ω and $0 \leq u \leq \phi$ on $\partial\Omega$.

Claim 5: *The function u defined in (5.11) belongs to $C^\infty(\Omega)$ and $\mathcal{Q}[u] = 0$ on Ω.*

Proof of the claim: Fix $x \in \Omega$ and a sequence $\{v_j\} \subset \mathscr{F}$ with $v_j(x) \to u(x)$. Up to replacing v_j with $\max\{v_1, \ldots, v_j\} \in \mathscr{F}$, we can assume that $v_j(x) \uparrow u(x)$. Pick a small ball $B \subset \Omega$ centered at x, and for each $j \in \mathbb{N}$ solve

$$\begin{cases} \mathcal{Q}[v'_j] = 0 & \text{on } B, \\ v'_j = v_j & \text{on } \partial B. \end{cases}$$

The existence of the unique $v'_j \in C^2(B) \cap C(\overline{B})$ follows by Case B above. From the comparison principle we deduce that $v_j \leq v'_j \leq \mathscr{G}$ on B, for each $j \in \mathbb{N}$; see [18].[1]

[1] Comparison in this case holds trivially: if $\max_B(v_j - v'_j) = c > 0$, the function $v'_j + c$ would touch from above v_j at some interior point x_0. However, $\mathcal{Q}[v'_j + c] < 0$, contradicting the fact that v_j is a subsolution at x_0.

Define the replacement $\tilde{v}_j$ of v_j to be the function

$$\tilde{v}_j = \begin{cases} v'_j & \text{on } B, \\ v_j & \text{on } \Omega \backslash B. \end{cases}$$

Then, $\tilde{v}_j$ still belongs to $\mathscr{F}$ and $\tilde{v}_j(x) \to u(x)$. Local gradient estimates [48, Corollary 1] and higher elliptic regularity imply $\tilde{v}_j \to v \leq u$ locally smoothly on B, where v is a function with $v(x) = u(x)$. We claim that $u \equiv v$ on B. Assume to the contrary that $u(p) > v(p)$ for some $p \in B$, let $w \in \mathscr{F}$ such that $w(p) > v(p)$ and consider $w_j = \max\{\tilde{v}_j, w\}$. Let $\tilde{w}_j$ be the replacement of w_j on B. Again elliptic estimates guarantee that $\tilde{w}_j \to \tilde{w} \leq \mathscr{G}$ on B locally smoothly. By construction, $\tilde{w} \geq v$ on B, with strict inequality at p but with equality at x, contradicting the maximum principle. ⊛

Claim 6: *The function u defined in* (5.11) *is continuous up to the boundary* $\partial\Omega$ *and* $u \equiv \phi$ *on* $\partial\Omega$.

Proof of the claim: Fix a point $x_0 \in \partial\Omega$, choose a positive $\varepsilon > 0$ and a large ball B_{r_0} centered at some fixed origin for which $x_0 \in B_{r_0-2}$. To simplify the notation, let us denote here the intersection of $\partial\Omega$ with a ball B_r of radius r by $\partial\Omega_r$, that is $\partial\Omega_r = \partial\Omega \cap B_r$. Parametrize now a neighborhood of $\partial\Omega_{r_0}$ by the smooth Fermi chart $[0, \rho) \times \partial\Omega_{r_0} \to \overline{\Omega}$ given by

$$(r, y) \mapsto y + r\nu(y),$$

where ν the unit normal to the boundary $\partial\Omega$ pointing toward Ω. Let $B \geq \|\mathscr{G}\|_\infty$ and consider functions $\phi_1, \phi_2 \in C^3(\partial\Omega)$ satisfying the following properties:

$$\begin{array}{lll} (i_1) & 0 \leq \phi_2 \leq \phi \quad \text{and} \quad \varepsilon + \phi \leq \phi_1 \leq \mathscr{G}, & \text{on } \partial\Omega, \\ (i_2) & |\phi_j - \phi| \leq 2\varepsilon, & \text{on } \partial\Omega_{r_0-2}, \\ (i_3) & \phi_2 = 0 \quad \text{and} \quad \phi_1 = \mathscr{G}, & \text{on } \partial\Omega \cap (\partial'_\infty \mathbb{H}^{n+1} \backslash B_{r_0-1}). \end{array}$$

By the construction of boundary gradient estimates in Case 1, there exists $\rho_0 < \rho$ (depending on ε) and C^2-smooth functions v_1 and v_2 on $U = [0, \rho_0] \times \partial\Omega_{r_0}$ with the following properties:

- $\mathcal{Q}[v_1] \leq 0$ on U, $\mathcal{Q}[v_2] \geq 0$ on $\{v_2 > 0\}$.
- It holds

$$v_1(\rho_0, y) = B, \quad v_1(0, y) = \phi_1(y), \quad \partial_r v_1(r, y) > 0, \quad \text{for each } (r, y) \in U,$$

and

$$v_1(r, y) \geq \mathscr{G}(r, y), \quad \text{for each } (r, y) \in [0, \rho_0) \times (\partial\Omega_{r_0} \backslash \partial\Omega_{r_0-1}).$$

- It holds

$$v_2(\rho_0, y) < 0, \quad v_2(0, y) = \phi_2 \quad \text{and} \quad \partial_r v_2(r, y) < 0, \quad \text{for each } (r, y) \in U.$$

Therefore,

$$v_2(r, y) < 0 \quad \text{on} \quad (0, \rho_0) \times (\partial\Omega_{r_0} \backslash \partial\Omega_{r_0-1}).$$

Pick now a smooth function $\eta : \partial\Omega_{r_0} \to [0, \rho_0]$ satisfying

$$\eta(y) = \rho_0 \quad \text{for} \quad y \in \partial\Omega_{r_0-1} \quad \text{and} \quad \eta(y) = 0 \quad \text{for} \quad y \in \partial\Omega_{r_0} \cap \partial B_{r_0},$$

and consider the region

$$\widetilde{U} = \big\{(r, y) \in U : 0 \leq r < \eta(y)\big\} \subset U.$$

Then, by construction,

$$v_2 < 0 < \mathscr{G} \leq v_1 \quad \text{on} \quad \partial\widetilde{U}\backslash\partial\Omega.$$

By the comparison principle, we have that $v \leq v_1$ on $\widetilde{U}$ for each $v \in \mathscr{F}$. Thus, $u \leq v_1$ on $\widetilde{U}$ and

$$\limsup_{x\to x_0} u(x) \leq \limsup_{x\to x_0} v_1(x) = \phi_1(x_0) \leq \phi(x_0) + 2\varepsilon.$$

On the other hand, by construction we have that $\{v_2 > 0\} \subset \widetilde{U}$. Therefore, the function v given by

$$v = \begin{cases} \max\{v_2, s\} & \text{on } \widetilde{U}, \\ s & \text{elsewhere on } \Omega, \end{cases}$$

is well defined on the entire Ω and $v \in \mathscr{F}$. Inequality $u \geq v$ implies

$$\liminf_{x\to x_0} u(x) \geq \lim_{x\to x_0} v(x) = \lim_{x\to x_0} v_2(x) = \phi_2(x_0) > \phi(x_0) - 2\varepsilon.$$

The continuity of u at x_0 follows by letting $\varepsilon \to 0$. ⊛

This conclude the proof of Theorem B(1). □

5.2 Proof of Theorem B. Part (2) - non-existence

Suppose that there exists a point $y \in \partial\Omega$ with $H_{\partial\Omega}(y) < 0$, and let $u \in C^\infty(\Omega) \cap C(\overline{\Omega})$ be a positive solution to $\mathcal{Q}[u] = 0$ on Ω. Our approach follows [47], and we split the argument into three steps. The main difference with [47] is Lemma 5.2 below: as observed in Remark 1.4(3), its use to construct a boundary data ϕ for which the Dirichlet problems is not solvable forces a lower bound on the oscillation of ϕ.

To achieve our goal, we need to compare u with appropriate supersolutions to (SE). Recall that a function r defined on an open subset of Ω is called a distance function if it is smooth and $|Dr| \equiv 1$. We start with the following:

Lemma 5.1 *Let r be a distance function defined on an open subset $U \subset \mathbb{R}^n$ and $\omega : (0, \infty) \to \mathbb{R}$ a smooth function. Then, $v = \omega(r) : U \to \mathbb{R}$ is a supersolution of (SE), if there exists a continuous function h defined on $(0, r)$ such that*

$$\omega' < 0, \quad \frac{\omega''}{\omega'[1 + (\omega')^2]} + h \geq 0 \quad \text{and} \quad \Delta r - \frac{f(v)}{\omega'(r)} \geq h(r) \geq 0 \quad \text{on } U. \tag{5.12}$$

Proof Consider the orthonormal frame $\{e_1 = Dr; e_2, \ldots, e_n\}$. By a straightforward computation, we deduce that

$$\sqrt{1 + (\omega'(r))^2}\, \mathcal{Q}[v] = \frac{\omega''(r)}{1 + (\omega'(r))^2} + \omega'(r)\Delta r - f(v) \leq \frac{\omega''(r)}{1 + (\omega'(r))^2} + h(r)\omega'(r),$$

from where the statement follows. □

Lemma 5.2 *For each positive number $\varepsilon > 0$, there exist positive constants $a_0 = a_0(\varepsilon, \operatorname{diam}(\Omega), n)$ and $c = c(\operatorname{diam}(\Omega), n)$ such that*

$$\sup_{\Omega\setminus B_a(y)} u \leq \varepsilon + \sup_{\partial\Omega\setminus B_a(y)} u - cf(\sup_{\partial\Omega\setminus B_a(y)} u)$$

for all $a < a_0$.

Proof Let a be a positive number. To simplify the notation let us set

$$d = 2\,\mathrm{diam}(\Omega), \quad U_a = \Omega \backslash B_a(y), \quad V_a = \partial\Omega \backslash B_a(y) \quad \text{and} \quad u_a^* = \sup_{V_a} u.$$

Denote by r the distance function $r(x) = |x - y|$ and let $v : U_a \to \mathbb{R}$ be the function given by

$$v = \omega(r) + u_a^*,$$

where ω is C^2-smooth on $(a, d]$ and continuous on $[a, d]$. Furthermore, we require for ω that

$$\omega \geq 0, \quad \omega(d) = 0, \quad \omega' < \frac{2d}{n-1} f(u_a^*) < 0 \quad \text{and} \quad \omega'(a^+) = -\infty. \tag{5.13}$$

From the monotonicity of f and Lemma 5.1, we easily see that

$$\Delta r - \frac{f(v)}{\omega'(r)} \geq \frac{n-1}{r} - \frac{f(u_a^*)}{\omega'(r)} \geq \frac{n-1}{r} - \frac{n-1}{2d} \geq \frac{n-1}{2r} > 0.$$

Therefore, $\mathcal{Q}[v] \leq 0$ provided that

$$\frac{\omega''}{\omega'[1 + (\omega')^2]} + \frac{n-1}{2r} \geq 0. \tag{5.14}$$

We first find a solution $\tilde{\omega}$ to (5.14) with the equality sign. To achieve this goal, consider the strictly decreasing diffeomorphism $F : (0, \infty) \to (0, \infty)$ given by

$$F(s) = \int_s^\infty \frac{\mathrm{d}\tau}{\tau(1 + \tau^2)} = \log \sqrt{1 + s^{-2}}. \tag{5.15}$$

By a direct computation we see that

$$\big(F(-\tilde{\omega}')\big)' = -\frac{\tilde{\omega}''}{\tilde{\omega}'[1 + (\tilde{\omega}')^2]} = \frac{n-1}{2r}.$$

Integrating on $[a, r]$ and using the fact $\tilde{\omega}'(a^+) = -\infty$, we get

$$-\tilde{\omega}'(r) = F^{-1}\left(\frac{n-1}{2} \log\left(\frac{r}{a}\right)\right).$$

Another integration on $[r, d]$ gives

$$\tilde{\omega}(r) = \int_r^d F^{-1}\left(\frac{n-1}{2} \log\left(\frac{t}{a}\right)\right) \mathrm{d}t. \tag{5.16}$$

Since $F^{-1}(t) \asymp t^{-1/2}$ as $t \to 0$, it follows that $\tilde{\omega}'$ is integrable in a neighborhood of a. Also, explicit computation gives $\tilde{\omega}(a) \to 0$ as $a \to 0$. We can therefore choose $a_0 = a_0(\varepsilon, d, n)$ small enough so that $\tilde{\omega}(a) < \varepsilon$ for each $a < a_0$. Summarizing, $\tilde{\omega}$ given in (5.16) solves (5.14). Choose now the function

$$\omega(r) = \tilde{\omega}(r) - \frac{2d}{n-1} f(u_a^*)(d - r).$$

Observe that ω satisfies both (5.14) and (5.13). Hence, $v = \omega(r) + u_a^*$ is a supersolution to (SE). We claim that $u \leq v$ on the closure of U_a. Indeed, assume by contradiction that $u - v$ has a positive maximum at some point x_0. By the strong maximum principle, x_0 is not an

interior point, and thus $x_0 \in \partial B_a(y)$ by the construction of v. However, along the segment ℓ given by

$$\ell(t) = x_0 + t\,Dr(x_0)$$

it holds

$$(u \circ \ell - v \circ \ell)'(t) = \langle Du, Dr\rangle(\ell(t)) - \omega'(a+t) \to \infty \quad \text{as} \quad t \to 0^+,$$

contradiction. From the inequality $u \le v$ on $\overline{U}_a$ we deduce

$$u(x) \le u_a^* + \tilde{\omega}(a) - \frac{2d^2}{n-1} f(u_a^*) < u_a^* + \varepsilon - \frac{2d^2}{n-1} f(u_a^*) \quad \text{for all} \ \ a < a_0,$$

concluding the proof of the lemma. □

Lemma 5.3 *For each $\varepsilon > 0$, there exists a positive constant $a_0 = a_0(\Omega, \varepsilon)$ such that*

$$\sup_{\Omega \cap B_a(y)} u \le \varepsilon + \sup_{\Omega \cap \partial B_a(y)} u \tag{5.17}$$

for all $a < a_0$.

Proof Define $r(x) = \operatorname{dist}(x, \partial\Omega)$, and choose a_0 small enough to guarantee that r is smooth on $\overline{\Omega} \cap B_{a_0}(y)$. Since $\Delta r(y) = -H_{\partial\Omega}(y)$, by continuity there exist $a_0, \theta > 0$ such that

$$\Delta r \ge 2\theta \qquad \text{on} \ \ \Omega \cap B_a(y).$$

Fix $a < a_0$, choose $\delta \in (0, a)$ and set

$$u_a^{**} = \sup_{\Omega \cap \partial B_a(y)} u, \qquad k = \theta^{-1} \sup_{\Omega} |f(u)|.$$

Consider now the function $v : \Omega \cap B_a(y) \to \mathbb{R}$ given by

$$v = u_a^{**} + \omega(r),$$

where ω is a C^2-smooth function on $(\delta, a]$ and continuous on $[\delta, a]$. Furthermore, we require ω to satisfy

$$\omega > 0 \quad \text{on} \ [\delta, a], \qquad \omega' \le -k \quad \text{on} \ (\delta, a] \qquad \text{and} \qquad \omega'(\delta^+) = -\infty. \tag{5.18}$$

Observe that

$$\Delta r - \frac{f(v)}{\omega'(r)} \ge 2\theta - \frac{\sup_\Omega |f(u)|}{k} \ge \theta \qquad \text{on} \ \ \Omega \cap B_a(y).$$

By Lemma 5.1, we deduce that $\mathcal{Q}[v] \le 0$ provided

$$\frac{\omega''}{\omega'[1 + (\omega')^2]} + \theta = 0$$

and the conditions (5.18) are satisfied. Consider the decreasing diffeomorphism $F : (0, \infty) \to (0, \infty)$ given in (5.15) to rewrite the last ODE in the form

$$(F(-\omega'))' = \theta.$$

Integrating on (δ, r) and using that $\omega'(\delta^+) = -\infty$ we get

$$-\omega'(r) = F^{-1}(\theta(r - \delta)) \ge F^{-1}(\theta a_0) \ge k,$$

where the last inequality holds if a_0 is small enough. Moreover, the last requirement in (5.18) is also satisfied. Integrating again and using the asymptotic behavior of F^{-1}, the function ω' is integrable in a neighborhood of δ and thus

$$\omega(r) = \int_r^{a_0} F^{-1}(\theta(t-\delta))\mathrm{d}t \in C([\delta, a]) \cap C^2((\delta, a]).$$

Consequently, all of the required assumptions on ω are satisfied. Making use of the comparison maximum principle as in the previous lemma, we obtain $u \leq v$ on $\Omega \cap B_a(y)\backslash B_\delta(y)$. Therefore,

$$u \leq u_a^{**} + \omega(\delta) = u_a^{**} + \int_\delta^{a_0} F^{-1}(\theta(t-\delta))\mathrm{d}t.$$

Changing variables from s to $t-\delta$ in the last integral, letting $\delta \to 0$ and using the monotone convergence theorem, we get

$$u \leq u_a^{**} + \int_0^{a_0} F^{-1}(\theta s)\mathrm{d}s \qquad \text{on } \Omega \cap B_a(y)\backslash\{y\}.$$

The integral on the right hand side is finite. By continuity, the same inequality also holds at the point y. Therefore, choosing a_0 sufficiently small, the estimate (5.17) holds. This concludes the proof of the lemma. □

We are now ready to complete the proof of Theorem B(2). Recall that we are dealing with a domain Ω with smooth boundary $\partial\Omega$, which at a point y has strictly negative mean curvature. Fix $\varepsilon > 0$ and let $a_0 > 0$ small enough so that both Lemmas 5.2 and 5.3 hold for $a < a_0$. This means that any positive solution $u \in C^\infty(\Omega) \cap C(\overline{\Omega})$ of (SE) must satisfy the estimate

$$u(y) \leq \varepsilon + \sup_{\Omega\cap\partial B_a(y)} u \leq 2\varepsilon + \sup_{\partial\Omega\backslash B_a(y)} u - cf(\sup_{\partial\Omega\backslash B_a(y)} u) \tag{5.19}$$

for each $a < a_0$. On the other hand, choose an arbitrary positive constant $c_0 > 0$ and a positive boundary datum $\phi \in C^\infty(\partial\Omega)$ satisfying

$$\phi \equiv c_0 > 0 \quad \text{on } \partial\Omega\backslash B_a(y) \quad \text{and} \quad \phi(y) > 2\varepsilon + c_0 - cf(c_0).$$

Then, from (5.19) it follows that $u(y) < \phi(y)$. Consequently, the Dirichlet problem $\mathcal{Q}[u] = 0$ with prescribed $u \equiv \phi$ on $\partial\Omega$ does not admit any solution $u \in C^\infty(\Omega) \cap C(\overline{\Omega})$. □

6 Uniqueness of the grim-reaper cylinder

In this section, we prove Theorem F. We begin with the following observation:

Lemma 6.1 *Let $M \subset \mathbb{H}^{n+1}$ be a properly immersed soliton with respect to $-\partial_0$ such that $\partial'_\infty M = \pi_1 \cup \pi_2$, where π_1 and π_2 are parallel hyperplanes in $\partial'_\infty\mathbb{H}^{n+1}$. Then M is contained in the open region $\mathcal{U}$ bounded by the parallel hyperplanes $\Pi_1, \Pi_2 \subset \mathbb{H}^{n+1}$ that meet $\partial'_\infty\mathbb{H}^{n+1}$ orthogonally and satisfy $\partial'_\infty\Pi_1 = \pi_1$, $\partial'_\infty\Pi_2 = \pi_2$, and by the half-cylinder $\mathcal{C}$ with $\partial'_\infty\mathcal{C} = \pi_1 \cup \pi_2$.*

Proof By Proposition 3.2, M is contained in the slab between Π_1 and Π_2. Again from the strong maximum principle M cannot touch $\Pi_1 \cap \Pi_2$. Next, consider a spherical barrier $\mathcal{S}$ centered at a point $q_\infty \in \partial'_\infty\mathbb{H}^{m+1}$ equidistant, with respect to the Euclidean metric, from

π_1 and π_2, and choose its radius to be small enough to satisfy $\mathcal{S} \cap M = \emptyset$. By increasing the radius, the strong maximum principle ensures that M lies above the half-sphere centered at q_∞ and tangent to $\Pi_1 \cup \Pi_2$. The conclusion follows since $\mathcal{C}$ is the envelope of such barriers for varying q_∞. □

6.1 Compactness and a maximum principle for varifolds

Let us set recall some important facts that we will need in the sequel.

Definition 6.2 Let $\{M_i\}_{i\in\mathbb{N}}$ be a sequence of properly embedded hypersurfaces in a Riemannian manifold (N, g). We say that $\{M_i\}_{i\in\mathbb{N}}$ has uniformly bounded area on compact subsets of N if

$$\limsup_{i\to\infty} |M_i \cap K|_{\mathrm{g}} < \infty$$

for any compact subset K of N.

The following well-known theorem in geometric measure theory holds; see, for example, [49, Theorem 42.7].

Theorem 6.3 *Let $\{M_i\}$ be a sequence of minimal hypersurfaces in $\Omega \subset \mathbb{R}^{n+1}$, where Ω is an open subset equipped not necessarily with the canonical metric, whose area is locally bounded. Then, a subsequence of $\{M_i\}$ converges weakly to a stationary integral varifold M_∞.*

Let us denote by

$$\mathcal{Z} = \left\{p \in \Omega : \limsup_{i\to\infty} |M_i \cap B_r(p)|_{\mathrm{g}} = \infty \text{ for every } r > 0\right\},$$

the set where the area blows up. Clearly $\mathcal{Z}$ is a closed set. It will be useful to have conditions that will imply that the set $\mathcal{Z}$ is empty. In this direction, White [54, Theorems 2.6 and 7.3] shows that under some natural conditions the set $\mathcal{Z}$ satisfies the same maximum principle as properly embedded minimal hypersurfaces without boundary.

Theorem 6.4 *Let (N, g) be a smooth Riemannian $(n+1)$-manifold and $\{M_i\}_{i\in\mathbb{N}}$ a sequence of properly embedded minimal hypersurfaces without boundary in (N, g). Suppose that the area blow up set $\mathcal{Z}$ of $\{M_i\}_{i\in\mathbb{N}}$ is contained in a closed $(n+1)$-dimensional region $P \subset N$ with smooth, connected boundary ∂P such that $\mathrm{g}\left(\mathbf{H}, \xi\right) \geq 0$, at every point of ∂P, where $\mathbf{H}$ is the mean curvature vector of ∂P and ξ is the unit normal to the hypersurface ∂P that points into P. If the set $\mathcal{Z}$ contains any point of ∂P, then it contains all of ∂P.*

Remark 6.5 The above theorem is a sub-case of a more general result. In fact the strong barrier principle holds for sequences of properly embedded hypersurfaces (possibly with boundaries) of Riemannian manifolds which are not necessarily minimal but they have uniformly bounded mean curvature. For more details, we refer to [54].

In the proof of Theorem F, we will need the following strong maximum principle which is due to Solomon and White [52].

Theorem 6.6 *Let (N^{n+1}, g) be a Riemannian manifold with connected non-empty boundary ∂N and that N^{n+1} is mean convex, that is, that $\mathrm{g}(\mathbf{H}, \xi) \geq 0$ on ∂N where $\mathbf{H}$ is the mean curvature vector of ∂N and where ξ is the unit inward pointing normal of ∂N. Let V be an n-dimensional stationary varifold in N^{n+1}. If $\mathrm{spt}V$ contains a point of ∂N, then it must contain all ∂N.*

6.2 The hyperbolic dynamic lemma

Let $M \subset \mathbb{H}^{n+1}$ be a conformal soliton with respect to $-\partial_0$ satisfying the GR-property. Without loss of generality we can choose coordinates so that π_1 and π_2 are, respectively, given by the equations

$$\{x_0 = 0, x_1 = a\} \quad \text{and} \quad \{x_0 = 0, x_1 = -a\},$$

where a is a positive constant. Recall that the soliton property is preserved if we act on M via isometries of the hyperbolic space which fix the vector $-\partial_0$. Therefore, if $v = (0, 0, v_2, \dots, v_m)$ is a vector of $\mathbb{H}^{n+1}$, then the hypersurface

$$M + v = (x_0, x_1, x_2 + v_2, \dots, x_m + v_m)$$

is again a soliton in $\mathbb{H}^{n+1}$ satisfying the GR-property.

Lemma 6.7 *Let $M \subset \mathbb{H}^{n+1}$ be a properly embedded soliton with respect to $-\partial_0$ satisfying the GR-property. Suppose that $\{v_i\}_{i\in\mathbb{N}} \subset \operatorname{span}\{\partial_2, \dots, \partial_m\}$ is a sequence of vectors and let $M_i = M + v_i$. Then, after passing to a subsequence, $\{M_i\}_{i\in\mathbb{N}}$ weakly converges to a connected stationary integral varifold M_∞ with $\partial_\infty M_\infty = \partial_\infty M$.*

Proof First, by Lemma 6.1 we know that M lies in the region $\mathcal{U}$. Let $\tau > 0$ be the constant guaranteed by the GR-property, set

$$\mathcal{U}_\tau = \mathcal{U} \cap \{\tau < x_0 < \sup_M x_0\},$$

and denote with $\mathcal{Z}$ the blow-up set of $\{M_i\}$. We split the proof in four steps.

Step 1: We show that the sequence $\{M_i\}_{i\in\mathbb{N}}$ has locally bounded area with respect to the Ilmanen metric outside $\overline{\mathcal{U}}_\tau$. Due to the GR-property,

$$M \setminus \overline{\mathcal{U}}_\tau = \big(\mathcal{W}_1 \cup \mathcal{W}_2\big) \cap \{x_0 < \tau\},$$

where the wing $\mathcal{W}_j$ is the image of the graph of the function $\varphi_j : \mathcal{H}_j^\tau \to \mathbb{R}$, $j \in \{1, 2\}$. To simplify notation, for fixed j we let

$$\mathcal{W} = \mathcal{W}_j, \quad \varphi = \varphi_j \quad \text{and} \quad \mathcal{H} = \mathcal{H}_j.$$

Let ν be a Euclidean unit normal to $\mathcal{H}$. Then,

$$\nu = \cos\theta\,\partial_1 + \sin\theta\,\partial_0,$$

where $\theta \in (-\pi/2, \pi/2)$. Notice that since $\mathcal{W} \subset \mathcal{U}$, there exists $C > 0$ such that

$$|\varphi(p)| < C \quad \text{for each} \quad p \in \mathcal{H}^\tau. \tag{6.1}$$

Let us introduce the coordinates

$$y_0 = \cos\theta x_1 + \sin\theta x_0, \quad y_1 = -\sin\theta x_1 + \cos\theta x_0, \quad y_k = x_k \text{ for } k \geq 2.$$

Thus, $y = (y_1, \dots, y_m)$ are coordinates on $\mathcal{H}$. Choose an Euclidean ball $B \subset \mathcal{H}^\tau$, consider the box $Q = [-C, C] \times B$ in coordinates $(y_0; y)$ and let

$$K = Q \cap \overline{\mathcal{U}};$$

see Fig. 5. By construction, K is compact subset with piecewise smooth boundary in $\mathbb{H}^{n+1}$, and $\mathcal{W} \cap K$ is the image of the graph of the function φ over the entire $\overline{B}$. We claim that the $g_I(n)$-area of $\{M_i\}_{i\in\mathbb{N}}$ on K is uniformly bounded. Since for varying B, C the sets K cover

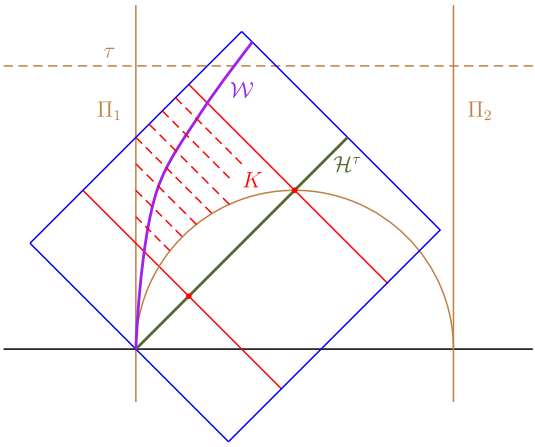

Fig. 5 The subset K

the region $\mathcal{U} \cap \{x_0 < \tau\}$, we deduce that the sequence $\{M_i\}$ has locally bounded area outside $\overline{\mathcal{U}}_\tau$. We use a calibration method.

Write for convenience

$$\mathrm{g_I}(n) = \lambda^2\, \mathrm{g}_{\mathbb{R}}, \qquad \text{with} \quad \lambda(y_0; y) = x_0^{-1}(y_0; y) e^{\frac{1}{n x_0(y_0; y)}},$$

and notice that the unit normals ξ and ξ_{I} along $\mathcal{W}$ with respect to the Euclidean and Ilmanen metric (say, pointing toward increasing y_0) are related by

$$\xi_{\mathrm{I}}(q) = \lambda^{-1}(q)\xi(q), \quad \text{for each} \quad q = (\varphi(y); y) \in \mathcal{W}.$$

Extend ξ on K to be constant along the y_0-direction and accordingly extend ξ_{I} on K by

$$\xi_{\mathrm{I}}(y_0; y) = \lambda^{-1}(y_0; y)\xi(y_0; y), \quad \text{for each} \quad (y_0; y) \in K.$$

We compute

$$\begin{aligned} \mathrm{div}_{\mathrm{g_I}} \xi_{\mathrm{I}} &= \mathrm{div}_{\mathrm{g}_{\mathbb{R}}} \xi_{\mathrm{I}} + n\, \mathrm{g}_{\mathbb{R}}(D \log \lambda, \xi_{\mathrm{I}}) \\ &= \lambda^{-1}\big(\mathrm{div}_{\mathrm{g}_{\mathbb{R}}} \xi + (n-1)\, \mathrm{g}_{\mathbb{R}}(D \log \lambda, \xi)\big). \end{aligned} \tag{6.2}$$

Let $p = (y_0; y) \in K$, and let $q = (\varphi(y); y)$. Since $\mathcal{W}$ is $\mathrm{g_I}(n)$-minimal and $|\xi_{\mathrm{I}}|_{\mathrm{g_I}} = 1$ by construction on the entire K, we have

$$\mathrm{div}_{\mathrm{g_I}} \xi_{\mathrm{I}}(q) = 0, \qquad \mathrm{div}_{\mathrm{g}_{\mathbb{R}}} \xi(p) = \mathrm{div}_{\mathrm{g}_{\mathbb{R}}} \xi(q).$$

Evaluating (6.2) at p and q and using the last two identities, we get

$$\mathrm{div}_{\mathrm{g_I}} \xi_{\mathrm{I}}(p) = \frac{n-1}{\lambda(p)} \Big(\mathrm{g}_{\mathbb{R}} \left(D_p \log \lambda, \xi(p) \right) - \mathrm{g}_{\mathbb{R}} \left(D_q \log \lambda, \xi(q) \right) \Big).$$

Since $\min_K x_0 > 0$, the function $\log \lambda$ is bounded in $C^1(K, \mathrm{g}_{\mathbb{R}})$. So there exists a constant C_K such that

$$|\mathrm{div}_{\mathrm{g_I}} \xi_{\mathrm{I}}| \le C_K \quad \text{on} \quad K.$$

Let K' be the region of K where $y_0 < \varphi(y)$. By the divergence theorem,

$$C_K |K|_{\mathrm{g_I}} \ge \int_{K'} \mathrm{div}_{\mathrm{g_I}} \xi_{\mathrm{I}} \mathrm{d}x_{\mathrm{g_I}} = |\mathcal{W} \cap K|_{\mathrm{g_I}} + \int_{\partial K' \cap \partial K} \mathrm{g_I}(\xi_{\mathrm{I}}, \eta) \mathrm{d}\sigma_{\mathrm{g_I}} \ge |\mathcal{W} \cap K|_{\mathrm{g_I}} - |\partial K|_{\mathrm{g_I}}.$$

Therefore,

$$|\mathcal{W} \cap K|_{\mathrm{g_I}} \leq |\partial K|_{\mathrm{g_I}} + C_K |K|_{\mathrm{g_I}}$$

is uniformly bounded, as claimed.

Step 2: From Step 1, $\mathcal{Z} \subset \overline{\mathcal{U}}_\tau$. Choose a spherical barrier $\mathscr{B}$ not intersecting $\overline{\mathcal{U}}_\tau$, and increase its radius up to touching $\mathcal{Z}$. Theorem 6.4 would imply that $\mathscr{B} \subset \mathcal{Z}$, a contradiction. Thus, $\mathcal{Z} = \emptyset$.

Step 3: From Steps 1 & 2, the sequence $\{M_i\}$ has locally bounded $\mathrm{g_I}(n)$-area. Hence, Theorem 6.3 guarantees the weak subconvergence of $\{M_i\}$ to a stationary integral varifold M_∞.

Step 4: By the GR-property, $\partial'_\infty M_i = \partial'_\infty M$ for each $i \in \mathbb{N}$. Every point $p \in \partial'_\infty \mathbb{H}^{m+1} \setminus \partial'_\infty M$ can be separated to M, hence to each M_i, by a small spherical barrier. Thus, $\partial'_\infty M_\infty \subset \partial'_\infty M$. On the other hand, on each Euclidean ball B centered at $p \in \partial'_\infty M$ the GR-property and the almost monotonicity formula for $\mathrm{g_I}(n)$-stationary varifolds guarantee a uniform lower bound for the $\mathrm{g_I}(n)$-area of $M_i \cap B$. Therefore, p belongs to $\mathrm{spt} M_\infty$.

This completes the proof. □

6.3 Proof of Theorem F

By Lemma 6.1 and since x_0 is bounded on M, we have that

$$M \subset \mathcal{U} \cap \{x_0 \leq \sup x_0\}.$$

Pick a grim-reaper cylinder $\mathscr{G}_h$ of height h whose symmetry axis is parallel to the hyperplanes Π_1, Π_2, and (Euclidean) equidistant to them. For h small enough, $\mathscr{G}_h \cap M = \emptyset$. We claim that we can increase h up to a limit value h^* in such a way that $\mathscr{G}_h \cap M = \emptyset$, for each $h < h^*$, and

$$\partial'_\infty \mathscr{G}_{h^*} = \pi_1 \cup \pi_2.$$

Suppose that this is not the case. Then, necessarily,

(i) $\mathrm{dist}_{\mathrm{g_I}}(\mathscr{G}_{h^*}, M) = 0$,
(ii) $\partial'_\infty \mathscr{G}_{h^*}$ is contained in the open slab between π_1 and π_2.

By (i) there exists a sequence $\{p_i\}_{i\in\mathbb{N}} \subset M$ such that $\mathrm{dist}(p_i, \mathscr{G}_{h^*}) \to 0$. Denote by p_i^k the x_k-component of p_i and define $v_i = (0, 0, p_i^2, \ldots, p_i^m)$. Since M is contained in $\mathcal{U}$ and because of (ii), it follows that there exists $\tau_0 > 0$ such that $p_i^0 > \tau_0$ for each $i \in \mathbb{N}$. Therefore, up to a subsequence, $p_i - v_i$ converges to a point $p_\ell \in \mathscr{G}_{h^*}$. Applying the hyperbolic dynamic Lemma 6.7 to $M_i = M + v_i$, we obtain a limiting $\mathrm{g_I}(n)$-stationary varifold M_∞ with

$$\partial'_\infty M_\infty = \pi_1 \cup \pi_2.$$

By construction $p_\ell \in M_\infty$ and M_∞ lies above $\mathcal{G}_{h^*}$. By Theorem 6.6 we get that $\mathscr{G}_{h^*} \subset M_\infty$, contradicting condition (ii). To conclude, pick a large grim-reaper cylinder $\mathscr{G}_s$ with the same axis as $\mathscr{G}_{h^*}$ and containing $\mathcal{U} \cap \{x_0 \leq \sup x_0\}$. Decreasing s and following the same argument as before, we can show that

$$\mathscr{G}_s \cap M = \emptyset,$$

for each $s > h^*$. Hence $M \equiv \mathscr{G}_{h^*}$ and this completes the proof. □

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References

1. Alías, L., De Lira, J.-H., Rigoli, M.: Mean curvature flow solitons in the presence of conformal vector fields. J. Geom. Anal. **30**, 1466–1529 (2020)
2. Anderson, M.: Complete minimal hypersurfaces in hyperbolic n-manifolds. Comment. Math. Helv. **58**, 264–290 (1983)
3. Anderson, M.: Complete minimal varieties in hyperbolic space. Invent. Math. **69**, 477–494 (1982)
4. Anderson, M.: The Dirichlet problem at infinity for manifolds of negative curvature. J. Differ. Geom. **18**, 701–722 (1983)
5. Arezzo, C., Sun, J.: Conformal solitons to the mean curvature flow and minimal submanifolds. Math. Nachr. **286**, 772–790 (2013)
6. Baker, C.: The mean curvature flow of submanifolds of high codimension, Ph.D. Thesis, Australian National University. ArXiv version at arXiv:1104.4409, (2010)
7. Bernstein, S.: Conditions nécessaires et suffisantes pour la possibilité du problème de Dirichlet. Acad. Sci. Paris **150**, 514–515 (1910)
8. Bianchini, B., Mari, L., Pucci, P., Rigoli, M.: Geometric Analysis of Quasilinear Inequalities on Complete Manifolds Maximum and Compact Support Principles and Detours on Manifolds. Cham, Birkhäuser (2021)
9. Bonorino, L., Castéras, J.-B., Klaser, P., Ripoll, J., Telichevesky, M.: On the asymptotic Dirichlet problem for a class of mean curvature type partial differential equations. Calc. Var. Partial Diff. Equ. **59**, 135 (2020)
10. Castéras, J.-B., Heinonen, E., Holopainen, I.: 1106–1130, Solvability of Minimal Graph Equation Under Pointwise Pinching Condition for Sectional Curvatures, 27. J. Geom, Anal (2017)
11. Castéras, J.B., Heinonen, E., Holopainen, I.: Dirichlet problem for f-minimal graphs. J. d Anal. Math. **138**(2), 917–50 (2019)
12. Castéras, J.-B., Holopainen, I., Ripoll, J.: Convexity at infinity in Cartan–Hadamard manifolds and applications to the asymptotic Dirichlet and Plateau problems. Math. Z. **290**, 221–250 (2018)
13. Clutterbuck, J., Schnürer, O., Schulze, F.: Stability of translating solutions to mean curvature flow. Calc. Var. Partial Diff. Equ. **29**, 281–293 (2007)
14. Cheng, X., Zhou, D.: Spectral properties and rigidity for self-expanding solutions of the mean curvature flows. Math. Ann. **371**(1–2), 371–389 (2018)
15. Colding, T.H., Minicozzi, W.P., II.: Generic mean curvature flow I; generic singularities. Ann. Math. **175**, 755–833 (2012)
16. Colding, T.H., Minicozzi, W.P., II.: Smooth compactness of self-shrinkers. Comment. Math. Helv. **87**, 463–475 (2012)
17. Colombo, G., Mari, L., Rigoli, M.: Remarks on mean curvature flow solitons in warped products. Discrete Contin. Dyn. Syst. Ser. S **13**(7), 1957–1991 (2020)
18. Crandall, M.G., Ishii, H., Lions, P.-L.: User's guide to viscosity solutions of second order partial differential equations. Bull. Am. Math. Soc. New Ser. **27**, 1–67 (1992)
19. Guan, B., Spruck, J.: Hypersurfaces of constant mean curvatures in hyperbolic space with prescribed asymptotic boundary at infinity. Am. J. Math. **122**, 1039–1060 (2000)

20. De Lellis, C.: The size of the singular set of area-minimizing currents, Advances in geometry and mathematical physics. Lectures given at the geometry and topology conference at Harvard University, Cambridge, MA, USA **2016**, 1–83 (2014)
21. Lin, F.: On the Dirichlet problem for minimal graphs in hyperbolic space. Invent. Math. **96**, 593–612 (1989)
22. Dajczer, M., Hinojosa, P.-A., De Lira, J.-H.: Killing graphs with prescribed mean curvature. Calc. Var. Partial. Differ. Equ. **33**, 231–248 (2008)
23. Dajczer, M., Ripoll, J.: An extension of a theorem of Serrin to graphs in warped products. J. Geom. Anal. **15**, 193–205 (2005)
24. Eberlein, P., O'Neill, B.: Visibility manifolds. Pac. J. Math. **46**, 45–109 (1973)
25. Eschenburg, J.-H.: Maximum principle for hypersurfaces. Manuscripta Math. **64**, 55–75 (1989)
26. Gama, E.-S., Martín, F.: Translating solitons of the mean curvature flow asymptotic to hyperplanes in $\mathbb{R}^{n+1}$. Int. Math. Res. Not. **24**, 10114–10153 (2020)
27. Gilbarg, D., Trudinger, N.: Elliptic Partial Differential Equations of Second Order, Grundlehren Der Mathematischen Wissenschaften, 224, 2. Springer-Verlag, Berlin (1983)
28. Gromov, M.: Isoperimetric of waists and concentration of maps. Geom. Funct. Anal. **13**, 178–205 (2003)
29. Huisken, G., Polden, A.: Geometric evolution equations for hypersurfaces, Calculus of variations and geometric evolution problems (Cetraro, 1996), Lecture Notes in Math., 1713, 45-84, Springer, Berlin, (1999)
30. Hungerbühler, N., Mettler, T.: Soliton solutions of the mean curvature flow and minimal hypersurfaces. Proc. Am. Math. Soc. **140**, 2117–2126 (2012)
31. Hungerbühler, N., Roost, B.: Mean curvature flow solitons, Analytic aspects of problems in Riemannian geometry: elliptic PDEs, solitons and computer imaging. in Selected papers based on the presentations of the conference, Landea, France **13**, pp. 129–158 (2011)
32. Hungerbühler, N., Smoczyk, K.: Soliton solutions for the mean curvature flow. Differ. Integral Equ. **13**, 1321–1345 (2000)
33. Ilmanen, T.: Elliptic regularization and partial regularity for motion by mean curvature. Mem. Amer. Math. Soc. **108**, 520 (1994)
34. Impera, D., Rimoldi, M., Savo, A.: Index and first Betti number of f-minimal hypersurfaces and self-shrinkers. Rev. Mat. Iberoam. **36**(3), 817–840 (2020)
35. Impera, D., Rimoldi, M.: Rigidity results and topology at infinity of translating solitons of the mean curvature flow. Commun. Contemp. Math. **19**(6), 21 (2017)
36. Jenkins, H., Serrin, J.: The Dirichlet problem for the minimal surface equation in higher dimensions. J. Reine Angew. Math. **229**, 170–187 (1968)
37. Jorge, L., Tomi, F.: The barrier principle for minimal submanifolds of arbitrary codimension. Ann. Global Anal. Geom. **24**, 261–267 (2003)
38. Ladyzhenskaya, O.A., Ural'tseva, N.N.: On Hölder continuity of solutions and their derivatives of linear and quasilinear elliptic and parabolic equations. Transl. Ser Am. Math. Soc. **61**, 207–269 (1967)
39. Lang, U.: The existence of complete minimizing hypersurfaces in hyperbolic manifolds. Int. J. Math. **6**(1), 45–58 (1995)
40. Lott, J.: Mean curvature flow in a Ricci flow background. Commun. Math. Phys. **313**, 517–533 (2012)
41. Magni, A., Mantegazza, C., Tsatis, E.: Flow by mean curvature inside a moving ambient space. J. Evol. Equ. **13**(3), 561–576 (2013)
42. Martín, F., Pérez García, J., Savas-Halilaj, A., Smoczyk, K.: A characterization of the grim reaper cylinder. J. Reine Angew. Math. **746**, 1–28 (2019)
43. Martín, F., Savas-Halilaj, A., Smoczyk, K.: On the topology of translating solitons of the mean curvature flow. Calc. Var. Partial Diff. Equ. **54**, 2853–2882 (2015)
44. Montiel, S.: Unicity of constant mean curvature hypersurface in some Riemannian manifolds. Indiana Univ. Math. J. **48**, 711–748 (1999)
45. Pucci, P., Serrin, J.: The Maximum Principle, Progress in Nonlinear Differential Equations and their Applications, 73. Birkhäuser Verlag, Basel (2007)
46. Ripoll, J., Telichevesky, M.: Regularity at infinity of Hadamard manifolds with respect to some elliptic operators and applications to asymptotic Dirichlet problems. Trans. Am. Math. Soc. **367**(3), 1523–1541 (2015)
47. Serrin, J.: The problem of Dirichlet for quasilinear elliptic differential equations with many independent variables. Philos. Trans. Roy. Soc. London Ser. A **264**, 413–496 (1969)
48. Simon, L.: Global estimates of Hölder continuity for a class of divergence-form elliptic equations. Arch. Ration. Mech. Anal. **56**, 253–272 (1974)
49. Simon, L.: Lectures on geometric measure theory, 3,: Proc. Cent. Math. Anal. Aust. Natl. Univ, Australian National University, Centre for Mathematical Analysis, Canberra, vii, 272 p (1983)

50. Smoczyk, K.: Mean curvature flow in higher codimension–introduction and survey. Glob. Diff. Geometry Springer Proceed. Math. **12**, 231–274 (2012)
51. Smoczyk, K.: A relation between mean curvature flow solitons and minimal submanifolds. Math. Nachr. **229**, 175–186 (2001)
52. Solomon, B., White, B.: A strong maximum principle for varifolds that are stationary with respect to even parametric elliptic functionals. Indiana Univ. Math. J. **38**, 683–691 (1989)
53. Telichevesky, M.: A note on minimal graphs over certain unbounded domains of Hadamard manifolds. Pac. J. Math. **281**, 243–255 (2016)
54. White, B.: Controlling area blow-up in minimal or bounded mean curvature varieties. J. Diff. Geom. **102**, 501–535 (2016)
55. White, B.: The maximum principle for minimal varieties of arbitrary codimension. Comm. Anal. Geom. **18**, 421–432 (2010)
56. Yamamoto, H.: Ricci-mean curvature flows in gradient shrinking Ricci solitons. Asian J. Math. **24**, 77–94 (2020)

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