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On some geometric properties of φ-static spaces

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Riassunto della tesi

Lo scopo di questa tesi è l'analisi di proprietà geometriche di alcune strutture, che sono denotate con φ-*static spaces*, che consistono in una mappa fra varietà Riemanniane e una funzione detta potenziale sulla varietà di partenza, le quali soddisfano un sistema accoppiato di PDE. Tali spazi rientrano come caso specifico di spazi statici, in quanto sono varietà Riemanniane tramite le quali è possibile costruire, tramite un prodotto *warped*, soluzioni statiche delle equazioni di Einstein della Relatività Generale dove il tensore di stress-energia è relativo a un campo φ dato da una mappa fra la varietà di partenza e una varietà Riemanniana fissata.
La prima parte del lavoro riguarderà la presentazione dei principali oggetti e metodi utilizzati nei capitoli successivi, al contempo fissando il formalismo. La parte successiva fornirà alcune condizioni di rigidit à che coinvolgeranno i tensori di curvatura algebrica, ovvero tensori 4 volte covarianti che condividono le stesse simmetrie del tensore di Riemann a livello algebrico. Nello specifico, si applicherà il metodo di Bochner, richiedendo una condizione di non negatività riguardante l'operatore di curvatura, allo scopo di ottenere rigidità di tensori di curvatura algebrica che siano armonici. Dopodiché, saranno dati alcuni risultati con scelte specifiche del tensore di curvatura algebrica.
Il capitolo finale sarà interamente dedicato allo studio dei φ-*static spaces*. L'analisi che ne verrà fatta seguirà approcci simili a quelli usati nel caso dei classici spazi statici vuoti. Nel dettaglio, si possono riottenere alcuni dei risultati già noti in questo contesto, come il fatto che la curvatura scalare sia costante - qui sarà costante una funzione scalare legata alla curvatura scalare e alla mappa tangente - o il fatto che le ipersuperfici di livello zero della funzione potenziale siano totalmente geodetiche sulla varietà di partenza. Successivamente, si daranno alcuni vincoli all'esistenza di tali spazi, così come si guarderà alla loro relazione con le varietà di Einstein armoniche - le quali sono varietà dove il tensore di Ricci è dato da un multiplo della metrica più una parte dipendente dalla mappa, richiedendo che questa sia armonica. Dopodiché, ci si concentrerà su quegli spazi in cui la funzione potenziale è data dalla divergenza di un campo vettoriale conforme sulla varietà di partenza. Nello specifico, si darà una parziale caratterizzazione di tali spazi su varietà compatte con bordo tramite un funzionale di bordo, dipendente dalla metrica sulla varietà di base e dalla mappa, nella cui definizione è coinvolto il suddetto campo vettoriale. Da ultimo, si considererà il caso in cui nella varietà Riemanniana vi sia, oltre a una struttura di φ-*static space*, anche un campo vettoriale conforme chiuso. In tal caso, il gradiente della funzione potenziale e il campo vettoriale saranno proporzionali e, come conseguenza, questo influenzerà sia la geometria della varietà di base, la quale localmente si spezzerà in un prodotto *warped* avente come fette varietà di Einstein armoniche, sia la mappa, che non dipenderà dal flusso del campo vettoriale e sará in alcuni casi costante.

Abstract

The purpose of the thesis is to analyze geometric properties of some structures, which are called φ-static spaces, involving a map between Riemannian manifolds together with a potential function on the base manifold, both satisfying a coupled system of PDEs. Such spaces are are indeed a special case of static spaces, since they are Riemannian manifolds from which one can construct, via a warped product, static solutions to the Einstein equations of General Relativity where the stress-energy tensor is related to a field φ given by a map from the starting manifold to a fixed Riemannian manifold.

The first part of this work will be about presenting the main objects and tools used in the subsequent chapters, at the same time fixing the formalism. The next part will provide some rigidity conditions involving algebraic curvature tensors, which are 4-covariant tensors sharing the same symmetries, at the algebraic level, of the Riemann curvature tensor. Namely, the Bochner technique will be applied, requiring a non-negativity condition dealing with the curvature operator, to obtain rigidity of harmonic algebraic curvature tensors. Then some results with specific choices of the algebraic curvature tensor will be given.

The final chapter will be entirely devoted to the study of φ-static spaces. The analysis will be done following similar approaches to the ones used for classical vacuum static spaces. Namely, one can recover some of the already known results in this context, such as the constancy of the scalar curvature - here it will be the constancy of a scalar function related to the scalar curvature and the tangent map - or the fact that the zero level set of the potential function is totally geodesic hypersurface on the base manifold. Then some constraints on the existence of such spaces will be given, as well as their relation with harmonic-Einstein manifolds - that are manifolds where the Ricci tensor is given by a multiple of the metric plus a quantity depending on the harmonic map. After that, there will be a focus on those spaces where the potential function is given by the divergence of a conformal Killing vector field on the base manifold: to be specific, there will be a partial characterization of such spaces on compact manifolds with boundary by means of a boundary functional involving the vector field and depending on both the metric of the base manifold and the map. Lastly, it will be considered the case where the base Riemannian manifold admits both a φ-static space structure and a closed conformal vector field. In this case, the gradient of the potential function and the vector field will be proportional and, as a consequence, this will impact both the geometry of the base manifold, which will locally split as a warped product with harmonic-Einstein slices, and on the map, not depending on the flow of the vector field and possibly being constant.

Contents

Introduction

The purpose of this thesis is to study the geometry of Riemannian spaces (M, g), $m = \dim(M)$, together with a smooth map $\varphi : (M, g) \to (N, g_N)$ which satisfy, for some $w \in C^\infty(M)$, $w \not\equiv 0$, the φ-static equation:

$$\begin{cases} \mathrm{Hess}(w) - w\left(\mathrm{Ric}^\varphi - \frac{S^\varphi}{m-1} g\right) = 0 \\ w\tau(\varphi) = -\mathrm{d}\varphi(\nabla w), \end{cases} \tag{1}$$

where

$$\mathrm{Ric}^\varphi = \mathrm{Ric} - \alpha\varphi^*\langle\, , \,\rangle_N, \tag{2}$$

for some $\alpha \neq 0$, is the φ-Ricci tensor,

$$S^\varphi = S - \alpha|\mathrm{d}\varphi|^2$$

is the φ-scalar curvature, given by the trace with respect to the metric of Ric^φ, and

$$\tau(\varphi) = \mathrm{tr}_g(\nabla \mathrm{d}\varphi)$$

is the tension field of the map φ. Such spaces are called *φ-static spaces* and they are, in fact, the generalization of the vacuum static spaces in the presence of a smooth map. We recall that vacuum static spaces are Riemannian manifolds (M, g_M) together with a smooth function $w \not\equiv 0$ satisfying

$$\mathrm{Hess}(w) - w\left(\mathrm{Ric} - \frac{S}{m-1} g_M\right) = 0. \tag{3}$$

They are studied in General Relativity as they give rise, if $w > 0$ on M, to warped product solutions of the Einstein equations

$$\widehat{\mathrm{Ric}} - \frac{1}{2}\hat{S}g + \Lambda\hat{g} = 0$$

on $\hat{M} = \mathbb{R} \times M$ of the form

$$\hat{g}(x, t) = -w^2(x)\mathrm{d}t^2 + g_M$$

so that the metric on the spacelike distribution orthogonal to the Killing vector field ∂_t doesn't depend on the coordinate t, and hence it is *static*.

The study of vacuum static spaces is strictly related to the study of metrics with prescribed scalar curvature, since equation (3) is equivalent to

$$\mathrm{Hess}(w) - (\Delta w)g_M - w\,\mathrm{Ric} = 0,$$

and the left hand side is the expression for $D\mathcal{S}_g^*(w)$, that is, the adjoint of the linearization of the scalar curvature operator

$$\mathcal{S} : \mathcal{M} \to C^\infty(M)$$

where $\mathcal{M}$ is the space of smooth Riemannian metrics on M. The existence of nontrivial solutions to (3) means that $\ker(D\mathcal{S}_g^*) \neq 0$ and has as a consequence the non surjectivity of

$$D\mathcal{S}_g : T_g\mathcal{M} \to C^\infty(M)$$

(see [18, 17] for further details). Recently, Herzlich [20] provided a class of solutions by considering Einstein manifolds admitting a conformal non Killing vector field and, by converse, Miao and Tam [26] showed that the manifolds admitting a solution to (3) given by the divergence of a conformal Killing vector field are, under certain assumptions, Einstein manifolds. They also provided a characterization of some vacuum static spaces as critical points of the metric dependent functional

$$\mathbf{F}_{X,\Omega}(g) := \int_{\Omega} G(X,\nu)\mathrm{d}A,$$

for a vector field X and a subset $\Omega \subset\subset M$, where $g \in \mathcal{M}$ is such that X is conformal with respect to it. The above functional is involved in the definition of the ADM mass (see [5]).

When one takes into account a stress-energy tensor of the Einstein equations of the form

$$T(\hat{\varphi}) = \alpha(\hat{\varphi}^* g_N - e(\hat{\varphi})g), \quad \alpha \in \mathbb{R},$$

where $\hat{\varphi} : (\hat{M}, \hat{g}) \to (N, g_N)$ is a smooth map and

$$e(\hat{\varphi}) = \frac{1}{2}|\mathrm{d}\hat{\varphi}|^2_{\hat{g}\times g_N}$$

is the energy density associated to $\hat{\varphi}$, the resulting Einstein equations

$$\widehat{\mathrm{Ric}} - \frac{1}{2}\hat{S}g + \Lambda\hat{g} = T(\hat{\varphi})$$

are easily obtained by considering the critical points of the functional, depending on $\Omega \subset\subset \hat{M}$,

$$\mathbf{A}_{\Omega}(\hat{g}, \hat{\varphi}) := \int_{\Omega} (\hat{S} - 2\alpha\, e(\hat{\varphi}) - 2\Lambda)\mathrm{d}\,\mathrm{Vol}_{\hat{g}}, \tag{4}$$

with respect to smooth variations of the metric compactly supported inside Ω. If one also considers the critical points of compactly supported variation of the same functional with respect to $\hat{\varphi}$, the resulting system is

$$\begin{cases} \widehat{\mathrm{Ric}} - \alpha\hat{\varphi}^* g_N = \frac{2\Lambda}{m-1}\hat{g} \\ \tau(\hat{\varphi}) = 0. \end{cases}$$

In case of a warped product and a map $\hat{\varphi}$ depending only on M, $\hat{\varphi} = \varphi \circ \pi_M$ with $\varphi : M \to (N, g_N)$, the above system is shown to be equivalent to (1). We will discuss it more in detail in Section 3.1. The φ-static equations (1), when the potential function $w > 0$ on M, with the choice of $f = -\log(w)$ are equivalent to the system

$$\begin{cases} \mathrm{Ric}^{\varphi} + \mathrm{Hess}(f) - \mathrm{d}f \otimes \mathrm{d}f = \frac{S^{\varphi}}{m-1}g \\ \tau(\varphi) = \mathrm{d}\varphi(\nabla f), \end{cases}$$

and thus provide a special case of *Einstein-type structures*, which are given, $(M, g), \varphi, \alpha$ as above, by the choice of $f, \lambda \in C^{\infty}(M)$ and $\mu \in \mathbb{R}$ such that

$$\begin{cases} \mathrm{Ric}^{\varphi} + \mathrm{Hess}(f) - \mu\mathrm{d}f \otimes \mathrm{d}f = \lambda g \\ \tau(\varphi) = \mathrm{d}\varphi(\nabla f). \end{cases}$$

Such structures were studied extensively in [4].

Let us here sketch the plan of the work. In chapter 1 we will first recall the main definitions of the curvature tensors, at the same time fixing the notation and the conventions. We will also deal with the algebraic curvature tensors, which are 4-covariant tensors sharing the same symmetries, at the algebraic level, of the Riemann curvature tensor. These tensors can be split orthogonally (with respect to a Riemannian metric) into three different components, namely a totally traceless one (the *Weyl* part), a Ricci traceless part and a total trace part. The main reason why we are interested in algebraic curvature tensors is that the main results presented in the second chapter

involve directly tensors of this kind. Introducing the Bianchi operator, we will then specialize our discussion on those algebraic curvature tensors for which this vanishes, and hence satisfying a sort of second Bianchi identity. Along the way, some other analogues of the usual curvature tensors, such as the Cotton tensor and the projective tensor, will be generalized to this setting.
After that, we will give some basic definitions regarding maps between Riemannian manifolds, from the energy density associated to a map to the tension field, which is a vector field along φ given by the trace with respect to the metric on the base manifold of the covariant derivative, extended to tensors along φ, of the tangent map. The maps with vanishing tension field are called harmonic and they are the critical points, in the space of smooth maps between two fixed Riemannian manifolds, of the local energy functional (see [15] for further details). Subsequently, some of the so called φ-curvature tensors will be introduced. The first φ-curvature tensor was introduced by Müller (see [28]) by coupling the standard Ricci tensor of the base manifold to the pullback of the metric on the target manifold by a constant, as in (2). The motivation for this was the analysis of a geometric flow given by the Ricci flow coupled with the harmonic map flow, that is a flow of smooth maps introduced by Eells and Sampson [16] to question the existence, given a map between two Riemannian manifolds, of a harmonic map homotopic to the given one. In this context, harmonic-Einstein manifolds are introduced as solitons of this flow with respect to a constant function, and thus satisfying

$$\begin{cases} \mathrm{Ric}^{\varphi} = \lambda g, \\ \tau(\varphi) = 0, \end{cases}$$

for some constant $\lambda \in \mathbb{R}$ (when the dimension of M is at least 3, assuming $\lambda \in C^{\infty}(M)$ it can be shown to be already constant by an analogue of the Schur Lemma). Then, referring to [4], we will define the φ-scalar curvature as the trace of the φ-Ricci tensor and the φ-Weyl tensor accordingly to the splitting of the Riemann curvature tensor into the Weyl, the Ricci and the scalar curvature parts, except for the fact that the φ counterparts are no longer orthogonal to each other. Along with these tensors, also the φ-Schouten and the φ-Cotton tensors will be defined in analogy to their usual definition. We notice that here there is no longer a proportionality between the divergence of the Weyl tensor and the Cotton tensor, as it holds in the standard case. This fact will affect our hypothesis on the φ-Weyl tensor in chapter 2, when we will have to assume, in order for W^{φ} to be harmonic, that both the φ-Cotton and the φ-Weyl divergence vanish.
The last part of this introductory chapter will be dedicated to performing the variation of some metric tensors and tensors deriving from a smooth map between Riemannian manifolds, both with respect to the metric on the base manifold and on the smooth map, thus fixing the metrics on the base and the target manifolds. As for the variation of tensors with respect to the metric, we will use a different formalism from the usual one, first choosing local orthonormal frames and coframes with respect to of the metrics, then finding the structure constants and their dependence on the variation of the local frames, and finding in turn the variation of the connection forms and the curvature tensors. As for the maps, we will follow the formalism of the work of Anselli [3] and take variations of a map by means of smooth vector fields along the same map and then applying the exponential map to them.

In the second Chapter, the main focus will be that of proving some results concerning algebraic curvature tensors on a Riemannian manifold. Namely, if they are harmonic (thus satisfying the second Bianchi identity and having zero divergence) and satisfy some integrability conditions, under certain non-negativity assumptions on the curvature they are forced to be parallel, possibly being a constant multiple of

$$\langle\, , \rangle \owedge \langle\, , \rangle$$

where $\langle\, , \rangle$ is the metric on M and $\owedge$ is the Kulkarni-Nomizu product. A key ingredient for the proof of the theorems will be a Bochner type formula for the squared norm of the algebraic curvature tensor, which will be given by the sum of the squared norm of its covariant derivative and another term, coming from the commutation relations of covariant derivatives of tensors, that by the Ricci identities will be equal to a quadratic term on the tensor itself. Then, the latter will be estimated under some assumptions on the curvature. To be specific, we will require the non-negativity of the sum of the smallest $\lfloor \frac{m-1}{2} \rfloor$ eigenvalues of the curvature operator. First found by Petersen and Wink [30] where they deal with generic harmonic p-forms, this hypothesis on the curvature yields the non negativity of the quadratic term in the Bochner formula of the harmonic algebraic curvature tensor. In general, this term can be estimated to be greater or equal to the sum of the smallest $\lfloor \frac{m-1}{2} \rfloor$ eigenvalues of the curvature operator times the squared norm of a tensor, defined in analogy to the projective curvature tensor, which vanishes if and only if an algebraic curvature tensor is proportional to

$$\langle\, ,\, \rangle \owedge \langle\, ,\, \rangle.$$

The last step in the proof of the results will be the application of a suitable maximum principle to conclude that, since the estimate of the laplacian is greater or equal than zero, then it must be zero and therefore the tensor must be parallel. In addition, assuming also the positivity somewhere of the sum of the eigenvalues of the curvature, then the pseudo-projective tensor will be forced to be zero, yielding the last part of the theorems. Notable applications are the choices of $T = \mathrm{Riem}$, which gives a generalization of a theorem by Tachibana [36] and of Petersen and Wink [31], where they require the Riemannian manifold to be Einstein, or $T = W$ the Weyl tensor, extending on Tran [38]. Notice that the method adopted is the so called *Bochner technique*, consisting in applying maximum principles to formulas involving the laplacian of the norm of a tensor, and was first developed by Bochner [7] and Yano [40].

In the first part of Chapter 2 we will consider a generic algebraic curvature tensor and, after some manipulations of the laplacian of its squared norm, we will give a Bochner type formula that can be simplified using the assumptions of the tensor being harmonic. Then, as for the quadratic term appearing in the formula, we will link it to a quadratic form where the curvature operator is applied to 2-forms with values in algebraic curvature tensors. By means of this link and using some elementary inequalities, we will be able to prove a result for generic algebraic curvature tensors in the compact case. In Section 2.2 we will consider a possibly non compact manifold and will give some conditions on the growth of the tensor under which we will be able to apply a maximum principle and gain the same conclusions as in the compact case, whereas in the last section we will specify our argument to the choice of $T = W^{\varphi}$, the φ-Weyl tensor defined in presence of a map $\varphi : (M, \langle\, ,\, \rangle) \to (N, \langle\, ,\, \rangle_N)$. This analysis is justified by the fact that this tensor encodes both properties of the geometry of the base manifold and of the map itself. When applied to this tensor, the results given previously will thus provide restraints on the geometry of M, which will be locally conformally flat or locally symmetric, but also on the map φ, forced to be relatively affine - meaning that $\langle \nabla\tau(\varphi), \mathrm{d}\varphi\rangle_N = 0$ - or even an homothety. Along the way we will also give some equivalent conditions for the harmonicity of W^{φ}.

Chapter 3 will be entirely dedicated to the analysis of the above defined φ-static spaces. First of all, in Section 3.1 we will give some basic properties of these structures. Namely, we will begin deriving the equations of the system (1) by requiring a metric $\hat{g}$ on a Lorentzian manifold $\hat{M}$ and a smooth map $\hat{\varphi} : \hat{M} \to (N, g_N)$ to be critical with respect to compactly supported variations of the functional (4) and then specializing the resulting equations in case $\hat{M}$ is a warped product of the form $M \times_w \mathbb{R}$ and when $\hat{\varphi} = \varphi \circ \pi_M$, where π_M is the canonical projection onto M. We will then move on and show some properties that all φ-static spaces share. Specifically, we will see that all non trivial φ-static spaces, or else all Riemannian manifolds (M, g) together with a map $\varphi : (M, g) \to (N, g_N)$ for which the kernel of $D\mathcal{S}^*$, where $\mathcal{S}$ will denote the φ-scalar curvature operator, have constant φ-scalar curvature. On top of that, we will see that if the potential function w has nonempty 0 level set, then this is a totally geodesic (regular) hypersurface of M. A rigidity condition for non trivial solutions to (1), resulting in the constancy of the map, will be given requiring a non-negative coupling constant and a bound of the sectional curvature of the target manifold related to the latter. We will also see that, in case a φ-static space is also harmonic-Einstein with the same choice of the coupling constant defining the φ-curvatures, then w is a "special concircular scalar field" - i.e. the hessian of w is a linear function on w times the metric - and an application of classical results such as that of Tashiro [37] will give some rigidity conditions on both the metric on M and the map φ, which in some cases - when M is compact, for instance - must be constant.

After that, the main subject of Section 3.2 will be a functional analogous to that defined in the work of Miao and Tam [26] and the purpose will be that of giving some conditions under which the critical points of the functional (in some suitable subspaces of the space of metrics on M times the space of smooth maps from M to a fixed Riemannian manifold), when the vector field X involved in the definition of the functional is conformal with respect to the critical metric, give rise to a φ-static space. Firstly, we will consider a functional whose critical points (under some requests on the variations of the metric and the map) are precisely the harmonic-Einstein manifolds. Then, we will give some formulas for a conformal Killing vector field X on a Riemannian manifold and, subsequently, apply them to express the adjoint of the linearization of the φ-scalar curvature functional evaluated at $\operatorname{div}(X)$. By means of the latter, we will then obtain that harmonic-Einstein manifolds satisfy (1) with the choice of $w = \operatorname{div}(X)$ but also the converse is true assuming a further condition on X. Then, we will perform the variation of the boundary functional involving the vector field and the φ-Einstein tensor, after which we will be able to give a partial characterization of φ-static spaces (and also of some harmonic-Einstein manifolds) as critical points of this functional. The last section of the chapter will be dedicated to describe the local geometry of Riemannian manifolds supporting a φ-static space structure and having a non trivial closed conformal vector field. We will begin showing that, assuming some conditions relating the vector field to the map φ, the gradient of the potential function and the closed conformal vector field are indeed proportional at all the points where $X \neq 0$. Then, we will recall a result of Montiel [27] showing that Riemannian manifolds admitting closed conformal vector fields locally split as warped products around the points when X is not zero. Having in hand this result, we will then give an expression for the φ-Ricci tensor and its restriction to a slice of the warped product with respect to the φ-Ricci tensor of the slice (when the map is here given by the restriction of φ to the slice). From this formula, requiring that the map is harmonic, we will obtain that all the regular level sets of the potential function are indeed harmonic-Einstein, which will be the last result presented here.

Chapter 1

Some basic results

This chapter is devoted to the presentation of some already known results concerning the Riemannian curvature tensors, as well as generic four times covariant tensors sharing the same algebraic symmetries as the Riemann tensor. After that, we will introduce maps between Riemannian manifolds and some tensors related to them - the so called *φ-curvature tensors* - and, finally, we will compute the variation of the curvature tensors and φ-curvature tensors with respect to the variation of the metric and of a map from the starting manifold to a fixed Riemannian manifold. Although the results presented are interesting on their own, we will collect them in order to simplify the reading of the next chapters and also to fix the notations and the conventions. Notice that if not explicitly stated otherwise, we adopt the Einstein summation convention on repeated indices.

1.1 The Riemannian curvature tensors

We consider a Riemannian manifold (M, g) of dimension m and choose a local orthonormal base for T^*M given by m 1-forms $\{\theta^i\}_{i=1}^m$, so that the metric can locally be written as

$$g = \delta_{ij}\theta^i \otimes \theta^j.$$

We will often denote the metric as $\langle \, , \, \rangle$, especially when dealing with its extension for generic (p, q) tensors. We also set $\{e_i\}_{i=1}^m$ as the dual base to $\{\theta^i\}$ with respect to the metric and the Levi-Civita connection forms as the unique 1-forms $\{\theta^i_j\}$ such that $\theta^i_j + \theta^j_i = 0$ and satisfying the *first structure equations*

$$\mathrm{d}\theta^i = -\theta^i_j \wedge \theta^j. \tag{1.1.1}$$

The Levi-Civita connection ∇ is then given on $\{e_i\}$ by

$$\nabla e_i = \theta^j_i \otimes e_j,$$

so that

$$\theta^j_i(e_k) = g(\nabla_{e_k} e_i, e_j),$$

and then extended by linearity, Leibniz rule and compatibility with the metric to arbitrary (p, q) tensors. If $X = X^i e_i$ is a smooth vector field,

$$\nabla X = (\mathrm{d}X^i) \otimes e_i + X^i \nabla e_i = (\mathrm{d}X^i + X^j \theta^i_j) \otimes e_i =: X^i_j \theta^j \otimes e_i,$$

whereas if $\omega = \omega_i \theta^i$ is a smooth 1-form

$$\nabla \omega = (\mathrm{d}\omega_i - \omega_j \theta^j_i) \otimes \theta^i =: \omega_{i,j} \theta^j \otimes \theta^i.$$

For a generic (p, q) tensor

$$T = T^{i_1 \dots i_p}_{j_1 \dots j_q} \theta^{j_1} \otimes \dots \otimes \theta^{j_q} \otimes e_{i_1} \otimes \dots \otimes e_{i_p},$$

the components of ∇T are

$$T^{i_1\dots i_p}_{j_1\dots j_q, j_{q+1}} = \left(dT^{i_1\dots i_p}_{j_1\dots j_q} - \sum_{l=1}^{q} T^{i_1\dots i_p}_{j_1\dots r \dots j_q}\theta^r_{j_l} + \sum_{l=1}^{p} T^{i_1\dots s\dots i_p}_{j_1\dots j_q}\theta^{i_l}_s \right)(e_{j_{q+1}}),$$

The curvature 2-forms Θ^i_j are defined as the unique 2-forms satisfying $\Theta^i_j + \Theta^j_i = 0$ and the *second structure equations*

$$\mathrm{d}\theta^i_j = -\theta^i_k \wedge \theta^k_j + \Theta^i_j. \tag{1.1.2}$$

The curvature tensor is defined as usual, as

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z,$$

in local components along an orthonormal frame

$$R^i_{jkl} = \theta^i(\nabla_{e_k}\nabla_{e_l}e_j - \nabla_{e_l}\nabla_{e_k}e_j - \nabla_{[e_k,e_l]}e_j),$$

and it is easily verified that

$$\Theta^i_j = \frac{1}{2}R^i_{jkl}\theta^k \wedge \theta^l. \tag{1.1.3}$$

We also denote the $(0,4)$ version of the curvature tensor as Riem by setting

$$\mathrm{Riem}(W,Z,X,Y) = \langle W, R(X,Y)Z\rangle$$

for every $X,Y,Z,W \in \mathfrak{X}(M)$, so that in a local orthonormal frame it writes

$$\mathrm{Riem} = R_{ijkt}\,\theta^i \otimes \theta^j \otimes \theta^k \otimes \theta^t,$$

where, since the manifold is Riemannian,

$$R_{ijkl} = \delta_{it}R^t_{jkl} = R^i_{jkl}.$$

Remark 1.1. In the sequel, when there is no ambiguity, we will write the components of the tensors as they were totally covariant. Indeed, this shouldn't be misleading, as working in the Riemannian context and on orthonormal frames yields the equivalence between the components of a tensor in every version of it - i.e. when one raises or lowers indices.

The Ricci tensor is given by the contraction of the first and the third indices of Riem,

$$\mathrm{Ric}(X,Y) = \mathrm{tr}_g[(Z,W) \mapsto \mathrm{Riem}(Z,X,W,Y)]$$

for every $X,Y \in \mathfrak{X}(M)$ and the scalar curvature is given by the trace

$$S = \mathrm{tr}_g\,\mathrm{Ric}\;.$$

With respect to a local orthonormal coframe the components of the Ricci tensor are $R_{ij} = R_{kikj}$, whereas $S = R_{ii} = R_{ijij}$. The Riemann tensor has the symmetries

$$R_{ijkt} = -R_{jikt} = R_{ktij} \qquad \forall\, 1 \le i,j,k,t \le m$$

and satisfies the first and second Bianchi identities

$$\begin{aligned} &R_{ijkt} + R_{itjk} + R_{iktj} = 0 && \forall\, 1 \le i,j,k,t \le m \\ &R_{ijkt,l} + R_{ijlk,t} + R_{ijtl,k} = 0 && \forall\, 1 \le i,j,k,t,l \le m\,. \end{aligned}$$

The Riemann tensor splits into three mutually orthogonal components under the action of $O(m)$. Namely, denoting with $\owedge$ the Kulkarni-Nomizu product between two 2-covariant symmetric tensors

$$(E \owedge F)_{ijkl} = E_{ik}F_{jl} + E_{jl}F_{ik} - E_{il}F_{jk} - E_{jk}F_{il}$$

and with

$$Z = \mathrm{Ric} - \frac{S}{m}\langle\, , \,\rangle$$

the traceless part of the Ricci tensor, then the Riemann tensor splits as

$$\mathrm{Riem} = W + V + U, \tag{1.1.4}$$

where

$$U = \frac{S}{2m(m-1)}\langle\, , \,\rangle \owedge \langle\, , \,\rangle$$

is the trace part,

$$V = \frac{1}{m-2} Z \owedge \langle\, , \,\rangle$$

the traceless Ricci part and

$$W = \mathrm{Riem} - V - U$$

is the Weyl tensor, which is totally traceless and can be shown to be conformally invariant in its $(1,3)$ version. A Riemannian manifold is called Einstein if the traceless Ricci tensor vanishes, i.e. if

$$\mathrm{Ric} = \frac{S}{m}\langle\, , \,\rangle.$$

If this occurs and $m \geq 3$, then the scalar curvature S is constant. If also $S = 0$, then $(M, \langle\, , \,\rangle)$ is Ricci flat and the Riemann tensor coincides with its Weyl part. Introducing the Schouten tensor

$$A = \mathrm{Ric} - \frac{S}{2(m-1)}\langle\, , \,\rangle, \tag{1.1.5}$$

the Riemann tensor can be further written as the sum

$$\mathrm{Riem} = W + \frac{1}{m-2} A \owedge \langle\, , \,\rangle.$$

The Cotton tensor C is defined from the covariant derivative of the Schouten tensor as

$$C(X,Y,Z) = \nabla_Z A(X,Y) - \nabla_Y A(X,Z),$$

or in components along a orthonormal frame as

$$C_{ijk} = A_{ij,k} - A_{ik,j} = R_{ij,k} - R_{ik,j} - \frac{1}{2(m-1)}(S_k\delta_{ij} - S_j\delta_{ik}). \tag{1.1.6}$$

Recalling that a 2-covariant symmetric tensor E is a Codazzi tensor if for every $X, Y, Z \in \mathfrak{X}(M)$

$$\nabla_Z E(X,Y) = \nabla_Y E(X,Z),$$

or in local notation if

$$E_{ij,k} = E_{ik,j},$$

the Cotton tensor measures the obstruction of the Schouten tensor from being Codazzi. The Cotton tensor is totally trace free, i.e.

$$C_{ttk} = C_{tjt} = C_{itt} = 0,$$

is antisymmetric in its last two indices and satisfies

$$C_{ijk} + C_{jki} + C_{kij} = 0.$$

Its divergences satisfy

$$C_{ijk,i} = 0 \qquad \text{and} \qquad C_{ijk,k} = C_{jik,k}.$$

A manifold is said to be locally conformally flat if, in a neighbourhood of each point, there exist a conformal deformation of the metric such that the conformal metric is flat. It is known that equivalent conditions for a manifold to be locally conformally flat are those provided by Weyl and Shouten in the following theorem (notice that, if $m < 4$, the Weyl tensor vanishes identically):

Theorem 1.2. *Let $(M, \langle\, ,\, \rangle)$ be a Riemannian manifold of dimension $m \geq 3$. $(M, \langle\, ,\, \rangle)$ is locally conformally flat if and only if*

$$C \equiv 0 \quad \text{if} \quad m \geq 3, \text{ and}$$
$$W \equiv 0 \quad \text{if} \quad m \geq 4.$$

A last tensor of mixed order on the Riemann tensor is the 2-symmetric and totally traceless Bach tensor B, defined on manifolds of dimension $m \geq 3$ and whose components are

$$B_{ij} = \frac{1}{m-2}(C_{ijk,k} + R_{kt}W_{ikjt}),$$

and plays an important role in general relativity (in the semi-Riemannian context) for it is also divergence-free in dimension $m = 4$. We here give, in the moving frame setting, some commutation rules for the covariant derivatives of tensor fields. Although we won't give a general formula, this can be induced from the following

Proposition 1.3. *Let $(M, \langle\, ,\, \rangle)$ be a Riemannian manifold and Q be a $(1,1)$ tensor on M, given in local components by Q^i_j. Then*

$$Q^i_{j,kt} = Q^i_{j,tk} - Q^s_j R^i_{skt} + Q^i_s R^s_{jkt}. \tag{1.1.7}$$

Proof. By the Leibniz rule, the coefficients $Q^i_{j,k}$ of ∇Q are defined by the relation

$$Q^i_{j,k}\theta^k = \mathrm{d}Q^i_j + Q^s_j\theta^i_s - Q^i_s\theta^s_j.$$

Applying to the above the differential operator d and using the relation for the coefficients of $\nabla^2 Q$,

$$Q^i_{j,kt}\theta^t = \mathrm{d}Q^i_{j,k} + Q^s_{j,k}\theta^i_s - Q^i_{s,k}\theta^s_j - Q^i_{j,s}\theta^s_k,$$

one has that

$$(Q^i_{j,kt}\theta^t - Q^s_{j,k}\theta^i_s + Q^i_{s,k}\theta^s_j + Q^i_{j,s}\theta^s_k) \wedge \theta^k + T^i_{j,k}\mathrm{d}\theta^k = \mathrm{d}Q^s_j \wedge \theta^i_s + Q^s_j\mathrm{d}\theta^i_s - \mathrm{d}Q^i_s \wedge \theta^s_j - Q^i_s\mathrm{d}\theta^s_j.$$

Using once again the relation for the coefficients of ∇Q, simplifying and using the second structure equations, we arrive to

$$Q^i_{j,kt}\theta^t \wedge \theta^k = Q^s_j\Theta^i_s - Q^i_s\Theta^s_j,$$

i.e., skew-symmetrizing and by (1.1.3), (1.1.7). $\square$

Remark 1.4. If we wanted commutation rules for the totally covariant version of Q, they would be

$$Q_{ij,kt} = Q_{ij,tk} + Q_{sj}R^s_{ikt} + Q_{is}R^s_{jkt},$$

which comply with those of its $(1,1)$ version, since

$$Q_{ij} = Q^i_j$$

and

$$R^s_{ikt} = R_{sikt} = -R^i_{skt}.$$

We may also recall that, for a given (p,q) tensor Q on M and for any $X \in \mathfrak{X}(M)$, the Lie derivative of Q in the direction of X is the (p,q) tensor

$$\mathcal{L}_X Q_x = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_0 (\Phi_{-t})_*(Q_{\Phi_t(x)}),$$

where Φ_t is the local flow of diffeomorphisms generated by X and $x \in M$. On a Riemannian manifold $(M, \langle\, ,\, \rangle)$, its expression can be given in terms of the Levi-Civita connection: in local components, indeed, it holds

$$(\mathcal{L}_X Q)^{i_1\ldots i_p}_{j_1\ldots j_q} = X^k Q^{i_1\ldots i_p}_{j_1\ldots j_q,k} - \sum_{s=1}^{p} X^{i_s}_k Q^{i_1\ldots k\ldots i_p}_{j_1\ldots j_q} + \sum_{t=1}^{q} X^k_{j_t} Q^{i_1\ldots i_p}_{j_1\ldots k\ldots j_q}. \tag{1.1.8}$$

1.1.1 The sectional curvatures and the curvature operator

The Riemannian curvature can be seen from another perspective introducing the curvature operator, which is a linear, self-adjoint endomorphism $\mathfrak{R}$ on the space $\wedge^2 M$ of 2-forms on M defined as follows: with respect to a local coframe $\{\theta^i\}$, for every 2-form $\omega = \omega_{ij}\,\theta^i \otimes \theta^j \equiv \frac{1}{2}\omega_{ij}\,\theta^i \wedge \theta^j$ we let $\mathfrak{R}\omega = (\mathfrak{R}\omega)_{kt}\,\theta^k \otimes \theta^t$ be given by

$$(\mathfrak{R}\omega)_{kt} = R_{ijkt}\omega_{ij}\,. \tag{1.1.9}$$

On the other hand, one can define the sectional curvatures by considering, for any $x \in M$ and for any 2-plane $\pi \subseteq T_x M$, the expression

$$\mathrm{Sect}(\pi) = \frac{\mathrm{Riem}(X, Y, X, Y)}{|X|^2|Y|^2 - \langle X, Y\rangle^2},$$

where $X, Y \in T_x M$ is any couple of tangent vectors such that $\pi = \mathrm{span}\{X, Y\}$. The value of the quotient appearing on the right-hand side does not depend on the choice of the basis $\{X, Y\}$. It is worth noticing that the sectional curvature can be regarded as a particular case of the curvature operator. Indeed, the values assumed by the quadratic form $\langle \mathfrak{R}\,\cdot\,, \cdot\,\rangle$ on decomposable 2-forms are related to the sectional curvatures of M up to normalization, as we can easily see. For any plane $\pi = \mathrm{span}\{X, Y\}$, considering $u = X_i\theta^i$ and $v = Y_i\theta^i$ the dual forms to X and Y with respect to the metric, and letting

$$\omega = \frac{1}{2}\omega_{ij}\theta^i \wedge \theta^j = \frac{1}{2}v \wedge u = \frac{1}{2}(Y_i X_j - X_i Y_j)\theta^i \wedge \theta^j\,,$$

then

$$R(X, Y, X, Y) = R_{ijkt}X_iY_jX_kY_t = \frac{1}{4}R_{ijkt}\omega_{ij}\omega_{kt}$$
$$|X|^2|Y|^2 - \langle X, Y\rangle^2 = X_iX_iY_jY_j - X_iY_iX_jY_j = \frac{1}{2}\omega_{ij}\omega_{ij}$$

so we have

$$\mathrm{Sect}(\pi) = \frac{1}{2}\frac{\langle \mathfrak{R}\omega, \omega\rangle}{|\omega|^2}\,. \tag{1.1.10}$$

A manifold $(M, \langle\,,\,\rangle)$ has constant sectional curvature if the sectional curvature doesn't depend on the choice of the plane nor on the choice of $x \in M$. In this case, $(M, \langle\,,\,\rangle)$ is in particular Einstein and locally conformally flat.

We recall that a Riemannian manifold is said to have nonnegative curvature operator if for every 2-form ω

$$\langle \mathfrak{R}\omega, \omega\rangle \geq 0,$$

and positive curvature operator if the above inequality is strict for every $\omega \in \wedge^2 M$. Similarly, (M, g) has nonnegative sectional curvature if for any $x \in M$ and for any 2-plane $\pi \subseteq T_x M$

$$\mathrm{Sect}(\pi) \geq 0,$$

being positive if the above inequality is strict of any choice of x and $\pi \subseteq T_x M$.

We now want to extend the definition of curvature positivity, and we do that in the following way. For every $x \in M$, we denote by $\{\lambda_\alpha(x)\}_{1\leq\alpha\leq\binom{m}{2}}$ the non-decreasing sequence of the eigenvalues of $\mathfrak{R}_x : \wedge_x^2 M \to \wedge_x^2 M$ repeated according to multiplicity. We also let $\{\omega^\alpha\}_\alpha$ be an orthonormal basis for $\wedge_x^2(M)$ consisting of eigenvectors of $\mathfrak{R}$ corresponding to $\{\lambda_\alpha\}_\alpha$. Then, in a local orthonormal frame

$$R_{ijkt} = \sum_\alpha \lambda_\alpha \omega_{ij}^\alpha \omega_{kt}^\alpha\,, \qquad \frac{1}{2}(\delta_{ik}\delta_{jt} - \delta_{it}\delta_{jk}) = \sum_\alpha \omega_{ij}^\alpha \omega_{kt}^\alpha\,. \tag{1.1.11}$$

Definition 1.1. Let M be a Riemannian manifold of dimension $m \geq 2$. For $k \in \{1, \ldots, \binom{m}{2}\}$, the k-th (normalized) partial trace of $\mathfrak{R}$ is the function

$$x \mapsto \mathfrak{R}^{(k)}(x) = \inf_{\substack{V \leq \wedge_x^2 M \\ \dim V = k}} \left(\frac{1}{k} \sum_{\alpha=1}^{k} \langle \mathfrak{R}\psi^\alpha, \psi^\alpha \rangle \right) \tag{1.1.12}$$

where $\{\psi^\alpha\}_{\alpha=1}^k$ is any orthonormal basis of V.

By standard linear algebra we have that the infimum in the RHS of (1.1.12) is attained when $V = \operatorname{span}\{\omega^1, \ldots, \omega^k\}$, so that

$$\mathfrak{R}^{(k)} = \frac{1}{k} \sum_{\alpha=1}^{k} \lambda_\alpha$$

for every $k \in \{1, \ldots, \binom{m}{2}\}$. In particular we observe that

$$\mathfrak{R}^{(h)} \geq \mathfrak{R}^{(k)} \qquad \text{for every } 1 \leq k \leq h \leq \binom{m}{2} \tag{1.1.13}$$

as a consequence of the following elementary inequality:

Lemma 1.5. *Let $N \geq 1$ and let $\{a_i\}_{1 \leq i \leq N}$ be a nondecreasing sequence of real numbers. Then*

$$\frac{1}{h} \sum_{i=1}^{h} a_i \geq \frac{1}{k} \sum_{i=1}^{k} a_i \qquad \textit{for every } 1 \leq k \leq h \leq N\,.$$

Proof. By induction, it suffices to prove the inequality in case $k < N$ and $h = k+1$. Since $a_{k+1} \geq a_i$ for $1 \leq i \leq k$, we have $a_{k+1} \geq \frac{1}{k} \sum_{i=1}^{k} a_i$ and then

$$\frac{1}{k+1} \sum_{i=1}^{k+1} a_i = \frac{k}{k+1} \frac{1}{k} \sum_{i=1}^{k} a_i + \frac{1}{k+1} a_{k+1} \geq \frac{k}{k+1} \frac{1}{k} \sum_{i=1}^{k} a_i + \frac{1}{k+1} \frac{1}{k} \sum_{i=1}^{k} a_i = \frac{1}{k} \sum_{i=1}^{k} a_i\,.$$

□

That being set, a Riemannian manifold is said to have k-nonnegative curvature operator if $\mathfrak{R}^{(k)} \geq 0$ (positive if it holds the strict inequality).
Similarly, we extend the notion of positivity also for the sectional curvatures. To do so, we first need the following definition:

Definition 1.2. Let M be a Riemannian manifold of dimension $m \geq 2$ and let $x \in M$. We say that two 2-planes $\pi_1, \pi_2 \leq T_x M$ are mutually orthogonal, and we write $\langle \pi_1, \pi_2 \rangle = 0$, if for some (equivalently, for any) choice of bases $\{X_1, Y_1\}$ and $\{X_2, Y_2\}$ of π_1 and π_2, respectively, the 2-forms

$$\omega_1 = \frac{1}{2} u_1 \wedge v_1 \qquad \text{and} \qquad \omega_2 = \frac{1}{2} u_2 \wedge v_2$$

are orthogonal with respect to the inner product on $\wedge_x^2 M$, where u_1, u_2, v_1, v_2 are the 1-forms metrically equivalent to X_1, X_2, Y_1, Y_2, respectively.

In particular, any two 2-planes $\pi_1, \pi_2 \leq T_x M$ are mutually orthogonal if either

(i) each one of them is contained in the orthogonal complement of the other, or

(ii) $\dim(\pi_1 \cap \pi_2) = 1$ and there exist three mutually orthogonal vectors $X, Y, Z \in T_x M$ such that $\pi_1 = \operatorname{span}\{X, Y\}$ and $\pi_2 = \operatorname{span}\{X, Z\}$.

Definition 1.3. Let M be a Riemannian manifold of dimension $m \geq 2$. For $k \in \{1, \ldots, \binom{m}{2}\}$, the k-th averaged lower bound on the sectional curvature is the function

$$x \mapsto \operatorname{Sect}^{(k)}(x) = \inf_{\{\pi_1, \ldots, \pi_k\}} \left(\frac{1}{k} \sum_{i=1}^{k} \operatorname{Sect}(\pi_i) \right)$$

where the infimum is taken with respect to $\{\pi_1, \ldots, \pi_k\}$ varying among all collections of k mutually orthogonal 2-planes in $T_x M$.

A manifold is said to have k-nonnegative sectional curvature if $\mathrm{Sect}^{(k)} \geq 0$ (positive if $\mathrm{Sect}^{(k)} > 0$).
Moreover, from the above definitions together with (1.1.10) and a further application of Lemma 1.5

$$\mathrm{Sect}^{(h)} \geq \mathrm{Sect}^{(k)} \geq \frac{1}{2}\mathfrak{R}^{(k)} \qquad \forall\, 1 \leq k \leq h \leq \binom{m}{2}. \tag{1.1.14}$$

In particular, for the (non-normalized) Ricci tensor we have

$$\mathrm{Ric} \geq (m-1)\,\mathrm{Sect}^{(m-1)}$$

and therefore

$$\mathrm{Ric} \geq (m-1)\,\mathrm{Sect}^{(k)} \geq \frac{m-1}{2}\mathfrak{R}^{(k)} \qquad \text{for any } \ 1 \leq k \leq m-1\,. \tag{1.1.15}$$

The last thing we want to point out about the k-th partial trace of the curvature operator, which will be used in the sequel, is that the non-negativity of $\mathfrak{R}^{(k)}$ for some $k < \binom{m}{2}$ implies an upper bound on $|\,\mathrm{Riem}\,|$ in terms of the scalar curvature S. To this aim, we first observe that $|\,\mathrm{Riem}\,|$ and S are equal, respectively, to the Hilbert-Schmidt norm and the trace of $\mathfrak{R}$, that is,

$$|\,\mathrm{Riem}\,|^2 = \sum_\alpha \lambda_\alpha^2\,, \qquad S = \sum_\alpha \lambda_\alpha\,. \tag{1.1.16}$$

This can be directly seen from (1.1.11). Then, we apply the following

Lemma 1.6. *Let $N \geq 1$ and let $\{a_i\}_{1\leq i \leq N}$ be a nondecreasing sequence of real numbers. If*

$$\sum_{i=1}^{k} a_i \geq 0 \tag{1.1.17}$$

for some $k \in \{1, \dots, N-1\}$, then

$$\sum_{i=1}^{N} a_i \geq \frac{1}{k}\left(\frac{1}{N}\sum_{i=1}^{N} a_i^2\right)^{1/2}.$$

Proof. We can find j such that $|a_j| \geq \sqrt{\frac{1}{N}\sum_{i=1}^N a_i^2}$. By (1.1.17), there exists $h \in \{1, \dots, k\}$ such that $a_i < 0$ if $i < h$ and $a_i \geq 0$ if $i \geq h$. Note that $a_i \geq 0$ for $i \geq k$. If $j \geq h$ then $a_j = |a_j|$, hence

$$\sum_{i=1}^{N} a_i = \sum_{i=1}^{k} a_i + \sum_{i=k+1}^{N} a_i \geq \sum_{i=1}^{k} a_i + a_\ell \geq a_\ell \geq a_j = |a_j|$$

for $\ell = \max\{j, k+1\}$. If $j < h$ then we observe that

$$(k-h+1)a_k \geq \sum_{i=h}^{k} a_i \geq -\sum_{i=1}^{h-1} a_i = \sum_{i=1}^{h-1} |a_i| \geq |a_j|$$

where the second inequality is a rewriting of (1.1.17), so

$$\sum_{i=1}^{N} a_i = \sum_{i=1}^{k} a_i + \sum_{i=k+1}^{N} a_i \geq \sum_{i=k+1}^{N} a_i \geq (N-k)a_k \geq \frac{N-k}{k-h+1}|a_j|\,.$$

In conclusion,

$$\sum_{i=1}^{N} a_i \geq \min\left\{1, \frac{N-k}{k-h+1}\right\}|a_j| \geq \min\left\{1, \frac{N-k}{k-h+1}\right\}\left(\frac{1}{N}\sum_{i=1}^{N} a_i^2\right)^{1/2}$$

and, since $1 \leq k < N$ and $1 \leq k-h+1 \leq k$, we have $\min\left\{1, \frac{N-k}{k-h+1}\right\} \geq \frac{1}{k}$. ☐

Corollary 1.7. *Let M be a Riemannian manifold of dimension $m \geq 2$. If $\mathfrak{R}^{(k)} \geq 0$ for some $1 \leq k < \binom{m}{2}$ then $k^2\binom{m}{2}S^2 \geq |\,\mathrm{Riem}\,|^2$.*

1.2 Algebraic curvature tensors

In this section we shift our attention to more generic 4-covariant tensor fields T which share the same algebraic symmetries of the Riemann curvature tensor. Namely, given a Riemannian manifold $(M, \langle\, , \rangle)$ of dimension $m > 2$, T is an algebraic curvature tensor if its components along a local orthonormal coframe $\{\theta^i\}_{i=1}^m$ on M

$$T = T_{ijkt}\, \theta^i \otimes \theta^j \otimes \theta^k \otimes \theta^t$$

satisfy

$$T_{ijkt} = -T_{jikt} = T_{ktij} \qquad \forall\, 1 \leq i, j, k, t \leq m\,, \tag{1.2.1}$$

$$T_{ijkt} + T_{iktj} + T_{itkj} = 0 \qquad \forall\, 1 \leq i, j, k, t \leq m\,. \tag{1.2.2}$$

We remark that (1.2.2) is a consequence of (1.2.1) if $m \leq 3$, see [6, page 46].

If T is a smooth algebraic tensor field, we say that T satisfies the second Bianchi identity if

$$T_{ijkt,l} + T_{ijlk,t} + T_{ijtl,k} = 0 \qquad \forall\, 1 \leq i, j, k, t, l \leq m\,. \tag{1.2.3}$$

More generally, we can define a first-order differential operator $B : T \mapsto B(T)$ on the bundle of algebraic curvature tensors of M by setting

$$B(T)(X, Y, Z, W, V) = (\nabla_V T)(X, Y, Z, W) + (\nabla_W T)(X, Y, V, Z) + (\nabla_Z T)(X, Y, W, V)$$

for every $X, Y, Z, W, V \in \mathfrak{X}(M)$. In local notation this reads as

$$B(T)_{ijktl} = T_{ijkt,l} + T_{ijlk,t} + T_{ijtl,k}$$

and T satisfies the second Bianchi identity if and only if $B(T) = 0$. We can also recover the notion of harmonicity of such tensors, similarly to the case of the Riemann and the Weyl tensors:

Definition 1.4. A smooth algebraic curvature tensor T is harmonic if $\operatorname{div} T = 0$ and $B(T) = 0$.

Following the case of the Riemann tensor, we let E_T denote the Ricci contraction of T defined by

$$E_T(X, Y) = \operatorname{tr}_g[(Z, W) \mapsto T(Z, X, W, Y)]$$

for every $X, Y \in \mathfrak{X}(M)$. In local notation, $E_T = E_{ij}\, \theta^i \otimes \theta^j$ with

$$E_{ij} = T_{kikj}\,.$$

We also set $S_T = \operatorname{tr}_g E_T$ and we denote $Z_T = E_T - \frac{S_T}{m}\langle\, , \rangle$ the traceless part of E_T. We say that an algebraic curvature tensor is totally traceless if all of its contractions with the metric tensor vanish - or, equivalently, if $E \equiv 0$. Any algebraic curvature tensor T can be orthogonally decomposed in a unique way as the sum

$$T = W_T + V_T + U_T \tag{1.2.4}$$

of a totally traceless Weyl part W_T and two additional terms V_T and U_T that are further irreducible with respect to the action of the orthogonal group $O(m)$. Explicitely (see [2]),

$$V_T = \frac{1}{m-2} Z_T \owedge \langle\, , \rangle\,, \qquad U_T = \frac{S_T}{2m(m-1)} \langle\, , \rangle \owedge \langle\, , \rangle\,, \qquad W_T = T - V_T - U_T\,. \tag{1.2.5}$$

Setting the analogous of the Schouten tensor as

$$A_T = E_T - \frac{S_T}{2(m-1)}\langle\, , \rangle \equiv Z_T + \frac{m-2}{2m(m-1)} S_T \langle\, , \rangle \tag{1.2.6}$$

we can also write

$$T = W_T + \frac{1}{m-2} A_T \owedge \langle\, , \rangle\,. \tag{1.2.7}$$

Note that W_T, V_T, U_T and $A_T \owedge \langle\, , \rangle$ also are algebraic curvature tensors. Moreover, if $m \leq 3$ then the Weyl part W_T of T is always zero, so that T is completely determined by its Ricci contraction E_T, see [6, observation 1.119.b)].

Remark 1.8. For ease of notation, here we drop the subscript $_T$ and we simply write E, S, Z, A, W, V, U instead of E_T, S_T, Z_T, A_T, W_T, V_T, U_T to denote the tensors associated to T as above. This won't cause ambiguity with the notation that we adopted for the Weyl curvature tensor (W) and scalar curvature (S) of the Riemannian manifold $(M, \langle\,,\,\rangle)$, since these geometric objects will not appear in this section. On the other hand, we reserve the notation R_{ijkt} and R_{ij} for the components of the Riemann and Ricci curvature tensors of $(M, \langle\,,\,\rangle)$.

Lemma 1.9. *For any algebraic curvature tensor T we have*

$$|T|^2 = |W|^2 + \frac{4}{m-2}|Z|^2 + \frac{2S^2}{m(m-1)} \tag{1.2.8}$$

$$|\nabla T|^2 = |\nabla W|^2 + \frac{4}{m-2}|\nabla Z|^2 + \frac{2|\nabla S|^2}{m(m-1)} \tag{1.2.9}$$

or, equivalently,

$$|T|^2 = |W|^2 + \frac{4}{m-2}|E|^2 - \frac{2S^2}{(m-1)(m-2)} \tag{1.2.10}$$

$$|\nabla T|^2 = |\nabla W|^2 + \frac{4}{m-2}|\nabla E|^2 - \frac{2|\nabla S|^2}{(m-1)(m-2)}\,. \tag{1.2.11}$$

Proof. By orthogonality of the decomposition $T = W + V + U$ we have $|T|^2 = |W|^2 + |V|^2 + |U|^2$, then a direct computation yields

$$|V|^2 = \frac{4}{m-2}|Z|^2\,, \qquad |U|^2 = \frac{2S^2}{m(m-1)}\,.$$

As for the second identity, observing that $\nabla\langle\,,\,\rangle = 0$, in local notation in an orthonormal frame

$$T_{ijkt,l} = W_{ijkt,l} + \frac{1}{m-2}(Z_{ik,l}\delta_{jt} + Z_{jt,l}\delta_{ik} - Z_{it,l}\delta_{jk} - Z_{jk,l}\delta_{it}) + \frac{S_l}{m(m-1)}(\delta_{ik}\delta_{jt} - \delta_{it}\delta_{jk}).$$

Then, after some simplifications due to the fact that ∇W and ∇Z are still traceless in their first four and two indices respectively,

$$\begin{aligned} T_{ijkt,l}T_{ijkt,l} = {} & W_{ijkt,l}W_{ijkt,l} + \\ & \frac{1}{(m-2)^2}(Z_{ik,l}\delta_{jt} + Z_{jt,l}\delta_{ik} - Z_{it,l}\delta_{jk} - Z_{jk,l}\delta_{it})(Z_{ik,l}\delta_{jt} + Z_{jt,l}\delta_{ik} - \\ & Z_{it,l}\delta_{jk} - Z_{jk,l}\delta_{it}) + \frac{|\nabla S|^2}{m^2(m-1)^2}(\delta_{ik}\delta_{jt} - \delta_{it}\delta_{jk})(\delta_{ik}\delta_{jt} - \delta_{it}\delta_{jk}) \\ = {} & |\nabla W|^2 + \frac{4}{m-2}|\nabla Z|^2 + \frac{2}{m(m-1)}|\nabla S|^2 \end{aligned}$$

The third and fourth identities are equivalent to the first two since

$$|E|^2 = |Z|^2 + \frac{S^2}{m}\,, \qquad |\nabla E|^2 = |\nabla Z|^2 + \frac{|\nabla S|^2}{m}\,.$$

□

To any algebraic curvature tensor T we can associate a 4-covariant tensor $P = P_T$ of local components

$$P_{ijkt} = T_{ijkt} - \frac{1}{m-1}(\delta_{ik}E_{jt} - \delta_{it}E_{jk})\,. \tag{1.2.12}$$

Note that P is not an algebraic curvature tensor. However, its definition is not accidental. In case $T = \mathrm{Riem}$ (thus, $E = \mathrm{Ric}$) the $(1,3)$ version $P^i_{\ jkt}\, e_i \otimes \theta^j \otimes \theta^k \otimes \theta^t$ of P, of local components

$$P^i_{\ jkt} = R^i_{\ jkt} - \frac{1}{m-1}(\delta^i_k R_{jt} - \delta^i_t R_{jk})\,,$$

is the projective curvature tensor, which is invariant under projective transformations and vanishes if and only if the manifold has constant sectional curvature. In general, we have:

Lemma 1.10. *Let T be an algebraic curvature tensor and let P be as in* (1.2.12). *Then*

$$|P|^2 = |T|^2 - \frac{2}{m-1}|E|^2 = |W|^2 + \frac{2m}{(m-2)(m-1)}|Z|^2\,. \tag{1.2.13}$$

In particular, $P = 0$ if and only if $T = \dfrac{S}{2m(m-1)}\langle\, ,\,\rangle \owedge \langle\, ,\,\rangle$.

Proof. By direct computation on an orthonormal frame,

$$\begin{aligned}|P|^2 &= |T|^2 - \frac{2T_{ijkt}(\delta_{ik}E_{jt} - \delta_{it}E_{jk})}{m-1} + \frac{(\delta_{ik}E_{jt} - \delta_{it}E_{jk})(\delta_{ik}E_{jt} - \delta_{it}E_{jk})}{(m-1)^2} \\ &= |T|^2 - \frac{2}{m-1}|E|^2\end{aligned}$$

and substituting (1.2.8) and $|E|^2 = |Z|^2 + \frac{1}{m}S^2$ we obtain

$$\begin{aligned}|P|^2 &= |W|^2 + \frac{4}{m-2}|Z|^2 + \frac{2S^2}{m(m-1)} - \frac{2}{m-1}\left(|Z|^2 + \frac{S^2}{m}\right) \\ &= |W|^2 + \frac{2m}{(m-2)(m-1)}|Z|^2\,.\end{aligned}$$

□

This tensor will play a role in the subsequent chapter, as it will be used to give an estimate on a quadratic term appearing in Bochner-type formulas for algebraic tensor fields.

1.2.1 Algebraic curvature tensors with $B(T) = 0$

The condition $B(T) = 0$ has many relevant implications, that we briefly describe with the aim of establishing Propositions 1.11 and 1.12 below. If that happens, indeed, the symmetries and relations between the actions of several first order differential operators on T and the above described tensor related to it are essentially those satisfied in the case $T = \mathrm{Riem}$, where the condition $B(T) = 0$ is always satisfied.

First, let us recall that a symmetric twice covariant tensor field E is a Codazzi tensor if

$$(\nabla_X E)(\,\cdot\,, Y) = (\nabla_Y E)(\,\cdot\,, X) \qquad \forall\, X, Y \in \mathfrak{X}(M)\,,$$

that is, if

$$E_{ij,k} - E_{ik,j} = 0 \qquad \forall\, 1 \leq i, j, k \leq m\,.$$

Let us assume that T satisfies $B(T) = 0$. Tracing (1.2.3) with respect to i and l we get

$$(\operatorname{div} T)_{jkt} = T_{ijkt,i} = E_{jt,k} - E_{jk,t}$$

hence $\operatorname{div} T = 0$ if and only if E is a Codazzi tensor. Tracing again with respect to j and t we obtain the Schur's identity

$$2E_{ik,i} = S_k\,, \qquad \text{that is,} \qquad 2\operatorname{div} E = \nabla S\,.$$

Schur's identity is equivalent to the Einstein-like tensor $G = E - \frac{1}{2}S\langle\, ,\,\rangle$ being divergence-free. Equivalently, the Cotton-like tensor C_T (that we here denote as C for the ease of notation) of local components

$$C_{ijk} = A_{ij,k} - A_{ik,j} \tag{1.2.14}$$

is totally trace-free,

$$C_{iji} = C_{iij} = C_{jii} = 0\,. \tag{1.2.15}$$

Writing

$$E_{ij,k} - E_{ik,j} = C_{ijk} + \frac{1}{2(m-1)}(S_k\delta_{ij} - S_j\delta_{ik})\,,$$

(1.2.15) implies that the right-hand side is the sum of two orthogonal covariant tensors, hence it is apparent that E is Codazzi if and only if $C = 0$ and $\nabla S = 0$. In particular,

$$|\operatorname{div} T|^2 = |C|^2 + \frac{|\nabla S|^2}{2(m-1)} . \tag{1.2.16}$$

Summarizing, (1.2.16) proves the validity of

Proposition 1.11. *Let M be a Riemannian manifold of dimension $m \geq 3$ and let T be a smooth algebraic curvature tensor satisfying the second Bianchi identity. Then*

$$\operatorname{div} T = 0 \qquad \Leftrightarrow \qquad C = 0 \ \ \textit{and} \ \ \nabla S = 0 .$$

If $\dim M = 3$ then the Weyl part of any algebraic curvature tensor vanishes. If $m \geq 4$ then, as a second relevant consequence of $B(T) = 0$, there is a tight relation between C, $B(W)$ and $\operatorname{div} W$, which allows to restate Proposition (1.11) in a different form, see Proposition 1.12 below. Writing (1.2.7) in local notation we have

$$W_{ijkt} = T_{ijkt} - \frac{1}{m-2}(A_{ik}\delta_{jt} + A_{jt}\delta_{ik} - A_{it}\delta_{jk} - A_{jk}\delta_{it})$$

then applying the operator B to both sides and using $B(T) = 0$ and $\nabla\langle\, ,\, \rangle = 0$ we get

$$B(W)_{ijktl} = -\frac{1}{m-2}\left(C_{ikl}\delta_{jt} + C_{ilt}\delta_{jk} + C_{itk}\delta_{jl} - C_{jkl}\delta_{it} - C_{jlt}\delta_{ik} - C_{jtk}\delta_{il}\right) . \tag{1.2.17}$$

We trace with respect to i and l. Since $W_{ijik,t} = W_{ijti,k} = 0$ as W is totally traceless, we get

$$W_{ijkt,i} = B(W)_{ijkti} = \frac{m-3}{m-2}C_{jtk} , \qquad \text{that is,} \qquad \operatorname{div} W = -\frac{m-3}{m-2}C . \tag{1.2.18}$$

Formulas (1.2.18) and (1.2.17) show that $C = 0$ amounts to $\operatorname{div} W = 0$ and implies $B(W) = 0$. The converse is also true. To see this, we compute $|B(W)|^2$. Note that we can write

$$|B(W)|^2 = \frac{2}{(m-2)^2}(X_{ijktl}X_{ijktl} - X_{ijktl}X_{jiktl})$$

with $X_{ijktl} = C_{ikl}\delta_{jt} + C_{ilt}\delta_{jk} + C_{itk}\delta_{jl}$. Then we have

$$\begin{aligned} X_{ijktl}X_{ijktl} &= 3C_{ikl}C_{ikl}\delta_{jt}\delta_{jt} + 2C_{ikl}C_{ilt}\delta_{jt}\delta_{jk} + 2C_{ikl}C_{itk}\delta_{jt}\delta_{jl} + 2C_{ilt}C_{itk}\delta_{jk}\delta_{jl} \\ &= 3mC_{ijl}C_{ijl} + 2C_{ikl}C_{ilk} + 2C_{ikj}C_{ijk} + 2C_{ijt}C_{itj} \\ &= 3(m-2)C_{ijk}C_{ijk} , \end{aligned}$$

where we have used the symmetry $C_{ijk} = -C_{ikj}$, and

$$\begin{aligned} X_{ijktl}X_{jiktl} &= (C_{ikl}\delta_{jt} + C_{ilt}\delta_{jk} + C_{itk}\delta_{jl})\delta_{it}C_{jkl} \\ &\quad + (C_{ikl}\delta_{jt} + C_{ilt}\delta_{jk} + C_{itk}\delta_{jl})\delta_{ik}C_{jlt} \\ &\quad + (C_{ikl}\delta_{jt} + C_{ilt}\delta_{jk} + C_{itk}\delta_{jl})\delta_{il}C_{jtk} \\ &= 3C_{ikl}C_{ikl} \end{aligned}$$

where we have also exploited (1.2.15). Summing up, we get

$$|B(W)|^2 = \frac{6(m-3)}{(m-2)^2}|C|^2 , \qquad |\operatorname{div} W|^2 = \frac{(m-3)^2}{(m-2)^2}|C|^2 .$$

In conclusion, we have the following

Proposition 1.12. *Let M be a Riemannian manifold of dimension $m \geq 4$ and let T be a smooth algebraic curvature tensor satisfying the second Bianchi identity. Then*

$$\operatorname{div} T = 0 \quad \Leftrightarrow \quad \operatorname{div} W = 0 \ \ \textit{and} \ \ \nabla S = 0$$

and

$$\operatorname{div} W = 0 \quad \Leftrightarrow \quad B(W) = 0 \quad \Leftrightarrow \quad C = 0 .$$

In particular, T is harmonic if and only if W is harmonic and S is constant.

1.3 Maps between Riemannian manifolds

Let $(M, \langle\,,\,\rangle)$ and $(N, \langle\,,\,\rangle_N)$ be Riemannian manifolds of dimensions $m \geq 2$ and n respectively, and let $\varphi : M \to N$ be smooth. We fix the indices ranges

$$1 \leq i, j, k, \cdots \leq m \text{ and } 1 \leq a, b, \cdots \leq n.$$

Having fixed local orthonormal bases $\{\theta^i\}$ and $\{E_a\}$ for T^*M and TN respectively, we let φ_i^a be the components of the tangent map $\mathrm{d}\varphi : TM \to TN$ viewed as a section of $T^*M \otimes \varphi^*TN$, so that

$$\mathrm{d}\varphi = \varphi_i^a \theta^i \otimes E_a.$$

The energy density of φ is then defined by

$$e(\varphi) = \frac{1}{2}|\mathrm{d}\varphi|^2 = \frac{1}{2}\operatorname{tr}_{\langle\,,\,\rangle}(\varphi^*\langle\,,\,\rangle_N) = \varphi_i^a\varphi_i^a. \tag{1.3.1}$$

Given a map φ, the Levi-Civita connection ∇ on $(M, \langle\,,\,\rangle)$ can be extended to the vector bundle φ^*TN. For any $v \in \Gamma(\varphi^*TN)$, its covariant derivative is locally given by

$$\nabla v := v_i^a \theta^i \otimes E_a,$$

where $v = v^a E_a$ and

$$v_i^a \theta^i = \mathrm{d}v^a + v^b \varphi^* \omega_b^a,$$

ω_b^a being the connection forms relative to $(N, \langle\,,\,\rangle_N)$. Notice that ∇v is then a section of $T^*M \otimes \varphi^*TN$. Via the Leibniz rule, ∇ can be further extended to any bundle of the type $T_qM \otimes \varphi^*TN$. For instance, given $\sigma \in \Gamma(T_2M \otimes \varphi^*TN)$, where locally

$$\sigma = \sigma_{ij}^a \theta^i \otimes \theta^j \otimes E_a,$$

its covariant derivative will be given by

$$\nabla\sigma = \sigma_{ij,k}^a \theta^k \otimes \theta^i \otimes \theta^j \otimes E_a,$$

where the coefficients $\sigma_{ij,k}^a$ obey to the rule

$$\sigma_{ij,k}^a \theta^k = \mathrm{d}\sigma_{ij}^a + \sigma_{ij}^b \varphi^*\omega_b^a - \sigma_{kj}^a \theta_i^k - \sigma_{ik}^a \theta_j^k.$$

From now on, we shall omit the pullback over φ as we will only consider tensors along φ.
Moving on, one can define the so called generalized second fundamental tensor as

$$\nabla \mathrm{d}\varphi = \varphi_{ij}^a \theta^j \otimes \theta^i \otimes E_a,$$

where the coefficients φ_{ij}^a are defined as

$$\varphi_{ij}^a = (\mathrm{d}\varphi_i^a - \varphi_k^a \theta_i^k + \varphi_i^b \omega_b^a)(e_j). \tag{1.3.2}$$

See also [2, Section 1.7] for further details. It can be easily shown that

$$\varphi_{ij}^a = \varphi_{ji}^a. \tag{1.3.3}$$

The tension field of a smooth map φ is a tensor field along φ defined as

$$\tau(\varphi) = \operatorname{tr}_{\langle\,,\,\rangle}(\nabla \mathrm{d}\varphi) = \varphi_{ii}^a E_a. \tag{1.3.4}$$

A map φ is said to be harmonic if the energy functional, defined on any relatively compact domain $\Omega \subseteq M$ as

$$E_\Omega(\varphi) = \int_\Omega e(\varphi),$$

is stationary with respect to variations of φ where the values of the map at the boundary of Ω are fixed. Standard variational arguments (see [15]) show that φ is harmonic if and only if the tension

field vanishes identically.
Given a smooth map, one can define its stress-energy tensor as the 2-symmetric tensor

$$T(\varphi) = \varphi^*\langle\, , \,\rangle_N - e(\varphi)\langle\, , \,\rangle. \tag{1.3.5}$$

A map φ is conservative if $T(\varphi)$ is divergence-free. It is easy to see that if a map is harmonic, it is also conservative: indeed, in local components

$$\begin{aligned}\mathrm{div}(T(\varphi))_i &= (\varphi^a_i \varphi^a_j)_j - \frac{1}{2}[(\varphi^a_k \varphi^a_k)\delta_{ij}]_j = \\ &= \varphi^a_{ij}\varphi^a_j + \varphi^a_i \varphi^a_{jj} - \varphi^a_k \varphi^a_{kj} = \varphi^a_{jj}\varphi^a_i,\end{aligned}$$

i.e.

$$\mathrm{div}(T(\varphi)) = \langle \tau(\varphi), \mathrm{d}\varphi\rangle_N.$$

Let us give a couple more concepts related to maps. A map is said to be conformal if, for some function $\lambda \in C^\infty(M), \lambda > 0$,

$$\varphi^*\langle\, , \,\rangle_N = \lambda\langle\, , \,\rangle_M, \tag{1.3.6}$$

and it is homothetic if in the above λ is a (possibly zero) constant. In the last case, in particular $|\mathrm{d}\varphi|^2$ is constant, since

$$\lambda = \frac{1}{m}|\mathrm{d}\varphi|^2.$$

Notice that if φ is homothetic and nontrivial, then it is an isometric immersion of M into N. Furthermore, if φ is homothetic it is also conservative, for the stress-energy tensor can be written as

$$T(\varphi) = \frac{1}{m}|\mathrm{d}\varphi|^2\langle\, , \,\rangle - \frac{1}{2}|\mathrm{d}\varphi|^2\langle\, , \,\rangle = -\frac{m-2}{2m}|\mathrm{d}\varphi|^2\langle\, , \,\rangle$$

and $|\mathrm{d}\varphi|^2$ is constant, whereas if φ is at the same time conformal and conservative it is also homothetic.
As further notions, a map φ is called affine if

$$\nabla \mathrm{d}\varphi = 0,$$

that is, if the generalized second fundamental tensor vanishes, and relatively affine if equivalently

$$\nabla \varphi^*\langle\, , \,\rangle_N = 0$$

or

$$\langle \nabla \mathrm{d}\varphi, \mathrm{d}\varphi\rangle_N = 0. \tag{1.3.7}$$

In local notation, being relatively affine means that

$$\varphi^a_{ij}\varphi^a_k = 0.$$

Every relatively affine map is also conservative and has constant $|\mathrm{d}\varphi|^2$, as it is easy to show, but it can be non-harmonic (and thus non-affine).
It is also worth showing the properties of $\nabla^2\mathrm{d}\varphi$ and the commutation relations of its covariant derivatives: according to the Leibniz rule, its local components obey to

$$\varphi^a_{ijk}\theta^k = \mathrm{d}\varphi^a_{ij} - \varphi^a_{kj}\theta^k_i - \varphi^a_{ik}\theta^k_j + \varphi^b_{ij}\omega^a_b,$$

and it can be shown that they satisfy the commutation relations

$$\varphi^a_{ijk} = \varphi^a_{jik} \qquad \text{and} \qquad \varphi^a_{ijk} = \varphi^a_{ikj} + R_{tijk}\varphi^a_t - {}^N R^a_{bcd}\varphi^b_i\varphi^c_j\varphi^d_k. \tag{1.3.8}$$

1.3.1 φ-curvatures

Let $\alpha \in \mathbb{R}\setminus\{0\}$ be fixed, and φ be a smooth map as above. The φ-Ricci tensor Ric^φ, first introduced by Müller in [28], is the 2-covariant tensor defined by

$$\mathrm{Ric}^\varphi = \mathrm{Ric} - \alpha\varphi^*\langle\, ,\, \rangle_N,$$

thus its components with respect to the coframes on M and N are given by

$$R^\varphi_{ij} = R_{ij} - \alpha\varphi^a_i\varphi^a_j. \tag{1.3.9}$$

The φ-scalar curvature, S^φ, is, as expected, the trace of Ric^φ,

$$S^\varphi = \mathrm{tr}_{\langle\, ,\, \rangle}(\mathrm{Ric}^\varphi) = R^\varphi_{ii} = R_{ii} - \alpha\varphi^a_i\varphi^a_i = S - \alpha|d\varphi|^2. \tag{1.3.10}$$

Similarly to the standard case, a Riemannian manifold $(M, \langle\, ,\, \rangle)$ together with a map $\varphi : M \to (N, \langle\, ,\, \rangle_N)$ is said to be harmonic Einstein if it holds

$$\begin{cases} \mathrm{Ric}^\varphi = \frac{S^\varphi}{m}\langle\, ,\, \rangle \quad \text{and} \\ \tau(\varphi) = 0, \end{cases} \tag{1.3.11}$$

i.e. if φ is harmonic and Ric^φ is proportional to the metric. We remark that, if this is the case, S^φ must be constant, provided that $m \geq 3$. One has indeed that

$$R^\varphi_{ij,j} = \frac{1}{2}S_i - \alpha\varphi^a_{ij}\varphi^a_j - \alpha\varphi^a_i\varphi^a_{jj} = \frac{1}{2}S^\varphi_i - \alpha\varphi^a_i\varphi^a_{jj} \tag{1.3.12}$$

by the definition of the φ-Ricci tensor and by the second contracted Bianchi identity, whereas on the other hand

$$R^\varphi_{ij,j} = \left(\frac{S^\varphi}{m}\delta_{ij}\right)_j = \frac{1}{m}S^\varphi_i,$$

hence

$$\frac{m-2}{2m}S^\varphi_i = \alpha\varphi^a_i\varphi^a_{jj}$$

that, if φ is harmonic, yields the constancy of S^φ.

The φ-Schouten tensor A^φ is defined by

$$A^\varphi = \mathrm{Ric}^\varphi - \frac{S^\varphi}{2(m-1)}\langle\, ,\, \rangle, \tag{1.3.13}$$

and is related to the usual Schouten tensor

$$A = \mathrm{Ric} - \frac{S}{2(m-1)}\langle\, ,\, \rangle$$

by the identity

$$A^\varphi = A - \alpha\Big(\varphi^*\langle\, ,\, \rangle_N - \frac{|d\varphi|^2}{2(m-1)}\langle\, ,\, \rangle\Big).$$

In dimension $m \geq 3$, the φ-Weyl tensor, W^φ, is given by

$$W^\varphi = \mathrm{Riem} - \frac{1}{m-2}A^\varphi \owedge \langle\, ,\, \rangle, \tag{1.3.14}$$

similarly to the definition of the standard Weyl tensor

$$W = \mathrm{Riem} - \frac{1}{m-2}A \owedge \langle\, ,\, \rangle.$$

Comparing W^φ with W,

$$W^\varphi = W + \frac{\alpha}{m-2}\Big(\varphi^*\langle\, ,\, \rangle_N - \frac{|d\varphi|^2}{2(m-1)}\langle\, ,\, \rangle\Big) \owedge \langle\, ,\, \rangle. \tag{1.3.15}$$

The tensor W^φ has the same symmetries of the Riemann tensor and the Weyl tensor but, unlike the Weyl tensor, generally is not totally trace free. Indeed, an immediate computation yields

$$W^\varphi_{ijit} = \alpha\varphi^a_j\varphi^a_t. \tag{1.3.16}$$

Having the same symmetries of Riem, the φ-Weyl tensor satisfies the first Bianchi-type identity

$$W^\varphi_{ijkt} + W^\varphi_{iktj} + W^\varphi_{itjk} = 0, \tag{1.3.17}$$

while the covariant derivative of W^φ satisfies the following "fake second Bianchi-type identity", analogous to the one valid for the Weyl tensor:

$$W^\varphi_{ijkt,l}+W^\varphi_{ijtl,k}+W^\varphi_{ijlk,t} = \frac{1}{m-2}\Big(C^\varphi_{ilk}\delta_{jt}+C^\varphi_{ikt}\delta_{jl}+C^\varphi_{itl}\delta_{jk}+C^\varphi_{jkl}\delta_{it}+C^\varphi_{jtk}\delta_{il}+C^\varphi_{jlt}\delta_{ik}\Big), \tag{1.3.18}$$

where C^φ_{ijk} are the components of the φ-Cotton tensor C^φ defined as the obstruction to A^φ to be Codazzi, that is,

$$C^\varphi_{ijk} = A^\varphi_{ij,k} - A^\varphi_{ik,j}. \tag{1.3.19}$$

The φ-Cotton tensor satisfies

$$\begin{aligned} &C^\varphi_{ikj} = -C^\varphi_{ijk};\\ &C^\varphi_{ijk} + C^\varphi_{jki} + C^\varphi_{kij} = 0;\\ &C^\varphi_{jji} = -C^\varphi_{jij} = \alpha\varphi^a_i\varphi^a_{jj}. \end{aligned} \tag{1.3.20}$$

Equation (1.3.18) comes from (1.3.14) by taking the covariant derivative, summing over the cyclic permutation of the last three indices, using the second Bianchi identity for the Riemann tensor and (1.3.19).

We set $\operatorname{div} W^\varphi$ for the 3-times covariant tensor of components

$$(\operatorname{div} W^\varphi)_{ijk} = W^\varphi_{ijkl,l}. \tag{1.3.21}$$

Differently from the standard case, the φ-Weyl divergence and the Cotton tensor are in general non proportional, as the following formula shows:

$$W^\varphi_{sjkt,s} = -\frac{m-3}{m-2}C^\varphi_{jkt} + \alpha(\varphi^a_{jk}\varphi^a_t - \varphi^a_{jt}\varphi^a_k) + \frac{\alpha}{m-2}\varphi^a_{ss}(\varphi^a_k\delta_{jt} - \varphi^a_t\delta_{jk}). \tag{1.3.22}$$

Equation (1.3.22) can be proved by contracting on the first and the last indices of (1.3.18) and substituting (1.3.16) and the trace of φ-Cotton in (1.3.20), resulting in

$$W^\varphi_{ijkt,i} - (\alpha\varphi^a_j\varphi^a_t)_k + (\alpha\varphi^a_j\varphi^a_k)_t = \frac{1}{m-2}\Big(\alpha\varphi^a_k\varphi^a_{ll}\delta_{jt} + C^\varphi_{jkt} - \alpha\varphi^a_t\varphi^a_{ll}\delta_{jk} + C^\varphi_{jkt} + mC^\varphi_{jtk} + C^\varphi_{jkt}\Big).$$

Some simplifications then lead to (1.3.22).

1.4 Variation of smooth tensor with respect to the metric and to a smooth map

In this section we compute the variation of some tensors, starting from the curvature tensors, on a Riemannian manifold (M, g) with respect to the variation of the metric and of a map $\varphi : M \to (N, g_N)$. Variation of tensors will come into play in the third chapter, when we will deal with the study of some functionals and their variation, that possibly characterizes some spaces such as φ-Static Spaces.
Let us start with the variation with respect to the metric. Here we want to follow a different approach than the usual one and consider first the variation of the section in the coframes bundle - which determines a variation of the metric in turn. Namely, we consider, for the metric g on the manifold M, a local orthonormal basis $\{\theta^i\}_{i=1}^m$ and we consider a variation of it by taking a smooth

map A from $(-\varepsilon, \varepsilon)$ with values in the isomorphisms of the coframe bundle, with the condition that $A(0)$ is the identity map. Such A can be locally represented by a matrix such that the new basis $\{\theta^i(t)\}$ is given by $\theta^i(t) = A^i_j(t)\theta^j$, so that $\theta^i(0) = \theta^i$. Since A is smooth, also $\{\theta^i(t)\}$ vary smoothly in t, and

$$\dot{\theta}^i(t) = \dot{A}^i_j(t)\theta^j = \dot{A}^i_j(A^{-1})^j_k\theta^k(t).$$

As a consequence, the resulting metric $g(t)$, locally given by $g_t = g(t) = \theta^i(t) \otimes \theta^i(t)$, is such that

$$\dot{g}(t) = (\dot{A}A^{-1})^i_j(t)\theta^j(t) \otimes \theta^i(t) + (\dot{A}A^{-1})^i_j(t)\theta^i(t) \otimes \theta^j(t) = \big[(\dot{A}A^{-1})^i_j(t) + (\dot{A}A^{-1})^j_i(t)\big]\theta^j(t) \otimes \theta^i(t).$$

Let us denote with $a^i_j(t) = (\dot{A}A^{-1})^i_j(t)$ and with

$$h = h_{ij}\theta^i(t) \otimes \theta^j(t) = (a^i_j + a^j_i)\theta^i(t) \otimes \theta^j(t),$$

so that the above equations read as

$$\dot{\theta}^i = a^i_j\theta^j, \tag{1.4.1}$$
$$\dot{g} = h. \tag{1.4.2}$$

Proposition 1.13. *With the above notation, let $\theta^i_j(t)$ be the Levi-Civita connection forms relative to the metric $g(t)$, $R^i_{jkl}(t) = \theta^i(R_t(e_k, e_l)e_j)$ be the components of the Riemann curvature tensor on the orthonormal base $\{e_i(t)\}$ dual to $\{\theta^i(t)\}$ with respect to $g(t)$, $R_{ij} = \mathrm{Ric}(e_i, e_j)$ be the components of the Ricci tensor and S be the scalar curvature. Then it holds:*

$$\dot{R} = \frac{1}{2}(h_{il,jk} + h_{jk,il} - h_{ik,jl} - h_{jl,ik} - h_{ti}R^t_{jkl} - h_{tj}R^t_{ikl})e_i \otimes \theta^j \otimes \theta^k \otimes \theta^l; \tag{1.4.3}$$
$$\dot{\mathrm{Riem}} = \frac{1}{2}(h_{il,jk} + h_{jk,il} - h_{ik,jl} - h_{jl,ik} + h_{ti}R^t_{jkl} - h_{tj}R^t_{ikl})\theta^i \otimes \theta^j \otimes \theta^k \otimes \theta^l; \tag{1.4.4}$$
$$\dot{\mathrm{Ric}} = \frac{1}{2}(h_{si,js} + h_{sj,is} - h_{ss,ij} - h_{ij,ss})\theta^i \otimes \theta^j; \tag{1.4.5}$$
$$\dot{S} = h_{ij,ij} - h_{ii,jj} - R_{ij}h_{ij}. \tag{1.4.6}$$

Proof. We begin by setting c^k_{ij} as the coefficients

$$c^k_{ij}(t) = g_t([e_i(t), e_j(t)], e_k(t)) = g_t(\nabla_{e_i}e_j - \nabla_{e_j}e_i, e_k),$$

so that the components of the Levi-Civita connection forms along the orthonormal coframe are

$$(\theta^i_j)_k = \theta^i_j(e_k) = g_t(\nabla_{e_k}e_j, e_i) = \frac{1}{2}(c^k_{ij} + c^j_{ik} - c^i_{jk}). \tag{1.4.7}$$

In order to obtain the variation of θ^i_j, we should first compute the variation of the $\{c^k_{ij}\}$. Making explicit the dependence on t, one has:

$$\begin{aligned}
c^k_{ij}e_k = [e_i, e_j] &= [(A^{-1})^l_i e_l(0), (A^{-1})^t_j e_t(0)] = (A^{-1})^l_i\Big(e_l(0)\big((A^{-1})^t_j\big)e_t(0) + (A^{-1})^t_j e_l(0)e_t(0)\Big) - \\
&- (A^{-1})^t_j\Big(e_t(0)\big((A^{-1})^l_i\big)e_l(0) + (A^{-1})^l_i e_t(0)e_l(0)\Big) = \\
&= (A^{-1})^l_i(A^{-1})^t_j[e_l(0), e_t(0)] + (A^{-1})^l_i \mathrm{d}(A^{-1})^t_j\big(e_l(0)\big)e_t(0) - (A^{-1})^t_j \mathrm{d}(A^{-1})^l_i\big(e_t(0)\big)e_l(0) = \\
&= (A^{-1})^l_i(A^{-1})^t_j c^s_{lt}(0)e_s(0) - (A^{-1}\mathrm{d}AA^{-1})^t_j\big(e_i\big)e_t(0) + (A^{-1}\mathrm{d}AA^{-1})^l_i\big(e_j\big)e_l(0) = \\
&= (A^{-1})^l_i(A^{-1})^t_j A^k_s c^s_{lt}(0)e_k - (\mathrm{d}AA^{-1})^k_j(e_i)e_k + (\mathrm{d}AA^{-1})^k_i(e_j)e_k,
\end{aligned}$$

and hence

$$c^k_{ij} = (A^{-1})^l_i(A^{-1})^t_j A^k_s c^s_{lt}(0) - \mathrm{d}A^k_s(e_i)(A^{-1})^s_j + \mathrm{d}A^k_s(e_j)(A^{-1})^s_i. \tag{1.4.8}$$

Towards the aim to compute the derivative in t of (1.4.8), we notice, from the fact that θ^i is dual to e_i, that

$$\dot{e}_i = -a^j_i e_j. \tag{1.4.9}$$

Moreover, we recall that $(\dot{A^{-1}}) = -A^{-1}\dot{A}A^{-1}$. We can now proceed to take the derivative in t:

$$\begin{aligned}\dot{c}^k_{ij} &= -(A^{-1}\dot{A}A^{-1})^l_i(A^{-1})^t_jA^k_sc^s_{lt}(0) - (A^{-1})^l_i(A^{-1}\dot{A}A^{-1})^t_jA^k_sc^s_{lt}(0) + (A^{-1})^l_i(A^{-1})^t_j\dot{A}^k_sc^s_{lt}(0)-\\ &\quad - \mathrm{d}\dot{A}^k_s(e_i)(A^{-1})^s_j - \mathrm{d}A^k_s(-a^r_ie_r)(A^{-1})^s_j - \mathrm{d}A^k_s(e_i)(-A^{-1}\dot{A}A^{-1})^s_j+\\ &\quad + \mathrm{d}\dot{A}^k_s(e_j)(A^{-1})^s_i + \mathrm{d}A^k_s(-a^r_je_r)(A^{-1})^s_i + \mathrm{d}A^k_s(e_j)(-A^{-1}\dot{A}A^{-1})^s_i =\\ &= -(A^{-1})^l_ra^r_i(A^{-1})^t_jA^k_sc^s_{lt}(0) - (A^{-1})^l_i(A^{-1})^t_ra^r_jA^k_sc^s_{lt}(0) + (A^{-1})^l_i(A^{-1})^t_ja^k_rA^r_sc^s_{lt}(0)-\\ &\quad - \mathrm{d}\dot{A}^k_s(e_i)(A^{-1})^s_j + a^r_i\mathrm{d}A^k_s(e_r)(A^{-1})^s_j + \mathrm{d}A^k_s(e_i)(A^{-1}\dot{A}A^{-1})^s_j+\\ &\quad + \mathrm{d}\dot{A}^k_s(e_j)(A^{-1})^s_i - a^r_j\mathrm{d}A^k_s(a^r_je_r)(A^{-1})^s_i - \mathrm{d}A^k_s(e_j)(A^{-1}\dot{A}A^{-1})^s_i.\end{aligned}$$

We can get rid of the terms $c^s_{lt}(0)$ in the above expression, by making use of (1.4.8) and thus expressing them in terms of $c^k_{ij}(t)$:

$$\begin{aligned}\dot{c}^k_{ij} &= -a^r_i\big(c^k_{rj} + \mathrm{d}A^k_s(e_r)(A^{-1})^s_j - \mathrm{d}A^k_s(e_j)(A^{-1})^s_r\big)-\\ &\quad - a^r_j\big(c^k_{ir} + \mathrm{d}A^k_s(e_i)(A^{-1})^s_r - \mathrm{d}A^k_s(e_r)(A^{-1})^s_i\big)+\\ &\quad + a^k_r\big(c^r_{ij} + \mathrm{d}A^r_s(e_i)(A^{-1})^s_j - \mathrm{d}A^r_s(e_j)(A^{-1})^s_i\big)-\\ &\quad - \mathrm{d}\dot{A}^k_s(e_i)(A^{-1})^s_j + a^r_i\mathrm{d}A^k_s(e_r)(A^{-1})^s_j + \mathrm{d}A^k_s(e_i)(A^{-1})^s_ra^r_j+\\ &\quad + \mathrm{d}\dot{A}^k_s(e_j)(A^{-1})^s_i - a^r_j\mathrm{d}A^k_s(a^r_je_r)(A^{-1})^s_i - \mathrm{d}A^k_s(e_j)(A^{-1})^s_ra^r_i =\\ &= -a^r_ic^k_{rj} - a^r_jc^k_{ir} + a^k_rc^r_{ij} + a^k_r\mathrm{d}A^r_s(e_i)(A^{-1})^s_j - a^k_r\mathrm{d}A^r_s(e_j)(A^{-1})^s_i-\\ &\quad - \mathrm{d}\dot{A}^k_s(e_i)(A^{-1})^s_j + \mathrm{d}\dot{A}^k_s(e_j)(A^{-1})^s_i =\\ &= -a^r_ic^k_{rj} - a^r_jc^k_{ir} + a^k_rc^r_{ij} - \mathrm{d}\,a^k_j(e_i) + \mathrm{d}\,a^k_i(e_j),\end{aligned}$$

where in the last equality we have exploited the fact that

$$\mathrm{d}\,a = \mathrm{d}(\dot{A}A^{-1}) = \mathrm{d}\dot{A}A^{-1} - \dot{A}A^{-1}\mathrm{d}AA^{-1} = \mathrm{d}\dot{A}A^{-1} - a\,\mathrm{d}AA^{-1}.$$

Recalling that, in this setting, the components of the covariant derivative of a are given by

$$a^i_{j,k}\theta^k = \mathrm{d}\,a^i_j + a^k_j\theta^i_k - a^i_k\theta^k_j,$$

one has that

$$\begin{aligned}\dot{c}^k_{ij} &= -a^l_ic^k_{lj} - a^t_jc^k_{it} + a^k_sc^s_{ij} - a^k_{j,i} + a^l_j\theta^k_l(e_i) - a^k_l\theta^l_j(e_i) + a^k_{i,j} - a^l_i\theta^k_l(e_j) + a^k_l\theta^l_i(e_j) =\\ &= -a^k_{j,i} + a^k_{i,j} + \frac{1}{2}a^l_j(c^i_{kl} + c^l_{ki} + c^k_{li}) - \frac{1}{2}a^l_i(c^j_{kl} + c^l_{kj} + c^k_{lj}),\end{aligned}$$

where the last equality follows from (1.4.7) after some simplifications.
Thanks to the above expression for $\dot{c}^k_{ij}$, we are now able to calculate the variation of the $\theta^i_j(t)$:

$$\begin{aligned}(\dot{\theta}^i_j)_k\theta^k &= \frac{d}{dt}\Big(\frac{1}{2}(c^k_{ij}(t) + c^j_{ik}(t) - c^i_{jk}(t))\theta^k(t)\Big) =\\ &= \frac{1}{2}\Big(\dot{c}^k_{ij} + \dot{c}^j_{ik} - \dot{c}^i_{jk}\Big)\theta^k + \frac{1}{2}(c^k_{ij} + c^j_{ik} - c^i_{jk})a^k_l\theta^l =\\ &= \frac{1}{2}\big[-a^k_{j,i} + a^k_{i,j} + \frac{1}{2}a^l_j(c^i_{kl} + c^l_{ki} + c^k_{li}) - \frac{1}{2}a^l_i(c^j_{kl} + c^l_{kj} + c^k_{lj})-\\ &\quad - a^j_{k,i} + a^j_{i,k} + \frac{1}{2}a^l_k(c^i_{jl} + c^l_{ji} + c^j_{li}) - \frac{1}{2}a^l_i(c^k_{jl} + c^l_{jk} + c^j_{lk})+\\ &\quad + a^i_{k,j} - a^i_{j,k} - \frac{1}{2}a^l_k(c^j_{il} + c^l_{ij} + c^i_{lj}) + \frac{1}{2}a^l_j(c^k_{il} + c^l_{ik} + c^i_{lk})\big]\theta^k+\\ &\quad + \frac{1}{2}(c^l_{ij} + c^j_{il} - c^i_{jl})a^l_k\theta^k.\end{aligned}$$

After some simplifications, we arrive to the expression

$$(\dot{\theta}^i_j)_k = \frac{1}{2}(h_{ik,j} - h_{jk,i} + a^j_{i,k} - a^i_{j,k}). \tag{1.4.10}$$

Notice that, since the difference between the connection forms $\theta^i_j(t+s)-\theta^i_j(t)$ defines a tensor on M, $(\dot\theta^i_j)_k$ are in fact the components of a rank 3 tensor.
Then, the variation of the curvature 2-form at $t=0$ can then be computed in terms of $\dot\theta^i_j$ by exploiting the second structure equations:

$$\begin{aligned}\dot\Theta^i_j &= \mathrm{d}\dot\theta^i_j + \dot\theta^i_k\wedge\theta^k_j + \theta^i_k\wedge\dot\theta^k_j = \mathrm{d}((\dot\theta^i_j)_k\theta^k) + (\dot\theta^i_k)_l\theta^l\wedge\theta^k_j + \theta^i_k\wedge((\dot\theta^k_j)_l\theta^l) = \\ &= \mathrm{d}(\dot\theta^i_j)_k\wedge\theta^k + (\dot\theta^i_j)_k(-\theta^k_l\wedge\theta^l) - (\dot\theta^i_l)_k\theta^l_j\wedge\theta^k + (\dot\theta^l_j)_k\theta^i_l\wedge\theta^k = \\ &= (\dot\theta^i_j)_{k,l}\theta^l\wedge\theta^k\end{aligned}$$

Since the curvature 2-forms are related to the Riemann tensor via

$$\Theta^i_j = \frac{1}{2}R^i_{jkl}\theta^k\wedge\theta^l,$$

one has that

$$\frac{1}{2}((\dot\theta^i_j)_{l,k}-(\dot\theta^i_j)_{k,l})\theta^k\wedge\theta^l = \dot\Theta^i_j = \frac{1}{2}(\dot R^i_{jkl}+R^i_{jtl}a^t_k+R^i_{jkt}a^t_l)\theta^k\wedge\theta^l,$$

therefore the variation of the components of the curvature tensor (at $t=0$) is given by the expression

$$\dot R^i_{jkl} = -R^i_{jkt}a^t_l + R^i_{jlt}a^t_k + (\dot\theta^i_j)_{l,k} - (\dot\theta^i_j)_{k,l}.$$

Substituting (1.4.10),

$$\dot R^i_{jkl} = -R^i_{jkt}a^t_l + R^i_{jlt}a^t_k + \frac{1}{2}(h_{il,jk}+h_{jk,il}-h_{ik,jl}-h_{jl,ik}) + \frac{1}{2}(a^i_t-a^t_i)R_{tjkl} - \frac{1}{2}(a^j_t-a^t_j)R_{tikl}. \quad (1.4.11)$$

At this point, the variation of the curvature tensor given by

$$\dot R = \dot R^i_{jkl}e_i\otimes\theta^j\otimes\theta^k\otimes\theta^l + R^i_{jkl}(\dot e_i\otimes\theta^j\otimes\theta^k\otimes\theta^l + e_i\otimes\dot\theta^j\otimes\theta^k\otimes\theta^l + e_i\otimes\theta^j\otimes\dot\theta^k\otimes\theta^l + e_i\otimes\theta^j\otimes\theta^k\otimes\dot\theta^l).$$

Inserting (1.4.11), (1.4.1) and (1.4.9), we arrive at (1.4.3). As regards the variation of the Riemann tensor (the 4-times covariant form of R) the computations are quite similar, recalling that at any time t

$$R^i_{jkl} = R_{ijkl},$$

and thus

$$\dot R^i_{jkl} = \dot R_{ijkl}.$$

Summing (1.4.11) over i and k, one obtains the variation of the components of the Ricci tensor,

$$\dot R_{ij} = \frac{1}{2}(h_{si,js}+h_{sj,is}-h_{ss,ij}-h_{ij,ss}) - R_{it}a^t_j - R_{tj}a^t_i \quad (1.4.12)$$

and thus

$$\dot{\mathrm{Ric}} = \dot R_{ij}\theta^i\otimes\theta^j + R_{ij}\dot\theta^i\otimes\theta^j + R_{ij}\theta^i\otimes\dot\theta^j$$

is precisely given by (1.4.5). Finally, as for the variation of the scalar curvature it is sufficient to consider the trace over i and j in (1.4.12), resulting in (1.4.6). □

Remark 1.14. Notice that, as evidenced from the proof of the above Proposition, different versions of a tensor have in general different variations, even in local components along orthonormal frames. In spite of this, when it turns to the variation of the components along orthonormal frames (in the Riemannian case), the variation does not depend on the version of the tensor, since we recall that in this setting every version of a tensor has the same components on an orthonormal frame. Because of this and because of the fact that tracing the components of the variation of a tensor equals considering the variation of the components of its trace on the same indices, we shall consider variation of components of tensors (along orthonormal coframes, otherwise this argument doesn't work), rather than of the tensor.

One can compute the variation of covariant derivative of generic tensors, and one way to compute it is the following. Let us consider, for instance, a 2-covariant tensor (the method can be generalized to any type of tensor) $T = T_{ij}\theta^i \otimes \theta^j$ for a local orthonormal coframe θ^i. Assume that the variation of T is known, say $\dot{T} = U$ for some $U \in T^0_2 M$. Then the coefficients of ∇T obey the rule

$$T_{ij,k}\theta^k = \mathrm{d}T_{ij} - T_{sj}\theta^s_i - T_{is}\theta^s_j,$$

where although not written explicitly we consider ∇ and θ as dependent on t, as above. Then, differentiating with respect to t we obtain

$$\dot{T}_{ij,k}\theta^k + T_{ij,k}\dot{\theta}_k = \mathrm{d}\dot{T}_{ij} - \dot{T}_{sj}\theta^s_i - T_{sj}\dot{\theta}^s_i - \dot{T}_{is}\theta^s_j - T_{is}\dot{\theta}^s_j, \tag{1.4.13}$$

where we have denoted with $\dot{T}_{ij,k}$ the variation of the components along an orthonormal coframe of the tensor T. We can group the three terms of the RHS in (1.4.13), since for the covariant derivative of the components $\dot{T}_{ij}$ it holds

$$(\dot{T}_{ij})_{,k}\theta^k = \mathrm{d}\dot{T}_{ij} - \dot{T}_{sj}\theta^s_i - \dot{T}_{is}\theta^s_j,$$

and arrive to

$$\dot{T}_{ij,k} + T_{ij,s}a^s_k = (\dot{T}_{ij})_{,k} - T_{sj}(\dot{\theta}^s_i)_k - T_{is}(\dot{\theta}^s_j)_k. \tag{1.4.14}$$

Now,

$$U_{ij} = (\dot{T})_{ij} = \dot{T}_{ij} + T_{sj}a^s_i + T_{is}a^s_j,$$

which yields, by differentiation,

$$U_{ij,k} = (\dot{T}_{ij})_{,k} + T_{sj,k}a^s_i + T_{is,k}a^s_j + T_{sj}a^s_{i,k} + T_{is}a^s_{j,k}$$

Inserting the latter into (1.4.14),

$$\dot{T}_{ij,k} + T_{ij,s}a^s_k + T_{sj,k}a^s_i + T_{is,k}a^s_j = U_{ij,k} - T_{sj}a^s_{i,k} - T_{is}a^s_{j,k} - T_{sj}(\dot{\theta}^s_i)_k - T_{is}(\dot{\theta}^s_j)_k.$$

At this point, we use the fact that

$$(\dot{\nabla T})_{ijk} = \dot{T}_{ij,k} + T_{ij,s}a^s_k + T_{sj,k}a^s_i + T_{is,k}a^s_j$$

and (1.4.10) to conclude that

$$(\dot{\nabla T})_{ijk} = U_{ij,k} - \frac{1}{2}T_{sj}(h_{sk,i} + h_{is,k} - h_{ki,s}) - \frac{1}{2}T_{is}(h_{sk,j} + h_{js,k} - h_{kj,s}). \tag{1.4.15}$$

Let us now consider the variation of a map between Riemannian manifolds. We denote with $\mathcal{F} = C^\infty(M, N)$, the space of smooth maps between two Riemannian manifolds (M, g) and (N, g_N). Then, considering a map $\varphi : M \to N$, the tangent space $T_\varphi\mathcal{F}$ can be identified with the space of section of the pullback bundle φ^*TN: i.e., to every variation $\varphi_t : M \to N$, $t \in (-\varepsilon, \varepsilon)$ such that $\varphi_0 = \varphi$, we can associate a vector along φ, $v \in \Gamma(\varphi^*TN)$, such that $\varphi_t(x) = \exp_{\varphi(x)}(tv_x)$. Let $\overline{M} = M \times (-\varepsilon, \varepsilon)$ and $\Phi : \overline{M} \to N$ be such that $\Phi(x,t) = \varphi_t(x)$. If we consider on $\overline{M}$ the product metric between g on M and the flat metric on $(-\varepsilon, \varepsilon)$, we can then extend the Levi-Civita connection for this Riemannian product to a connection on tensors along Φ, which we denote by $\overline{\nabla}$. Notice that, under the identification $M = M \times \{0\} \subset \overline{M}$, the connection $\overline{\nabla}$ restricted to $TM \otimes \varphi^*TN \subset T\overline{M} \otimes \Phi^*TN$ coincides with ∇. The same is valid if, denoting with $M_t = M \times \{t\}$ and with ${}^t\nabla$ the connection on tensors along φ_t, one considers $\overline{\nabla}$ restricted to $TM_t \otimes \varphi_t^*TN \subset T\overline{M} \otimes \Phi^*TN$. We choose an orthonormal coframe on M, $\{\theta^i\}$, and an orthonormal frame on $\overline{M}$ such that

$$\begin{aligned} \bar{\theta}^{m+1} &= \mathrm{d}t \qquad \text{the coordinate on} \quad (-\varepsilon, \varepsilon), \\ \bar{\theta}^i &= \theta^i, \quad i, j, k = 1 \ldots m, \end{aligned}$$

where with abuse of notation we have written θ^i instead of its pullback under the projection of $\overline{M}$ onto M. It is immediate to see that the only non-vanishing components of the Riemann tensor on $\overline{M}$ are

$$\overline{R}_{ijkl} = R_{ijkl}.$$

Then,

$$\left.\frac{d}{dt}\right|_0 \varphi_t = (\overline{\nabla}_{\partial_t}\Phi)\big|_{t=0} = v, \tag{1.4.16}$$

hence, in components,

$$\Phi^a_{m+1}\big|_{t=0} = v^a.$$

We shall also take into account the variation of the covariant derivatives of φ. As one expects, their variation can be linked to the variation of the map, as it is shown in the following Proposition:

Proposition 1.15. *With the above notation,*

$$\left.\frac{d}{dt}\right|_0 (\varphi_t) = v, \tag{1.4.17}$$

$$\left.\frac{d}{dt}\right|_0 (\mathrm{d}\varphi_t) = \nabla v, \tag{1.4.18}$$

$$\left.\frac{d}{dt}\right|_0 ({}^t\nabla \mathrm{d}\varphi_t) = \nabla^2 v + {}^N\!R(\cdot, \mathrm{d}\varphi, v, \mathrm{d}\varphi). \tag{1.4.19}$$

Proof. Equation (1.4.17) is immediate. As for the others, we make use of the above notation and notice that

$$\left.\frac{d}{dt}\right|_{t=0} (\mathrm{d}\varphi_t) = (\overline{\nabla}_{\partial_t}\mathrm{d}\Phi)\big|_{TM\otimes\varphi^*TN},$$

and therefore

$$\left.\frac{d}{dt}\right|_{t=0} (\mathrm{d}\varphi_t) = \Phi^a_{i\,m+1}\theta^i \otimes E_a\big|_{t=0} = \Phi^a_{m+1\,i}\theta^i \otimes E_a\big|_{t=0} = v^a_i\theta^i \otimes E_a = \nabla v,$$

thus proving (1.4.18). Notice that we exploited the symmetry between the two indices in the covariant derivative of the tangent map with respect to the connection of the bas manifold, as in (1.3.3). Regarding $\nabla \mathrm{d}\varphi$, we have

$$\left.\frac{d}{dt}\right|_{t=0} ({}^t\nabla \mathrm{d}\varphi_t) = (\overline{\nabla}_{\partial_t}\overline{\nabla}\mathrm{d}\Phi)\big|_{TM\otimes\varphi^*TN}.$$

By the commutation relations in (1.3.8),

$$\begin{aligned}\left.\frac{d}{dt}\right|_{t=0} ({}^t\nabla \mathrm{d}\varphi_t) = \Phi^a_{ij\,m+1}\theta^i \otimes \theta^j \otimes E_a\big|_{t=0} &= [\Phi^a_{i\,m+1\,j} + \overline{R}_{lij\,m+1}\Phi^a_l \\ &- {}^N\!R^a_{bcd}\Phi^b_i\Phi^c_j\Phi^d_{m+1}]_{t=0}\theta^i \otimes \theta^j \otimes E_a = \\ &= (v^a_{ij} + {}^N\!R^a_{bcd}\varphi^b_i v^c \varphi^d_j)\theta^i \otimes \theta^j \otimes E_a,\end{aligned}$$

proving (1.4.19). □

Remark 1.16. Similarly to the variation of tensors with respect to the metric, we can also consider the variation of the components of tensors along φ with respect to φ. As for the variation of the components of $\mathrm{d}\varphi$ on the frame $\theta^i \otimes E_a$, φ^a_i, we have that

$$\left.\frac{d}{dt}\right|_{t=0} (\varphi_t)^a_i = \left.\frac{d}{dt}\Phi^a_i\right|_{t=0} = (\mathrm{d}\Phi^a_i(\partial_t))|_{t=0} = \left(\Phi^a_{i\,m+1} + \Phi^a_\alpha\theta^\alpha_i(\partial_t) - \Phi^b_i\Phi^*\omega^a_b(\partial_t)\right)\big|_{t=0},$$

where $1 \le \alpha \le m+1$. On one hand, since $\overline{M}$ has a product metric, $\theta^{m+1}_i = 0$ and the θ^j_is do not depend no the t coordinate, and hence

$$\theta^\alpha_i(\partial_t) = 0.$$

On the other,

$$(\Phi^*\omega^a_b(\partial_t))\big|_{t=0} = (\Phi^c_{m+1}\omega^a_b(E_c))\big|_{t=0} = \omega^a_b(v).$$

Therefore, we have

$$\left.\frac{d}{dt}\right|_{t=0}(\varphi_t)^a_i = v^a_i - \varphi^b_i\omega^a_b(v). \tag{1.4.20}$$

Analogously, using the commutation rules (1.3.8),

$$\left.\frac{d}{dt}\right|_{t=0}(\varphi_t)^a_{ij} = v^a_{ij} + {}^N R^a_{bcd}\varphi^b_i v^c\varphi^d_j - \varphi^b_{ij}\omega^a_b(v). \tag{1.4.21}$$

Chapter 2

Rigidity results for maps between Riemannian manifolds

This chapter is devoted to the presentation of some rigidity conditions involving generic algebraic curvature tensors, which we introduced in the previous chapter as 4-times covariant tensors sharing the same symmetries, at the algebraic level (thus not involving the covariant derivatives) of the Riemann curvature tensor. In particular, we are going to show that, under certain conditions involving the curvature operator, a harmonic algebraic curvature tensor T must be parallel and possibly forced to be of the form

$$T = \lambda \langle \, , \, \rangle \owedge \langle \, , \, \rangle,$$

for some $\lambda \in \mathbb{R}$. To be more specific, we have:

Theorem 2.1. *Let $(M^m, \langle \, , \, \rangle)$ be a compact m-dimensional Riemannian manifold with $\lfloor \frac{m-1}{2} \rfloor$-nonnegative curvature operator $\mathfrak{R}$ and let T be an algebraic curvature tensor. Assume that T is harmonic. Then T is parallel. If M has $\lfloor \frac{m-1}{2} \rfloor$-positive curvature operator at some point then T is a constant multiple of $\langle \, , \, \rangle \owedge \langle \, , \, \rangle$.*

Theorem 2.1 with the choice of $T = \mathrm{Riem}$, together with the application of a classification Theorem by Noronha in [29], yields the following:

Theorem 2.2. *Let M be a compact Riemannian manifold of dimension $m \geq 3$ with $\mathfrak{R}^{(\lfloor \frac{m-1}{2} \rfloor)} \geq 0$ and harmonic curvature. Then M is locally symmetric. If $\mathfrak{R}^{(\lfloor \frac{m-1}{2} \rfloor)} > 0$ at some point then M is a quotient of $\mathbb{S}^m$.*

Theorem 2.2 provides a generalization of a Theorem by Tachibana [36], who shows that if a Riemannian manifold has positive curvature operator, then it is a quotient of $\mathbb{S}^m$. Notice that Theorem 2.2 also generalizes results of Tran [38] and Petersen and Wink [30, 31].

The validity of Theorem 2.1 will be showed by making use of the so called "Bochner technique". First applied by S. Bochner and K. Yano, [7, 8], it consists in evaluating the laplacian of the squared norm of a tensor field and, after having given some estimates, an application of some maximum principles gives some constraints on the covariant derivatives of such tensor. In the specific case of an algebraic curvature tensor T we will show that, if it is harmonic, it satisfies

$$\frac{1}{2}\Delta |T|^2 = |\nabla T|^2 + \frac{1}{2}\langle \Gamma T, T \rangle, \tag{2.0.1}$$

where Γ is a special case of a self-adjoint endomorphism defined by Lichnerowicz in [24] for generic tensors in $T^0_q M$, $q > 0$. Then, estimates on the quadratic term $\langle \Gamma T, T \rangle$ via the orthogonal splitting into its irreducible components as in (1.2.4) under the hypothesis of non-negativity of $\mathfrak{R}^{\lfloor \frac{m-1}{2} \rfloor}$ and the application of the divergence theorem yields the conclusions of Theorem 2.1. Notice that the condition on the curvature operator was first identified in [30], where they use it to give estimates on the laplacian of the squared norm of harmonic p-forms, and can be applied to any harmonic

algebraic curvature tensor whose Ricci contraction is proportional to the metric. Here, instead, we will provide a lower bound on $\langle \Gamma T, T\rangle$ by only requiring that T is harmonic.
In Section 2.1 we will provide this estimate and then prove, in the compact setting, the rigidity Theorem 2.1. Then we will discuss the complete case (possibly non-compact). Similar conclusions to those of Theorem 2.1 can be deduced, but here we have to make some assumptions on the growth of T and on the geometry of M, in such a way to recover, in some sense, the possibility to apply a maximum principle to $|T|^2$ and hence to infer its constancy in view of (2.0.1). This will be done in Section 2.2. In the last section we will specialize our discussion to maps between manifolds, and find a suitable tensor, namely the φ-Weyl tensor defined in the previous chapter as

$$W^{\varphi} = \mathrm{Riem} - \frac{1}{m-2} A^{\varphi} \owedge \langle\, ,\, \rangle,$$

that provides a link between the geometry of the base manifold $(M, \langle\, ,\, \rangle)$ and of the map $\varphi : M \to (N, \langle\, ,\, \rangle_N)$, and then apply to it our arguments for generic algebraic curvature tensors. For instance, in the compact case we will prove the following

Theorem 2.3. *Let $(M, \langle\, ,\, \rangle)$ be a compact m-dimensional manifold with $\lfloor \frac{m-1}{2} \rfloor$-nonnegative curvature operator and let $\varphi : M \to (N, \langle\, ,\, \rangle_N)$ be a smooth map from M to a fixed Riemannian manifold $(N, \langle\, ,\, \rangle_N)$. Then, if W^{φ} is harmonic, φ must be relatively affine and $(M, \langle\, ,\, \rangle)$ is either locally symmetric or locally conformally flat and φ is an homothety.*

As for the complete case, an application of a classification result due to G. Carron and H. Herzlich [11] for complete locally conformally flat manifolds with non-negative Ricci curvature yields, in case S^{φ} is constant, rigidity for the base manifold.

Notice that here we focus our attention to the φ curvatures and applications of 2.1 and the analogous results in the complete non-compact case to the φ-Weyl tensor. For a detailed treatment on the generalization of the Tachibana theorem and therefore to the applications in case $T = \mathrm{Riem}$, we refer the reader to [13].

2.1 Rigidity of harmonic algebraic curvature tensors on compact manifolds

This section is devoted to the proof of Theorem 2.1, that we here restate:

Theorem 2.4. *Let $(M^m, \langle\, ,\, \rangle)$ be a compact m-dimensional Riemannian manifold with $\lfloor \frac{m-1}{2} \rfloor$-nonnegative curvature operator $\mathfrak{R}$ and let T be an algebraic curvature tensor. Assume that*

$$T \text{ satisfies the second Bianchi identity and } \operatorname{div} T = 0.$$

Then T is parallel. Moreover, if the curvature operator is $\lfloor \frac{m-1}{2} \rfloor$-positive at some point, then T is a constant multiple of $\langle\, ,\, \rangle \owedge \langle\, ,\, \rangle$.

We will prove Theorem 2.4 at the end of the section. Indeed, we begin by showing the validity of a Bochner-type formula for generic algebraic curvature tensors in Proposition 2.5, and then in Theorem 2.7 we will give a sharp estimate on the quadratic term arising in the expression of the formula itself. In what follows until the end of the section, we will use the same notation as in Section 1.2 for algebraic curvature tensors: indeed, there will be no ambiguity, since the only curvature tensors coming into play here are the Riemann and the Ricci tensors.
Before stating Proposition 2.5, we set $\Gamma = \Gamma_q : T^0_q M \to T^0_q M$ as the self-adjoint endomorphism introduced by Lichnerowicz, whose action can be described in the following way: in a local orthonormal coframe $\{\theta^i\}$, for every q-covariant tensor $Q = Q_{i_1 \dots i_q}\, \theta^{i_1} \otimes \cdots \otimes \theta^{i_q}$ the components of $\Gamma Q = (\Gamma Q)_{i_1 \dots i_q}\, \theta^{i_1} \otimes \cdots \otimes \theta^{i_q}$ are given by

$$(\Gamma Q)_{i_1 \dots i_q} = \sum_{l=1}^{q} R_{i_l j} Q_{i_1 \dots j \dots i_q} - \sum_{1 \le l \neq h \le q} R_{i_l j i_h t} Q_{i_1 \dots j \dots t \dots i_q} \tag{2.1.1}$$

where on the right-hand side j and t occupy the l-th and h-th places, respectively, among the indices of Q (note that in the second term we do not necessarily have $l < h$). If Q is twice continuously differentiable, then

$$(\Gamma Q)_{i_1 \dots i_q} = \sum_{h=1}^{q} \left(Q_{i_1 \dots t \dots i_q, i_h t} - Q_{i_1 \dots t \dots i_q, t i_h} \right) . \tag{2.1.2}$$

Indeed, by the Ricci identities we have

$$Q_{i_1 \dots t \dots i_q, i_h t} - Q_{i_1 \dots t \dots i_q, t i_h} = \sum_{l \neq h, l=1}^{q} Q_{i_1 \dots j \dots t \dots i_q} R_{j i_l i_h t} + Q_{i_1 \dots j \dots i_q} R_{j i_h} ,$$

thus summing over h we obtain (2.1.1). As we will often apply $\Gamma = \Gamma_4$ in the case of an algebraic curvature tensor T, it is worth noticing that in this case (2.1.1), thanks to the symmetries of T, becomes

$$\begin{aligned}(\Gamma T)_{ijkl} = {} & R_{is}T_{sjkl} + R_{js}T_{iskl} + R_{ks}T_{ijsl} + R_{ls}T_{ijks} - 2R_{isjt}T_{stkl} - 2R_{iskt}T_{sjtl} \\ & - 2R_{islt}T_{sjkt} - 2R_{jskt}T_{istl} - 2R_{jslt}T_{iskt} - 2R_{kslt}T_{ijst}.\end{aligned} \tag{2.1.3}$$

We are now ready to state and prove the following:

Proposition 2.5. *Let $(M, \langle \, , \, \rangle)$ be a Riemannian manifold and let T be a smooth algebraic curvature tensor. Then*

$$\frac{1}{2}\Delta|T|^2 = |\nabla T|^2 + \frac{1}{2}\langle \Gamma T, T\rangle - \frac{1}{3}|B(T)|^2 - 2|\operatorname{div} T|^2 + \operatorname{div} X(T) \tag{2.1.4}$$

where $X(T)$ is the vector field whose components are given by

$$X(T)_i = T_{sjkt} B(T)_{sjkti} + 2T_{ijkt}(\operatorname{div} T)_{jkt} . \tag{2.1.5}$$

Remark 2.6. In particular, if T is harmonic (i.e., T satisfies the second Bianchi identity and $\operatorname{div} T = 0$) then (2.1.4) reduces to

$$\frac{1}{2}\Delta|T|^2 = |\nabla T|^2 + \frac{1}{2}\langle \Gamma T, T\rangle . \tag{2.1.6}$$

Proof. We compute

$$\frac{1}{2}\Delta|T|^2 = \operatorname{div}(\nabla|T|^2) = (T_{ijkt}T_{ijkt,l})_{,l} = T_{ijkt,l}T_{ijkt,l} + T_{ijkt}T_{ijkt,ll}$$

and $T_{ijkt,l}T_{ijkt,l} = |\nabla T|^2$. Looking at the second term, we rewrite

$$T_{ijkt,ll} = T_{ijkt,ll} + T_{ijlk,tl} + T_{ijtl,kl} - T_{ijlk,tl} - T_{ijtl,kl} = B(T)_{ijktl,l} + T_{ijkl,tl} - T_{ijtl,kl}$$

so that, using the symmetry $T_{ijtk} = -T_{ijkt}$,

$$T_{ijkt}T_{ijkt,ll} = T_{ijkt}B(T)_{ijktl,l} + 2T_{ijkt}T_{ijkl,tl} .$$

We further rewrite

$$T_{ijkl,tl} = T_{ijkl,lt} + T_{ijkl,tl} - T_{ijkl,lt}$$

and summing up we obtain

$$\frac{1}{2}\Delta|T|^2 = |\nabla T|^2 + T_{ijkt}B(T)_{ijktl,l} + 2T_{ijkt}T_{ijkl,lt} + 2T_{ijkt}(T_{ijkl,tl} - T_{ijkl,lt}) . \tag{2.1.7}$$

"Integrating by parts" we get

$$\begin{aligned} T_{ijkt}B(T)_{ijktl,l} &= \left(T_{ijkt}B(T)_{ijktl}\right)_{,l} - T_{ijkt,l}B(T)_{ijktl} \\ &= \left(T_{ijkt}B(T)_{ijktl}\,\right)_{,l} - \frac{1}{3}|B(T)|^2 , \\ T_{ijkt}T_{ijkl,lt} &= \left(T_{ijkt}T_{ijkl,l}\right)_{,t} - T_{ijkt,t}T_{ijkl,l} \\ &= \left(T_{ijkt}T_{ijkl,l}\right)_{,t} - |\operatorname{div} T|^2 \end{aligned}$$

hence

$$T_{ijkt}B(T)_{ijktl,l} + 2T_{ijkt}T_{ijkl,lt} = \operatorname{div} X(T) - \frac{1}{3}|B(T)|^2 - 2|\operatorname{div} T|^2 \,. \tag{2.1.8}$$

On the other hand, by the symmetries of T, renaming the indices and using (2.1.2) we have

$$\begin{aligned}
4T_{ijkt}(T_{ijkl,tl} - T_{ijkl,lt}) &= T_{ijkt}(T_{ijkl,tl} - T_{ijkl,lt}) + T_{ijtk}(T_{ijlk,tl} - T_{ijlk,lt}) \\
&\quad + T_{ktij}(T_{klij,tl} - T_{klij,lt}) + T_{tkij}(T_{lkij,tl} - T_{lkij,lt}) \\
&= T_{ijkt}(T_{ijkl,tl} - T_{ijkl,lt}) + T_{ijkt}(T_{ijlt,kl} - T_{ijlt,lk}) \\
&\quad + T_{ijkt}(T_{ilkt,jl} - T_{klij,lj}) + T_{ijtk}(T_{ljtk,il} - T_{ljtk,li}) \\
&= \langle \Gamma T, T \rangle \,.
\end{aligned}$$

Substituting this and (2.1.8) into (2.1.7) we obtain the desired conclusion. □

Having in hand the desired Bochner-type formula (2.1.4), the next step is to find an estimate for the quantity $\langle \Gamma T, T \rangle$. Our goal is now to prove the following Theorem:

Theorem 2.7. *Let M be a Riemannian manifold of dimension $m \geq 2$ with $\mathfrak{R}^{(\lfloor \frac{m-1}{2} \rfloor)} \geq a(x)$ for some function $a : M \to \mathbb{R}$ and let T be a smooth algebraic curvature tensor. Then*

$$\frac{1}{2}\Delta|T|^2 \geq |\nabla T|^2 + (m-1)a(x)|P|^2 - \frac{1}{3}|B(T)|^2 - 2|\operatorname{div} T|^2 + \operatorname{div} X(T) \tag{2.1.9}$$

where P and $X(T)$ are as in (1.2.12) *and* (2.1.5)*.*

The proof of Theorem 2.7 is split into several lemmas and propositions, for the sake of clarity of the exposition. The scheme of the proof is the following: in the first step, given by Proposition 2.8, we split the quadratic term $\langle \Gamma T, \tilde{T} \rangle$ as the sum of two analogous quadratic terms in W and Z, thus decomposing, in some sense, the expression in accordance with the orthogonal splitting of T. We will afterwards treat the terms $\langle \Gamma W, W \rangle$ and $\langle \Gamma Z, Z \rangle$ separately and give lower bounds on them exploiting the non-negativity of $\mathfrak{R}^{\lfloor \frac{m-1}{2} \rfloor}$. As for $\langle \Gamma Z, Z \rangle$, in Proposition 2.14 we will show, with the help of Lemma 2.13, that a lower bound on the weighted $\lfloor \frac{m}{2} \rfloor$-sectional curvature yields a lower bound on $\langle \Gamma Z, Z \rangle$: namely, if $\mathrm{Sect}^{\lfloor \frac{m}{2} \rfloor} \geq C$, then

$$\langle \Gamma Z, Z \rangle \geq 2mC|Z|^2.$$

On the other hand, in order to estimate $\langle \Gamma W, W \rangle$ we will link Γ to $\mathfrak{R}$ through a map that sends a 4-covariant tensor Q to a 2-form with values in T_4M, $\hat{Q}$, similarly to what Petersen and Wink did in [30], and we will show that, extending the curvature operator to the space of 2-forms with values in T_4M, then

$$\langle \mathfrak{R}^{T_4M}\hat{Q}, \hat{V} \rangle = \langle \Gamma Q, V \rangle,$$

for any $Q, V \in T_4M$. In Proposition 2.15 we will express the norm of $\hat{W}$ with respect to the norm of W, finding that

$$|\hat{W}|^2 = 2(m-1)|W|^2.$$

After that, in Proposition 2.17 we will explain $\langle \mathfrak{R}^{T_4M}\hat{W}, \hat{W} \rangle$ by splitting the curvature operator in an orthonormal base of 2-forms and, thanks to the estimate proven in Lemma 2.16 we will be able to apply an elementary inequality shown in Lemma 2.13 to conclude that, if $\mathfrak{R}^{\lfloor \frac{m-1}{2} \rfloor} \geq C$, then

$$\langle \Gamma W, W \rangle \geq 2(m-1)C|W|^2.$$

Theorem 2.18 will put together the lower bounds in Propositions 2.14 and 2.17, using the expression for the norm of the projective-like tensor P in 1.10, to obtain that, in case $\mathfrak{R}^{\lfloor \frac{m-1}{2} \rfloor} \geq C$, then

$$\langle \Gamma T, T \rangle \geq 2(m-1)C|P|^2.$$

Then it will be only a matter of estimating the quadratic term in the Bochner formula proven in 2.5 by means of Theorem 2.18 to obtain the conclusion of Theorem 2.7.

Proposition 2.8. *Let T, $\tilde{T}$ be algebraic tensor fields. Then*

$$\langle \Gamma T, \tilde{T} \rangle = \langle \Gamma W, \tilde{W} \rangle + \frac{4}{m-2} \langle \Gamma Z, \tilde{Z} \rangle \tag{2.1.10}$$

where W, $\tilde{W}$ are the Weyl parts of T, $\tilde{T}$ and Z, $\tilde{Z}$ are the traceless parts of their respective Ricci contractions E, $\tilde{E}$.

The proof of Proposition 2.8 is essentially a long computation, that we split into the proofs of several lemmas.

Lemma 2.9. *Let $T, \tilde{T}$ be algebraic curvature tensors. Then*

$$\langle \Gamma T, \tilde{T} \rangle = 4 R_{is} T_{sjkt} \tilde{T}_{ijkt} - 4 R_{isjl} T_{slkt} \tilde{T}_{ijkt} - 8 R_{iskl} T_{sjlt} \tilde{T}_{ijkt} \,. \tag{2.1.11}$$

Proof. From (2.1.3) we have

$$\begin{aligned}(\Gamma T)_{ijkl} = {} & R_{is} T_{sjkl} + R_{js} T_{iskl} + R_{ks} T_{ijsl} + R_{ls} T_{ijks} \\ & - 2 R_{isjt} T_{stkl} - 2 R_{iskt} T_{sjtl} - 2 R_{islt} T_{sjkt} - 2 R_{jskt} T_{istl} - 2 R_{jslt} T_{iskt} - 2 R_{kslt} T_{ijst} .\end{aligned}$$

We contract with $\tilde{T}_{ijkl}$. Renaming the indices and using the symmetries of T, $\tilde{T}$ we get

$$\begin{aligned}& R_{is} T_{sjkl} \tilde{T}_{ijkl} + R_{js} T_{iskl} \tilde{T}_{ijkl} + R_{ks} T_{ijsl} \tilde{T}_{ijkl} + R_{ls} T_{ijks} \tilde{T}_{ijkl} \\ = {} & R_{is} T_{sjkl} \tilde{T}_{ijkl} + R_{is} T_{jskl} \tilde{T}_{jikl} + R_{is} T_{klsj} \tilde{T}_{klij} + R_{is} T_{lkjs} \tilde{T}_{lkji} \\ = {} & 4 R_{is} T_{sjkt} \tilde{T}_{ijkt} \,, \\ & 2 R_{isjt} T_{stkl} \tilde{T}_{ijkl} + 2 R_{kslt} T_{ijst} \tilde{T}_{ijkl} \\ = {} & 2 R_{isjt} T_{stkl} \tilde{T}_{ijkl} + 2 R_{isjt} T_{klst} \tilde{T}_{klij} \\ = {} & 4 R_{isjt} T_{stkl} \tilde{T}_{ijkl} \,, \\ & 2 R_{iskt} T_{sjtl} \tilde{T}_{ijkl} + 2 R_{islt} T_{sjkt} \tilde{T}_{ijkl} + 2 R_{jskt} T_{istl} \tilde{T}_{ijkl} + 2 R_{jslt} T_{iskt} \tilde{T}_{ijkl} \\ = {} & 2 R_{iskt} T_{sjtl} \tilde{T}_{ijkl} + 2 R_{iskt} T_{sjlt} \tilde{T}_{ijlk} + 2 R_{iskt} T_{jstl} \tilde{T}_{jikl} + 2 R_{iskt} T_{jslt} \tilde{T}_{jilk} \\ = {} & 8 R_{iskt} T_{sjtl} \tilde{T}_{ijkl} \,.\end{aligned}$$

Summing up we obtain (2.1.11). ☐

Next, we show that the traceless and the trace components of an algebraic curvature tensor are orthogonal with respect to the bilinear form $\langle \Gamma \cdot, \cdot \rangle$:

Lemma 2.10. *Let W be a totally traceless algebraic curvature tensor and E a symmetric 2-covariant tensor. Then*

$$\langle \Gamma W, E \owedge \langle\, ,\, \rangle \rangle = 0 \,. \tag{2.1.12}$$

Proof. We apply (2.1.11) with $T = W$ and $\tilde{T} = E \owedge \langle\, ,\, \rangle$. We write

$$(E \owedge \langle\, ,\, \rangle)_{ijkt} = E_{ik} \delta_{jt} + E_{jt} \delta_{ik} - E_{it} \delta_{jk} - E_{jk} \delta_{it} \,.$$

We separately compute the three terms in (2.1.11). Since W is totally traceless, we have

$$\begin{aligned}R_{is} W_{sjkt} (E \owedge \langle\, ,\, \rangle)_{ijkt} &= R_{is} W_{sjkt} E_{jt} \delta_{ik} - R_{is} W_{sjkt} E_{jk} \delta_{it} \\ &= R_{is} W_{sjit} E_{jt} - R_{is} W_{sjki} E_{jk} \\ &= 2 R_{is} W_{sjit} E_{jt} \,, \\ R_{isjl} W_{slkt} (E \owedge \langle\, ,\, \rangle)_{ijkt} &= R_{isjl} W_{slkt} (E_{ik} \delta_{jt} - E_{it} \delta_{jk}) + R_{isjl} W_{slkt} (E_{jt} \delta_{ik} - E_{jk} \delta_{it}) \\ &= 2 R_{isjl} W_{slkt} E_{ik} \delta_{jt} + 2 R_{isjl} W_{slkt} E_{jt} \delta_{ik} \\ &= 2 R_{isjl} W_{slkj} E_{ik} + 2 R_{isjl} W_{slit} E_{jt} \,, \\ R_{iskl} W_{sjlt} (E \owedge \langle\, ,\, \rangle)_{ijkt} &= R_{iskl} W_{sjlt} E_{jt} \delta_{ik} - R_{iskl} W_{sjlt} E_{it} \delta_{jk} - R_{iskl} W_{sjlt} E_{jk} \delta_{it} \\ &= R_{sl} W_{sjlt} E_{jt} - R_{iskl} W_{sklt} E_{it} - R_{iskl} W_{sjli} E_{jk} \,.\end{aligned}$$

Summing up, we obtain

$$\begin{aligned}\frac{1}{8}\langle \Gamma W, E \owedge g\rangle = & R_{is}W_{sjit}E_{jt} - R_{isjl}W_{slkj}E_{ik} - R_{isjl}W_{slit}E_{jt}\\ & - R_{sl}W_{sjlt}E_{jt} + R_{iskl}W_{sklt}E_{it} + R_{iskl}W_{sjli}E_{jk}\,.\end{aligned}$$

A few algebraic manipulations renaming the indices and exploiting the symmetries of Ric, Riem, W or E yield

$$\begin{aligned}R_{is}W_{sjit}E_{jt} &= R_{ls}W_{sjlt}E_{jt} = R_{sl}W_{sjlt}E_{jt}\,,\\ R_{isjl}W_{slkj}E_{ik} &= R_{jlis}W_{lsjk}E_{ik} = R_{isjl}W_{slit}E_{jt}\\ R_{iskl}W_{sklt}E_{it} &= R_{klis}W_{ltsk}E_{ti} = R_{iskl}W_{sjli}E_{jk}\,,\end{aligned}$$

Substituting and manipulating a little more we get

$$\begin{aligned}\frac{1}{16}\langle \Gamma W, E \owedge \langle\,,\,\rangle\rangle &= R_{iskl}W_{sklt}E_{it} - R_{isjl}W_{slkj}E_{ik}\\ &= R_{iskl}W_{sklt}E_{it} + R_{isjl}W_{sljk}E_{ik}\\ &\equiv R_{iskl}W_{sklt}E_{it} + R_{islk}W_{sklt}E_{it}\\ &= (R_{iskl} + R_{islk})W_{sklt}E_{it}\\ &= 0\,.\end{aligned}$$

□

In the next two lemmas we relate the bilinear form $\langle \Gamma\cdot,\cdot\rangle$ on the trace part of algebraic curvature tensors to their Ricci contraction, with the help of Lemma 2.9, and we observe that the bilinear form loses information on the total trace of algebraic curvature tensors.

Lemma 2.11. *Let $E, \tilde{E}$ be symmetric 2-covariant tensors and let $Z, \tilde{Z}$ be their respective traceless parts. Then*

$$\langle \Gamma Z, \tilde{Z}\rangle = \langle \Gamma E, \tilde{E}\rangle = 2R_{is}E_{sj}\tilde{E}_{ij} - 2R_{isjl}E_{ls}\tilde{E}_{ij}\,. \tag{2.1.13}$$

Proof. From (2.1.1) we have

$$(\Gamma E)_{ij} = R_{is}E_{sj} + R_{js}E_{is} - 2R_{isjl}E_{sl}\,. \tag{2.1.14}$$

Contracting with $\tilde{E}_{ij}$ we get the second equality (2.1.13). In case $E = \langle\,,\,\rangle$, from (2.1.14) we deduce $\Gamma\langle\,,\,\rangle = 0$. As for the first equality in (2.1.13),

$$\begin{aligned}\langle \Gamma E, \tilde{E}\rangle &= \langle \Gamma Z, \tilde{Z}\rangle + \langle \Gamma Z, \frac{\tilde{S}}{m}\langle\,,\,\rangle\rangle + \langle \Gamma \frac{S}{m}\langle\,,\,\rangle, \tilde{E}\rangle\\ &= \langle \Gamma Z, \tilde{Z}\rangle + \langle Z, \Gamma \frac{\tilde{S}}{m}\langle\,,\,\rangle\rangle + \langle \Gamma \frac{S}{m}\langle\,,\,\rangle, \tilde{E}\rangle\\ &= \langle \Gamma Z, \tilde{Z}\rangle\,,\end{aligned}$$

where we have used the self-adjointness of Γ on T^0_2M. □

Lemma 2.12. *Let $E, \tilde{E}$ be symmetric 2-covariant tensors and let $Z, \tilde{Z}$ be their respective traceless parts. Then*

$$\langle \Gamma(E \owedge \langle\,,\,\rangle), \tilde{E} \owedge \langle\,,\,\rangle\rangle = 4(m-2)\langle \Gamma E, \tilde{E}\rangle = 4(m-2)\langle \Gamma Z, \tilde{Z}\rangle\,. \tag{2.1.15}$$

Proof. We separately compute the three terms in (2.1.11) for $T = E \owedge \langle\,,\,\rangle$, $\tilde{T} = \tilde{E} \owedge \langle\,,\,\rangle$. We have

$$\begin{aligned}(E \owedge g)_{slkt}(\tilde{E} \owedge \langle\,,\,\rangle)_{ijkt} &= E_{sk}(\tilde{E} \owedge \langle\,,\,\rangle)_{ijkl} - E_{st}(\tilde{E} \owedge \langle\,,\,\rangle)_{ijlt} + E_{lt}(\tilde{E} \owedge \langle\,,\,\rangle)_{ijst} - E_{lk}(\tilde{E} \owedge \langle\,,\,\rangle)_{ijks}\\ &= 2E_{sk}(\tilde{E} \owedge \langle\,,\,\rangle)_{ijkl} - 2E_{lk}(\tilde{E} \owedge \langle\,,\,\rangle)_{ijks}\\ &= 2E_{sk}\tilde{E}_{ik}\delta_{jl} + 2E_{si}\tilde{E}_{jl} - 2E_{sk}\tilde{E}_{jk}\delta_{il} - 2E_{sj}\tilde{E}_{il}\\ &\quad - 2E_{lk}\tilde{E}_{ik}\delta_{js} - 2E_{li}\tilde{E}_{js} + 2E_{lk}\tilde{E}_{jk}\delta_{is} + 2E_{lj}\tilde{E}_{is}\,.\end{aligned}$$

Hence,

$$
\begin{aligned}
R_{is}(E \owedge \langle\, , \rangle)_{sjkt}(\tilde{E} \owedge \langle\, , \rangle)_{ijkt} &= R_{is}\delta_{lj}(E \owedge \langle\, , \rangle)_{slkt}(\tilde{E} \owedge \langle\, , \rangle)_{ijkt} \\
&= 2mR_{is}E_{sk}\tilde{E}_{ik} + 2R_{is}E_{si}\tilde{E}_{jj} - 2R_{is}E_{sk}\tilde{E}_{ik} - 2R_{is}E_{sl}\tilde{E}_{il} \\
&\quad - 2R_{il}E_{lk}\tilde{E}_{ik} - 2R_{is}E_{ji}\tilde{E}_{js} + 2R_{ii}E_{jk}\tilde{E}_{jk} + 2R_{is}\tilde{E}_{is}E_{jj} \\
&= 2(m-4)R_{ij}E_{jk}\tilde{E}_{ik} + 2R_{ij}E_{ij}\tilde{E}_{kk} + 2R_{ij}\tilde{E}_{ij}E_{kk} + 2R_{ii}E_{jk}\tilde{E}_{jk}\,, \\
R_{isjl}(E \owedge \langle\, , \rangle)_{slkt}(\tilde{E} \owedge \langle\, , \rangle)_{ijkt} &= 4R_{sj}E_{sk}\tilde{E}_{jk} + 4R_{islj}E_{sj}\tilde{E}_{il}\,.
\end{aligned}
$$

Similarly

$$
\begin{aligned}
(E \owedge \langle\, , \rangle)_{sjlt}(\tilde{E} \owedge \langle\, , \rangle)_{ijkt} &= E_{sl}(\tilde{E} \owedge \langle\, , \rangle)_{ijkj} - E_{st}(\tilde{E} \owedge \langle\, , \rangle)_{ilkt} - E_{jl}(\tilde{E} \owedge \langle\, , \rangle)_{ijks} \\
&\quad + E_{jt}\delta_{sl}(\tilde{E} \owedge \langle\, , \rangle)_{ijkt} \\
&= E_{sl}\tilde{E}_{ik}\delta_{jj} + E_{sl}\tilde{E}_{jj}\delta_{ik} - E_{sl}\tilde{E}_{ij}\delta_{jk} - E_{sl}\tilde{E}_{jk}\delta_{ij} \\
&\quad - E_{st}\tilde{E}_{ik}\delta_{lt} - E_{st}\tilde{E}_{lt}\delta_{ik} + E_{st}\tilde{E}_{it}\delta_{lk} + E_{st}\tilde{E}_{lk}\delta_{it} \\
&\quad - E_{jl}\tilde{E}_{ik}\delta_{js} - E_{jl}\tilde{E}_{js}\delta_{ik} + E_{jl}\tilde{E}_{is}\delta_{jk} + E_{jl}\tilde{E}_{jk}\delta_{is} \\
&\quad + E_{jt}\tilde{E}_{ik}\delta_{sl}\delta_{jt} + E_{jt}\tilde{E}_{jt}\delta_{sl}\delta_{ik} - E_{jt}\tilde{E}_{it}\delta_{sl}\delta_{jk} - E_{jt}\tilde{E}_{jk}\delta_{sl}\delta_{it} \\
&= mE_{sl}\tilde{E}_{ik} + E_{sl}\tilde{E}_{jj}\delta_{ik} - E_{sl}\tilde{E}_{ik} - E_{sl}\tilde{E}_{ik} \\
&\quad - E_{sl}\tilde{E}_{ik} - E_{st}\tilde{E}_{lt}\delta_{ik} + E_{st}\tilde{E}_{it}\delta_{lk} + E_{si}\tilde{E}_{lk} \\
&\quad - E_{sl}\tilde{E}_{ik} - E_{lj}\tilde{E}_{js}\delta_{ik} + E_{kl}\tilde{E}_{is} + E_{lj}\tilde{E}_{jk}\delta_{is} \\
&\quad + \tilde{E}_{ik}E_{jj}\delta_{sl} + E_{jt}\tilde{E}_{jt}\delta_{sl}\delta_{ik} - E_{kt}\tilde{E}_{it}\delta_{sl} - E_{ij}\tilde{E}_{jk}\delta_{sl}
\end{aligned}
$$

that is

$$
\begin{aligned}
(E \owedge \langle\, , \rangle)_{sjlt}(\tilde{E} \owedge \langle\, , \rangle)_{ijkt} &= (m-4)E_{sl}\tilde{E}_{ik} - E_{st}\tilde{E}_{lt}\delta_{ik} - E_{lj}\tilde{E}_{js}\delta_{ik} - E_{kt}\tilde{E}_{it}\delta_{sl} - E_{ij}\tilde{E}_{jk}\delta_{sl} \\
&\quad + E_{sl}\tilde{E}_{jj}\delta_{ik} + \tilde{E}_{ik}E_{jj}\delta_{sl} + E_{jt}\tilde{E}_{jt}\delta_{sl}\delta_{ik} + E_{st}\tilde{E}_{it}\delta_{lk} + E_{si}\tilde{E}_{lk} + E_{kl}\tilde{E}_{is} + E_{lj}\tilde{E}_{jk}\delta_{is}\,.
\end{aligned}
$$

We contract with R_{iskl} to get

$$
\begin{aligned}
R_{iskl}(E \owedge \langle\, , \rangle)_{sjlt}(\tilde{E} \owedge \langle\, , \rangle)_{ijkt} &= (m-4)R_{iskl}E_{sl}\tilde{E}_{ik} - 4R_{sl}E_{st}\tilde{E}_{lt} \\
&\quad + R_{sl}E_{sl}\tilde{E}_{jj} + R_{ik}\tilde{E}_{ik}E_{jj} + R_{ii}E_{jt}\tilde{E}_{jt}
\end{aligned}
$$

Summing up,

$$
\begin{aligned}
\langle \Gamma(E \owedge \langle\, , \rangle), \tilde{E} \owedge \langle\, , \rangle\rangle &= 8(m-4)R_{ij}E_{jk}\tilde{E}_{ik} + 8R_{ij}E_{ij}\tilde{E}_{kk} + 8R_{ij}\tilde{E}_{ij}E_{kk} + 8R_{ii}E_{jk}\tilde{E}_{jk} \\
&\quad - 16R_{sj}E_{sk}\tilde{E}_{jk} - 16R_{islj}E_{sj}\tilde{E}_{il} - 8(m-4)R_{iskl}E_{sl}\tilde{E}_{ik} + 32R_{sl}E_{st}\tilde{E}_{lt} \\
&\quad - 8R_{sl}E_{sl}\tilde{E}_{jj} - 8R_{ik}\tilde{E}_{ik}E_{jj} - 8R_{ii}E_{jt}\tilde{E}_{jt} \\
&= 8(m-2)R_{ij}E_{jk}\tilde{E}_{ik} - 8(m-2)R_{iskl}E_{sl}\tilde{E}_{ik}
\end{aligned}
$$

and by (2.1.13) we obtain (2.1.15). □

We are now able to prove the splitting in $\langle \Gamma T, \tilde{T}\rangle$ as in (2.1.10):

Proof of Proposition 2.8. As in (1.2.7), we write

$$
T = W + \frac{1}{m-2}A \owedge \langle\, , \rangle\,, \qquad \tilde{T} = \tilde{W} + \frac{1}{m-2}\tilde{A} \owedge \langle\, , \rangle
$$

where A, $\tilde{A}$ are the Schouten-like tensors associated to T, $\tilde{T}$ as in (1.2.6). Then, by Lemma 2.10 with the choices $E = \frac{1}{m-2}A$, $\tilde{E} = \frac{1}{m-2}\tilde{A}$ to obtain

$$
\langle \Gamma T, \tilde{T}\rangle = \langle \Gamma W, \tilde{W}\rangle + \frac{1}{(m-2)^2}\langle \Gamma(A \owedge \langle\, , \rangle), \tilde{A} \owedge \langle\, , \rangle\rangle.
$$

Next, by Lemma 2.12, noting that Z, $\tilde{Z}$ are also the traceless parts of A, $\tilde{A}$, we get

$$\langle \Gamma T, \tilde{T} \rangle = \langle \Gamma W, \tilde{W} \rangle + \frac{4}{m-2} \langle \Gamma Z, \tilde{Z} \rangle,$$

that is, (2.1.10). $\square$

We now proceed to prove that a lower bound on $\mathfrak{R}^{(\lfloor \frac{m-1}{2} \rfloor)}$ ensures a lower bound on $\langle \Gamma T, T \rangle$ for any algebraic curvature tensor T. To this purpose, we shall introduce the map we mentioned above, which provides a link between the endomorphism Γ and the curvature operator $\mathfrak{R}$. Namely, adopting the notation used by Petersen and Wink in [30], to any q-covariant tensor Q we associate a $T^0_q M$-valued 2-form $\hat{Q}$ of local components

$$\hat{Q}_{i_1 \dots i_q sr} = \frac{1}{2} \sum_{l=1}^{q} Q_{i_1 \dots s \dots i_q} \delta_{i_l r} - \frac{1}{2} \sum_{l=1}^{q} Q_{i_1 \dots r \dots i_q} \delta_{i_l s} \,. \tag{2.1.16}$$

Any symmetry that Q may enjoy is inherited by $\hat{Q}$ in its first q indices.

The curvature operator $\mathfrak{R}$ can be extended to a self-adjoint endomorphism

$$\mathfrak{R}^{T^0_q M} : T^0_q M \otimes \wedge^2 M \to T^0_q M \otimes \wedge^2 M$$

on the bundle $T^0_q M \otimes \wedge^2 M$ of $T^0_q M$-valued 2-forms, where self-adjointness is intended with respect to the inner product on $T^0_{q+2} M \supseteq T^0_q M \otimes \wedge^2 M$. Given a local orthonormal coframe $\{\theta^i\}$ on M, for any section $\omega = \omega_{i_1 \dots i_q sr} \, \theta^{i_1} \otimes \dots \otimes \theta^{i_q} \otimes \theta^s \otimes \theta^r$ of $T^0_q M \otimes \wedge^2 M$ the tensor $\mathfrak{R}^{T^0_q M} \omega$ is locally defined by

$$(\mathfrak{R}^{T^0_q M} \omega)_{i_1 \dots i_q kt} = R_{srkt} \omega_{i_1 \dots i_q sr} \,.$$

We have

$$(\mathfrak{R}^{T^0_q M} \hat{Q})_{i_1 \dots i_q kt} = \sum_{l=1}^{q} R_{s i_l kt} Q_{i_1 \dots s \dots i_q} = - \sum_{l=1}^{q} R_{i_l jkt} Q_{i_1 \dots j \dots i_q}$$

and if V is another tensor field of type $(0, q)$ then

$$\langle \mathfrak{R}^{T^0_q M} \hat{Q}, \hat{V} \rangle = - \sum_{l,h=1}^{q} R_{i_l jkt} Q_{i_1 \dots j \dots i_q} V_{i_1 \dots k \dots i_q} \delta_{i_h t} \,,$$

where on the right-hand side j occupies the l-th place among the indices of Q and k occupies the h-th place among the indices of V. Splitting the cases $h = l$ and $h \neq l$, we get

$$\langle \mathfrak{R}^{T^0_q M} \hat{Q}, \hat{V} \rangle = \sum_{l=1}^{q} R_{jk} Q_{i_1 \dots j \dots i_q} V_{i_1 \dots k \dots i_q} - \sum_{1 \leq l \neq h \leq q} R_{i_l jkt} Q_{i_1 \dots j \dots t \dots i_q} V_{i_1 \dots k \dots i_q}$$

and renaming $k = i_h$ it is apparent that

$$\langle \mathfrak{R}^{T^0_q M} \hat{Q}, \hat{V} \rangle = \langle \Gamma Q, V \rangle \,. \tag{2.1.17}$$

This shows that a lower bound on the quadratic form $\langle \mathfrak{R} \cdot, \cdot \rangle$ implies a lower bound on $\langle \Gamma \cdot, \cdot \rangle$. The next Lemma provides an elementary inequality that will be used in the following Propositions.

Lemma 2.13. *Let $N \geq 2$ be a positive integer and let $\{a_i\}_{1 \leq i \leq N}$, $\{b_i\}_{1 \leq i \leq N}$ be sequences of non-negative real numbers such that*

$$a_i \leq a_{i+1} \quad \text{for } 1 \leq i < N \qquad \text{and} \qquad b_i \geq 0 \quad \text{for } i = 1, \dots, N \,. \tag{2.1.18}$$

Let $1 \leq k < N$ be a real number such that

$$b_i \leq \frac{1}{k} \sum_{j=1}^{N} b_j \qquad \text{for } i = 1, \dots, N \,. \tag{2.1.19}$$

Then

$$\sum_{i=1}^{N} a_i b_i \geq \frac{1}{\lfloor k \rfloor} \sum_{i=1}^{\lfloor k \rfloor} a_i \sum_{j=1}^{N} b_j \,. \tag{2.1.20}$$

Proof. Let us assume first that k is an integer. Then, we separately estimate

$$\sum_{i=k+1}^{N} a_i b_i \geq a_{k+1} \sum_{i=k+1}^{N} b_i \,,$$
$$\sum_{i=1}^{k} a_i b_i = \sum_{i=1}^{k} (a_i - a_{k+1}) b_i + a_{k+1} \sum_{i=1}^{k} b_i \geq \frac{1}{k} \sum_{i=1}^{k} (a_i - a_{k+1}) \sum_{j=1}^{N} b_j + a_{k+1} \sum_{i=1}^{k} b_i \,,$$

where we have used (2.1.19) and the fact that $a_i - a_{k+1} \leq 0$ for $i \leq k$. Summing up,

$$\sum_{i=1}^{N} a_i b_i \geq \left[\frac{1}{k} \sum_{i=1}^{k} (a_i - a_{k+1}) + a_{k+1} \right] \sum_{j=1}^{N} b_j = \left(\frac{1}{k} \sum_{i=1}^{k} a_i \right) \sum_{j=1}^{N} b_j \,,$$

i.e.

$$\sum_{i=1}^{N} a_i b_i \geq \left(\frac{1}{k} \sum_{i=1}^{k} a_i \right) \sum_{j=1}^{N} b_j \,. \tag{2.1.21}$$

If k is a real number then (2.1.19) holds with k replaced by $\lfloor k \rfloor$, which is an integer in the range $\{1, \ldots, N-1\}$. Hence it holds (2.1.21), thus proving (2.1.20). □

We are now able to estimate the term $\langle \Gamma Z, Z \rangle$, where Z is any 2-covariant, symmetric and traceless tensor:

Proposition 2.14. *Let $x \in M$, $C \in \mathbb{R}$ and assume that for every collection $\{\pi_1, \ldots, \pi_{\lfloor \frac{m}{2} \rfloor}\}$ of mutually orthogonal 2-dimensional subspaces of $T_x M$ it holds*

$$\frac{1}{\lfloor \frac{m}{2} \rfloor} \sum_{i=1}^{\lfloor \frac{m}{2} \rfloor} \mathrm{Sect}(\pi_i) \geq C \,. \tag{2.1.22}$$

Then for any traceless symmetric 2-covariant tensor Z we have

$$\langle \Gamma Z, Z \rangle \geq 2mC|Z|^2 \qquad \text{at } x. \tag{2.1.23}$$

Proof. Since Z is a symmetric tensor, we can consider an orthonormal base $\{e_i\}$ for TM consisting of eigenvectors of Z. Let us denote with $\{\zeta_i\}_{1 \leq i \leq m}$ the corresponding eigenvalues. From (2.1.14) we obtain (no summation is intended on i)

$$(\Gamma Z)_{ii} = 2R_{ii}\zeta_i - 2\sum_{j=1}^{m} R_{ijij}\zeta_j = 2\sum_{j=1}^{m} R_{ijij}(\zeta_i - \zeta_j) \qquad \text{for } i = 1, \ldots, m$$

hence

$$\langle \Gamma Z, Z \rangle = \sum_{i=1}^{m} (\Gamma Z)_{ii} Z_{ii} = \sum_{i=1}^{m} (\Gamma Z)_{ii}\zeta_i = 2\sum_{i,j=1}^{m} R_{ijij}\zeta_i(\zeta_i - \zeta_j)$$

that can be rewritten, by renaming indices and the symmetries of Riem, as

$$\langle \Gamma Z, Z \rangle = \sum_{i,j=1}^{m} R_{ijij}\zeta_i(\zeta_i - \zeta_j) + \sum_{i,j=1}^{m} R_{jiji}\zeta_j(\zeta_j - \zeta_i) = \sum_{i,j=1}^{m} R_{ijij}(\zeta_i - \zeta_j)^2 \,. \tag{2.1.24}$$

We now want to apply Lemma 2.13 to give a lower bound on $\langle \Gamma Z, Z \rangle$. Since $\sum_{i=1}^{m} \zeta_i = 0$, we have $\sum_{i,j=1}^{m} \zeta_i \zeta_j = 0$ and therefore

$$\begin{aligned} 2 \sum_{1 \leq i < j \leq m} (\zeta_i - \zeta_j)^2 &= \sum_{i,j=1}^{m} (\zeta_i - \zeta_j)^2 \\ &= m\sum_{i=1}^{m} \zeta_i^2 + m\sum_{j=1}^{m} \zeta_j^2 - 2\sum_{i,j=1}^{m} \zeta_i\zeta_j = 2m\sum_{i=1}^{m} \zeta_i^2 = 2m|Z|^2 \,. \end{aligned} \tag{2.1.25}$$

Moreover, for any $1 \leq k < t \leq m$ we have

$$(\zeta_k - \zeta_t)^2 \leq 2(\zeta_k^2 + \zeta_t^2) \leq 2|Z|^2 = \frac{2}{m} \sum_{1\leq i<j\leq m} (\zeta_i - \zeta_j)^2 \,, \tag{2.1.26}$$

where we have used (2.1.25). If we order the set $\{(i,j) \in \mathbb{N}\times\mathbb{N} : 1 \leq i < j \leq m\}$ as a sequence $\{(i_\alpha, j_\alpha)\}_{1\leq\alpha\leq\binom{m}{2}}$ so that

$$R_{i_\alpha j_\alpha i_\alpha j_\alpha} \leq R_{i_\beta j_\beta i_\beta j_\beta} \qquad \forall\, 1 \leq \alpha \leq \beta \leq \binom{m}{2}$$

and we set $a_\alpha = R_{i_\alpha j_\alpha i_\alpha j_\alpha}$, for every $1 \leq \alpha \leq \binom{m}{2}$, the a_α are as in (2.1.18). Then, denoting with $b_\alpha = (\zeta_{i_\alpha} - \zeta_{j_\alpha})^2 \geq 0$, (2.1.26) reads as

$$b_\alpha \leq \frac{2}{m} \sum_{\beta=1}^{\binom{m}{2}} b_\beta \qquad \forall\, 1 \leq \alpha \leq \binom{m}{2}, \tag{2.1.27}$$

hence the condition (2.1.19) is satisfied. Since (2.1.24) and (2.1.25) can be expressed as

$$\langle \Gamma Z, Z\rangle = 2 \sum_{1\leq i<j\leq m} R_{ijij}(\zeta_i - \zeta_j)^2 = 2\sum_{\alpha=1}^{\binom{m}{2}} a_\alpha b_\alpha \,, \qquad \sum_{\alpha=1}^{\binom{m}{2}} b_\alpha = m|Z|^2 \,,$$

we can then apply Lemma 2.13 with $N = \binom{m}{2}$, $k = \frac{m}{2} < \binom{m}{2}$ in order to get

$$\langle \Gamma Z, Z\rangle \geq \frac{2}{\lfloor \frac{m}{2} \rfloor} \sum_{\alpha=1}^{\lfloor \frac{m}{2} \rfloor} a_\alpha \sum_{\beta=1}^{\binom{m}{2}} b_\beta = 2m \frac{1}{\lfloor \frac{m}{2} \rfloor} \sum_{\alpha=1}^{\lfloor \frac{m}{2} \rfloor} R_{i_\alpha j_\alpha i_\alpha j_\alpha} |Z|^2 \,.$$

Using the hypothesis (2.1.22) we conclude

$$\langle \Gamma Z, Z\rangle \geq 2mC|Z|^2 \,.$$

$\square$

Next, we give a lower bound on $\langle \Gamma W, W\rangle$, where W is any traceless algebraic curvature tensor. In particular, the next two lemmas will provide the right conditions to apply Lemma 2.13 in the subsequent Proposition 2.17.

Lemma 2.15 ([36]). *Let T be an algebraic curvature tensor. Then*

$$|\hat{T}|^2 = 2(m-1)|P|^2 \tag{2.1.28}$$

where P is the tensor defined in (1.2.12) *and $\hat{T}$ is defined as in* (2.1.16)*. In particular, if $T = W$ is totally traceless then*

$$|\hat{W}|^2 = 2(m-1)|W|^2 \,. \tag{2.1.29}$$

Proof. From the defining formula (2.1.16) we have

$$2\hat{T}_{ijktsr} = T_{sjkt}\delta_{ir} + T_{iskt}\delta_{jr} + T_{ijst}\delta_{kr} + T_{ijks}\delta_{tr} - T_{rjkt}\delta_{is} - T_{irkt}\delta_{js} - T_{ijrt}\delta_{ks} - T_{ijkr}\delta_{ts} \,.$$

A direct computation, using the symmetries of T, yields

$$\hat{T}_{ijktsr}\hat{T}_{ijktsr} = \hat{T}_{ijktsr}(T_{sjkt}\delta_{ir} + T_{iskt}\delta_{jr} + T_{ijst}\delta_{kr} + T_{ijks}\delta_{tr}) = 4\hat{T}_{ijktsr}T_{sjkt}\delta_{ir}$$

and then

$$\begin{aligned}
2\hat{T}_{ijktsr}T_{sjkt}\delta_{ir} &= T_{sjkt}T_{sjkt}\delta_{ir}\delta_{ir} + T_{iskt}T_{sjkt}\delta_{jr}\delta_{ir} + T_{ijst}T_{sjkt}\delta_{kr}\delta_{ir} + T_{ijks}T_{sjkt}\delta_{tr}\delta_{ir} \\
&\quad - T_{rjkt}T_{sjkt}\delta_{is}\delta_{ir} - T_{irkt}T_{sjkt}\delta_{js}\delta_{ir} - T_{ijrt}T_{sjkt}\delta_{ks}\delta_{ir} - T_{ijkr}T_{sjkt}\delta_{ts}\delta_{ir} \\
&= mT_{sjkt}T_{sjkt} + T_{iskt}T_{sikt} + T_{ijst}T_{sjit} + T_{ijks}T_{sjki} - T_{rjkt}T_{rjkt} - 2E_{jt}E_{jt} \\
&= (m-1)|T|^2 - 2|E|^2 + T_{iskt}(T_{sikt} + T_{ksit} + T_{tski}) \\
&= (m-1)|T|^2 - 2|E|^2 + T_{iskt}(T_{sikt} + T_{ksit} + T_{ikst}) \\
&= (m-1)|T|^2 - 2|E|^2
\end{aligned}$$

where in the last equality we have used the fact that T satisfies the first Bianchi identity. The conclusion then follows by (1.2.13). $\square$

Lemma 2.16 ([30])**.** *Let T be an algebraic curvature tensor and ω a 2-form. Then*

$$\langle\langle\hat{T},\omega\rangle_{\Lambda^2(M)},\langle\hat{T},\omega\rangle_{\Lambda^2(M)}\rangle\leq 4\langle\omega,\omega\rangle_{\Lambda^2(M)}\langle T,T\rangle\,, \tag{2.1.30}$$

where $\langle\,,\,\rangle_{\Lambda^2(M)}$ here denotes the contraction of the last two indices of T with ω.

Proof. Choosing an orthonormal coframe, (2.1.30) can be written as

$$\omega_{ij}\omega_{kt}\hat{T}_{abcdij}\hat{T}_{abcdkt}\leq 4\omega_{ij}\omega_{ij}T_{abcd}T_{abcd}\,. \tag{2.1.31}$$

We observe that

$$\omega_{ij}\widehat{T}_{abcdij}=\omega_{ia}T_{ibcd}+\omega_{ib}T_{aicd}+\omega_{ic}T_{abid}+\omega_{id}T_{abci}\,. \tag{2.1.32}$$

Then, we assume that the coframe $\{\theta^i\}$ is chosen so that ω can be expressed as

$$\omega=\omega_{12}\,\theta^1\wedge\theta^2+\omega_{34}\,\theta^3\wedge\theta^4+\cdots+\omega_{2k-1,2k}\,\theta^{2k-1}\wedge\theta^{2k}$$

with $k=\lfloor\frac{m}{2}\rfloor$. For every $1\leq a\leq m$, set

$$a'=\begin{cases}a-1 & \text{if } a\leq 2k,\ a \text{ even},\\ a+1 & \text{if } a\leq 2k,\ a \text{ odd},\\ a & \text{otherwise.}\end{cases}$$

Then, (2.1.32) rewrites as (no summation is intended over repeated indices on the RHS)

$$\omega_{ij}\widehat{T}_{abcdij}=\omega_{a'a}T_{a'bcd}+\omega_{b'b}T_{ab'cd}+\omega_{c'c}T_{abc'd}+\omega_{d'd}T_{abcd'}\,. \tag{2.1.33}$$

By Cauchy's inequality we can bound

$$(\omega_{ij}\widehat{T}_{abcdij})^2\leq\left(\omega^2_{a'a}+\omega^2_{b'b}+\omega^2_{c'c}+\omega^2_{d'd}\right)\left(T^2_{a'bcd}+T^2_{ab'cd}+T^2_{abc'd}+T^2_{abcd'}\right)$$

but in fact we also have the more effective bound

$$(\omega_{ij}\widehat{T}_{abcdij})^2\leq\left(\sum_{i,j=1}^m\omega^2_{ij}\right)\left(T^2_{a'bcd}+T^2_{ab'cd}+T^2_{abc'd}+T^2_{abcd'}\right)\,. \tag{2.1.34}$$

Since for every $1\leq i,j\leq m$ we have $\omega^2_{ij}=\omega^2_{ji}$, to justify deduction of (2.1.34) from (2.1.33) one observes that, up to dropping out vanishing terms from the RHS of (2.1.33), for every $i\neq j$ there are at most two sets, amongst $\{a,a'\}$, $\{b,b'\}$, $\{c,c'\}$ and $\{d,d'\}$, that coincide with $\{i,j\}$. Indeed, if $a=b$ then $a'=b'$, and $\omega_{a'a}=\omega_{b'b}$ while $T_{a'bcd}=T_{b'acd}=-T_{ab'cd}$; if $a=b'$ then $b=a'$ and $T_{a'bcd}=0=T_{ab'cd}$. Hence, for any a,b,c,d we have

$$\omega_{a'a}T_{a'bcd}+\omega_{b'b}T_{ab'cd}\neq 0\qquad\Rightarrow\qquad\{a,a'\}\cap\{b,b'\}=\varnothing$$

and similarly

$$\omega_{c'c}T_{abc'd}+\omega_{d'd}T_{abcd'}\neq 0\qquad\Rightarrow\qquad\{c,c'\}\cap\{d,d'\}=\varnothing\,.$$

Summing over all tuples (a,b,c,d), we obtain

$$\omega_{ij}\omega_{kt}\widehat{T}_{abcdij}\widehat{T}_{abcdkt}\leq\omega_{ij}\omega_{ij}\sum_{a,b,c,d=1}^m\left(T^2_{a'bcd}+T^2_{ab'cd}+T^2_{abc'd}+T^2_{abcd'}\right)=4\omega_{ij}\omega_{ij}T_{abcd}T_{abcd}$$

where equality follows since the map $a\mapsto a'$ is a bijection of $\{1,\dots,m\}$ into itself, so that

$$\sum_{a,b,c,d=1}^m T^2_{a'bcd}=\sum_{a,b,c,d=1}^m T^2_{abcd}$$

and similarly for the other terms. $\square$

Proposition 2.17 ([30]). *Let $x \in M$, $C \in \mathbb{R}$ and assume that*

$$\mathfrak{R}^{(\lfloor \frac{m-1}{2} \rfloor)}(x) \geq C\,.$$

Then for every totally traceless algebraic curvature tensor W we have

$$\langle \Gamma W, W\rangle \geq 2(m-1)C|W|^2 \qquad \text{at } x.$$

Proof. Let $\{\omega^\alpha\}_\alpha$ be an orthonormal basis of $\wedge^2_x M$ consisting of eigenvectors of $\mathfrak{R}$ with corresponding eigenvalues $\{\lambda_\alpha\}_\alpha$. Then, with respect to any local orthonormal coframe $\{\theta^i\}$ we have

$$R_{ijkt} = \sum_\alpha \lambda_\alpha \omega^\alpha_{ij}\omega^\alpha_{kt}\,, \qquad \delta_{ik}\delta_{jt} - \delta_{it}\delta_{jk} = 2\sum_\alpha \omega^\alpha_{ij}\omega^\alpha_{kt}\,. \tag{2.1.35}$$

From (2.1.17) then we have

$$\langle \Gamma W, W\rangle = \langle \mathfrak{R}^{T^0_4 M}\hat{W}, \hat{W}\rangle = \sum_{\alpha=1}^{\binom{m}{2}} \lambda_\alpha \omega^\alpha_{ij}\omega^\alpha_{kt}\hat{W}_{abcdij}\hat{W}_{abcdkt} = \sum_{\alpha=1}^{\binom{m}{2}} \lambda_\alpha c_\alpha \tag{2.1.36}$$

where we have set $c_\alpha = \omega^\alpha_{ij}\omega^\alpha_{kt}\hat{W}_{abcdij}\hat{W}_{abcdkt}$. By (2.1.29) and the second in (2.1.35) we have

$$2(m-1)|W|^2 = |\hat{W}|^2 = \delta_{ik}\delta_{jt}\hat{W}_{abcdij}\hat{W}_{abcdkt} = \frac{1}{2}(\delta_{ik}\delta_{jt} - \delta_{it}\delta_{jk})\hat{W}_{abcdij}\hat{W}_{abcdkt} = \sum_{\alpha=1}^{\binom{m}{2}} c_\alpha$$

and then by (2.1.30) for every α

$$c_\alpha \leq 4|W|^2 = \frac{2}{m-1}|\hat{W}|^2 = \frac{2}{m-1}\sum_{\beta=1}^{\binom{m}{2}} c_\beta \qquad \forall\, 1 \leq \alpha \leq \binom{m}{2}\,. \tag{2.1.37}$$

Applying Lemma 2.13 we obtain the desired conclusion. ☐

Theorem 2.18. *Let $x \in M$, $C \in \mathbb{R}$ and assume that*

$$\mathfrak{R}^{(\lfloor \frac{m-1}{2} \rfloor)}(x) \geq C\,. \tag{2.1.38}$$

Then for every algebraic curvature tensor T we have

$$\langle \Gamma T, T\rangle \geq 2(m-1)C|P|^2 \qquad \text{at } x, \tag{2.1.39}$$

where P is the tensor defined in (1.2.12).

Proof. First, recall from (2.1.10) that

$$\langle \Gamma T, T\rangle = \langle \Gamma W, W\rangle + \frac{4}{m-2}\langle \Gamma Z, Z\rangle\,. \tag{2.1.40}$$

By Proposition 2.17 we have

$$\langle \Gamma W, W\rangle \geq 2(m-1)C|W|^2\,. \tag{2.1.41}$$

By (1.1.13), from (2.1.38) we deduce $\mathfrak{R}^{(\lfloor \frac{m}{2} \rfloor)}(x) \geq C$ and then by (1.1.14)

$$\frac{1}{\lfloor \frac{m}{2} \rfloor}\sum_{i=1}^{\lfloor \frac{m}{2} \rfloor} \mathrm{Sect}(\pi_i) \geq \frac{C}{2}$$

for every set $\{\pi_1, \dots, \pi_{\lfloor \frac{m}{2} \rfloor}\}$ of mutually orthogonal 2-planes in T_xM. Then, by Proposition 2.14

$$\langle \Gamma Z, Z\rangle \geq mC|Z|^2\,. \tag{2.1.42}$$

Putting together (2.1.40), (2.1.41), (2.1.42) and using (1.2.13) we conclude

$$\langle \Gamma T, T\rangle \geq C\left[2(m-1)|W|^2 + \frac{4m}{m-2}|Z|^2\right] = 2(m-1)C|P|^2\,.$$

☐

Theorem 2.7 is a direct consequence of Proposition 2.5 and Theorem 2.18. We are now able to prove Theorem 2.4:

Proof of Theorem 2.4. From Theorem 2.7 and from the harmonicity of T, we have that

$$\frac{1}{2}\Delta|T|^2 \geq |\nabla T|^2 + (m-1)\mathfrak{R}^{(\lfloor\frac{m-1}{2}\rfloor)}|P|^2$$

By compactness of M, the subharmonic function $|T|^2$ must be constant, and hence

$$\nabla T \equiv 0 \qquad \text{and} \qquad \mathfrak{R}^{(\lfloor\frac{m-1}{2}\rfloor)}|P|^2 \equiv 0.$$

Therefore T is parallel and, if $\mathfrak{R}^{(\lfloor\frac{m-1}{2}\rfloor)} > 0$ at some point, P must be 0, which means, from Lemma 1.10, that T is a constant multiple of $\langle\, ,\, \rangle \owedge \langle\, ,\, \rangle$. □

2.2 The complete case

The main subject this section is the same as above - i.e. harmonic algebraic curvature tensors - but here we lift the hypothesis of M being compact. Clearly, the Bochner formula in (2.1.4) and the estimate in Theorem 2.7 are still valid, but in complete manifolds subharmonic functions can be non-constant, even if they are bounded. To argue similarly to the Proof of Theorem 2.4, we therefore need to make further assumptions both on the growth of the algebraic curvature tensor involved and on the geometry of M. To this end, we recall and refine some results concerning complete manifolds. We first have the following mean value inequalities for subharmonic functions due to P. Li, [22], and Li-Schoen, [21]. Notice that, whenever the curvature operator is $\lfloor\frac{m-1}{2}\rfloor$ non-negative, the Ricci tensor is non-negative in turn (see (1.1.15)), so the assumptions made in the following Propositions are compatible with Theorem 2.7 and Theorems 2.26, 2.27 below.

Proposition 2.19 ([22], Theorem 4)**.** *Let $(M, \langle\, ,\, \rangle)$ be a complete Riemannian manifold with* $\mathrm{Ric} \geq 0$. *Let $f \in L^\infty(M)$ be a subharmonic function. Then for any $x \in M$*

$$\lim_{R\to+\infty}\frac{1}{|B_R(x)|}\int_{B_R(x)} f = \sup_M f\,. \tag{2.2.1}$$

Proposition 2.20 ([21], Theorem 2.1)**.** *Let $(M^m, \langle\, ,\, \rangle)$ be a complete Riemannian manifold with* $\mathrm{Ric} \geq -(m-1)\kappa^2$. *Let $R > 0$, $x \in M$ and let $f \geq 0$ be a subharmonic function defined on $B_R(x)$. There exists a constant $C = C(m,p) > 0$ such that*

$$\sup_{B_{(1-\tau)R}(x)} f^p \leq \tau^{-C(1+\kappa R)}\frac{1}{|B_R(x)|}\int_{B_R(x)} f^p \tag{2.2.2}$$

for every $\tau \in (0, 1/2)$.

Corollary 2.21. *Let $(M, \langle\, ,\, \rangle)$ be a complete Riemannian manifold with* $\mathrm{Ric} \geq 0$. *Let $f \geq 0$ be a nonnegative subharmonic function. Then for any $p \in [1, +\infty)$*

$$\lim_{R\to+\infty}\frac{1}{|B_R(x)|}\int_{B_R(x)} f^p = \sup_M f^p\,. \tag{2.2.3}$$

Proof. First observe that if $f \geq 0$ is subharmonic, then f^p is also subharmonic for any $p \geq 1$. If f is bounded then f^p is also bounded and the conclusion follows by Proposition 2.19. If f is unbounded then by Proposition 2.20 both sides of (2.2.3) equal $+\infty$ and the conclusion follows. □

The next Proposition 2.22, together with its proof, rephrases in general terms an observation contained in [12].

Proposition 2.22 ([12])**.** *Let $(M, \langle\, ,\, \rangle)$ be a complete Riemannian manifold with* $\mathrm{Ric} \geq 0$. *Let $f \in L^\infty(M)$ be a subharmonic function. Then for any $x \in M$*

$$\lim_{R\to+\infty}\frac{R^2}{|B_R(x)|}\int_{B_R(x)} \Delta f = 0\,. \tag{2.2.4}$$

Proof. Let r denote the distance function from x. By the Laplacian comparison theorem we have

$$\Delta r^2 \leq 2m \tag{2.2.5}$$

where $m = \dim M$. Define $h = \sup_M f - f \geq 0$. Green's identities give

$$\int_{B_R(x)} \left(1 - \frac{r^2}{R^2}\right) \Delta h + \frac{1}{R^2} \int_{B_R(x)} h \Delta r^2 = \frac{1}{R^2} \int_{\partial B_R(x)} h \langle \nabla r^2, \nu \rangle \geq 0$$

for almost every $R > 0$. Since $h \geq 0$ and $-\Delta h = \Delta f \geq 0$, by (2.2.5) we estimate

$$\frac{2m}{R^2} \int_{B_R(x)} h \geq \frac{1}{R^2} \int_{B_R(x)} h \Delta r^2 \geq \int_{B_R(x)} \left(1 - \frac{r^2}{R^2}\right) \Delta f \geq \frac{3}{4} \int_{B_{R/2}(x)} \Delta f \,.$$

Hence, dividing by $|B_{R/2}(x)|$ we have

$$\frac{1}{|B_{R/2}(x)|} \int_{B_R(x)} h \geq \frac{3}{8m} \frac{R^2}{|B_{R/2}(x)|} \int_{B_{R/2}(x)} \Delta f \geq 0 \,. \tag{2.2.6}$$

By Bishop-Gromov theorem we also have $|B_{R/2}(x)| \geq 2^m |B_R(x)|$, hence

$$\frac{2^m}{|B_R(x)|} \int_{B_R(x)} h \geq \frac{1}{|B_{R/2}(x)|} \int_{B_R(x)} h \tag{2.2.7}$$

and by Proposition 2.19

$$\frac{1}{|B_R(x)|} \int_{B_R(x)} h = \sup_M f - \frac{1}{|B_R(x)|} \int_{B_R(x)} f \to 0 \qquad \text{as } R \to +\infty \,. \tag{2.2.8}$$

Putting together (2.2.6), (2.2.7) and (2.2.8) we obtain (2.2.4). □

Let $(M, \langle\, ,\, \rangle)$ be a complete Riemannian manifold. For every $x \in M$, $R > 0$ and for every measurable function ψ on $B_R(x)$ we define

$$\psi_{x,R} = \frac{1}{|B_R(x)|} \int_{B_R(x)} \psi$$

whenever the RHS of this inequality happens to be well defined. From the work of P. Buser, [9], combined with Cheeger's inequality it is known (see for instance L. Saloff-Coste, [34, page 439]) that geodesic balls of a complete Riemannian manifold with non-negative Ricci curvature support the following Poincaré inequality.

Proposition 2.23 ([9],[34]). *Let $(M^m, \langle\, ,\, \rangle)$ be a complete Riemannian manifold with* Ric ≥ 0. *Then, there exists $C = C(m) > 0$ such that for every $x \in M$ and $R > 0$*

$$\int_{B_R(x)} |f - f_{x,R}|^2 \leq CR^2 \int_{B_R(x)} |\nabla f|^2 \qquad \forall\, f \in C^\infty(B_R(x)) \,. \tag{2.2.9}$$

We are now in the position to prove the next Liouville-type theorem.

Theorem 2.24. *Let $(M, \langle\, ,\, \rangle)$ be a complete, non-compact Riemannian manifold with* Ric ≥ 0. *Let $a \geq 0$ be a measurable function on M and let $0 \leq f$ be a function of class $L^\infty(M)$ satisfying*

$$\Delta f \geq af \qquad \text{on } M \,.$$

Assume that for some $x \in M$ one of the following conditions is satisfied:

i) $f_{x,R} \to 0$ as $R \to +\infty$,

ii) for some constant $C_0 > 0$ and for some compact set $K \subsetneq M$

$$a \geq \frac{C_0^2}{r^2} \qquad \text{on } M \setminus K \tag{2.2.10}$$

where r is the distance function from x, *iii) for some constants $C_1, C_2 > 0$*

$$a_{x,R} \geq \frac{C_1}{R^2}\,, \qquad \frac{1}{|B_R(x)|}\int_{B_R(x)} |a - a_{x,R}|^2 \leq \frac{C_2}{R^4} \tag{2.2.11}$$

for all sufficiently large $R > 0$.

Then $f \equiv 0$.

Proof. Since $a \geq 0$ and $f \geq 0$, we have that f is a bounded, nonnegative subharmonic function on M. Hence, by Proposition 2.19 the limit

$$\ell = \lim_{R\to+\infty} f_{x,R}$$

exists and equals $\sup_M f \in [0, +\infty)$, and by Proposition 2.22 we also have

$$\lim_{R\to+\infty} R^2 (af)_{x,R} = 0\,. \tag{2.2.12}$$

Since $f \geq 0$, the desired conclusion $f \equiv 0$ is equivalent to having $\ell = 0$. Note that this is, in turn, equivalent to i). Hence let us assume, by contradiction, that $\ell > 0$. Then we must be in either case ii) or iii). In both cases we aim at showing that (2.2.12) cannot hold, hence concluding the proof by contradiction.

If ii) is in force, then fix $R_0 > 0$ large enough so that $K \subsetneq B_{R_0}(x)$. For every $R > R_0$ we have, using (2.2.10),

$$R^2(af)_{x,R} \geq \frac{R^2}{|B_R(x)|}\int_{B_R(x)\setminus K} af \geq \frac{C_0}{|B_R(x)|}\int_{B_R(x)\setminus K} f = C_0 f_{x,R} - \frac{C_0}{|B_R(x)|}\int_K f\,.$$

M has infinite volume as it is a complete noncompact manifold with $\mathrm{Ric} \geq 0$, see for instance [35, page 25], hence letting $R \to +\infty$ in the above inequality we obtain

$$\liminf_{R\to+\infty} R^2(af)_{x,R} \geq C_0 \ell > 0\,,$$

contradicting (2.2.12).

If iii) is in force, then writing $a = a_{x,R} + (a - a_{x,R})$ and $f = f_{x,R} + (f - f_{x,R})$ one has

$$(af)_{x,R} = a_{x,R} f_{x,R} + \frac{1}{|B_R(x)|}\int_{B_R(x)} (a - a_{x,R})(f - f_{x,R})$$

for every $R > 0$. Using Cauchy-Schwarz inequality together with (2.2.11) we further estimate

$$R^2(af)_{x,R} \geq C_1 f_{x,R} - \sqrt{C_2}\left(\frac{1}{|B_R(x)|}\int_{B_R(x)} |f - f_{x,R}|^2\right)^{1/2}\,. \tag{2.2.13}$$

The function f^2 is also bounded and subharmonic. In particular,

$$\Delta f^2 = 2f\Delta f + 2|\nabla f|^2 \geq 2|\nabla f|^2$$

and by Proposition 2.22 we get

$$\lim_{R\to+\infty} \frac{R^2}{|B_R|}\int_{B_R} |\nabla f|^2 = 0\,.$$

Hence, by Proposition 2.23

$$\lim_{R\to+\infty} \frac{1}{|B_R(x)|}\int_{B_R(x)} |f - f_{x,R}|^2 = 0 \tag{2.2.14}$$

and by (2.2.13) and (2.2.14) we obtain

$$\liminf_{R\to+\infty} R^2(af)_{x,R} \geq C_1 \ell > 0\,,$$

again contradicting (2.2.12). $\square$

We remark that if M is a complete *parabolic* Riemannian manifold, in the sense of the subsequent Definition 2.1, then the analogue of Theorem 2.24 holds with less restrictive conditions on a and f and no requirements on the Ricci tensor.

Definition 2.1. We say that a complete Riemannian manifold M is parabolic if every upper bounded subharmonic function on M is constant.

This terminology originates from the complex analytic classification of (noncompact) Riemann surfaces, where the function theoretic property expressed by Definition 2.1 distinguishes the parabolic from the hyperbolic ones, see [1, Section IV.1.6]. For M a complete Riemannian manifold of any dimension, M is parabolic in the sense of Definition 2.1 if and only if it does not admit any positive Green's function. A sufficient condition for parabolicity, which is also necessary for manifolds with non-negative Ricci curvature, see [23, 39], is that

$$\limsup_{R\to+\infty}\int_1^R \frac{t}{|B_t|}\,\mathrm{d}t = +\infty \tag{2.2.15}$$

where $|B_t|$ is the volume of the geodesic ball B_t of radius t centered at a fixed point $o \in M$.

Theorem 2.25. *Let $(M, \langle\,,\,\rangle)$ be a complete, parabolic Riemannian manifold. Let $a \geq 0$ be a measurable function on M and let $0 \leq f \in L^\infty(M)$ satisfy*

$$\Delta f \geq af \qquad \text{on } M. \tag{2.2.16}$$

Then f is constant. Moreover, if $a > 0$ somewhere then $f \equiv 0$.

Proof. The function f is bounded and subharmonic, hence it is constant by parabolicity of M and from (2.2.16) it follows that $af \equiv 0$. If there exists $x \in M$ such that $a(x) > 0$, then $f(x) = 0$ and thus $f \equiv 0$ on M. ☐

We are now ready to give analogous results to Theorem 2.4 in the complete non-compact case:

Theorem 2.26. *Let M be a complete Riemannian manifold of dimension $m \geq 3$ satisfying $\mathfrak{R}^{(\lfloor\frac{m-1}{2}\rfloor)} \geq 0$. If T is a harmonic algebraic curvature tensor on M such that*

$$\lim_{R\to+\infty}\frac{1}{|B_R(x)|}\int_{B_R(x)}|W_T|^p + |Z_T|^p = 0$$

for some $x \in M$ and $p \in [1, +\infty)$, where W_T and Z_T are the Weyl part of T and the traceless part of the Ricci contraction of T. Then T is a constant multiple of $\langle\,,\,\rangle \owedge \langle\,,\,\rangle$.

Proof. Let $T = W_T + V_T + U_T$ be the orthogonal decomposition of T given by (1.2.4)-(1.2.5). From Proposition 1.12 we see that the total trace S_T of T is constant, hence $U_T = \frac{S_T}{2m(m-1)}\langle\,,\,\rangle \owedge \langle\,,\,\rangle$ is parallel. By linearity, the tensor field $T' = W_T + V_T \equiv T - U_T$ is again a harmonic algebraic curvature tensor and its standard orthogonal decomposition $T' = W_{T'} + V_{T'} + U_{T'}$ is given by $W_{T'} = W_T$, $V_{T'} = V_T$, $U_{T'} = 0$. In particular, the traceless part $Z_{T'}$ of the Ricci contraction of T' coincides with the analogous tensor Z_T associated to T. By Theorem 2.7 we have

$$\frac{1}{2}\Delta|T'|^2 \geq |\nabla T'|^2 + (m-1)\mathfrak{R}^{(\lfloor\frac{m-1}{2}\rfloor)}|P_{T'}|^2 \geq 0$$

where $P_{T'}$ is the pseudo-projective curvature tensor associated to T' according to (1.2.12), which coincides with the one associated to T.

Since T is harmonic, the tensor T' is also harmonic and by Theorem 2.7 we have

$$\frac{1}{2}\Delta|T'|^2 \geq |\nabla T'|^2 + (m-1)\mathfrak{R}^{(\lfloor\frac{m-1}{2}\rfloor)}|P_{T'}|^2 \qquad \text{on } M. \tag{2.2.17}$$

At any point where $|T'| \neq 0$ we have

$$\frac{1}{2}\Delta|T'|^2 = \mathrm{div}(|T'|\nabla|T'|) = |T'|\Delta|T'| + |\nabla|T'||^2 \qquad \text{and} \qquad |\nabla|T'||^2 \leq |\nabla T'|^2$$

hence

$$|T'|\Delta|T'| \geq |\nabla T'|^2 + (m-1)\mathfrak{R}^{(\lfloor\frac{m-1}{2}\rfloor)}|P_{T'}|^2 - |\nabla|T'||^2 \geq (m-1)\mathfrak{R}^{(\lfloor\frac{m-1}{2}\rfloor)}|P_{T'}|^2,$$

and in particular

$$\Delta|T'| \geq 0\,.$$

Since $|T'| \geq 0$ on M, any point where $|T'| = 0$ is a global mininum point for $|T'|$, hence $\Delta|T'| \geq 0$ holds in the weak sense on the whole M, i.e. $|T'|$ is subharmonic. Namely,

$$|T'|\Delta|T'| \geq (m-1)\mathfrak{R}^{(\lfloor\frac{m-1}{2}\rfloor)}|P_{T'}|^2 \tag{2.2.18}$$

pointwise on $\{|T'| > 0\}$ and in the weak sense on M. Note that

$$|T'|^2 = |W_{T'}|^2 + |V_{T'}|^2 = |W_T|^2 + |V_T|^2 = |W_T|^2 + \frac{4}{m-2}|Z_T|^2 \leq \left(|W_T| + \frac{2}{\sqrt{m-2}}|Z_T|\right)^2 \tag{2.2.19}$$

thus, since $(a+b)^p \leq 2^{p-1}(a^p + b^p)$ for any $a, b \geq 0$ and $p \geq 1$,

$$|T'|^p \leq 2^{p-1}|W_T|^p + \frac{2^p}{(m-2)^{(p-1)/2}}|Z_T|^p\,.$$

In particular, under the assumptions of the present theorem we have

$$\lim_{R\to+\infty}\frac{1}{|B_R(x)|}\int_{B_R(x)}|T'|^p = 0$$

and by Corollary 2.21 we get $T' \equiv 0$ on M, that is, $T \equiv U_T$. $\square$

Theorem 2.27. *Let M be a complete Riemannian manifold of dimension $m \geq 3$. Assume that $\mathfrak{R}^{(\lfloor\frac{m-1}{2}\rfloor)} \geq 0$ and that either*

- *(a) $\mathfrak{R}^{(\lfloor\frac{m-1}{2}\rfloor)} > 0$ somewhere on M and* (2.2.15) *is satisfied for some $o \in M$, or*
- *(b) $\mathfrak{R}^{(\lfloor\frac{m-1}{2}\rfloor)} \geq a$ for some measurable function $a \geq 0$ satisfying ii) or iii) in Theorem 2.24.*

If T is a harmonic algebraic curvature tensor on M such that

$$\limsup_{R\to+\infty}\frac{1}{|B_R(x)|}\int_{B_R(x)}|W_T|^p + |Z_T|^p < +\infty \tag{2.2.20}$$

for some $x \in M$ and $p \in [1, +\infty)$, then T is a constant multiple of $\langle\,,\,\rangle \owedge \langle\,,\,\rangle$.

Proof. Letting T' and $P_{T'}$ be as in the proof of Theorem 2.26, by (1.2.13) and (2.2.19) we compute

$$|P_{T'}|^2 = |W_T|^2 + \frac{2m}{(m-2)(m-1)}|Z_T|^2 \geq \frac{1}{2}|T'|^2$$

and therefore from (2.2.18) we get

$$\Delta|T'| \geq \frac{(m-1)}{2}a|T'|\,.$$

It is easy to check (using for instance Propositions 1.11 and 1.12) that the algebraic curvature tensor fields W_T and $Z_T \owedge \langle\,,\,\rangle$ are both harmonic, hence $|W_T|$ and $|Z_T|$ are subharmonic functions. So, the lim sup in (2.2.20) is in fact a limit and in particular

$$\lim_{R\to+\infty}\frac{1}{|B_R(x)|}\int_{B_R(x)}|W_T|^p + |Z_T|^p = \sup_M|W_T|^p + \sup_M|Z_T|^p\,.$$

Hence, in this setting (2.2.20) is equivalent to boundedness of the non-scalar part $W_T + V_T = W_T + \frac{4}{m-2}Z_T \owedge \langle\,,\,\rangle$ of T, so we infer $|T'| \equiv 0$ applying Theorem 2.25 or Theorem 2.24, depending on which assumption among (a) and (b) is in force. This shows that T is a scalar multiple of $\langle\,,\,\rangle \owedge \langle\,,\,\rangle$, and the conclusion follows since the total trace S_T of T is constant. $\square$

2.3 Rigidity for maps between Riemannian manifolds

In this section, given a smooth map φ between two Riemannian manifolds $(M, \langle\, ,\, \rangle)$ and $(N, \langle\, ,\, \rangle_N)$, we provide some rigidity results both on the map and the base Riemannian manifold $(M, \langle\, ,\, \rangle)$. To this aim, we consider a tensor that, by its definition, encodes information about both φ and $(M, \langle\, ,\, \rangle)$, that is, the φ-Weyl tensor defined in the previous chapter as

$$W^{\varphi} = \mathrm{Riem} - \frac{1}{m-2} A^{\varphi} \owedge \langle\, ,\, \rangle.$$

Denoting with

$$F = \varphi^{*}\langle\, ,\, \rangle_N - \frac{|\mathrm{d}\varphi|^2}{2(m-1)} \langle\, ,\, \rangle, \tag{2.3.1}$$

then

$$W^{\varphi} = W + \alpha F \owedge \langle\, ,\, \rangle, \tag{2.3.2}$$

where the two components W and $F \owedge \langle\, ,\, \rangle$ of the sum are orthogonal to each other and related only to the geometry of $(M, \langle\, ,\, \rangle)$ and to the map φ respectively. Therefore, imposing conditions on W^{φ} means in fact imposing restrictions on M and φ. Namely, here we consider the case of W^{φ} being harmonic, which allows us to apply the results proven for generic algebraic curvature tensors in Sections 2.1 and 2.2. Notice that we don't consider the "trivial" case for W^{φ}, i.e. when the Weyl component vanish (if $m = \dim M \leq 3$). After having made some considerations about the hypothesis of W^{φ} being harmonic, we then deal with bot the compact and complete non-compact cases, with the help of classification theorems of locally conformally flat manifolds with non-negative Ricci curvature.

Let us find some equivalent condition to the harmonicity of the φ-Weyl tensor. We recall that an algebraic curvature tensor is harmonic if and only if it satisfies the second Bianchi identity and has vanishing divergence. Therefore, from the "fake second Bianchi identity" for W^{φ} in (1.3.18)

$$B(W^{\varphi})_{ijktl} = \frac{1}{m-2}\Big(C^{\varphi}_{ilk}\delta_{jt} + C^{\varphi}_{ikt}\delta_{jl} + C^{\varphi}_{itl}\delta_{jk} + C^{\varphi}_{jkl}\delta_{it} + C^{\varphi}_{jtk}\delta_{il} + C^{\varphi}_{jlt}\delta_{ik}\Big),$$

and computing the norm $B(W^{\varphi})$ as

$$|B(W^{\varphi})|^2 = \frac{6(m-3)}{(m-2)^2}|C^{\varphi}|^2 + \frac{12}{(m-2)^2}|\operatorname{tr} C^{\varphi}|^2,$$

The second Bianchi identity is satisfied if and only if the φ-Cotton tensor vanishes. In other words, W^{φ} is harmonic if and only if

$$C^{\varphi} = 0 \quad \text{and} \quad \operatorname{div} W^{\varphi} = 0. \tag{2.3.3}$$

Notice that this differs from the standard case, where $\operatorname{div} W$ is proportional to C and then saying that W harmonic is equivalent to having vanishing divergence. We can try to find another equivalent condition, that "decouples" W^{φ} being harmonic to a condition on the base manifold and a condition on the map φ. To be specific, we are going to prove that (2.3.3) is in fact also equivalent to

$$i)\operatorname{div} W \equiv 0, \quad ii)\operatorname{div} F \equiv 0, \quad iii) F \text{ is a Codazzi tensor} \tag{2.3.4}$$

where F is defined as in (2.3.1), or in local components by

$$F_{ij} = (\varphi^a_i \varphi^a_j - \frac{|d\varphi|^2}{2(m-1)}\delta_{ij}).$$

The interesting fact in the set of assumptions (2.3.4) is that (2.3.4) *ii)* and *iii)* are expressed in terms of properties of the map

$$\varphi : (M, \langle\, ,\, \rangle) \to (N, \langle\, ,\, \rangle_N)$$

while (2.3.4) $i)$ is on the geometry of $(M,\langle\,,\,\rangle)$ alone; it tells us that $(M,\langle\,,\,\rangle)$ is conformally harmonic.
To prove the equivalence of (2.3.3) with (2.3.4), we first observe that a simple computation shows

$$\operatorname{div} F = 0 \quad \text{in and only if} \quad \varphi^a_{ti}\varphi^a_t = -\frac{m-1}{m-2}\varphi^a_{ss}\varphi^a_i \tag{2.3.5}$$

and

$$F \text{ is Codazzi if and only if } \varphi^a_{jk}\varphi^a_t - \varphi^a_{jt}\varphi^a_k = \frac{1}{m-1}\varphi^a_s(\varphi^a_{sk}\delta_{jt} - \varphi^a_{st}\delta_{jk}). \tag{2.3.6}$$

Furthermore, we have the validity of the following

Lemma 2.28. *Let C be the usual Cotton tensor, $m \geq 3$. Then*

$$\big|\operatorname{div} W^\varphi - \frac{1}{m-2}C^\varphi\big|^2 = |C|^2 + 2\frac{m-1}{(m-2)^2}|\operatorname{div} F|^2. \tag{2.3.7}$$

Proof. From (2.3.2) we have

$$W^\varphi_{ijkt} = W_{ijkt} + \frac{\alpha}{m-2}(F_{ik}\delta_{jt} + F_{jt}\delta_{ik} - F_{it}\delta_{jk} - F_{jk}\delta_{it}) \tag{2.3.8}$$

and from (1.3.19)

$$C^\varphi_{ijk} = C_{ijk} - \alpha F_{ij,k} + \alpha F_{ik,j}. \tag{2.3.9}$$

Taking the divegence of (2.3.8) we have

$$\begin{aligned} W^\varphi_{tijk,t} =& W_{tijk,t} + \frac{\alpha}{m-2}(F_{tj,t}\delta_{ik} + F_{ik,t}\delta_{tj} - F_{tk,t}\delta_{ij} - F_{ij,t}\delta_{tk}) = \\ =& W_{tijk,t} + \frac{\alpha}{m-2}(F_{ik,j} - F_{ij,k}) + \frac{\alpha}{m-2}(F_{tj,t}\delta_{ik} - F_{tk,t}\delta_{ij}). \end{aligned}$$

Hence, using (2.3.9), we get

$$W^\varphi_{tijk,t} - \frac{1}{m-2}C^\varphi_{ijk} = W_{tijk,t} - \frac{1}{m-2}C_{ijk} + \frac{\alpha}{m-2}(F_{tj,t}\delta_{ik} - F_{tk,t}\delta_{ij}).$$

Thus from the identity

$$W_{tijk,t} = \frac{m-3}{m-2}C_{ikj} \tag{2.3.10}$$

we can further rewrite the above in the form

$$W^\varphi_{tijk,t} - \frac{1}{m-2}C^\varphi_{ijk} = C_{ikj} + \frac{\alpha}{m-2}(F_{tj,t}\delta_{ik} - F_{tk,t}\delta_{ij}) \tag{2.3.11}$$

Using the fact that C is totally trace free, an immediate computation using (2.3.11) gives the validity of (2.3.7). $\square$

We are now ready to prove

Proposition 2.29. *Let $m \geq 4$. Then conditions* (2.3.3) *and* (2.3.4) *are equivalent.*

Proof. We assume (2.3.3). Then by (2.3.7) $C \equiv 0$ and $\operatorname{div} F \equiv 0$. By (2.3.10) $C \equiv 0$ is equivalent to $\operatorname{div} W \equiv 0$. Furthermore, by (1.3.22)

$$0 = \alpha(\varphi^a_{jk}\varphi^a_t - \varphi^a_{jt}\varphi^a_k) + \frac{\alpha}{m-2}\varphi^a_{ss}(\varphi^a_k\delta_{jt} - \varphi^a_t\delta_{jk}) \tag{2.3.12}$$

and $C^\varphi \equiv 0$ implies $0 = C^\varphi_{ssk} = \alpha\varphi^a_{ss}\varphi^a_k$. From (2.3.12) we then have

$$\varphi^a_{jk}\varphi^a_t = \varphi^a_{jt}\varphi^a_k. \tag{2.3.13}$$

Tracing with respect to j and t

$$\varphi^a_{tk}\varphi^a_t = \varphi^a_{tt}\varphi^a_k \tag{2.3.14}$$

and we realize that the equality in (2.3.6) is satisfied, thus F is a Codazzi tensor. We have thus shown that (2.3.4) holds. Viceversa, assume the latter. By (2.3.10), $C \equiv 0$ and therefore (2.3.7) gives

$$\operatorname{div} W^{\varphi} = \frac{C^{\varphi}}{m-2}. \tag{2.3.15}$$

Inserting (2.3.15) into (1.3.22) yields

$$C^{\varphi}_{jkt} = \alpha(\varphi^a_{jk}\varphi^a_t - \varphi^a_{jt}\varphi^a_k) + \frac{\alpha}{m-2}\varphi^a_{ss}(\varphi^a_k\delta_{jt} - \varphi^a_t\delta_{jk}). \tag{2.3.16}$$

By (2.3.6), since F is Codazzi

$$\varphi^a_{jk}\varphi^a_t - \varphi^a_{jt}\varphi^a_k == \frac{1}{m-1}\varphi^a_s(\varphi^a_{sk}\delta_{jt} - \varphi^a_{st}\delta_{jk})$$

and, since $\operatorname{div} F = 0$, by (2.3.5)

$$\varphi^a_{ti}\varphi^a_t = -\frac{m-1}{m-2}\varphi^a_{ss}\varphi^a_i \tag{2.3.17}$$

Inserting these informations in (2.3.16) we obtain $C^{\varphi} \equiv 0$ and by (2.3.15), $\operatorname{div} W^{\varphi} \equiv 0$, that is, (2.3.3) holds true $\square$

Remark 2.30. Since $\operatorname{div} F = 0$, by (2.3.17) φ conservative is equivalent to $|d\varphi|^2$ constant.

It seems worth specifying formula (2.1.4) to the case $T = W^{\varphi}$, since it it interesting on its own. We have that

$$\frac{1}{2}\Delta|W^{\varphi}|^2 = |\nabla W^{\varphi}|^2 + \frac{1}{2}\langle \Gamma W^{\varphi}, W^{\varphi}\rangle - \frac{1}{3}|B(W^{\varphi})|^2 - 2|\operatorname{div} W^{\varphi}|^2 + \operatorname{div} X(W^{\varphi}) \tag{2.3.18}$$

From the second fake Bianchi identity (1.3.18), we can evaluate $|B(W^{\varphi})|^2$ in terms of the φ-Cotton tensor:

$$\begin{aligned}|B(W^{\varphi})|^2 &= \frac{1}{(m-2)^2}\Big(C^{\varphi}_{ilk}\delta_{jt} + C^{\varphi}_{ikt}\delta_{jl} + C^{\varphi}_{itl}\delta_{jk} + C^{\varphi}_{jkl}\delta_{it} + C^{\varphi}_{jtk}\delta_{il} + C^{\varphi}_{jlt}\delta_{ik}\Big)\times \\ &\quad \Big(C^{\varphi}_{ilk}\delta_{jt} + C^{\varphi}_{ikt}\delta_{jl} + C^{\varphi}_{itl}\delta_{jk} + C^{\varphi}_{jkl}\delta_{it} + C^{\varphi}_{jtk}\delta_{il} + C^{\varphi}_{jlt}\delta_{ik}\Big) \\ &= \frac{1}{(m-2)^2}\big(6(m-3)|C^{\varphi}|^2 + 12|\operatorname{tr} C^{\varphi}|^2\big) = \frac{6(m-3)}{(m-2)^2}|C^{\varphi}|^2 + \frac{12}{(m-2)^2}|\langle \tau(\varphi), \mathrm{d}\varphi\rangle_N|^2,\end{aligned}$$

where in the last equality we have made use of the third in (1.3.20) for the trace of the φ-Cotton tensor. Then, since

$$X(W^{\varphi})_i = W^{\varphi}_{sjkl}(B(W^{\varphi}))_{sjkli} + 2W^{\varphi}_{ijkl}(\operatorname{div} W^{\varphi})_{jkl},$$

again from (1.3.18) we infer

$$X(W^{\varphi}) = \frac{1}{m-2}\big(-4\alpha\varphi^a_s\varphi^a_l C^{\varphi}_{sli} - 2W^{\varphi}_{ijkl}C^{\varphi}_{jkl}\big) + 2W^{\varphi}_{ijkl}(\operatorname{div} W^{\varphi})_{jkl}.$$

Putting all together into (2.3.18), we obtain

$$\begin{aligned}\frac{1}{2}\Delta|W^{\varphi}|^2 &= |\nabla W^{\varphi}|^2 + \frac{1}{2}\langle \Gamma W^{\varphi}, W^{\varphi}\rangle - 2|\operatorname{div} W^{\varphi}|^2 - \frac{2(m-3)}{(m-2)^2}|C^{\varphi}|^2 - \frac{4}{(m-2)^2}|\langle \tau(\varphi), \mathrm{d}\varphi\rangle_N|^2 \\ &\quad + 2\{W^{\varphi}_{tijk}(\operatorname{div} W^{\varphi})_{ijk}\}_{,t} - \frac{4}{m-2}\big(\alpha\varphi^a_s\varphi^a_l C^{\varphi}_{sli}\big)_{,i} - \frac{2}{m-2}\big(W^{\varphi}_{ijkl}C^{\varphi}_{jkl}\big)_{,i}.\end{aligned} \tag{2.3.19}$$

We observe that, when φ is constant, from the known relation

$$\operatorname{div} W = -\frac{m-3}{m-2}C,$$

equation (2.3.19) reduces to

$$\frac{1}{2}\Delta|W|^2 = |\nabla W|^2 + \frac{1}{2}\langle \Gamma W, W\rangle - 2\frac{m-2}{m-3}|\operatorname{div} W|^2 + 2\frac{m-2}{m-3}\Big\{W_{tijk}(\operatorname{div} W)_{ijk}\Big\}_t. \qquad (2.3.20)$$

We can now discuss the restrictions on the geometry of $(M, \langle\, ,\,\rangle)$ and φ provided by the harmonicity of W^φ. Beginning with the compact case, we have the following result (which is in fact Theorem 2.3):

Theorem 2.31. *Let $(M, \langle\, ,\,\rangle)$ be a compact Riemannian manifold of dimension $m \geq 4$ and let $\varphi : M \to (N, \langle\, ,\,\rangle_N)$ be a smooth map from M to a Riemannian manifold $(N, \langle\, ,\,\rangle_N)$. Assume that $(M, \langle\, ,\,\rangle)$ has $\lfloor \frac{m-1}{2} \rfloor$-nonnegative curvature operator. Then, if W^φ is harmonic, φ must be relatively affine and $(M, \langle\, ,\,\rangle)$ is either locally symmetric or locally conformally flat. If $\mathfrak{R}^{\lfloor \frac{m-1}{2} \rfloor}$ is somewhere positive, $(M, \langle\, ,\,\rangle)$ is conformally equivalent to a quotient of $\mathbb{S}^m$ and φ is an homothety.*

The proof of Theorem 2.31 makes use of a result by Noronha [29], which classifies compact locally conformally flat manifolds with non-negative Ricci curvature, that we here recall:

Theorem 2.32 ([29], Theorem 1 and Proposition 4.2)**.** *Let M be a locally conformally flat, compact manifold with* Ric ≥ 0 *and dimension $m \geq 3$. Then the universal cover of M is either*

(i) globally conformally equivalent to $\mathbb{S}^m$, or

(ii) isometric to $\mathbb{S}^{m-1} \times \mathbb{R}$ or $\mathbb{R}^m$.

If M is also locally symmetric, then its universal cover is isometric to either $\mathbb{S}^m$, $\mathbb{S}^{m-1} \times \mathbb{R}$ or $\mathbb{R}^m$ (that is, the conformal equivalence in (i) *can be strengthened to isometry).*

We also need the following Theorem that combines results by Głodek [19] and Derdziński and Roter [14], which we are going to prove:

Theorem 2.33. *Let $(M, \langle\, ,\,\rangle)$ be a conformally symmetric Riemannian manifold of dimension $m \geq 4$. Then either $W = 0$ or $(M, \langle\, ,\,\rangle)$ is locally symmetric.*

First, we show the following

Lemma 2.34. *Let $(M, \langle\, ,\,\rangle)$ have vanishing Cotton tensor. Then,* Ric *satisfies*

$$R_{tj}W_{tikl} + R_{tk}W_{tilj} + R_{tl}W_{tijk} = 0. \qquad (2.3.21)$$

Proof. Since $C = 0$,

$$R_{ij,k} - R_{ik,j} = \frac{1}{2(m-1)}(S_k\delta_{ij} - S_j\delta_{ik}).$$

Taking the covariant derivative, we have

$$R_{ij,kl} - R_{ik,jl} = \frac{1}{2(m-1)}(S_{kl}\delta_{ij} - S_{jl}\delta_{ik})$$

and, summing on cyclic permutations of j, k, and l we obtain, since $\nabla^2 S$ is symmetric,

$$R_{ij,kl} - R_{ik,jl} + R_{ik,lj} - R_{il,kj} + R_{il,jk} - R_{ij,lk} = 0$$

We can regroup the terms so that we are able to use the Ricci identities to get that

$$R_{tj}R_{tikl} + R_{it}R_{tjkl} + R_{tk}R_{tilj} + R_{it}R_{tklj} + R_{tl}R_{tijk} + R_{it}R_{tljk} = 0$$

and, by the Bianchi identity, this reduces to

$$R_{tj}R_{tikl} + R_{tk}R_{tilj} + R_{tl}R_{tijk} = 0.$$

If we write the Riemann tensor according to its splitting

$$\mathrm{Riem} = W + \frac{1}{m-2}\mathrm{Ric} \owedge \langle\, ,\,\rangle - \frac{S}{2(m-1)(m-2)}\langle\, ,\,\rangle \owedge \langle\, ,\,\rangle,$$

some simplifications lead to (2.3.21). ☐

Proof of Theorem 2.33. Since by hypothesis $\nabla W = 0$, its second covariant derivative must vanish in turn. We can then apply the Ricci identities to

$$W_{ijkl,st} - W_{ijkl,ts} = 0$$

to obtain that

$$W_{rjkl}R_{rist} + W_{irkl}R_{rjst} + W_{ijrl}R_{rkst} + W_{ijkr}R_{rlst} = 0. \tag{2.3.22}$$

Then, we consider the covariant derivative of (2.3.22) that, since W is parallel, equals to

$$0 = W_{rjkl}R_{rist,u} + W_{irkl}R_{rjst,u} + W_{ijrl}R_{rkst,u} + W_{ijkr}R_{rlst,u}. \tag{2.3.23}$$

On the other hand, since $\nabla W = 0$, in particular $\operatorname{div} W = 0$ or, equivalently, Cotton tensor vanishes (see for instance (1.2.18) for generic algebraic curvature tensors). Therefore, by its very definition (1.1.6),

$$R_{ij,k} - R_{ik,j} = \frac{1}{2(m-1)}(S_k\delta_{ij} - S_j\delta_{ik}). \tag{2.3.24}$$

From the second contracted Bianchi identity,

$$R_{ijkt,t} = R_{ki,j} - R_{kj,i} = \frac{1}{2(m-1)}(S_j\delta_{ki} - S_i\delta_{kj}).$$

hence tracing (2.3.23) with respect to t and u then leads to

$$0 = W_{rjkl}(S_i\delta_{sr} - S_r\delta_{si}) + W_{irkl}(S_j\delta_{sr} - S_r\delta_{sj}) + W_{ijrl}(S_k\delta_{sr} - S_r\delta_{sk}) + W_{ijkr}(S_l\delta_{sr} - S_r\delta_{sl}). \tag{2.3.25}$$

Now we can trace with respect to i and s in order to obtain that

$$S_iW_{ijkl} = 0. \tag{2.3.26}$$

If we contract (2.3.25) with S_i, by (2.3.26) we thus obtain

$$|\nabla S|^2 W_{rjkl} = 0.$$

Then, either $W = 0$, i.e. the manifold is locally conformally flat, or S is constant. Suppose that $W \neq 0$. By (2.3.24), we infer that the Ricci tensor must be Codazzi, i.e.

$$R_{ij,k} = R_{ik,j}. \tag{2.3.27}$$

The covariant derivative of the curvature tensor, if the Weyl tensor and the scalar curvature are parallel, is given by

$$R_{ijkl,s} = \frac{1}{m-2}(R_{ik,s}\delta_{jl} + R_{jl,s}\delta_{ik} - R_{il,s}\delta_{jk} - R_{jk,s}\delta_{il}), \tag{2.3.28}$$

thus, inserting into (2.3.23),

$$\begin{aligned}0 = {} & W_{rjkl}(R_{rs,u}\delta_{it} + R_{it,u}\delta_{rs} - R_{rt,u}\delta_{is} - R_{is,u}\delta_{rt}) + W_{irkl}(R_{rs,u}\delta_{jt} + R_{jt,u}\delta_{rs} - R_{rt,u}\delta_{js} - R_{js,u}\delta_{rt}) \\ & + W_{ijrl}(R_{rs,u}\delta_{kt} + R_{kt,u}\delta_{rs} - R_{rt,u}\delta_{ks} - R_{ks,u}\delta_{rt}) + W_{ijkr}(R_{rs,u}\delta_{lt} + R_{lt,u}\delta_{rs} - R_{rt,u}\delta_{ls} - R_{ls,u}\delta_{rt})\end{aligned} \tag{2.3.29}$$

We now contract (2.3.29) on i and s to obtain, using the symmetries and the Bianchi identity for W, that

$$(m-2)W_{rjkl}R_{rt,u} = -W_{itkl}R_{ji,u} + W_{ijrl}R_{ri,u}\delta_{kt} - W_{ijtl}R_{ki,u} + W_{ijkr}R_{ri,u}\delta_{lt} - W_{ijkt}R_{li,u}. \tag{2.3.30}$$

If we further contract on k and u,

$$(m-2)W_{rjul}R_{rt,u} = -W_{itul}R_{ji,u} + W_{ijrl}R_{ri,t} - W_{ijtl}R_{ui,u} + W_{ijur}R_{ri,u}\delta_{lt} - W_{ijut}R_{li,u}$$

which, since

$$R_{ui,u} = \frac{1}{2}S_i = 0 \quad \text{and} \quad W_{ijur}R_{ri,u} = W_{ijur}R_{ru,i} = 0,$$

reduces to

$$(m-2)W_{rjul}R_{rt,u} = -W_{itul}R_{ji,u} + W_{ijrl}R_{ri,t} - W_{ijut}R_{li,u} \tag{2.3.31}$$

On the other hand, differentiating (2.3.21),

$$R_{tj,u}W_{tikl} + R_{tk,u}W_{tilj} + R_{tl,u}W_{tijk} = 0 \tag{2.3.32}$$

and, contracting on j and u and using the Codazzi symmetry of the Ricci tensor,

$$R_{tu,k}W_{tiul} = R_{tu,l}W_{tiuk} = R_{tu,i}W_{tluk}. \tag{2.3.33}$$

Going back to (2.3.31), renaming the indices and again by (2.3.27)

$$(m-2)W_{rjul}R_{ru,t} = -W_{rtul}R_{ru,j} + W_{rjul}R_{ru,t} - W_{rjut}R_{ru,l}.$$

using (2.3.33), we arrive to

$$(m-3)W_{rjul}R_{ru,t} = 0,$$

which, since $m \geq 4$, provides

$$W_{rjul}R_{ru,t} = 0. \tag{2.3.34}$$

Substituting (2.3.34) into (2.3.30), we get

$$(m-2)W_{rjkl}R_{rt,u} = -W_{itkl}R_{ji,u} - W_{ijtl}R_{ki,u} - W_{ijkt}R_{li,u},$$

that we rewrite, renaming the indices, as

$$(m-2)W_{rjkl}R_{rt,u} = -W_{rtkl}R_{rj,u} - W_{rjtl}R_{rk,u} - W_{rjkt}R_{rl,u}.$$

Now we make use of (2.3.32) on the second term of the right hand side to get

$$(m-1)W_{rjkl}R_{rt,u} = -W_{rtkl}R_{rj,u},$$

which can be applied twice to obtain that

$$W_{rtkl}R_{rj,u} = (m-1)^2 W_{rtkl}R_{rj,u},$$

and therefore

$$W_{rjkl}R_{ri,s} = 0. \tag{2.3.35}$$

We substitute (2.3.35) into (2.3.29), resulting in

$$\begin{aligned}0 = {} & W_{sjkl}R_{it,u} - W_{tjkl}R_{is,u} + W_{iskl}R_{jt,u} - W_{itkl}R_{js,u} \\ & W_{ijsl}R_{kt,u} - W_{ijtl}R_{ks,u} + W_{ijks}R_{lt,u} - W_{ijkt}R_{ls,u}.\end{aligned}$$

It is now only a matter of contracting with $R_{it,u}$ and using (2.3.35) to obtain that

$$W_{sjkl}|\nabla \operatorname{Ric}|^2 = 0,$$

therefore proving that the Ricci tensor is parallel. Together with $\nabla W = 0$ and $\nabla S = 0$, this guarantees that $\nabla \operatorname{Riem} = 0$, i.e. $(M, \langle\, ,\, \rangle)$ is locally symmetric. $\square$

We can now proceed to prove Theorem 2.31:

Proof. Since M is compact and W^φ harmonic, we can apply Theorem 2.4 with the choice of $T = W^\varphi$ to deduce that

$$\nabla W^\varphi \equiv 0 \quad \text{and} \quad \mathfrak{R}^{(\lfloor \frac{m-1}{2} \rfloor)}|P_{W^\varphi}|^2 \equiv 0.$$

Then, we orthogonally split W^{φ} as the sum

$$W^{\varphi} = W_{W^{\varphi}} + V_{W^{\varphi}} + U_{W^{\varphi}} = W + \frac{\alpha^2}{m-2} Z \owedge \langle\, ,\, \rangle + \frac{\alpha^2 |\mathrm{d}\varphi|^2}{2m(m-1)} \langle\, ,\, \rangle \owedge \langle\, ,\, \rangle, \tag{2.3.36}$$

where we have set

$$Z = \varphi^* \langle\, ,\, \rangle_N - \frac{|\mathrm{d}\varphi|^2}{m} \langle\, ,\, \rangle.$$

We can now use the expression for the norm of the covariant derivative of an algebraic curvature tensor in (1.2.9) applied to $T = W^{\varphi}$ in order to have

$$0 = |\nabla W^{\varphi}|^2 = |\nabla W|^2 + \frac{4\alpha^2}{m-2} |\nabla Z|^2 + \frac{2\alpha^2 \left|\nabla(|\mathrm{d}\varphi|^2)\right|^2}{m(m-1)}. \tag{2.3.37}$$

Therefore, $|\mathrm{d}\varphi|^2$ must be constant and Z parallel. As a consequence, also

$$\nabla \varphi^* \langle\, ,\, \rangle_N = 0,$$

i.e. the map is relatively affine - or equivalently, as we recall,

$$\langle \nabla \mathrm{d}\varphi, \mathrm{d}\varphi \rangle_N = 0.$$

From the fact that $\nabla W = 0$ we obtain, by Theorem 2.33 that either $(M, \langle\, ,\, \rangle)$ is locally conformally flat, or locally symmetric.
Moreover, if we assume that $\mathfrak{R}^{(\lfloor \frac{m-1}{2} \rfloor)} > 0$ at some point, then by Theorem 2.4 W^{φ} is a constant multiple of $\langle\, ,\, \rangle \owedge \langle\, ,\, \rangle$. In particular, from (2.3.36), the Weyl tensor vanishes and $\varphi^* \langle\, ,\, \rangle_N$ is a constant multiple of $\langle\, ,\, \rangle$, i.e. φ is homothetic. Finally, from the classification theorem by Noronha 2.32, M must have a universal cover globally conformally equivalent to $\mathbb{S}^m$. $\square$

In the complete case, we can still gain rigidity of both $(M, \langle\, ,\, \rangle)$ and φ by making further assumptions on the geometry of M or on the growth of W^{φ}, as the following Theorems show. Namely, we will show the validity of the following:

Theorem 2.35. *Let $(M^m, \langle\, ,\, \rangle)$ be a complete Riemannian manifold of dimension $m \geq 4$, with $\lfloor \frac{m-1}{2} \rfloor$-nonnegative curvature operator and let $\varphi : M \to (N, \langle\, ,\, \rangle_N)$ be a smooth map. Assume that W^{φ} is harmonic (or, equivalently, $C^{\varphi} = 0$ and $\varphi^* \langle\, ,\, \rangle_N$ is Codazzi). If*

$$\lim_{R \to +\infty} \frac{1}{|B_R(x)|} \int_{B_R(x)} |W|^p + \left| \varphi^* \langle\, ,\, \rangle_N - \frac{1}{m} |d\varphi|^2 \langle\, ,\, \rangle \right|^p = 0$$

for some $p \in [1, +\infty)$, then M is locally conformally flat and the map φ is homothetic. Moreover, if S^{φ} is constant, then M is isometric to a quotient of either $\mathbb{S}^m$, $\mathbb{S}^{m-1} \times \mathbb{R}$ or $\mathbb{R}^m$. Assuming that $\mathfrak{R}^{(\lfloor \frac{m-1}{2} \rfloor)} > 0$ somewhere, then M is isometric to a quotient of $\mathbb{S}^m$.

Theorem 2.36. *Let $(M^m, \langle\, ,\, \rangle)$ be a complete Riemannian manifold of dimension $m \geq 4$, with $\mathfrak{R}^{\lfloor \frac{m-1}{2} \rfloor} \geq 0$ and that satisfies either*

(a) $\mathfrak{R}^{(\lfloor \frac{m-1}{2} \rfloor)} > 0$ somewhere on M and (2.2.15) for some $o \in M$, or

(b) $\mathfrak{R}^{(\lfloor \frac{m-1}{2} \rfloor)} \geq a$ for some measurable function $a \geq 0$ satisfying ii) or iii) in Theorem 2.24.

Let also $\varphi : M \to (N, \langle\, ,\, \rangle_N)$ for which W^{φ} is harmonic, and let

$$\lim_{R \to +\infty} \frac{1}{|B_R(x)|} \int_{B_R(x)} |W|^p + \left| \varphi^* \langle\, ,\, \rangle_N - \frac{1}{m} |d\varphi|^2 \langle\, ,\, \rangle \right|^p < +\infty$$

for some $p \in [1, +\infty)$. Then M is locally conformally flat and the map φ is homothetic. Moreover, if S^{φ} is constant, then M is isometric to a quotient of $\mathbb{S}^m$.

Notice that the additional assumptions of the above theorems are the ones that make possible to apply Theorem 2.26 or 2.27 to the tensor $T = W^\varphi$. Moreover, to the purpose of proving Theorems 2.35 and 2.36, we will also need the following Theorem by Carron and Herzlich [11], which classifies locally conformally flat manifolds with $\mathrm{Ric} \geq 0$ when the manifold is not assumed to be compact but only to be complete - thus a more general case than the one proven by Noronha.

Theorem 2.37 ([11]). *Let M be a locally conformally flat, complete Riemannian manifold with $\mathrm{Ric} \geq 0$ and dimension $m \geq 3$. Then the universal cover of M is either*

(i) isometric to $\mathbb{R}^m$,

(ii) isometric to $\mathbb{S}^{m-1} \times \mathbb{R}$,

(iii) globally conformally equivalent to $\mathbb{S}^m$, or

(iv) non-flat and globally conformally equivalent to $\mathbb{R}^m$.

Under the additional assumption of constant scalar curvature, the above can be strengthened to the subsequent proposition:

Proposition 2.38. *Let (M, g) be a complete and locally conformally flat Riemannian manifold of dimension $m \geq 3$ with $\mathrm{Ric} \geq 0$ and constant scalar curvature S.*

i) If $S = 0$ then M is flat.

ii) If $S > 0$ then M is either isometric to a quotient of $\mathbb{R} \times \mathbb{S}^{m-1}_{S/(m-1)(m-2)}$ or conformally equivalent to a quotient of $\mathbb{S}^m$.

In particular, if $\mathrm{Ric} > 0$ at some point then M is conformally equivalent to a quotient of $\mathbb{S}^m$.

Proof. i) If $S = 0$ then $\mathrm{Ric} \equiv 0$, and since we also have $W \equiv 0$ we conclude that $\mathrm{Riem} \equiv 0$.

ii) If $S > 0$, by the work of Zhu [41] and Theorem 2.37, we know that the universal cover $(\tilde{M}, \tilde{g})$ of (M, g) satisfies one of the following:

a) $(\tilde{M}, \tilde{g})$ is isometric to $\mathbb{R} \times \mathbb{S}^{m-1}_{S/(m-1)(m-2)}$,

b) $(\tilde{M}, \tilde{g})$ is conformally equivalent to $\mathbb{S}^m$,

c) $(\tilde{M}, \tilde{g})$ is conformally equivalent to $\mathbb{R}^m$.

We repeat the argument of Theorem 1.1 of [32] to show that c) cannot occur. Suppose, by contradiction, that c) holds. Then $\tilde{M} = \mathbb{R}^m$ and $\tilde{g} = u^{\frac{4}{m-2}} g_{\mathbb{R}^m}$ for some $0 < u \in C^\infty(\mathbb{R}^m)$ satisfying the Yamabe equation

$$c_m \Delta_{g_{\mathbb{R}^m}} u + S u^{\frac{m+2}{m-2}} = 0 \qquad \text{on } \mathbb{R}^m , \tag{2.3.38}$$

where $g_{\mathbb{R}^m}$ is the canonical Euclidean metric on $\mathbb{R}^m$. Since S is a positive constant, by the celebrated work of Caffarelli, Gidas and Spruck [10, Corollary 8.2] it follows that u is radially symmetric around some point $x_0 \in \mathbb{R}^m$ and has the expression

$$u(x) = A(B + |x - x_0|^2)^{-\frac{m-2}{2}}$$

for some positive constants A, B only depending on m and S. In particular, $(\tilde{M}, \tilde{g})$ is an m-sphere of constant curvature with one point removed, hence it is not complete. But $(\tilde{M}, \tilde{g})$ is the universal cover of the complete Riemannian manifold (M, g), contradiction. □

We can now prove the theorems stated above:

Proof of Theorem 2.35. From the hypotheses, we can apply Theorem 2.26 to $T = W^\varphi$, which has

$$W_{W^\varphi} = W \qquad \text{and} \qquad Z_{W^\varphi} = \alpha \left(\varphi^* \langle \, , \, \rangle_N - \frac{1}{m} |d\varphi|^2 \langle \, , \, \rangle \right) ,$$

to obtain that W^φ is a constant multiple of $\langle \, , \, \rangle \owedge \langle \, , \, \rangle$. Therefore, M is locally conformally flat and φ is homothetic.

If we further assume that S^φ is constant, then also S is constant since $|d\varphi|^2$ is constant too. In particular, by Proposition 2.38 we have that one of the following cases occurs:

a) M is a quotient of $\mathbb{R}^m$,

b) M is a quotient of $\mathbb{S}^{m-1} \times \mathbb{R}$,

c) M is conformally equivalent to a quotient of $\mathbb{S}^m$.

If c) is in force, then M is necessarily compact, so we can apply Theorem 2.4 to $T = \mathrm{Riem}$, which is harmonic in this case, to deduce that M is locally symmetric. Since we also know that M is locally conformally flat and we have $\mathrm{Ric} \geq 0$ as a consequence of $\mathfrak{R}^{(\lfloor \frac{m-1}{2} \rfloor)} \geq 0$, by a Theorem by Noronha [29, Theorem 1 and Proposition 4.2] we conclude that M is in fact *isometric* to a quotient of $\mathbb{S}^m$, and this proves the first part of the thesis. If $\mathfrak{R}^{(\lfloor \frac{m-1}{2} \rfloor)} > 0$ at some point $x \in M$, then we also have $\mathrm{Ric} > 0$ at x, so alternatives a) and b) are ruled out and the only possibility is that M is a quotient of $\mathbb{S}^m$. $\square$

Proof of Theorem 2.36. The first claim is a mere application of Theorem 2.27 to the case of $T = W^\varphi$. As for the second part, we argue as in the proof of Theorem 2.35, noticing that assuming either (a) or (b) yields $\mathfrak{R}^{\lfloor \frac{m-1}{2} \rfloor} \geq 0$ somewhere, hence M must be isometric to a quotient of a sphere. $\square$

Slightly modifying the argument, we can assume S^φ constant and remove the bound on $|W|^p$ in the hypotheses of Theorem 2.36, resulting in the following Theorem:

Theorem 2.39. *Let $(M^m, \langle\, , \rangle)$ be a complete Riemannian manifold of dimension $m \geq 4$, with $\mathfrak{R}^{\lfloor \frac{m-1}{2} \rfloor} \geq 0$ and that satisfies either*

(a) $\mathfrak{R}^{(\lfloor \frac{m-1}{2} \rfloor)} > 0$ somewhere on M and (2.2.15) for some $o \in M$, or

(b) $\mathfrak{R}^{(\lfloor \frac{m-1}{2} \rfloor)} \geq a$ for some measurable function $a \geq 0$ satisfying ii) or iii) in Theorem 2.24.

Let also $\varphi : M \to (N, \langle\, , \rangle_N)$ for which W^φ is harmonic and S^φ is constant. Assume that

$$\lim_{R \to +\infty} \frac{1}{|B_R(x)|} \int_{B_R(x)} \left| \varphi^* \langle\, , \rangle_N - \frac{1}{m} |d\varphi|^2 \langle\, , \rangle \right|^p < +\infty$$

for some $p \in [1, +\infty)$. Then M is isometric to a quotient of $\mathbb{S}^m$ and the map φ is homothetic.

Proof. Let us consider

$$V_{W^\varphi} = \frac{\alpha}{m-2} \left(\varphi^* \langle\, , \rangle_N - \frac{1}{m} |d\varphi|^2 \langle\, , \rangle \right).$$

Since W^φ is harmonic, V_{W^φ} is harmonic too and satisfies the hypotheses of Theorem 2.27. Therefore, V_{W^φ} is a constant multiple of $\langle\, , \rangle \owedge \langle\, , \rangle$. In particular, it must vanish and, as a consequence, $|d\varphi|^2$ is constant and φ is homothetic. Concerning the scalar curvature, from the constancy of $|d\varphi|^2$ we obtain that S is constant and, since $|W| \leq |\,\mathrm{Riem}\,|$, by Corollary 1.7 we see that $|W|$ is bounded on M. A further application of Theorem 2.27 to $T = W$ yields the locally conformally flatness of M. The conclusion is the same as that in the proof of Theorem 2.36 $\square$

Chapter 3

φ-Static spaces

The main subject of this chapter are Riemannian manifolds (M, g), with $\dim(M) \geq 3$, together with a map $\varphi : (M, g) \to (N, g_N)$ for some Riemannian manifold (N, g_N) and satisfying - having fixed, for some $\alpha \neq 0$, $\mathrm{Ric}^\varphi := \mathrm{Ric} - \alpha\varphi^* g_N$ - the equations

$$\begin{cases} \mathrm{Hess}(w) - w\left(\mathrm{Ric}^\varphi - \frac{S^\varphi}{m-1} g\right) = 0 \\ w\tau(\varphi) = -\mathrm{d}\varphi(\nabla w). \end{cases} \tag{3.0.1}$$

If φ is a constant map, then they reduce to

$$\mathrm{Hess}(w) - w\left(\mathrm{Ric} - \frac{S}{m-1} g\right) = 0, \tag{3.0.2}$$

which is called *vacuum static equation* and is widely studied in the literature. The terminology, as well as the derivation of these structures, comes from General Relativity, as already discussed in the Introduction. In the first section we will deal with the derivation of these structures from Lorentzian manifolds given by warped products between the time dimension and an m-dimensional Riemannian manifold. On top of that, we will obtain some first results concerning manifolds carrying these structures, from the constancy of the φ-scalar curvature to the relations between harmonic-Einstein manifolds and φ-static spaces. In the second section we will examine two functionals, both depending on the metric on the base manifold and on the smooth map to a fixed Riemannian manifold, whose critical points are, under some conditions, precisely the harmonic-Einstein manifolds and the φ-static spaces. Lastly, in the third section we will discuss the local geometry of φ-static spaces in presence of a closed conformal vector field that, under further assumptions relating X to the smooth map, lead to a local splitting of M to a warped product with harmonic-Einstein fibers.

3.1 A derivation and first properties

In this section we will first derive the φ-static space structure from Lorentzian warped products which satisfy the Einstein equations with a cosmological constant. Later on, we will discuss some first properties valid for every φ-static space. Namely, we begin by showing that φ-static spaces always have constant φ-scalar curvature. After that, Theorem 3.3 will provide a rigidity result for the vacuum static structure in the compact case as well as for the smooth map φ, given a bound on the geometry of the target manifold. To conclude, we characterize the complete manifolds which support both an harmonic-Einstein structure and a φ-static one.

3.1.1 A derivation from General Relativity

Let us consider, given a $(m+1)$-dimensional manifold $\hat{M}$, the space $\hat{\mathcal{M}}$ of smooth Lorentzian metrics on $\hat{M}$ with signature $-, + \cdots +$, m plus signs and, given a Riemannian manifold (N, g_N), the space $\hat{\mathcal{P}} = C^\infty(\hat{M}, (N, g_N))$. Let us also consider, for every $\Omega \subset\subset \hat{M}$ the functional on the space $\hat{\mathcal{M}} \times \hat{\mathcal{P}}$, defined as

$$\mathbf{A}_\Omega(\hat{g}, \hat{\varphi}) := \int_\Omega (\hat{S} - \alpha \operatorname{tr}_{\hat{g}} \hat{\varphi}^* g_N - V(\hat{\varphi})) \mathrm{d}\operatorname{Vol}_{\hat{g}}, \tag{3.1.1}$$

for some $\alpha \in \mathbb{R} \setminus \{0\}$, where $\hat{S}$ is the scalar curvature of $(\hat{M}, \hat{g})$ and $V : (N, g_N) \to \mathbb{R}$ a smooth function. The Euler-Lagrange equations of the above functional with respect to compactly supported variations of $\hat{g}$ and $\hat{\varphi}$ can be computed by considering the variation of $\mathbf{A}_\Omega$ evaluated at $(\hat{g}, \hat{\varphi})$ under both the variation of $\hat{g}$ and $\hat{\varphi}$, and then imposing it to be zero. We first consider the variation with respect to the metric on $\hat{M}$. To this purpose, we choose an orthonormal coframe $\{\hat{\theta}^\alpha\}_{\alpha=0\ldots m}$, for which the metric $\hat{g}$ writes as

$$\hat{g} = \eta_{\alpha\beta} \hat{\theta}^\alpha \otimes \hat{\theta}^\beta,$$

where $\eta = \operatorname{diag}(-1, 1, \ldots, 1)$. Now, the calculations in Proposition 1.13 can be adapted in this setting, where here

$$\dot{\hat{\theta}}^\alpha(t) = a^\alpha_\beta(t) \hat{\theta}^\beta(t)$$

and

$$\hat{g}_t = \hat{h}_{\alpha\beta}(t) \hat{\theta}^\alpha(t) \otimes \hat{\theta}^\beta(t) = (\eta_{\gamma\beta} a^\gamma_\alpha(t) + \eta_{\alpha\gamma} a^\gamma_\beta(t)) \hat{\theta}^\alpha(t) \otimes \hat{\theta}^\beta(t).$$

Hence, with the usual notation for raised and lowered indices, from (1.4.6)

$$\dot{\hat{S}} = \hat{h}_{\alpha\beta,}{}^{\alpha\beta} - \hat{h}_\alpha{}^\alpha{}_{,\beta}{}^\beta - \hat{R}_{\alpha\beta} \hat{h}^{\alpha\beta}.$$

Moreover, since $\hat{\varphi}^* g_N$ doesn't vary under the variation of $\hat{g}$,

$$\frac{d}{dt}(\operatorname{tr}_{\hat{g}}(\hat{\varphi}^* g_N)) = \frac{d}{dt}(\hat{g}^{-1}, \hat{\varphi}^* g_N) = -\hat{h}^{\alpha\beta} \hat{\varphi}^a_\alpha \hat{\varphi}^a_\beta$$

and, because $V(\hat{\varphi})$ in independent of the metric on M, its variation vanishes. It remains to evaluate the variation of the volume form, which is

$$\frac{d}{dt}(\mathrm{d}\operatorname{Vol}_{\hat{g}}) = \frac{d}{dt}(\hat{\theta}^0 \wedge \hat{\theta}^1 \wedge \cdots \wedge \hat{\theta}^m) = a^0_0 \hat{\theta}^0 \wedge \hat{\theta}^1 \wedge \cdots \wedge \hat{\theta}^m + \cdots + a^m_m \hat{\theta}^0 \wedge \hat{\theta}^1 \wedge \cdots \wedge \hat{\theta}^m = \frac{1}{2} \operatorname{tr}_{\hat{g}}(\hat{h}) \mathrm{d}\operatorname{Vol}_{\hat{g}}.$$

Summarizing we have, using the divergence theorem,

$$\begin{aligned}
\frac{d}{dt}\mathbf{A}_\Omega(\hat{g}, \hat{\varphi}) &= \int_\Omega \Big(\hat{h}_{\alpha\beta,}{}^{\alpha\beta} - \hat{h}_\alpha{}^\alpha{}_{,\beta}{}^\beta - \hat{R}_{\alpha\beta}\hat{h}^{\alpha\beta} + \alpha \hat{h}^{\alpha\beta} \hat{\varphi}^a_\alpha \hat{\varphi}^a_\beta + \frac{1}{2} \operatorname{tr}_{\hat{g}} \hat{h} (\hat{S} - \alpha \hat{\varphi}^* g_N - V(\hat{\varphi}))\Big) \mathrm{d}\operatorname{Vol}_{\hat{g}} \\
&= \int_\Omega \Big(- \hat{R}_{\alpha\beta}\hat{h}^{\alpha\beta} + \alpha \hat{h}^{\alpha\beta} \hat{\varphi}^a_\alpha \hat{\varphi}^a_\beta + \frac{1}{2}(\hat{S} - \alpha \hat{\varphi}^* g_N - V(\hat{\varphi})) \eta_{\alpha\beta} \hat{h}^{\alpha\beta}\Big) \mathrm{d}\operatorname{Vol}_{\hat{g}} \\
&= \int_\Omega \Big(- \hat{R}_{\alpha\beta} + \alpha \hat{\varphi}^a_\alpha \hat{\varphi}^a_\beta + \frac{1}{2}(\hat{S} - \alpha \hat{\varphi}^* g_N - V(\hat{\varphi})) \eta_{\alpha\beta}\Big) \hat{h}^{\alpha\beta} \mathrm{d}\operatorname{Vol}_{\hat{g}}.
\end{aligned} \tag{3.1.2}$$

Imposing it to be 0 for any $\hat{h} \in T_{\hat{g}}\hat{\mathcal{M}}$ (with compact support inside Ω) yields

$$\widehat{\operatorname{Ric}} - \frac{1}{2}\hat{S}g + \frac{1}{2}V(\hat{\varphi})\hat{g} = \alpha\left(\hat{\varphi}^* g_N - \frac{1}{2}|\mathrm{d}\hat{\varphi}|^2 \hat{g}\right). \tag{3.1.3}$$

which can be interpreted as Einstein equation with a stress-energy tensor (the stress-energy tensor of the map $\hat{\varphi}$ as defined in (1.3.5))

$$T = \alpha\left(\hat{\varphi}^* g_N - \frac{1}{2}|\mathrm{d}\hat{\varphi}|^2 \hat{g}\right),$$

and where $V : (N, g_N) \to \mathbb{R}$ plays the role of a self-interacting potential. As for the variation with respect to the map $\hat{\varphi}$, the only map-dependent terms inside the integral in (3.1.1) are

$$\operatorname{tr}_{\hat{g}}(\hat{\varphi}^* g_N)$$

and

$$V(\hat{\varphi}).$$

To evaluate the variations, we adapt the notation of Proposition 1.15 to this setting, having on one hand

$$\begin{aligned}\frac{d}{dt}\operatorname{tr}_{\hat{g}}(\hat{\varphi}^* g_N) &= \frac{d}{dt}\operatorname{tr}_{\hat{g}\times g_N}(\mathrm{d}\hat{\varphi}\otimes \mathrm{d}\hat{\varphi})\\ &= 2\operatorname{tr}_{\hat{g}\times g_N}(\mathrm{d}\hat{\varphi}\otimes \frac{d}{dt}\mathrm{d}\hat{\varphi}) = 2\operatorname{tr}_{\hat{g}\times g_N}(\mathrm{d}\hat{\varphi}\otimes \nabla\hat{v}),\end{aligned}$$

whereas on the other

$$\frac{d}{dt}V(\hat{\varphi}) = \operatorname{tr}_{g_N}\left(\nabla V(\hat{\varphi})\otimes \frac{d}{dt}\hat{\varphi}\right) = \operatorname{tr}_{g_N}(\nabla V(\hat{\varphi})\otimes \hat{v}).$$

Putting all together and using the fact that $\hat{v}$ has compact support inside Ω, we can use the divergence theorem on

$$\operatorname{div}((\operatorname{tr}_{g_N}\mathrm{d}\hat{\varphi}\otimes\hat{v})^\sharp) = \operatorname{tr}_{\hat{g}\times g_N}(\mathrm{d}\hat{\varphi}\otimes\nabla\hat{v}) + (\eta^{\alpha\beta}\hat{\varphi}^a_{\alpha\beta}\hat{v}^a) = \operatorname{tr}_{\hat{g}\times g_N}(\mathrm{d}\hat{\varphi}\otimes\nabla\hat{v}) + \operatorname{tr}_{g_N}(\tau(\hat{\varphi})\otimes\hat{v}),$$

where $\tau(\hat{\varphi})$ is the tension field of the map $\hat{\varphi}$, in order to obtain

$$\begin{aligned}\frac{d}{dt}\mathbf{A}(\hat{g},\hat{\varphi}) &= \int_\Omega (-2\alpha\operatorname{tr}_{\hat{g}\times g_N}(\mathrm{d}\hat{\varphi}\otimes\nabla\hat{v}) - \operatorname{tr}_{g_N}(\nabla V(\hat{\varphi})\otimes\hat{v}))\mathrm{d}\operatorname{Vol}_{\hat{g}}\\ &= \int_\Omega \big(2\alpha\operatorname{tr}_{g_N}(\tau(\hat{\varphi})\otimes\hat{v}) - \operatorname{tr}_{g_N}(\nabla V(\hat{\varphi})\otimes\hat{v})\big)\mathrm{d}\operatorname{Vol}_{\hat{g}},\end{aligned}$$

which vanishes for an arbitrary $\hat{v}\in T_{\hat{\varphi}}\mathcal{P}$ if and only if

$$2\alpha\tau(\hat{\varphi}) = \nabla V(\hat{\varphi}). \tag{3.1.4}$$

At this point, using (3.1.3) we can express $\hat{S}$ in terms of $\hat{\varphi}$,

$$\hat{S} = \alpha|\mathrm{d}\hat{\varphi}|^2 + \frac{m+1}{m-1}V(\hat{\varphi})$$

and substitute it back into the field equations (3.1.3) and (3.1.4) to obtain

$$\begin{cases}\widehat{\operatorname{Ric}} - \alpha\hat{\varphi}^* g_N = \frac{V(\hat{\varphi})}{m-1}\hat{g}\\ \tau(\hat{\varphi}) = \frac{1}{2\alpha}\nabla V(\hat{\varphi}).\end{cases} \tag{3.1.5}$$

Let us now put ourselves in the case where

$$\hat{M} = \mathbb{R}\times M,$$

with the warped product metric of the form

$$\hat{g} = -e^{-2\hat{f}}dt^2 + g_M, \tag{3.1.6}$$

t being the standard coordinate on $\mathbb{R}$, g_M a Riemannian metric on M, $\hat{f} = f\circ\pi_M, f: M\to\mathbb{R}$ smooth, and suppose that $\hat{\varphi} = \varphi\circ\pi_M, \varphi: (M, g_M)\to(N, g_N)$ smooth. Then we can split system (3.1.5) according to the warped product, thus obtaining a system for the Riemannian tensors on (M, g_M), the warping function f and the map φ. To this purpose, we can choose an orthonormal coframe by

$$\begin{aligned}\hat{\theta}^0 &= e^{-\hat{f}}\mathrm{d}t\\ \hat{\theta}^i &= \pi_M^*\theta^i, \quad i = 1,\dots,m\end{aligned}$$

where $\{\theta^i\}$ is an orthonormal coframe for (M, g_M). Since the connection forms $\hat{\theta}^\alpha_\beta$ of $(\hat{M},\hat{g})$ satisfy

$$\eta_{\alpha\gamma}\hat{\theta}^\gamma_\beta + \eta_{\beta\gamma}\hat{\theta}^\gamma_\alpha = 0,$$

we infer that

$$\begin{aligned}
\hat{\theta}^0_0 &= \hat{\theta}^i_i = 0;\\
\hat{\theta}^i_0 &= \hat{\theta}^0_i;\\
\hat{\theta}^i_j &= -\hat{\theta}^j_i.
\end{aligned}$$

Then we exploit the first structure equations, to get

$$-\hat{\theta}^0_\alpha \wedge \hat{\theta}^\alpha = \mathrm{d}\hat{\theta}^0 = -e^{-\hat{f}}\hat{f}_\alpha\hat{\theta}^\alpha \wedge \mathrm{d}t = \hat{f}_j\hat{\theta}^0 \wedge \hat{\theta}^j \tag{3.1.7}$$

$$-\hat{\theta}^i_\alpha \wedge \hat{\theta}^\alpha = \mathrm{d}\hat{\theta}^i = \mathrm{d}\theta^i = -\theta^i_j \wedge \theta^j, \tag{3.1.8}$$

where we have made an abuse of notation. On one hand, evaluating (3.1.7) in $(e_i, \cdot)$ we obtain that

$$-\hat{\theta}^0_j(e_i) \wedge \hat{\theta}^j + \hat{\theta}^0_i = -\hat{f}_i\hat{\theta}^0, \tag{3.1.9}$$

and hence, contracting with e_j,

$$(\hat{\theta}^0_i)_j = (\hat{\theta}^0_j)_i. \tag{3.1.10}$$

On the other, we have that

$$-\hat{\theta}^i_j + \hat{\theta}^i_0(e_j)\hat{\theta}^0 = -\theta^i_j + \theta^i_k(e_j) \wedge \theta^k,$$

from which we deduce, evaluating in e_0, that

$$\hat{\theta}^i_0(e_j) = (\hat{\theta}^i_j - \theta^i_j)(e_0),$$

so that, in particular,

$$(\hat{\theta}^i_0)_j = -(\hat{\theta}^j_0)_i.$$

At this point, from the fact that $\hat{\theta}^i_0 = \hat{\theta}^0_i$ and by (3.1.10) we infer that

$$\hat{\theta}^0_i(e_j) = 0.$$

Therefore, from (3.1.9) we get

$$\hat{\theta}^i_0 = -\hat{f}_i\hat{\theta}^0.$$

As a consequence, from (3.1.8) we have that

$$(\hat{\theta}^i_j - \theta^i_j) \wedge \hat{\theta}^j = 0.$$

We then make use of the Cartan's lemma and exploit the antisymmetricity of $\hat{\theta}^i_j - \theta^i_j$ on i and j to see that, denoting with

$$b^i_{jk} = b^i_{kj} = (\hat{\theta}^i_j - \theta^i_j)(e_k),$$

they satisfy

$$b^i_{jk} = -b^j_{ik} = -b^j_{ki} = b^k_{ji} = b^k_{ij} = -b^i_{kj} = -b^i_{jk},$$

and hence

$$\hat{\theta}^i_j = \theta^i_j.$$

Summarizing, we have

$$\begin{aligned}
\hat{\theta}^0_0 &= 0;\\
\hat{\theta}^0_i &= \hat{\theta}^i_0 = -\hat{f}_i\hat{\theta}^0;\\
\hat{\theta}^i_j &= -\hat{\theta}^j_i = \theta^i_j.
\end{aligned}$$

Next, we make use of the second structure equations, where we set $\hat{\Theta}^\alpha_\beta$ as the curvature forms of $(\hat{M}, \hat{g})$:

$$
\begin{aligned}
\hat{\Theta}^i_0 = \hat{\Theta}^0_i &= \mathrm{d}\hat{\theta}^0_i + \hat{\theta}^0_t \wedge \hat{\theta}^t_i = -\mathrm{d}\hat{f}_i \wedge \hat{\theta}^0 - \hat{f}_i \mathrm{d}\hat{\theta}^0 + \hat{\theta}^0_t \wedge \hat{\theta}^t_i \\
&= -(\hat{f}_{i\alpha}\hat{\theta}^\alpha + \hat{f}_\alpha \hat{\theta}^\alpha_i) \wedge \hat{\theta}^0 + \hat{f}_i \hat{\theta}^0_\alpha \wedge \hat{\theta}^\alpha - \hat{f}_t \hat{\theta}^0 \wedge \hat{\theta}^t_i \\
&= (\hat{f}_{ij} - \hat{f}_i \hat{f}_j)\hat{\theta}^0 \wedge \hat{\theta}^j \\
\hat{\Theta}^i_j &= \mathrm{d}\hat{\theta}^i_j + \hat{\theta}^i_\alpha \wedge \hat{\theta}^\alpha_j = \mathrm{d}\hat{\theta}^i_j + \hat{\theta}^i_k \wedge \hat{\theta}^k_j = \Theta^i_j.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&\hat{R}^0_{i0j} = \hat{R}^i_{00j} = -\hat{R}^0_{ij0} = f_{ij} - f_i f_j; \\
&\hat{R}^0_{ijk} = 0; \\
&\hat{R}^i_{jkl} = R^i_{jkl},
\end{aligned}
$$

where we have omitted the pullbacks, and consequently

$$
\begin{aligned}
&\hat{R}_{00} = \hat{R}^i_{0i0} = -\Delta_M f + |\nabla^M f|^2_{g_M}; \\
&\hat{R}_{0i} = 0; \\
&\hat{R}_{ij} = \hat{R}^0_{i0j} + \hat{R}^k_{ikj} = f_{ij} - f_i f_j + R_{ij},
\end{aligned}
$$

from which

$$
\hat{S} = \operatorname{tr}_{\hat{g}} \widehat{\mathrm{Ric}} = 2(\Delta_M f - |\nabla f|^2_{g_M}) + S.
$$

Moreover, since

$$
\mathrm{d}\hat{\varphi} = \hat{\varphi}^a_i \theta^i,
$$

$$
\begin{aligned}
\hat{\varphi}^a_{0\alpha}\hat{\theta}^\alpha &= \mathrm{d}\hat{\varphi}^a_0 - \hat{\varphi}^a_\beta \wedge \hat{\theta}^\beta_0 + \hat{\theta}^b_0 \omega^a_b \\
&= -\hat{\varphi}^a_i \wedge \hat{\theta}^i_0 = \hat{\varphi}^a_i f_i \hat{\theta}^0 \\
\hat{\varphi}^a_{i\alpha}\hat{\theta}^\alpha &= \mathrm{d}\hat{\varphi}^a_i - \hat{\varphi}^a_\beta \wedge \hat{\theta}^\beta_i + \hat{\theta}^b_i \omega^a_b \\
&= \varphi^a_{ij} \wedge \hat{\theta}^j,
\end{aligned}
$$

and hence

$$
\begin{aligned}
&\hat{\varphi}^a_{00} = \varphi^a_i f_i = \mathrm{d}\varphi(\nabla f) \\
&\hat{\varphi}^a_{0i} = 0 \\
&\hat{\varphi}^a_{ij} = \varphi^a_{ij},
\end{aligned}
$$

so that tracing

$$
\tau(\hat{\varphi}) = -\mathrm{d}\varphi(\nabla f) + {}^M\tau(\varphi).
$$

Substituting the previous relations into (3.1.5) we obtain the splitting

$$
\begin{cases}
\Delta_M f - |\nabla f|^2 = \frac{V(\varphi)}{m-1} \\
\mathrm{Ric} - \alpha\varphi^* g_N + \mathrm{Hess}(f) - \mathrm{d}f \otimes \mathrm{d}f = \frac{V(\varphi)}{m-1} g_M \\
{}^M\tau(\varphi) = \mathrm{d}\varphi(\nabla f) + \frac{1}{2\alpha}\nabla V(\varphi).
\end{cases}
\tag{3.1.11}
$$

Note that the first equation in the above system is the "time" component of the first equation in (3.1.5). Tracing the second equation in (3.1.11) (on g_M) and substituting the first, we see that, on M,

$$
S^\varphi = S - \alpha|\mathrm{d}\varphi|^2 = V(\varphi).
$$

Operating the change of variable

$$
w = e^{-f} \tag{3.1.12}
$$

system (3.1.11) becomes

$$\begin{cases} \Delta w = -\frac{S^\varphi}{m-1} w \\ w \operatorname{Ric}^\varphi - \operatorname{Hess}(w) = w \frac{S^\varphi}{m-1} g_M \\ w\tau(\varphi) + \mathrm{d}\varphi(\nabla w) = w \frac{1}{2\alpha} \nabla V(\varphi). \end{cases} \tag{3.1.13}$$

If we let $V : N \to \mathbb{R}$ be a constant, then also S^φ is constant and the system (3.1.13) becomes

$$\begin{cases} \operatorname{Hess}(w) - w \left(\operatorname{Ric}^\varphi - \frac{S^\varphi}{m-1} g \right) = 0 \\ w\tau(\varphi) = -\mathrm{d}\varphi(\nabla w) \end{cases} \tag{3.1.14}$$

where we omitted, since it is redundant, the first equation in (3.1.13). Notice that, in view of the Einstein equation (3.1.3), $V(\varphi) = S^\varphi = 2\Lambda$, where Λ stands for the cosmological constant. We are now in the position to give the following

Definition 3.1. Let $(M, g), (N, g_N)$ be Riemannian manifolds and $\varphi : M \to N$ smooth. We say that (M, g) is a φ-static space if for some $\alpha \neq 0$ and $w \in C^\infty(M), w \not\equiv 0$ w is a solution of (3.1.14) on M.

Note that when φ is constant Definition 3.1 reduces to the definition of a vacuum static space. Moreover, this definition takes into account sign changing functions w, which do not derive from smooth semi-riemannian metrics on $\hat{M}$ (or where we allow singularities on the Lorentzian metric). Here we also allow S^φ to be non-constant. However, as we show in Proposition 3.1 below, if (3.1.14) holds for some $w \not\equiv 0$, then S^φ turns out to be constant. We will often consider the system (3.1.14) in its equivalent form

$$\begin{cases} \operatorname{Hess}(w) - (\Delta w) g - w \operatorname{Ric}^\varphi = 0 \\ w\tau(\varphi) + \mathrm{d}\varphi(\nabla w) = 0. \end{cases} \tag{3.1.15}$$

3.1.2 First properties and rigidity conditions

We now show that, if (M, g) is a φ-static space, where $\varphi : M \to (N, g_N)$, then S^φ must be constant:

Proposition 3.1. *Let (M, g) be a Riemannian manifold, possibly with smooth boundary, and φ be a smooth map between M and a Riemannian manifold (N, g_N). Assume that there exists $w \not\equiv 0$ such that*

$$\begin{cases} \operatorname{Hess}(w) - (\Delta w) g - w \operatorname{Ric}^\varphi = 0 \\ w\tau(\varphi) + \mathrm{d}\varphi(\nabla w) = 0. \end{cases} \tag{3.1.16}$$

Then

i) S^φ must be constant;

ii) if M is compact, then $S^\varphi > 0$ provided that either M is without boundary and w is non constant, or $\partial M \neq \varnothing$ and $w = 0$ on ∂M;

iii) $\{w = 0\}$ is a union of embedded, totally geodesic hypersurfaces of M where $|\nabla w|$ is locally constant and nowhere vanishing.

Proof. (i) Taking the divergence of the first in (3.1.16), in local coordinates we have

$$\begin{aligned} 0 &= (w_{ij} - w_{ss}\delta_{ij} - wR^\varphi_{ij})_{,j} = w_{ijj} - w_{ssi} - w_j R^\varphi_{ij} - wR^\varphi_{ij,j} \\ &= w_t R_{ti} - w_j R^\varphi_{ij} - w \left(\frac{1}{2} S^\varphi_i - \alpha \varphi^a_{jj} \varphi^a_i \right) \\ &= \alpha w_j \varphi^a_j \varphi^a_i + \frac{1}{2} w S^\varphi_i + \alpha w \varphi^a_{jj} \varphi^a_i \\ &= \frac{1}{2} w S^\varphi_i + \alpha (w_j \varphi^a_j + w \varphi^a_{jj}) \varphi^a_i \\ &= \frac{1}{2} w S^\varphi_i . \end{aligned}$$

Here we have used the commutation relations for the covariant derivatives of w, as well as (1.3.12) and the second equation in (3.1.16). On the other hand, since w satisfies

$$\Delta w = -\frac{S^{\varphi}}{m-1} w, \tag{3.1.17}$$

if $\{w = 0\}$ contains an open non empty set, it would vanish identically by unique continuation principle. Therefore, the interior of $\{w = 0\}$ is empty and, as a consequence, $\nabla S^{\varphi} = 0$ on the closure of $M \setminus \{w = 0\}$, i.e. on the whole M, leading to S^{φ} being constant.
(ii) If M is compact and either of the assumptions is satisfied then w is not constant on M, so using (3.1.17) and integrating by parts - in case $\partial M \neq \varnothing$, notice that we have $w|_{\partial M} = 0$ - we get

$$\frac{S^{\varphi}}{m-1}\int_M w^2 = -\int_M w\Delta w = \int_M |\nabla w|^2 > 0,$$

implying that $S^{\varphi} > 0$.
(iii) We first prove that $|\nabla w| \neq 0$ everywhere on $\{w = 0\}$ by showing that the set

$$M_0 = \{x \in M : w(x) = 0\,,\ \nabla w(x) = 0\}$$

is both open and closed in M, hence empty by connectedness of M and assumption $w \not\equiv 0$. M_0 is clearly closed by continuity of w and ∇w, so we only have to show that it is also open. Let $x \in M_0$ be given. Then $w = 0$ along all geodesic curves issuing from x and contained in M. Indeed, if $\sigma : [0,T) \to M$, $T > 0$, is a unit speed geodesic such that $\sigma(0) = x$ then by (3.1.14) and (3.1.17) the function $h = w \circ \sigma : [0,T] \to \mathbb{R}$ is a solution of the Cauchy problem

$$\begin{cases} h'' &= Ah \qquad \text{on } (0,T) \\ h'(0) &= 0 \\ h(0) &= 0 \end{cases}$$

where $A(t) = \mathrm{Ric}^{\varphi}_g(\dot\sigma(t), \dot\sigma(t)) - \frac{1}{m-1}S^{\varphi}_g(\sigma(t))$, and since A is continuous we have $h \equiv 0$ on $[0,T]$.

If $x \in M_0 \setminus \partial M$ this shows that $w \equiv 0$, and therefore $\nabla w \equiv 0$, on a whole neighbourhood of x in M, so that x is an interior point for M_0.

If $x \in M_0 \cap \partial M$ then the image of the exponential map at x may not contain a whole M-neighbourhood of x (for instance, this happens if ∂M is not convex at x), however we overcome the obstacle by a suitable "triangulation". First, observe that for any sufficiently small $\varepsilon > 0$ the exponential map $\exp_x$ at x is defined at least on the small open solid cone (with tip removed)

$$C_{\varepsilon} = \{v \in T_xM : |v| < \varepsilon\,,\ \langle v, \mathbf{n}_x\rangle > |v|/2\},$$

where $\mathbf{n}_x$ denotes the inward normal to ∂M at x, and $\exp_x : C_{\varepsilon} \to \exp_x(C_{\varepsilon})$ is a diffeomorphism. By the same reasoning as above we have $w \equiv 0$, and therefore $\nabla w \equiv 0$, on the open set $\exp_x(C_{\varepsilon})$, so $\exp_x(C_{\varepsilon}) \subseteq M_0$. Now, let $U \subseteq \partial M$ be a small enough neighbourhood of x in ∂M and $\delta > 0$ a small enough parameter so that the normal exponential map

$$\begin{array}{rccc} \exp_{\mathbf{n},U} : & U \times [0,\delta) & \to & M \\ & (y,t) & \mapsto & \exp_y(t\mathbf{n}_y) \end{array}$$

is a diffeormorphism onto its image $V := \exp_{\mathbf{n},U}(U \times [0,\delta))$. Then V is a neighbourhood of x in M and for $\varepsilon > 0$ small enough we have $\exp_x(C_{\varepsilon}) \subseteq B_{\varepsilon}(x) \subseteq V$. Let $\pi_1 : U \times [0,\delta) \to U$ denote the canonical projection onto the first factor. Then $U_1 = \pi_1(\exp^{-1}_{\mathbf{n},U}(\exp_x(C_{\varepsilon})))$ is an open subset of U (hence, of ∂M) since $\exp_x(C_{\varepsilon})$ is open in M and π_1 is an open map. By construction, U_1 is the subset of points in U which are joined by a normal geodesic of length $< \delta$ to some point of $\exp_x(C_{\varepsilon})$. We also have $x \in U_1$, since for every $t \in (0,\varepsilon)$ we have $t\mathbf{n}_x \in C_{\varepsilon}$ and therefore $x = \pi_1(\exp^{-1}_{\mathbf{n},U}(\exp_x(t\mathbf{n}_x)) \in U_1$. So U_1 is a neighbourhood of x in ∂M and $V_1 := \exp_{\mathbf{n},U}(U_1 \times [0,\delta))$ is an open neighbourhood of x in M. Every point of V_1 lies on a geodesic curve passing through some point of $\exp_x(C_{\varepsilon}) \subseteq M_0$, hence by the same reasoning as above we have $w \equiv 0$ on V_1. Since V_1 is open and ∂M is smooth we also have $\nabla w \equiv 0$ on V_1. So V_1 is a neighbourhood of x contained in M_0, as desired.

So far, we have proved that $\nabla w \neq 0$ everywhere on $\{w = 0\}$. Then 0 is a regular value for w and the set $\Sigma = \{w = 0\}$ is a union of embedded hypersurfaces of M. By substituting (3.1.17) into (3.1.16) we have $\operatorname{Hess} w = 0$ on Σ. Since the differential of $|\nabla w|^2$ in M and the second fundamental form II_Σ of Σ in M in the direction of $-\nabla w/|\nabla w|$ are given by

$$\mathrm{d}|\nabla w|^2 = 2\operatorname{Hess} w(\nabla w, \cdot\,) \qquad \text{and} \qquad \mathrm{II}_\Sigma = \frac{1}{|\nabla w|}\operatorname{Hess} w_{|T\Sigma\times T\Sigma}$$

we see that $|\nabla w|$ is locally constant on Σ, whose components are in turn totally geodesic in M. □

Remark 3.2. Notice that, if we assume a weak sign condition, e.g. $w \geq 0$ on a compact manifold without boundary satisfying (3.1.14), then w must be constant. Indeed, if $S^\varphi \neq 0$ integrating

$$\Delta w = -\frac{S^\varphi}{m-1} w$$

on M, we have

$$0 = -\frac{S^\varphi}{m-1}\int_M w,$$

which imply, by the weak sign condition, that $w \equiv 0$. Otherwise, if $S^\varphi = 0$, then w is harmonic and therefore constant on M.

In some cases, where we have bounds on the geometry of the target manifold according to the coupling constant giving rise to the φ-curvatures on (M, g), then the map φ is forced to be constant and therefore the φ-curvatures reduce to the standard ones. This is the case of the following

Theorem 3.3. *Let (M^m, g), $m \geq 3$, be a compact Riemannian manifold without boundary, (N^n, g_N) a second Riemannian manifold and $\alpha \in \mathbb{R}$. Let $w \in C^\infty(M)$, w non constant, and $\varphi \in C^\infty(M, N)$ satisfy*

$$\begin{cases} w\operatorname{Ric}^\varphi - \operatorname{Hess} w + (\Delta w)\, g &= 0 \\ w\tau(\varphi) + \mathrm{d}\varphi(\nabla w) &= 0\,. \end{cases} \tag{3.1.18}$$

If $\alpha \geq 0$ and the sectional curvatures of N satisfy $\sec_N \leq \frac{\alpha}{m-1}$, then φ is constant.

In order to prove it, we shall first need the subsequent Lemma, providing us with a Bochner-type formula that later on will be integrated on M.

Lemma 3.4. *Let (M, g), (N, g_N) be Riemannian manifolds and $w \in C^\infty(M)$, $\varphi \in C^\infty(M, N)$ satisfy* (3.1.18). *Then*

$$\frac{1}{2}\operatorname{div}(w\nabla|\mathrm{d}\varphi|^2) = w|\nabla \mathrm{d}\varphi|^2 + w|\tau(\varphi)|^2 - |\mathrm{d}\varphi|^2\Delta w + wQ(\mathrm{d}\varphi) \tag{3.1.19}$$

where, in components along an orthonormal coframe,

$$Q(\mathrm{d}\varphi) = (\alpha g^N_{ad} g^N_{bc} - {}^N R_{abcd})\varphi^a_i \varphi^b_j \varphi^c_i \varphi^d_j\,.$$

Moreover, if $\alpha \geq 0$ and $\sec_N \leq \frac{\alpha}{m-1}$ then $Q(\mathrm{d}\varphi) \geq 0$.

Proof. Let $\{e_i\}$ and $\{E_a\}$ be local orthonormal frames for TM and TN, respectively. A straightforward computation yields

$$\begin{aligned} \frac{1}{2}\operatorname{div}(w\nabla|\mathrm{d}\varphi|^2) &= (w\varphi^a_i\varphi^a_{ji})_{,j} \\ &= w\varphi^a_{ij}\varphi^a_{ji} + w_j\varphi^a_{ji}\varphi^a_i + w\varphi^a_i\varphi^a_{jij} \\ &= w\varphi^a_{ij}\varphi^a_{ij} + w_j\varphi^a_{ji}\varphi^a_i + w\varphi^a_i(\varphi^a_{jji} + R_{ji}\varphi^a_j - {}^N R^a_{bcd}\varphi^b_j\varphi^c_i\varphi^d_j) \end{aligned}$$

where we applied (a contraction of) (1.3.8). At this point, differentiating the second in (3.1.18),

$$0 = (w_j\varphi^a_j + w\varphi^a_{jj})_i = (w_j\varphi^a_{ji} + w\varphi^a_{jji}) + (w_{ji}\varphi^a_j + w_i\varphi^a_{jj})$$

and therefore

$$\begin{aligned}\frac{1}{2}\operatorname{div}(w\nabla|\mathrm{d}\varphi|^2) &= w|\nabla\mathrm{d}\varphi|^2 + (w_j\varphi^a_{ji} + w\varphi^a_{jji})\varphi^a_i + wR_{ij}\varphi^a_i\varphi^a_j - w\,{}^N\!R^a_{bcd}\varphi^a_i\varphi^b_j\varphi^c_i\varphi^d_j \\ &= w|\nabla\mathrm{d}\varphi|^2 - (w_{ji}\varphi^a_j + w_i\varphi^a_{jj})\varphi^a_i + wR_{ij}\varphi^a_i\varphi^a_j - w\,{}^N\!R^a_{bcd}\varphi^a_i\varphi^b_j\varphi^c_i\varphi^d_j \\ &= w|\nabla\mathrm{d}\varphi|^2 + (wR_{ij} - w_{ij})\varphi^a_i\varphi^a_j - w_i\varphi^a_i\varphi^a_{jj} - w\,{}^N\!R^a_{bcd}\varphi^a_i\varphi^b_j\varphi^c_i\varphi^d_j.\end{aligned}$$

Again by (3.1.18) we have $wR_{ij} - w_{ij} = \alpha\, w\,\varphi^b_i\varphi^b_j - (\Delta w)\,g_{ij}$ and $-w_i\varphi^a_i = w\varphi^a_{ii}$. Substituting these expressions into the above formula we finally obtain

$$\frac{1}{2}\operatorname{div}(w\nabla|\mathrm{d}\varphi|^2) = w|\nabla\mathrm{d}\varphi|^2 + w|\tau(\varphi)|^2 - |\mathrm{d}\varphi|^2\Delta w + \alpha w\varphi^a_i\varphi^b_j\varphi^b_i\varphi^a_j - wR^N_{abcd}\varphi^a_i\varphi^b_j\varphi^c_i\varphi^d_j\,.$$

We now show that $Q(\mathrm{d}\varphi) \geq 0$ under assumptions

$$\alpha \geq 0\,, \qquad \sec_N \leq \frac{\alpha}{m-1}\,. \tag{3.1.20}$$

Let us set $Y_i = \mathrm{d}\varphi(e_i) = \varphi^a_i E_a$ for $1 \leq i \leq m$. Then for each $1 \leq i, j \leq m$ we have

$$\varphi^a_i\varphi^b_j\varphi^b_i\varphi^a_j = (g_N(Y_i, Y_j))^2\,, \qquad {}^N\!R_{abcd}\varphi^a_i\varphi^b_j\varphi^c_i\varphi^d_j = {}^N\mathrm{Riem}(Y_i, Y_j, Y_i, Y_j)$$

(no summation over i or j is intended in the above formulas). For each pair (i,j) we let κ_{ij} be the sectional curvature in N of some 2-plane containing Y_i and Y_j (which is clearly uniquely determined in case Y_i and Y_j are linearly independent, otherwise the coefficients can be set to 0). Then, we have

$${}^N\mathrm{Riem}(Y_i, Y_j, Y_i, Y_j) = \kappa_{ij}\left[g_N(Y_i, Y_i)g_N(Y_j, Y_j) - (g_N(Y_i, Y_j))^2\right]\,.$$

For ease of notation let us set $c_{ij} = g_N(Y_i, Y_j)$ for each $1 \leq i, j \leq m$. Then from the above observations we have

$$Q(\mathrm{d}\varphi) = \alpha\sum_{i,j=1}^m c_{ij}^2 - \sum_{i,j=1}^m \kappa_{ij}(c_{ii}c_{jj} - c_{ij}^2)\,. \tag{3.1.21}$$

Noting that

$$\sum_{1\leq i<j\leq m}(c_{ii}^2 + c_{jj}^2) = \frac{1}{2}\sum_{i\neq j=1}^m (c_{ii}^2 + c_{jj}^2) = \frac{1}{2}\left(\sum_{i,j=1}^m (c_{ii}^2 + c_{jj}^2) - \sum_{i=j=1}^m (2c_{ii}^2)\right) = (m-1)\sum_{i=1}^m c_{ii}^2$$

we can express

$$\sum_{1\leq i,j\leq m} c_{ij}^2 = \sum_{i=1}^m c_{ii}^2 + 2\sum_{1\leq i<j\leq m} c_{ij}^2 = \sum_{1\leq i<j\leq m}\left(\frac{1}{m-1}(c_{ii}^2 + c_{jj}^2) + 2c_{ij}^2\right)$$

Since we also have $c_{ii}c_{jj} - c_{ij}^2 = 0$ whenever $i = j$, we can restate (3.1.21) as

$$Q(\mathrm{d}\varphi) = \sum_{1\leq i<j\leq m}\left[\frac{\alpha}{m-1}(c_{ii}^2 + c_{jj}^2) + 2\alpha c_{ij}^2 - 2\kappa_{ij}(c_{ii}c_{jj} - c_{ij}^2)\right]\,. \tag{3.1.22}$$

Now, since $c_{ii}c_{jj} - c_{ij}^2 \geq 0$, from the assumptions on the sectional curvature in (3.1.20)

$$-2\kappa_{ij}(c_{ii}c_{jj} - c_{ij}^2) \geq -\frac{2\alpha}{m-1}(c_{ii}c_{jj} - c_{ij}^2).$$

Therefore, substituting into (3.1.22) we obtain

$$\begin{aligned}Q(\mathrm{d}\varphi) &\geq \sum_{1\leq i<j\leq m}\left[\frac{\alpha}{m-1}(c_{ii}^2 + c_{jj}^2 - 2c_{ii}c_{jj}) + 2\frac{m}{m-1}\alpha c_{ij}^2\right] \\ &= \sum_{1\leq i<j\leq m}\left[\frac{\alpha}{m-1}(c_{ii} - c_{jj})^2 + 2\frac{m}{m-1}\alpha c_{ij}^2\right] \geq 0.\end{aligned}$$

□

Proof of Theorem 3.3. By (3.1.19) we have

$$\frac{1}{2}\operatorname{div}(w\nabla|\mathrm{d}\varphi|^2) = w|\nabla \mathrm{d}\varphi|^2 + w|\tau(\varphi)|^2 + \frac{S^\varphi}{m-1}w|\mathrm{d}\varphi|^2 + wQ(\mathrm{d}\varphi)$$

with $Q(\mathrm{d}\varphi) \geq 0$ and $S^\varphi > 0$ by Proposition 3.1. So have $\operatorname{div}(w\nabla|\mathrm{d}\varphi|^2) \geq 0$ on $\Omega^+ = \{w > 0\}$, with equality if and only if $\mathrm{d}\varphi \equiv 0$ on Ω^+, and similarly $\operatorname{div}(w\nabla|\mathrm{d}\varphi|^2) \leq 0$ on $\Omega^- = \{w < 0\}$ with equality if and only if $\mathrm{d}\varphi \equiv 0$. Applying the divergence theorem on both sets Ω^+ and Ω^- and using that $w\nabla|\mathrm{d}\varphi|^2$ vanishes on the regular submanifold $\{w = 0\} = \partial\Omega^+ = \partial\Omega^-$ we conclude that $\mathrm{d}\varphi \equiv 0$ on M, that is, φ is constant. □

It is worth noticing that, if a vacuum static space supports also an harmonic-Einstein structure with respect to the same φ and α as in (3.1.14), then we have restrictions on the underlying manifold, as the next Theorem shows:

Theorem 3.5. *Let (M, g) be a complete Riemannian manifold of dimension $m \geq 3$ satisfying* (3.1.14) *for some $\varphi : M \to (N, g_N)$, $\alpha \neq 0$ and a non-constant $w \in C^\infty(M)$. Assume that (M, g) is harmonic-Einstein with the same choice of the map and of the coupling constant. Then*

i) If M is compact without boundary, then (M, g) is isometric to the standard sphere $\mathbb{S}^m$ of sectional curvature $\frac{S^\varphi}{m(m-1)}$ and φ is constant;

ii) If (M, g) is compact with boundary, under the condition of

$$\partial M = \{x \in M : w(x) = 0\},$$

then (M, g) is isometric to a closed hemisphere of $\mathbb{S}^m(\frac{S^\varphi}{m(m-1)})$ with the canonical metric and φ is constant;

iii) If M is non-compact, then $S^\varphi \leq 0$. Moreover, if w has one critical point, then it is isometric to a hyperbolic space of constant curvature $\frac{S^\varphi}{m(m-1)}$ and φ is constant. Otherwise, w has no critical point, (M, g) splits as a warped product $\mathbb{R} \times_f P$, where P is a complete $(m-1)$-dimensional Riemannian manifold and φ is constant along the flow generated by ∇w - i.e. along the curves $\mathbb{R} \times \{p\}, p \in P$.

Proof. Recalling the harmonic-Einstein equation (1.3.11), that is

$$\begin{cases} \mathrm{Ric}^\varphi = \frac{S^\varphi}{m}\langle\, ,\, \rangle \\ \tau(\varphi) = 0, \end{cases}$$

and substituting it into (3.1.14), we obtain

$$\begin{cases} \mathrm{Hess}(w) + \frac{S^\varphi}{m(m-1)}wg = 0; \\ \mathrm{d}\varphi(\nabla w) = 0. \end{cases} \tag{3.1.23}$$

(i) If M is compact without boundary, then we consider the first in (3.1.23) which, by a classical result of Obata (see for instance [37]), yields the fact that (M, g) is a sphere of constant sectional curvature given by $\frac{S^\varphi}{m(m-1)}$. Therefore, tracing the riemannian tensor gives $S = S^\varphi$, from which follows that $|\mathrm{d}\varphi|^2 \equiv 0$, i.e. φ is constant.
(ii) Since the boundary coincides with $\{x \in M : w(x) = 0\}$, we can apply Proposition 3.1 to infer that $\{w = 0\}$ is totally geodesic on M. Then we can apply a result of Reilly [33, Lemma 3], obtaining that (M, g) is an hemisphere of constant sectional curvature $\frac{S^\varphi}{m(m-1)}$. Reasoning as before, we conclude that φ must be constant also in this case.
(iii) If M is complete non-compact, then $S^\varphi \leq 0$, otherwise - see [37, Theorem 2] - (M, g) would be isometric to a sphere and therefore compact. Now, again in view of the classification theorem in [37], if w has at most one critical point. If w has one critical point, then it is isometric to the hyperbolic space of sectional curvature $\frac{S^\varphi}{m(m-1)}$, and hence the scalar curvature $S = S^\varphi$, yielding φ constant. Otherwise, if w has no critical point, then it is a warped product $\mathbb{R} \times_f P$, for some $f : \mathbb{R} \to \mathbb{R}$. Moreover, by the second in (3.1.23) we have that φ must be constant along the flow generated by ∇w. Notice that, in this case, f is either a constant, if $S^\varphi = 0$, or it takes the form $f(t) = A\exp(ct)$ or else $f(t) = A\cosh(ct)$, where A is an arbitrary constant and $c = \sqrt{\frac{-S^\varphi}{m(m-1)}}$. □

To conclude, we give a formula valid on every φ-static space:

Proposition 3.6. *Let $(M, \langle\, ,\, \rangle)$ be a φ-static space with $w \not\equiv 0$. Setting T for the traceless φ-Ricci tensor, we have*

$$\operatorname{div}(T(\nabla w,)^\sharp) = w|T|^2 + \alpha w|\tau(\varphi)|^2 \quad \text{on } M. \tag{3.1.24}$$

Equivalently,

$$\operatorname{div}\left(\frac{1}{w}\left(\nabla|\nabla w|^2 + \frac{S^\varphi}{m(m-1)}\nabla w^2\right)\right) = 2w|T|^2 + 2\alpha w|\tau(\varphi)|^2 \text{ on } M \setminus \Sigma \tag{3.1.25}$$

where as above $\Sigma = \{x \in M : w(x) = 0\}$. If M is compact then

$$\int_M w|T|^2 + \alpha \int_M w|\tau(\varphi)|^2 = 0. \tag{3.1.26}$$

Proof. We compute the divergence of $T(\nabla w,)^\sharp$ and we use the φ-Schur's identity together with the constancy of S^φ and (3.1.14) in the equivalent form (3.1.15) to get

$$\begin{aligned}(w_i T_{ij})_i &= w_{ij}T_{ij} + w_j T_{ij,i} = T_{ij}T_{ij} + w_j R^\varphi_{ij,i} \\ &= w|T|^2 - \alpha\varphi^a_{ii}\varphi^a_i w_j \\ &= w|T|^2 + \alpha w|\tau(\varphi)|^2,\end{aligned}$$

that is, (3.1.24). Rewriting the first in (3.1.14) in the form

$$\operatorname{Hess} w + \frac{S^\varphi}{m(m-1)} w\langle\, ,\, \rangle = wT$$

we obtain

$$wT(\nabla w,)^\sharp = \operatorname{Hess}(\nabla w,)^\sharp + \frac{S^\varphi}{m(m-1)} w\nabla w = \frac{1}{2}\left(\nabla|\nabla w|^2 + \frac{S^\varphi}{m(m-1)}\nabla w^2\right)$$

proving (3.1.25). Finally, if M is compact integrating (3.1.24) on M and applying the divergence theorem we obtain (3.1.26). □

3.2 A variational characterization

First, we will introduce the φ-scalar curvature operator, and we will show that the spaces with non-trivial kernel of the adjoint of $D\mathcal{S}$ (for a choice of $(g, \varphi) \in \mathcal{M} \times \mathcal{P}$) are precisely the φ-static spaces. After having provided a characterization of harmonic-Einstein manifolds, we will consider another functional, defined on compact manifolds with boundary, by the choice of a vector field $X \in \mathfrak{X}(M)$. With the help of this functional, Theorem 3.12 will provide a description of some φ-static spaces, with the potential given by the divergence of a vector field X conformal with respect to the metric on M plus some further assumptions. Namely, they will be given by the choices of (g, φ) such that the functional is stationary with respect to variations leaving the φ-scalar curvature invariant and fixed at the boundary.

3.2.1 The total φ-scalar curvature functional and harmonic-Einstein manifolds

Let us consider $\mathcal{M}$ the space of smooth metrics on a differential manifold M and $\mathcal{P} := C^\infty(M, (N, g_N))$ the space of smooth maps between M and a differential manifold N with a fixed Riemannian metric g_N. We then define, having fixed a constant $\alpha \neq 0$, the φ-scalar curvature operator

$$\mathcal{S} : \mathcal{M} \times \mathcal{P} \to C^\infty(M)$$

by

$$\mathcal{S}(g, \varphi) = S^\varphi.$$

In the next Lemma we will compute the differential of $\mathcal{S}$ at (g, φ) by considering its variation with respect to the variation of both the metric and the smooth map:

Lemma 3.7. *Let $\varphi : (M, g) \to (N, g_N)$ be a smooth map between Riemannian manifolds, and let $\mathcal{S} : \mathcal{M} \times \mathcal{P} \to C^\infty(M)$ be the φ-scalar curvature operator, where $\mathcal{M}$ and $\mathcal{P}$ are defined as above. Then the variation of the φ-scalar curvature is given by*

$$D\mathcal{S}_{(g,\varphi)}(h, v) = \operatorname{div}(\operatorname{div} h) - \Delta(\operatorname{tr} h) - \langle \operatorname{Ric}^\varphi, h\rangle - 2\alpha\langle \nabla v, \mathrm{d}\varphi\rangle, \tag{3.2.1}$$

where h and v are elements of the tangent spaces $T_g\mathcal{M}$ and $T_\varphi\mathcal{P}$ respectively, and $\langle\, ,\, \rangle$ is the inner product on $TM \times \varphi^*TN$.

Proof. Let us consider a curve $(g(t), \varphi(t))$on $\mathcal{M}\times\mathcal{P}$ such that $g(0) = g$, $\varphi(0) = \varphi$ and $(\dot g(0), \dot\varphi(0)) = (h, v)$, where h is a 2-covariant symmetric tensor on M and v is a section of φ^*TN. Then, we have

$$D\mathcal{S}_{(g,\varphi)}(h, v) = \left.\frac{d}{dt}\right|_0 \mathcal{S}(g(t), \varphi(t)) = D_g\mathcal{S}_{(g,\varphi)}(h) + D_\varphi\mathcal{S}_{(g,\varphi)}(v), \tag{3.2.2}$$

where $D_g\mathcal{S}$ and $D_\varphi\mathcal{S}$ stand for the differentials of $\mathcal{S}$ with respect to the metric component and the map component respectively. As for the first term, we consider a curve on $\mathcal{M} \times \mathcal{P}$ such that $\varphi_t \equiv \varphi$. Therefore, exploiting (1.4.6),

$$D_g\mathcal{S}_{(g,\varphi)}(h) = \left.\frac{d}{dt}\right|_0 \Big(S^\varphi(g(t))\Big) = \operatorname{div}(\operatorname{div} h) - \Delta(\operatorname{tr} h) - \langle \operatorname{Ric}, h\rangle - \alpha \left.\frac{d}{dt}\right|_0 \Big(|\mathrm{d}\varphi|^2(g(t))\Big).$$

Since both φ and g_N are independent of $g(t)$, their variation is 0, so that

$$\left.\frac{d}{dt}\right|_0 \big(|\mathrm{d}\varphi|^2(g(t))\big) = \left.\frac{d}{dt}\right|_0 \operatorname{tr}_{g(t)} \varphi^* g_N = \varphi^* g_N \left.\frac{d}{dt}\right|_0 (e_r(t), e_r(t)) = -2\varphi^a_s\varphi^a_r a^s_r = -\varphi^a_s\varphi^a_r h_{rs},$$

hence

$$D_g\mathcal{S}_{(g,\varphi)}(h) = \operatorname{div}(\operatorname{div} h) - \Delta(\operatorname{tr} h) - \langle \operatorname{Ric}, h\rangle + \alpha\varphi^a_s\varphi^a_r h_{rs} = \operatorname{div}(\operatorname{div} h) - \Delta(\operatorname{tr} h) - \langle \operatorname{Ric}^\varphi, h\rangle. \tag{3.2.3}$$

On the other hand, we consider a curve on $\mathcal{M} \times \mathcal{P}$ such that $g_t \equiv g$. By (1.4.18), since the trace on $TM \times \varphi^*TN$ does not change along the curve,

$$\begin{aligned}
\left.\frac{d}{dt}\right|_0 \operatorname{tr}_g \varphi_t^* g_N &= \left.\frac{d}{dt}\right|_0 \operatorname{tr}(\mathrm{d}\varphi_t(\cdot) \otimes \mathrm{d}\varphi_t(\cdot)) \\
&= 2\operatorname{tr}\Big(\mathrm{d}\varphi \otimes \left.\frac{d}{dt}\right|_0 (\mathrm{d}\varphi_t)\Big) \\
&= 2\operatorname{tr}(\mathrm{d}\varphi \otimes \nabla v),
\end{aligned}$$

and therefore

$$D_\varphi\mathcal{S}_{(g,\varphi)}(v) = \left.\frac{d}{dt}\right|_0 \Big(S^\varphi(\varphi(t))\Big) = -\alpha \left.\frac{d}{dt}\right|_0 (|\mathrm{d}\varphi|^2) = -2\alpha\varphi^a_i v^a_i. \tag{3.2.4}$$

Substituting (3.2.3) and (3.2.4) into (3.2.2) we obtain (3.2.1). □

We are now able to consider the formal L^2-adjoint of the differential of the scalar curvature operator, which we denote by

$$(D\mathcal{S})^*_{(g,\varphi)} : C^\infty(M) \to \Gamma(S_2(M)) \times \Gamma(\varphi^*TN),$$

in order to relate it to the definition of φ-static spaces.

Proposition 3.8. *Let M be a smooth manifold and (N, g_N) be a Riemannian manifold. Then, with the above notation - for a choice of $\alpha \neq 0$ - (M, g) together with $\varphi : M \to (N, g_N)$ is a φ-static space with potential $w \in C^\infty(M)$ if and only if $w \in \ker(D\mathcal{S}^*)(g, \varphi)$.*

Proof. To prove the proposition, it is sufficient to evaluate the formal adjoint of $D\mathcal{S}$ at a given point (g,φ) of $\mathcal{M}\times\mathcal{P}$. Its expression is given integrating by parts (and getting rid of the divergence terms) $D\mathcal{S}_{(g,\varphi)}(h,v)\cdot w$, for $w\in C^\infty(M)$:

$$\begin{aligned}
D\mathcal{S}_{(g,\varphi)}(h,v)\cdot w &= h_{ij,ji}w - h_{ii,jj}w - R^{\varphi}_{ij}h_{ij}w - 2\alpha\varphi^a_i v^a_i w\\
&= (h_{ij,j}w)_i - h_{ij,j}w_i - (h_{ii,j}w)_j + h_{ii,j}w_j - R^{\varphi}_{ij}h_{ij}w\\
&\quad - 2\alpha(v^a\varphi^a_i w)_i + \alpha v^a\varphi^a_{ii}w + \alpha v^a\varphi^a_i w_i\\
&= (h_{ij,j}w)_i - (h_{ij}w_i)_j + h_{ij}w_{ij} - (h_{ii,j}w)_j + (h_{ii}w_j)_j - h_{ii}w_{jj}\\
&\quad - R^{\varphi}_{ij}h_{ij}w - 2\alpha(v^a\varphi^a_i w)_i + 2\alpha v^a(\varphi^a_{ii}w + \varphi^a_i w_i)\\
&= (h_{ij,j}w)_i - (h_{ij}w_i)_j - (h_{ii,j}w)_j + (h_{ii}w_j)_j - 2\alpha(v^a\varphi^a_i w)_i\\
&\quad + h_{ij}(w_{ij} - \delta_{ij}\Delta w - wR^{\varphi}_{ij}) + 2\alpha v^a(\varphi^a_{ii}w + \varphi^a_i w_i),
\end{aligned}$$

so that

$$\begin{aligned}
\Big\langle (D\mathcal{S})^*_{(g,\varphi)}w,(h,v)\Big\rangle_{S_2(M)\times\varphi^*TN} &= \langle (D_g\mathcal{S})^*,h\rangle_{S_2(M)} + \langle (D_\varphi\mathcal{S})^*,v\rangle_{\varphi^*TN}\\
&= (w_{ij} - \delta_{ij}\Delta w - wR^{\varphi}_{ij})h_{ij} + 2\alpha(\varphi^a_{ii}w + \varphi^a_i w_i)v^a\\
&= \langle \operatorname{Hess} w - \Delta(w)g - w\operatorname{Ric}^{\varphi},h\rangle_{S_2(M)}\\
&\quad + \langle 2\alpha(\tau(\varphi)w + \mathrm{d}\varphi(\nabla w)),v\rangle_{\varphi^*TN},
\end{aligned}\tag{3.2.5}$$

where we have denoted with

$$(D_g\mathcal{S})^*w = \operatorname{Hess} w - \Delta(w)g - w\operatorname{Ric}^{\varphi}\tag{3.2.6}$$

and

$$(D_\varphi\mathcal{S})^*w = 2\alpha(\tau(\varphi)w + \mathrm{d}\varphi(\nabla w))\tag{3.2.7}$$

the components of $(D\mathcal{S})^*w$ along $S_2(M)$ and φ^*TN respectively. To conclude, we notice that, since the expression is the same (with the exception of a constant) to the left hand side of the system (3.1.15), then it vanishes if and only if (3.1.15) holds, i.e. if and only if (M,g,φ) is a φ-static space with potential w. $\square$

We now introduce the total scalar curvature functional, and relate its critical points to φ-Einstein manifolds. Given a differential manifold M, a target Riemannian manifold (N,g_N), and letting $\mathcal{M}$ and $\mathcal{P}$ be as above, for every relatively compact domain $\Omega\subset M$

$$\mathbf{S}_\Omega:\mathcal{M}\times\mathcal{P}\to\mathbb{R}$$

is defined by

$$\mathbf{S}_\Omega(g,\varphi) := \int_\Omega \mathcal{S}(g,\varphi)\mathrm{d}V_g,\tag{3.2.8}$$

where $\mathcal{S}(g,\varphi) = S^{\varphi} = S - \alpha\varphi^*g_N$ for a fixed $\alpha\in\mathbb{R}\setminus\{0\}$. By standard variational arguments we can show that the harmonic-Einstein manifolds are those for which, for every open relatively compact domain with smooth boundary Ω, S_Ω is stationary with respect to compactly supported variation of (g,φ) such that the volume element remains unaltered. By compactly supported we mean variations (g_t,φ_t), $t\in(-\varepsilon,\varepsilon)$, of (g,φ) on $\mathcal{M}\times\mathcal{P}$ such that, for every t, g_t coincides with g and φ_t with φ outside of a set compactly contained in Ω.

Remark 3.9. Notice that, since the variation of the volume element

$$\mathrm{d}V(t) = \theta^1(t)\wedge\cdots\wedge\theta^m(t)$$

with respect to t is given by

$$\begin{aligned}
\frac{d}{dt}\mathrm{d}V(t) &= \dot{\theta}^1(t)\wedge\cdots\wedge\theta^m(t) + \cdots + \theta^1(t)\wedge\cdots\wedge\dot{\theta}^m(t)\\
&= a^1_1\theta^1\wedge\cdots\wedge\theta^m + \cdots + a^m_m\theta^1\wedge\cdots\wedge\theta^m\\
&= \operatorname{tr}(a)\mathrm{d}V(t)\\
&= \frac{1}{2}\operatorname{tr}(h)\mathrm{d}V(t),
\end{aligned}$$

where a and h are as in (1.4.1) and (1.4.2), the variation leaves the volume element unchanged if and only if $\operatorname{tr}(h) = 0$.

Proposition 3.10. *Let M be a differentiable manifold of dimension $m \geq 2$, and let $\mathcal{M}$ and $\mathcal{P}$ be given as above, for a fixed Riemannian manifold (N, g_N). Then (M, g, φ) is harmonic-Einstein with respect to some $\alpha \in \mathbb{R} \setminus \{0\}$ if and only if, for every relatively compact domain with smooth boundary Ω, the functional $\mathbf{S}_\Omega$ defined as in* (3.2.8) *is stationary with respect to compactly supported variations which are locally volume preserving.*

Proof. Let us evaluate the variation of $\mathbf{S}_\Omega$ by considering the derivative in t at $t = 0$ of $\mathbf{S}_\Omega(g_t, \varphi_t)$, with the help of (3.2.1):

$$\begin{aligned}\left.\frac{d}{dt}\right|_0 \mathbf{S}_\Omega(g_t, \varphi_t) &= \int_\Omega \Big[\left.\frac{d}{dt}\right|_0 \mathcal{S}(g_t, \varphi_t)\mathrm{d}V + S^\varphi \left.\frac{d}{dt}\right|_0 (\mathrm{d}V(t))\Big] \\ &= \int_\Omega \Big[\operatorname{div}(\operatorname{div} h) - \Delta(\operatorname{tr} h) - \langle \operatorname{Ric}^\varphi, h\rangle - 2\alpha\langle \nabla v, \mathrm{d}\varphi\rangle + \frac{1}{2} S^\varphi \operatorname{tr}(h)\Big]\mathrm{d}V\end{aligned}$$

which under the hypothesis of $\operatorname{tr}(h) = 0$ reduces to

$$\left.\frac{d}{dt}\right|_0 \mathbf{S}_\Omega(g_t, \varphi_t) = \int_\Omega \Big[\operatorname{div}(\operatorname{div} h) - \langle \operatorname{Ric}^\varphi, h\rangle - 2\alpha\langle \nabla v, \mathrm{d}\varphi\rangle\Big]\mathrm{d}V.$$

Then, applying the divergence theorem, since the variations have compact support in Ω, we arrive to the expression

$$\left.\frac{d}{dt}\right|_0 \mathbf{S}_\Omega(g_t, \varphi_t) = \int_\Omega \Big[- \langle \operatorname{Ric}^\varphi, h\rangle + 2\alpha\langle v, \tau(\varphi)\rangle_N\Big]\mathrm{d}V. \tag{3.2.9}$$

At this point, if we assume (M, g) to be harmonic-Einstein with respect to φ, then $\tau(\varphi) = 0$ and $\operatorname{Ric}^\varphi$ is proportional to the metric. Therefore, since $\operatorname{tr} h = 0$, $\langle \operatorname{Ric}^\varphi, h\rangle = 0$ and the variation vanishes. On the other hand, let us assume that (3.2.9) vanishes. We can consider a cutoff function ψ, which is identically 1 on an open set Ω' compactly contained in Ω, with $0 \leq \psi \leq 1$ and with support compactly contained in Ω. Then, considering $h = 0$ and $v = \psi\,\tau(\varphi)$,

$$0 = \int_\Omega 2\psi\alpha|\tau(\varphi)|^2\mathrm{d}V \geq \int_{\Omega'} 2\alpha|\tau(\varphi)|^2\mathrm{d}V,$$

so that $\tau(\varphi)$ vanishes for every $\Omega' \subset\subset \Omega$ and therefore on Ω. Moreover, with the choice of $v = 0$ and

$$h = -\psi\left(\operatorname{Ric}^\varphi - \frac{S^\varphi}{m} g\right)$$

we get

$$0 = \int_\Omega \psi\left(|\operatorname{Ric}^\varphi|^2 - \frac{1}{m}(S^\varphi)^2\right)\mathrm{d}V \geq \int_{\Omega'} \left(|\operatorname{Ric}^\varphi|^2 - \frac{1}{m}(S^\varphi)^2\right)\mathrm{d}V = \int_{\Omega'} \left|\operatorname{Ric}^\varphi - \frac{S^\varphi}{m} g\right|^2 \mathrm{d}V,$$

and hence also $\operatorname{Ric}^\varphi - \frac{S^\varphi}{m}$ vanishes on Ω. Since Ω is chosen arbitrarily, then

$$\begin{cases}\operatorname{Ric}^\varphi = \frac{S^\varphi}{m} g \\ \tau(\varphi) = 0\end{cases} \quad \text{on } M,$$

meaning that (M, g) is harmonic-Einstein. □

The former characterization of harmonic-Einstein manifold may be considered a bit artificial, but in the compact case it can be made simpler, as the following proposition shows:

Proposition 3.11. *Let M be a compact manifold without boundary of dimension $m \geq 3$ and let $(g, \varphi) \in \mathcal{M} \times \mathcal{P}$. Then (M, g, φ) is harmonic-Einstein if and only if the functional*

$$\mathbf{S} := \mathbf{S}_M$$

is stationary with respect to variations of (g, φ) leaving the total volume unchanged.

Proof. From Remark 3.9, a variation of the metric on a compact manifold leaves the volume invariant if and only if

$$\int_M \operatorname{tr} h \, \mathrm{d}V = 0. \tag{3.2.10}$$

The variation of $\mathbf{S}$ is given by

$$\begin{aligned} D\mathbf{S}_{(g,\varphi)}(h,v) &= \int_M \Big[\operatorname{div}(\operatorname{div} h) - \Delta(\operatorname{tr} h) - \langle \operatorname{Ric}^\varphi, h\rangle - 2\alpha\langle \nabla v, \mathrm{d}\varphi\rangle + \frac{1}{2}S^\varphi \operatorname{tr}(h)\Big]\mathrm{d}V \\ &= \int_M \Big[-\langle \operatorname{Ric}^\varphi - \frac{1}{2}S^\varphi g, h\rangle + 2\alpha\langle v, \tau(\varphi)\rangle_N\Big]\mathrm{d}V, \end{aligned}$$

where we have used the divergence theorem. Suppose that (M,g) is harmonic-Einstein. Then S^φ is constant, $\operatorname{Ric}^\varphi = \frac{1}{m}S^\varphi$ and $\tau(\varphi) = 0$. Therefore

$$D\mathbf{S}_{(h,\varphi)}(h,v) = \frac{m-2}{2m}S^\varphi \int_M \operatorname{tr} h \, \mathrm{d}V = 0$$

by (3.2.10). In order to show the converse, we assume that

$$D\mathbf{S}_{(h,\varphi)}(h,v) = 0$$

for every $(h,v) \in \Gamma(S_2(M) \times \varphi^*TN)$ such that (3.2.10) is satisfied. Now, with the choice of $(h,v) = (-\operatorname{Ric}^\varphi + \frac{1}{m}S^\varphi g, 0)$, h satisfies (3.2.10) and hence

$$0 = \int_M \operatorname{Ric}^\varphi - \frac{1}{m}(S^\varphi)^2 \mathrm{d}V = \int_M \left|\operatorname{Ric}^\varphi - \frac{1}{m}S^\varphi g\right|^2 \mathrm{d}V,$$

concluding that $\operatorname{Ric}^\varphi = \frac{1}{m}S^\varphi$ in M. On the other hand, with $(h,v) = (0,\tau(\varphi))$,

$$0 = \int_M 2\alpha|\tau(\varphi)|^2 \mathrm{d}V,$$

and therefore also $\tau(\varphi) = 0$, proving that (M,g) is harmonic-Einstein. □

3.2.2 A boundary functional and a partial characterization of φ-static spaces

Let us consider, on a compact differential manifold M of dimension $m \geq 3$ and with $\partial M \neq \varnothing$, the functional $\mathbf{F} : \mathcal{M} \times \mathcal{P} \to C^\infty(M)$ - $\mathcal{M}$ and $\mathcal{P}$ as above - defined as follows:

$$\mathbf{F}_X(g,\varphi) = \int_{\partial M} G(X,\nu)\mathrm{d}A, \tag{3.2.11}$$

where X is a vector field on M,

$$G = \operatorname{Ric}^\varphi - \frac{1}{2}S^\varphi g$$

is the φ-Einstein tensor, ν is the outward unit normal to ∂M with respect to g and $\mathrm{d}A$ denotes the area element on ∂M.

The goal of this subsection is to give a characterization of the critical points of $\mathbf{F}_X$ restricted to some subsets of $\mathcal{M} \times \mathcal{P}$. Namely, we consider the spaces

$$\mathcal{Q}_0 = \{(g,\varphi) \in \mathcal{M} \times \mathcal{P} : \mathcal{S}(g,\varphi) = \lambda \text{ for a fixed } \lambda \in \mathbb{R}\}$$
$$\mathcal{Q}_1 = \{(g,\varphi) \in \mathcal{Q}_0 : (g,\varphi) = (g_0,\varphi_0) \text{ and } \mathrm{d}\varphi(\nu) = \mathrm{d}\varphi_0(\nu) \text{ on } \partial M, (g_0,\varphi_0) \in \mathcal{M} \times \mathcal{P}\}.$$

The following result is a generalization, in the context of φ-curvatures, of Theorem 1.1 in [26]:

Theorem 3.12. *Let $\varphi_0 : (M,g_0) \to (N,g_N)$ be a smooth map between Riemannian manifolds, where $m = \dim M \geq 3$, and let X be a vector field on M which is conformal with respect to g_0. Let $\mathbf{F}_X$ be the functional defined in* (3.2.11)*. Then, assuming that the first eigenvalue of*

$$(m-1)\Delta + S^\varphi$$

is positive,

(i) If $\mathrm{div}_{g_0}(X) \in \ker(D\mathcal{S})^*$*, then* $(D\mathbf{F}_X|_{\mathcal{Q}_1})_{(g_0,\varphi_0)} = 0$*, whereas the converse is true assuming that*

$$\mathrm{tr}_{g_0} \mathcal{L}_X \left(\varphi^* g_N - \frac{1}{2}|\mathrm{d}\varphi|^2 g \right) = 0; \tag{3.2.12}$$

(ii) Assume that $X \in \ker(\mathrm{d}\varphi)$*. If* g_0 *is harmonic-Einstein, then* $(D\mathbf{F}_X|_{\mathcal{Q}_0})_{(g_0,\varphi_0)} = 0$*, whereas the converse is true assuming the existence of a function* u *such that* $\mathrm{div}_{g_0} X - X(u)$ *has a fixed sign on a dense subset of* M *which is satisfied if* $\mathrm{div}_{g_0} X$ *does not vanish on* M*.*

First, in Proposition 3.13 we compute the Laplacian and the Hessian of the divergence of a conformal vector field. After having derived an expression for $(D\mathcal{S})^* w$ in case $w = \mathrm{div}\, X$ for a conformal vector field X, in Propositions 3.16 and 3.17 we will investigate the relation between harmonic-Einstein manifolds and φ-Static Spaces when X belongs to the kernel of the tangent map of φ. Subsequently, in Lemma 3.21 we will derive an expression for the differential of $\mathbf{F}_X$ at a point $(g_0, \varphi_0) \in \mathcal{M} \times \mathcal{P}$ such that X is conformal with respect to g_0, after which we will conclude by giving the proof of Theorem 3.12.

As said above, we now proceed to compute the laplacian and the hessian of the divergence of a smooth vector field which is conformal with respect to the metric g on M:

Proposition 3.13. *Let* (M, g) *be a Riemannian manifold of dimension* $m \geq 3$ *and suppose that* X *is a conformal vector field on* M*. Set*

$$w = \mathrm{div}\, X. \tag{3.2.13}$$

Then

$$\mathrm{Hess}(w) = \frac{S}{(m-1)(m-2)} w\, g + \frac{m}{2(m-1)(m-2)} \langle \nabla S, X \rangle g - \frac{m}{m-2} \mathcal{L}_X \mathrm{Ric} \tag{3.2.14}$$

and

$$\Delta w = -\frac{S}{m-1} w - \frac{m}{2(m-1)} \langle \nabla S, X \rangle. \tag{3.2.15}$$

Proof. To the aim of proving (3.2.14), we first establish (3.2.15). Since X is conformal

$$X^i_j + X^j_i = \frac{2}{m} w \delta^i_j. \tag{3.2.16}$$

Contracting (3.2.16) with the Ricci tensor gives

$$R_{ij} X^i_j = \frac{w}{m} S. \tag{3.2.17}$$

From the definition of w,

$$\Delta w = \Delta\, \mathrm{div}\, X \tag{3.2.18}$$

and using the commutation relation

$$X^i_{jk} - X^i_{kj} = X^t R_{tijk} \tag{3.2.19}$$

we compute

$$\begin{aligned} \Delta w = \Delta\, \mathrm{div}\, X = (X^i_i)_{jj} = (X^i_{ij})_j = (X^i_{ji} + X^k R_{kiij})_j \\ = X^i_{jij} - (R_{ij} X^i)_j = (X^i_j)_{ij} - R_{ij,j} X^i - R_{ij} X^i_j. \end{aligned}$$

With the aid of (3.2.16), (3.2.17) and Schur's identity the latter equality can be written in the form

$$\Delta w = \left(-X^j_i + \frac{2}{m} w \delta_{ij} \right)_{ij} - \frac{1}{2} S_i X^i - \frac{S}{m} w$$

that is,

$$\Delta w = -X^j_{iij} + \frac{2}{m} \Delta w - \frac{1}{2} S_i X^i - \frac{S}{m} w. \tag{3.2.20}$$

Using the commutation relation

$$X^i_{jkl} - X^i_{jlk} = X^t_j R_{tikl} + X^i_t R_{tjkl}$$

we obtain

$$X^j_{iij} = X^j_{iji} + R_{kjij} X^k_i + R_{kiij} X^j_k = X^j_{iji} R_{ki} X^k_i - R_{kj} X^j_k = X^j_{iji} \tag{3.2.21}$$

and inserting into (3.2.20) we infer

$$\Delta w = -(X^j_{ij})_i + \frac{2}{m}\Delta w - \frac{1}{2} S_i X^i - \frac{S}{m} w.$$

Using again (3.2.16), (3.2.19), Schur's identity and (3.2.17) we get

$$\begin{aligned}
\Delta w &= -(X^j_{ji} + R_{kjij} X^k)_i + \frac{2}{m}\Delta w - \frac{1}{2} S_i X^i - \frac{S}{m} w \\
&= -X^j_{jii} - (R_{ki} X^k)_i + \frac{2}{m}\Delta w - \frac{1}{2} S_i X^i - \frac{S}{m} w \\
&= -\Delta w - R_{ki,i} X^k - R_{ki} X^k_i + \frac{2}{m}\Delta w - \frac{1}{2} S_i X^i - \frac{S}{m} w \\
&= -\Delta w - S_i X^i - \frac{2}{m} S w + \frac{2}{m}\Delta w,
\end{aligned}$$

that is, (3.2.15).

Let us turn to $\mathrm{Hess}(w)$. We want to express its components in terms of the components of the (rough) laplacian of the covariant derivative of X so that, after symmetrizing, we can use (3.2.16) to lead us back to the laplacian of w, where we can exploit (3.2.15). In order to do so, we shift the indices i and j of

$$\big(\mathrm{Hess}(w)\big)_{ij} = w_{ij} = X^l_{lij}$$

in the first position, exploiting the fact that X is conformal Killing and the commutation relations for covariant derivatives of vector fields.

$$\begin{aligned}
w_{ij} = X^l_{lij} &= X^l_{ilj} + (X^t R_{tllj} + X^t R_{tlli})_{,j} \\
&= X^l_{ijl} + X^t_i R_{tllj} + X^l_t R_{tilj} - X^t_j R_{ti} - X^t R_{ti,j} \\
&= (-X^i_l + \frac{2}{m} X^t_t \delta_{il})_{jl} - X^t_i R_{tj} - X^t_j R_{ti} - X^t R_{ti,j} + X^l_t R_{tilj} \\
&= -X^i_{ljl} + \frac{2}{m} X^t_{tji} - X^t_i R_{tj} - X^t_j R_{ti} - X^t R_{ti,j} + X^l_t R_{tilj} \\
&= -X^i_{jll} - (X^t R_{tilj})_l + \frac{2}{m} X^t_{tji} - X^t_i R_{tj} - X^t_j R_{ti} - X^t R_{ti,j} + X^l_t R_{tilj}.
\end{aligned}$$

Next, for the term $(X^t R_{itlj})_l = X^t_l R_{tilj} + X^t R_{tilj,l}$ we use the second Bianchi identity to arrive at

$$\begin{aligned}
w_{ij} &= -X^i_{jll} - X^t_l R_{tilj} - X^t (R_{ij,t} - R_{tj,i}) + \frac{2}{m} X^t_{tji} - X^t_i R_{tj} - X^t_j R_{ti} - X^t R_{ti,j} + X^l_t R_{tilj} \\
&= -X^i_{jll} + \frac{2}{m} X^t_{tji} - X^t_l R_{tilj} + X^l_t R_{tilj} - X^t R_{ij,t} + X^t R_{tj,i} - X^t R_{ti,j} - X^t_i R_{tj} - X^t_j R_{ti} \\
&= -X^i_{jll} + \frac{2}{m} X^t_{tji} - X^t_l R_{ltij} + X^t (R_{tj,i} - R_{ti,j}) - X^t R_{ij,t} - X^t_i R_{tj} - X^t_j R_{it},
\end{aligned}$$

where in the last equality we have renamed the indices and used the first Bianchi identity. Now, since $w_{ij} = w_{ji}$, symmetrizing with respect to the indices i and j we obtain

$$\begin{aligned}
X^l_{lij} &= -\frac{1}{2}(X^i_j + X^j_i)_{ll} + \frac{2}{m} X^l_{lij} - X^t R_{ij,t} - X^t_i R_{tj} - X^t_j R_{it} \\
&= -\frac{1}{m} X^k_{kll} \delta_{ij} + \frac{2}{m} X^l_{lij} - (\mathcal{L}_X \, \mathrm{Ric})_{ij}.
\end{aligned}$$

Hence, rearranging the terms and using (3.2.15) we obtain (3.2.14). □

Remark 3.14. Notice that, since

$$\mathcal{L}_X(Sg) = \mathcal{L}_X S\, g + S\mathcal{L}_X g = \langle \nabla S, X\rangle g + \frac{2S}{m} wg,$$

equation (3.2.14) can be rewritten as

$$\operatorname{Hess}(w) = \frac{m}{m-2}\left[\frac{1}{2(m-1)}\mathcal{L}_X(S\,g) - \mathcal{L}_X \operatorname{Ric}\right] = -\frac{m}{m-2}\mathcal{L}_X A, \tag{3.2.22}$$

where A is the Schouten tensor defined in (1.1.5).

In the next Lemma we will compute the adjoint of $D\mathcal{S}_{(g,\varphi)}$ on $w = \operatorname{div} X$, when X is a conformal vector field on (M, g):

Lemma 3.15. *Let (M, g) be a Riemannian manifold of dimension $m \geq 3$ and let $\varphi : M \to (N, g_N)$. Let X be a conformal vector field and denote with $w = \operatorname{div} X$. Then*

$$(D\mathcal{S}_{(g,\varphi)})^* w = ((D\mathcal{S}_g)^* w, (D\mathcal{S}_\varphi)^* w),$$

where

$$(D\mathcal{S}_g)^* w = \frac{1}{2}\langle \nabla S^\varphi, X\rangle g - \frac{m}{m-2}\mathcal{L}_X T - w\,T - \frac{m}{m-2}\alpha\mathcal{L}_X\left(\varphi^* g_N - \frac{|\mathrm{d}\varphi|^2}{2} g\right) \tag{3.2.23}$$

and

$$(D\mathcal{S}_\varphi)^* w = 2\alpha\frac{m}{m-2}\left(w\tau(\varphi) + \nabla_X\tau(\varphi) + \operatorname{tr}_g({}^N R(\mathrm{d}\varphi(\cdot), \mathrm{d}\varphi(X))\mathrm{d}\varphi(\cdot)) - \Delta(\mathrm{d}\varphi(X))\right), \tag{3.2.24}$$

where Δ denotes the rough laplacian acting on vectors along φ, in local coordinates

$$\left[\Delta(\mathrm{d}\varphi(X))\right]^a = (\varphi^a_t X^t)_{ll}.$$

Proof. From (3.2.22), if we substitute the expressions for the φ-Ricci tensor and the φ-scalar curvature, we obtain

$$\operatorname{Hess} w = -\frac{m}{m-2}\mathcal{L}_X\left(\operatorname{Ric}^\varphi + \alpha\varphi^* g_N - \frac{S^\varphi}{2(m-1)}g - \alpha\frac{|\mathrm{d}\varphi|^2}{2(m-1)}g\right)$$

which, with respect to the traceless Ricci tensor $T = \overset{\circ}{\operatorname{Ric}}{}^\varphi$ is

$$\operatorname{Hess} w = -\frac{m}{m-2}\mathcal{L}_X T - \alpha\frac{m}{m-2}\mathcal{L}_X\left(\varphi^* g_N - \frac{|\mathrm{d}\varphi|^2}{2(m-1)}g\right) - \frac{1}{2(m-1)}\mathcal{L}_X(S^\varphi g). \tag{3.2.25}$$

Then, from (3.2.15),

$$-\Delta(w)\, g = \frac{1}{m-1}\left(S\operatorname{div}(X) g + \frac{m}{2}\nabla_X(S)\, g\right) = \frac{m}{2(m-1)}\mathcal{L}_X(S\,g) = \frac{m}{2(m-1)}\mathcal{L}_X(S^\varphi\, g + \alpha|\mathrm{d}\varphi|^2\, g), \tag{3.2.26}$$

whereas, on the other hand,

$$w\operatorname{Ric}^\varphi = wT + w\frac{S^\varphi}{m} g = wT + \frac{1}{2}S^\varphi\mathcal{L}_X g = wT + \frac{1}{2}\mathcal{L}_X(S^\varphi g) - \frac{1}{2}\mathcal{L}_X(S^\varphi) g. \tag{3.2.27}$$

Substituting (3.2.25), (3.2.26) and (3.2.27) into the expression for $(D\mathcal{S}_g)^*$ in (3.2.6), we arrive to (3.2.23).

As for (3.2.24), we consider the rough laplacian of the quantity $\mathrm{d}\varphi(X)$ seen as a vector field along φ. We have

$$\begin{aligned}
(\varphi^a_s X^s)_{,tt} &= \varphi^a_{stt}X^s + 2\varphi^a_{st}X^s_t + \varphi^a_s X^s_{tt} = \\
&= \varphi^a_{stt}X^s + \frac{2}{m}w\varphi^a_{tt} + \varphi^a_s X^s_{tt} = \\
&= \varphi^a_{tst}X^s + \frac{2}{m}w\varphi^a_{tt} + \varphi^a_s\Big(-X^t_s + \frac{2}{m}w\delta_{ts}\Big)_t = \\
&= \varphi^a_{tts}X^s + {}^N R^a_{bcd}\varphi^b_t\varphi^c_t\varphi^d_s X^s + X^s\varphi^a_i R_{is} + \frac{2}{m}w\varphi^a_{tt} + \frac{2}{m}\varphi^a_s w_s - \varphi^a_s X^t_{st}.
\end{aligned}$$

Recalling that

$$X^t_{st} = X^t_{ts} + X^i R_{is} = w_s + X^i R_{is},$$

we thus have

$$(\varphi^a_s X^s)_{,tt} = \varphi^a_{tts} X^s + {}^N\!R^a_{bcd}\varphi^b_t\varphi^c_t\varphi^d_s X^s + \frac{2}{m}w\varphi^a_{tt} - \frac{m-2}{m}\varphi^a_s w_s.$$

Rearranging the terms, we have

$$\frac{m-2}{m}\varphi^a_s w_s = \frac{2}{m}w\varphi^a_{tt} + \varphi^a_{tts}X^s + {}^N\!R^a_{bcd}\varphi^b_t\varphi^c_t\varphi^d_s X^s - (\varphi^a_s X^s)_{,tt}.$$

Therefore, from (3.2.7),

$$2\alpha[w\tau(\varphi) + \mathrm{d}\varphi(\nabla w)] = 2\alpha\frac{m}{m-2}(w\varphi^a_{tt} + \varphi^a_{tts}X^s + {}^N\!R^a_{bcd}\varphi^b_t\varphi^c_t\varphi^d_s X^s - (\varphi^a_s X^s)_{,tt}),$$

i.e. (3.2.24). □

With the aid of (3.2.23) and (3.2.24), we are now able to prove the following Proposition, which extends to the setting of φ-curvatures a result of Herzlich [20], reproved by Miao and Tam [26]:

Proposition 3.16. *Let (M, g) be a harmonic-Einstein manifold of dimension $m \geq 3$ with respect to φ and $\alpha \neq 0$ as in Definition 1.3.11. Let X be a conformal vector field such that $X \in \ker \mathrm{d}\varphi$. Then*

$$w = \operatorname{div} X \in \ker(D\mathcal{S}_{(g,\varphi)})^*.$$

Proof. From the previous Lemma, we have to show that both (3.2.23) and (3.2.24) vanish under the harmonic-Einstein condition. We begin with (3.2.23), which using the hypothesis of $\mathrm{d}\varphi(X) = 0$ reduces to

$$(D\mathcal{S}_g)^* w = \frac{1}{2}\langle \nabla S^\varphi, X\rangle g - \frac{m}{m-2}\mathcal{L}_X T - wT. \tag{3.2.28}$$

Then, since (M, g) is harmonic-Einstein, in particular we have that

$$T = 0$$

and that S^φ is constant, hence (3.2.28) vanishes. As regards (3.2.24), again using the fact that $\mathrm{d}\varphi(X) = 0$ we reduce it to

$$(D\mathcal{S}_\varphi)^* w = 2\alpha\frac{m}{m-2}\big[w\tau(\varphi) + \nabla_X\tau(\varphi)\big], \tag{3.2.29}$$

and by the fact that harmonic-Einstein manifolds have vanishing tension field, (3.2.28) is equal to zero in turn, thus proving the thesis. □

We now give a partial converse to Proposition 3.16:

Proposition 3.17. *Let (M, g) be a compact manifold of dimension $m \geq 3$ with boundary $\partial M \neq \varnothing$. Assume that X is a conformal, non-Killing vector field on M satisfying $X \in \ker \mathrm{d}\varphi$ and suppose that $w = \operatorname{div} X$ is a solution of the system* (3.1.15)*. Furthermore, assume that*

$$\langle X, \nu\rangle \leq 0 \quad \text{on } \partial M \tag{3.2.30}$$

where ν is the outward unit normal to ∂M and that, for some $v \in C^\infty(M)$,

$$\langle X, \nabla v\rangle > \operatorname{div} X \quad \text{on a dense subset of } M. \tag{3.2.31}$$

Then

$$\begin{cases} \mathrm{Ric}^\varphi = \frac{S^\varphi}{m}\, g \\ \tau(\varphi) = 0, \end{cases} \tag{3.2.32}$$

that is, (M, g) is harmonic-Einstein.

Remark 3.18. Note that $w \not\equiv 0$ since X is non-Killing and Proposition 3.1 implies that S^{φ} is constant.

Proof. Since w satisfies the first equation in (3.1.15), it satisfies $(D\mathcal{S}_g)^* w = 0$. Using the fact that $X \in \ker \mathrm{d}\varphi$, we consider the expression in (3.2.28) together with constancy of S^{φ} to get

$$0 = wT_{ij} + \frac{m}{m-2} X^t T_{ij,t} + \frac{m}{m-2} X^t_i T_{tj} + \frac{m}{m-2} X^t_j T_{it}.$$

Contracting with T_{ij} and using the symmetry of T and conformality of X we obtain

$$\begin{aligned}
0 &= w|T|^2 + \frac{m}{2(m-2)}\langle X, \nabla |T|^2\rangle + \frac{m}{m-2} X^t_i T_{ij} T_{jt} + \frac{m}{m-2} X^t_j T_{ji} T_{it} = \\
&= w|T|^2 + \frac{m}{2(m-2)}\langle X, \nabla |T|^2\rangle + \frac{2m}{m-2} X^t_i T_{ij} T_{jt} = \\
&= w|T|^2 + \frac{m}{2(m-2)}\langle X, \nabla |T|^2\rangle + \frac{m}{m-2} (X^t_i + X^i_t) T_{ij} T_{jt} = \\
&= w|T|^2 + \frac{m}{2(m-2)}\langle X, \nabla |T|^2\rangle + \frac{2}{m-2} w|T|^2,
\end{aligned}$$

that is

$$0 = |T|^2 \operatorname{div}(X) + \frac{1}{2}\langle X, \nabla |T|^2\rangle. \tag{3.2.33}$$

For any $\psi \in C^2(M)$, using (3.2.33) we have

$$\begin{aligned}
\operatorname{div}\left(\psi |T|^2 X\right) &= \psi |T|^2 \operatorname{div} X + \psi \langle X, \nabla |T|^2\rangle + |T|^2 \langle X, \nabla \psi\rangle = \\
&= -\psi |T|^2 \operatorname{div} X + |T|^2 \langle X, \nabla \psi\rangle.
\end{aligned}$$

Choosing $\psi = e^v$ and integrating on M we obtain

$$\int_{\partial M} e^v |T|^2 \langle X, \nu\rangle = \int_M |T|^2 (\langle X, \nabla v\rangle - \operatorname{div} X) e^v$$

Thus (3.2.31) and (3.2.30) imply that $T \equiv 0$ on M, that is, the first equation in (3.2.32). We now show that φ is harmonic. Towards this aim, we observe that in the present assumptions it holds (3.2.29):

$$(D\mathcal{S}_{\varphi})^* w = 2\alpha \frac{m}{m-2} [w\tau(\varphi) + \langle \nabla \tau(\varphi), X\rangle]. \tag{3.2.34}$$

Since $(D\mathcal{S}_{\varphi})^* w = 0$ from (3.1.15), in orthonormal components we have

$$0 = w\varphi^a_{tt} + \varphi^a_{tts} X^s.$$

Contracting with the tension field φ^a_{kk} and using $w = \operatorname{div} X$ we get

$$0 = |\tau(\varphi)|^2 \operatorname{div} X + \frac{1}{2}\langle X, \nabla |\tau(\varphi)|^2\rangle. \tag{3.2.35}$$

Then, for any $\psi \in C^2(M)$ we have

$$\begin{aligned}
\operatorname{div}(\psi |\tau(\varphi)|^2 X) &= \psi |\tau(\varphi)|^2 \operatorname{div} X + \psi \langle X, \nabla |\tau(\varphi)|^2\rangle + |\tau(\varphi)|^2 \langle X, \nabla \psi\rangle = \\
&= -\psi |\tau(\varphi)|^2 \operatorname{div} X + |\tau(\varphi)|^2 \langle X, \nabla \psi\rangle.
\end{aligned}$$

With the choice of $\psi = e^v$ similarly to what we did above, integrating we deduce that $\tau(\varphi) \equiv 0$. $\square$

Before moving on towards the proof of Theorem 3.12, we present two similar results in the compact case that make use of the expression for $(D\mathcal{S})^*_{(g,\varphi)} w$ given in Lemma 3.15, when w is the divergence of a conformal vector field on (M, g), but relaxing the hypothesis of X belonging to the kernel of the tangent map $\mathrm{d}\varphi$. More specifically, we will assume that $\mathcal{L}_X(\varphi^* g_N) = \psi g$ for some constant or function ψ. This condition is obviously satisfied in case $\mathcal{L}_X(\varphi^* g_N) = 0$. This happens for instance when the image of the covariant derivative $\nabla Y : TM \to TN$ of the vector field

$Y = \mathrm{d}\varphi(X) : M \to TN$ is orthogonal to the image of $\mathrm{d}\varphi : TM \to TN$. In particular this is true if $Y \equiv 0$, that is, $X \in \ker \mathrm{d}\varphi$. Assuming conformality of X, another case in which this condition holds is that where $\varphi : (M, g) \to (N, g_N)$ is a weakly conformal map, that is, $\varphi^* g_N = wg$ for some $0 \leq w \in C^\infty(M)$. In this case it is satisfied with the choice of $\psi = \langle X, \nabla w\rangle + (2/m)w \operatorname{div} X$.
If we are in presence of a vacuum static structure, under some additional assumptions we have:

Proposition 3.19. *Let (M, g) be a compact Riemannian manifold without boundary, $\varphi : (M, g) \to (N, g_N)$ a smooth map and $\alpha \in \mathbb{R} \setminus \{0\}$ a coupling constant. Suppose that M supports a conformal, non-Killing vector field X such that*

a) $\mathcal{L}_X(\varphi^* g_N) = \psi g$ for some $\psi \in C^\infty(M)$

b) the function $w = \operatorname{div} X$ is a solution of

$$\begin{cases} \operatorname{Hess} w - (\Delta w)g - w \operatorname{Ric}^\varphi = 0 \\ w\tau(\varphi) + \mathrm{d}\varphi(\nabla w) = 0 \end{cases} \tag{3.2.36}$$

c) $\langle X, \nabla w\rangle \leq 0$ on $\{u = 0\}$.

Then (M, g) is a round sphere and φ is constant.

Proof. We know from Proposition 3.1 that, if w is a nontrivial solution of (3.2.36), then S^φ_g is constant. Then, exploiting the expression in (3.2.23), we have

$$0 = \frac{m}{m-2}\mathcal{L}_X T + w\,T + \frac{m}{m-2}\alpha\mathcal{L}_X\left(\varphi^* g_N - \frac{|\mathrm{d}\varphi|^2}{2}g\right),$$

which under hypothesis (a), since

$$\begin{aligned} \mathcal{L}_X(|\mathrm{d}\varphi|^2 g) &= |\mathrm{d}\varphi|^2_k X_k g + |\mathrm{d}\varphi|^2 \frac{2}{m} wg \\ &= [(\varphi^a_i\varphi^a_i)_k X_k + 2\varphi^a_i\varphi^a_j X_{i,j}]g = \operatorname{tr}_g(\mathcal{L}_X\varphi^* g_N)g = m\psi g, \end{aligned}$$

reduces to

$$0 = \frac{m}{m-2}\mathcal{L}_X T + w\,T - \frac{m}{2}\alpha\psi g. \tag{3.2.37}$$

Tracing the latter we obtain

$$0 = -\frac{m^2}{2}\alpha\psi,$$

which yields $\psi \equiv 0$, i.e.

$$\mathcal{L}_X\varphi^* g_N = 0.$$

Therefore, again from (3.2.37),

$$0 = \frac{m}{m-2}\mathcal{L}_X T + w\,T.$$

Then we proceed as in the proof of Proposition 3.17, to show that it holds (3.2.33), that is

$$0 = |T|^2 w + \frac{1}{2}\langle X, \nabla|T|^2\rangle,$$

from which we obtain that

$$0 = \frac{1}{2}\operatorname{div}(|T|^2 X) + \frac{1}{2}|T|^2 w.$$

Let Ω be any connected component of $\{w \neq 0\}$. Suppose, to fix ideas, that Ω is a connected component of $\{w > 0\}$. Integrating the above identity over Ω and applying the divergence theorem (note that $\partial\Omega$ is contained in $\{w = 0\}$, which is a regular level set for w, and that the outward pointing normal on $\partial\Omega$ is $-\nabla w/|\nabla w|$)

$$0 \leq \int_\Omega w|T|^2 = -\int_\Omega \operatorname{div}(|T|^2 X) = \int_{\partial\Omega} \frac{|T|^2}{|\nabla w|}\langle X, \nabla w\rangle \leq 0$$

where the last inequality follows by the assumption that $\langle X, \nabla w\rangle \leq 0$ on $\{w=0\} \supseteq \partial\Omega$. Hence, the integral of $w|T|^2$ over Ω must vanish, and since $w>0$ on Ω we infer that $T \equiv 0$ on Ω. If Ω is a connected component of $\{w<0\}$ then a similar argument shows that the integral over Ω of the non-positive function $w|T|^2$ vanishes (note that in this case the outward normal on $\partial\Omega$ is $\nabla w/|\nabla w|$) and again we deduce that $T \equiv 0$ on Ω. Hence, $T \equiv 0$ on the dense set $\{w \neq 0\}$ and by continuity we get that $T \equiv 0$ on M. But then w is a non-constant solution of

$$\operatorname{Hess} w + \frac{S^{\varphi}}{m(m-1)} w = 0$$

where $S^{\varphi} > 0$ is constant, and by a classical theorem of Obata we conclude that (M,g) is a round sphere of constant sectional curvature $\frac{1}{m(m-1)} S^{\varphi}$. In particular, the scalar curvature S of (M,g) equals S^{φ}, so $\alpha|\mathrm{d}\varphi|^2 = S - S^{\varphi} \equiv 0$ on M and φ is constant. □

On the other hand, if we are on a harmonic-Einstein manifold, we have:

Proposition 3.20. *Let (M,g) be a compact Riemannian manifold without boundary, harmonic-Einstein with respect to $\varphi : (M,g) \to (N,g_N)$ and $\alpha \in \mathbb{R}\setminus\{0\}$. Suppose that M supports a conformal, non-Killing vector field X such that*

$$\mathcal{L}_X(\varphi^* g_N) = \psi g \text{ for some } \psi \in \mathbb{R}.$$

Then (M,g) is isometric to the round sphere and φ is constant.

Proof. Let us consider the expression for $(D\mathcal{S}_g)^* w$, where $w = \operatorname{div} X$, in (3.2.23). Since (M,g) is harmonic-Einstein, it reduces to

$$(D\mathcal{S}_g)^* w = -\alpha \frac{m}{m-2} \mathcal{L}_X(\varphi^* g_N - \frac{1}{2}|\mathrm{d}\varphi|^2 g)$$

that, from the hypothesis, is

$$(D\mathcal{S}_g)^* w = \alpha \frac{m}{2} \psi g.$$

Substituting the expression for $(D\mathcal{S}_g)^* w$ and using the fact that

$$\operatorname{Ric}^{\varphi} = \frac{S^{\varphi}}{m} g,$$

we have

$$\operatorname{Hess}(w) = \left(-\frac{S^{\varphi}}{m(m-1)} w + \alpha \frac{m}{2}\psi\right) g.$$

Now, from [37, Theorem 2], we have that (M,g) must be isometric to a sphere of constant sectional curvature $\frac{S^{\varphi}}{m(m-1)}$, and hence $\alpha|\mathrm{d}\varphi|^2 = S - S^{\varphi} = 0$, from which we deduce that φ must be constant. □

Let us come back to the steps in the direction of proving Theorem 3.12. The last ingredient is the following formula for the differential of $\mathbf{F}_X$:

Lemma 3.21. *Let (M,g_0) and (N,g_N) be Riemannian manifolds, where $m = \dim M \geq 3$, and let $\varphi_0 : (M,g_0) \to (N,g_N)$ be smooth. Then, given the functional $\mathbf{F}_X$ defined as in (3.2.11), for a vector field X on M which is conformal Killing with respect to g_0, its differential in (g_0,φ_0) evaluated in $(h,v) \in \Gamma(S_2(M)) \times \Gamma(\varphi^*TN)$ takes the form*

$$\begin{aligned}(D\mathbf{F}_X)_{(g_0,\varphi_0)}(h,v) = \frac{m-2}{2m}&\int_M \Big(\big\langle (D\mathcal{S})^*(\operatorname{div} X),(h,v)\big\rangle - (\operatorname{div} X)\cdot D\mathcal{S}(h,v)\Big)\mathrm{d}V \\ &+ \int_{\partial M} \frac{1}{2}\langle (G,-2\alpha\tau(\varphi)),(h,v)\rangle\langle X,\nu\rangle \mathrm{d}A \\ &+ \int_{\partial M} \alpha\left(-\frac{1}{2}h_{ss}\varphi^a_i\varphi^a_j X_i + h_{jk}\varphi^a_i\varphi^a_k X_i - v^a_j\varphi^a_i X_i + v^a(\varphi^a_i X_i)_j\right)\nu_j \mathrm{d}A, \quad (3.2.38)\end{aligned}$$

where $\langle\,,\,\rangle$ stands for either the inner product on $S_2(M)\times\varphi^*TN$ or the metric g on M.

Proof. In order to prove (3.2.38), it is convenient to separate

$$(D\mathbf{F}_X)_{(g_0,\varphi_0)} = (D_g\mathbf{F}_X)_{(g_0,\varphi_0)} + (D_\varphi\mathbf{F}_X)_{(g_0,\varphi_0)}$$

as we did to evaluate $D\mathcal{S}$ in Lemma 3.7.
We thus begin to find an expression for $(D_g\mathbf{F}_X)_{(g_0,\varphi_0)}$. To this aim, we first rewrite $\mathbf{F}_X(g)$ as

$$\mathbf{F}_X(g) = \int_{\partial M} G(X,\nu)\mathrm{d}A = \int_M \mathrm{div}(G(X,\cdot))\mathrm{d}V = \int_M (G_{ij}X_i)_j\mathrm{d}V = \int_M (G_{ij,j}X_i + G_{ij}X_{i,j})\mathrm{d}V.$$

Then, considering a curve $g : (-\varepsilon,\varepsilon) \to \mathcal{M}$ such that $g(0) = g_0$ and $\dot{g}(0) = h$,

$$\left.\frac{d}{dt}\right|_0 \mathbf{F}_X(g) = \int_M \Big((\dot{G_{ij,j}})X_i + G_{ij,j}(\dot{X_i}) + (\dot{G_{ij}})X_{i,j} + G_{ij}(\dot{X_{i,j}})\Big)\mathrm{d}V + \\ + \int_M (G_{ij,j}X_i + G_{ij}X_{i,j})\mathrm{d}\dot{V}. \quad (3.2.39)$$

We begin by evaluating the variation of G_{ij}: from (1.4.6) and (1.4.12) for the variation of the scalar curvature and of the components of the Ricci tensor,

$$\begin{aligned}
\dot{G}_{ij} &= \dot{R}_{ij} - \frac{1}{2}\dot{S}\delta_{ij} - \alpha \left.\frac{d}{dt}\right|_0 \left(\varphi^* g_N(e_i,e_j) - \frac{1}{2}\varphi^* g_N(e_s,e_s)\delta_{ij}\right) \\
&= \frac{1}{2}(h_{si,js} + h_{sj,is} - h_{ss,ij} - h_{ij,ss}) - R_{it}a^t_j - R_{tj}a^t_i - \frac{1}{2}(h_{st,st} - h_{ss,tt} - R_{st}h_{st}) \\
&\quad + \alpha(\varphi^a_i\varphi^a_t a^t_j + \varphi^a_t\varphi^a_j a^t_i - \frac{1}{2}\varphi^a_s\varphi^a_t h_{st}\delta_{ij}) \\
&= \frac{1}{2}(h_{si,js} + h_{sj,is} - h_{ss,ij} - h_{ij,ss}) - R^\varphi_{it}a^t_j - R^\varphi_{tj}a^t_i - \frac{1}{2}(h_{st,st} - h_{ss,tt} - R^\varphi_{st}h_{st}).
\end{aligned}$$

Contracting with $X_{i,j}$, symmetrizing in the indices (since $\dot{G}_{ij}$ is symmetric) and exploiting the fact that it is conformal with respect to g_0,

$$\begin{aligned}
\dot{G}_{ij}X_{i,j} &= \frac{1}{2}\dot{G}_{ij}(X_{i,j} + X_{j,i}) = \frac{1}{m}\,\mathrm{tr}(\dot{G})\,\mathrm{div}(X) \\
&= \frac{1}{m}\Big[(h_{st,ts} - h_{ss,tt} - R^\varphi_{st}h_{st}) - \frac{m}{2}(h_{st,st} - h_{ss,tt} - R^\varphi_{st}h_{st})\Big]\,\mathrm{div}(X) \\
&= -\frac{m-2}{2m}\,\mathrm{div}(X)(h_{st,st} - h_{ss,tt} - R^\varphi_{st}h_{st})
\end{aligned}$$

Hence

$$\dot{G}_{ij}X_{i,j} = -\frac{m-2}{2m}\,\mathrm{div}(X)D_g\mathcal{S}(h). \quad (3.2.40)$$

We then consider X, which doesn't vary under g, so that

$$\dot{X}^i = (\dot{\theta^i(X)}) = \dot{\theta}^i(X) = a^i_jX^j, \quad (3.2.41)$$

and we can then compute $\dot{X}_{i,j}$: differentiating in t the defining equation

$$X_{i,j}\theta^j = \mathrm{d}X_i - X_k\theta^k_i$$

we have

$$\dot{X}_{i,j}\theta^j + X_{i,j}a^j_k\theta^k = \mathrm{d}\dot{X}_i - \dot{X}_k\theta^k_i - X_k\dot{\theta}^k_i = (\dot{X}_i)_{,j}\theta^j - X_k\dot{\theta}^k_i,$$

where we made use of the relation defining the coefficients of the covariant derivative of X_i. At this point, we substitute (3.2.41), (1.4.10) and contract with e_j to obtain

$$\begin{aligned}
\dot{X}_{i,j} &= -X_{i,k}a^k_j + (a^i_{k,j}X_k + a^i_kX_{k,j}) - X_k(\dot{\theta}^k_i)_j \\
&= -X_{i,k}a^k_j + (a^i_{k,j}X_k + a^i_kX_{k,j}) - \frac{1}{2}X_k(h_{kj,i} - h_{ij,k} + a^i_{k,j} - a^k_{i,j}) \\
&= X_{k,j}a^i_k - X_{i,k}a^k_j + \frac{1}{2}X_kh_{ij,k} + \frac{1}{2}X_k(h_{ki,j} - h_{kj,i}).
\end{aligned}$$

Next, we can contract with the φ-Einstein tensor and symmetrize to get

$$
\begin{aligned}
G_{ij}\dot{X}_{i,j} &= G_{ij}\frac{1}{2}(\dot{X}_{i,j}+\dot{X}_{j,i}) \\
&= \frac{1}{2}G_{ij}(-X_{i,k}a^k_j - X_{j,k}a^k_i + X_{k,j}a^i_k + X_{k,i}a^j_k + X_k h_{ij,k})
\end{aligned}
$$

Next, we use the conformality of X and the divergence theorem in order to have it expressed in terms of $\mathcal{L}_X G$:

$$
\begin{aligned}
G_{ij}\dot{X}_{i,j} = {} & \frac{1}{2}G_{ij}(-X_{i,k}a^k_j - X_{j,k}a^k_i - X_{j,k}a^i_k + \frac{2}{m}\operatorname{div}(X)\delta_{jk}a^i_k - X_{i,k}a^j_k + \frac{2}{m}\operatorname{div}(X)\delta_{ik}a^j_k) \\
& + \frac{1}{2}(G_{ij}X_k h_{ij})_k - \frac{1}{2}(G_{ij,k}X_k + G_{ij}\operatorname{div}(X))h_{ij} \\
= {} & -\frac{1}{2}(G_{ij}X_{i,k}h_{kj} + G_{ij}X_{j,k}h_{ki}) + \frac{1}{m}G_{ij}h_{ij}\operatorname{div}(X) + \frac{1}{2}(G_{ij}X_k h_{ij})_k - \frac{1}{2}(G_{ij,k}X_k + G_{ij}\operatorname{div}(X))h_{ij} \\
= {} & -\frac{1}{2}(G_{tj}X_{t,i}h_{ij} + G_{it}X_{t,j}h_{ji} + G_{ij,k}X_k) - \frac{m-2}{2m}G_{ij}h_{ij}\operatorname{div}(X) + \frac{1}{2}(G_{ij}X_k h_{ij})_k.
\end{aligned}
$$

In other terms,

$$
G_{ij}\dot{X}_{i,j} = \langle -\frac{1}{2}\mathcal{L}_X G - \frac{m-2}{2m}\operatorname{div}(X)G, h\rangle + \frac{1}{2}\operatorname{div}(\langle G,h\rangle X). \tag{3.2.42}
$$

Next, we consider

$$
G_{ij,j} = R^\varphi_{ij,j} - \frac{1}{2}S^\varphi_i = \left(\frac{1}{2}S^\varphi_i - \alpha\varphi^a_{jj}\varphi^a_i\right) - \frac{1}{2}S^\varphi_i = -\alpha\varphi^a_{jj}\varphi^a_i. \tag{3.2.43}
$$

As for the term $G_{ij,j}\dot{X}_i$, we have

$$
G_{ij,j}\dot{X}_i = -\alpha\varphi^a_{jj}\varphi^a_i X_k a^i_k, \tag{3.2.44}
$$

whereas for $\dot{G}_{ij,j}X_i$ we first have to compute $\dot{G}_{ij,j}$. To this aim, we consider $\mathrm{d}\varphi$, which doesn't change under the variation of the metric. Hence

$$
0 = \dot{\mathrm{d}\varphi} = \dot{\varphi}^a_i\theta^i\otimes E_a + \varphi^a_j a^j_i\theta^i\otimes E_a,
$$

yielding

$$
\dot{\varphi}^a_i = -\varphi^a_j a^j_i. \tag{3.2.45}
$$

Afterwards, similarly to what we did for $\dot{X}_{i,j}$, from (1.3.2)

$$
\begin{aligned}
\dot{\varphi}^a_{ij} &= (\mathrm{d}\varphi^a_i - \varphi^a_k\theta^k_i + \varphi^b_i\omega^a_b)(\dot{e}_j) + (\mathrm{d}\dot{\varphi}^a_i - \dot{\varphi}^a_k\theta^k_i - \varphi^a_k\dot{\theta}^k_i + \dot{\varphi}^b_i\omega^a_b)(e_j) \\
&= -\varphi^a_{ik}a^k_j + (\dot{\varphi}^a_i)_j - \frac{1}{2}\varphi^a_k(h_{kj,i} - h_{ij,k} + a^i_{k,j} - a^k_{i,j}) \\
&= -\varphi^a_{ik}a^k_j - (\varphi^a_k a^k_i)_j - \frac{1}{2}\varphi^a_k(h_{kj,i} - h_{ij,k} + a^i_{k,j} - a^k_{i,j}) \\
&= -\varphi^a_{ik}a^k_j - \varphi^a_{kj}a^k_i + \frac{1}{2}\varphi^a_k h_{ij,k} - \frac{1}{2}(h_{kj,i} + h_{ki,j}),
\end{aligned}
$$

where we have used (3.2.45). Tracing,

$$
\dot{\varphi}^a_{jj} = -\varphi^a_{ij}h_{ij} + \frac{1}{2}\varphi^a_k h_{ss,k} - \varphi^a_k h_{kj,j}. \tag{3.2.46}
$$

Therefore, using (3.2.45) and (3.2.46) we have

$$
\begin{aligned}
\dot{G}_{ij,j}X_i &= -\alpha(\dot{\varphi}^a_{ll}\varphi^a_i X_i + \varphi^a_{ll}\dot{\varphi}^a_i X_i) = \\
&= -\alpha\Big[(-\varphi^a_{st}h_{st} + \frac{1}{2}\varphi^a_k h_{ss,k} - \varphi^a_k h_{kj,j})\varphi^a_i - \varphi^a_{ll}\varphi^a_j a^j_i\Big]X_i
\end{aligned} \tag{3.2.47}
$$

We are now able to substitute (3.2.40), (3.2.42), (3.2.43), (3.2.44) and (3.2.47) into (3.2.39), recalling from Remark 3.9 that

$$\mathrm{d}\dot{V} = \frac{1}{2}\operatorname{tr}_g(h)\mathrm{d}V :$$

$$\begin{aligned}
\frac{d}{dt}\Big|_0 \mathbf{F}_X(g) &= \int_M \Big[-\alpha\Big((-\varphi^a_{st}h_{st} + \frac{1}{2}\varphi^a_k h_{ss,k} - \varphi^a_k h_{kj,j})\varphi^a_i - \varphi^a_{ll}\varphi^a_j a^j_i \Big) X_i \\
&\quad - \alpha\varphi^a_{jj}\varphi^a_i X_k a^i_k - \frac{m-2}{2m}\operatorname{div}(X) D_g\mathcal{S}(h) + \langle -\frac{1}{2}\mathcal{L}_X G - \frac{m-2}{2m}\operatorname{div}(X)G, h\rangle + \frac{1}{2}\operatorname{div}(\langle G,h\rangle X)\Big]\mathrm{d}V + \\
&\quad + \int_M \frac{1}{2}(-\alpha\varphi^a_{jj}\varphi^a_i X_i + \frac{1}{2}G_{ij}(X_{i,j}+X_{j,i}))h_{ss}\mathrm{d}V \\
&= \int_M \Big[-\alpha(-\varphi^a_{kj}h_{kj} + \frac{1}{2}\varphi^a_k h_{ss,k} - \varphi^a_k h_{kj,j})\varphi^a_i X_i \\
&\quad - \frac{m-2}{2m}\operatorname{div}(X) D_g\mathcal{S}(h) + \langle -\frac{1}{2}\mathcal{L}_X G - \frac{m-2}{2m}\operatorname{div}(X)G, h\rangle + \frac{1}{2}\operatorname{div}(\langle G,h\rangle X)\Big]\mathrm{d}V + \\
&\quad + \int_M \frac{1}{2}(-\alpha\varphi^a_{jj}\varphi^a_i X_i - \frac{m-2}{2m}S^\varphi\operatorname{div}(X))h_{ss}\mathrm{d}V
\end{aligned}$$

At this point we can use the divergence theorem on the terms

$$\begin{aligned}
-\alpha\frac{1}{2}\varphi^a_k h_{ss,k}\varphi^a_i X_i &= -\frac{\alpha}{2}(\varphi^a_k\varphi^a_i X_i h_{ss})_k + \frac{1}{2}\alpha(\varphi^a_{kk}\varphi^a_i X_i + \varphi^a_k\varphi^a_{ik}X_i + \varphi^a_k\varphi^a_i X_{i,k})h_{ss} \\
&= -\frac{\alpha}{2}(\varphi^a_k\varphi^a_i X_i h_{ss})_k + \frac{1}{2}\alpha(\varphi^a_{kk}\varphi^a_i X_i + \frac{1}{2}\mathcal{L}_X|\mathrm{d}\varphi|^2 + \frac{1}{m}|\mathrm{d}\varphi|^2\operatorname{div}(X))h_{ss} \\
&= -\frac{\alpha}{2}(\varphi^a_k\varphi^a_i X_i h_{ss})_k + \frac{1}{2}\alpha\varphi^a_{kk}\varphi^a_i X_i h_{ss} + \frac{1}{4}\alpha(\mathcal{L}_X|\mathrm{d}\varphi|^2\delta_{st} + |\mathrm{d}\varphi|^2(\mathcal{L}_X g)_{st})h_{st} \\
&= -\frac{\alpha}{2}(\varphi^a_k\varphi^a_i X_i h_{ss})_k + \frac{1}{2}\alpha\varphi^a_{kk}\varphi^a_i X_i h_{ss} + \frac{1}{4}\alpha\langle\mathcal{L}_X(|\mathrm{d}\varphi|^2 g), h\rangle
\end{aligned}$$

and

$$\begin{aligned}
\alpha(\varphi^a_{kj}h_{kj} + \varphi^a_k h_{kj,j})\varphi^a_i X_i &= \alpha(\varphi^a_k\varphi^a_i X_i h_{kj})_j - \alpha(\varphi^a_k\varphi^a_{ij}X_i h_{kj} + \varphi^a_k\varphi^a_i X_{i,j}h_{kj}) \\
&= \alpha(\varphi^a_k\varphi^a_i X_i h_{kj})_j - \frac{1}{2}\alpha(\varphi^a_k\varphi^a_{ji}X_i h_{kj} + \varphi^a_j\varphi^a_{ki}X_i h_{kj} + \varphi^a_k\varphi^a_i X_{i,j}h_{kj} + \varphi^a_j\varphi^a_i X_{i,k}h_{kj}) \\
&= \alpha(\varphi^a_k\varphi^a_i X_i h_{kj})_j - \frac{1}{2}\alpha\langle\mathcal{L}_X\varphi^* g_N, h\rangle
\end{aligned}$$

and substitute to get

$$\begin{aligned}
\frac{d}{dt}\Big|_0 \mathbf{F}_X(g) &= \int_M \Big[\alpha\Big(-\frac{1}{2}(\varphi^a_k\varphi^a_i X_i h_{ss})_k + \frac{1}{4}\langle\mathcal{L}_X(|\mathrm{d}\varphi|^2 g), h\rangle + (\varphi^a_k\varphi^a_i X_i h_{kj})_j - \frac{1}{2}\langle\mathcal{L}_X\varphi^* g_N, h\rangle\Big) \\
&\quad - \frac{m-2}{4m}S^\varphi\operatorname{div}(X)h_{ss} - \frac{m-2}{2m}\operatorname{div}(X)D_g\mathcal{S}(h) + \langle -\frac{1}{2}\mathcal{L}_X G - \frac{m-2}{2m}\operatorname{div}(X)G, h\rangle \\
&\quad + \frac{1}{2}\operatorname{div}(\langle G,h\rangle X)\Big]\mathrm{d}V \\
&= \int_M \Big[\Big(-\frac{1}{2}\alpha(\operatorname{div}(\operatorname{tr}(h)\varphi^* g_N(X,\cdot)^\sharp) + \alpha\operatorname{div}(h(\varphi^* g_N(X,\cdot)^\sharp,\cdot)^\sharp) + \frac{1}{2}\operatorname{div}(\langle G,h\rangle X)\Big) \\
&\quad - \frac{m-2}{2m}\operatorname{div}(X)D_g\mathcal{S}(h) + \Big\langle -\frac{1}{2}\alpha\mathcal{L}_X\varphi^* g_N + \frac{1}{4}\alpha\mathcal{L}_X(|\mathrm{d}\varphi|^2 g) - \frac{m-2}{4m}S^\varphi\operatorname{div}(X)g \\
&\quad - \frac{1}{2}\mathcal{L}_X G - \frac{m-2}{2m}\operatorname{div}(X)G, h\Big\rangle\Big]\mathrm{d}V.
\end{aligned}$$

We now replace instead of G the φ-Einstein tensor T exploiting the relation

$$G = \operatorname{Ric}^\varphi - \frac{1}{2}S^\varphi g = T + \frac{1}{m}S^\varphi g - \frac{1}{2}S^\varphi g = T - \frac{m-2}{2m}S^\varphi g,$$

which yields also

$$\mathcal{L}_X G = \mathcal{L}_X T - \frac{m-2}{2m}\mathcal{L}_X(S^\varphi g) = \mathcal{L}_X T - \frac{m-2}{2m}X(S^\varphi)g - \frac{m-2}{m^2}S^\varphi\operatorname{div}(X)g,$$

into the previous expression to obtain

$$\begin{aligned}\frac{d}{dt}\Big|_0 \mathbf{F}_X(g) = \int_M \Big[\Big(&-\frac{1}{2}\alpha(\operatorname{div}(\operatorname{tr}(h)\varphi^* g_N(X,\cdot)^\sharp) + \alpha \operatorname{div}(h(\varphi^* g_N(X,\cdot)^\sharp,\cdot)^\sharp) + \frac{1}{2}\operatorname{div}(\langle G,h\rangle X)\Big)\\
&-\frac{m-2}{2m}\operatorname{div}(X) D_g\mathcal{S}(h) + \Big\langle -\frac{1}{2}\mathcal{L}_X T + \frac{m-2}{4m}X(S^\varphi)g + \frac{m-2}{2m^2}S^\varphi \operatorname{div}(X)g\\
&-\frac{m-2}{2m}\operatorname{div}(X)T + \frac{(m-2)^2}{4m^2}\operatorname{div}(X)S^\varphi g - \frac{m-2}{4m}S^\varphi \operatorname{div}(X)g - \frac{1}{2}\alpha\mathcal{L}_X(\varphi^* g_N - \frac{1}{2}|\mathrm{d}\varphi|^2 g)\Big]\mathrm{d}V.\end{aligned}$$

Simplifying and by (3.2.23), we arrive to

$$\begin{aligned}\frac{d}{dt}\Big|_0 \mathbf{F}_X(g) = \int_M \Big[\Big(&-\frac{1}{2}\alpha(\operatorname{div}(\operatorname{tr}(h)\varphi^* g_N(X,\cdot)^\sharp) + \alpha \operatorname{div}(h(\varphi^* g_N(X,\cdot)^\sharp,\cdot)^\sharp) + \frac{1}{2}\operatorname{div}(\langle G,h\rangle X)\Big)\\
&+\frac{m-2}{2m}\Big(\langle (D_g\mathcal{S})^*(\operatorname{div}(X)), h\rangle - D_g\mathcal{S}(h)\operatorname{div}(X)\Big)\Big]\mathrm{d}V.\end{aligned} \tag{3.2.48}$$

We now turn to $D_\varphi \mathbf{F}_X$. To this purpose, we consider a curve $\varphi : (-\varepsilon,\varepsilon) \to \mathcal{P}$ such that $\varphi(0) = \varphi_0$ and with the same setting as Proposition 1.15. We again rewrite, as above,

$$\mathbf{F}_X(g,\varphi) = \int_{\partial M} G(X,\nu)\mathrm{d}A = \int_M \operatorname{div}(G(X,\cdot))\mathrm{d}V,$$

which, in local orthonormal coordinates, is

$$\mathbf{F}_X(g,\varphi) = \int_M (G_{ij}X_i)_j \mathrm{d}V = \int_M (G_{ij,j}X_i + G_{ij}X_{i,j})\mathrm{d}V,$$

that by (3.2.43) and the conformality of X we can write, for every $t \in (-\varepsilon,\varepsilon)$, as

$$\mathbf{F}_X(g,\varphi) = \int_M \Big(-\alpha\varphi^a_{jj}\varphi^a_i X_i - \frac{m-2}{2m}S^\varphi \operatorname{div}(X)\Big)\mathrm{d}V.$$

We now exploit (1.4.20) and (1.4.21) to get

$$\begin{aligned}\frac{d}{dt}\Big|_0 \mathbf{F}_X &= \int_M \Big(-\alpha(v^a_{jj} + {}^N\!R^a_{bcd}\varphi^b_j v^c \varphi^d_j - \varphi^b_{jj}\omega^a_b(v))\varphi^a_i X_i - \alpha\varphi^a_{jj}(v^a_i - \varphi^b_i\omega^a_b(v))X_i\\
&\quad - \frac{m-2}{2m}D_\varphi\mathcal{S}(v)\operatorname{div}(X)\Big)\mathrm{d}V\\
&= \int_M \Big(-\alpha(v^a_{jj} + {}^N\!R^a_{bcd}\varphi^b_j v^c\varphi^d_j)\varphi^a_i X_i - \alpha\varphi^a_{jj}v^a_i X_i - \frac{m-2}{2m}D_\varphi\mathcal{S}(v)\operatorname{div}(X)\Big)\mathrm{d}V.\end{aligned}$$

Next, we use the divergence theorem on the quantities

$$v^a_{jj}\varphi^a_i X_i = (v^a_j\varphi^a_i X_i)_j - (v^a(\varphi^a_i X_i)_j)_j + v^a(\varphi^a_i X_i)_{jj}$$

and

$$\varphi^a_{jj}v^a_i X_i = (v^a\varphi^a_{jj}X_i)_i - v^a\varphi^a_{jji}X_i - v^a\varphi^a_{jj}\operatorname{div}(X),$$

and substitute into the expression for $d/dt(\mathbf{F}_X)$ in order to obtain

$$\begin{aligned}\frac{d}{dt}\Big|_0 \mathbf{F}_X = \int_M \Big(&-\alpha\big((v^a_j\varphi^a_i X_i)_j - (v^a(\varphi^a_i X_i)_j)_j + v^a(\varphi^a_i X_i)_{jj} + {}^N\!R_{abcd}\varphi^d_j v^a\varphi^b_j\varphi^c_i X_i\\
&+ (v^a\varphi^a_{jj}X_i)_i - v^a\varphi^a_{jji}X_i - v^a\varphi^a_{jj}\operatorname{div}(X)\big) - \frac{m-2}{2m}D_\varphi\mathcal{S}(v)\operatorname{div}(X)\Big)\mathrm{d}V.\end{aligned}$$

Substituting (3.2.24), we come to

$$\frac{d}{dt}\bigg|_0 \mathbf{F}_X = \int_M \Big(-\alpha((v^a_j\varphi^a_i X_i)_j - (v^a(\varphi^a_i X_i)_j)_j - (v^a\varphi^a_{jj}X_i)_i) \\ + \frac{m-2}{2m}(\langle (D_\varphi S)^*(\operatorname{div}(X)), v\rangle - D_\varphi S(v)\operatorname{div}(X))\Big)\mathrm{d}V. \tag{3.2.49}$$

To conclude, formula (3.2.38) follows putting together (3.2.48) and (3.2.49), and then using the divergence theorem. □

Proof of Theorem 3.12. (i). First, we notice that $D\mathcal{S}|_{\mathcal{Q}_1} = 0$ by the very definition of $\mathcal{Q}_1$. Moreover, from the condition $\mathrm{d}\varphi(\nu) = \mathrm{d}\varphi_0(\nu)$, we infer that (by considering a variation with respect to the map φ, and using (1.4.18)), the elements $v \in T_{\varphi_0}\mathcal{P}$ of the projection onto the second component of $T_{(g_0,\varphi_0)}\mathcal{Q}_1$ must satisfy

$$\nabla_\nu v = 0,$$

which in components reads as

$$v^a_i \nu_i = 0.$$

If we also remark that $(h, v) = (0, 0)$ at ∂M as elements of $T_{(g_0,\varphi_0)}\mathcal{Q}_1$ and, we can write (3.2.38) as

$$(D\mathbf{F}_X)_{(g_0,\varphi_0)}(h, v) = \frac{m-2}{2m}\int_M \big\langle (D\mathcal{S})^*(\operatorname{div} X), (h, v)\big\rangle \mathrm{d}V.$$

Therefore, assuming that $(D\mathcal{S}_{(g_0,\varphi_0)})^*(\operatorname{div} X) = 0$, then $(D\mathbf{F}_X|_{\mathcal{Q}_1})_{(g_0,\varphi_0)} = 0$.
In order to show the converse, we claim that, for every variation $(\hat{g}_t, \varphi_t)$ of (g_0, φ_0), there exists a conformal deformation of $\hat{g}_t$,

$$g_t = u_t^{\frac{4}{m-2}}\hat{g}_t$$

for some $u_t \in C^\infty(M)$ satisfying

$$u_0 \equiv 1 \text{ on } M$$

and

$$u_t = 1 \text{ on } \partial M,$$

such that the φ-scalar curvature $\mathcal{S}(g_t, \varphi_t)$ is constant in t - and hence constant, in our assumptions. To see this, we recall that, denoting with $\hat{S}(t) = \mathcal{S}(\hat{g}_t)$ and $S(t) = \mathcal{S}(g_t)$

$$S(t)u^{\frac{m+2}{m-2}} = \hat{S}(t)u - 4\frac{m-1}{m-2}\Delta u.$$

Since $u^{\frac{4}{m-2}}|\mathrm{d}\varphi_t|^2 = |\mathrm{d}\hat{\varphi}_t|^2$, we then have

$$S^\varphi u^{\frac{m+2}{m-2}} = \hat{S}^\varphi(t)u - 4\frac{m-1}{m-2}\Delta u,$$

where $\hat{S}^\varphi(t) = \mathcal{S}(\hat{g}_t, \varphi_t)$ and $S^\varphi = S^\varphi(t) = \mathcal{S}(g_t, \varphi_t)$.
Thus, repeating the same argument of [25, Proposition 1], using the positivity of the first eigenvalue of $(m-1)\Delta + S^\varphi$, u_t exists at least for t small enough and is smooth in t. Moreover, its derivative satisfies

$$h = \dot{g}(0) = \frac{4}{m-2}\dot{u}(0)g(0) + \hat{h}, \tag{3.2.50}$$

where $\hat{h} = \frac{d}{dt}\big|_0 \hat{g}$. Now, we choose arbitrary $(\hat{h}, v)$ with compact support inside $M \setminus \partial M$, in order to satisfy $\nabla_\nu v = 0$ on ∂M. Then, by the condition (3.2.12), since $\dot{u}(0) = 0$ on ∂M

$$\operatorname{tr}_{g_0}\big((D\mathcal{S})^*(\operatorname{div}_{g_0} X)\big) = \operatorname{tr}_{g_0}\Big(-\frac{m}{m-2}\mathcal{L}_X T - \operatorname{div}(X)T - \frac{m}{m-2}\mathcal{L}_X\big(\varphi^* g_N - \frac{1}{2}|\mathrm{d}\varphi|^2 g\big)\Big) = 0.$$

Therefore, by (3.2.23), substituting (3.2.50) into (3.2.38) yields

$$(D\mathbf{F}_X)_{(g_0,\varphi_0)}(h, v) = \frac{m-2}{2m}\int_M \big\langle (D\mathcal{S})^*(\operatorname{div} X), (\hat{h}, v)\big\rangle \mathrm{d}V.$$

Assuming that $(D\mathbf{F}_X|_{\mathcal{Q}_1})_{(g_0,\varphi_0)} = 0$, we readily obtain that $\operatorname{div} X \in \ker(D\mathcal{S})^*_{(g_0,\varphi_0)}$.
(ii). Under the assumption of $X \in \ker(\mathrm{d}\varphi)$, (3.2.38) becomes

$$
\begin{aligned}
(D\mathbf{F}_X)_{(g_0,\varphi_0)}(h,v) = \frac{m-2}{2m}\int_M \Big(\big\langle (D\mathcal{S})^*(\operatorname{div} X),(h,v)\big\rangle - (\operatorname{div} X)\cdot D\mathcal{S}(h,v)\Big)\mathrm{d}V \\
+ \int_{\partial M}\frac{1}{2}\langle (G,-2\alpha\tau(\varphi)),(h,v)\rangle\langle X,\nu\rangle \mathrm{d}A \quad (3.2.51)
\end{aligned}
$$

Supposing that (g_0,φ_0) is harmonic-Einstein, then by Proposition 3.16, which holds under the condition $X\in\ker(\mathrm{d}\varphi)$, $\operatorname{div} X \in \ker(D\mathcal{S})^*_{(g_0,\varphi_0)}$, hence $(D\mathbf{F}_X|_{\mathcal{Q}_0})_{(g_0,\varphi_0)} = 0$ by (3.2.51).
To show the partial converse to this, we first notice that, by (i), $\operatorname{div} X \in \ker(D\mathcal{S})^*_{(g_0,\varphi_0)}$. Therefore, repeating the same construction of (i), for any $(\hat{h},v) \in \Gamma(S_2(M)\times\varphi^*TN)$, imposing that $(D\mathbf{F}_X|_{\mathcal{Q}_0})_{(g_0,\varphi_0)} = 0$ yields, by (3.2.51),

$$
0 = \frac{1}{2}\int_{\partial M} \langle (G,-2\alpha\tau(\varphi_0)),(\hat{h},v)\rangle\langle X,\nu\rangle \mathrm{d}A.
$$

Now, taking any extension f of $\langle X,\nu\rangle$ and considering $(\hat{h},v) = f(G,-2\alpha\tau(\varphi_0))$,

$$
(|G|^2 + 4\alpha^2|\tau(\varphi_0)|^2)\langle X,\nu\rangle^2 = 0 \text{ on } \partial M,
$$

hence, from the hypotheses we can apply Proposition 3.17 to conclude that (g_0,φ_0) is harmonic-Einstein. □

3.3 φ-static spaces and closed conformal vector fields

In this section we will investigate φ-static spaces in presence of a closed conformal vector field X on (M,g). A closed conformal vector field is a conformal vector field such that the corresponding 1-form is closed. From the first structure equations and the Cartan lemma, it is easy to see that it is equivalent to the symmetricity of the tensor ∇X. Hence, since X is also conformal, it must satisfy

$$
\nabla X^\flat = \frac{1}{m}\operatorname{div}(X) g,
$$

which can be written in local orthonormal components as

$$
X_{i,j} = \frac{1}{m}X_{s,s}\delta_{ij}. \quad (3.3.1)
$$

As we will show, the presence of such vector field, plus some additional assumptions relating the vector field to the map $\varphi : M \to (N,g_N)$, will affect the geometry of the regular level sets of w. Namely, our goal is to prove the following Theorem:

Theorem 3.22. *Let (M,g) be a complete Riemannian manifold of dimension $m \geq 2$ admitting a closed conformal non-killing vector field X, and let $\varphi : (M,g) \to (N,g_N)$ be a smooth harmonic map such that $\varphi^* g_N(X,\cdot) = \lambda X^\flat$, $\lambda\in\mathbb{R}$, and*

$$
\operatorname{Ric}^\varphi(X,X) \neq \frac{S^\varphi - \alpha\lambda}{m}|X|^2 \text{ on } M.
$$

In addition, suppose that $w \not\equiv 0$ satisfies the φ-static equation

$$
\begin{cases}
\operatorname{Hess}(w) = w\left(\operatorname{Ric}^\varphi - \frac{S^\varphi}{m-1}g\right) \\
w\tau(\varphi) + \mathrm{d}\varphi(\nabla w) = 0.
\end{cases} \quad (3.3.2)
$$

Then every regular level set of w is harmonic Einstein with respect to the restriction of φ (which is constant along the flow of ∇w) and to the same coupling constant $\alpha \neq 0$. Moreover, if X has at least one zero, then φ is constant and (M,g) is isometric to either $\mathbb{R}^m$ or $\mathbb{S}^m$ equipped with some rotationally symmetric metric with respect to the poles coinciding with the zeroes of X.

A key ingredient is a result of Montiel [27, Proposition 2], stated below, which characterizes the local (or global, in some cases) geometry of a Riemannian manifold (M, g) admitting a non-Killing closed conformal vector field X. In particular, (M, g) is forced to be either locally a warped product, or a rotationally symmetric metric on $\mathbb{R}^m$ or $\mathbb{S}^m$, depending on the number of zeroes of X. We will combine this with Proposition 3.23, that will provide us a formula from which we will deduce (under some further hypotheses) that ∇w and X are indeed parallel. This will enable us to compare the local geometry of the level surface of w to that of the leaves of X, at least when $\nabla w \neq 0$, and conclude (exploiting (3.1.14)) that the leaves have the φ-Ricci tensor proportional to the metric. Then, by the fact that φ is assumed to be harmonic, we conclude that also the restriction to the leaves is harmonic, from which it follows the Einstein-harmonicity of such leaves. We begin with the following

Proposition 3.23. *Let $(M, \langle\, , \,\rangle)$ be a φ-static space of dimension $m \geq 2$ and let X be a closed conformal vector field on M such that X is an eigenvector of $\varphi^* g_N$ relative to a constant eigenvalue λ. Let w be a solution to the vacuum-static equation* (3.1.14)*. Then*

$$\Big(X(\operatorname{div} X) + \big(\frac{S^\varphi}{m-1} + \alpha\lambda\big)|X|^2\Big)\big(|X|^2\nabla w - X(w)X\big) \equiv 0 \text{ on } M. \tag{3.3.3}$$

Proof. Let us set

$$\psi = \frac{1}{m}\operatorname{div} X \tag{3.3.4}$$

so that (3.3.1) simply writes

$$X_{i,j} = \psi\delta_{ij}. \tag{3.3.5}$$

We covariantly differentiate the latter to obtain

$$X_{i,jk} = \psi_k\delta_{ij}.$$

Skew-symmetrizing in the last two indices, by the Ricci equations we get

$$X_t R_{tijk} = X_{i,jk} - X_{i,kj} = \psi_k\delta_{ij} - \psi_j\delta_{ik}. \tag{3.3.6}$$

Tracing on i and k yields the relation

$$X_t R_{tk} = -(m-1)\psi_k. \tag{3.3.7}$$

On the other hand, by contracting (3.3.6) with X_i we deduce that

$$\psi_k X_j = \psi_j X_k. \tag{3.3.8}$$

Contracting again with X_k, we have

$$|X|^2\nabla\psi = X(\psi)X. \tag{3.3.9}$$

We now compute

$$\nabla(X(w)) \text{ and } \operatorname{Hess}(X(w)).$$

Using (3.3.7), the hypothesis

$$\varphi^* g_N(X) = \lambda X^\flat, \quad \lambda \in \mathbb{R}, \tag{3.3.10}$$

and (3.3.5) we have

$$\begin{aligned}(X_s w_s)_i &= X_{si}w_s + X_s w_{si} = \psi w_i + wX^s R^\varphi_{si} - \frac{S^\varphi}{m-1}wX_i \\ &= \psi w_i - (m-1)w\psi_i - \alpha\lambda w X_i - \frac{S^\varphi}{m-1}wX_i\end{aligned}$$

Covariantly differentiating the latter and using again (3.3.5) we obtain

$$(X_s w_s)_{ij} = \psi_j w_i + \psi w_{ij} - (m-1)w_j\psi_i - (m-1)\psi_{ij} - \alpha\lambda w_j X_i - \alpha\lambda\psi\delta_{ij} - \frac{S^\varphi}{m-1}w_j X_i - \frac{S^\varphi}{m-1}w\psi\delta_{ij}.$$

At this point, we skew symmetrize the latter obtaining that,

$$\psi_j w_i - (m-1)w_j\psi_i - \alpha\lambda w_j X_i - \frac{S^\varphi}{m-1}w_j X_i = \psi_i w_j - (m-1)w_i\psi_j - \alpha\lambda w_i X_j - \frac{S^\varphi}{m-1}w_i X_j.$$

which can be simplified to

$$m\psi_j w_i - \alpha\lambda w_j X_i - \frac{S^\varphi}{m-1}w_j X_i = m\psi_i w_j - \alpha\lambda w_i X_j - \frac{S^\varphi}{m-1}w_i X_j. \tag{3.3.11}$$

Next we use (3.3.9) into the right hand side of (3.3.11) multiplied by $|X|^2$ to obtain

$$m|X|^2\psi_i w_j - \big(\frac{S^\varphi}{m-1} + \alpha\lambda\big)|X|^2 w_i X_j = mX(\psi)X_i w_j - \big(\frac{S^\varphi}{m-1} + \alpha\lambda\big)|X|^2 X_j w_i.$$

Similarly with the left hand side of (3.3.11)

$$m|X|^2\psi_j w_i - \big(\frac{S^\varphi}{m-1} + \alpha\lambda\big)|X|^2 w_j X_i = mX(\psi)X_j w_i - \big(\frac{S^\varphi}{m-1} + \alpha\lambda\big)|X|^2 X_i w_j.$$

By (3.3.11) we then deduce the equality

$$mX(\psi)X_i w_j - \big(\frac{S^\varphi}{m-1} + \alpha\lambda\big)|X|^2 X_j w_i = mX(\psi)X_j w_i - \big(\frac{S^\varphi}{m-1} + \alpha\lambda\big)|X|^2 X_i w_j$$

so that multiplying by X^i we deduce

$$\begin{aligned} mX(\psi)|X|^2 w_j - \big(\frac{S^\varphi}{m-1} + \alpha\lambda\big)|X|^2 X(w)X_j = \\ = mX(\psi)X(w)X_j - \big(\frac{S^\varphi}{m-1} + \alpha\lambda\big)|X|^4 w_j. \end{aligned} \tag{3.3.12}$$

Rearranging (3.3.12) we infer

$$\Big(X(\operatorname{div} X) + \big(\frac{S^\varphi}{m-1} + \alpha\lambda\big)|X|^2\Big)\big(|X|^2 w_j - X(w)X_j\big) = 0,$$

that is, (3.3.3). □

In the next Proposition, making an assumption on Ric^φ evaluated at X, we can guarantee the proportionality between X and ∇w:

Proposition 3.24. *In the assumptions of Proposition 3.23 let*

$$A = {}^cZ(X) = \{x \in M : X_x \neq 0\}.$$

If, denoting the traceless φ-Ricci tensor as T,

$$T(X,X) \neq -\frac{\alpha}{m}\varphi^* g_N(X,X) \text{ on } A, \tag{3.3.13}$$

then

$$|X|^2\nabla w - X(w)X \equiv 0 \text{ on } M. \tag{3.3.14}$$

Proof. By (3.3.7) and (3.3.10),

$$T(X,X) = \mathrm{Ric}(X,X) - \alpha\varphi^* g_N(X,X) - \frac{S^\varphi}{m}|X|^2 = -(m-1)X(\psi) - \alpha\lambda|X|^2 - \frac{S^\varphi}{m}|X|^2.$$

Therefore, condition (3.3.13) reads as

$$-\frac{m-1}{m}X(\operatorname{div}(X)) - \frac{S^\varphi}{m}|X|^2 \neq \alpha\frac{m-1}{m}\lambda|X|^2$$

that, rearranging, is

$$X(\operatorname{div}(X)) + \Big(\frac{S^\varphi}{m-1} + \alpha\lambda\Big)|X|^2 \neq 0.$$

By (3.3.3), we then deduce that, if $X \neq 0$, then

$$|X|^2\nabla w - X(w)X = 0.$$

On the other hand, if $X = 0$ then the latter expression is already zero, thus proving (3.3.14). □

We here recall the result of Montiel concerning manifolds admitting closed conformal vector fields:

Proposition 3.25 ([27], Proposition 2)**.** *Let (M, g) be a complete Riemannian manifold of dimension $m \geq 2$ admitting a nontrivial closed conformal vector field X. Then, X has at most two zeroes and, depending on the number of zeroes of X, the following alternatives hold:*

(a) If X has no zeroes, then the universal cover of (M, g) is a warped product $\mathbb{R} \times_h P$, where P is a complete simply connected riemannian manifold of dimension $m-1$ and h is a function defined on $\mathbb{R}$. Moreover, h is invariant by the action of the projection of the group of deck isometries into $\mathrm{Iso}(\mathbb{R})$ and X is the projection on M of $h(s)\partial_s$.

(b) If X has one zero, then $M = \mathbb{R}^m$ and g is a rotationally invariant metric expressed in terms of polar coordinates, on $\mathbb{R}^m \setminus \{0\} = \mathbb{R}^+ \times \mathbb{S}^{m-1} \ni (r, v)$, as

$$g = \mathrm{d}r^2 + h(r)^2 \mathrm{d}\sigma^2_{m-1},$$

where $\mathrm{d}\sigma^2_{m-1}$ is the round metric of curvature one on $\mathbb{S}^{m-1}$ and h is the positive restriction to $\mathbb{R}^+$ of an odd differentiable function with $h'(0) = 1$. Moreover, $X(r, v) = h(r)v$.

(c) If X has two zeroes, then $M = \mathbb{S}^m$ and g is a rotationally invariant metric That is, considering the two antipodal zeroes of X $\{a, -a\} \in \mathbb{S}^m \subset \mathbb{R}^{m+1}$ and given the relative polar coordinates $(\theta, p) \in (0, \pi) \times \mathbb{S}^{m-1}$, $\mathbb{S}^{m-1} \subset \mathbb{S}^m$ being the equator relative to the poles $\{a, -a\}$, such that $\mathbb{S}^m \setminus \{a, -a\} \ni x = a\cos(\theta) + p\sin(\theta)$,

$$g = \mathrm{d}\theta^2 + h(\theta)^2 \mathrm{d}\sigma^2_{m-1}.$$

Here $\mathrm{d}\sigma^2_{m-1}$ denotes again the round metric of curvature one on $\mathbb{S}^{m-1}$, whereas h is the positive restriction to $(0, \pi)$ of a 2π-periodic odd differentiable function with $h'(0) = 1$. In addition, $X = f(\theta)(a\sin(\theta) - p\cos(\theta))$.

Having in hand the former Theorem, we can in any case split locally (M, g) (having removed the possible zeroes of X) as a warped product $g = \mathrm{d}s^2 + h(s)^2 g_P$, where $X = h(s)\partial_s$. Before moving on towards the proof of Theorem 3.22, it is then convenient to express the φ-Ricci tensor of (M, g) with respect to the φ-Ricci tensor of a slice (where we consider the restriction of φ to the slice for the definition of the φ-Ricci tensor on (P, g_P)) and h. Notice that the hypothesis of Proposition 3.23 on X being an eigenvector of the pullback of g_N reads locally as ∂_s being an eigenvector relative to the same eigenvalue.

Lemma 3.26. *Let (M, g) be a warped product $M = I \times_h P$, where I is a complete flat 1-dimensional manifold, with the warped product metric $g = \mathrm{d}s^2 + h(s)^2 g_P$, where $h : I \to R$ is smooth and g_P is a metric on the $(m-1)$-dimensional manifold P. Let us consider also a smooth map $\varphi : (M, g) \to (N, g_N)$, assuming that ∂_s is an eigenvalue of $\varphi^* g_N$ relative to a constant eigenvalue λ, and denote with $\hat{\varphi} = \varphi|_{P_s}$, for a slice $P_s = \{s\} \times P, s \in I$. Then, letting Ric^φ and $\mathrm{Ric}^{\hat{\varphi}}_s$ be the φ-Ricci tensor relative to the same $\alpha \neq 0$ of (M, g) and (P_s, g_P) respectively, it holds*

$$\mathrm{Ric}^\varphi = \mathrm{Ric}^{\hat{\varphi}}_s - \big((m-2)(h'h^{-1})^2 + h''h^{-1}\big)h^2 g_P - \big((m-1)h''h^{-1} + \alpha\lambda\big)\theta^1 \otimes \theta^1 \quad \text{on } P_s. \tag{3.3.15}$$

Proof. Let us consider an orthonormal base for (P, g_P) given by $\hat{\theta}^i$, $i = 2 \dots m$. The corresponding orthonormal base on (M, g) is given by

$$\theta^1 = \mathrm{d}s,$$
$$\theta^i = h\hat{\theta}^i,$$

where for the easy of notation we have omitted the pullbacks. Let us compute the connection forms θ^s_t, $s, t = 1, \dots, m$, of (M, g) with respect to those of (P, g_P), which we denote with $\hat{\theta}^i_j$. Using the first structure equations we have

$$-\theta^1_s \wedge \theta^s = \mathrm{d}\theta^1 = \mathrm{d}(\mathrm{d}s) = 0, \tag{3.3.16}$$

$$-\theta^i_s \wedge \theta^s = \mathrm{d}\theta^i = \mathrm{d}(h\hat{\theta}^i) = h'h^{-1}\theta^i - \hat{\theta}^i_j \wedge \theta^j. \tag{3.3.17}$$

Evaluating (3.3.16) on (e_i, e_j) we get

$$(\theta_i^1)_j = (\theta_j^1)_i, \tag{3.3.18}$$

whereas evaluating (3.3.17) on (e_j, e_k) yields

$$(\theta_j^i - \hat{\theta}_j^i)(e_k) = (\theta_k^i - \hat{\theta}_k^i)(e_j).$$

Denoting with $b_{jk}^i = (\theta_j^i - \hat{\theta}_j^i)(e_k)$ since they are by construction antisymmetric on i and j and by the above equation also symmetric on j and k, they are easily shown to be 0. As a consequence, evaluating (3.3.17) on $(\cdot, e_j)$ gives

$$\theta_j^i - \hat{\theta}_j^i = (\theta_1^i)_j\theta^1 - h'h^{-1}\delta_{ij}\theta^1$$

By (3.3.18), skew-symmetrizing the latter we obtain

$$\theta_j^i = \hat{\theta}_j^i, \tag{3.3.19}$$

whereas symmetrizing we infer that

$$\theta_1^i = h'h^{-1}\theta^i. \tag{3.3.20}$$

We can now compute the curvature forms by means of the second structure equations. Exploiting (3.3.20), we have

$$-\theta_s^i \wedge \theta_1^s + \Theta_1^i = \mathrm{d}\theta_1^i = \mathrm{d}(h'h^{-1}\theta^i) = (h''h^{-1} - (h'h^{-1})^2)\theta^1 \wedge \theta^i - h'h^{-1}\theta_s^i \wedge \theta^s,$$

which can be simplified, using (3.3.20), to

$$\Theta_1^i = (h''h^{-1} - (h'h^{-1})^2)\theta^1 \wedge \theta^i - h'h^{-1}\theta_1^i \wedge \theta^1.$$

Substituting again (3.3.20), it reduces to

$$\Theta_1^i = h''h^{-1}\theta^1 \wedge \theta^i.$$

As for the other curvature forms, we make use of (3.3.19) to obtain

$$\begin{aligned}\Theta_j^i &= \mathrm{d}\theta_j^i + \theta_s^i \wedge \theta_j^s = \mathrm{d}\hat{\theta}_j^i + \theta_s^i \wedge \theta_j^s \\ &= \hat{\Theta}_j^i + \theta_0^i \wedge \theta_j^0 = \hat{\Theta}_j^i - (h'h^{-1})^2\theta^i \wedge \theta^j.\end{aligned}$$

At the level of components of the curvature tensor $R_{tuv}^s = \Theta_t^s(e_u, e_v)$, the above relations read as

$$R_{ijk1} = 0; \tag{3.3.21}$$

$$R_{i1k1} = -h''h^{-1}\delta_{ik}; \tag{3.3.22}$$

$$R_{ijkl} = h^{-2}\hat{R}_{ijkl} - (h'h^{-1})^2(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}). \tag{3.3.23}$$

Tracing, we obtain

$$\begin{aligned}R_{11} &= -(m-1)h''h^{-1}; \\ R_{i1} &= 0; \\ R_{ik} &= R_{isks} = -h''h^{-1}\delta_{ik} + h^{-2}\hat{R}_{ik} - (h'h^{-1})^2(m-2)\delta_{ik} \\ &= h^{-2}\hat{R}_{ik} - \big((m-2)(h'h^{-1})^2 + h''h^{-1}\big)\delta_{ik}.\end{aligned}$$

In global notation, we can write (again omitting the pullbacks)

$$\mathrm{Ric} = \mathrm{Ric}_s - \big((m-2)(h'h^{-1})^2 + h''h^{-1}\big)h^2 g_P - (m-1)h''h^{-1}\theta^1 \otimes \theta^1, \tag{3.3.24}$$

where Ric_s denotes the Ricci tensor of (P_s, g_P). As for $\varphi^* g_N$, we notice that, assuming ∂_s to be an eigenvector relative to λ, then it can be written as

$$\varphi^* g_N = \lambda\theta^1 \otimes \theta^1 + \pi^*\hat{\varphi}^* g_N,$$

and hence, by (3.3.24), (omitting the pullback π^* as above)

$$\begin{aligned}\mathrm{Ric} - \alpha\varphi^* g_N &= \mathrm{Ric} - \alpha(\lambda\theta^1 \otimes \theta^1 + \hat{\varphi}^* g_N) \\ &= \mathrm{Ric}_s^{\hat{\varphi}} - \big((m-2)(h'h^{-1})^2 + h''h^{-1}\big)h^2 g_P - \big((m-1)h''h^{-1} + \alpha\lambda\big)\theta^1 \otimes \theta^1,\end{aligned}$$

that is, (3.3.15). □

Let us now consider the tension field of $\hat{\varphi} : (P_s, g_P) \to (N, g_N)$ and express it with respect to the tension field of φ:

Lemma 3.27. *In the same setting of Lemma 3.26, we have that*

$$\tau(\hat{\varphi}) = h^2\big(\tau(\varphi) - \nabla \mathrm{d}\varphi(\partial_s, \partial_s) - (m-1)h'h^{-1}\mathrm{d}\varphi(\partial_s)\big). \tag{3.3.25}$$

Proof. In components along an adapted orthonormal frame as in the proof of Lemma 3.26 we have, for $i = 2 \dots m$,

$$\hat{\varphi}^a_i = \omega^a(\mathrm{d}\hat{\varphi}(\hat{e}_i)) = \omega^a(\mathrm{d}\varphi(he_i)|_{P_s}) = h\varphi^a_i|_{P_s}.$$

Differentiating on P_s, recalling that h depends only on s, we have (t varying on $1 \dots m$)

$$\begin{aligned}
\hat{\varphi}^a_{ij}\hat{\theta}^j &= \mathrm{d}\hat{\varphi} - \hat{\varphi}^a_j\hat{\theta}^j_i + \hat{\varphi}^b_i\varphi^*\omega^a_b \\
&= h\mathrm{d}\varphi^a_i - h\varphi^a_j\theta^j_i + h\varphi^b_i\varphi^*\omega^a_b \\
&= h\mathrm{d}\varphi^a_i - h\varphi^a_t\theta^t_i + h\varphi^b_i\varphi^*\omega^a_b + h\varphi^a_1\theta^1_i \\
&= h\varphi^a_{ij}\theta^j - h'\varphi^a_1\theta^i,
\end{aligned}$$

where we have used (3.3.19) and (3.3.20). Hence

$$\hat{\varphi}^a_{ij} = h^2(\varphi^a_{ij} - h'h^{-1}\varphi^a_1\delta_{ij}). \tag{3.3.26}$$

Tracing on i and j,

$$\tau(\hat{\varphi}) = h^2\Big(\sum_{i=2\dots m} \nabla\mathrm{d}\varphi(e_i, e_i) - (m-1)h'h^{-1}\mathrm{d}\varphi(e_1)\Big),$$

and therefore

$$\tau(\hat{\varphi}) = h^2(\tau(\varphi) - \nabla\mathrm{d}\varphi(e_1, e_1) - (m-1)h'h^{-1}\mathrm{d}\varphi(e_1)),$$

that is, (3.3.25). □

We now have all the ingredients to prove the following Proposition, inspecting the local geometry of the leaves of the foliation of X:

Proposition 3.28. *Let $\varphi : (M, g) \to (N, g_N)$ be a smooth map between Riemannian manifolds and let X be a nontrivial closed conformal vector field on X such that*

$$\varphi^* g_N(X, \cdot) = \lambda X^\flat, \quad \lambda \in \mathbb{R}$$

and, when $X \neq 0$,

$$T(X, X) \neq -\frac{\alpha}{m}\varphi^* g_N(X, X). \tag{3.3.27}$$

Assume also that (3.3.2) admits a solution $w \not\equiv 0$. Then the metric g on $M \setminus \{p : X_p = 0\}$ splits locally as a warped product $I \times_h P$,

$$g = \mathrm{d}s^2 + h(s)^2 g_P,$$

and on every slice P_s where $w \neq 0$, w is constant, the φ-Ricci tensor satisfies

$$\mathrm{Ric}^\varphi = \Big(\frac{w_1}{w}h'h^{-1} + \frac{S^\varphi}{m-1}\Big)h^2 g_P - \Big((m-1)h''h^{-1} + \alpha\lambda\Big)\theta^1 \otimes \theta^1 \tag{3.3.28}$$

and $\hat{\varphi} = \varphi|_{P_s}$ is conservative. Moreover, if it holds

$$\nabla\mathrm{d}\varphi(X, X) = -\left(\frac{X(w)}{w} + (m-1)\psi\right)\mathrm{d}\varphi(X) \tag{3.3.29}$$

then (P_s, g_P) is harmonic-Einstein with respect to $\hat{\varphi}$ (and the same α as that of (M, g)).

Proof. Making use of the same notation of the previous propositions and lemmas, by Proposition 3.24 we have, in coordinates, where $X \neq 0$,

$$w_s = \frac{X(w)}{|X|^2} X_s,$$

meaning that ∇w is parallel to X whenever it does not vanish. Let us denote with $\rho = \frac{X(w)}{|X|^2}$. Then,

$$\begin{aligned}
\rho_t &= \frac{X_{s,t} w_s + X_s w_{st}}{|X|^2} - \frac{X(w)}{|X|^4} 2\psi X_t \\
&= \frac{\psi}{|X|^2} w_t + \frac{1}{|X|^2} X_s \Big(w R^{\varphi}_{st} - w \frac{S^{\varphi}}{m-1} \delta_{st} \Big) - 2 \frac{\psi \rho}{|X|^2} X_t \\
&= -\frac{\psi \rho}{|X|^2} X_t + \frac{w}{|X|^2} \Big(X_s R_{st} - \alpha X_s \varphi^a_s \varphi^a_t - \frac{S^{\varphi}}{m-1} X_t \Big) \\
&= -\frac{\psi \rho}{|X|^2} X_t + \frac{w}{|X|^2} \Big(-(m-1) \frac{X(\psi)}{|X|^2} X_t - \alpha \lambda X_t - \frac{S^{\varphi}}{m-1} X_t \Big) \\
&= -\frac{1}{|X|^2} \Big(\frac{\psi X(w)}{|X|^2} + (m-1) w \frac{X(\psi)}{|X|^2} + \alpha \lambda w + \frac{S^{\varphi}}{m-1} w \Big) X_t.
\end{aligned}$$

As a consequence, when computing the hessian of w we have

$$\begin{aligned}
w_{st} &= \rho_t X_s + \rho \psi \delta_{st} \\
&= -\frac{1}{|X|^2} \Big(\frac{\psi X(w)}{|X|^2} + (m-1) w \frac{X(\psi)}{|X|^2} + \alpha \lambda w + \frac{S^{\varphi}}{m-1} w \Big) X_s X_t + \rho \psi \delta_{st}.
\end{aligned}$$

Substituting the first in (3.3.2),

$$w \operatorname{Ric}^{\varphi} = -\frac{1}{|X|^2} \Big(\frac{\psi X(w)}{|X|^2} + (m-1) w \frac{X(\psi)}{|X|^2} + \alpha \lambda w + \frac{S^{\varphi}}{m-1} w \Big) X^{\flat} \otimes X^{\flat} + \Big(\frac{X(w)}{|X|^2} \psi + w \frac{S^{\varphi}}{m-1} \Big) g.$$

We can now make some substitutions in local adapted coordinates as in 3.26, so that

$$h = |X|, \quad h' = \psi,$$

and

$$X^{\flat} = h \theta^1,$$

resulting in

$$\begin{aligned}
w \operatorname{Ric}^{\varphi} &= (w_1 h^{-1} h' + w \frac{S^{\varphi}}{m-1}) \delta_{ij} \theta^i \otimes \theta^j - \big((m-1) w h'' h^{-1} + \alpha \lambda w \big) \theta^1 \otimes \theta^1 \\
&= (w_1 h^{-1} h' + w \frac{S^{\varphi}}{m-1}) h^2 g_P - \big((m-1) w h'' h^{-1} + \alpha \lambda w \big) \theta^1 \otimes \theta^1,
\end{aligned}$$

which readily gives (3.3.28). We can now combine the latter with the expression for $\operatorname{Ric}^{\hat{\varphi}}_s$ in (3.3.15) in order to get

$$\operatorname{Ric}^{\hat{\varphi}}_s = (\frac{w_1}{w} h' h + \frac{S^{\varphi}}{m-1} h^2 + (m-2)(h')^2 + h'' h) g_P. \tag{3.3.30}$$

As a consequence, since w locally depends only on s, we see that the proportionality function is constant on P_s, and hence $S^{\hat{\varphi}}_s$ is constant. Moreover,

$$R^{\hat{\varphi}}_{ij,j} - \frac{1}{2} S^{\hat{\varphi}}_i = -\alpha \hat{\varphi}^a_i \hat{\varphi}^a_{jj} = 0,$$

meaning that $\hat{\varphi}$ is conservative.
Let us suppose that it holds (3.3.29). Then, in the local orthonormal frame adapted to the warped product, the condition reads as

$$\nabla \mathrm{d}\varphi(\partial_s, \partial_s) = -\Big(\frac{w_1}{w} + (m-1) h' h^{-1} \Big) \mathrm{d}\varphi(\partial_s).$$

By the second in (3.3.2)

$$\tau(\varphi) = -\frac{1}{w}\mathrm{d}\varphi(\nabla w) = -\frac{w_1}{w}\mathrm{d}\varphi(\partial_s)$$

and we thus have

$$\nabla\mathrm{d}\varphi(\partial_s, \partial_s) = \tau(\varphi) - (m-1)h'h^{-1}\mathrm{d}\varphi(\partial_s).$$

Substituting into (3.3.25), we have that

$$\tau(\hat{\varphi}) = 0,$$

and thus (P_s, g_P) is harmonic-Einstein.

□

Remark 3.29. The condition (3.3.29) is true when one assumes that the map doesn't depend on the flow of X. Indeed, this means that, in the adapted frame

$$\mathrm{d}\varphi(\partial_s) = 0$$

or, in local components,

$$\varphi_1^a = 0,$$

and hence

$$\varphi_{11}^a = \mathrm{d}\varphi_1^a(e_1) - \varphi_s^a\theta_1^s(e_1) + \varphi_1^b\varphi^*\omega_b^a(e_1) = 0.$$

Thus

$$\nabla\mathrm{d}\varphi(\partial_s, \partial_s) = 0$$

and (3.3.29) is satisfied.
If w is non-constant, this is equivalent to requiring that φ is harmonic. Indeed, in this case the critical points of w are a zero measure set - otherwise, if the critical set of w contains an open set, then w would be locally constant and hence constant by the unique continuation principle. Therefore, from the second in (3.3.2) we get, dividing by ρ, that

$$\mathrm{d}\varphi(X) = 0$$

on the points where $\nabla w \neq 0$ and hence on all M by continuity. The converse is true again by the second in (3.3.2) and by the fact that the critical points are at most a zero measure set.

With the help of the Previous proposition, we are able to prove the main theorem of this section:

Proof of Theorem 3.22. Let $c \neq 0$ be a regular value for w, and denote with $\Sigma_c = \{x \in M : w(x) = c\}$ the corresponding (possibly disconnected) regular level set. Then, from the hypotheses and applying Proposition 3.24 we have that ∇w and X are proportional to each other when $X \neq 0$, leading to the fact that locally the level set coincides with a leave of the distribution of X. Moreover, a regular level set of w cannot contain any zero of X, since the zeroes of X are at most two by Proposition 3.25 and they constitute separate leaves of the distribution. Therefore, from Proposition 3.28 Σ_c is globally a disjoint union of leaves of X, which are harmonic-Einstein with respect to the metric g_P and hence with respect to the restriction of the metric g to each leaf $\mathcal{F}_p$, $p \in M$ such that $w(p) = c$, of the foliation of X. Indeed, $g|_{\mathcal{F}_p} = h^2 g_P$ but h^2 is constant on $\mathcal{F}_p$ (and therefore locally constant on Σ_c).
As for $c = 0$, we already know by Proposition 3.1 that Σ_0 is a regular level set and that it is the union of totally geodesic hypersurfaces on M. Moreover, for each $p \in \Sigma_0$ there exists a neighbourhood U_p such that $\nabla w \neq 0$. As a consequence, we can consider equation (3.3.28) on the adapted frame as above, which gives

$$\begin{cases} R_{ij}^{\varphi} = 0, \text{ for } i \neq j \geq 2; \\ R_{ii}^{\varphi} = \left(\frac{w_1}{w}h'h^{-1} + \frac{S^{\varphi}}{m-1}\right), i \geq 2. \end{cases}$$

By this, we have that Ric^{φ} is diagonal on $\Sigma_c \cap U_p$ and there exists the limit

$$\lim_{x \to p} \frac{w_1}{w}h'h^{-1} =: \beta(p).$$

Therefore, by (3.3.30) and by continuity we have

$$\mathrm{Ric}^{\hat{\varphi}}_s = (\beta h^2 + \frac{S^\varphi}{m-1}h^2 + (m-2)(h'h^{-1})^2 + h''h^{-1})g_P.$$

From this and by the fact that, given (3.3.25) and the harmonicity of the map (and the fact that, as a consequence, $\mathrm{d}\varphi(X) = 0$), it holds

$$\tau(\hat{\varphi}) = 0 \text{ on } \Sigma_0,$$

it follows that (Σ_0, g_P) is harmonic-Einstein and hence, arguing as above, $(\Sigma_0, g|_{\Sigma_0})$ is harmonic-Einstein as well.
To conclude, we show that if X has one or two zeroes then φ must be constant. To see this, let us consider any two points p_1 and p_2 of M where $X \neq 0$ and $p \in M$ where $X_p = 0$. Since φ is constant along the flow of X - see the previous remark - and hence along the s-curves, taking two such curves originating from p and going to p_1 and p_2 respectively we see that $\varphi(p_1) = \varphi(p) = \varphi(p_2)$. As a consequence, in this case the connected components of the regular level sets of w are Einstein. □

Remark 3.30. As far as condition (3.3.13) is concerned, from which follows that ∇w and X are proportional one another, in the local notation of the splitting as a warped product it is equivalent to requiring that

$$h'' + \left(\frac{S^\varphi}{m-1} + \alpha\lambda\right) h \neq 0.$$

By converse, supposing that this doesn't hold on an open set A, then h (and therefore X) on A has a specific expression:

$$\begin{cases} h(s) = A\left(\sqrt{-\left(\frac{S^\varphi}{m-1} + \alpha\lambda\right)}s\right) + B\left(-\sqrt{-\left(\frac{S^\varphi}{m-1} + \alpha\lambda\right)}s\right) & \text{if } \frac{S^\varphi}{m-1} + \alpha\lambda < 0, \\ h(s) = As + B & \text{if } \frac{S^\varphi}{m-1} + \alpha\lambda = 0, \\ h(s) = A\sin\left(\sqrt{\left(\frac{S^\varphi}{m-1} + \alpha\lambda\right)}s\right) + B\cos\left(\sqrt{\left(\frac{S^\varphi}{m-1} + \alpha\lambda\right)}s\right) & \text{if } \frac{S^\varphi}{m-1} + \alpha\lambda > 0, \end{cases}$$

where A, B are some real constants. A special case is when

$$\mathrm{Ric}^\varphi = \frac{S^\varphi}{m}g,$$

meaning that both ∇w and X are closed conformal vector fields on M, which leads to more symmetry on the metric, provided that ∇w and X are not proportional. This is the case, for instance, of constant sectional curvature metrics. This means that ∇w and X are forced to be proportional only if there is a "preferred coordinate" that differs from the others.

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