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The Huang–Yang Formula for the Low-Density Fermi Gas: Upper Bound

Emanuela L. Giacomelli[1] | Christian Hainzl[2] | Phan Thành Nam[2] | Robert Seiringer[3]

[1]University of Milan, Department of Mathematics "Federigo Enriques", Milan, Italy | [2]LMU Munich, Department of Mathematics, Munich, Germany | [3]Institute of Science and Technology Austria, Klosterneuburg, Austria

Correspondence: Emanuela L. Giacomelli (emanuela.giacomelli@unimi.it)

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ABSTRACT

We study the ground state energy of a gas of spin $1/2$ fermions with repulsive short-range interactions. We derive an upper bound that agrees, at low density ϱ, with the Huang–Yang conjecture. The latter captures the first three terms in an asymptotic low-density expansion, and in particular the Huang–Yang correction term of order $\varrho^{7/3}$. Our trial state is constructed using an adaptation of the bosonic Bogoliubov theory to the Fermi system, where the correlation structure of fermionic particles is incorporated by quasi-bosonic Bogoliubov transformations. In the latter, it is important to consider a modified zero-energy scattering equation that takes into account the presence of the Fermi sea, in the spirit of the Bethe–Goldstone equation.

1 | Introduction and Main Result

Establishing asymptotic formulas for the ground state energy of dilute gases has been an active area in Mathematical Physics in recent years, partly motivated by the advances in the physics of cold atoms and molecules. In the bosonic case, the validity of the Lee–Huang–Yang formula [1] for the first two terms in a low-density expansion was recently established [2–5], following earlier work on the first term in [6, 7]. In this work, we are concerned with the analogous question for fermions, for which the Huang–Yang formula [8] gives the first *three* terms in an asymptotic expansion at low density. We shall establish the validity of this formula, at least as an upper bound.

We consider a system of N fermions with spin $1/2$ in a box $\Lambda := [-L/2, L/2]^3 \subset \mathbb{R}^3$ with periodic boundary conditions. The interaction between particles is described by the periodization $V : \Lambda \to \mathbb{R}$ of a potential $V_\infty : \mathbb{R}^3 \to \mathbb{R}$, that is, $V(x) = \sum_{z\in\mathbb{Z}^3} V_\infty(x + Lz)$, where we assume that V_∞ is nonnegative, radial and compactly supported. The Hamiltonian describing the system is given by

$$H_N := -\sum_{j=1}^{N} \Delta_{x_j} + \sum_{1\le i<j\le N} V(x_i - x_j), \tag{1.1}$$

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acting on $\bigwedge^N L^2(\Lambda, \mathbb{C}^2)$, the Hilbert space of antisymmetric square-integrable functions of N space-spin variables (x_i, σ_i), with $x_i \in \Lambda$ and $\sigma_i \in \{\uparrow, \downarrow\}$. Since H_N is spin-independent, if $N = N_\uparrow + N_\downarrow$, then H_N leaves invariant the subspace $\mathfrak{h}(N_\uparrow, N_\downarrow) \subset \bigwedge^N L^2(\Lambda, \mathbb{C}^2)$ consisting of N-body wave functions with exactly N_σ particles of spin $\sigma \in \{\uparrow, \downarrow\}$.

The Hamiltonian H_N is bounded from below and can be defined as a self-adjoint operator by Friedrichs' method. The ground state energy in the subspace $\mathfrak{h}(N_\uparrow, N_\downarrow)$ is given by

$$E_L(N_\uparrow, N_\downarrow) = \inf_{\psi \in \mathfrak{h}(N_\uparrow, N_\downarrow)} \frac{\langle \psi, H_N \psi \rangle}{\langle \psi, \psi \rangle}, \tag{1.2}$$

and the ground state energy density in the thermodynamic limit is

$$e(\varrho_\uparrow, \varrho_\downarrow) = \lim_{\substack{L \to \infty \\ N_\sigma / L^3 \to \varrho_\sigma, \sigma \in \{\uparrow, \downarrow\}}} \frac{E_L(N_\uparrow, N_\downarrow)}{L^3}. \tag{1.3}$$

It is well-known that the thermodynamic limit exists and is independent of the boundary conditions [9, 10].

By using a pseudopotential method, Huang and Yang [8] predicted that in the low-density limit $\varrho_\uparrow = \varrho_\downarrow = \varrho/2 \to 0$,

$$e(\varrho/2, \varrho/2) = \frac{3}{5}(3\pi^2)^{\frac{2}{3}} \varrho^{\frac{5}{3}} + 2\pi a \varrho^2 + \frac{4}{35}(11 - 2\log 2)(9\pi)^{\frac{2}{3}} a^2 \varrho^{7/3} + o(\varrho^{\frac{7}{3}})_{\varrho \to 0}, \tag{1.4}$$

where $a > 0$ is the s-wave scattering length of the interaction potential V_∞, defined by $8\pi a = \int_{\mathbb{R}^3} V_\infty (1 - \varphi_\infty)$ with $\varphi_\infty : \mathbb{R}^3 \to \mathbb{R}$ the solution of the zero-scattering equation

$$2\Delta \varphi_\infty + V_\infty (1 - \varphi_\infty) = 0, \qquad \varphi_\infty(x) \to 0 \quad \text{as} \quad |x| \to \infty. \tag{1.5}$$

The Huang–Yang conjecture was derived from a consideration of hard-sphere interactions, but it is expected that the formula extends to a large class of repulsive and short-range interaction potentials. The formula (1.4) is of great interest in the theory of dilute quantum gases, since it exhibits a remarkable universality, with the energy depending on the interaction potential only via its scattering length, at least to the order considered. We refer to [11] for an experimental verification.

On the mathematical side, for a large class of repulsive interactions, including hard spheres, the validity of the first two terms of (1.4) was proved in [12]; more precisely, [12, Thm. 1] states that

$$e(\varrho_\uparrow, \varrho_\downarrow) = \frac{3}{5}(6\pi^2)^{2/3} \left(\varrho_\uparrow^{5/3} + \varrho_\downarrow^{5/3} \right) + 8\pi a \varrho_\uparrow \varrho_\downarrow + o(\varrho^2)_{\varrho \to 0}. \tag{1.6}$$

The leading order term comes from the kinetic energy of a non-interacting Fermi gas, while the second-order term provides an effective description of the interaction between components of different spins. (In the case of fermions without spin, this second term vanishes; see [13, 14] for the corresponding result in this case.) An analogue of (1.6) for the thermodynamic pressure at positive temperature was derived in [15].

An alternative proof of (1.6) was recently given in [16], by applying elements of bosonic Bogoliubov theory to Fermi systems. The method in [16] was later extended in [17] to derive an optimal error estimate $O(\varrho^{7/3})$ in the upper bound, and will also be of crucial importance in our analysis here. (See also [18] for an alternative approach.) In fact, in the present article we shall prove an upper bound containing the precise Huang–Yang correction of order $\varrho^{7/3}$. We impose the following conditions on the interaction potential.

Assumption 1.1. The interaction potential $V_\infty \in L^2(\mathbb{R}^3)$ is nonnegative, radial and compactly supported.

In order to state our main result, we shall need the function $F : \mathbb{R}_+ \to \mathbb{R}_+$ defined as

$$F(x) = \frac{(6\pi^2)^{1/3}}{35} \Bigg(16 x^{7/3} \ln x - 48 \left(x^{7/3} + 1 \right) \ln(1 + x^{1/3})$$
$$+ 6 \left(15 x^{1/3} - 4 x^{2/3} + 33 x + 33 x^{4/3} - 4 x^{5/3} + 15 x^2 \right)$$

$$+21\left(1-6x^{2/3}+5x+5x^{4/3}-6x^{5/3}+x^{7/3}\right)\ln\frac{|1-x^{1/3}|}{1+x^{1/3}}\Bigg). \tag{1.7}$$

It is a continuous, increasing function, with $F(1)=\frac{48}{35}(11-2\log 2)(6\pi^2)^{\frac{1}{3}}$.

Theorem 1.2 (Upper bound). *Let V_∞ be as in Assumption 1.1, and let $a>0$ be its scattering length. In the low-density limit $\varrho_\uparrow+\varrho_\downarrow=\varrho\to 0$, the ground state energy density defined in (1.3) satisfies*

$$e(\varrho_\uparrow,\varrho_\downarrow)\le\frac{3}{5}(6\pi^2)^{\frac{2}{3}}\left(\varrho_\uparrow^{\frac{5}{3}}+\varrho_\downarrow^{\frac{5}{3}}\right)+8\pi a\varrho_\uparrow\varrho_\downarrow+a^2\varrho_\uparrow^{\frac{7}{3}}F\left(\frac{\varrho_\downarrow}{\varrho_\uparrow}\right)+O(\varrho^{\frac{7}{3}+\frac{1}{9}}), \tag{1.8}$$

where F is defined in (1.7). In particular, if $\varrho_\uparrow=\varrho_\downarrow=\varrho/2\to 0$, we have

$$e(\varrho/2,\varrho/2)\le\frac{3}{5}(3\pi^2)^{\frac{2}{3}}\varrho^{\frac{5}{3}}+2\pi a\varrho^2+\frac{4}{35}(11-2\log 2)(9\pi)^{\frac{2}{3}}a^2\varrho^{7/3}+O(\varrho^{\frac{7}{3}+\frac{1}{9}}). \tag{1.9}$$

As we will see in the proof, the contribution $a^2\varrho_\uparrow^{\frac{7}{3}}F(\varrho_\downarrow/\varrho_\uparrow)$ in (1.8) comes from the expression

$$\frac{a^2}{8\pi^7}\int_{\mathbb{R}^3}\mathrm{d}p\int_{|r|\le k_F^\uparrow}\mathrm{d}r\int_{|r'|\le k_F^\downarrow}\mathrm{d}r'\left(\frac{1}{2|p|^2}-\frac{\mathbb{1}_{\{|r+p|>k_F^\uparrow\}}\mathbb{1}_{\{|r'-p|>k_F^\downarrow\}}}{|r+p|^2-|r|^2+|r'-p|^2-|r'|^2}\right), \tag{1.10}$$

with $k_F^\sigma=(6\pi^2)^{\frac{1}{3}}\varrho_\sigma^{1/3}$ the Fermi momenta of the two spin components. In particular, the roles of $\varrho_\uparrow$ and $\varrho_\downarrow$ are symmetric. In fact, $\varrho_\uparrow^{\frac{7}{3}}F\left(\varrho_\downarrow/\varrho_\uparrow\right)=\varrho_\downarrow^{\frac{7}{3}}F\left(\varrho_\uparrow/\varrho_\downarrow\right)$ since $F(x^{-1})=x^{-7/3}F(x)$ for $x>0$. The explicit form of F in (1.7) has appeared in the physics literature in [19] (see also [20, 21]).

The Huang–Yang correction in (1.4) can be interpreted as the fermionic analogue of the Lee–Huang–Yang correction for the energy of a dilute Bose gas, in the sense that it captures the next to leading order contribution of the interactions, beyond the ϱ^2 term. The validity of the Lee–Huang–Yang correction was proved in [3, 4] (lower bound) and [2, 5] (upper bound) (see also [22–24] for extensions to the free energy at positive temperature). In the analysis of the Bose gas, an important role is played by Bogoliubov's theory [25], where the correlation between particles is captured by unitary transformations containing quadratic expressions in the creation and annihilation operators on Fock space. The solution of the zero-energy scattering equation (1.5) naturally enters in these transformations.

Our proof of Theorem 1.2 is based on an adaptation of the bosonic Bogoliubov theory to Fermi systems. We will interpret suitable pairs of fermions as bosons, and then construct a trial state using the corresponding quasi-bosonic Bogoliubov transformations. This bosonization method goes back to heuristic arguments in [26, 27] in the attempt to explain the random phase approximation proposed in the 1950s [28–31]. In the context of the high-density Fermi gas, the bosonization method has been recently made rigorous to study the correlation energy in the mean-field regime [32–37].

For the low-density Fermi gas, the bosonization method was used in [16] to give a new proof of (1.6), and was improved further in [17] to obtain an optimal error estimate $O(\varrho^{7/3})$ in the upper bound. However, to obtain the precise Huang–Yang correction of order $\varrho^{7/3}$, we need to go substantially beyond the existing analysis. We will in fact introduce two quasi-bosonic Bogoliubov transformations to extract the correlation contribution of the particles. The first Bogoliubov transformation is closely related to the construction in [17], and involves the solution to the zero-energy scattering equation in (1.5). In the present article, this transformation carries an additional momentum cut-off to ensure that all relevant error terms are of order $o(\varrho^{7/3})$. The main new tool is the second Bogoliubov transformation, which contains the solution of a modified scattering equation (related to the Bethe–Goldstone equation [38]) taking into account the presence of the Fermi sea at low momentum. The main ingredients of the proof will be explained in the next section.

Our analysis can be extended in a straightforward way to higher spin fermions or, equivalently, more than two species of spinless fermions. For simplicity, we stick to the case of spin $1/2$ here. We also remark that the method of the present article can be used to derive a lower bound with the optimal error $O(\varrho^{7/3})$; see [39] for details.

Note added in proof. After the completion of the present article, the matching lower bound was established in [40]. Subsequently, the authors of [41] studied the free energy of the dilute Fermi gas for temperatures $T < \varrho^{2/3+1/6}$ and proved an analogue of the Huang–Yang formula in this case. Several ingredients of the present work, in particular the analysis of the Bethe–Goldstone equation explaining the Huang–Yang correction and factorization techniques yielding estimates uniform in volume, play an essential role in these subsequent developments.

Organization of the paper. In the next section, we shall explain the main structure of the proof. In Sections 2.1 and 2.2 we introduce the fermionic Fock space and the particle–hole transformation, which gives a convenient representation of the correlation Hamiltonian in Proposition 2.3. In Section 2.3 we construct the trial state in terms of two quasi-bosonic unitary transformations T_1 and T_2, defined in Definition 2.7. A heuristic discussion of the effect of the unitary transformations on the correlation Hamiltonian, and a sketch of the key estimates required, is then given in Section 2.4. Section 3 contains some preliminary bounds that in particular will allow to reduce the correlation Hamiltonian to a simpler, effective one, as formulated in Proposition 3.6. The main technical analysis needed in the proof of Theorem 1.2 is done in Sections 4 and 5, respectively, where the effect of the two unitary transformations T_1 and T_2 on the various operators of relevance is investigated. The main results of these two sections are summarized in Propositions 4.1 and 5.1, respectively. Finally, in Section 6 we combine the results of the previous sections, and complete the proof of Theorem 1.2. The appendix contains various auxiliary results needed in the proof.

2 | Main Ingredients in the Proof

In this section we shall explain in detail the construction of the trial state and the main strategy of the proof of Theorem 1.2. We will briefly recall the Fock space formalism and the particle–hole transformation, which will be convenient in order to focus on the excitations around the Fermi sea. Then we will introduce quasi-bosonic transformations, which are our main tool to impose the precise correlations among particles needed to capture the energy to the desired order. Finally, we will explain heuristically some key estimates in the computation of the energy, leading to the upper bound in Theorem 1.2.

2.1 | The Fermionic Fock Space

It will be convenient to work in second quantization. The fermionic Fock space is given by

$$\mathcal{F}_{\mathrm{f}} = \bigoplus_{n\geq 0} \mathcal{F}_{\mathrm{f}}^{(n)}, \quad \mathcal{F}_{\mathrm{f}}^{(n)} = \bigwedge^{n} L^2(\Lambda, \mathbb{C}^2).$$

Any $\psi \in \mathcal{F}_{\mathrm{f}}$ is of the form $\psi = (\psi^{(0)}, \psi^{(1)}, \dots, \psi^{(n)}, \dots)$ with $\psi^{(n)} = \psi^{(n)}((x_1,\sigma_1), \dots, (x_n, \sigma_n)) \in \mathcal{F}_{\mathrm{f}}^{(n)}$. The vacuum state will be denoted by $\Omega = (1, 0, 0, \dots)$.

For any $f \in L^2(\Lambda, \mathbb{C}^2)$, we denote by $a^*(f)$ and $a(f)$ the creation and annihilation operators, respectively. These operators satisfy the canonical commutation relations

$$\{a(f), a^*(g)\} = \langle f, g\rangle_{L^2(\Lambda;\mathbb{C}^2)}, \quad \{a(f), a(g)\} = \{a^*(f), a^*(g)\} = 0,$$

for any $f, g \in L^2(\Lambda; \mathbb{C}^2)$, where $\{A, B\} = AB + BA$. As a consequence, these operators are bounded, with

$$\|a(f)\| = \|a^*(f)\| = \|f\|_{L^2(\Lambda,\mathbb{C}^2)}. \tag{2.1}$$

For $\sigma \in \{\uparrow, \downarrow\}$ and $k \in \Lambda^* := (2\pi/L)\mathbb{Z}^3$, we denote $(\delta_\sigma f_k)(x, \sigma') = \delta_{\sigma,\sigma'} L^{-3/2} e^{ik\cdot x}$ and

$$\hat{a}_{k,\sigma} = a(\delta_\sigma f_k) = \frac{1}{L^{3/2}} \int_\Lambda dx\, a_{x,\sigma} e^{-ikx}, \tag{2.2}$$

where we introduced the operator-valued distributions $a_{x,\sigma} = a(\delta_{x,\sigma})$ with $\delta_{x,\sigma}(y,\sigma') = \delta_{\sigma,\sigma'}\delta(x-y)$. Furthermore, we denote by $\mathcal{N}_\sigma$, $\sigma \in \{\uparrow,\downarrow\}$, the number operators counting the number of particles of spin σ, which can be written as

$$\mathcal{N}_\sigma = \sum_{k\in\Lambda^*} \hat{a}^*_{k,\sigma}\hat{a}_{k,\sigma} = \int_\Lambda dx\, a^*_{x,\sigma}a_{x,\sigma},$$

and the total number operator $\mathcal{N} = \sum_{\sigma\in\{\uparrow,\downarrow\}}\mathcal{N}_\sigma$.

The Hamiltonian H_N in (1.1) can be extended to the operator on Fock space

$$\mathcal{H} = \sum_{\sigma\in\{\uparrow,\downarrow\}}\int_\Lambda dx\, \nabla_x a^*_{x,\sigma}\nabla_x a_{x,\sigma} + \frac{1}{2}\sum_{\sigma,\sigma'\in\{\uparrow,\downarrow\}}\int_{\Lambda\times\Lambda} dxdy\, V(x-y)a^*_{x,\sigma}a^*_{y,\sigma'}a_{y,\sigma'}a_{x,\sigma}.$$

That is, H_N is the restriction of $\mathcal{H}$ to $\mathcal{F}_{\mathrm{f}}^{(N)} \subset \mathcal{F}_{\mathrm{f}}$. Equivalently, we can write

$$\mathcal{H} = \sum_{\sigma\in\{\uparrow,\downarrow\}}\sum_{k\in\Lambda^*}|k|^2\hat{a}^*_{k,\sigma}\hat{a}_{k,\sigma} + \frac{1}{2L^3}\sum_{\sigma,\sigma'\in\{\uparrow,\downarrow\}}\sum_{k,p,q\in\Lambda^*}\hat{V}(k)\hat{a}^*_{p+k,\sigma}\hat{a}^*_{q-k,\sigma'}\hat{a}_{q,\sigma'}\hat{a}_{p,\sigma},$$

where $\hat{V}$ are the Fourier coefficients of V. Here and in the following, we use the convention

$$\hat{f}(p) = \int_\Lambda dx\, f(x)e^{-ip\cdot x}, \qquad f(x) = \frac{1}{L^3}\sum_{p\in\Lambda^*}\hat{f}(p)e^{ip\cdot x},$$

for the Fourier transform in a box Λ. (The Fourier transform in $\mathbb{R}^3$ will instead be denoted by $\mathcal{F}$ in the following.)

Finally, we can identity $\mathfrak{h}(N_\uparrow,N_\downarrow)$ with the subspace $\mathcal{F}_{\mathrm{f}}(N_\uparrow,N_\downarrow) \subset \mathcal{F}_{\mathrm{f}}$ where $\mathcal{N}_\sigma = N_\sigma$ for $\sigma\in\{\uparrow,\downarrow\}$ (which is left invariant by $\mathcal{H}$), and write the ground state energy in (1.2) as

$$E_L(N_\uparrow,N_\downarrow) = \inf_{\psi\in\mathcal{F}_{\mathrm{f}}(N_\uparrow,N_\downarrow)}\frac{\langle\psi,\mathcal{H}\psi\rangle}{\langle\psi,\psi\rangle}.$$

2.2 | The Particle–Hole Transformation

For a non-interacting system, the ground state is given by the free Fermi gas state ψ_{FFG}, which is simply the Slater determinant of the plane waves with momentum inside the Fermi sphere(s), namely

$$\psi_{\mathrm{FFG}} = \prod_{\sigma\in\{\uparrow,\downarrow\}}\prod_{k\in\mathcal{B}_F^\sigma}\hat{a}^*_{k,\sigma}\Omega,$$

where $\mathcal{B}_F^\sigma := \left\{k\in(2\pi/L)\mathbb{Z}^3 \;\middle|\; |k|\le k_F^\sigma\right\}$ is the Fermi ball of spin σ, with $k_F^\sigma = (6\pi^2)^{\frac{1}{3}}\varrho_\sigma^{\frac{1}{3}} + o(1)_{L\to\infty}$.

$$\lim_{L\to\infty}\frac{E_{\mathrm{FFG}}}{L^3} = \frac{3}{5}(6\pi^2)^{\frac{2}{3}}\left(\varrho_\uparrow^{\frac{5}{3}} + \varrho_\downarrow^{\frac{5}{3}}\right) + \hat{V}(0)\varrho_\uparrow\varrho_\downarrow + O(\varrho^{\frac{8}{3}}), \tag{2.3}$$

for small ϱ, where the shorthand notation $\lim_{L\to\infty}$ stands for the thermodynamic limit.

We will denote by υ the one-particle reduced density matrix of ψ_{FFG}, which is an orthogonal projection with integral kernel given by

$$\upsilon_{\sigma,\sigma'}(x;y) = \frac{\delta_{\sigma,\sigma'}}{L^3}\sum_{k\in\mathcal{B}_F^\sigma}e^{ik\cdot(x-y)}. \tag{2.4}$$

Note that $v_{\sigma,\sigma'}(x;y)$ is real-valued. We shall also define $u = 1 - v$, with integral kernel

$$u_{\sigma,\sigma'}(x;y) = \frac{\delta_{\sigma,\sigma'}}{L^3} \sum_{k \notin \mathcal{B}_F^\sigma} e^{ik\cdot(x-y)}, \tag{2.5}$$

satisfying $uv = vu = 0$. We will write $u_\sigma = u_{\sigma,\sigma}$ and $v_\sigma = v_{\sigma,\sigma}$ for simplicity. Furthermore, we define $u_{x,\sigma}(\cdot) := u_\sigma(\cdot\,;x)$ and $v_{x,\sigma}(\cdot) := v_\sigma(\cdot\,;x)$.

As in [16, 17] we factor out E_{FFG} by a unitary particle–hole transformation.

Definition 2.1 (Particle–hole transformation). Let $u, v : L^2(\Lambda;\mathbb{C}^2) \to L^2(\Lambda;\mathbb{C}^2)$ be given in (2.4)–(2.5). The particle–hole transformation $R : \mathcal{F}_{\mathrm{f}} \to \mathcal{F}_{\mathrm{f}}$ is a unitary operator such that the following properties hold:

i. The state $R\Omega$ is such that $(R\Omega)^{(n)} = 0$ whenever $n \neq N$ and $(R\Omega)^{(N)} = \Psi_{\mathrm{FFG}}$.

ii.

$$R^* a^*_{x,\sigma} R = a^*_\sigma(u_x) + a_\sigma(v_x), \tag{2.6}$$

where

$$a^*_\sigma(u_x) = \int_\Lambda dy\, u_\sigma(y;x) a^*_{y,\sigma}, \qquad a_\sigma(v_x) = \int_\Lambda dy\, v_\sigma(y;x) a_{y,\sigma}. \tag{2.7}$$

Remark 2.2. As explained in [42], the particle–hole transformation R is a particular example of a fermionic Bogoliubov transformation, which allows to focus on the excitations around the free Fermi gas. Equivalently, the transformation R can be defined by

$$R^* \hat{a}_{k,\sigma} R = \begin{cases} \hat{a}_{k,\sigma} & \text{if } k \notin \mathcal{B}_F^\sigma, \\ \hat{a}^*_{-k,\sigma} & \text{if } k \in \mathcal{B}_F^\sigma, \end{cases} \tag{2.8}$$

which justifies the name *particle–hole transformation*.

Note that we define the particle–hole transformation slightly differently from the one used in the previous work [16, 17], in order to preserve translation-invariance. Since $R\Omega = \psi_{\mathrm{FFG}}$ we have $E_{\mathrm{FFG}} = \langle R\Omega, \mathcal{H} R\Omega\rangle$. With the aid of (2.6) we can compute $R^*\mathcal{H}R$ explicitly. Proceeding as in [16] (using (2.6) and normal-ordering all the terms), we obtain

Proposition 2.3 (Conjugation of $\mathcal{H}$ by R). *Let V_∞ be as in Assumption 1.1. Let $\psi \in \mathcal{F}_{\mathrm{f}}$ be a normalized state satisfying $\mathcal{N}_\sigma \psi = N_\sigma \psi$ for $\sigma \in \{\uparrow,\downarrow\}$. Then*

$$\langle \psi, \mathcal{H}\psi\rangle = E_{\mathrm{FFG}} + \langle R^*\psi, \mathcal{H}_{\mathrm{corr}} R^*\psi\rangle, \tag{2.9}$$

where E_{FFG} is the energy of the free Fermi gas introduced above (2.3) and $\mathcal{H}_{\mathrm{corr}} = \mathbb{H}_0 + \mathbb{X} + \sum_{i=1}^4 \mathbb{Q}_i$ is the correlation Hamiltonian given by

$$\mathbb{H}_0 = \sum_\sigma \sum_k \big||k|^2 - (k_F^\sigma)^2\big| \hat{a}^*_{k,\sigma}\hat{a}_{k,\sigma},$$

$$\mathbb{X} = \sum_\sigma \int_{\Lambda^2} dxdy\, V(x-y) v_\sigma(x-y)\big(a^*_\sigma(u_x)a_\sigma(u_y) - a^*_\sigma(v_y)a_\sigma(v_x)\big),$$

$$\mathbb{Q}_1 = \sum_{\sigma,\sigma'} \int_{\Lambda^2} dxdy\, V(x-y) a^*_\sigma(u_x) a^*_\sigma(v_x) a_{\sigma'}(v_y) a_{\sigma'}(u_y)$$

$$+\frac{1}{2}\sum_{\sigma,\sigma'}\int_{\Lambda^2} dxdy\, V(x-y)\Big(a^*_\sigma(v_x)a^*_{\sigma'}(v_y)a_{\sigma'}(v_y)a_\sigma(v_x) - 2a^*_\sigma(u_x)a^*_{\sigma'}(v_y)a_{\sigma'}(v_y)a_\sigma(u_x)\Big),$$

$$\mathbb{Q}_2 = \frac{1}{2}\sum_{\sigma,\sigma'}\int_{\Lambda^2} dxdy\, V(x-y)a^*_\sigma(u_x)a^*_{\sigma'}(u_y)a^*_{\sigma'}(v_y)a^*_\sigma(v_x) + \text{h.c.},$$

$$\mathbb{Q}_3 = \sum_{\sigma,\sigma'}\int_{\Lambda^2} dxdy\, V(x-y)\Big(a^*_\sigma(u_x)a^*_{\sigma'}(v_y)a^*_\sigma(v_x)a_{\sigma'}(v_y) - a^*_\sigma(u_x)a^*_{\sigma'}(u_y)a^*_\sigma(v_x)a_{\sigma'}(u_y)\Big) + \text{h.c.},$$

$$\mathbb{Q}_4 = \frac{1}{2}\sum_{\sigma,\sigma'}\int_{\Lambda^2} dxdy\, V(x-y)a^*_\sigma(u_x)a^*_{\sigma'}(u_y)a_{\sigma'}(u_y)a_\sigma(u_x). \tag{2.10}$$

In the following, it will be convenient to further decompose $\mathbb{Q}_2 = \mathbb{Q}_2^{\parallel} + \mathbb{Q}_2^{\uparrow\downarrow}$ into a part $\mathbb{Q}_2^{\parallel}$ involving interactions of particles with the same spin, and $\mathbb{Q}_2^{\uparrow\downarrow}$ involving interactions of particles of opposite spin, that is,

$$\mathbb{Q}_2^{\parallel} = \frac{1}{2}\sum_{\sigma}\int_{\Lambda^2} dxdy\, V(x-y)a^*_\sigma(u_x)a^*_\sigma(u_y)a^*_\sigma(v_y)a^*_\sigma(v_x) + \text{h.c.}$$

$$\mathbb{Q}_2^{\uparrow\downarrow} = \frac{1}{2}\sum_{\sigma\neq\sigma'}\int_{\Lambda^2} dxdy\, V(x-y)a^*_\sigma(u_x)a^*_{\sigma'}(u_y)a^*_{\sigma'}(v_y)a^*_\sigma(v_x) + \text{h.c.} \tag{2.11}$$

For our trial state, the main contribution to the energy will come from the effective correlation Hamiltonian

$$\mathcal{H}^{\text{eff}}_{\text{corr}} := \mathbb{H}_0 + \mathbb{Q}_2^{\uparrow\downarrow} + \mathbb{Q}_4. \tag{2.12}$$

(see Section 3.2 for more details).

2.3 | Trial State and Quasi-Bosonic Bogoliubov Transformations

The trial state we are going to use in order to prove Theorem 1.2 is of the form

$$\psi_{\text{trial}} := RT_1T_2\Omega, \tag{2.13}$$

where R is the particle–hole transformation defined in Definition 2.1, $\Omega \in \mathcal{F}_{\text{f}}$ is the vacuum state, and T_1, T_2 are certain unitary transformations that will be defined below.

The main idea of our approach is to describe the low energy excitations around the Fermi ball by pairs of fermionic particles that display an approximate bosonic behavior. To do that we introduce the quasi-bosonic annihilation operators

$$b_{p,k,\sigma} = \hat{u}_\sigma(p+k)\hat{v}_\sigma(k)\hat{a}_{p+k,\sigma}\hat{a}_{-k,\sigma} \qquad \text{and} \qquad b_{p,\sigma} = \sum_{k\in\Lambda^*} b_{p,k,\sigma}, \tag{2.14}$$

and their adjoints, the corresponding creation operators, where $\hat{u}_\sigma, \hat{v}_\sigma$ are the Fourier coefficients of the kernels introduced in (2.5). In particular,

$$\hat{u}_\sigma(k) = \begin{cases} 0 & \text{if } |k| \leq k_F^\sigma, \\ 1 & \text{if } |k| > k_F^\sigma, \end{cases} \qquad \hat{v}_\sigma(k) = \begin{cases} 1 & \text{if } |k| \leq k_F^\sigma, \\ 0 & \text{if } |k| > k_F^\sigma. \end{cases} \tag{2.15}$$

Remark 2.4. Unlike in [16, 17], we use here the sharp projections $\hat{u}, \hat{v}$ (instead of regularized ones), which will help to simplify and improve many error terms.

The unitary transformations T_1 and T_2 are defined as exponentials of expressions quadratic in the b, b^* operators. The two transformations correspond to two different regimes of the momentum p of the bosonic quasi-particle: for high-momenta (with respect to k_F^σ) we use a "less refined" expression which helps to extract the leading term $8\pi a\varrho_\uparrow\varrho_\downarrow$ of the interaction energy, and at the same time renormalizes the interaction potential, while for low momenta we use a "more refined" expression which allows us to extract the correct energy of order $\varrho^{7/3}$. In order to introduce the momentum splitting, we choose two smooth functions $\mathbb{R}^3 \to \mathbb{R}$, denoted by $\mathcal{F}(\chi_{<,\infty})$ and $\mathcal{F}(\chi_{>,\infty})$, respectively, satisfying

$$\mathcal{F}(\chi_{<,\infty}) + \mathcal{F}(\chi_{>,\infty}) = 1, \quad \mathcal{F}(\chi_{<,\infty})(p) = \begin{cases} 1 & \text{if } |p| < 4\varrho^{\frac{1}{3}-\gamma}, \\ 0 & \text{if } |p| > 5\varrho^{\frac{1}{3}-\gamma}. \end{cases} \tag{2.16}$$

Here $0 < \gamma < 1/3$ is a parameter which will be optimized over at the end; we shall in fact choose $\gamma = 1/9$. From $\mathcal{F}(\chi_{<,\infty})$ we can construct the periodic function $\chi_< : \Lambda \to \mathbb{C}$ using the Fourier coefficients $\{\mathcal{F}(\chi_{<,\infty}(p)\}_{p\in\Lambda^*}$, which satisfies the uniform bound $\|\chi_<\|_{L^1(\Lambda)} \le C$ (see Lemma A.5). In the following we will also use the notation $\hat{\chi}_>(p) = 1 - \hat{\chi}_<(p)$.

The operators T_1 and T_2 will implement the desired correlation structure between particles by using two different expressions both related to the zero-energy scattering Equation (1.5).

Definition 2.5 (Scattering solutions). We define the periodic function $\varphi : \Lambda \to \mathbb{C}$ by $\hat{\varphi}(p) = \mathcal{F}(\varphi_\infty)(p)$ for $0 \neq p \in \Lambda^*$, $\hat{\varphi}(0) = 0$, where $\varphi_\infty : \mathbb{R}^3 \to \mathbb{R}$ is the solution of the zero-scattering equation (1.5). Moreover, for $\varepsilon = \varrho^{2/3+\delta}$ with a parameter $\delta > 0$, we define

$$\hat{\eta}^\varepsilon_{r,r'}(p) := \frac{8\pi a}{\lambda_{r,p} + \lambda_{r',-p} + 2\varepsilon}, \quad \lambda_{r,p} := |r+p|^2 - |r|^2, \tag{2.17}$$

where a denotes the scattering length of V_∞.

Remark 2.6. We will see in Appendix A that φ is well defined. In fact $\|\varphi\|_{L^\infty(\Lambda)} \le C$ uniformly in L but $\|\varphi\|_{L^2(\Lambda)}$ diverges as $L \to \infty$ (see Remark A.3).

Definition 2.7 (Quasi-bosonic transformations). For a parameter $\lambda \in \mathbb{R}$, we define

$$T_{1;\lambda} = \exp\left(\lambda(B_1 - B_1^*)\right), \quad B_1 = \frac{1}{L^3}\sum_{p\in\Lambda^*} \hat{\varphi}(p)\hat{\chi}_>(p)\, b_{p,\uparrow} b_{-p,\downarrow}, \tag{2.18}$$

$$T_{2;\lambda} = \exp\left(\lambda(B_2 - B_2^*)\right), \quad B_2 = \frac{1}{L^3}\sum_{p,r,r'\in\Lambda^*} \hat{\eta}^\varepsilon_{r,r'}(p)\hat{\chi}_<(p) b_{p,r,\uparrow} b_{-p,r',\downarrow}, \tag{2.19}$$

where $\hat{\varphi}$ and $\hat{\eta}^\varepsilon_{r,r'}$ are given in Definition 2.5. We also write T_1 and T_2 in place of $T_{1;1}$ and $T_{2;1}$, respectively.

The cut-off $\hat{\chi}_>(p)$ helps to regularize B_1 since the function $\tilde{\varphi} : \Lambda \to \mathbb{R}$ with $\hat{\tilde{\varphi}}(p) = \hat{\varphi}(p)\hat{\chi}_>(p)$ is uniformly bounded in $L^1(\Lambda) \cap L^\infty(\Lambda)$ (see Lemma A.2) as $L \to \infty$. In the expression for B_2 in (2.19), we only use $\hat{\eta}^\varepsilon_{r,r'}(p)$ for momenta $r \in \mathcal{B}_F^\uparrow$, $r' \in \mathcal{B}_F^\downarrow$, $r + p \notin \mathcal{B}_F^\uparrow$, $r' - p \notin \mathcal{B}_F^\downarrow$, where $\lambda_{r,p} = |r+p|^2 - |r|^2 > 0$ and $\lambda_{r',-p} = |r'-p|^2 - |r'|^2 > 0$. The parameter $\varepsilon = \varrho^{2/3+\delta}$ in the denominator of $\hat{\eta}^\varepsilon_{r,r'}(p)$ is introduced for technical reasons to avoid logarithmic divergences otherwise encountered in integrals. At the end we shall choose δ to satisfy $16/63 \le \delta \le 8/9$.

Remark 2.8 (Relation to the Bethe–Goldstone equation). From our analysis, it may seem more natural to consider a trial state of the form $\tilde{\psi}_{\text{trial}} := RT\Omega$, with the unitary T given by

$$T = \exp\left(\frac{1}{L^3}\sum_{p,r,r'} \hat{\varphi}_{r,r'}(p) b_{p,r,\uparrow} b_{-p,r',\downarrow} - \text{h.c.}\right), \tag{2.20}$$

where $\hat{\varphi}_{r,r'}$ is chosen to satisfy the equation

$$2(|p|^2 + p\cdot(r - r'))\hat{\varphi}_{r,r'}(p) = \mathcal{F}(V_\infty(1 - \varphi_{r,r'}))(p), \tag{2.21}$$

(in the infinity volume limit). Note that (2.21) reduces to (1.5) for $r = r' = 0$. Equation (2.21) can be interpreted as describing the scattering (at zero energy) of two particles at relative momentum p, with initial momenta r, r' within the Fermi ball that is fully occupied. From the definition of $b_{p,r,\uparrow}$ and $b_{-p,r',\downarrow}$, the function $\varphi_{r,r'}(p)$ appearing in (2.20) is naturally supported on the set where $|r+p| \geq k_F^{\uparrow}, |r'-p| \geq k_F^{\downarrow}$. Note that the effective kinetic energy

$$2|p|^2 + 2p \cdot (r - r') = \lambda_{r,p} + \lambda_{r',-p},$$

describes the difference between the kinetic energy of the two particles with initial momenta r, r' and excited momenta $r+p, r'-p$, and is positive. Equation (2.21) plays a prominent role in the physics literature. With the help of $G_{r,r'}(p) := (\lambda_{r,p} + \lambda_{r',-p})\hat{\varphi}_{r,r'}(p)$, it can be rewritten in the form

$$G_{r,r'}(p) = \mathcal{F}(V_\infty)(p) - \int_{|r+q|\geq k_F^{\uparrow}, |r'-q|\geq k_F^{\downarrow}} dq\, \frac{\mathcal{F}(V_\infty)(p-q)}{\lambda_{r,q} + \lambda_{r',-q}} G_{r,r'}(q), \tag{2.22}$$

which is known as the *Bethe–Goldstone* Equation, see [38, Equation 2.2].

For technical reasons, we find it simpler not to work with $\widetilde{\psi}_{\text{trial}} = RT\Omega$, but rather split the unitary into two parts, T_1 and T_2. Note that the $\hat{\varphi}$ in the definition of T_1 depends only on the momentum p; in this sense T_1 is "less refined" but it is sufficient to renormalize the interaction in $\mathbb{Q}_2^{\uparrow\downarrow}$ into a softer one with integral given by $8\pi a$. To be precise, the transformation T_1 allows us to replace the original potential V in $\mathbb{Q}_2^{\uparrow\downarrow}$ by the softer potential $8\pi a\chi_<$, which is supported in small momenta ($|p| \leq 5\varrho^{1/3-\gamma} \ll 1$), and consequently the scattering length of $8\pi a\chi_<$ is well approximated by $(8\pi)^{-1}$ times its integral. On the other hand, the transformation T_2 is "more refined" since the $\hat{\eta}^{\varepsilon}_{r,r'}$ in T_2 depends on p, r, r', but it is also simpler than T since the function $\hat{\eta}^{\varepsilon}_{r,r'}$ is explicit.

Note that on the support of $\chi_>(p)$, the term $p \cdot (r - r')$ is subleading compared to p^2, hence it is natural to replace $\varphi_{r,r'}$ by the usual scattering solution φ in T_1. On the other hand, on the support of $\chi_<$ and for $r \in \mathcal{B}_F^{\uparrow}, r' \in \mathcal{B}_F^{\downarrow}$, $\mathcal{F}(V_\infty(1 - \varphi_{r,r'}))(p)$ on the right-hand side of (2.21) can be replaced by $8\pi a$ to leading order, naturally reducing $\hat{\varphi}_{r,r'}$ to $\hat{\eta}^{\varepsilon}_{r,r'}$ in this regime.

Remark 2.9 (Configuration space representation). In the calculation below it will be convenient to estimate the error terms in configuration space. For this reason, we will often write

$$B_1 = \int dz dz'\, \widetilde{\varphi}(z - z') a_\uparrow(u_z) a_\uparrow(v_z) a_\downarrow(u_{z'}) a_\downarrow(v_{z'}), \tag{2.23}$$

where

$$\widehat{\widetilde{\varphi}}(p) = \mathcal{F}(\varphi_\infty)(p)\widehat{\chi}_>(p). \tag{2.24}$$

Useful bounds on the function $\widetilde{\varphi}$ will be given in Lemma A.2.

Writing the unitary T_2 in configuration space is a bit more complicated, since $\hat{\eta}^{\varepsilon}_{r,r'}(p)$ depends on all the momenta involved in the definition of the transformation. First of all we notice that in the definition of B_2 in (2.19), the coefficients $\hat{u}_\uparrow(r+p)$ and $\hat{u}_\downarrow(r'-p)$ appearing in (2.14) can be replaced by $\hat{u}^<_\uparrow(r+p)$ and $\hat{u}^<_\downarrow(r'-p)$ with

$$\hat{u}^<_\sigma(k) := \begin{cases} 1 & \text{if } k_F^\sigma < |k| \leq 6\varrho^{\frac{1}{3}-\gamma}, \\ 0 & \text{if } |k| < k_F^\sigma, \text{ or } |k| > 6\varrho^{\frac{1}{3}-\gamma}, \end{cases} \tag{2.25}$$

since $|p| \leq 5\varrho^{1/3-\gamma}$ and both r and r' are bounded by (a constant times) $\varrho^{1/3}$, hence $|r+p|, |r'-p| < 6\varrho^{1/3-\gamma}$ for ϱ small. Moreover, since both $\lambda_{r,p}$ and $\lambda_{r',-p}$ appearing in the dominator in the definition of $\hat{\eta}^{\varepsilon}_{r,r'}$ in (2.17) are positive, we can write

$$\hat{\eta}^{\varepsilon}_{r,r'}(p) = 8\pi a \int_0^\infty dt\, e^{-(|r+p|^2 - |r|^2 + |r'-p|^2 - |r'|^2 + 2\varepsilon)t},$$

for $r+p \notin \mathcal{B}_F^\uparrow$, $r'-p \notin \mathcal{B}_F^\downarrow$, $r \in \mathcal{B}_F^\uparrow$ and $r' \in \mathcal{B}_F^\downarrow$. Therefore, introducing the functions

$$\hat{v}_\sigma^t(\,\cdot\,) := e^{t|\cdot|^2}\hat{v}_\sigma(\,\cdot\,), \qquad \hat{u}_\sigma^t(\,\cdot\,) := e^{-t|\cdot|^2}\hat{u}_\sigma^<(\,\cdot\,), \tag{2.26}$$

with $\hat{u}_\sigma^<$ and $\hat{v}_\sigma$ defined in (2.25) and (2.4), respectively, we can write B_2 in configuration space as

$$B_2 = 8\pi a \int_0^\infty dt\, e^{-2t\varepsilon} \int dz dz'\, \chi_<(z-z') a_\uparrow(u_z^t) a_\uparrow(v_z^t) a_\downarrow(u_{z'}^t) a_\downarrow(v_{z'}^t). \tag{2.27}$$

A big advantage of choosing the explicit function (2.17) (as opposed to the solution of (2.21)) is to allow for this representation in configuration space.

Remark 2.10. The transformation T_1 is related to the unitary transformation used in [17, Definition 4.3] (see also Remark 4.4 and Remark 4.5 in [17]). In [17, Definition 4.3], T_1 contains a cut-off around $|x| \simeq \varrho^{-1/3}$ in configuration space, which is essentially equivalent to a cut-off around $|p| \simeq \varrho^{1/3}$ in momentum space. In the present article we choose a weaker localization with a momentum cut-off around $|p| \simeq \varrho^{1/3-\gamma}$ in order to obtain better error estimates of order $o(\varrho^{7/3})$.

We conclude this section by showing that the trial state introduced in (2.13) is admissible, that is, is a state with $N_\uparrow$ particles with spin $\uparrow$ and $N_\downarrow$ particles with spin $\downarrow$. To see this, note that $R\Omega$ is clearly admissible, and RB_jR commutes with $\mathcal{N}_\sigma$ for $j \in \{1,2\}$ and $\sigma \in \{\uparrow,\downarrow\}$, which can be deduced from the definitions in a straightforward way. Hence

$$\mathcal{N}_\sigma \psi_{\text{trial}} = \mathcal{N}_\sigma R T_1 T_2 \Omega = R T_1 T_2 R \mathcal{N}_\sigma R\Omega = N_\sigma \psi_{\text{trial}}.$$

2.4 | Key Estimates

Using the trial state (2.13) and Proposition 2.3, we obtain as an upper bound to the ground state energy

$$E_L(N_\uparrow, N_\downarrow) \le E_{\text{FFG}} + \langle \Omega, T_2^* T_1^* \mathcal{H}_{\text{corr}} T_1 T_2 \Omega\rangle \simeq E_{\text{FFG}} + \langle \Omega, T_2^* T_1^* \mathcal{H}_{\text{corr}}^{\text{eff}} T_1 T_2 \Omega\rangle,$$

where $\mathcal{H}_{\text{corr}}^{\text{eff}} = \mathbb{H}_0 + \mathbb{Q}_2^{\uparrow\downarrow} + \mathbb{Q}_4$ was defined in (2.12). We shall now explain some key estimates in the computation of $\langle \Omega, T_2^* T_1^* \mathcal{H}_{\text{corr}}^{\text{eff}} T_1 T_2 \Omega\rangle$. In the first step, the unitary operator T_1 introduced in (2.18) is responsible for extracting the leading order of the interaction energy given by $8\pi a \varrho_\uparrow \varrho_\downarrow$. Moreover, via T_1 we also renormalize the interaction, effectively replacing $\mathbb{Q}_2^{\uparrow\downarrow} + \mathbb{Q}_4$ by

$$\mathbb{Q}_{2;<} = \frac{8\pi a}{L^3} \sum_p \hat{\chi}_<(p) b_{p,\uparrow} b_{-p,\downarrow} + \text{h.c.}. \tag{2.28}$$

At this point the correlation operator to leading order reduces to $\mathbb{H}_0 + \mathbb{Q}_{2;<}$. To extract the next order correction to the correlation energy we then conjugate $\mathbb{H}_0 + \mathbb{Q}_{2;<}$ by the unitary operator T_2 in the second step.

Step 1: We conjugate the effective correlation operator with the unitary T_1 by writing

$$\begin{aligned} T_1^* \mathcal{H}_{\text{corr}}^{\text{eff}} T_1 &= \mathcal{H}_{\text{corr}}^{\text{eff}} + \int_0^1 d\lambda\, \partial_\lambda\, T_{1;\lambda}^* \mathcal{H}_{\text{corr}}^{\text{eff}} T_{1;\lambda} \\ &= \mathcal{H}_{\text{corr}}^{\text{eff}} + \int_0^1 d\lambda\, T_{1;\lambda}^* [\mathcal{H}_{\text{corr}}^{\text{eff}}, B_1 - B_1^*] T_{1;\lambda}, \end{aligned}$$

recalling the notation $T_{1;\lambda} = \exp(\lambda(B_1 - B_1^*))$ introduced in (2.18). To leading order, we shall see that

$$[\mathbb{H}_0 + \mathbb{Q}_4, B_1 - B_1^*] \simeq -\frac{1}{L^3} \sum_p \left(2|p|^2 \hat{\varphi}(p) \hat{\chi}_>(p) + \hat{V} *_\Sigma (\hat{\varphi}\hat{\chi}_>)(p)\right) b_{p,\uparrow} b_{-p,\downarrow} + \text{h.c.},$$

where $*_\sum$ denotes the "discrete convolution" for functions defined on momentum space, namely

$$(\hat{f} *_\sum \hat{g})(p) = \frac{1}{L^3} \sum_{q\in\Lambda^*} \hat{f}(p-q)\hat{g}(q) = \widehat{(fg)}(p).$$

This explains the choice of the function $\hat{\varphi}\hat{\chi}_>$ appearing in B_1: From the zero-energy scattering equation (1.5) we deduce that $\hat{V} *_\sum \hat{\varphi}(p) \simeq \hat{V}(p) - 2|p|^2\hat{\varphi}(p)$. The operator $\mathbb{Q}_2^{\uparrow\downarrow}$ in (2.10) can equivalently be written as

$$\mathbb{Q}_2^{\uparrow\downarrow} = \frac{1}{L^3} \sum_p \hat{V}(p) b_{p,\uparrow} b_{-p,\downarrow} + \text{h.c.}$$

Since $2|p|^2\hat{\varphi}(p) \simeq 8\pi a$ for small p, we conclude that

$$\mathbb{Q}_2^{\uparrow\downarrow} + [\mathbb{H}_0 + \mathbb{Q}_4, B_1 - B_1^*] \simeq \mathbb{Q}_{2;<} + \widetilde{\mathbb{Q}}_{2;<}, \tag{2.29}$$

where $\mathbb{Q}_{2;<}$ is defined in (2.28) and

$$\widetilde{\mathbb{Q}}_{2;<} = \frac{1}{L^3} \sum_p \hat{V} *_\sum (\hat{\varphi}\hat{\chi}_<)(p) b_{p,\uparrow} b_{-p,\downarrow} + \text{h.c.}. \tag{2.30}$$

We emphasize that $\mathbb{Q}_{2;<}$ is a renormalized version of $\mathbb{Q}_2^{\uparrow\downarrow}$, where after taking the 2-body scattering process into account, we have the softer interaction potential $8\pi a\hat{\chi}_<$ instead of $\hat{V}$. The operator $\widetilde{\mathbb{Q}}_{2;<}$ will be important only in the next order of the Duhamel expansion, where it leads via a commutator to a term relevant for the correct Huang–Yang correction of order $\varrho^{7/3}$, as will be demonstrated below. In fact, applying (2.29) we can extend the above Duhamel expansion as

$$\begin{aligned} T_1^*(\mathbb{H}_0 + \mathbb{Q}_4 + \mathbb{Q}_2^{\uparrow\downarrow})T_1 &= \mathbb{H}_0 + \mathbb{Q}_4 + \mathbb{Q}_2^{\uparrow\downarrow} + [\mathbb{H}_0 + \mathbb{Q}_4, B_1 - B_1^*] \\ &+ \int_0^1 d\lambda\, (1-\lambda) T_{1;\lambda}^* [[\mathbb{H}_0 + \mathbb{Q}_4, B_1 - B_1^*], B_1 - B_1^*] T_{1;\lambda} + \int_0^1 d\lambda\, T_{1;\lambda}^* [\mathbb{Q}_2^{\uparrow\downarrow}, B_1 - B_1^*] T_{1;\lambda} \\ &\simeq \mathbb{H}_0 + \mathbb{Q}_4 + \mathbb{Q}_{2;<} + \widetilde{\mathbb{Q}}_{2;<} + \int_0^1 d\lambda\, T_{1;\lambda}^* [\lambda \mathbb{Q}_2^{\uparrow\downarrow} + (1-\lambda)\big(\mathbb{Q}_{2;<} + \widetilde{\mathbb{Q}}_{2;<}\big), B_1 - B_1^*] T_{1;\lambda}. \end{aligned} \tag{2.31}$$

Computing the commutator in the last integrand and normal-ordering the expression leads to constant term, which is the leading contribution. The integration over λ then just gives a factor $1/2$. We shall see that

$$\frac{1}{2}[\mathbb{Q}_2^{\uparrow\downarrow} + \mathbb{Q}_{2;<} + \widetilde{\mathbb{Q}}_{2;<}, B_1 - B_1^*] \simeq -\varrho_\uparrow\varrho_\downarrow \sum_{p\in\Lambda^*} \hat{V}(p)\hat{\varphi}(p) + (8\pi a)^2 \varrho_\uparrow\varrho_\downarrow \sum_{0\neq p\in\Lambda^*} \frac{\hat{\chi}_<(p)}{2|p|^2}.$$

The first term on the right-hand side naturally combines with $\hat{V}(0)\varrho_\uparrow\varrho_\downarrow$ from (2.3) to yield the scattering length, since

$$\hat{V}(0) - \frac{1}{L^3} \sum_{p\in\Lambda^*} \hat{V}(p)\hat{\varphi}(p) = (\hat{V} - \hat{V} *_\sum \hat{\varphi})(0) \simeq \int_{\mathbb{R}^3} V_\infty (1 - \varphi_\infty) = 8\pi a.$$

Therefore, we obtain the upper bound

$$\begin{aligned} \frac{E_L(N_\uparrow, N_\downarrow)}{L^3} &\leq \frac{3}{5}(6\pi^2)^{\frac{2}{3}} \left(\varrho_\uparrow^{\frac{5}{3}} + \varrho_\downarrow^{\frac{5}{3}} \right) + 8\pi a \varrho_\uparrow\varrho_\downarrow + \frac{(8\pi a)^2 \varrho_\uparrow\varrho_\downarrow}{L^3} \sum_{0\neq p\in\Lambda^*} \frac{(\hat{\chi}_<(p))^2}{2|p|^2} \\ &+ \frac{1}{L^3}\langle T_2\Omega, (\mathbb{H}_0 + \mathbb{Q}_4 + \mathbb{Q}_{2;<}) T_2\Omega\rangle + o(\varrho^{\frac{7}{3}}). \end{aligned} \tag{2.32}$$

The rigorous justification of this statement is the of Proposition 4.1.

Step 2: To complete the proof, we still have to compute the last term in (2.32). We will see that $\mathbb{Q}_4$ is negligible, that is, that $L^{-3}\langle T_2\Omega, \mathbb{Q}_4 T_2\Omega\rangle = o(\varrho^{7/3})$ (see Proposition 5.3). It therefore remains to conjugate the final effective correlation

Hamiltonian $\mathbb{H}_0 + \mathbb{Q}_{2;<}$ under T_2. As above we can write

$$T_2^*(\mathbb{H}_0 + \mathbb{Q}_{2;<})T_2 = \mathbb{H}_0 + T_2^*\mathbb{Q}_{2;<}T_2 + \int_0^1 d\lambda\, T_{2;\lambda}^*[\mathbb{H}_0, B_2 - B_2^*]T_{2;\lambda}, \tag{2.33}$$

where $T_{2;\lambda} = \exp(\lambda(B_2 - B_2^*))$. From the definition of the unitary operator T_2 (see (2.19) and (2.17)), we find that

$$[\mathbb{H}_0, B_2 - B_2^*] \simeq -\mathbb{Q}_{2;<}. \tag{2.34}$$

As a consequence, inserting (2.34) in (2.33) we obtain

$$T_2^*(\mathbb{H}_0 + \mathbb{Q}_{2;<})T_2 \simeq \mathbb{H}_0 + T_2^*\mathbb{Q}_{2;<}T_2 - \int_0^1 d\lambda\, T_{2;\lambda}^*\mathbb{Q}_{2;<}T_{2;\lambda} = \mathbb{H}_0 + \int_0^1 \lambda d\lambda\ T_{2;\lambda}^*[\mathbb{Q}_{2;<}, B_2 - B_2^*]T_{2;\lambda}.$$

Computing the commutator and normal-ordering leads to a constant contribution in the last expression, which is the leading term and given by

$$\frac{1}{2}[\mathbb{Q}_{2;<}, B_2 - B_2^*] \simeq -\frac{(8\pi a)^2}{L^6}\sum_{p,r,r'\in\Lambda^*}\frac{\hat{u}_\uparrow(r+p)\hat{u}_\downarrow(r'-p)\hat{v}_\uparrow(r)\hat{v}_\downarrow(r')}{|r+p|^2 - |r|^2 + |r'-p|^2 - |r'|^2 + 2\varepsilon}(\hat{\chi}_<(p))^2, \tag{2.35}$$

where $\hat{u}_\sigma$ and $\hat{v}_\sigma$ are given in (2.15). The rigorous justification of this formula is contained in Proposition 5.5. Combining both transformations we find that the energy per unit volume satisfies

$$\begin{aligned}\frac{E_L(N_\uparrow, N_\downarrow)}{L^3} \le\ & \frac{3}{5}(6\pi^2)^{\frac{2}{3}}\left(\varrho_\uparrow^{\frac{5}{3}} + \varrho_\downarrow^{\frac{5}{3}}\right) + 8\pi a\varrho_\uparrow\varrho_\downarrow \\ & - \frac{(8\pi a)^2}{L^9}\sum_{p,r,r'\in\Lambda^*}\left(\frac{\hat{u}_\uparrow(r+p)\hat{u}_\downarrow(r'-p)\hat{v}_\uparrow(r)\hat{v}_\downarrow(r')}{|r+p|^2 - |r|^2 + |r'-p|^2 - |r'|^2 + 2\varepsilon} - \frac{1}{2|p|^2}\right)(\hat{\chi}_<(p))^2 + o(\varrho^{\frac{7}{3}}).\end{aligned}$$

To conclude we can remove the ε and the momentum cut-off in the third term (see Lemma 6.1) to get, in the thermodynamic limit,

$$\begin{aligned}e(\varrho_\uparrow, \varrho_\downarrow) \le\ & \frac{3}{5}(6\pi^2)^{\frac{2}{3}}\left(\varrho_\uparrow^{\frac{5}{3}} + \varrho_\downarrow^{\frac{5}{3}}\right) + 8\pi a\varrho_\uparrow\varrho_\downarrow \\ & - \frac{(8\pi a)^2}{(2\pi)^9}\int_{\mathbb{R}^3} dp \int_{B_F^\uparrow} dr \int_{B_F^\downarrow} dr' \left(\frac{\chi^c_{B_F^\uparrow}(r+p)\chi^c_{B_F^\downarrow}(r'-p)}{|r+p|^2 - |r|^2 + |r'-p|^2 - |r'|^2} - \frac{1}{2|p|^2}\right) + o(\varrho^{\frac{7}{3}}).\end{aligned}$$

The evaluation of the last integral yields the expression $a^2\varrho_\uparrow^{\frac{7}{3}}F(\varrho_\downarrow/\varrho_\uparrow)$ in Theorem 1.2, as will be demonstrated in Appendix B.

3 | Preliminary Bounds and Effective Correlation Operator

In this section, we collect some useful preliminary bounds that we will use frequently in our analysis. We also identify the effective correlation operator from which we will extract the desired expression for the energy in the upper bound in Theorem 1.2. As usual, we will denote by C generic constants in the following, which may have different values at different occurrences.

3.1 | Some Operator Bounds

In this subsection we collect some general operator inequalities. For $g \in L^1(\Lambda)$, define

$$b_\sigma(g) := \int_\Lambda dz\, g(z) a_\sigma(u_z) a_\sigma(v_z), \tag{3.1}$$

where we recall the notation (2.7). Similarly, we define

$$b_{j,\sigma}(g) = \int_\Lambda dz\, g(z) a_\sigma(u_z) a_\sigma(\partial_j v_z), \qquad j = 1,2,3. \tag{3.2}$$

Lemma 3.1 (Bounds for $b_\sigma(\widetilde{\varphi}_z)$, $b_{j,\sigma}(\widetilde{\varphi}_z)$). *For $\widetilde{\varphi}$ defined in (2.24) let $\widetilde{\varphi}_z(z') = \varphi(z - z')$. Then*

$$\|b_\sigma(\widetilde{\varphi}_z)\| \le C\varrho^{\frac{1}{3}+\frac{\gamma}{2}}, \qquad \|b_{j,\sigma}(\widetilde{\varphi}_z)\| \le C\varrho^{\frac{2}{3}+\frac{\gamma}{2}}, \qquad j = 1,2,3 \tag{3.3}$$

uniformly in $z \in \Lambda$.

Proof. The proof is analogous to the one of [17, Lemma 4.8]. We have

$$\begin{aligned} b_\sigma(\widetilde{\varphi}_z) &= \int_\Lambda dz'\, \widetilde{\varphi}(z - z') a_\sigma(u_{z'}) a_\sigma(v_{z'}) = \frac{1}{L^3} \sum_{p,k} \hat{\varphi}(p) \widehat{\chi}_>(p) e^{ip\cdot z} \hat{u}_\sigma(k+p) \hat{v}_\sigma(k) \hat{a}_{k+p,\sigma} \hat{a}_{-k,\sigma} \\ &= \frac{1}{L^3} \sum_{p,k} \hat{\varphi}(p) \widehat{\chi}_>(p) e^{ip\cdot z} \hat{v}_\sigma(k) \mathbb{1}_{\{|k+p| \ge 3\varrho^{1/3-\gamma}\}} \hat{a}_{k+p,\sigma} \hat{a}_{-k,\sigma}, \end{aligned} \tag{3.4}$$

where we used that the conditions $|p| \ge 4\varrho^{1/3-\gamma}$ and $|k| < k_F^\sigma$ imply that $|k+p| \ge 3\varrho^{1/3-\gamma}$ for small ϱ, and hence $\hat{u}_\sigma(k + p) = 1$. We multiply and divide the expression on the right hand side of (3.4) by $|k+p|^2$ and decompose $b_\sigma(\widetilde{\varphi}_z)$ as a sum of three terms. Writing them in configuration space, we obtain

$$\begin{aligned} b_\sigma(\widetilde{\varphi}_z) = &-\int dz'\, \Delta\widetilde{\varphi}(z - z') a_\sigma(\widetilde{u}_{z'}) a_\sigma(v_{z'}) + 2\sum_{j=1}^3 \int dz'\, \partial_j \widetilde{\varphi}(z - z') a_\sigma(\widetilde{u}_{z'}) a_\sigma(\partial_j v_{z'}) \\ &- \int dz'\, \widetilde{\varphi}(z - z') a_\sigma(\widetilde{u}_{z'}) a_\sigma(\Delta v_{z'}), \end{aligned}$$

where

$$\widetilde{u}_x(y) := \frac{1}{L^3} \sum_{p\in\Lambda^*} \frac{1}{|p|^2} \mathbb{1}_{\{|p| \ge 3\varrho^{1/3-\gamma}\}} e^{ip\cdot(y-x)}. \tag{3.5}$$

Using (2.1) we thus obtain

$$\|b_\sigma(\widetilde{\varphi}_z)\| \le \|v_\sigma\|_2 \|\widetilde{u}\|_2 \|\Delta\widetilde{\varphi}\|_1 + 2\|\nabla v_\sigma\|_2 \|\widetilde{u}\|_2 \|\nabla\widetilde{\varphi}\|_1 + \|\Delta v_\sigma\|_2 \|\widetilde{u}\|_2 \|\widetilde{\varphi}\|_1 \le C\varrho^{\frac{1}{3}+\frac{\gamma}{2}},$$

where in last step we used (A.3) to bound the various norms involving $\widetilde{\varphi}$, as well as $\|\widetilde{u}\|_2 \le C\varrho^{\gamma/2-1/6}$ and $\|\partial^n v_\sigma\|_2 \le C\varrho^{1/2+n/3}$.

For the proof of the second estimate in (3.3) we can proceed in the same way. The extra factor $\varrho^{1/3}$ arises from the additional derivative hitting v_σ. □

Lemma 3.2. *For any $g \in L^1(\Lambda) \cap L^2(\Lambda)$, we have*

$$\|b_\sigma^*(g)\psi\|^2 \le \|b_\sigma(g)\psi\|^2 + C\varrho\|g\|_2^2 \qquad \forall \psi \in \mathcal{F},\ \|\psi\| = 1. \tag{3.6}$$

Proof. The proof of (3.6) can be done as in [16, Proposition 5.1, Eq. (5.10)]. The idea is to write

$$\|b^*_\sigma(g)\psi\|^2 = \|b_\sigma(g)\psi\|^2 - \langle\psi, [b^*_\sigma(g), b_\sigma(g)]\psi\rangle,$$

and to use the positivity of the (normal-ordered) quadratic terms in the commutator above, obtaining

$$-[b^*_\sigma(g_z), b_\sigma(g_z)] \le \frac{1}{L^6}\sum_{k,r} |\hat{g}(k)|^2 \hat{v}_\sigma(r)\hat{u}_\sigma(r+k).$$

Bounding $\hat{u}_\sigma$ by 1, the result follows. □

Next, we turn to the number operator $\mathcal{N}$ on Fock space, which, after the particle–hole transformation R, measures the number of excitations above the Fermi sea. We have the following estimates concerning its behavior under the quasi-bosonic transformations $T_{1;\lambda}$ and $T_{2;\lambda}$ defined in Section 2.3.

Proposition 3.3 (Estimate for $\mathcal{N}$ – Part I). *Let $\lambda \in [0,1]$ and $0 < \gamma < 1/3$. For any normalized $\psi \in \mathcal{F}_{\mathrm{f}}$, we have*

$$\langle T_{1;\lambda}\psi, \mathcal{N} T_{1;\lambda}\psi\rangle \le C\langle\psi, \mathcal{N}\psi\rangle + CL^3\varrho^{\frac{5}{3}+\gamma}. \tag{3.7}$$

Proof. The proof is very similar to the one in [17, Proposition 4.11]. We compute

$$\begin{aligned}\partial_\lambda T^*_{1;\lambda}\mathcal{N} T_{1;\lambda} &= T^*_{1;\lambda}[\mathcal{N}, B_1 - B_1^*]T_{1;\lambda} = -4T^*_{1;\lambda}\big(B_1 + B_1^*\big)T_{1;\lambda}\\ &= 4T^*_{1;\lambda}\bigg(\int dz\, b_\downarrow(\widetilde{\varphi}_z)a_\uparrow(v_z)a_\uparrow(u_z) + \mathrm{h.c.}\bigg)T_{1;\lambda},\end{aligned}$$

where we used the definition 3.1 and the form of B_1 in (2.23). Using that $\|a_\uparrow(v_z)\| = \|v_\uparrow\|_2 \le \varrho^{1/2}$ as well as the bound (3.3) from Lemma 3.1, we obtain with the help of the Cauchy–Schwarz inequality

$$|\partial_\lambda\langle T_{1;\lambda}\psi, \mathcal{N} T_{1;\lambda}\psi\rangle| \le C\varrho^{\frac{5}{6}+\frac{\gamma}{2}}\int dz\, \|a_\uparrow(u_z)T_{1;\lambda}\psi\| \le CL^{\frac{3}{2}}\varrho^{\frac{5}{6}+\frac{\gamma}{2}}\|\mathcal{N}^{\frac{1}{2}}T_{1;\lambda}\psi\|.$$

The bound (3.7) then follows from Grönwall's Lemma. □

Before considering the analogous problem for $T_{2;\lambda}$ we shall prove the following Lemma.

Lemma 3.4 (Integration over t). *Let $0 < \gamma < 1/3$ and $0 < \delta \le 8\gamma$. For v^t_σ and u^t_σ defined in (2.26), we have*

$$\int_0^\infty dt\, e^{-2t\varepsilon}e^{2t(k_F^\sigma)^2}\|u^t_\sigma\|_2^2 \le C\varrho^{\frac{1}{3}-\gamma}, \tag{3.8}$$

and

$$\int_0^\infty dt\, e^{-2t\varepsilon}e^{-2t(k_F^\sigma)^2}\|v^t_\sigma\|_2^2 \le C\varrho^{\frac{1}{3}-\frac{\delta}{8}}. \tag{3.9}$$

The proof shows that the factor 8 can be replaced by any other number. The choice 8 guarantees that the final error bound is independent of it. In the following we will often use

$$\int dt\, e^{-2t\varepsilon}\|u^t_\sigma\|_2\|v^t_\sigma\|_2 \le C\varrho^{\frac{1}{3}-\frac{\gamma}{2}-\frac{\delta}{16}}, \tag{3.10}$$

which follows from Lemma 3.4 by the Cauchy–Schwarz inequality.

Proof. We start with proving (3.8). Writing the L^2-norm as a sum in momentum space, we have

$$\int_0^\infty dt\, e^{-2t\epsilon} e^{2t(k_F^\sigma)^2} \|u_\sigma^t\|_2^2 = \frac{1}{2L^3} \sum_k \frac{|\hat{u}_\sigma^<(k)|^2}{|k|^2 - (k_F^\sigma)^2 + \epsilon}.$$

Since $\hat{u}_\sigma^<(k)$ is supported on $k_F^\sigma < |k| \le 6\varrho^{\frac{1}{3}-\gamma}$ (see (2.25)), the contribution to the sum from $|k| \ge 2k_F^\sigma$ is bounded by $C\varrho^{1/3-\gamma}$. For the remaining part, we bound the sum by the corresponding integral and obtain

$$\begin{aligned}\frac{1}{2L^3} \sum_{k_F^\uparrow < |k| < 2k_F^\sigma} \frac{1}{(|k| - k_F^\sigma)(|k| + k_F^\sigma) + \epsilon} &\le C \int_{k_F^\sigma \le |k| \le 2k_F^\sigma} dk \frac{1}{(|k| - k_F^\sigma)(|k| + k_F^\sigma) + \epsilon} \\ &\le C k_F^\sigma \log\left(1 + \frac{(k_F^\sigma)^2}{\epsilon}\right).\end{aligned}$$

The last factor grows only logarithmically with $(k_F^\sigma)^2/\epsilon \lesssim \varrho^{-\delta}$, hence we can bound it by $C\varrho^{-\delta/8}$. This term is negligible compared to the one of order $\varrho^{1/3-\gamma}$ above, and we arrive at (3.8). For (3.9) we have analogously

$$\int_0^\infty dt\, e^{-2t\epsilon} e^{-2t(k_F^\uparrow)^2} \|v_\uparrow^t\|_2^2 = \frac{1}{2L^3} \sum_k \frac{|\hat{v}_\uparrow(k)|^2}{(k_F^\uparrow)^2 - |k|^2 + \epsilon}.$$

The sum can be bounded very similarly as above, with the result that

$$\frac{1}{2L^3} \sum_k \frac{|\hat{v}_\sigma(k)|^2}{(k_F^\sigma)^2 - |k|^2 + \epsilon} \le C \int_{0 \le |k| \le k_F^\sigma} dk \frac{1}{(k_F^\sigma - |k|)(k_F^\sigma + |k|) + \epsilon} \le C k_F^\sigma \log\left(1 + \frac{(k_F^\sigma)^2}{\epsilon}\right) \le C\varrho^{\frac{1}{3} - \frac{\delta}{8}}.$$

$\square$

Proposition 3.5 (Estimate for $\mathcal{N}$ – Part II). *Let $0 < \delta \le 8\gamma$ for some $0 < \gamma < 1/3$, and $\lambda \in [0,1]$. For any normalized $\psi \in \mathcal{F}_\mathrm{f}$, we have*

$$|\partial_\lambda \langle T_{2;\lambda}\psi, \mathcal{N} T_{2;\lambda}\psi\rangle| \le C L^{\frac{3}{2}} \varrho^{\frac{5}{6} - \frac{\gamma}{2} - \frac{\delta}{16}} \|\mathcal{N}^{\frac{1}{2}} T_{2;\lambda}\psi\|. \tag{3.11}$$

Furthermore,

$$\langle T_{1;\lambda} T_2 \Omega, \mathcal{N} T_{1;\lambda} T_2 \Omega\rangle \le C L^3 \varrho^{\frac{5}{3} - \gamma - \frac{\delta}{8}}. \tag{3.12}$$

Proof. As in Proposition 3.3, we compute

$$\partial_\lambda T_{2;\lambda}^* \mathcal{N} T_{2,\lambda} = T_{2;\lambda}^* [\mathcal{N}, B_2 - B_2^*] T_{2,\lambda} = -4 T_{2;\lambda}^* \big(B_2 + B_2^*\big) T_{2,\lambda}.$$

In particular, inserting the expression (2.27) for B_2, we have

$$\partial_\lambda T_{2;\lambda}^* \mathcal{N} T_{2;\lambda} = 4 T_{2;\lambda}^* \left(8\pi a \int_0^\infty dt\, e^{-2t\epsilon} \int dx dy\, \chi_<(x-y) a_\uparrow(u_x^t) a_\uparrow(v_x^t) a_\downarrow(v_y^t) a_\downarrow(u_y^t) + \text{h.c.}\right) T_{2;\lambda},$$

where u_σ^t and v_σ^t are defined in (2.26). We can then estimate

$$\begin{aligned}|\partial_\lambda \langle T_{2;\lambda}\psi, \mathcal{N} T_{2;\lambda}\psi\rangle| &\le C \int dt\, e^{-2t\epsilon} \|u_\uparrow^t\|_2 \|v_\uparrow^t\|_2 \|v_\downarrow^t\|_2 \int dx dy\, |\chi_<(x-y)| \|a_\downarrow(u_y^t) T_{2;\lambda}\psi\| \\ &\le C L^{\frac{3}{2}} \varrho^{\frac{1}{2}} \|\mathcal{N}^{\frac{1}{2}} T_{2;\lambda}\psi\| \int dt\, e^{-2t\epsilon} \|\hat{u}_\uparrow^t\|_2 \|\hat{v}_\uparrow^t\|_2,\end{aligned} \tag{3.13}$$

where we used that $\|\chi_<\|_1 \le C$, $\|v^t_\downarrow\|_2 \le C\varrho^{1/2} e^{t(k_F^\downarrow)^2}$, and that

$$\int dy\, \|a_\sigma(u^t_y)\psi\|^2 = \sum_k |\hat{u}^t_\sigma(k)|^2 \langle \psi, \hat{a}^*_{k,\sigma}\hat{a}_{k,\sigma}\psi\rangle \le \sum_{|k|>k_F^\sigma} e^{-2t|k|^2}\langle \psi, \hat{a}^*_{k,\sigma}\hat{a}_{k,\sigma}\psi\rangle \le e^{-2t(k_F^\sigma)^2}\langle \psi, \mathcal{N}\psi\rangle, \tag{3.14}$$

along with the Cauchy–Schwarz inequality. The desired bound (3.11) then follows from (3.10).
Grönwall's Lemma readily implies (3.12) for $\lambda = 0$. The general case then follows directly from (3.7). □

3.2 | Simplified Correlation Operator

In this section, we collect some bounds already discussed in [16, 17], which allow to reduce the correlation operator $\mathcal{H}_{\mathrm{corr}} = \mathbb{H}_0 + \mathbb{X} + \sum_{i=1}^4 \mathbb{Q}_i$ defined in (2.10) to a simplified one as far as the energy of our trial state $\psi_{\mathrm{trial}} = RT_1T_2\Omega$ is concerned. We shall see that we can ignore the contributions of $\mathbb{X}$, $\mathbb{Q}_1$, $\mathbb{Q}_2^{\parallel}$ and $\mathbb{Q}_3$.

3.2.1 | Estimates for $\mathbb{X}$ and $\mathbb{Q}_1$

Proceeding as in the proof of [16, Proposition 3.3], it is easy to see that

$$|\langle \psi, \mathbb{X}\psi\rangle| \le C\varrho\langle \psi, \mathcal{N}\psi\rangle, \qquad |\langle \psi, \mathbb{Q}_1\psi\rangle| \le C\varrho\langle \psi, \mathcal{N}\psi\rangle, \qquad \forall \psi \in \mathcal{F}_{\mathrm{f}}. \tag{3.15}$$

This estimate guarantees that $\langle T_1T_2\Omega, (\mathbb{X} + \mathbb{Q}_1)T_1T_2\Omega\rangle = L^3 o(\varrho^{7/3})$, as a direct consequence of the bound for the number operator in (3.12), as long as $\gamma + \delta/8 < 1/3$.

3.2.2 | Estimate for $\mathbb{Q}_3$

We note that the state $T_1T_2\Omega$ is such that in the n-particle sector of the Fock space $(T_1T_2\Omega)^{(n)} = 0$ unless $n = 4k$ for $k \in \mathbb{N}$. This is a consequence of the fact that both B_1 and B_2 either create or annihilate 4 particles. Since $\mathbb{Q}_3$ changes the particle number by ± 2, we immediately conclude that $\langle T_1T_2\Omega, \mathbb{Q}_3T_1T_2\Omega\rangle = 0$.

3.2.3 | Estimate for $\mathbb{Q}_2^{\parallel}$

We shall now argue, similarly as above, that also $\langle T_1T_2\Omega, \mathbb{Q}_2^{\parallel}T_1T_2\Omega\rangle = 0$. This follows from the fact that $T_1T_2\Omega$ and $\mathbb{Q}_2^{\parallel}T_1T_2\Omega$ are orthogonal, since $\mathbb{Q}_2^{\parallel}$ creates or annihilates 4 particles of the same spin, while in the state $T_1T_2\Omega$ the number of particles in each spin component is the same, since both T_1 and T_2 create particle (and annihilate) pairs of opposite spin simultaneously.
In summary, we obtain the following simplification of the correlation Hamiltonian.

Proposition 3.6 (Effective correlation operator). *We have*

$$\langle T_1T_2\Omega, \mathcal{H}_{\mathrm{corr}}T_1T_2\Omega\rangle \le \langle T_1T_2\Omega, \mathcal{H}^{\mathrm{eff}}_{\mathrm{corr}}T_1T_2\Omega\rangle + CL^3\varrho^{\frac{8}{3}-\gamma-\frac{\delta}{8}},$$

where $\mathcal{H}^{\mathrm{eff}}_{\mathrm{corr}} = \mathbb{H}_0 + \mathbb{Q}_2^{\uparrow\downarrow} + \mathbb{Q}_4$ *(with the various terms defined in (2.10)).*

4 | First Quasi-Bosonic Bogoliubov Transformation T_1

In this section we perform the first step discussed in Section 2.4, that is, we conjugate the effective correlation operator $\mathcal{H}^{\mathrm{eff}}_{\mathrm{corr}} = \mathbb{H}_0 + \mathbb{Q}_2^{\uparrow\downarrow} + \mathbb{Q}_4$ in Proposition 3.6 with the unitary T_1 introduced in (2.18). The main result of this section is the following rigorous version of (2.32). Throughout this section, we shall denote by $o(1)_{L\to\infty}$ terms that vanish in the thermodynamic limit (and likewise $o(L^\alpha)_{L\to\infty}$ for those that vanish after dividing by L^α). We shall not specify their ϱ-dependence since we take $L\to\infty$ before considering the low density limit.

Proposition 4.1 (Conjugation by T_1). *For any $\gamma \in (0,\frac{1}{6})$, $\delta \in (0, 8\gamma]$ with $2\gamma + \frac{\delta}{16} \le \frac{1}{3}$ we have*

$$\frac{1}{L^3}\langle T_1T_2\Omega, \mathcal{H}^{\mathrm{eff}}_{\mathrm{corr}} T_1T_2\Omega\rangle \le \varrho_\uparrow\varrho_\downarrow\left(8\pi a - \hat{V}(0) + \frac{1}{L^3}\sum_{0\neq p\in\Lambda^*}\frac{(8\pi a)^2(\hat{\chi}_<(p))^2}{2|p|^2}\right)$$
$$+\frac{1}{L^3}\langle T_2\Omega, (\mathbb{H}_0 + \mathbb{Q}_{2;<} + \mathcal{E}_{\mathbb{H}_0} + C\varrho^{\frac{2}{3}+2\gamma-\frac{\delta}{8}}\mathbb{H}_0 + C\mathbb{Q}_4)T_2\Omega\rangle + C\varrho^{\frac{8}{3}-2\gamma} + o(1)_{L\to\infty}, \quad (4.1)$$

with $\mathbb{Q}_{2;<}$ defined in (2.28) and

$$\mathcal{E}_{\mathbb{H}_0} = -\frac{2}{L^3}\sum_{k,s,s'\in\Lambda^*} k\cdot(s-s')\hat{\varphi}(k)\hat{\chi}_>(k)\, b_{k,s,\uparrow}b_{-k,s',\downarrow} + \text{h.c.}. \quad (4.2)$$

The operator $\mathbb{Q}_{2;<}$ can be interpreted as a renormalized version of $\mathbb{Q}_2^{\uparrow\downarrow}$, where the effect of the 2-body scattering process have been taken into account, leading to the softer interaction potential $8\pi a\hat{\chi}_<$ instead of $\hat{V}$. This operator, together with the kinetic energy operator $\mathbb{H}_0$, allows us to extract the full constant term of order $\varrho^{7/3}$ after conjugation with T_2. The error term $\langle T_2\Omega, (\mathcal{E}_{\mathbb{H}_0} + C\varrho^{\frac{2}{3}+2\gamma-\frac{\delta}{8}}\mathbb{H}_0 + C\mathbb{Q}_4)T_2\Omega\rangle$ will be estimated in the next section.

To prove Proposition 4.1, we first consider the conjugation of the operators $\mathbb{H}_0$, $\mathbb{Q}_4$ and $\mathbb{Q}_2^{\uparrow\downarrow}$ with respect to the unitary T_1 separately in Sections 4.1–4.3. Then in Section 4.4 we use the zero-energy scattering equation (1.5) to justify (2.29), effectively leading to $\mathbb{Q}_{2;<}$ as a renormalization of $\mathbb{Q}_2^{\uparrow\downarrow}$. In Section 4.5 we prove propagation bounds for the operators $\mathbb{H}_0$ and $\mathbb{Q}_4$, which help to control some of the error terms. Finally, in Section 4.6 we collect all the estimates and complete the proof of Proposition 4.1.

In the following we will often use the expression (2.23) for T_1 in configuration space, together with several estimates on $\tilde{\varphi}$ from Lemma A.2.

4.1 | Conjugation of $\mathbb{H}_0$

First, let us consider the conjugation of $\mathbb{H}_0$ by T_1. Our goal is to extract a (quasi-bosonic) quadratic term which helps to renormalize the interaction potential in $\mathbb{Q}_2^{\uparrow\downarrow}$.

Proposition 4.2 (Conjugation of $\mathbb{H}_0$ by T_1). *Let $\lambda\in[0,1]$ and $\gamma\in(0,1/3)$. Under the same assumptions as in Theorem 1.2*

$$\partial_\lambda T^*_{1;\lambda}\mathbb{H}_0 T_{1;\lambda} = T^*_{1;\lambda}(\mathbb{T}_1 + \mathcal{E}_{\mathbb{H}_0})T_{1;\lambda}, \quad (4.3)$$

where $\mathcal{E}_{\mathbb{H}_0}$ is given in (4.2), and

$$\mathbb{T}_1 = -\frac{2}{L^3}\sum_{k\in\Lambda^*}|k|^2\hat{\varphi}(k)\hat{\chi}_>(k)\, b_{k,\uparrow}b_{-k,\downarrow} + \text{h.c.}. \quad (4.4)$$

Furthermore, for any $\psi\in\mathcal{F}_{\mathrm{f}}$

$$\left|\partial_\lambda\langle T_{1;\lambda}\psi, \mathcal{E}_{\mathbb{H}_0}T_{1;\lambda}\psi\rangle\right| \le C\varrho^{1+\gamma}\|\mathbb{H}_0^{\frac{1}{2}}T_{1;\lambda}\psi\|\|\mathcal{N}^{\frac{1}{2}}T_{1;\lambda}\psi\| + C\varrho^{\frac{4}{3}+\gamma}\langle T_{1;\lambda}\psi, \mathcal{N}T_{1;\lambda}\psi\rangle. \quad (4.5)$$

Remark 4.3. In our analysis, $\mathcal{E}_{\mathbb{H}_0}$ gives rise to an error term. In order to estimate it, in Section 4.6 (see (4.71)–(4.72)), we shall apply Duhamel's formula again, in the form

$$T^*_{1;\lambda}\mathcal{E}_{\mathbb{H}_0}T_{1;\lambda} = \mathcal{E}_{\mathbb{H}_0} + \int_0^\lambda d\lambda'\, \partial_{\lambda'} T^*_{1;\lambda'}\mathcal{E}_{\mathbb{H}_0}T_{1;\lambda'},$$

and use (4.5) to estimate the last term above.

Remark 4.4 Comparison with [17]. The proof of Proposition 4.2 is similar to that in [17, Proposition 5.1 and Lemma 5.2]. However, here we do not need to insert a smooth cut-off in the projection inside or outside the Fermi ball, which simplifies and improves many estimates.

Proof of Proposition 4.2. The equality in (4.3) is a straightforward consequence of

$$\partial_\lambda T^*_{1;\lambda}\mathbb{H}_0 T_{1;\lambda} = T^*_{1;\lambda}[\mathbb{H}_0, B_1 - B_1^*]T_{1;\lambda},$$

inserting the definitions of $\mathbb{H}_0$ and B_1 in (2.10) and (2.18), respectively, and computing the commutator. We shall skip the details. To prove (4.5), we compute

$$\partial_\lambda T^*_{1;\lambda}\mathcal{E}_{\mathbb{H}_0}T_{1;\lambda} = T^*_{1;\lambda}[\mathcal{E}_{\mathbb{H}_0}, B_1 - B_1^*]T_{1;\lambda} = T^*_{1;\lambda}[\mathcal{E}_{\mathbb{H}_0}, B_1]T_{1;\lambda} + \text{h.c.}.$$

For simplicity, we only consider the first term involving $k \cdot s$ in the definition (4.2) of $\mathcal{E}_{\mathbb{H}_0}$, that is,

$$\mathcal{E}_{\mathbb{H}_0;1} := -\frac{2}{L^3}\sum_{k,s,s'\in\Lambda^*} k\cdot s\,\hat{\varphi}(k)\hat{\chi}_>(k)\, b_{k,s,\uparrow}b_{-k,s',\downarrow} + \text{h.c.}, \tag{4.6}$$

the estimate for the other term involving $k\cdot s'$ is analogous. For the calculation of the commutator $[\mathcal{E}_{\mathbb{H}_0;1}, B_1]$, we need to compute

$$\begin{aligned}
&[\hat{a}^*_{-s',\downarrow}\hat{a}^*_{s'-k,\downarrow}\hat{a}^*_{-s,\uparrow}\hat{a}^*_{s+k,\uparrow}, \hat{a}_{r+p,\uparrow}\hat{a}_{-r,\uparrow}\hat{a}_{r'-p,\downarrow}\hat{a}_{-r',\downarrow}] \\
&= \hat{a}^*_{s+k,\uparrow}\hat{a}^*_{s'-k,\downarrow}\hat{a}_{r+p,\uparrow}\hat{a}_{r'-p,\downarrow}(\delta_{s,r}\delta_{s',r'} - \delta_{s,r}\hat{a}^*_{-s',\downarrow}\hat{a}_{-r',\downarrow} - \delta_{s',r'}\hat{a}^*_{-s,\uparrow}\hat{a}_{-r,\uparrow}) \\
&\quad + (\delta_{s+k,r+p}\hat{a}^*_{s'-k,\downarrow}\hat{a}_{r'-p,\downarrow} + \delta_{s'-k,r'-p}\hat{a}^*_{s+k,\uparrow}\hat{a}_{r+p,\uparrow} - \delta_{s'-k,r'-p}\delta_{r+p,s+k})\hat{a}_{-r,\uparrow}\hat{a}_{-r',\downarrow}\hat{a}^*_{-s',\downarrow}\hat{a}^*_{-s,\uparrow}
\end{aligned} \tag{4.7}$$

Correspondingly, we have $[\mathcal{E}_{\mathbb{H}_0;1}, B_1] = \sum_{j=1}^6 \mathrm{I}_j$, with

$$\begin{aligned}
\mathrm{I}_1 = \frac{1}{L^6}\sum_{k,p,r,r'} & k\cdot r\hat{\varphi}(k)\hat{\chi}_>(k)\hat{v}_\uparrow(r)\hat{\varphi}(p)\hat{\chi}_>(p)\hat{v}_\downarrow(r')\hat{u}_\uparrow(r+k)\hat{u}_\downarrow(r'-k)\hat{u}_\uparrow(r+p)\hat{u}_\downarrow(r'-p) \\
&\times \hat{a}^*_{r+k,\uparrow}\hat{a}^*_{r'-k,\downarrow}\hat{a}_{r+p,\uparrow}\hat{a}_{r'-p,\downarrow},
\end{aligned} \tag{4.8}$$

$$\begin{aligned}
\mathrm{I}_2 = -\frac{1}{L^6}\sum_{k,p,r,r',s} & k\cdot r\hat{\varphi}(k)\hat{\chi}_>(k)\hat{v}_\uparrow(r)\hat{\varphi}(p)\hat{\chi}_>(p)\hat{v}_\downarrow(r')\hat{v}_\downarrow(s)\hat{u}_\uparrow(r+k)\hat{u}_\downarrow(s-k)\hat{u}_\uparrow(r+p)\hat{u}_\downarrow(r'-p) \\
&\times \hat{a}^*_{r+k,\uparrow}\hat{a}^*_{s-k,\downarrow}\hat{a}^*_{-s,\downarrow}\hat{a}_{-r',\downarrow}\hat{a}_{r+p,\uparrow}\hat{a}_{r'-p,\downarrow},
\end{aligned} \tag{4.9}$$

$$\begin{aligned}
\mathrm{I}_3 = -\frac{1}{L^6}\sum_{k,p,r,r',s} & k\cdot s\hat{\varphi}(k)\hat{\chi}_>(k)\hat{v}_\uparrow(s)\hat{\varphi}(p)\hat{\chi}_>(p)\hat{v}_\downarrow(r')\hat{v}_\uparrow(r)\hat{u}_\uparrow(s+k)\hat{u}_\downarrow(r'-k)\hat{u}_\uparrow(p+r)\hat{u}_\downarrow(r'-p) \\
&\times \hat{a}^*_{s+k,\uparrow}\hat{a}^*_{r'-k,\downarrow}\hat{a}^*_{-s,\uparrow}\hat{a}_{-r,\uparrow}\hat{a}_{r+p,\uparrow}\hat{a}_{r'-p,\downarrow},
\end{aligned} \tag{4.10}$$

$$\begin{aligned}
\mathrm{I}_4 = \frac{1}{L^6}\sum_{k,p,r,r',s} & k\cdot(r+p-k)\hat{\varphi}(k)\hat{\chi}_>(k)\hat{\varphi}(p)\hat{\chi}_>(p)\hat{u}_\uparrow(r+p)\hat{u}_\downarrow(s-k)\hat{u}_\downarrow(r'-p) \\
&\times \hat{v}_\uparrow(r+p-k)\hat{v}_\downarrow(s)\hat{v}_\uparrow(r)\hat{v}_\downarrow(r')\hat{a}^*_{s-k,\downarrow}\hat{a}_{r'-p,\downarrow}\hat{a}_{-r,\uparrow}\hat{a}_{-r',\downarrow}\hat{a}^*_{-s,\downarrow}\hat{a}^*_{-r-p+k,\uparrow},
\end{aligned} \tag{4.11}$$

$$\mathrm{I}_5 = \frac{1}{L^6} \sum_{k,p,r,r',s} k \cdot s\hat{\varphi}(k)\hat{\chi}_>(k)\hat{\varphi}(p)\hat{\chi}_>(p)\hat{u}_\uparrow(r+p)\hat{u}_\uparrow(s+k)\hat{u}_\downarrow(r'-p)$$
$$\times \hat{v}_\uparrow(s)\hat{v}_\downarrow(r')\hat{v}_\uparrow(r)\hat{v}_\downarrow(r'+k-p)\hat{a}^*_{s+k,\uparrow}\hat{a}_{r+p,\uparrow}\hat{a}_{-r,\uparrow}\hat{a}_{-r',\downarrow}\hat{a}^*_{-r'-k+p,\downarrow}\hat{a}^*_{-s,\uparrow}, \tag{4.12}$$

and

$$\mathrm{I}_6 = -\frac{1}{L^6} \sum_{k,p,r,r'} k \cdot (r+p-k)\hat{\varphi}(k)\hat{\chi}_>(k)\hat{\varphi}(p)\hat{\chi}_>(p)\hat{u}_\uparrow(r+p)\hat{u}_\downarrow(r'-p)\hat{v}_\uparrow(r+p-k)\hat{v}_\downarrow(r'-p+k)$$
$$\times \hat{v}_\uparrow(r)\hat{v}_\downarrow(r')\,\hat{a}_{-r,\uparrow}\hat{a}_{-r',\downarrow}\hat{a}^*_{-r'+p-k,\downarrow}\hat{a}^*_{-r-p+k,\uparrow}. \tag{4.13}$$

In the following, we shall estimate all these terms. We shall repeatedly use the bound

$$\|a_\sigma(\partial^n v_x)\| \leq \||\cdot|^n \hat{v}_\sigma\|_2 \leq C\varrho^{\frac{1}{2}+\frac{n}{3}}, \tag{4.14}$$

for the fermionic creation and annihilation operators. Moreover, we shall employ the bounds in Lemma 3.1 in Lemma A.2. We will estimate the expectation value of all the terms in a general state $\psi \in \mathcal{F}_{\mathrm{f}}$. We first consider the error terms coming from the second line on the right hand side in (4.7), that is, I_1, I_2 and I_3. It is convenient to divide them all into two parts, which corresponds to an integration by parts in configuration space: this allows us to get a factor $\varrho^{1/3}$ when the derivative hits a v_σ, and to use $\mathbb{H}_0$ instead of $\mathcal{N}$ to estimate the error terms when the derivative hits a u_σ. We start by estimating the term I_1 in (4.8). Writing $k \cdot r = -r^2 + r \cdot (r+k)$, we split I_1 into two terms, which we denote by $\mathrm{I}_{1;a}$ and $\mathrm{I}_{1;b}$, respectively. In $\mathrm{I}_{1;b}$ it is convenient to replace $\hat{u}_\uparrow(r+k)$ by $\hat{v}^>_\uparrow(r+k)$ with $\hat{v}^>$ defined as

$$\hat{v}^>(k) = \begin{cases} 1 & \text{if } |k| > 3\varrho^{\frac{1}{3}-\gamma}, \\ 0 & \text{if } |k| \leq 3\varrho^{\frac{1}{3}-\gamma}. \end{cases} \tag{4.15}$$

This can be done since $|k| \geq 4\varrho^{1/3-\gamma}$ and $|r| \leq k_F^\uparrow$ for all the terms in the sum. The same argument applies to I_2 and I_3. To estimate I_1, we rewrite both $\mathrm{I}_{1;a}$ and $\mathrm{I}_{1;b}$ in configuration space. We have

$$\mathrm{I}_{1;a} = \int dxdydzdz'\, \tilde{\varphi}(x-y)\tilde{\varphi}(z-z')\big(\Delta v_\uparrow\big)(x;z)v_\downarrow(y;z')a^*_\uparrow(u_x)a^*_\downarrow(u_y)a_\uparrow(u_z)a_\downarrow(u_{z'}).$$

For fixed y and z, we can use (4.14) and $0 \leq \hat{u}_\uparrow \leq 1$ to bound

$$\left\| \int dx\, \tilde{\varphi}(x-y)\big(\Delta v_\uparrow\big)(x;z)a_\uparrow(u_x)\right\| \leq \|\tilde{\varphi}_y \Delta v_{z,\uparrow}\|_2,$$

where we used the notation $\tilde{\varphi}_y(\,\cdot\,) = \tilde{\varphi}(y - \,\cdot\,)$, $\Delta v_{z,\uparrow}(\,\cdot\,) = \Delta v_\uparrow(\,\cdot\,; z)$, as in Lemma 3.1 and the discussion after (2.5). Similarly,

$$\left\| \int dz'\, \tilde{\varphi}(z-z')v_\downarrow(y;z')a_\downarrow(u_{z'})\right\| \leq \|\tilde{\varphi}_z v_{y,\downarrow}\|_2. \tag{4.16}$$

With the aid of the Cauchy–Schwarz's inequality, we hence find that

$$|\langle\psi, \mathrm{I}_{1;a}\psi\rangle| \leq \int dydz\, \|\tilde{\varphi}_y \Delta v_{z,\uparrow}\|_2 \|\tilde{\varphi}_z v_{y,\downarrow}\|_2 \|a_\downarrow(u_y)\psi\| \|a_\uparrow(u_z)\psi\|$$
$$\leq \left(\int dydzdw\, |\tilde{\varphi}(w-y)|^2 |\Delta v_\uparrow(z;w)|^2 \|a_\downarrow(u_y)\psi\|^2\right)^{\frac{1}{2}} \left(\int dydzdw\, |\tilde{\varphi}(z-w)|^2 |v_\downarrow(y;w)|^2 \|a_\uparrow(u_z)\psi\|^2\right)^{\frac{1}{2}}.$$

Thus, using also Lemma A.2, we obtain

$$|\langle\psi,\mathrm{I}_{1;a}\psi\rangle| \le \|\widetilde{\varphi}\|_2^2\|\Delta v_\uparrow\|_2\|v_\downarrow\|_2\langle\psi,\mathcal{N}\psi\rangle \le C\varrho^{\frac{4}{3}+\gamma}\langle\psi,\mathcal{N}\psi\rangle. \tag{4.17}$$

The estimate of the term $\mathrm{I}_{1;b}$ is similar. We have

$$\mathrm{I}_{1;b} = \sum_{\ell=1}^{3}\int dxdydzdz'\,\widetilde{\varphi}(x-y)\widetilde{\varphi}(z-z')\partial_\ell v_\uparrow(x;z)v_\downarrow(y;z')a_\uparrow^*(\partial_\ell \nu_x^>)a_\downarrow^*(u_y)a_\uparrow(u_z)a_\downarrow(u_{z'}),$$

where $\nu_x^>(\cdot) = \nu^>(\cdot - x)$. Proceeding as above, we can bound

$$\left\|\int dz\,\widetilde{\varphi}(z-z')\partial_\ell v_\uparrow(x;z)a_\uparrow(u_z)\right\| \le \|\widetilde{\varphi}_{z'}\partial_\ell v_{x,\uparrow}\|_2. \tag{4.18}$$

Using also (4.16) and that

$$\sum_{\ell=1}^{3}\int dx\,\|a_\uparrow(\partial_\ell \nu_x^>)\psi\|^2 = \sum_k |k|^2|\hat{\nu}_\uparrow^>(k)|^2\|\hat{a}_{k,\uparrow}\psi\|^2 \le C\langle\psi,\mathbb{H}_0\psi\rangle, \tag{4.19}$$

we obtain the bound

$$|\langle\psi,\mathrm{I}_{1;b}\psi\rangle| \le C\|\widetilde{\varphi}\|_2^2\|\nabla v_\uparrow\|_2\|v_\downarrow\|_2\|\mathbb{H}_0^{\frac{1}{2}}\psi\|\|\mathcal{N}^{\frac{1}{2}}\psi\| \le C\varrho^{1+\gamma}\|\mathbb{H}_0^{\frac{1}{2}}\psi\|\|\mathcal{N}^{\frac{1}{2}}\psi\|. \tag{4.20}$$

We now estimate the term I_2 defined in (4.9). Similarly as above, we write $k\cdot r = -r^2 + r\cdot(r+k)$ and correspondingly we split I_2 as a sum of two different terms, $\mathrm{I}_{2;a}$ and $\mathrm{I}_{2;b}$. In order to bound $\mathrm{I}_{2;a}$, we rewrite it partially in configuration space. Recalling the definition of b_σ in (3.1), we find

$$\langle\psi,\mathrm{I}_{2;a}\psi\rangle = \frac{1}{L^3}\sum_r |r|^2\hat{v}_\uparrow(r)\left\|\int dx\,e^{-ir\cdot x}b_\downarrow(\widetilde{\varphi}_x)a_\uparrow(u_x)\psi\right\|^2.$$

We can bound $|r|^2\hat{v}_\uparrow(r) \le C\varrho^{2/3}$ for all $r\in\Lambda^*$. From the bounds in Lemma 3.1, we find

$$\begin{aligned}\frac{1}{L^3}\sum_r\left\|\int dx\,e^{-ir\cdot x}b_\downarrow(\widetilde{\varphi}_x)a_\uparrow(u_x)\psi\right\|^2 &= \int dx\,\|b_\downarrow(\widetilde{\varphi}_x)a_\uparrow(u_x)\psi\|^2\\ &\le \int dx\,\|b_\downarrow(\widetilde{\varphi}_x)\|^2\|a_\uparrow(u_x)\psi\|^2 \le C\varrho^{\frac{2}{3}+\gamma}\langle\psi,\mathcal{N}\psi\rangle,\end{aligned}$$

and hence we obtain

$$|\langle\psi,\mathrm{I}_{2;a}\psi\rangle| \le C\varrho^{\frac{4}{3}+\gamma}\langle\psi,\mathcal{N}\psi\rangle. \tag{4.21}$$

The estimate of $\mathrm{I}_{2;b}$ can be done analogously as the one of $\mathrm{I}_{2;a}$, using in addition (4.19). We omit the details, and directly write the final estimate as

$$|\langle\psi,\mathrm{I}_{2;b}\psi\rangle| \le C\varrho^{1+\gamma}\|\mathbb{H}_0^{\frac{1}{2}}\psi\|\|\mathcal{N}^{\frac{1}{2}}\psi\|. \tag{4.22}$$

Next we consider I_3 in (4.10). Again we split the term in two, $\mathrm{I}_{3;a}$ and $\mathrm{I}_{3;b}$, by writing $k = r' - (r'-k)$. The term $\mathrm{I}_{3;a}$ can then be written as

$$\mathrm{I}_{3;a} = -\sum_{\ell=1}^{3}\sum_{r'} r'_\ell\hat{v}_\downarrow(r')\int dydz'\,e^{ir'\cdot(y-z')}a_\downarrow^*(u_y)b_{\ell,\uparrow}^*(\widetilde{\varphi}_y)b_\uparrow(\widetilde{\varphi}_{z'})a_\downarrow(u_{z'}),$$

where we used the operators introduced in (3.1) and (3.2). Using the bounds in Lemma 3.1 and that $|r'_\ell\hat{v}_\downarrow(r')| \le C\varrho^{1/3}$, we get with the aid of the Cauchy–Schwarz inequality

$$|\langle\psi, \mathrm{I}_{3;a}\psi\rangle|$$

$$\leq C\varrho^{\frac{1}{3}}\sum_{\ell=1}^{3}\left(\frac{1}{L^3}\sum_{r'}\left\|\int dy\, e^{-ir'\cdot y}\, b_{\ell,\uparrow}(\widetilde{\varphi}_y)a_\downarrow(u_y)\psi\right\|^2\right)^{\frac{1}{2}}\left(\frac{1}{L^3}\sum_{r'}\left\|\int dz' e^{-ir'\cdot z'}\, b_\uparrow(\widetilde{\varphi}_{z'})a_\downarrow(u_{z'})\psi\right\|^2\right)^{\frac{1}{2}}$$

$$= C\varrho^{\frac{1}{3}}\sum_{\ell=1}^{3}\left(\int dy\,\|b_{\ell,\uparrow}(\widetilde{\varphi}_y)a_\downarrow(u_y)\psi\|^2\right)^{\frac{1}{2}}\left(\int dz'\,\|b_\uparrow(\widetilde{\varphi}_{z'})a_\downarrow(u_{z'})\psi\|^2\right)^{\frac{1}{2}} \leq C\varrho^{\frac{4}{3}+\gamma}\langle\psi, \mathcal{N}\psi\rangle. \tag{4.23}$$

The analysis of $\mathrm{I}_{3;b}$ can be done similarly as the one for $\mathrm{I}_{3;a}$, using also the inequality in (4.19). The result is

$$|\langle\psi, \mathrm{I}_{3;b}\psi\rangle| \leq C\varrho^{1+\gamma}\|\mathbb{H}_0^{\frac{1}{2}}\psi\|\|\mathcal{N}^{\frac{1}{2}}\psi\|. \tag{4.24}$$

The next error term we estimate is I_4 in (4.11). As above, it is convenient to rewrite it in configuration space. Since the constraints $|p| \geq 4\varrho^{1/3-\gamma}$ and $|r| \leq k_F^\uparrow$ imply that $|r+p| > 3\varrho^{1/3-\gamma}$, we have that $\hat{u}_\uparrow(r+p) = 1$ in (4.11), and hence

$$\mathrm{I}_4 = \sum_{\ell=1}^{3}\int dxdydz\,\partial_\ell\widetilde{\varphi}(x-y)\widetilde{\varphi}(x-z)a_\downarrow^*(u_y)a_\uparrow(v_x)a_\downarrow^*(v_z)a_\downarrow(v_y)a_\uparrow^*(\partial_\ell v_x)a_\downarrow(u_z).$$

Using Lemma A.2 and (4.14), we can bound

$$|\langle\psi, \mathrm{I}_4\psi\rangle| \leq C\varrho^{2+\frac{1}{3}}\sum_{\ell=1}^{3}\int dxdydz\,|\partial_\ell\widetilde{\varphi}(x-y)||\widetilde{\varphi}(x-z)|\|a_\downarrow(u_y)\psi\|\|a_\downarrow(u_z)\psi\|$$

$$\leq C\varrho^{2+\frac{1}{3}}\sum_{\ell=1}^{3}\|\partial_\ell\widetilde{\varphi}\|_1\|\widetilde{\varphi}\|_1\langle\psi, \mathcal{N}\psi\rangle \leq C\varrho^{\frac{4}{3}+3\gamma}\langle\psi, \mathcal{N}\psi\rangle. \tag{4.25}$$

The estimate for I_5 can be done similarly, we omit the details. Also for this term we obtain

$$|\langle\psi, \mathrm{I}_5\psi\rangle| \leq C\varrho^{\frac{4}{3}+3\gamma}\langle\psi, \mathcal{N}\psi\rangle. \tag{4.26}$$

Finally, we consider the last term I_6 in (4.13). Similarly as above, we have $\hat{u}_\uparrow(r+p) = \hat{u}_\downarrow(r'-p) = 1$ for all the terms in the sum. To estimate it, we shall write

$$-\hat{a}_{-r,\uparrow}\hat{a}_{-r',\downarrow}\hat{a}^*_{-r'+p-k,\downarrow}\hat{a}^*_{-r-p+k,\uparrow} = \hat{a}^*_{-r'+p-k,\downarrow}\hat{a}_{-r,\uparrow}\hat{a}^*_{-r-p+k,\uparrow}\hat{a}_{-r',\downarrow} - \delta_{p,k}\hat{a}_{-r,\uparrow}\hat{a}^*_{-r,\uparrow},$$

and correspondingly split I_6 in two terms, denoted by $\mathrm{I}_{6;a}$ and $\mathrm{I}_{6;b}$. The last one vanishes,

$$\mathrm{I}_{6;b} = -\frac{1}{L^6}\sum_{p,r,r'} p\cdot r\left|\hat{\varphi}(p)\widehat{\chi}_>(p)\right|^2\hat{v}_\downarrow(r')\hat{v}_\uparrow(r)\hat{a}_{-r,\uparrow}\hat{a}^*_{-r,\uparrow} = 0,$$

since the sum over p is zero by symmetry. The term $\mathrm{I}_{6;a}$ we can write in configuration space as

$$\mathrm{I}_{6;a} = \sum_{\ell=1}^{3}\int dxdy\,\widetilde{\varphi}(x-y)\partial_\ell\widetilde{\varphi}(x-y)\widetilde{\varphi}(x-y)a_\downarrow^*(v_y)a_\uparrow(v_x)a_\uparrow^*(\partial_\ell v_x)a_\downarrow(v_y). \tag{4.27}$$

We use $\widetilde{\varphi}\partial_\ell\widetilde{\varphi} = \frac{1}{2}\partial_\ell\widetilde{\varphi}^2$ and integrate by parts in x. The derivate then hits either $a_\uparrow(v_x)$ or $a_\uparrow^*(\partial_\ell v_x)$, and using (4.14) we can bound

$$|\langle\psi, \mathrm{I}_{6;a}\psi\rangle| \leq C\varrho^{1+\frac{2}{3}}\int dxdy\,|\widetilde{\varphi}(x-y)|^2\|a_\downarrow(v_y)\psi\|^2 \leq C\varrho^{\frac{4}{3}+\gamma}\langle\psi, \mathcal{N}\psi\rangle, \tag{4.28}$$

where we used Lemma A.2 in the last step. Collecting all the estimates, this concludes the proof. □

4.2 | Conjugation of $\mathbb{Q}_4$

We now conjugate $\mathbb{Q}_4$ by T_1 (see (2.10) and (2.18) for their definition).

Proposition 4.5 (Conjugation of $\mathbb{Q}_4$ by T_1). *Let $\lambda \in [0,1]$ and $\gamma \in (0,1/3)$. Under the same assumptions of Theorem 1.2*

$$\partial_\lambda T_{1;\lambda}^* \mathbb{Q}_4 T_{1;\lambda} = T_{1;\lambda}^* (\mathbb{T}_2 + \mathcal{E}_{\mathbb{Q}_4}) T_{1;\lambda}, \tag{4.29}$$

where

$$\mathbb{T}_2 := -\frac{1}{L^3} \sum_{k \in \Lambda^*} \hat{V} *_{\sum} (\hat{\varphi}\hat{\chi}_>)(k)\, b_{k,\uparrow} b_{-k,\downarrow} + \text{h.c.}, \tag{4.30}$$

and $\mathcal{E}_{\mathbb{Q}_4}$ is such that for any $\psi \in \mathcal{F}_{\mathrm{f}}$

$$|\langle \psi, \mathcal{E}_{\mathbb{Q}_4} \psi \rangle| \leq C \varrho^{\frac{5}{6}+\frac{\gamma}{2}} \| \mathcal{N}^{\frac{1}{2}} \psi \| \| \mathbb{Q}_4^{\frac{1}{2}} \psi \|. \tag{4.31}$$

Proof. We start by computing

$$\partial_\lambda T_{1;\lambda}^* \mathbb{Q}_4 T_{1;\lambda} = T_{1;\lambda}^* [\mathbb{Q}_4, B_1] T_{1;\lambda} + \text{h.c.},$$

with

$$[\mathbb{Q}_4, B_1] = \frac{1}{2L^6} \sum_{\sigma,\sigma'} \sum_{k,p,s,s'} \hat{V}(k)\hat{\varphi}(p)\hat{\chi}_>(p)\hat{u}_\sigma(s+k)\hat{u}_{\sigma'}(s'-k)\hat{u}_\sigma(s)\hat{u}_{\sigma'}(s')\, [\hat{a}^*_{s+k,\sigma}\hat{a}^*_{s'-k,\sigma'}\hat{a}_{s',\sigma'}\hat{a}_{s,\sigma}, b_{p,\uparrow}b_{-p,\downarrow}].$$

Recall from (2.14) that $b_{p,\sigma} = \sum_r \hat{u}_\sigma(p+r)\hat{v}_\sigma(r)\hat{a}_{p+r,\sigma}\hat{a}_{-r,\sigma}$. Hence we need to compute

$$\begin{aligned}
&[\hat{a}^*_{s+k,\sigma}\hat{a}^*_{s'-k,\sigma'}\hat{a}_{s',\sigma'}\hat{a}_{s,\sigma}, \hat{a}_{p+r,\uparrow}\hat{a}_{-r,\uparrow}\hat{a}_{-p+r',\downarrow}\hat{a}_{-r',\downarrow}] = -\delta_{s+k,r+p}\delta_{\sigma,\uparrow}\hat{a}^*_{s'-k,\sigma'}\hat{a}_{r'-p,\downarrow}\hat{a}_{-r',\downarrow}\hat{a}_{-r,\uparrow}\hat{a}_{s',\sigma'}\hat{a}_{s,\sigma} \\
&+ \delta_{s'-k,r+p}\delta_{\sigma',\uparrow}\hat{a}^*_{s+k,\sigma}\hat{a}_{r'-p,\downarrow}\hat{a}_{-r',\downarrow}\hat{a}_{-r,\uparrow}\hat{a}_{s',\sigma'}\hat{a}_{s,\sigma} - \delta_{s'-k,r'-p}\delta_{\sigma',\downarrow}\hat{a}^*_{s+k,\sigma}\hat{a}_{r+p,\uparrow}\hat{a}_{-r',\downarrow}\hat{a}_{-r,\uparrow}\hat{a}_{s',\sigma'}\hat{a}_{s,\sigma} \\
&+ \delta_{s+k,r'-p}\delta_{\sigma,\downarrow}\hat{a}^*_{s'-k,\sigma'}\hat{a}_{r+p,\uparrow}\hat{a}_{-r',\downarrow}\hat{a}_{-r,\uparrow}\hat{a}_{s',\sigma'}\hat{a}_{s,\sigma} + \delta_{s+k,r+p}\delta_{s'-k,r'-p}\delta_{\sigma,\uparrow}\delta_{\sigma',\downarrow}\hat{a}_{-r',\downarrow}\hat{a}_{-r,\uparrow}\hat{a}_{s',\sigma'}\hat{a}_{s,\sigma} \\
&- \delta_{s+k,r'-p}\delta_{s'-k,r+p}\delta_{\sigma,\downarrow}\delta_{\sigma',\uparrow}\hat{a}_{-r',\downarrow}\hat{a}_{-r,\uparrow}\hat{a}_{s',\sigma'}\hat{a}_{s,\sigma}.
\end{aligned}$$

Using that $\hat{V}(k) = \hat{V}(-k)$, $\hat{\varphi}(p)\hat{\chi}_>(p) = \hat{\varphi}(-p)\hat{\chi}_>(-p)$ and performing a suitable change of variables, we can combine the first two terms in the commutator above, as well as the third and the forth and the last two. Therefore, we find that

$$[\mathbb{Q}_4, B_1] = \mathrm{I}_1 + \mathrm{I}_2 + \mathrm{I}_3,$$

with

$$\begin{aligned}
\mathrm{I}_1 = \frac{1}{L^6} \sum_{\sigma} \sum_{k,p,r,r',s} &\hat{V}(k)\hat{\varphi}(p)\hat{\chi}_>(p)\hat{u}_\uparrow(r+p)\hat{u}_\downarrow(r'-p)\hat{u}_\uparrow(r+p-k)\hat{u}_\sigma(s-k)\hat{u}_\sigma(s)\hat{v}_\uparrow(r)\hat{v}_\downarrow(r') \\
&\times \hat{a}^*_{s-k,\sigma}\hat{a}_{r'-p,\downarrow}\hat{a}_{-r,\uparrow}\hat{a}_{-r',\downarrow}\hat{a}_{s,\sigma}\hat{a}_{r+p-k,\uparrow},
\end{aligned} \tag{4.32}$$

$$\begin{aligned}
\mathrm{I}_2 = \frac{1}{L^6} \sum_{\sigma} \sum_{k,p,r,r',s} &\hat{V}(k)\hat{\varphi}(p)\hat{\chi}_>(p)\hat{u}_\uparrow(r+p)\hat{u}_\downarrow(r'-p)\hat{u}_\downarrow(r'-p-k)\hat{u}_\sigma(s-k)\hat{u}_\sigma(s)\hat{v}_\uparrow(r)\hat{v}_\downarrow(r') \\
&\times \hat{a}^*_{s-k,\sigma}\hat{a}_{r+p,\uparrow}\hat{a}_{-r',\downarrow}\hat{a}_{-r,\uparrow}\hat{a}_{s,\sigma}\hat{a}_{r'-p-k,\downarrow},
\end{aligned}$$

$$\mathrm{I}_3 = -\frac{1}{L^6}\sum_{k,q,r,r'} \hat{V}(k)\hat{\varphi}(p)\hat{\chi}_>(p)\hat{u}_\uparrow(r+p)\hat{u}_\downarrow(r'-p)\hat{u}_\uparrow(r+p+k)\hat{u}_\downarrow(r'-p-k)\hat{v}_\uparrow(r)\hat{v}_\downarrow(r') \times \hat{a}_{r+p+k,\uparrow}\hat{a}_{-r,\uparrow}\hat{a}_{r'-p-k,\downarrow}\hat{a}_{-r',\downarrow}.$$

In I_3 we have $\hat{u}_\uparrow(r+p) = \hat{u}_\downarrow(r'-p) = 1$ due to the constraints on p, r, r' and, with another change of variables, we find that $\mathrm{I}_3 + \mathrm{I}_3^* = \mathbb{T}_2$ defined in (4.30). In other words, $\mathcal{E}_{\mathbb{Q}_4} = \mathrm{I}_1 + \mathrm{I}_2 + \mathrm{h.c.}$.

In the following we only consider I_1, the bound on I_2 works in exactly the same way. Using again that $\hat{u}_\uparrow(r+p) = 1$ in (4.32), we can write in configuration space

$$\mathrm{I}_1 = -\sum_\sigma \int dxdy\, V(x-y)a_\sigma^*(u_y)b_\downarrow(\widetilde{\varphi}_x)a_\uparrow(v_x)a_\sigma(u_y)a_\uparrow(u_x), \tag{4.33}$$

where we recall the definition of the operator b_σ in (3.1). Using $\|a_\downarrow(v_x)\| = \|v\|_2 \le \varrho^{1/2}$, Lemma 3.1 as well as the Cauchy–Schwarz inequality, we obtain

$$|\langle\psi, \mathrm{I}_1\psi\rangle| \le C\varrho^{\frac{5}{6}+\frac{\gamma}{2}}\sum_\sigma\int dxdy\, V(x-y)\|a_\sigma(u_y)\psi\|\|a_\sigma(u_y)a_\uparrow(u_x)\psi\| \le C\varrho^{\frac{5}{6}+\frac{\gamma}{2}}\|\mathcal{N}^{\frac{1}{2}}\psi\|\|\mathbb{Q}_4^{\frac{1}{2}}\psi\|. \tag{4.34}$$

This yields (4.31). □

4.3 | Conjugation of $\mathbb{Q}_2^{\uparrow\downarrow}$

In the next proposition we conjugate the operator $\mathbb{Q}_2^{\uparrow\downarrow}$ introduced in (2.11) with respect to the unitary operator T_1 defined in (2.18).

Proposition 4.6 (Conjugation of $\mathbb{Q}_2^{\uparrow\downarrow}$ by T_1). *Let $\lambda \in [0,1]$ and $\gamma \in (0, 1/3)$. Under the same assumptions of Theorem 1.2*

$$\partial_\lambda T_{1;\lambda}^*\mathbb{Q}_2^{\uparrow\downarrow}T_{1;\lambda} = -2\varrho_\uparrow\varrho_\downarrow\sum_{p\in\Lambda^*}\hat{V}(p)\hat{\varphi}(p)\hat{\chi}_>(p) + T_{1;\lambda}^*\mathcal{E}_{\mathbb{Q}_2^{\uparrow\downarrow}}T_{1;\lambda},$$

with $\mathcal{E}_{\mathbb{Q}_2^{\uparrow\downarrow}}$ such that for any $\psi \in \mathcal{F}_\mathrm{f}$

$$|\langle\psi, \mathcal{E}_{\mathbb{Q}_2^{\uparrow\downarrow}}\psi\rangle| \le C\varrho^{\frac{5}{6}+\frac{\gamma}{2}}\|\mathbb{Q}_4^{\frac{1}{2}}\psi\|\|\mathcal{N}^{\frac{1}{2}}\psi\| + C\varrho\langle\psi, \mathcal{N}\psi\rangle. \tag{4.35}$$

Proof. As in the previous propositions, we have

$$\partial_\lambda T_{1;\lambda}^*\mathbb{Q}_2^{\uparrow\downarrow}T_{1;\lambda} = T_{1;\lambda}^*[\mathbb{Q}_2^{\uparrow\downarrow}, B_1]T_{1;\lambda} + \mathrm{h.c.}. \tag{4.36}$$

For the computation of $[\mathbb{Q}_2^{\uparrow\downarrow}, B_1]$ we shall use that

$$\mathbb{Q}_2^{\uparrow\downarrow} = \frac{1}{L^3}\sum_{k,s,s'}\hat{V}(k)\hat{v}_\uparrow(s')\hat{u}_\uparrow(s'-k)\hat{v}_\downarrow(s)\hat{u}_\downarrow(s+k)\hat{a}^*_{-s',\uparrow}\hat{a}^*_{s'-k,\uparrow}\hat{a}^*_{-s,\downarrow}\hat{a}^*_{s+k,\downarrow} + \mathrm{h.c.},$$

and hence we need

$$\begin{aligned}
&[\hat{a}^*_{-s',\uparrow}\hat{a}^*_{s'-k,\uparrow}\hat{a}^*_{-s,\downarrow}\hat{a}^*_{s+k,\downarrow}, \hat{a}_{r+p,\uparrow}\hat{a}_{-r,\uparrow}\hat{a}_{r'-p,\downarrow}\hat{a}_{-r',\downarrow}]\\
&= \hat{a}^*_{s'-k,\uparrow}\hat{a}^*_{s+k,\downarrow}\hat{a}_{r+p,\uparrow}\hat{a}_{r'-p,\downarrow}\Big(\delta_{s',r}\hat{a}^*_{-s,\downarrow}\hat{a}_{-r',\downarrow} + \delta_{s,r'}\hat{a}^*_{-s',\uparrow}\hat{a}_{-r,\uparrow} - \delta_{s',r}\delta_{s,r'}\Big)\\
&\quad - \Big(\delta_{s'-k,r+p}\hat{a}^*_{s+k,\downarrow}\hat{a}_{r'-p,\downarrow} + \delta_{s+k,r'-p}\hat{a}^*_{s'-k,\uparrow}\hat{a}_{r+p,\uparrow} - \delta_{s'-k,r+p}\delta_{s+k,r'-p}\Big)\hat{a}_{-r,\uparrow}\hat{a}_{-r',\downarrow}\hat{a}^*_{-s',\uparrow}\hat{a}^*_{-s,\downarrow}.
\end{aligned} \tag{4.37}$$

Correspondingly, from the six terms on the right-hand side we obtain $[\mathbb{Q}_2^{\uparrow\downarrow}, B_1] = \sum_{j=1}^{6} \mathrm{I}_j$, and we shall now estimate these terms. The main term comes from normal-ordering of I_6. As in the previous proofs, in our estimates we often use that $\|V\|_1 \le C$, together with the bounds proved in Lemma 3.1, Lemma A.2 and (4.14) with $n = 0$.

We start by considering I_1. It is convenient to rewrite the term partially in configuration space, as

$$\mathrm{I}_1 = -\frac{1}{L^3} \sum_r \hat{v}_\uparrow(r) \int dxdydz\, e^{ir\cdot(x-z)} V(x-y) a_\downarrow^*(u_y) a_\uparrow^*(u_x) a_\downarrow^*(v_y) b_\downarrow(\widetilde{\varphi}_z) a_\uparrow(u_z), \tag{4.38}$$

where we used again the operator b_σ introduced in (3.1). Using that $0 \le \hat{v}_\uparrow(r) \le 1$ and the Cauchy–Schwarz inequality, we get

$$\begin{aligned}
&|\langle \psi, \mathrm{I}_1 \psi\rangle| \\
&\le \left(\frac{1}{L^3} \sum_r \left\| \int dxdy\, V(x-y) e^{-ir\cdot x} a_\downarrow(v_y) a_\uparrow(u_x) a_\downarrow(u_y)\psi \right\|^2 \right)^{\frac{1}{2}} \left(\frac{1}{L^3} \sum_r \left\| \int dz\, e^{-ir\cdot z} b_\downarrow(\widetilde{\varphi}_z) a_\uparrow(u_z)\psi \right\|^2 \right)^{\frac{1}{2}}.
\end{aligned}$$

We can further bound

$$\begin{aligned}
&\frac{1}{L^3} \sum_r \left\| \int dxdy\, V(x-y) e^{-ir\cdot x} a_\downarrow(v_y) a_\uparrow(u_x) a_\downarrow(u_y)\psi \right\|^2 \\
&\le \int dxdydy' V(x-y)V(x-y') \|a_\downarrow(v_y) a_\uparrow(u_x) a_\downarrow(u_y)\psi\| \|a_\downarrow(v_{y'}) a_\uparrow(u_x) a_\downarrow(u_{y'})\psi\| \\
&\le C\varrho \int dxdydy'\, V(x-y)V(x-y') \|a_\uparrow(u_x) a_\downarrow(u_y)\psi\|^2 \le C\varrho \langle \psi, \mathbb{Q}_4 \psi\rangle,
\end{aligned}$$

where we used (4.14) and $\|V\|_1 \le C$. Using Lemma 3.1 we can also estimate

$$\frac{1}{L^3} \sum_r \left\| \int dz\, e^{-ir\cdot z} b_\downarrow(\widetilde{\varphi}_z) a_\uparrow(u_z)\psi \right\|^2 = \int dz\, \left\| b_\downarrow(\widetilde{\varphi}_z) a_\uparrow(u_z)\psi \right\|^2 \le C\varrho^{\frac{2}{3}+\gamma} \langle \psi, \mathcal{N}\psi\rangle.$$

In combination, we thus obtain

$$|\langle \psi, \mathrm{I}_1 \psi\rangle| \le C\varrho^{\frac{5}{6}+\frac{\gamma}{2}} \|\mathbb{Q}_4^{\frac{1}{2}}\psi\| \|\mathcal{N}^{\frac{1}{2}}\psi\|. \tag{4.39}$$

The term I_2 in can be estimated similarly, with the same outcome, and we omit the details.

Next we consider I_3. Similarly as for the other terms, we rewrite it in configuration space as

$$\langle \psi, \mathrm{I}_3 \psi\rangle = \int dxdydz'\, V(x-y) v_\downarrow(y;z') \left\langle a_\uparrow(u_x) a_\downarrow(u_y)\psi, \left(\int dz\, \widetilde{\varphi}(z-z') v_\uparrow(x;z) a_\uparrow(u_z) \right) a_\downarrow(u_{z'})\psi \right\rangle. \tag{4.40}$$

By the Cauchy–Schwarz's inequality, we have

$$\begin{aligned}
|\langle \psi, \mathrm{I}_3 \psi\rangle| &\le \left(\int dxdydz'\, V(x-y) |v_\downarrow(y;z')|^2 \|a_\uparrow(u_x) a_\downarrow(u_y)\psi\|^2 \right)^{\frac{1}{2}} \\
&\qquad \times \left(\int dxdydz'\, V(x-y) \left\| \int dz\, \widetilde{\varphi}(z-z') v_\uparrow(x;z) a_\uparrow(u_z) \right\|^2 \|a_\downarrow(u_{z'})\psi\|^2 \right)^{\frac{1}{2}}.
\end{aligned}$$

The first factor is bounded by $\|v_\downarrow\|_2\|\mathbb{Q}_4^{\frac{1}{2}}\psi\|$. To bound the second, we proceed as in (4.16), that is, using (4.14) and $0 \le \hat{u}_\sigma \le 1$, to obtain

$$\begin{aligned}|\langle\psi, \mathrm{I}_3\psi\rangle| &\le \|v_\downarrow\|_2\|\mathbb{Q}_4^{\frac{1}{2}}\psi\|\left(\int dxdydz'dw\, V(x-y)|\widetilde{\varphi}(z'-w)|^2|v_\uparrow(x;w)|^2\|a_\downarrow(u_{z'})\psi\|^2\right)^{\frac{1}{2}}\\ &\le \|V\|_1^{\frac{1}{2}}\|v_\uparrow\|_2\|v_\downarrow\|_2\|\widetilde{\varphi}\|_2\|\mathbb{Q}_4^{\frac{1}{2}}\psi\|\|\mathcal{N}^{\frac{1}{2}}\psi\| \le C\varrho^{\frac{5}{6}+\frac{\gamma}{2}}\|\mathbb{Q}_4^{\frac{1}{2}}\psi\|\|\mathcal{N}^{\frac{1}{2}}\psi\|.\end{aligned}$$

Here we have used Lemma A.2 in the last step.

The next term we consider is I_4. Before estimating it, we note that from the constraint $|p| \ge 4\varrho^{1/3-\gamma}$ and $|r| \le k_F^\uparrow$, we get that $|r+p| > 3\varrho^{1/3-\gamma}$. Hence $\hat{u}_\uparrow(r+p) = 1$. Rewriting I_4 in configuration space, we then get

$$\mathrm{I}_4 = \int dxdydz\, V(x-y)\widetilde{\varphi}(y-z)a_\downarrow^*(u_x)a_\downarrow(u_z)a_\uparrow(v_y)a_\downarrow(v_z)a_\downarrow^*(v_x)a_\uparrow^*(v_y). \tag{4.41}$$

Hence, using again Lemma A.2 and the fact that $a_\downarrow(u_z)$ commutes with $a_\uparrow(v_y)a_\downarrow(v_z)a_\downarrow^*(v_x)a_\uparrow^*(v_y)$,

$$\begin{aligned}|\langle\psi, \mathrm{I}_4\psi\rangle| &\le C\varrho^2\int dxdydz\, V(x-y)|\widetilde{\varphi}(y-z)|\|a_\downarrow(u_x)\psi\|\|a_\downarrow(u_z)\psi\|\\ &\le C\varrho^2\|V\|_1\|\widetilde{\varphi}\|_1\langle\psi, \mathcal{N}\psi\rangle \le C\varrho^{\frac{4}{3}+2\gamma}\langle\psi, \mathcal{N}\psi\rangle.\end{aligned} \tag{4.42}$$

The term I_5 can be estimated in the same way, obtaining the same bound.

Finally, we consider the term I_6, from which we extract the leading contribution. We start by normal-ordering the terms, using

$$\hat{a}_{-r,\uparrow}\hat{a}_{-r',\downarrow}\hat{a}^*_{-s',\uparrow}\hat{a}^*_{-s,\downarrow} = -\left(\hat{a}^*_{-s',\uparrow}\hat{a}_{-r,\uparrow} - \delta_{r,s'}\right)\left(\hat{a}^*_{-s,\downarrow}\hat{a}_{-r',\downarrow} - \delta_{s,r'}\right). \tag{4.43}$$

This way, we obtain

$$\begin{aligned}\mathrm{I}_6 = &-\varrho_\uparrow\varrho_\downarrow\sum_p \hat{V}(p)\hat{\varphi}(p)\widehat{\chi}_>(p)\\ &+\frac{1}{L^3}\sum_{p,r}\hat{V}(p)\hat{\varphi}(p)\widehat{\chi}_>(p)\left(\varrho_\downarrow\hat{v}_\uparrow(r)\hat{a}^*_{r,\uparrow}\hat{a}_{r,\uparrow} + \varrho_\uparrow\hat{v}_\downarrow(r)\hat{a}^*_{r,\downarrow}\hat{a}_{r,\downarrow}\right)\\ &+\frac{1}{L^6}\sum_{k,p,r,r'}\hat{V}(k)\hat{\varphi}(p)\widehat{\chi}_>(p)\hat{v}_\uparrow(k+r+p)\hat{v}_\downarrow(r'-p-k)\hat{v}_\uparrow(r)\hat{v}_\downarrow(r')\hat{a}^*_{-k-r-p,\uparrow}\hat{a}^*_{p+k-r',\downarrow}\hat{a}_{-r,\uparrow}\hat{a}_{-r',\downarrow},\end{aligned} \tag{4.44}$$

where we have used again that $\hat{u}_\uparrow(r+p) = \hat{u}_\downarrow(r'-p) = 1$ in all the summands. The main contribution is the constant first term. Since

$$\left|\frac{1}{L^3}\sum_p \hat{V}(p)\hat{\varphi}(p)\widehat{\chi}_>(p)\right| = \left|\int_\Lambda dx\, V(x)\widetilde{\varphi}(x)\right| \le \|V\|_{L^1(\Lambda)}\|\widetilde{\varphi}\|_{L^\infty(\Lambda)} \le C,$$

the term in the second line is bounded by $C\varrho\mathcal{N}$. To bound the term in the last line in (4.44), we rewrite it in configuration space as

$$-\int dxdy\, V(x-y)\widetilde{\varphi}(x-y)a_\downarrow^*(v_x)a_\uparrow^*(v_y)a_\uparrow(v_y)a_\downarrow(v_x). \tag{4.45}$$

Hence also this term is bounded by $C\varrho\mathcal{N}$, using that $\|V\widetilde{\varphi}\|_1 \le \|V\|_1\|\widetilde{\varphi}\|_\infty \le C$ because of (A.2). Combining all the bounds the proof of Proposition 4.6 is complete. $\square$

4.4 | Renormalization of $\mathbb{Q}_2^{\uparrow\downarrow}$ by Scattering Equation

In this subsection we will combine the "$bb + b^*b^*$ operators" from Propositions 4.2 and 4.5 with $\mathbb{Q}_2^{\uparrow\downarrow}$ in order to obtain a renormalized interaction. Recall the definitions of $\mathbb{T}_1$ and $\mathbb{T}_2$ in (4.4) and in (4.30), as well as the ones of $\mathbb{Q}_{2;<}$ and $\widetilde{\mathbb{Q}}_{2;<}$ in (2.28) and (2.30). Note also that, as all these operators, $\mathbb{Q}_2^{\uparrow\downarrow}$ in (2.11) is quadratic in the b's, namely,

$$\mathbb{Q}_2^{\uparrow\downarrow} = \frac{1}{L^3}\sum_p \hat{V}(p) b_{p,\uparrow} b_{-p,\downarrow} + \text{h.c.}.$$

Proposition 4.7 (Scattering equation cancellation). *Let $\gamma \in (0, 1/3)$. Under the same assumptions of Theorem 1.2,*

$$\mathbb{T}_1 + \mathbb{T}_2 + \mathbb{Q}_2^{\uparrow\downarrow} = \mathbb{Q}_{2;<} + \widetilde{\mathbb{Q}}_{2;<} + \mathcal{E}_{\text{scatt}}, \tag{4.46}$$

where $\mathcal{E}_{\text{scatt}}$ is such that for any normalized $\psi \in \mathcal{F}_{\text{f}}$

$$|\langle \psi, \mathcal{E}_{\text{scatt}}\psi\rangle| \le C\left(\varrho^{\frac{5}{3}-2\gamma} + o(1)_{L\to\infty}\right)\langle\psi, \mathcal{N}\psi\rangle + CL^{\frac{3}{2}}\left(\varrho^{\frac{13}{6}-\frac{7}{2}\gamma} + o(1)_{L\to\infty}\right)\|\mathcal{N}^{\frac{1}{2}}\psi\|. \tag{4.47}$$

Proof. Using $\hat{\chi}_> + \hat{\chi}_< = 1$, a simple computation yields

$$\mathcal{E}_{\text{scatt}} = \frac{1}{L^3}\sum_p \hat{h}(p) b_{p,\uparrow} b_{-p,\downarrow} + \text{h.c.},$$

with

$$\hat{h}(p) = \hat{V}(p) - (\hat{V} *_\Sigma \hat{\varphi})(p) - 8\pi a \hat{\chi}_<(p) - 2p^2\hat{\varphi}(p)\hat{\chi}_>(p). \tag{4.48}$$

In the following we will show that, as a consequence of the zero-energy scattering equation (1.5), $\hat{h}$ is suitably small (for large L and small ϱ). We shall write $\hat{h} = \hat{h}_1 + \hat{h}_2$, with

$$\hat{h}_1(p) = \left(\hat{V}(p) - (\hat{V} *_\Sigma \hat{\varphi})(p) - 2p^2\hat{\varphi}(p)\right)\hat{\chi}_>(p). \tag{4.49}$$

According to (1.5), $\hat{V}(p) - 2p^2\hat{\varphi}(p) = (\mathcal{F}(V_\infty\varphi_\infty)(p)$, hence $\hat{h}_1$ takes into account the finite-size effect of the periodic box, and vanishes in the thermodynamic limit. In Lemma A.4, we shall in fact show that the periodic function h_1 with Fourier coefficients $\hat{h}_1$ satisfies $\|h_1\|_{L^1(\Lambda)} \to 0$ and $\|h_1\|_{L^2(\Lambda)} \to 0$ as $L \to \infty$.
In configuration space we can write

$$\text{I} := \frac{1}{L^3}\sum_p \hat{h}_1(p) b_{p,\uparrow} b_{-p,\downarrow} = \int dxdy\, h_1(x-y) a_\uparrow(u_x) a_\uparrow(v_x) a_\downarrow(u_y) a_\downarrow(v_y) = \int dy\, b_\uparrow((h_1)_y) a_\downarrow(u_y) a_\downarrow(v_y),$$

where we used the notation b_σ introduced in (3.1). With the aid of Lemma 3.2 and the Cauchy–Schwarz inequality, we can estimate

$$\begin{aligned}
|\langle\psi, \text{I}\psi\rangle| &\le \|\hat{v}_\downarrow\|_2\left(\int dy\, \|b_\uparrow((h_1)_y)\psi\|\|a_\downarrow(u_y)\psi\| + CL^{\frac{3}{2}}\varrho^{\frac{1}{2}}\|h_1\|_2\|\mathcal{N}^{\frac{1}{2}}\psi\|\right) \\
&\le \|\hat{v}_\downarrow\|_2\|\hat{v}_\uparrow\|_2\int dxdy\, |h_1(x-y)|\|a_\uparrow(u_x)\psi\|\|a_\downarrow(u_y)\psi\| + CL^{\frac{3}{2}}\varrho\|h_1\|_2\|\mathcal{N}^{\frac{1}{2}}\psi\| \\
&\le C\varrho\|h_1\|_1\langle\psi, \mathcal{N}\psi\rangle + CL^{\frac{3}{2}}\varrho\|h_1\|_2\|\mathcal{N}^{\frac{1}{2}}\psi\|.
\end{aligned} \tag{4.50}$$

This term is thus $o(L^3)_{L\to\infty}$ in the thermodynamic limit.

We now consider the remaining term

$$\mathrm{II} := \frac{1}{L^3}\sum_p \hat{h}_2(p) b_{p,\uparrow} b_{-p,\downarrow}\,, \qquad \hat{h}_2(p) = \Big(\hat{V}(p) - (\hat{V} *_\Sigma \hat{\varphi})(p) - 8\pi a\Big)\hat{\chi}_<(p). \tag{4.51}$$

To start, we note that

$$\frac{1}{L^3}\sum_p \hat{h}_2(p) b_{p,\uparrow} b_{-p,\downarrow} = \frac{1}{L^3}\sum_{p,r,r'} \hat{h}_2(p)\hat{u}_\uparrow^<(r+p)\hat{u}_\downarrow^<(r'-p)\hat{v}_\uparrow(r)\hat{v}_\downarrow(r')\hat{a}_{r+p,\uparrow}\hat{a}_{-r,\uparrow}\hat{a}_{r'-p,\downarrow}\hat{a}_{-r',\downarrow},$$

that is, we can replace $\hat{u}_\uparrow(r+p)$ and $\hat{u}_\downarrow(r'-p)$ by $\hat{u}_\uparrow^<(r+p)$ and $\hat{u}_\downarrow^<(r'-p)$, with $\hat{u}_\sigma^<$ defined in (2.25), due to the support properties of $\hat{\chi}_<$ and $\hat{v}_\sigma$. Using also that

$$\sum_r \hat{u}_\sigma^<(r+p)\hat{v}_\sigma(r)\hat{a}_{r+p,\sigma}\hat{a}_{-r,\sigma} = \int dx\, e^{-ip\cdot x} a_\sigma(u_x^<) a_\sigma(v_x),$$

we can write

$$\langle\psi, \mathrm{II}\psi\rangle = \frac{1}{L^3}\sum_p \hat{h}_2(p)\Big\langle\Big(\int dx\, e^{-ip\cdot x} a_\uparrow^*(u_x^<) a_\uparrow^*(v_x)\psi\Big), \Big(\int dy\, e^{-ip\cdot y} a_\downarrow(v_y) a_\downarrow(u_y^<)\psi\Big)\Big\rangle,$$

and hence, by the Cauchy–Schwarz inequality,

$$|\langle\psi, \mathrm{II}\psi\rangle| \le \|\hat{h}_2\|_\infty \Big(\int dx\, \|a_\uparrow^*(u_x^<) a_\uparrow^*(v_x)\psi\|^2\Big)^{1/2}\Big(\int dy\, \|a_\downarrow(v_y) a_\downarrow(u_y^<)\psi\|^2\Big)^{1/2}.$$

The last factor can simply be bounded by $\varrho^{\frac{1}{2}}\langle\psi, \mathcal{N}\psi\rangle^{1/2}$. In the first factor, we first normal and write $a_\uparrow(u_x^<)a_\uparrow^*(u_x^<) = -a_\uparrow^*(u_x^<)a_\uparrow(u_x^<) + \|u^<\|_2^2$, leading to the bound $\varrho^{\frac{1}{2}}\langle\psi, \mathcal{N}\psi\rangle^{1/2} + \varrho^{\frac{1}{2}}\|u^<\|_2 L^{3/2}$. Since $\|u^<\|_2^2 \le C\varrho^{1-3\gamma}$, this yields

$$|\langle\psi, \mathrm{II}\psi\rangle| \le C\|\hat{h}_2\|_\infty\Big(\varrho\langle\psi, \mathcal{N}\psi\rangle + \varrho^{\frac{3}{2}(1-\gamma)} L^{3/2}\langle\psi, \mathcal{N}\psi\rangle^{1/2}\Big).$$

We are left with bounding $\sup_p |\hat{h}_2(p)|$. Note that $8\pi a = \int_{\mathbb{R}^3} V_\infty(1-\varphi_\infty) = \hat{V}(0) - \int_{\mathbb{R}^3} V_\infty\varphi_\infty$. Since V_∞ is assumed to have compact support, $|\hat{V}(p) - \hat{V}(0)| \le C|p|^2$. Using $V(x) = V(-x)$ and also (A.4), we see that

$$(\hat{V} *_\Sigma \hat{\varphi})(p) = \int_\Lambda dx\, V(x)\varphi(x)e^{ip\cdot x} = \int_{\mathbb{R}^3} V_\infty(x)\varphi_\infty(x)e^{ip\cdot x} + o(1)_{L\to\infty}, \tag{4.52}$$

and hence $|(\hat{V} *_\Sigma \hat{\varphi})(p) - \int_{\mathbb{R}^3} V_\infty\varphi_\infty| \le C|p|^2 + o(1)_{L\to\infty}$. Here note that φ_∞ is bounded by Lemma A.1. Altogether, this shows that $|\hat{h}_2(p)| \le C|p|^2|\hat{\chi}_<(p)| + o(1)_{L\to\infty} \le C\varrho^{2/3-2\gamma} + o(1)_{L\to\infty}$, and thus completes the proof. □

Proposition 4.8 (Propagation estimates for $\mathbb{Q}_{2;<}$ and $\widetilde{\mathbb{Q}}_{2;<}$). *Let $\lambda \in [0,1]$ and $\gamma \in (0, 1/3)$. Under the same assumptions as in Theorem 1.2*

$$T_{1;\lambda}^*(\mathbb{Q}_{2;<} + \widetilde{\mathbb{Q}}_{2;<})T_{1;\lambda} = \mathbb{Q}_{2;<} + \widetilde{\mathbb{Q}}_{2;<} - 2\lambda\varrho_\uparrow\varrho_\downarrow\sum_{p\in\Lambda^*}\Big(8\pi a\hat{\chi}_<(p) + \hat{V} *_\Sigma (\hat{\chi}_<\hat{\varphi})(p)\Big)\hat{\varphi}(p)\hat{\chi}_>(p) \tag{4.53}$$

$$+ \int_0^\lambda d\lambda'\, T_{1;\lambda'}^*\mathcal{E}_{\mathbb{Q}_{2;<}+\widetilde{\mathbb{Q}}_{2;<}}T_{1;\lambda'}, \tag{4.54}$$

where the error term $\mathcal{E}_{\mathbb{Q}_{2;<}+\widetilde{\mathbb{Q}}_{2;<}}$ is such that for any $\psi \in \mathcal{F}_\mathrm{f}$

$$|\langle\psi, \mathcal{E}_{\mathbb{Q}_{2;<}+\widetilde{\mathbb{Q}}_{2;<}}\psi\rangle| \le C\varrho^{\frac{4}{3}-\gamma}\langle\psi, \mathcal{N}\psi\rangle + C\varrho^{\frac{7}{6}-\frac{\gamma}{2}}\|\mathbb{Q}_4^{\frac{1}{2}}\psi\|\|\mathcal{N}^{\frac{1}{2}}\psi\|.$$

Furthermore, for any normalized $\psi \in \mathcal{F}_\mathrm{f}$, we have

$$|\langle\psi, \widetilde{\mathbb{Q}}_{2;<}\psi\rangle| \le CL^{3/2}\varrho^{\frac{4}{3}-\gamma}\|\mathbb{Q}_4^{\frac{1}{2}}\psi\|, \tag{4.55}$$

$$|\langle\psi, \mathbb{Q}_{2;<}\psi\rangle| \le CL^{\frac{3}{2}}\varrho^{\frac{3}{2}(1-\gamma)}\|\mathcal{N}^{\frac{1}{2}}\psi\|. \tag{4.56}$$

Proof of Proposition 4.8. By Duhamel's formula,

$$T^*_{1;\lambda}(\mathbb{Q}_{2;<} + \widetilde{\mathbb{Q}}_{2;<})T_{1;\lambda} = \mathbb{Q}_{2;<} + \widetilde{\mathbb{Q}}_{2;<} + \int_0^\lambda d\lambda'\, T^*_{1;\lambda'}[\mathbb{Q}_{2;<} + \widetilde{\mathbb{Q}}_{2;<}, B_1 - B_1^*]T_{1;\lambda'}. \tag{4.57}$$

We start by computing $[\mathbb{Q}_{2;<}, B_1 - B_1^*]$. The terms have the same structure as in Proposition 4.6, the only difference is that $\hat{V}(k)$ has to be replaced by $8\pi a\widehat{\chi}_<(k)$ everywhere. We again write them as

$$[\mathbb{Q}_{2;<}, B_1 - B_1^*] = \sum_{j=1}^{6} \mathrm{I}_j + \text{h.c.}.$$

As in Proposition 4.6, putting I_6 in normal order, we get the constant term. All the other terms are error terms, which we estimate in expectation in a state $\psi \in \mathcal{F}_\mathrm{f}$. In some error terms it will again be convenient to replace $\hat{u}_\sigma$ by $\hat{u}_\sigma^<$, with $\hat{u}_\sigma^<$ defined in (2.25), which is possible by the compact support of $\widehat{\chi}_<$. In the estimates below, we often use the bounds in Lemma 3.1, Lemma A.2 and Lemma A.5 together with (2.1) and (4.14).

For I_1, we have, analogously to (4.38),

$$\mathrm{I}_1 = -\frac{8\pi a}{L^3}\sum_r \hat{v}_\uparrow(r)\int dxdydz\, e^{ir\cdot(x-z)}\chi_<(x-y)a^*_\downarrow(u^<_y)a^*_\uparrow(u^<_x)a^*_\downarrow(v_y)b_\downarrow(\widetilde{\varphi}_z)a_\uparrow(u_z). \tag{4.58}$$

Using $0 \le v_\uparrow(r) \le 1$ and the Cauchy–Schwarz inequality, we can bound

$$\begin{aligned}|\langle\psi, \mathrm{I}_1\psi\rangle| &\le 8\pi a\left(\int dx\left\|\int dy\,\chi_<(x-y)a_\downarrow(v_y)a_\uparrow(u^<_x)a_\downarrow(u^<_y)\psi\right\|^2\right)^{1/2}\left(\int dz\|b_\downarrow(\widetilde{\varphi}_z)a_\uparrow(u_z)\psi\|^2\right)^{1/2}\\ &\le 8\pi a\|\hat{v}_\downarrow\|_2\|\hat{u}^<_\downarrow\|_2\|\chi_<\|_1\varrho^{\frac{1}{3}+\frac{\gamma}{2}}\langle\psi, \mathcal{N}\psi\rangle \le C\varrho^{\frac{4}{3}-\gamma}\langle\psi, \mathcal{N}\psi\rangle.\end{aligned} \tag{4.59}$$

The term I_2 can be estimated in the same way, with the same result.

Next we consider I_3. In analogy with (4.40), it is given by

$$\langle\psi, \mathrm{I}_3\psi\rangle = 8\pi a\int dxdydz'\,\chi_<(x-y)v_\downarrow(y;z')\left\langle a_\uparrow(u^<_x)a_\downarrow(u^<_y)\psi, \left(\int dz\,\widetilde{\varphi}(z-z')v_\uparrow(x;z)a_\uparrow(u_z)\right)a_\downarrow(u_{z'})\psi\right\rangle.$$

Using (4.16) and the Cauchy–Schwarz inequality, we obtain the bound

$$\begin{aligned}|\langle\psi, \mathrm{I}_3\psi\rangle| &\le 8\pi a\left(\int dxdydz'|\chi_<(x-y)||v_\downarrow(y;z')|^2\|a_\uparrow(u^<_x)a_\downarrow(u^<_y)\psi\|^2\right)^{\frac{1}{2}}\\ &\quad\times\left(\int dxdydz'dw\,|\chi_<(x-y)||\widetilde{\varphi}(z'-w)|^2|v_\uparrow(x;w)|^2\|a_\downarrow(u_{z'})\psi\|^2\right)^{\frac{1}{2}}\\ &\le 8\pi a\|\chi_<\|_1\|v_\uparrow\|_2\|v_\downarrow\|_2\|u^<_\uparrow\|_2\|\widetilde{\varphi}\|_2\langle\psi, \mathcal{N}\psi\rangle \le C\varrho^{\frac{4}{3}-\gamma}\langle\psi, \mathcal{N}\psi\rangle,\end{aligned} \tag{4.60}$$

where we used the bounds mentioned at the beginning of the proof.

The term I_4 can be written, similarly to (4.41), as

$$\mathrm{I}_4 = 8\pi a\int dxdydz\,\chi_<(x-y)\widetilde{\varphi}(y-z)a^*_\downarrow(u_x)a_\downarrow(u_z)a_\uparrow(v_y)a_\downarrow(v_z)a^*_\downarrow(v_x)a^*_\uparrow(v_y).$$

Hence

$$|\langle \psi, \mathrm{I}_4 \psi \rangle| \leq C\varrho^2 \|\chi_<\|_1 \|\widetilde{\varphi}\|_1 \langle \psi, \mathcal{N}\psi\rangle = C\varrho^{\frac{4}{3}+2\gamma} \langle \psi, \mathcal{N}\psi\rangle, \tag{4.61}$$

where used that $a_\downarrow(u_x)$ commutes with $a_\uparrow(v_y)a_\downarrow(v_z)a^*_\downarrow(v_x)a^*_\uparrow(v_y)$, as well as Lemmas A.2 and A.5 in the last step. The term I_5 can be estimated in the same way, with the same result.

Finally, we consider the last term I_6 coming from the commutator $[\mathbb{Q}_{2;<}, B_1 - B_1]$, which equals (4.44) with $\hat{V}$ replaced by $8\pi a \widehat{\chi}_<$. The term in the first line is the desired constant. For the term in the second line, we use

$$\left| \frac{1}{L^3} \sum_p \widehat{\chi}_<(p)\hat{\varphi}(p)\widehat{\chi}_>(p) \right| \leq \frac{C}{L^3} \sum_{p\neq 0} \frac{\widehat{\chi}_<(p)}{|p|^2} \leq C\varrho^{\frac{1}{3}-\gamma}, \tag{4.62}$$

which follows from the fact that $p^2\hat{\varphi}(p)$ is bounded, as a consequence of (1.5). Hence the expectation value of this term is bounded by $C\varrho^{\frac{4}{3}-\gamma}\langle \psi, \mathcal{N}\psi\rangle$. In the last term corresponding to the third line of (4.44), we observe that the summation over p is restricted to $|p| \leq 7\varrho^{1/3-\gamma}$, due to the constraints on k, r, r' and $r+p+k, r'-p-k$ in the summands. In other words, we can add for free the cut-off function $\widetilde{\chi}$ given by

$$\widetilde{\chi}(p) = \begin{cases} 1 & \text{if } |p| \leq 7\varrho^{\frac{1}{3}-\gamma}, \\ 0 & \text{if } |p| > 7\varrho^{\frac{1}{3}-\gamma}, \end{cases}$$

arriving at

$$\frac{8\pi a}{L^6} \sum_{k,p,r,r'} \widehat{\chi}_<(k)\hat{\varphi}(p)\widehat{\chi}_>(p)\widetilde{\chi}(p)\hat{v}_\uparrow(r)\hat{v}_\downarrow(r')\hat{v}_\uparrow(k+r+p)\hat{v}_\downarrow(r'-p-k)\hat{a}^*_{-k-r-p,\uparrow}\hat{a}^*_{p+k-r',\downarrow}\hat{a}_{-r,\uparrow}\hat{a}_{-r',\downarrow}.$$

Introducing the function $\widetilde{\varphi}_<$ with Fourier coefficients $\widehat{\chi}_>(p)\hat{\varphi}(p)\widetilde{\chi}(p)$ and rewriting the term in configuration space as in (4.45), we obtain the bound $C\varrho\|\chi_<\widetilde{\varphi}_<\|_1\mathcal{N}$. We have $\|\chi_<\widetilde{\varphi}_<\|_1 \leq \|\chi_<\|_1\|\widetilde{\varphi}_<\|_\infty \leq C\varrho^{\frac{1}{3}-\gamma}$ using the same bound as in (4.62). Altogether, we have thus shown that

$$[\mathbb{Q}_{2;<}, B_1 - B_1^*] = -16\pi a\varrho_\uparrow\varrho_\downarrow \sum_p \widehat{\chi}_<(p)\hat{\varphi}(p)\widehat{\chi}_>(p) + \mathcal{E}_{\mathbb{Q}_{2;<}},$$

with $\mathcal{E}_{\mathbb{Q}_{2;<}}$ such that

$$|\langle \psi, \mathcal{E}_{\mathbb{Q}_{2;<}}\psi\rangle| \leq C\varrho^{\frac{4}{3}-\gamma}\langle \psi, \mathcal{N}\psi\rangle.$$

We now consider $[\widetilde{\mathbb{Q}}_{2;<}, B_1 - B_1^*]$. Again all the terms in the commutator are of the same type as the ones in $[\mathbb{Q}_2, B_1 - B_1^*]$ in Proposition 4.6, now with $\hat{V}$ replaced by $\hat{V} *_\Sigma (\widehat{\chi}_<\hat{\varphi})$. They can then be treated exactly in the same way as the corresponding error terms in Proposition 4.6, using that

$$\sup_{x\in\Lambda} \left| \frac{1}{L^3} \sum_p \hat{\varphi}(p)\widehat{\chi}_<(p)e^{ip\cdot x} \right| \leq C\varrho^{\frac{1}{3}-\gamma}, \tag{4.63}$$

as in (4.62). We thus get that

$$[\widetilde{\mathbb{Q}}_{2;<}, B_1 - B_1^*] = -2\varrho_\uparrow\varrho_\downarrow \sum_{p\in\Lambda^*} \hat{V} *_\Sigma (\widehat{\chi}_<\hat{\varphi})(p)\hat{\varphi}(p)\widehat{\chi}_>(p) + \mathcal{E}_{\widetilde{\mathbb{Q}}_{2;<}},$$

with $\mathcal{E}_{\widetilde{\mathbb{Q}}_{2;<}}$ such that

$$|\langle \psi, \mathcal{E}_{\widetilde{\mathbb{Q}}_{2;<}}\psi\rangle| \leq C\varrho^{\frac{7}{6}-\frac{\gamma}{2}}\|\mathbb{Q}_4^{\frac{1}{2}}\psi\|\|\mathcal{N}^{\frac{1}{2}}\psi\| + C\varrho^{\frac{4}{3}-\gamma}\langle \psi, \mathcal{N}\psi\rangle.$$

This proves (4.54).

Finally, we discuss the bounds in (4.55) and (4.56). Rewriting $\widetilde{\mathbb{Q}}_{2;<}$ in configuration space, we have

$$\widetilde{\mathbb{Q}}_{2;<} = \int dxdy\, V(x-y)\varphi_<(x-y)a_\uparrow(u_x)a_\uparrow(v_x)a_\downarrow(u_y)a_\downarrow(v_y) + \text{h.c.},$$

where $\varphi_<$ is the function with Fourier coefficients $\hat{\varphi}(p)\widehat{\chi}_<(p)$. The bound in (4.55) is then a consequence of $\|\varphi_<\|_\infty \leq C\varrho^{1/3-\gamma}$ as shown in (4.63). Indeed, via the Cauchy–Schwarz inequality, we get

$$|\langle\psi, \widetilde{\mathbb{Q}}_{2;<}\psi\rangle| \leq C\varrho^{\frac{4}{3}-\gamma}\int dxdy\, V(x-y)\|a_\uparrow(u_x)a_\downarrow(u_y)\psi\| \leq CL^{3/2}\varrho^{\frac{4}{3}-\gamma}\|\mathbb{Q}_4^{\frac{1}{2}}\psi\|.$$

Similarly, we can write $\mathbb{Q}_{2;<}$ in configuration space as

$$\mathbb{Q}_{2;<} = 8\pi a\int dxdy\ \chi_<(x-y)a_\uparrow(u_x^<)a_\downarrow(u_y^<)a_\downarrow(v_y)a_\uparrow(v_x) + \text{h.c.},$$

where we again could replace $\hat{u}_\sigma$ by $\hat{u}_\sigma^<$ defined in (2.25). We then find

$$|\langle\psi, \mathbb{Q}_{2;<}\psi\rangle| \leq 16\pi a\|u_\uparrow^<\|_2\|v_\uparrow\|_2\|v_\downarrow\|_2\int dxdy\,|\chi_<(x-y)|\|a_\downarrow(u_y^<)\psi\| \leq CL^{\frac{3}{2}}\varrho^{\frac{3}{2}(1-\gamma)}\|\mathcal{N}^{\frac{1}{2}}\psi\|. \tag{4.64}$$

This completes the proof of Proposition 4.8. □

4.5 | Propagation of the Estimates for $\mathbb{H}_0$ and $\mathbb{Q}_4$

In this subsection we obtain propagation estimates for $\mathbb{H}_0$ and $\mathbb{Q}_4$, which will be useful to estimate some of the error terms.

Proposition 4.9 (Propagation of the estimates for $\mathbb{H}_0$ and $\mathbb{Q}_4$). *Let $\lambda \in [0,1]$, $\gamma \in (0, 1/6)$, $\delta \in (0, 8\gamma]$ with $2\gamma + \frac{\delta}{16} \leq \frac{1}{3}$. Under the same assumptions as in Theorem 1.2*

$$\langle T_{1;\lambda}T_2\Omega, (\mathbb{H}_0 + \mathbb{Q}_4)T_{1;\lambda}T_2\Omega\rangle \leq C\langle T_2\Omega, (\mathbb{H}_0 + \mathbb{Q}_4)T_2\Omega\rangle + CL^3\varrho^2 + o(L^3)_{L\to\infty}.$$

Note that since $\mathbb{H}_0$ and $\mathbb{Q}_4$ are nonnegative, the above estimate gives two separate bounds for $\langle T_{1;\lambda}T_2\Omega, \mathbb{H}_0T_{1;\lambda}T_2\Omega\rangle$ and $\langle T_{1;\lambda}T_2\Omega, \mathbb{Q}_4T_{1;\lambda}T_2\Omega\rangle$. The term $\langle T_2\Omega, (\mathbb{H}_0 + \mathbb{Q}_4)T_2\Omega\rangle$ on the right-hand side is actually small compared to $L^3\varrho^2$, as will be proved later in Propositions 5.2 and 5.3.

Proof. Let $\xi_\lambda = T_{1;\lambda}T_2\Omega$. We prove the propagation estimate by Grönwall's Lemma. By Propositions 4.2 and 4.5, we have

$$\partial_\lambda\langle\xi_\lambda, (\mathbb{H}_0 + \mathbb{Q}_4)\xi_\lambda\rangle = \langle\xi_\lambda, (\mathbb{T}_1 + \mathbb{T}_2 + \mathcal{E}_{\mathbb{H}_0} + \mathcal{E}_{\mathbb{Q}_4})\xi_\lambda\rangle, \tag{4.65}$$

where the four terms on the right-hand side are defined in (4.4), (4.30), (4.2) and (4.29), respectively. Equation (4.31) in Proposition 4.5 gives a bound on $\mathcal{E}_{\mathbb{Q}_4}$. For the term $\mathcal{E}_{\mathbb{H}_0}$ in (4.2), we first consider the terms involving $k\cdot s$ in (4.6) and write, as in the proof of Proposition 4.2, $k = (k+s) - s$. Correspondingly, writing the resulting expression in configuration space and recalling the definition of b_σ in (3.1), we find

$$-\frac{2}{L^3}\sum_{k,s,s'\in\Lambda^*} k\cdot s\hat{\varphi}(k)\widehat{\chi}_>(k)\, b_{k,s,\uparrow}b_{-k,s',\downarrow} = 2\sum_{\ell=1}^{3}\int dx\, b_\downarrow(\widetilde{\varphi}_x)\Big(a_\uparrow(\partial_\ell v_x)a_\uparrow(\partial_\ell \nu_x^>) - a_\uparrow(\partial_\ell^2 v_x)a_\uparrow(u_x)\Big), \tag{4.66}$$

where in the first term we again replaced $u_\uparrow$ by $\nu^>$ defined in (4.15), which is possible due the presence of the cutoff $\chi_>$. Using Lemma 3.1 together with the bounds in (4.14), we can estimate via the Cauchy–Schwarz inequality

$$\int dx\,\|b_\downarrow(\widetilde{\varphi}_x)a_\uparrow(\partial_j^2 v_x)a_\uparrow(u_x)\xi_\lambda\| \leq CL^{\frac{3}{2}}\varrho^{\frac{3}{2}+\frac{\gamma}{2}}\|\mathcal{N}^{\frac{1}{2}}\xi_\lambda\|. \tag{4.67}$$

For the first term on the right-hand side of (4.66), we can proceed as in (4.19) to bound

$$\int dx\, \|b_{\downarrow}(\widetilde{\varphi}_x)a_{\uparrow}(\partial_\ell v_x)a_{\uparrow}(\partial_\ell v_x^{>})\xi_\lambda\| \leq CL^{\frac{3}{2}}\varrho^{\frac{7}{6}+\frac{\gamma}{2}}\langle \xi_\lambda, \mathbb{H}_0\xi_\lambda\rangle^{1/2}. \tag{4.68}$$

For the terms involving $k \cdot s'$ in $\mathcal{E}_{\mathbb{H}_0}$ in (4.2) we can proceed in the same way, obtaining the same bound.

We now consider the expectation of $\mathbb{T}_1 + \mathbb{T}_2$ in (4.65). From Proposition 4.7, we know that $\mathbb{T}_1 + \mathbb{T}_2 = -\mathbb{Q}_2^{\uparrow\downarrow} + \mathbb{Q}_{2;<} + \widetilde{\mathbb{Q}}_{2;<} + \mathcal{E}_{\mathrm{scatt}}$, with $\mathcal{E}_{\mathrm{scatt}}$ bounded as in (4.47), and Equations (4.55) and (4.56) from Proposition 4.8 give a bound on $\mathbb{Q}_{2;<} + \widetilde{\mathbb{Q}}_{2;<}$. It thus remains to give a bound on $\mathbb{Q}_2^{\uparrow\downarrow}$. By the Cauchy–Schwarz inequality and (4.14),

$$|\langle \xi_\lambda, \mathbb{Q}_2^{\uparrow\downarrow}\xi_\lambda\rangle| \leq C\varrho \int dxdy\, V(x-y)\|a_{\uparrow}(u_x)a_{\downarrow}(u_y)\xi_\lambda\| \leq CL^{\frac{3}{2}}\varrho\|\mathbb{Q}_4^{\frac{1}{2}}\xi_\lambda\| \leq CL^3\varrho^2 + \langle \xi_\lambda, \mathbb{Q}_4\xi_\lambda\rangle,$$

where we used $V \geq 0$ and $\|V\|_{L^1(\Lambda)} \leq C$. Therefore,

$$\begin{aligned}|\langle \xi_\lambda, (\mathbb{T}_1 + \mathbb{T}_2)\xi_\lambda\rangle| &\leq CL^3\varrho^2 + C\langle \xi_\lambda, \mathbb{Q}_4\xi_\lambda\rangle + C\left(\varrho^{\frac{5}{3}-2\gamma} + o(1)_{L\to\infty}\right)\langle \xi_\lambda, \mathcal{N}\xi_\lambda\rangle \\ &\quad + CL^{\frac{3}{2}}\left(\varrho^{\frac{3}{2}(1-\gamma)} + \varrho^{\frac{13}{6}-\frac{7}{2}\gamma} + o(1)_{L\to\infty}\right)\|\mathcal{N}^{\frac{1}{2}}\xi_\lambda\|.\end{aligned} \tag{4.69}$$

Inserting the bounds (4.31), (4.67), (4.68) and (4.69) in (4.65), we find

$$\begin{aligned}\left|\partial_\lambda\langle \xi_\lambda, (\mathbb{H}_0 + \mathbb{Q}_4)\xi_\lambda\rangle\right| &\leq C\langle \xi_\lambda, (\mathbb{H}_0 + \mathbb{Q}_4)\xi_\lambda\rangle + CL^3\varrho^2 + C\left(\varrho^{\frac{5}{3}-2\gamma} + o(1)_{L\to\infty}\right)\langle \xi_\lambda, \mathcal{N}\xi_\lambda\rangle \\ &\quad + CL^{\frac{3}{2}}\left(\varrho^{\frac{3}{2}(1-\gamma)} + \varrho^{\frac{13}{6}-\frac{7}{2}\gamma} + o(1)_{L\to\infty}\right)\|\mathcal{N}^{\frac{1}{2}}\xi_\lambda\|.\end{aligned}$$

Using the estimate for the number operator in (3.12) together with the constraints on the parameters $\gamma < 1/3$, $\delta \leq 8\gamma$ and $2\gamma + \frac{\delta}{16} \leq \frac{1}{3}$, we conclude that

$$\left|\partial_\lambda\langle \xi_\lambda, (\mathbb{H}_0 + \mathbb{Q}_4)\xi_\lambda\rangle\right| \leq C\langle \xi_\lambda, (\mathbb{H}_0 + \mathbb{Q}_4)\xi_\lambda\rangle + CL^3\varrho^2 + o(L^3)_{L\to\infty}.$$

The desired result then follows from Grönwall's Lemma. $\square$

4.6 | Conclusion of Proposition 4.1

Now we have all the necessary ingredients in order to give the proof of Proposition 4.1. Recall the definition of $\mathcal{H}^{\mathrm{eff}}_{\mathrm{corr}} = \mathbb{H}_0 + \mathbb{Q}_2^{\uparrow\downarrow} + \mathbb{Q}_4$ in (2.12). We start by writing

$$\langle T_1T_2\Omega, (\mathbb{H}_0 + \mathbb{Q}_4)T_1T_2\Omega\rangle = \langle T_2\Omega, (\mathbb{H}_0 + \mathbb{Q}_4)T_2\Omega\rangle + \int_0^1 d\lambda\, \partial_\lambda\langle T_{1;\lambda}T_2\Omega, (\mathbb{H}_0 + \mathbb{Q}_4)T_{1;\lambda}T_2\Omega\rangle. \tag{4.70}$$

From Proposition 4.2 and Remark 4.3,

$$\begin{aligned}&\int_0^1 d\lambda\, \partial_\lambda\langle T_{1;\lambda}T_2\Omega, \mathbb{H}_0T_{1;\lambda}T_2\Omega\rangle \\ &\quad = \int_0^1 d\lambda\, \langle T_{1;\lambda}T_2\Omega, \mathbb{T}_1T_{1;\lambda}T_2\Omega\rangle + \langle T_2\Omega, \mathcal{E}_{\mathbb{H}_0}T_2\Omega\rangle + \int_0^1 d\lambda\,(1-\lambda)\partial_\lambda\langle T_{1;\lambda}T_2\Omega, \mathcal{E}_{\mathbb{H}_0}T_{1;\lambda}T_2\Omega\rangle,\end{aligned} \tag{4.71}$$

where $\mathbb{T}_1$ is defined in (4.4), and the last error term can be controlled by combining the bound (4.5) from Proposition 4.2 with Propositions 3.5 and 4.9 as

$$\left|\partial_\lambda \langle T_{1;\lambda} T_2 \Omega, \mathcal{E}_{\mathbb{H}_0} T_{1;\lambda} T_2 \Omega\rangle\right| \leq C\varrho^{1+\gamma} \|\mathbb{H}_0^{\frac{1}{2}} T_{1;\lambda} T_2 \Omega\| \|\mathcal{N}^{\frac{1}{2}} T_{1;\lambda} T_2 \Omega\| + C\varrho^{\frac{4}{3}+\gamma} \langle T_{1;\lambda} T_2 \Omega, \mathcal{N} T_{1;\lambda} T_2 \Omega\rangle$$
$$\leq C L^{3/2} \varrho^{\frac{17}{6}+\frac{\gamma}{2}-\frac{\delta}{16}} \|(\mathbb{H}_0 + \mathbb{Q}_4)^{\frac{1}{2}} T_2 \Omega\| + C L^3 \left(\varrho^{\frac{17}{6}+\frac{\gamma}{2}-\frac{\delta}{16}} + \varrho^{3-\frac{\delta}{8}}\right) + o(L^3)_{L\to\infty}. \tag{4.72}$$

From Proposition 4.5 we get

$$\int_0^1 d\lambda\, \partial_\lambda \langle T_{1;\lambda} T_2 \Omega, \mathbb{Q}_4 T_{1;\lambda} T_2 \Omega\rangle = \int_0^1 d\lambda\, \langle T_{1;\lambda} T_2 \Omega, \left(\mathbb{T}_2 + \mathcal{E}_{\mathbb{Q}_4}\right) T_{1;\lambda} T_2 \Omega\rangle, \tag{4.73}$$

where $\mathbb{T}_2$ is defined in (4.30) and the error term $\mathcal{E}_{\mathbb{Q}_4}$ can be controlled, again with Proposition 3.5 and 4.9, as

$$|\langle T_{1;\lambda} T_2 \Omega, \mathcal{E}_{\mathbb{Q}_4} T_{1;\lambda} T_2 \Omega\rangle| \leq C\varrho^{\frac{5}{6}+\frac{\gamma}{2}} \|\mathcal{N}^{\frac{1}{2}} T_{1;\lambda} T_2 \Omega\| \|\mathbb{Q}_4^{\frac{1}{2}} T_{1;\lambda} T_2 \Omega\|$$
$$\leq C\varrho^{\frac{5}{3}-\frac{\delta}{16}} L^{3/2} \left(\|(\mathbb{H}_0 + \mathbb{Q}_4)^{\frac{1}{2}} T_2 \Omega\| + L^{3/2} \varrho + o(L^{3/2})_{L\to\infty}\right). \tag{4.74}$$

The terms $\mathbb{T}_1$ and $\mathbb{T}_2$ can be combined as in Proposition 4.7 to $\mathbb{T}_1 + \mathbb{T}_2 = -\mathbb{Q}_2^{\uparrow\downarrow} + \mathbb{Q}_{2;<} + \widetilde{\mathbb{Q}}_{2;<} + \mathcal{E}_{\mathrm{scatt}}$ with $\mathbb{Q}_{2;<}$ and $\widetilde{\mathbb{Q}}_{2;<}$ defined in (2.28) and (2.30), and $\mathcal{E}_{\mathrm{scatt}}$ satisfying

$$|\langle T_{1;\lambda} T_2 \Omega, \mathcal{E}_{\mathrm{scatt}} T_{1;\lambda} T_2 \Omega\rangle| \leq C L^3 \varrho^{3-4\gamma-\frac{\delta}{16}} + o(L^3)_{L\to\infty}, \tag{4.75}$$

where we used the estimate in (4.47) together with Proposition 3.5. Therefore, combining (4.70)–(4.75), we obtain

$$\langle T_1 T_2 \Omega, \mathcal{H}_{\mathrm{corr}}^{\mathrm{eff}} T_1 T_2 \Omega\rangle = \langle T_2 \Omega, (\mathbb{H}_0 + \mathbb{Q}_4 + \mathcal{E}_{\mathbb{H}_0}) T_2 \Omega\rangle + \int_0^1 d\lambda\, \langle T_{1;\lambda} T_2 \Omega, (\mathbb{Q}_{2;<} + \widetilde{\mathbb{Q}}_{2;<}) T_{1;\lambda} T_2 \Omega\rangle$$
$$+ \langle T_1 T_2 \Omega, \mathbb{Q}_2^{\uparrow\downarrow} T_1 T_2 \Omega\rangle - \int_0^1 d\lambda\, \langle T_{1;\lambda} T_2 \Omega, \mathbb{Q}_2^{\uparrow\downarrow} T_{1;\lambda} T_2 \Omega\rangle + \mathcal{E}, \tag{4.76}$$

with

$$|\mathcal{E}| \leq C\varrho^{\frac{5}{3}-\frac{\delta}{16}} L^{3/2} \|(\mathbb{H}_0 + \mathbb{Q}_4)^{\frac{1}{2}} T_2 \Omega\| + C L^3 \varrho^{\frac{8}{3}-\frac{\delta}{16}} + C L^3 \varrho^{3-4\gamma-\frac{\delta}{16}} + o(L^3)_{L\to\infty}. \tag{4.77}$$

The terms involving $\mathbb{Q}_2^{\uparrow\downarrow}$ in the second line on the right-hand of (4.76) above can be written as

$$\langle T_1 T_2 \Omega, \mathbb{Q}_2^{\uparrow\downarrow} T_1 T_2 \Omega\rangle - \int_0^1 d\lambda\, \langle T_{1;\lambda} T_2 \Omega, \mathbb{Q}_2^{\uparrow\downarrow} T_{1;\lambda} T_2 \Omega\rangle = \int_0^1 d\lambda\, \lambda\, \partial_\lambda \langle T_{1;\lambda} T_2 \Omega, \mathbb{Q}_2^{\uparrow\downarrow} T_{1;\lambda} T_2 \Omega\rangle,$$

which, according to Proposition 4.6, equals

$$-\varrho_\uparrow \varrho_\downarrow \sum_{p\in\Lambda^*} \hat{V}(p) \hat{\varphi}(p) \hat{\chi}_>(p) + \int_0^1 d\lambda\, \lambda\, \langle T_{1;\lambda} T_2 \Omega, \mathcal{E}_{\mathbb{Q}_2^{\uparrow\downarrow}} T_{1;\lambda} T_2 \Omega\rangle, \tag{4.78}$$

with

$$|\langle T_{1;\lambda} T_2 \Omega, \mathcal{E}_{\mathbb{Q}_2^{\uparrow\downarrow}} T_{1;\lambda} T_2 \Omega\rangle| \leq C L^3 \varrho^{\frac{8}{3}-\gamma-\frac{\delta}{8}} + C\varrho^{\frac{5}{3}-\frac{\delta}{16}} L^{3/2} \|(\mathbb{H}_0 + \mathbb{Q}_4)^{\frac{1}{2}} T_2 \Omega\| + o(L^3)_{L\to\infty}, \tag{4.79}$$

for all $\lambda \in [0,1]$, where we used again Proposition 3.5 and Proposition 4.9. Next we consider the contribution coming from $\mathbb{Q}_{2;<} + \widetilde{\mathbb{Q}}_{2;<}$ in the first line on the right-hand side of (4.76). By Proposition 4.8,

$$\int_0^1 d\lambda\, \langle T_{1;\lambda}T_2\Omega, (\mathbb{Q}_{2;<} + \widetilde{\mathbb{Q}}_{2;<})T_{1;\lambda}T_2\Omega\rangle = \langle T_2\Omega, (\mathbb{Q}_{2;<} + \widetilde{\mathbb{Q}}_{2;<})T_2\Omega\rangle - 8\pi a \varrho_\uparrow \varrho_\downarrow \sum_{p\in\Lambda^*} \hat{\chi}_<(p)\hat{\varphi}(p)\hat{\chi}_>(p)$$

$$- \varrho_\uparrow\varrho_\downarrow \sum_{p\in\Lambda^*} \hat{V} *_\Sigma (\hat{\chi}_<\hat{\varphi})(p)\hat{\varphi}(p)\hat{\chi}_>(p) + \int_0^1 d\lambda\,(1-\lambda)\langle T_{1;\lambda}T_2\Omega, \mathcal{E}_{\mathbb{Q}_{2;<}+\widetilde{\mathbb{Q}}_{2;<}} T_{1;\lambda}T_2\Omega\rangle,$$

with

$$|\langle T_{1;\lambda}T_2\Omega, \mathcal{E}_{\mathbb{Q}_{2;<}+\widetilde{\mathbb{Q}}_{2;<}} T_{1;\lambda}T_2\Omega\rangle| \leq CL^3\varrho^{3-2\gamma-\frac{\delta}{8}} + CL^{3/2}\varrho^{2-\gamma-\frac{\delta}{16}}\|(\mathbb{H}_0+\mathbb{Q}_4)^{\frac{1}{2}}T_2\Omega\| + o(L^3)_{L\to\infty},$$

where we used again the bounds in Propositions 3.5 and 4.9. We can further bound the term $\widetilde{\mathbb{Q}}_{2;<}$ on the right hand side above using (4.55). Therefore, combining all the estimates, we proved that

$$\begin{aligned}
\langle T_1T_2\Omega, \mathcal{H}^{\mathrm{eff}}_{\mathrm{corr}} T_1T_2\Omega\rangle \leq{}& -\varrho_\uparrow\varrho_\downarrow \sum_{p\in\Lambda^*} \Big(\hat{V}(p) + \hat{V} *_\Sigma (\hat{\chi}_<\hat{\varphi})(p) + 8\pi a\hat{\chi}_<(p)\Big)\hat{\varphi}(p)\hat{\chi}_>(p) \\
&+ \langle T_2\Omega, \Big(\mathbb{H}_0 + \mathbb{Q}_4 + \mathbb{Q}_{2;<} + \mathcal{E}_{\mathbb{H}_0}\Big)T_2\Omega\rangle \\
&+ C\varrho^{\frac{5}{3}-\frac{\delta}{16}}L^{3/2}\|(\mathbb{H}_0+\mathbb{Q}_4)^{\frac{1}{2}}T_2\Omega\| + CL^{3/2}\varrho^{\frac{4}{3}-\gamma}\|\mathbb{Q}_4^{\frac{1}{2}}T_2\Omega\| \\
&+ CL^3\varrho^{\frac{8}{3}-\gamma-\frac{\delta}{8}} + CL^3\varrho^{3-4\gamma-\frac{\delta}{16}} + o(L^3)_{L\to\infty}.
\end{aligned} \tag{4.80}$$

To complete the proof of Proposition 4.1, we need the following Lemma, evaluating the constant contribution in the first line on the right-hand side of (4.80).

Lemma 4.10 (Constant contribution to $8\pi a\varrho_\uparrow\varrho_\downarrow$). *Let V be as in Assumption 1.1. Let $\hat{\chi}_<, \hat{\chi}_>$ be defined as in (2.16), with $0<\gamma<1/3$, and let φ be as in Definition (2.5). Then*

$$\begin{aligned}
&\frac{1}{L^3}\sum_{p\in\Lambda^*}\Big(\hat{V}(p) + \hat{V} *_\Sigma (\hat{\chi}_<\hat{\varphi})(p) + 8\pi a\hat{\chi}_<(p)\Big)\hat{\varphi}(p)\hat{\chi}_>(p) \\
&= \hat{V}(0) - 8\pi a - \frac{1}{L^3}\sum_{0\neq p\in\Lambda^*}\frac{(8\pi a)^2(\hat{\chi}_<(p))^2}{2|p|^2} + \mathcal{E}_{\mathrm{const}},
\end{aligned} \tag{4.81}$$

with $|\mathcal{E}_{\mathrm{const}}| \leq C\varrho^{\frac{2}{3}-2\gamma} + o(1)_{L\to\infty}$.

Proof. Using $\hat{\chi}_< + \hat{\chi}_> = 1$, we can rewrite the left-hand side of (4.81) (multiplied by L^3) as

$$\sum_p \hat{V}(p)\hat{\varphi}(p) - \sum_p \hat{h}_2(p)\hat{\varphi}(p) - \sum_p \hat{V} *_\Sigma (\hat{\chi}_<\hat{\varphi})(p)\hat{\varphi}(p)\hat{\chi}_<(p) - 8\pi a\sum_p \hat{\varphi}(p)(\hat{\chi}_<(p))^2, \tag{4.82}$$

where we used the definition of $\hat{h}_2(p) = (\hat{V}(p) - (\hat{V} *_\Sigma \hat{\varphi})(p) - 8\pi a)\hat{\chi}_<(p)$ in (4.51). As in (4.52), we have $L^{-3}\sum_p \hat{V}(p)\hat{\varphi}(p) = \int_{\mathbb{R}^3} V_\infty\varphi_\infty + o(1)_{L\to\infty} = \hat{V}(0) - 8\pi a + o(1)_{L\to\infty}$. At the end of the proof of Proposition 4.7, we also showed that $|\hat{h}_2(p)| \leq C|p|^2|\hat{\chi}_<(p)| + o(1)_{L\to\infty}$. The zero-energy scattering equation (1.5) implies that $p^2\hat{\varphi}(p)$ is bounded, hence

$$\left|\frac{1}{L^3}\sum_p \hat{h}_2(p)\hat{\varphi}(p)\right| \leq \frac{C}{L^3}\sum_p \hat{\chi}_<(p) + o(1)_{L\to\infty}\frac{1}{L^3}\sum_{0\neq|p|\leq 5\varrho^{1/3-\gamma}}\frac{1}{p^2} \leq C\varrho^{1-3\gamma} + o(1)_{L\to\infty}.$$

Using that $\|\hat{V}\|_\infty \le C$, the third term in (4.82) is bounded as

$$\left| \frac{1}{L^6} \sum_{q,p} \hat{V}(p-q)\hat{\chi}_<(q)\hat{\varphi}(q)\hat{\chi}_<(p)\hat{\varphi}(p) \right| \le C\left(\frac{1}{L^3} \sum_{p\neq 0} \frac{\hat{\chi}_<(p)}{|p|^2} \right)^2 \le C\varrho^{\frac{2}{3}-2\gamma}. \tag{4.83}$$

Finally, an argument as at the end of the proof of Proposition 4.7 shows at $|p^2\hat{\varphi}(p) - 4\pi a| \le Cp^2$, hence

$$\sum_p \hat{\varphi}(p)(\hat{\chi}_<(p))^2 = 4\pi a \sum_{p\neq 0} \frac{(\hat{\chi}_<(p))^2}{2|p|^2} + \mathcal{E},$$

with $|\mathcal{E}| \le C\sum_p \chi_<(p)^2 \le CL^3\varrho^{1-3\gamma}$. This completes the proof. $\square$

Proposition 4.1 follows from combining (4.80) with (4.81) and a simple Cauchy–Schwarz inequality.

5 | Second Quasi-Bosonic Bogoliubov Transformation T_2

In this section we complete the second step discussed in Section 2.4. After the conjugation with T_1, the new effective correlation operator is $\mathbb{H}_0 + \mathbb{Q}_{2;<}$. By conjugating this operator with T_2, we will extract the full constant of order $\varrho^{7/3}$. The main result of this section is a rigorous version of (2.35), plus estimates on the conjugations of $\mathbb{H}_0$, $\mathbb{Q}_4$ and $\mathcal{E}_{\mathbb{H}_0}$, which are needed to control the error terms in (4.1) from Proposition 4.1.

Recall that $\mathbb{H}_0$ and $\mathbb{Q}_4$ are defined in (2.10), while $\mathbb{Q}_{2;<}$ and $\mathcal{E}_{\mathbb{H}_0}$ are given in (2.28) and (4.2), respectively.

Proposition 5.1 (Conjugation by T_2). *Let $0 < \gamma < \frac{1}{6}$, $0 < \delta \le 8\gamma$ with $2\gamma + \frac{\delta}{16} \le \frac{1}{3}$. Then*

$$\langle T_2\Omega, (\mathbb{H}_0 + \mathbb{Q}_{2;<})T_2\Omega\rangle = -\frac{(8\pi a)^2}{L^6} \sum_{p,r,r'\in\Lambda^*} \frac{\hat{u}_\uparrow(r+p)\hat{u}_\downarrow(r'-p)\hat{v}_\uparrow(r)\hat{v}_\downarrow(r')}{|r+p|^2 - |r|^2 + |r'-p|^2 - |r'|^2 + 2\varepsilon}(\hat{\chi}_<(p))^2 + \mathcal{E}_{T_2}, \tag{5.1}$$

with

$$|\mathcal{E}_{T_2}| \le CL^3\left(\varrho^{\frac{7}{3}-\gamma+\frac{7}{8}\delta} + \varrho^{3-3\gamma-\frac{3}{16}\delta} \right).$$

Moreover,

$$\langle T_{2;\lambda}\Omega, \mathbb{H}_0 T_{2;\lambda}\Omega\rangle \le CL^3\varrho^{\frac{7}{3}-2\gamma-\frac{\delta}{16}}, \tag{5.2}$$

$$\langle T_{2;\lambda}\Omega, \mathbb{Q}_4 T_{2;\lambda}\Omega\rangle \le CL^3\varrho^{\frac{8}{3}-2\gamma}, \tag{5.3}$$

$$|\langle T_2\Omega, \mathcal{E}_{\mathbb{H}_0} T_2\Omega\rangle| \le CL^3\left(\varrho^{\frac{7}{3}+\gamma} + \varrho^{3-2\gamma-\frac{\delta}{4}} \right). \tag{5.4}$$

We will study the individual terms $\langle T_{2;\lambda}\Omega, \mathbb{H}_0 T_{2;\lambda}\Omega\rangle$, $\langle T_{2;\lambda}\Omega, \mathbb{Q}_4 T_{2;\lambda}\Omega\rangle$, $\langle T_2\Omega, \mathcal{E}_{\mathbb{H}_0} T_2\Omega\rangle$ and $\langle T_{2;\lambda}\Omega, \mathbb{Q}_{2;<} T_{2;\lambda}\Omega\rangle$ in Sections 5.1–5.4, and then complete the proof of Proposition 5.1 in Section 5.5. In the proof, it will be very helpful to use the configuration space representation of B_2 in (2.27).

5.1 | Conjugation of $\mathbb{H}_0$

In this subsection we investigate $T_2^*\mathbb{H}_0 T_2$, and in particular validate (2.34). Moreover, we also prove a rough estimate for $\langle T_2\Omega, \mathbb{H}_0 T_2\Omega\rangle$, which is helpful to control the corresponding error term in Proposition 4.1.

Proposition 5.2 (Conjugation of $\mathbb{H}_0$ by T_2). *Let $\lambda \in [0,1]$, $0<\gamma<\frac{1}{6}$, $0<\delta\le 8\gamma$ with $2\gamma+\frac{\delta}{16}\le\frac{1}{3}$. Under the same assumptions as in Theorem 1.2,*

$$\partial_\lambda T^*_{2;\lambda}\mathbb{H}_0 T_{2;\lambda} = -T^*_{2;\lambda}(\mathbb{Q}_{2;<}+\mathfrak{e}_{\mathbb{H}_0})T_{2;\lambda}, \tag{5.5}$$

where $\mathfrak{e}_{\mathbb{H}_0}$ is such that for any normalized $\psi\in\mathcal{F}_\mathrm{f}$

$$|\langle\psi,\mathfrak{e}_{\mathbb{H}_0}\psi\rangle| \le CL^{\frac{3}{2}}\varrho^{\frac{3}{2}-\frac{\gamma}{2}+\frac{15}{16}\delta}\|\mathcal{N}^{\frac{1}{2}}\psi\|. \tag{5.6}$$

Moreover, for any normalized $\psi\in\mathcal{F}_\mathrm{f}$

$$\langle T_{2;\lambda}\psi,\mathbb{H}_0T_{2;\lambda}\psi\rangle \le \langle\psi,\mathbb{H}_0\psi\rangle + CL^{\frac{3}{2}}\varrho^{\frac{3}{2}(1-\gamma)}\|\mathcal{N}^{\frac{1}{2}}\psi\|. \tag{5.7}$$

Applying the bounds to $\psi=T_{2;\lambda}\Omega$ and $\psi=\Omega$, respectively, we obtain from (3.12) (with $\lambda=0$)

$$|\langle T_{2;\lambda}\Omega,\mathfrak{e}_{\mathbb{H}_0}T_{2;\lambda}\Omega\rangle| \le CL^3\varrho^{\frac{7}{3}-\gamma+\frac{7}{8}\delta}, \qquad \langle T_{2;\lambda}\Omega,\mathbb{H}_0T_{2;\lambda}\Omega\rangle \le CL^3\varrho^{\frac{7}{3}-2\gamma-\frac{\delta}{16}}. \tag{5.8}$$

Proof. We have

$$\partial_\lambda T^*_{2;\lambda}\mathbb{H}_0T_{2;\lambda} = T^*_{2;\lambda}[\mathbb{H}_0, B_2-B_2^*]T_{2;\lambda'}. \tag{5.9}$$

An explicit computation, using that for $s+k\notin\mathcal{B}_F^\uparrow$, $s'-k\notin\mathcal{B}_F^\downarrow$, $s\in\mathcal{B}_F^\uparrow$, $s'\in\mathcal{B}_F^\downarrow$,

$$-\big(|s+k|^2-|s|^2+|s'-k|^2-|s'|^2\big)\hat\eta^\varepsilon_{s,s'}(k) = -8\pi a + 2\varepsilon\hat\eta^\varepsilon_{s,s'}(k),$$

by the definition of $\hat\eta^\varepsilon_{s,s'}$ in (2.17), gives $[\mathbb{H}_0,B_2-B_2^*]=-\mathbb{Q}_{2;<}+2\varepsilon(B_2+B_2^*)$. We have already estimated the expectation value of $B_2+B_2^*$ in the proof of Proposition 3.5. Recalling that $\varepsilon=\varrho^{\frac{2}{3}+\delta}$ and applying this bound, we directly obtain (5.6). To prove (5.7) we use Grönwall's Lemma together with the bound (4.56) on $\mathbb{Q}_{2;<}$. $\square$

5.2 | Conjugation of $\mathbb{Q}_4$

In this subsection we conjugate $\mathbb{Q}_4$ with respect to $T_{2;\lambda}$. We will show that $\langle T_{2;\lambda}\Omega,\mathbb{Q}_4T_{2;\lambda}\Omega\rangle$ does not contribute to the energy to order $\varrho^{7/3}$.

Proposition 5.3 (Conjugation of $\mathbb{Q}_4$ by T_2). *Let $\lambda\in[0,1]$, $0<\gamma<\frac{1}{6}$, $0<\delta\le 8\gamma$ with $2\gamma+\frac{\delta}{16}\le\frac{1}{3}$. Under the same assumptions as in Theorem 1.2*

$$\langle T_{2;\lambda}\Omega,\mathbb{Q}_4T_{2;\lambda}\Omega\rangle \le CL^3\varrho^{\frac{8}{3}-2\gamma}. \tag{5.10}$$

Proof. For the proof we find it convenient to split the operator $\mathbb{Q}_4$ according to the support of the momenta of each of the $\hat u_\sigma$. To make this precise, we introduce smooth functions $\hat\zeta^<,\hat\zeta^>:\mathbb{R}^3\to[0,1]$ with $\hat\zeta^<+\hat\zeta^>=1$ such that

$$\hat\zeta^<(k) := \begin{cases} 1 & \text{if } |k|\le 5\max\{k_F^\uparrow,k_F^\downarrow\}, \\ 0 & \text{if } |k|>6\max\{k_F^\uparrow,k_F^\downarrow\}. \end{cases} \tag{5.11}$$

We denote $\zeta^<:\Lambda\to\mathbb{R}$ the periodic function with Fourier coefficients $\hat\zeta^<(k)$ for $k\in(2\pi/L)\mathbb{Z}^3$, and also $\hat u_\sigma^<(k)=\hat u_\sigma(k)\hat\zeta^<_\uparrow(k)$, $\hat u_\sigma^>(k)=\hat u_\sigma(k)\hat\zeta^>_\uparrow(k)$. Accordingly, we split

$$\mathbb{Q}_4=\mathbb{Q}_4^>+\mathbb{Q}_4^<,$$

where

$$\mathbb{Q}_4^{>} = \sum_{\sigma,\sigma'} \int dxdy\, V(x-y) a_\sigma^*(u_x^>) a_{\sigma'}^*(u_y^>) a_{\sigma'}(u_y^>) a_\sigma(u_x^>),$$

and $\mathbb{Q}_4^<$ contains all the other possible terms. Specifically, $\mathbb{Q}_4^< = \sum_{j=0}^3 \mathbb{Q}_{4;j}^<$, where $\mathbb{Q}_{4;j}^<$ contains all the terms that have j functions $u_\sigma^>$ and $4-j$ functions $u_\sigma^<$.

Let $\xi_\lambda = T_{2;\lambda}\Omega$. Using that $\|a_\sigma(u_x^<)\| \le C\varrho^{1/2}$, it is easy to see that

$$|\langle \xi_\lambda, (\mathbb{Q}_{4;0}^< + \mathbb{Q}_{4;1}^< + \mathbb{Q}_{4;2}^{<,a})\xi_\lambda\rangle| \le C\varrho\langle \xi_\lambda, \mathcal{N}\xi_\lambda\rangle, \tag{5.12}$$

where $\mathbb{Q}_{4;2}^{<,a}$ denotes the part of $\mathbb{Q}_{4;2}^<$ with one $a_\sigma(u^>)$ and one $a_\sigma^*(u^>)$. The other contributions to $\mathbb{Q}_4^<$ can be bounded via the Cauchy–Schwarz inequality as

$$|\langle \xi_\lambda, (\mathbb{Q}_{4;2}^{<,b} + \mathbb{Q}_{4;3}^<)\xi_\lambda\rangle| \le C\varrho^{\frac{1}{2}} \|\mathcal{N}^{\frac{1}{2}}\xi_\lambda\| \|(\mathbb{Q}_4^>)^{\frac{1}{2}}\xi_\lambda\|. \tag{5.13}$$

In particular, from (5.12) and (5.13) we find that

$$|\langle \xi_\lambda, \mathbb{Q}_4^<\xi_\lambda\rangle| \le C\varrho\langle \xi_\lambda, \mathcal{N}\xi_\lambda\rangle + C\langle \xi_\lambda, \mathbb{Q}_4^>\xi_\lambda\rangle. \tag{5.14}$$

We are thus left with estimating $\mathbb{Q}_4^>$, and we will use Grönwall's Lemma. We start by computing

$$\partial_\lambda T_{2;\lambda}^* \mathbb{Q}_4^> T_{2;\lambda} = T_{2;\lambda}^*[\mathbb{Q}_4^>, B_2 - B_2^*] T_{2;\lambda}.$$

The terms we find from the commutator above are of the same type as the ones in Proposition 4.5. Explicitly, $[\mathbb{Q}_4^>, B_2 - B_2^*] = \mathrm{I}_1 + \mathrm{I}_2 + \mathrm{I}_3 + \mathrm{h.c.}$ with

$$\begin{aligned}\mathrm{I}_1 = \frac{8\pi a}{L^6}\sum_\sigma \sum_{k,p,r,r',s} &\hat V(k)\hat\chi_<(p)\hat\eta^\varepsilon_{r,r'}(p)\hat u_\uparrow^>(r+p)\hat u_\downarrow(r'-p)\hat u_\uparrow^>(r+p-k)\hat u_\sigma^>(s-k)\hat u_\sigma^>(s)\hat v_\uparrow(r)\hat v_\downarrow(r') \\ &\times \hat a^*_{s-k,\sigma}\hat a_{r'-p,\downarrow}\hat a_{-r,\uparrow}\hat a_{-r',\downarrow}\hat a_{s,\sigma}\hat a_{r+p-k,\uparrow},\end{aligned} \tag{5.15}$$

$$\begin{aligned}\mathrm{I}_2 = \frac{1}{L^6}\sum_\sigma \sum_{k,p,r,r',s} &\hat V(k)\hat\chi_<(p)\hat\eta^\varepsilon_{r,r'}(p)\hat u_\uparrow(r+p)\hat u_\downarrow^>(r'-p)\hat u_\downarrow^>(r'-p-k)\hat u_\sigma^>(s-k)\hat u_\sigma^>(s)\hat v_\uparrow(r)\hat v_\downarrow(r') \\ &\times \hat a^*_{s-k,\sigma}\hat a_{r+p,\uparrow}\hat a_{-r',\downarrow}\hat a_{-r,\uparrow}\hat a_{s,\sigma}\hat a_{r'-p-k,\downarrow},\end{aligned}$$

$$\begin{aligned}\mathrm{I}_3 = -\frac{1}{L^6}\sum_{k,q,r,r'} &\hat V(k)\hat\chi_<(p)\hat\eta^\varepsilon_{r,r'}(p)\hat u_\uparrow^>(r+p)\hat u_\downarrow^>(r'-p)\hat u_\uparrow^>(r+p+k)\hat u_\downarrow^>(r'-p-k)\hat v_\uparrow(r)\hat v_\downarrow(r') \\ &\times \hat a_{r+p+k,\uparrow}\hat a_{-r,\uparrow}\hat a_{r'-p-k,\downarrow}\hat a_{-r',\downarrow}.\end{aligned} \tag{5.16}$$

In what follows we will often use that $\|V\|_1 \le C$, the bounds in Lemma A.5 and Lemma A.6 and that

$$\|a_\sigma(v_x^t)\| = \|v_\sigma^t\|_2 \le \varrho^{\frac{1}{2}} e^{t(k_F^\sigma)^2}, \tag{5.17}$$

with v_σ^t defined in (2.26).

We start by considering I_1. Since $\hat u_\uparrow^>(r+p) = \hat\zeta^>(r+p)\hat u_\uparrow(r+p) = \hat\zeta^>(r+p)$, the definition of $\hat\zeta^>$ implies that $|r+p| \ge 5\max\{k_F^\uparrow, k_F^\downarrow\}$, and hence $|p-r'| \ge 3\max\{k_F^\downarrow, k_F^\uparrow\}$ since $|r| \le k_F^\uparrow$ and $|r'| \le k_F^\downarrow$. Moreover, also $|r'-p| \le 6\varrho^{1/3-\gamma}$ and hence we can replace $\hat u_\downarrow(r'-p)$ in (5.15) by $\hat\eta(r'-p)$ with $\hat\eta$ the characteristic function of the set $3\max\{k_F^\uparrow, k_F^\downarrow\} \le |k| \le 6\varrho^{1/3-\gamma}$. Proceeding then as in Remark 2.9 we can rewrite I_1 in configuration space as

$$\mathrm{I}_1 = 8\pi a\sum_\sigma \int_0^\infty dt\, e^{-2t\varepsilon}\int dxdydzdz'\, V(x-y)\chi_<(z-z')\zeta^t(z;y)\, a_\sigma^*(u_y^>)a_\uparrow(v_z^t)a_\downarrow(\eta_{z'}^t)a_\downarrow(v_{z'}^t)a_\downarrow(u_x^>)a_\sigma(u_y^>), \tag{5.18}$$

with

$$\zeta^t(z;y) := \frac{1}{L^3}\sum_k \zeta^>(k)e^{-t|k|^2}e^{ik\cdot(z-y)}, \qquad \eta^t(x;y) = \frac{1}{L^3}\sum_k \hat\eta(k)e^{-t|k|^2}e^{ik\cdot(x-y)}. \tag{5.19}$$

We shall write the right-hand side of (5.18) as $8\pi a\int_0^\infty dt\, e^{-2t\varepsilon}\,\mathrm{I}_1^t$. With the aid of (5.17), Lemma A.6 and the Cauchy–Schwarz inequality, we find

$$\begin{aligned}
|\langle\xi_\lambda,\mathrm{I}_1^t\xi_\lambda\rangle| &\le C\sum_\sigma\int dxdydzdz'\,V(x-y)|\chi_<(z-z')||\zeta^t(z;y)|\|\hat v_\downarrow^t\|_2\|\hat\eta^t\|_2\|\hat v_\uparrow^t\|_2\|a_\sigma(u_y^>)\xi_\lambda\|\|a_\downarrow(u_x^>)a_\sigma(u_y^>)\xi_\lambda\|\\
&\le C\|\chi_<\|_1\|\zeta^t\|_1\|\hat v_\uparrow^t\|_2\|\hat v_\downarrow^t\|_2\|\hat\eta^t\|_2\sum_\sigma\int dxdy\,V(x-y)\|a_\sigma(u_y^>)\xi_\lambda\|\|a_\downarrow(u_x^>)a_\sigma(u_y^>)\xi_\lambda\|\\
&\le C\varrho^{\frac12}e^{t(k_F^\uparrow)^2}\|\hat v_\downarrow^t\|_2\|\hat\eta^t\|_2\|\mathcal{N}^{\frac12}\xi_\lambda\|\|(\mathbb{Q}_4^>)^{\frac12}\xi_\lambda\|.
\end{aligned}$$

For the integration over t, we can use the Cauchy–Schwarz's inequality to get

$$\int_0^\infty dt\,e^{-2t\varepsilon}e^{t(k_F^\uparrow)^2}\|\hat\eta^t\|_2\|\hat v_\downarrow^t\|_2 \le \left(\int_0^\infty dt\,e^{-2t\varepsilon}e^{2t(k_F^\uparrow)^2}e^{2t(k_F^\downarrow)^2}\|\hat\eta_\downarrow^t\|_2^2\right)^{\frac12}\left(\int_0^\infty dt\,e^{-2t\varepsilon}e^{-2t(k_F^\downarrow)^2}\|\hat v_\downarrow^t\|_2^2\right)^{\frac12}.$$

The second integral above is bounded in (3.9). The first equals

$$\frac12\frac{1}{L^3}\sum_{3\max\{k_F^\uparrow,k_F^\downarrow\}\le|k|\le 6\varrho^{1/3-\gamma}}\frac{1}{|k|^2-(k_F^\downarrow)^2-(k_F^\uparrow)^2+\varepsilon}\le C\varrho^{\frac13-\gamma}. \tag{5.20}$$

In combination, we have thus shown that

$$|\langle\xi_\lambda,\mathrm{I}_1\xi_\lambda\rangle|\le C\varrho^{\frac56-\frac{\gamma}{2}-\frac{\delta}{16}}\|\mathcal{N}^{\frac12}\xi_\lambda\|\|(\mathbb{Q}_4^>)^{\frac12}\xi_\lambda\|\le C\varrho^{\frac53-\gamma-\frac{\delta}{8}}\langle\xi_\lambda,\mathcal{N}\xi_\lambda\rangle + C\langle\xi_\lambda,\mathbb{Q}_4^>\xi_\lambda\rangle. \tag{5.21}$$

The term I_2 can be bounded in the same way.

To conclude the analysis of $\mathbb{Q}_4^>$, we now consider I_3 in (5.16). Arguing as above, we can replace $\hat u_\uparrow^>(r+p)\hat u_\downarrow^>(r'-p)$ by $\hat\eta(r+p)\hat\eta(r'-p)$. In configuration space, we then obtain

$$\mathrm{I}_3 = 8\pi a\int_0^\infty dt\,e^{-2t\varepsilon}\int dxdydzdz'\,V(x-y)\chi_<(z-z')\eta^t(z;x)\eta^t(z';y)\,a_\uparrow(u_x^>)a_\uparrow(v_z^t)a_\downarrow(u_y^>)a_\downarrow(v_{z'}^t),$$

with η^t as in (5.19). Writing $\mathrm{I}_3 = \int_0^\infty dt\,e^{-2t\varepsilon}\,\mathrm{I}_3^t$ and using (5.17), we can estimate

$$|\langle\xi_\lambda,\mathrm{I}_3^t\xi_\lambda\rangle|\le C\varrho e^{t(k_F^\uparrow)^2}e^{t(k_F^\downarrow)^2}\int dxdydzdz'V(x-y)|\chi_<(z-z')||\eta^t(z;x)||\eta^t(z';y)|\|a_\uparrow(u_x^>)a_\downarrow(u_y^>)\xi_\lambda\|.$$

With

$$\left|\int dzdz'\,\chi_<(z-z')\eta^t(z;x)\eta^t(z';y)\right|\le C\|\chi_<\|_1\|\eta^t\|_2^2\le C\|\eta^t\|_2^2,$$

and the Cauchy–Schwarz inequality we then get

$$|\langle\xi_\lambda,\mathrm{I}_3\xi_\lambda\rangle|\le CL^{\frac32}\varrho\left(\int_0^\infty dt\,e^{-2t\varepsilon}e^{t(k_F^\uparrow)^2}e^{t(k_F^\downarrow)^2}\|\hat\eta^t\|_2^2\right)\|(\mathbb{Q}_4^>)^{\frac12}\xi_\lambda\|.$$

The integral is again bounded as in (5.20). Inserting this bound, we obtain

$$|\langle\xi_\lambda,\mathrm{I}_3\xi_\lambda\rangle|\le CL^{\frac32}\varrho^{\frac43-\gamma}\|(\mathbb{Q}_4^>)^{\frac12}\xi_\lambda\|\le CL^3\varrho^{\frac83-2\gamma}+\langle\xi_\lambda,\mathbb{Q}_4^>\xi_\lambda\rangle. \tag{5.22}$$

Combining (5.21) and (5.22), we get

$$
\begin{aligned}
|\partial_\lambda \langle T_{2;\lambda}\Omega, \mathbb{Q}_4^{>} T_{2;\lambda}\Omega\rangle| &\leq CL^3 \varrho^{\frac{8}{3}-2\gamma} + C\varrho^{\frac{5}{3}-\gamma-\frac{\delta}{8}} \langle T_{2;\lambda}\Omega, \mathcal{N} T_{2;\lambda}\Omega\rangle + C\langle T_{2;\lambda}\Omega, \mathbb{Q}_4^{>} T_{2;\lambda}\Omega\rangle \\
&\leq CL^3 \varrho^{\frac{8}{3}-2\gamma} + C\langle T_{2;\lambda}\Omega, \mathbb{Q}_4^{>} T_{2;\lambda}\Omega\rangle,
\end{aligned}
$$

where we used the bound on the number operator $\mathcal{N}$ from Proposition 3.5 (Equation (3.12) for $\lambda = 0$) and the constraints $0 < \gamma < 1/3$ and $\delta \leq 8\gamma$ in the last estimate. Therefore, by Grönwall's Lemma, for any $\lambda \in [0,1]$, we have

$$
|\langle T_{2;\lambda}\Omega, \mathbb{Q}_4^{>} T_{2;\lambda}\Omega\rangle| \leq CL^3 \varrho^{\frac{8}{3}-2\gamma}.
$$

Inserting this estimate in (5.14) and using again the bound for the number operator from Proposition 3.5, we find (5.10). □

5.3 | Conjugation of $\mathcal{E}_{\mathbb{H}_0}$

The purpose of this subsection is to show that $\langle T_2\Omega, \mathcal{E}_{\mathbb{H}_0} T_2\Omega\rangle = L^3 o(\varrho^{\frac{7}{3}})$, with $\mathcal{E}_{\mathbb{H}_0}$ defined in (4.2) the kinetic error term coming from Proposition 4.2 from the conjugation of $\mathbb{H}_0$ by T_1.

Proposition 5.4 (Final estimate for $\mathcal{E}_{\mathbb{H}_0}$). *Let $0 < \gamma < \frac{1}{6}$, $0 < \delta \leq 8\gamma$ with $2\gamma + \frac{\delta}{16} \leq \frac{1}{3}$. Under the same assumptions of Theorem 1.2*

$$
|\langle T_2\Omega, \mathcal{E}_{\mathbb{H}_0} T_2\Omega\rangle| \leq CL^3 \left(\varrho^{\frac{7}{3}+\gamma} + \varrho^{3-2\gamma-\frac{\delta}{4}} \right). \tag{5.23}
$$

Proof. Again we use Grönwall's Lemma, and compute

$$
\partial_\lambda T_{2;\lambda}^* \mathcal{E}_{\mathbb{H}_0} T_{2;\lambda} = T_{2;\lambda} [\mathcal{E}_{\mathbb{H}_0}, B_2 - B_2^*] T_{2;\lambda}.
$$

The terms in $[\mathcal{E}_{\mathbb{H}_0}, B_2 - B_2^*]$ are of the same type as those in Proposition 4.2, see (4.8)–(4.13). The only difference is that each of them has $\hat{\eta}^{\varepsilon}_{r,r'}(p)\hat{\chi}_<(p)$ in place of $\hat{\varphi}(p)\hat{\chi}_>(p)$, which distinguishes B_1 from B_2. As in the proof of Proposition 4.2, it is enough to prove the bound in (5.23) with $\mathcal{E}_{\mathbb{H}_0}$ replaced by $\mathcal{E}_{\mathbb{H}_0;1}$, defined in 4.6. We shall again write $[\mathcal{E}_{\mathbb{H}_0;1}, B_2] = \sum_{j=1}^{6} \mathrm{I}_j$ and consider the terms individually.

We will estimate the expectation value of all the terms in the state $\xi_\lambda = T_{2;\lambda}\Omega$. In the estimates we will often use the bounds in Lemma A.2, Lemma 3.1, Lemma A.5 as well as (4.14) and (5.17). Similarly to (3.14), we shall also need that

$$
\int dx\, \|a_\sigma(v_x^t)\xi_\lambda\|^2 \leq e^{2t(k_F^\sigma)^2} \langle \xi_\lambda, \mathcal{N}\xi_\lambda\rangle. \tag{5.24}
$$

Moreover, we will often use the constraints on p, r, r' to replace $\hat{u}_\uparrow(r+p)$ and $\hat{u}_\downarrow(r'-p)$ by $\hat{u}_\uparrow^<(r+p)$ and $\hat{u}_\downarrow^<(r'-p)$, respectively, with $\hat{u}_\sigma^<$ defined as in (2.25). Similarly as in Proposition 4.2, it is convenient to split I_1, I_2 and I_3 into a sum of two terms each.

Proceeding as in the proof of Proposition 4.2, taking into account Remark 2.9, we have for the analogue of (4.8)

$$
\mathrm{I}_1 = 8\pi a \int_0^\infty dt\, e^{-2t\varepsilon} \left(\mathrm{I}_{1;a}^t + \mathrm{I}_{1;b}^t \right), \tag{5.25}
$$

with

$$
\mathrm{I}_{1;a}^t = \int dxdydzdz'\, \widetilde{\varphi}(x-y)\chi_<(z-z')v_\downarrow^t(y;z')\Delta v_\uparrow^t(x;z)\, a_\uparrow^*(u_x)a_\downarrow^*(u_y)a_\downarrow(u_{z'}^t)a_\uparrow(u_z^t),
$$

and

$$\mathrm{I}^t_{1;b} = \sum_{\ell=1}^{3} \int dxdydzdz'\, \widetilde{\varphi}(x-y)\chi_<(z-z')v^t_\downarrow(y;z')\partial_\ell v^t_\uparrow(x;z)a^*_\uparrow(\partial_\ell \nu^>_x)a^*_\downarrow(u_y)a_\downarrow(u^t_{z'})a_\uparrow(u^t_z),$$

with $\nu^>$ defined in (4.15). With the aid of Cauchy–Schwarz inequality, together with bounds as in (4.16) and (4.18), using that $0 \le \hat{u}^t_\downarrow \le e^{-t(k_F^\downarrow)^2}$ and $0 \le \hat{u}_\uparrow \le 1$, we obtain for the first term

$$\begin{aligned}
|\langle \xi_\lambda, \mathrm{I}^t_{1;a}\xi_\lambda\rangle| &\le \left(\int dydz \left\| \int dx\, \widetilde{\varphi}(x-y)\Delta v^t_\uparrow(x;z)a_\uparrow(u_x)\right\|^2 \|a_\downarrow(u_y)\xi_\lambda\|^2\right)^{\frac12} \\
&\quad\times\left(\int dydz \left\| \int dz'\, \chi_<(z-z')v^t_\downarrow(y;z')a_\downarrow(u^t_{z'})\right\|^2 \|a_\uparrow(u^t_z)\xi_\lambda\|^2\right)^{\frac12} \\
&\le e^{-t(k_F^\downarrow)^2}\left(\int dydzdz'\, |\widetilde{\varphi}(z'-y)|^2|\Delta v^t_\uparrow(z;z')|^2\|a_\downarrow(u_y)\xi_\lambda\|^2\right)^{\frac12} \\
&\quad\times\left(\int dydzdz'\, |\chi_<(z-z')|^2|v^t_\downarrow(y;z')|^2\|a_\uparrow(u^t_z)\xi_\lambda\|^2\right)^{\frac12}.
\end{aligned}$$

To bound the terms, we shall use Lemma A.2 and $\|\chi_<\|_2 \le C\varrho^{1/2-(3/2)\gamma}$, as well as (3.14). This way, we obtain

$$|\langle \xi_\lambda, \mathrm{I}^t_{1;a}\xi_\lambda\rangle| \le e^{-t\sum_\sigma (k_F^\sigma)^2}\|\widetilde{\varphi}\|_2\|\chi_<\|_2\|\Delta v^t_\uparrow\|_2\|v^t_\downarrow\|_2\langle \xi_\lambda, \mathcal{N}\xi_\lambda\rangle \le C\varrho^{1-\gamma}e^{-t\sum_\sigma (k_F^\sigma)^2}\|v^t_\uparrow\|_2\|v^t_\downarrow\|_2\langle \xi_\lambda, \mathcal{N}\xi_\lambda\rangle.$$

The Cauchy–Schwarz inequality and (3.9) for the integral with respect to t then yields

$$\int_0^\infty dt\, e^{-2t\varepsilon}|\langle \xi_\lambda, \mathrm{I}^t_{1;a}\xi_\lambda\rangle| \le C\varrho^{\frac43-\gamma-\frac{\delta}{8}}\langle \xi_\lambda, \mathcal{N}\xi_\lambda\rangle. \tag{5.26}$$

The term $\mathrm{I}_{1;b}$ can be bounded very similarly, using in addition (4.19). We obtain

$$\int_0^\infty dt\, e^{-2t\varepsilon}|\langle \xi_\lambda, \mathrm{I}^t_{1;b}\xi_\lambda\rangle| \le C\varrho^{1-\gamma-\frac{\delta}{8}}\|\mathbb{H}_0^{\frac12}\xi_\lambda\|\|\mathcal{N}^{\frac12}\xi_\lambda\|. \tag{5.27}$$

Next we consider the term I_2. Proceeding as above, we find again (5.25) with I_1 replaced by I_2 everywhere, and

$$\mathrm{I}^t_{2;a} = \frac{1}{L^3}\sum_r |r|^2\hat{v}^t_\uparrow(r)\left(\int dx\, e^{ir\cdot x}a^*_\uparrow(u_x)b^*_\downarrow(\widetilde{\varphi}_x)\right)\left(\int dz\, e^{-ir\cdot z}b^t_\downarrow(\chi_{<z})a_\uparrow(u^t_z)\right),$$

where we used the notation in (3.1) and introduced in addition

$$b^t_\sigma(\chi_{<z}) := \int dz'\, \chi_<(z-z')a_\sigma(u^t_{z'})a_\sigma(v^t_{z'}). \tag{5.28}$$

Using $|r|^2\hat{v}^t_\uparrow(r) \le C\varrho^{2/3}e^{t(k_F^\uparrow)^2}$ and the Cauchy–Schwarz inequality, we obtain

$$|\langle \xi_\lambda, \mathrm{I}^t_{2;a}\xi_\lambda\rangle| \le C\varrho^{\frac23}e^{t(k_F^\uparrow)^2}\sqrt{\int dx \|b_\downarrow(\widetilde{\varphi}_x)a_\uparrow(u_x)\xi_\lambda\|^2}\sqrt{\int dz \left\|b^t_\downarrow(\chi_{<z})a_\uparrow(u^t_z)\xi_\lambda\right\|^2}.$$

Using the bound from Lemma 3.1 for $\|b_\downarrow(\widetilde{\varphi}_x)\|$, we find

$$\int dx\, \|b_\downarrow(\widetilde{\varphi}_x)a_\uparrow(u_x)\xi_\lambda\|^2 \le C\varrho^{\frac23+\gamma}\langle \xi_\lambda, \mathcal{N}\xi_\lambda\rangle.$$

Similarly, from

$$\|b^t_\downarrow(\chi_{<z})\| \leq \int dz' |\chi_<(z-z')| \|a_\downarrow(u^t_{z'})\| \|a_\downarrow(v^t_{z'})\| \leq \|\chi_<\|_1 \|u^t_\downarrow\|_2 \|v^t_\downarrow\|_2 \leq C\|u^t_\downarrow\|_2 \|v^t_\downarrow\|_2, \tag{5.29}$$

together with (5.24), we find that

$$\int dz \left\| b^t_\downarrow(\chi_{<z}) a_\uparrow(u^t_z)\xi_\lambda \right\|^2 \leq C e^{-2t(k_F^\uparrow)^2} \|u^t_\downarrow\|_2^2 \|v^t_\downarrow\|_2^2 \langle \xi_\lambda, \mathcal{N}\xi_\lambda\rangle.$$

Using (3.10) for the integral with respect to t, we obtain

$$\int_0^\infty dt\, e^{-2t\varepsilon} |\langle \xi_\lambda, \mathrm{I}^t_{2;a}\xi_\lambda\rangle| \leq C\varrho^{1+\frac{\gamma}{2}} \left(\int_0^\infty dt\, e^{-2t\varepsilon} \|\hat{u}^t_\downarrow\|_2 \|\hat{v}^t_\downarrow\|_2 \right) \langle \xi_\lambda, \mathcal{N}\xi_\lambda\rangle = C\varrho^{\frac{4}{3}-\frac{\delta}{16}} \langle \xi_\lambda, \mathcal{N}\xi_\lambda\rangle.$$

The estimate for $\mathrm{I}^t_{2;b}$ is similar. It is given by

$$\mathrm{I}^t_{2;b} = \frac{1}{L^3} \sum_{\ell=1}^3 \sum_r r_\ell \hat{v}^t_\uparrow(r) \int dxdz\, e^{ir\cdot(x-z)} a^*_\uparrow(\partial_\ell \nu^>_x) b^*_\downarrow(\widetilde{\varphi}_x) b^t_\downarrow(\chi_{<z}) a_\uparrow(u^t_z).$$

Using (4.19) and proceeding as above, we obtain

$$\int_0^\infty dt\, e^{-2t\varepsilon} |\langle \xi_\lambda, \mathrm{I}^t_{2;b}\xi_\lambda\rangle| \leq C\varrho^{1-\frac{\delta}{16}} \|\mathcal{N}^{\frac{1}{2}}\xi_\lambda\| \|\mathbb{H}_0^{\frac{1}{2}}\xi_\lambda\|.$$

The term I_3 can be estimated in the same way as I_2, obtaining the same final bound. We omit the details.

We now consider the term I_4. In contrast to the corresponding one in Proposition 4.2 in (4.11), to estimate this term we rewrite it in normal order using that

$$\begin{aligned} &\hat{a}^*_{s-k,\downarrow}\hat{a}_{r'-p,\downarrow}\hat{a}_{-r,\uparrow}\hat{a}_{-r',\downarrow}\hat{a}^*_{-s,\downarrow}\hat{a}^*_{-r-p+k,\uparrow} = \hat{a}^*_{s-k,\downarrow}\hat{a}^*_{-s,\downarrow}\hat{a}^*_{-r-p+k,\uparrow}\hat{a}_{-r,\uparrow}\hat{a}_{-r',\downarrow}\hat{a}_{r'-p,\downarrow} \\ &+ \delta_{p,k}\hat{a}^*_{s-k,\downarrow}\hat{a}^*_{-s,\downarrow}\hat{a}_{-r',\downarrow}\hat{a}_{r'-p,\downarrow} - \delta_{r',s}\hat{a}^*_{s-k,\downarrow}\hat{a}^*_{-r-p+k,\uparrow}\hat{a}_{-r,\uparrow}\hat{a}_{r'-p,\downarrow} + \delta_{r',s}\delta_{p,k}\hat{a}^*_{s-k,\downarrow}\hat{a}_{r'-p,\downarrow}. \end{aligned} \tag{5.30}$$

Correspondingly, we have $\mathrm{I}_4 = 8\pi a \int_0^\infty dt\, e^{-2t\varepsilon}(\mathrm{I}^t_{4;a} + \mathrm{I}^t_{4;b} + \mathrm{I}^t_{4;c} + \mathrm{I}^t_{4;d})$. The first term equals

$$\mathrm{I}^t_{4;a} = \sum_{\ell=1}^3 \frac{1}{L^3} \sum_k \hat{u}^t_\uparrow(k) \int dydz\, e^{ik\cdot(z-y)} a^*_\uparrow(\partial_\ell v_y) b^*_\downarrow(\partial_\ell \widetilde{\varphi}_y) b^t_\downarrow(\chi_{<z}) a_\uparrow(v^t_z),$$

where we used again the notations introduced in (3.1) and (5.28). Since $0 \leq \hat{u}^t_\uparrow(k) \leq e^{-t(k_F^\uparrow)^2}$, the Cauchy–Schwarz inequality yields

$$|\langle \xi_\lambda, \mathrm{I}^t_{4;a}\xi_\lambda\rangle| \leq e^{-t(k_F^\uparrow)^2} \sum_{\ell=1}^3 \sqrt{\int dy \left\| b_\downarrow(\partial_\ell \widetilde{\varphi}_y) a_\uparrow(\partial_\ell v_y)\xi_\lambda \right\|^2} \sqrt{\int dz \left\| b^t_\downarrow(\chi_{<z}) a_\uparrow(v^t_z)\xi_\lambda \right\|^2}.$$

The last factor can be bounded with the aid of (5.29) and (5.24). For the first factor, we use

$$\begin{aligned} \left\| b_\downarrow(\partial_\ell \widetilde{\varphi}_y) a_\uparrow(\partial_\ell v_y)\xi_\lambda \right\| &\leq \int dx\, |\partial_\ell \widetilde{\varphi}(x-y)| \|a_\downarrow(u_x) a_\downarrow(v_x) a_\uparrow(\partial_\ell v_y)\xi_\lambda\| \\ &\leq \|v_\downarrow\|_2 \|\partial_\ell v_\uparrow\|_2 \int dx\, |\partial_\ell \widetilde{\varphi}(x-y)| \|a_\downarrow(u_x)\xi_\lambda\|, \end{aligned}$$

and hence

$$\begin{aligned} &\int dy \left\| b_\downarrow(\partial_\ell \widetilde{\varphi}_y) a_\uparrow(\partial_\ell v_y)\xi_\lambda \right\|^2 \\ &\leq C\varrho^{\frac{8}{3}} \int dydxdx'\, |\partial_\ell \widetilde{\varphi}(y-x)| |\partial_\ell \widetilde{\varphi}(y-x')| \|a_\downarrow(u_x)\xi_\lambda\| \|a_\downarrow(u_{x'})\xi_\lambda\| \leq C\varrho^{\frac{8}{3}} \|\partial_\ell \widetilde{\varphi}\|_1^2 \langle \xi_\lambda, \mathcal{N}\xi_\lambda\rangle. \end{aligned}$$

In combination with Lemma A.2 and (3.10), this yields

$$\int_0^\infty dt\, e^{-2t\varepsilon} |\langle \xi_\lambda, \mathrm{I}^t_{4;a}\xi_\lambda\rangle| \le C\varrho^{\frac{4}{3}+\frac{\gamma}{2}-\frac{\delta}{16}}\langle \xi_\lambda, \mathcal{N}\xi_\lambda\rangle. \tag{5.31}$$

Next we consider $\mathrm{I}^t_{4;b}$, which is given by

$$\mathrm{I}^t_{4;b} = \sum_{\ell=1}^{3} \int dydz\, u^t_\uparrow(y;z)\partial_\ell v^t_\uparrow(y;z) b^*_\downarrow(\partial_\ell \widetilde{\varphi}_y) b^t_\downarrow(\chi_{<z}),$$

and can be bounded as

$$|\langle \xi_\lambda, \mathrm{I}^t_{4;b}\xi_\lambda\rangle| \le \|v_\downarrow\|_2 \|v^t_\downarrow\|_2 \sum_{\ell=1}^{3}\int dxdydzdz'\, |\partial_\ell\widetilde{\varphi}(x-y)||\chi_<(z-z')||u^t_\uparrow(y;z)||\partial_\ell v^t_\uparrow(y;z)|\|a_\downarrow(u_x)\xi_\lambda\|\|a_\downarrow(u^t_{z'})\xi_\lambda\|.$$

With (3.14), $\|\partial^n_\ell v^t_\sigma\|_2 \le C\varrho^{n/3}\|v^t_\sigma\|_2$, $\|v^t_\sigma\|_2 \le C\varrho^{1/2}e^{t(k^\sigma_F)^2}$, Lemma A.2 and Cauchy–Schwarz we thus obtain

$$|\langle \xi_\lambda, \mathrm{I}^t_{4;b}\xi_\lambda\rangle| \le e^{-t(k_F^\downarrow)^2} \sum_{\ell=1}^{3} \|v_\downarrow\|_2\|v^t_\downarrow\|_2 \|\partial_\ell\widetilde{\varphi}\|_1 \|\chi_<\|_1 \|u^t_\uparrow\|_2 \|\partial_\ell v^t_\uparrow\|_2 \langle \xi_\lambda, \mathcal{N}\xi_\lambda\rangle \le C\varrho^{1+\gamma}\|v^t_\uparrow\|_2\|u^t_\uparrow\|_2\langle \xi_\lambda, \mathcal{N}\xi_\lambda\rangle.$$

In combination with (3.10) this yields the same bound as in (5.31) after integration over t. The same applies to $\mathrm{I}^t_{4;c}$ and we omit the details. The last term in I_4 is

$$\mathrm{I}^t_{4;d} = \frac{1}{L^6}\sum_{\ell=1}^{3}\sum_{p,r,r'} p_\ell \hat{\varphi}(p)\widehat{\chi}_>(p)\widehat{\chi}_<(p)\hat{u}^t_\uparrow(r+p) r_\ell \hat{v}^t_\uparrow(r)\hat{u}^t_\downarrow(r')\hat{v}^t_\downarrow(r'+p)\hat{a}^*_{r',\downarrow}\hat{a}_{r',\downarrow}.$$

Since $|p_\ell\hat{\varphi}(p)\widehat{\chi}_>(p)\widehat{\chi}_<(p)| \le \|\partial_\ell\widetilde{\varphi}\|_1\|\chi_<\|_1$ and

$$\frac{1}{L^6}\sum_{p,r} \hat{u}^t_\uparrow(r+p)|r_\ell|\hat{v}^t_\uparrow(r)\hat{u}^t_\downarrow(r')\hat{v}^t_\downarrow(r'+p) \le C\varrho^{4/3}\|u^t_\uparrow\|_2\|v^t_\uparrow\|_2,$$

for all $r' \in \Lambda^*$, we can again obtain a bound of the same kind. Altogether, we have thus shown that

$$|\langle \xi_\lambda, \mathrm{I}_4\xi_\lambda\rangle| \le C\varrho^{\frac{4}{3}+\frac{\gamma}{2}-\frac{\delta}{16}}\langle \xi_\lambda, \mathcal{N}\xi_\lambda\rangle. \tag{5.32}$$

The term I_5 can be estimated very similarly, obtaining the same result.

To conclude, we consider the last error term I_6, which corresponds to (4.13) in Proposition 4.2. We shall again start by writing it in normal order, using

$$\hat{a}_{-r,\uparrow}\hat{a}_{-r',\downarrow}\hat{a}^*_{-r'+p-k,\downarrow}\hat{a}^*_{-r-p+k,\uparrow} = \delta_{p,k}\left(1-\hat{a}^*_{-r,\uparrow}\hat{a}_{-r,\uparrow} - \hat{a}^*_{-r',\downarrow}\hat{a}_{-r',\downarrow}\right) + \hat{a}^*_{-r'+p-k,\downarrow}\hat{a}^*_{-r-p+k,\uparrow}\hat{a}_{-r,\uparrow}\hat{a}_{-r',\downarrow},$$

and correspondingly obtain 4 terms, which we write as $\mathrm{I}_6 = \mathrm{I}_{6;a} + 8\pi a\int_0^\infty dt\, e^{-2t\varepsilon}(\mathrm{I}^t_{6;b} + \mathrm{I}^t_{6;c} + \mathrm{I}^t_{6;d})$. The term $\mathrm{I}_{6;a}$ is a constant, given by

$$\mathrm{I}_{6;a} = \frac{1}{L^6}\sum_{p,r,r'} \widehat{\chi}_<(p)\hat{\eta}^\varepsilon_{r,r'}(p)\widehat{\chi}_>(p)\hat{\varphi}(p) p\cdot r \hat{v}_\uparrow(r)\hat{v}_\downarrow(r').$$

To arrive at this expression, we used that support properties of the various functions imply that $\hat{u}_\uparrow(r+p)\hat{u}_\downarrow(r'-p) = 1$. If we replace $\hat{\eta}^\varepsilon_{r,r'}(p)$ by $1/(2|p|^2)$, the resulting sum over p is zero by symmetry. Hence we can also write

$$\mathrm{I}_{6;a} = \frac{1}{L^6}\sum_{p,r,r'} \widehat{\chi}_<(p)\left(\hat{\eta}^\varepsilon_{r,r'}(p) - \frac{1}{2|p|^2}\right)\widehat{\chi}_>(p)\hat{\varphi}(p) p\cdot r\hat{v}_\uparrow(r)\hat{v}_\downarrow(r').$$

On the support of $\hat{\chi}_<(p)\hat{\chi}_>(p)$, that is, for $4\varrho^{1/3-\gamma} \leq |p| \leq 5\varrho^{1/3-\gamma}$, one readily checks that $|\hat{\eta}^{\varepsilon}_{r,r'}(p) - 1/(2|p|^2)| \leq C\varrho^{1/3}/|p|^3$ for $r \in \mathcal{B}_F^{\uparrow}$ and $r' \in \mathcal{B}_F^{\downarrow}$ (and $\varepsilon \lesssim \varrho^{2/3}$). Moreover, as already used earlier, $|\hat{\varphi}(p)| \leq C/|p|^2$ as a consequence of (1.5). Hence

$$|\mathrm{I}_{6;a}| \leq C\varrho^{\frac{1}{3}} \frac{1}{L^6} \sum_{p,r,r'} \hat{\chi}_<(p) \frac{1}{|p|^4} \hat{\chi}_>(p)|r|\hat{v}_\uparrow(r)\hat{v}_\downarrow(r') \leq CL^3\varrho^{\frac{8}{3}} \int_{4\varrho^{1/3-\gamma}\leq|p|\leq 5\varrho^{1/3-\gamma}} \frac{dp}{|p|^4} = CL^3\varrho^{7/3+\gamma}. \tag{5.33}$$

The term $\mathrm{I}^t_{6;b}$ is given by

$$\mathrm{I}^t_{6;b} = -\frac{1}{L^6} \sum_{p,r,r'} p \cdot r\hat{\varphi}(p)\hat{\chi}_>(p)\hat{\chi}_<(p)\hat{u}^t_\uparrow(r+p)\hat{u}^t_\downarrow(r'-p)\hat{v}^t_\uparrow(r)\hat{v}^t_\downarrow(r')\hat{a}^*_{-r,\uparrow}\hat{a}_{-r,\uparrow}.$$

Proceeding as with $\mathrm{I}_{4;d}$ above, we obtain

$$|\langle \xi_\lambda, \mathrm{I}^t_{6;b}\xi_\lambda\rangle| \leq C\varrho^{1+\gamma}\|u^t_\uparrow\|_2\|u^t_\downarrow\|_2 e^{t(k_F^\uparrow)^2}e^{t(k_F^\downarrow)^2}\langle \xi_\lambda, \mathcal{N}\xi_\lambda\rangle.$$

The integral over t can then be bounded with the aid of (3.8), with the result that

$$\int_0^\infty dt\, e^{-2t\varepsilon}|\langle \xi_\lambda, \mathrm{I}^t_{6;b}\xi_\lambda\rangle| \leq C\varrho^{\frac{4}{3}}\langle \xi_\lambda, \mathcal{N}\xi_\lambda\rangle. \tag{5.34}$$

The same bounds holds for $\mathrm{I}^t_{6;c}$.

The last term to study is $\mathrm{I}^t_{6;d}$, which in configuration space can be written as

$$\mathrm{I}^t_{6;d} = \sum_{\ell=1}^3 \int dxdydzdz'\, \partial_\ell\tilde{\varphi}(x-y)\chi_<(z-z')u^t_\uparrow(y;z)u^t_\downarrow(x;z')a^*_\downarrow(v_x)a^*_\uparrow(\partial_\ell v_y)a_\downarrow(v^t_{z'})a_\uparrow(v^t_z).$$

Using (4.14), (5.17) and (5.24) together with the Cauchy–Schwarz inequality we can bound

$$\begin{aligned}
&|\langle \xi_\lambda, \mathrm{I}_{6;d}\xi_\lambda\rangle| \\
&\leq \sum_{\ell=1}^3 \|\partial_\ell v_\uparrow\|_2\|v^t_\downarrow\|_2 \int dxdydzdz'\, |\partial_\ell\tilde{\varphi}(x-y)||\chi_<(z-z')||u^t_\uparrow(y;z)||u^t_\downarrow(x;z')|\|a_\downarrow(v_x)\xi_\lambda\|\|a_\uparrow(v^t_z)\xi_\lambda\| \\
&\leq Ce^{t(k_F^\uparrow)^2} \sum_{\ell=1}^3 \|\partial_\ell v_\uparrow\|_2\|v^t_\downarrow\|_2\|\partial_\ell\tilde{\varphi}\|_1\|\chi_<\|_1\, \|\hat{u}^t_\uparrow\|_2\|\hat{u}^t_\downarrow\|_2\langle \xi_\lambda, \mathcal{N}\xi_\lambda\rangle \\
&\leq C\varrho^{1+\gamma}e^{t(k_F^\downarrow)^2}e^{t(k_F^\uparrow)^2}\|\hat{u}^t_\uparrow\|_2\|\hat{u}^t_\downarrow\|_2\langle \xi_\lambda, \mathcal{N}\xi_\lambda\rangle,
\end{aligned}$$

where we also used the bounds from Lemma A.2 and Lemma A.5 in the last step. Again we can bound the integral over t using (3.8) to conclude that (5.34) also holds for $\mathrm{I}^t_{6;d}$. Combining all the bounds above, we obtain

$$|\langle \xi_\lambda, \mathrm{I}_6\xi_\lambda\rangle| \leq CL^3\varrho^{\frac{7}{3}+\gamma} + C\varrho^{\frac{4}{3}}\langle \xi_\lambda, \mathcal{N}\xi_\lambda\rangle. \tag{5.35}$$

Combining all these bounds with (3.12) (for $\lambda = 0$) and the second bound in (5.8), and using Grönwall's Lemma, we obtain (5.23). □

5.4 | Conjugation of $\mathbb{Q}_{2;<}$

In the next proposition we finally take into account the conjugation of $\mathbb{Q}_{2;<}$ by T_2. The contribution from $\langle T_2\Omega, \mathbb{Q}_{2;<}T_2\Omega\rangle$, when combined with the main contribution of $\langle T_2\Omega, \mathbb{H}_0 T_2\Omega\rangle$ in Proposition 5.2, will give the correct energy of order $\varrho^{7/3}$.

Proposition 5.5 (Conjugation of $\mathbb{Q}_{2;<}$ by T_2). *Let $0 < \gamma < \frac{1}{6}$, $\delta \le 8\gamma$, $2\gamma + \frac{\delta}{16} \le \frac{1}{3}$. Under the same assumptions as in Theorem 1.2*

$$\partial_\lambda T^*_{2;\lambda}\mathbb{Q}_{2;<}T_{2;\lambda} = -\frac{16\pi a}{L^6}\sum_{p,r,r'\in\frac{2\pi}{L}\mathbb{Z}^3}(\hat{\chi}_<(p))^2\hat{\eta}^\varepsilon_{r,r'}(p)\hat{u}_\uparrow(r+p)\hat{u}_\downarrow(r'-p)\hat{v}_\uparrow(r)\hat{v}_\downarrow(r') + T^*_{2;\lambda}\mathcal{E}_{\mathbb{Q}_{2;<}}T_{2;\lambda}, \tag{5.36}$$

where $\mathcal{E}_{\mathbb{Q}_{2;<}}$ is such that for any $\psi \in \mathcal{F}_{\mathrm{f}}$

$$|\langle\psi, \mathcal{E}_{\mathbb{Q}_{2;<}}\psi\rangle| \le C\varrho^{\frac{4}{3}-2\gamma-\frac{\delta}{16}}\langle\psi, \mathcal{N}\psi\rangle. \tag{5.37}$$

Combining (5.37) for $\psi = T_{2;\lambda}\Omega$ with (3.12) (for $\lambda = 0$), we get for any $\lambda \in [0,1]$

$$|\langle T_{2;\lambda}\Omega, \mathcal{E}_{\mathbb{Q}_{2;<}}T_{2;\lambda}\Omega\rangle| \le CL^3\varrho^{3(1-\gamma)-\frac{3}{16}\delta}. \tag{5.38}$$

Proof. Since $\partial_\lambda T^*_{2;\lambda}\mathbb{Q}_{2;<}T_{2;\lambda} = T^*_{2;\lambda}[\mathbb{Q}_{2;<}, B_2 - B_2^*]T_{2;\lambda}$, we have to compute the commutator $[\mathbb{Q}_{2;<}, B_2 - B_2^*]$. The structure of the various terms is the same as those in the commutator in (4.36) in Proposition 4.6. Following the same strategy, we have

$$[\mathbb{Q}_{2;<}, B_2 - B_2^*] = (8\pi a)^2\sum_{j=1}^{6}\int_0^\infty dt\, e^{-2t\varepsilon}\,\mathrm{I}^t_j + \text{h.c.}.$$

The leading order constant term comes from normal ordering I^t_6, all the other terms are error terms, which we now estimate individually. In all the error terms, we can replace each $\hat{u}_\sigma$ by $\hat{u}^<_\sigma$ defined in (2.25) as a consequence of the constraints on p, k, r, r', which will be convenient for the estimates.

The first term to consider is the analogue of (4.38),

$$\mathrm{I}^t_1 = \frac{1}{L^3}\sum_k \hat{v}^t_\uparrow(k)\int dxdz\, e^{ik\cdot(x-z)}a^*_\uparrow(u_x)b^{<*}_\downarrow(\chi_{<x})b^t_\downarrow(\chi_{<z})a_\uparrow(u^t_z),$$

where we used the definition (5.28) and introduced

$$b^<_\sigma(\chi_{<y}) = \int dx\,\chi_<(x-y)a_\sigma(u^<_x)a_\sigma(v_x). \tag{5.39}$$

By the Cauchy–Schwarz inequality and $0 \le \hat{v}^t_\uparrow(k) \le e^{t(k^\uparrow_F)^2}$,

$$|\langle\psi, \mathrm{I}^t_1\psi\rangle| \le e^{t(k^\uparrow_F)^2}\left(\int dx\,\left\|b^<_\downarrow(\chi_{<x})a_\uparrow(u_x)\psi\right\|^2\right)^{1/2}\left(\int dz\,\left\|b^t_\downarrow(\chi_{<z})a_\uparrow(u^t_z)\psi\right\|^2\right)^{1/2}.$$

From the bounds (5.29) and $\|b^<_\downarrow(\chi_{<y})\| \le \|\chi_<\|_1\|v_\downarrow\|_2\|u^<_\downarrow\|_2 \le C\varrho^{1-\frac{3}{2}\gamma}$, we find with (3.14) that

$$|\langle\psi, \mathrm{I}^t_1\psi\rangle| \le C\varrho^{1-\frac{3}{2}\gamma}\|u^t_\downarrow\|_2\|v^t_\downarrow\|_2\langle\psi, \mathcal{N}\psi\rangle. \tag{5.40}$$

The integral over t is bounded in (3.10), resulting in

$$\int_0^\infty dt\, e^{-2t\varepsilon}|\langle\psi, \mathrm{I}^t_1\psi\rangle| \le C\varrho^{\frac{4}{3}-2\gamma-\frac{\delta}{16}}\langle\psi, \mathcal{N}\psi\rangle. \tag{5.41}$$

The estimate of I_2 works in the same way, with the same result.

Next we consider the analogue of (4.40), given by

$$\mathrm{I}_3^t = \int dxdydzdz'\ \chi_<(x-y)\chi_<(z-z')v_\downarrow^t(y;z')v_\uparrow^t(x;z)a_\uparrow^*(u_x^<)a_\downarrow^*(u_y^<)a_\downarrow(u_{z'}^t)a_\uparrow(u_z^t).$$

Using the Cauchy–Schwarz inequality together with the bound (3.14), we can estimate

$$\begin{aligned}|\langle\psi,\mathrm{I}_3\psi\rangle| &\le \|u_\downarrow^t\|_2\|u_\downarrow^<\|_2\int dxdydzdz'\ |\chi_<(x-y)||\chi_<(z-z')||v_\downarrow^t(y;z')||v_\uparrow^t(x;z)|\|a_\uparrow(u_x^<)\psi\|\|a_\uparrow(u_z^t)\psi\| \\ &\le e^{-t(k_F^\uparrow)^2}\|u_\downarrow^t\|_2\|u_\downarrow^<\|_2\|\chi_<\|_1^2\|v_\downarrow^t\|_2\|v_\uparrow^t\|_2\langle\psi,\mathcal{N}\psi\rangle \le C\varrho^{1-\frac{3}{2}\gamma}\|u_\downarrow^t\|_2\|v_\downarrow^t\|_2\langle\psi,\mathcal{N}\psi\rangle,\end{aligned}$$

where we also used Lemma A.5 in the last step. Applying again (3.10) for the integration over t shows that (5.41) also holds for I_3^t.

The term I_4^t is the analogue of (4.41). Similarly as for the term I_4 in the proof of Proposition 5.4, it is convenient to write I_4^t in normal order. Using (5.30), we can write $\mathrm{I}_4^t = \mathrm{I}_{4;a}^t + \mathrm{I}_{4;b}^t + \mathrm{I}_{4;c}^t + \mathrm{I}_{4;d}^t$, and the individual terms can be analyzed in a similar way as the corresponding terms in Proposition 5.4. The first term equals

$$\mathrm{I}_{4;a}^t = \frac{1}{L^3}\sum_k \hat{u}_\uparrow^t(k)\int dydz\, e^{ik\cdot(y-z)}a_\uparrow^*(v_y)b_\downarrow^{<*}(\chi_{<y})b_\downarrow^t(\chi_{<z})a_\uparrow(v_z^t),$$

where we used again the operators introduced in (5.28) and (5.39). Using $0\le\hat{u}_\uparrow^t(k)\le e^{-t(k_F^\uparrow)^2}$ and the Cauchy–Schwarz inequality, we find

$$|\langle\psi,\mathrm{I}_{4;a}\psi\rangle| \le e^{-t(k_F^\uparrow)^2}\left(\int dy\left\|b_\downarrow^<(\chi_{<y})a_\uparrow(v_y)\psi\right\|^2\right)^{\frac{1}{2}}\left(\int dz\left\|b_\downarrow^t(\chi_{<z})a_\uparrow(v_z^t)\psi\right\|^2\right)^{\frac{1}{2}}.$$

We can further bound $\|b_\downarrow^<(\chi_{<y})a_\uparrow(v_y)\psi\|\le\|v_\uparrow\|_2\|v_\downarrow\|_2\int dx\,|\chi_<(x-y)|\|a_\downarrow(u_x^<)\psi\|$ and similarly for $\|b_\downarrow^t(\chi_{<z})a_\uparrow(v_z^t)\psi\|$, with the result that

$$|\langle\psi,\mathrm{I}_{4;a}\psi\rangle| \le \varrho\|\chi_<\|_1^2e^{-t(k_F^\uparrow)^2-t(k_F^\downarrow)^2}\|v_\uparrow^t\|_2\|v_\downarrow^t\|_2\langle\psi,\mathcal{N}\psi\rangle.$$

where we also used again (3.14). Applying now (3.9) for the integration over t, we arrive at

$$\int_0^\infty dt\, e^{-2t\varepsilon}|\langle\psi,\mathrm{I}_{4;a}^t\psi\rangle| \le C\varrho^{\frac{4}{3}-\frac{\delta}{8}}\langle\psi,\mathcal{N}\psi\rangle.$$

The term $\mathrm{I}_{4;b}^t$ is given by

$$\mathrm{I}_{4;b}^t = \int dxdydzdz'\ \chi_<(x-y)\chi_<(z-z')u_\uparrow^t(z;y)v_\downarrow^t(z';x)a_\downarrow^*(u_x^<)a_\uparrow^*(v_y)a_\downarrow(u_{z'}^t)a_\uparrow(v_z^t).$$

Similarly as above, we can bound it with the aid of the Cauchy–Schwarz inequality as

$$\begin{aligned}|\langle\psi,\mathrm{I}_{4;b}\psi\rangle| &\le \|v_\uparrow^t\|_2\|v_\downarrow\|_2\int dxdydzdz'\ |\chi_<(x-y)||\chi_<(z-z')||u_\uparrow^t(z;y)||v_\downarrow^t(z';x)|\|a_\downarrow(u_x^<)\psi\|\|a_\downarrow(u_{z'}^t)\psi\| \\ &\le e^{-t(k_F^\downarrow)^2}\|v_\uparrow^t\|_2\|v_\downarrow\|_2\|v_\downarrow^t\|_2\|u_\uparrow^t\|_2\|\chi_<\|_1^2\langle\psi,\mathcal{N}\psi\rangle \le C\varrho\|u_\uparrow^t\|_2\|v_\uparrow^t\|_2\langle\psi,\mathcal{N}\psi\rangle.\end{aligned}$$

With (3.10) to bound the integral over t we therefore obtain

$$\int_0^\infty dt\, e^{-2t\varepsilon}|\langle\psi,\mathrm{I}_{4;b}^t\psi\rangle| \le C\varrho^{\frac{4}{3}-\frac{\gamma}{2}-\frac{\delta}{16}}\langle\psi,\mathcal{N}\psi\rangle. \tag{5.42}$$

The estimate for the term $\mathrm{I}_{4;c}^t$, which is given by

$$\mathrm{I}_{4;c}^t = \int dxdydzdz'\ \chi_<(x-y)\chi_<(z-z')u_\uparrow^t(z;y)v_\uparrow^t(z;y)a_\downarrow^*(u_x^<)a_\downarrow^*(v_x)a_\downarrow(u_{z'}^t)a_\downarrow(v_{z'}^t),$$

can be done in a similar way, obtaining the same result. Finally, the term

$$\mathrm{I}^t_{4;d} = \int dxdydzdz' \,\chi_<(x-y)\chi_<(z-z')u^t_\uparrow(z;y)v^t_\downarrow(z';x)v^t_\uparrow(z;y)a^*_\downarrow(u^<_x)a_\downarrow(u^t_{z'}),$$

can be estimated similarly as $\mathrm{I}^t_{4;a}$ above, using also that $|v^t_\downarrow(z';x)| \leq \varrho e^{t(k^\downarrow_F)^2}$, with the result that

$$|\langle\psi, \mathrm{I}^t_{4;d}\psi\rangle| \leq \varrho\|\chi_<\|^2_1\|u^t_\uparrow\|_2\|v^t_\uparrow\|_2\langle\psi, \mathcal{N}\psi\rangle.$$

Again (3.10) then yields the validity of the bound (5.42) also for $\mathrm{I}^t_{4;d}$. Combining the bounds, we conclude that for $\delta \leq 8\gamma$,

$$\int_0^\infty dt\, e^{-2t\varepsilon}|\langle\psi, \mathrm{I}^t_4\psi\rangle| \leq C\varrho^{\frac{4}{3}-\frac{\gamma}{2}-\frac{\delta}{16}}\langle\psi, \mathcal{N}\psi\rangle. \tag{5.43}$$

The estimate for the term I^t_5 can be done in the same way, obtaining the same result.

Finally, we analyze I^t_6, which is the analogue of (4.44). As already mentioned at the beginning of the proof, we shall put it in normal order in order to extract the constant contribution. Using (4.43), we obtain $\mathrm{I}^t_6 = \mathrm{I}^t_{6;a} + \mathrm{I}^t_{6;b} + \mathrm{I}^t_{6;c}$, where the first is the desired constant,

$$(8\pi a)^2\int_0^\infty dt\, e^{-2t\varepsilon}\,\mathrm{I}^t_{6;a} = -\frac{8\pi a}{L^6}\sum_{p,r,r'}(\hat{\chi}_<(p))^2\hat{\eta}^\varepsilon_{r,r'}(p)\hat{u}_\uparrow(r+p)\hat{u}_\downarrow(r'-p)\hat{v}_\uparrow(r)\hat{v}_\downarrow(r').$$

The second term equals

$$\mathrm{I}^t_{6;b} = \frac{1}{L^6}\sum_{p,r,r'}\hat{\chi}_<(p)^2\hat{u}^t_\uparrow(r+p)\hat{u}^t_\downarrow(r'-p)\hat{v}^t_\uparrow(r)\hat{v}^t_\downarrow(r')\Big(\hat{a}^*_{-r,\uparrow}\hat{a}_{-r,\uparrow} + \hat{a}^*_{-r',\downarrow}\hat{a}_{-r',\downarrow}\Big),$$

and is thus bounded by $\varrho\|u^t_\uparrow\|_2\|u^t_\downarrow\|_2 e^{t(k^\uparrow_F)^2+t(k^\downarrow_F)^2}\mathcal{N}$, using $0 \leq \hat{\chi}_<(p) \leq 1$. The third term we can write in configuration space as

$$\mathrm{I}^t_{6;c} = \int dxdydzdz'\,\chi_<(x-y)\chi_<(z-z')u^t_\uparrow(z;x)u^t_\downarrow(z';y)a^*_\uparrow(v_x)a^*_\downarrow(v_y)a_\uparrow(v^t_z)a_\downarrow(v^t_{z'}).$$

Proceeding similarly as for $\mathrm{I}^t_{4;b}$ above, we can bound it as

$$|\langle\psi, \mathrm{I}^t_{6;c}\psi\rangle| \leq \|\chi_<\|^2_1\|v_\downarrow\|_2\|v^t_\uparrow\|_2\|u^t_\uparrow\|_2\|u^t_\downarrow\|_2 e^{t(k^\downarrow_F)^2}\langle\psi, \mathcal{N}\psi\rangle \leq C\varrho\|u^t_\uparrow\|_2\|u^t_\downarrow\|_2 e^{t(k^\uparrow_F)^2+t(k^\downarrow_F)^2}\langle\psi, \mathcal{N}\psi\rangle.$$

For the integration over t, we can again use Lemma 3.4 to conclude that

$$\int_0^\infty dt\, e^{-2t\varepsilon}|\langle\psi, (\mathrm{I}^t_{6;b} + \mathrm{I}^t_{6;c})\psi\rangle| \leq C\varrho^{\frac{4}{3}-\gamma}\langle\psi, \mathcal{N}\psi\rangle. \tag{5.44}$$

Combining all the estimates we proved (5.37). □

5.5 | Conclusion of Proposition 5.1

Proof of Proposition 5.1. We shall now combine the results of Propositions 5.2–5.5 in order to prove Proposition 5.1. The bounds (5.2), (5.3) and (5.4) were already proved in Proposition 5.2, Proposition 5.3 and Proposition 5.4, respectively. Moreover, combining Proposition 5.2 and Proposition 5.5 we obtain

$$\langle T_2\Omega, (\mathbb{H}_0 + \mathbb{Q}_{2;<})T_2\Omega\rangle = \int_0^1 d\lambda\,\lambda\,\partial_\lambda\langle T_{2;\lambda}\Omega, \mathbb{Q}_{2;<}T_{2;\lambda}\Omega\rangle + \int_0^1 d\lambda\,\langle T_{2,\lambda}\Omega\,\mathfrak{e}_{\mathbb{H}_0}T_{2;\lambda}\Omega\rangle$$

$$= -\frac{8\pi a}{L^6}\sum_{p,r,r'}(\hat{\chi}_<(p))^2\hat{\eta}^\varepsilon_{r,r'}(p)\hat{u}_\uparrow(r+p)\hat{u}_\downarrow(r'-p)\hat{v}_\uparrow(r)\hat{v}_\downarrow(r')$$

$$+\int_0^1 d\lambda\,\langle T_{2,\lambda}\Omega,(\mathfrak{e}_{\mathbb{H}_0}+\lambda\mathcal{E}_{\mathbb{Q}_{2;<}})T_{2;\lambda}\Omega\rangle.$$

Inserting the definition of $\hat{\eta}^\varepsilon_{r,r'}$ in (2.17), the desired estimate (5.1) follows from (5.8) and (5.38). □

6 | Conclusion of Theorem 1.2

We shall now complete the proof of Theorem 1.2. As already explained in Section 2.4, the starting point is the bound

$$E_L(N_\uparrow,N_\downarrow)\le E_{\mathrm{FFG}}+\langle\Omega,T_2^*T_1^*\mathcal{H}_{\mathrm{corr}}T_1T_2\Omega\rangle,$$

using the trial state (2.13) and Proposition 2.3. From Proposition 3.6 we further obtain

$$\langle T_1T_2\Omega,\mathcal{H}_{\mathrm{corr}}T_1T_2\Omega\rangle\le\langle T_1T_2\Omega,\mathcal{H}^{\mathrm{eff}}_{\mathrm{corr}}T_1T_2\Omega\rangle+CL^3\varrho^{\frac{8}{3}-\gamma-\frac{\delta}{8}}.$$

Combining now Propositions 4.1 and 5.1 and taking the thermodynamic limit we find that

$$\begin{aligned}e(\varrho_\uparrow,\varrho_\downarrow)\le{}&\frac{3}{5}(6\pi^2)^{\frac{2}{3}}\left(\varrho_\uparrow^{\frac{5}{3}}+\varrho_\downarrow^{\frac{5}{3}}\right)+8\pi a\varrho_\uparrow\varrho_\downarrow\\&+\frac{(8\pi a)^2}{(2\pi)^9}\int dp\,dr\,dr'\left(\frac{1}{2|p|^2}-\frac{\hat{u}_\uparrow(r+p)\hat{u}_\downarrow(r'-p)}{|r+p|^2-|r|^2+|r'-p|^2-|r'|^2+2\varepsilon}\right)(\hat{\chi}_<(p))^2\hat{v}_\uparrow(r)\hat{v}_\downarrow(r')\\&+\mathcal{E}_{T_1+T_2},\end{aligned}\tag{6.1}$$

with

$$|\mathcal{E}_{T_1+T_2}|\le C\left(\varrho^{\frac{7}{3}+\gamma}+\varrho^{\frac{8}{3}-2\gamma}+\varrho^{\frac{7}{3}-\gamma+\frac{7}{8}\delta}+\varrho^{3-3\gamma-\frac{3}{16}\delta}\right).$$

To obtain (6.1), we also used (2.3) and that $(2\pi)^{-3}\int dr\,\hat{v}_\sigma(r)=\varrho_\sigma$. Recall the conditions on the parameters, $0<\gamma<\frac{1}{3}$, $0<\delta\le 8\gamma$, and $2\gamma+\frac{\delta}{16}\le\frac{1}{3}$. By choosing $\gamma=1/9$ and $16/63\le\delta\le 8/9$, we fulfill these conditions and find that $|\mathcal{E}_{T_1+T_2}|\le C\varrho^{\frac{7}{3}+\frac{1}{9}}$.

To complete the proof of Theorem 1.2, we still have to remove the parameter $2\varepsilon=2\varrho^{2/3+\delta}$ in (6.1), which was used to have strictly positive lower bound on the relevant denominator, and the cut-off $(\hat{\chi}_<(p))^2$ which restricts the integration to $|p|\lesssim\varrho^{1/3-\gamma}$. This is the of the following lemma.

Lemma 6.1 (Removing the gap and the cut-off). *We have*

$$\begin{aligned}&\int dp\,dr\,dr'\left(\frac{\hat{u}_\uparrow(r+p)\hat{u}_\downarrow(r'-p)}{|r+p|^2-|r|^2+|r'-p|^2-|r'|^2+2\varrho^{\frac{2}{3}+\delta}}-\frac{1}{2|p|^2}\right)(\hat{\chi}_<(p))^2\hat{v}_\uparrow(r)\hat{v}_\downarrow(r')\\&=\int dp\,dr\,dr'\left(\frac{\hat{u}_\uparrow(r+p)\hat{u}_\downarrow(r'-p)}{|r+p|^2-|r|^2+|r'-p|^2-|r'|^2}-\frac{1}{2|p|^2}\right)\hat{v}_\uparrow(r)\hat{v}_\downarrow(r')+O(\varrho^{\frac{7}{3}+\delta})+O(\varrho^{\frac{7}{3}+\gamma}).\end{aligned}\tag{6.2}$$

Proof. Recall the notation $\lambda_{r,p}=|r+p|^2-|r|^2$. We split the proof in two parts. We first prove that

$$\int dpdrdr' \frac{\hat{u}_\uparrow(r+p)\hat{u}_\downarrow(r'-p)\hat{v}_\uparrow(r)\hat{v}_\downarrow(r')}{\lambda_{r,p}+\lambda_{r',-p}+2\varrho^{\frac{2}{3}+\delta}}(\hat{\chi}_<(p))^2$$

$$= \int dpdrdr' \frac{\hat{u}_\uparrow(r+p)\hat{u}_\downarrow(r'-p)\hat{v}_\uparrow(r)\hat{v}_\downarrow(r')}{\lambda_{r,p}+\lambda_{r',-p}}(\hat{\chi}_<(p))^2 + O(\varrho^{\frac{7}{3}+\delta}). \tag{6.3}$$

The difference between the two terms in (6.3) is given by

$$\mathrm{I} := 2\varrho^{\frac{2}{3}+\delta} \int dpdrdr' \frac{\hat{u}_\uparrow(r+p)\hat{u}_\downarrow(r'-p)\hat{v}_\uparrow(r)\hat{v}_\downarrow(r')}{(\lambda_{r,p}+\lambda_{r',-p})(\lambda_{r,p}+\lambda_{r',-p}+2\varrho^{\frac{2}{3}+\delta})}(\hat{\chi}_<(p))^2. \tag{6.4}$$

Without loss of generality let us assume that $k_F^\uparrow \geq k_F^\downarrow$. Rescaling the variables respect to $k_F^\uparrow$, we get

$$|\mathrm{I}| \leq C\varrho^{\frac{7}{3}+\delta} \int_{\substack{|r|<1<|r+p| \\ |r'|<k_F^\downarrow/k_F^\uparrow<|r'-p|}} dpdrdr' \frac{1}{(\lambda_{r,p}+\lambda_{r',-p})^2} \leq C\varrho^{\frac{7}{3}+\delta}, \tag{6.5}$$

where the boundedness of the integral follows from Lemma C.1 in Appendix C. To conclude it remains to show that

$$\mathrm{II} := \int dpdrdr' \left(\frac{\hat{u}_\uparrow(r+p)\hat{u}_\downarrow(r'-p)}{\lambda_{r,p}+\lambda_{r',-p}} - \frac{1}{2|p|^2} \right) \hat{v}_\uparrow(r)\hat{v}_\downarrow(r')\big(1-\hat{\chi}_<(p)^2\big) = O(\varrho^{\frac{7}{3}+\gamma}).$$

The integration over p runs over $|p| \geq 4\varrho^{1/3-\gamma}$ only, hence we have $\hat{u}_\uparrow(r+p)\hat{u}_\downarrow(r'-p) = 1$. Moreover, using that

$$\frac{1}{\lambda_{r,p}+\lambda_{r',-p}} - \frac{1}{2|p|^2} = \frac{p\cdot(r'-r)}{2|p|^4} + \frac{\big(p\cdot(r'-r)\big)^2}{|p|^4(\lambda_{r,p}+\lambda_{r',-p})}, \tag{6.6}$$

we can write

$$\mathrm{II} = \int dpdrdr' \frac{\big(p\cdot(r'-r)\big)^2}{|p|^4(\lambda_{r,p}+\lambda_{r',-p})} \hat{v}_\uparrow(r)\hat{v}_\downarrow(r')\big(1-\hat{\chi}_<(p)^2\big).$$

since the first term in (6.6) integrates to zero by symmetry. On the support of $1-\chi_<(p)^2$, $\lambda_{r,p}+\lambda_{r',-p}$ is bounded from below by Cp^2, hence

$$\mathrm{II} \leq C\varrho^{\frac{8}{3}} \int_{|p|>4\varrho^{\frac{1}{3}-\gamma}} \frac{dp}{|p|^4} = C\varrho^{\frac{7}{3}+\gamma}.$$

This completes the proof of the lemma. □

Applying Lemma 6.1 to the bound in (6.1) completes the proof of Theorem 1.2. To get the explicit form (1.7) of the Huang–Yang correction one has to compute the integral on the right-hand side of (6.2). This will be carried out in Appendix B.

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Appendix A: Useful Bounds on Scattering Solution and Cut-Off Functions

In this section we prove bounds on the solution of the zero-energy scattering equation (1.5), and on the cut-off functions we use in our analysis.

A.1 | Bounds on Scattering Solution

Let us start this section by collecting some well-known estimates for the scattering solution φ_∞.

Lemma A.1 (Properties of φ_∞). *Let V_∞ be as in Assumption 1.1, and φ_∞ the solution of (1.5). Then*

$$0 \le \varphi_\infty(x) \le \min\left\{1, \frac{a}{|x|}\right\} \quad \forall x \in \mathbb{R}^3, \qquad \varphi_\infty(x) = \frac{a}{|x|} \quad \forall x \in \mathbb{R}^3 \setminus (\mathrm{supp} V_\infty), \tag{A.1}$$

where $a = (8\pi)^{-1}\int_{\mathbb{R}^3} V_\infty(1-\varphi_\infty)$ is the scattering length of V_∞.

For the proof of Lemma A.1 we refer to [43] or [44]. Next we investigate $\tilde\varphi$ defined in (2.24), a periodized version of φ_∞ regularized by a momentum cut-off. Recall also the definition of φ in Definition 2.5.

Lemma A.2 (Bounds for $\tilde\varphi$). *Let $0 < \gamma < 1/3$. Let V_∞ as in Assumption 1.1 and let $\tilde\varphi$ be as in (2.24). Then the following bounds hold uniformly for L large,*

$$\|\tilde\varphi\|_{L^\infty(\Lambda)} \le C, \qquad \|\tilde\varphi\|_{L^2(\Lambda)} \le C\varrho^{-\frac{1}{6}+\frac{\gamma}{2}}, \tag{A.2}$$

and

$$\|\tilde\varphi\|_{L^1(\Lambda)} \le C\varrho^{-\frac{2}{3}+2\gamma}, \qquad \|\nabla\tilde\varphi\|_{L^1(\Lambda)} \le C\varrho^{-\frac{1}{3}+\gamma}, \qquad \|\Delta\tilde\varphi\|_{L^1(\Lambda)} \le C. \tag{A.3}$$

Furthermore, with φ as in Definition 2.5, for any $R > 0$

$$\lim_{L\to\infty}\sup_{|x|\le R}|\varphi_\infty(x) - \varphi(x)| = 0. \tag{A.4}$$

Proof. We start by proving (A.2). Let $f_\infty := 1-\varphi_\infty$. By the definition (2.24) and the zero-energy scattering Equation (1.5) we have that

$$|\widetilde{\varphi}(x)| \leq \frac{1}{L^3}\sum_p |\hat{\varphi}(p)| \leq \frac{C}{L^3}\sum_{p\neq 0}\frac{|\mathcal{F}(V_\infty f_\infty)(p)|}{|p|^2} \leq \frac{C}{L^3}\sum_{0\neq|p|\leq 1}\frac{1}{|p|^2}+\frac{C}{L^3}\sum_{|p|>1}\frac{|\mathcal{F}(V_\infty f_\infty)(p)|}{|p|^2},$$

where we used that $V_\infty f_\infty \in L^1(\mathbb{R}^3)$ because of Lemma A.1. Assumption 1.1 implies that also $V_\infty f_\infty \in L^2(\mathbb{R}^3)$ and the function has compact support. Hence, for large enough L, the $L^2(\Lambda)$-norm of its periodization equals the $L^2(\mathbb{R}^3)$ norm, and in particular,

$$\frac{1}{L^3}\sum_{p\in\Lambda^*}|\mathcal{F}(V_\infty f_\infty)(p)|^2 = \int_{\mathbb{R}^3} dx\,|V_\infty(x) f_\infty(x)|^2 \leq C\|V_\infty\|^2_{L^2(\mathbb{R}^3)} \leq C. \tag{A.5}$$

Therefore, by Cauchy–Schwarz and using that $\frac{1}{L^3}\sum_{|p|>1}\frac{1}{|p|^4}\leq C$, we can conclude that $|\widetilde{\varphi}(x)|\leq C$. For the L^2-norm, we have

$$\|\widetilde{\varphi}\|^2_{L^2(\Lambda)} = \frac{1}{L^3}\sum_p |\hat{\varphi}(p)|^2|\hat{\chi}_>(p)|^2 \leq \frac{C}{L^3}\sum_{|p|\geq 4\varrho^{1/3-\gamma}}\frac{1}{|p|^4}\leq C\varrho^{-\frac{1}{3}+\gamma},$$

where we used again that $|\hat{\varphi}(p)|\leq C/p^2$.

Remark A.3. The proof above actually shows that φ in Definition 2.5 is uniformly bounded, that is, $\|\varphi\|_{L^\infty(\Lambda)}\leq C$. However, $\|\varphi\|_{L^2(\Lambda)}$ diverges as $L\to\infty$.

In the following, we bound the L^1 norms of $\widetilde{\varphi}$ and its derivatives. With the aid of (1.5) and (2.16) we can write

$$\hat{\widetilde{\varphi}}(p) = \mathcal{F}(V_\infty f_\infty * g),$$

where $g:\mathbb{R}^3\to\mathbb{C}$ has Fourier transform $\mathcal{F}(g)(p)=\mathcal{F}(\chi_{>,\infty})(p)/(2|p|^2)$. The function $\widetilde{\varphi}$ is thus the periodization of $V_\infty f_\infty * g$, and hence

$$\|\widetilde{\varphi}\|_{L^1(\Lambda)} \leq \|V_\infty f_\infty * g\|_{L^1(\mathbb{R}^3)} \leq \|V_\infty f_\infty\|_{L^1(\mathbb{R}^3)}\|g\|_{L^1(\mathbb{R}^3)}.$$

It is a simple exercise, using the smoothness and support properties of $\chi_{>\infty}$, to check that $\|g\|_{L^1(\mathbb{R}^3)}\leq C\varrho^{-2/3+2\gamma}$, yielding the first bound in (A.3). The second bound on $\nabla\widetilde{\varphi}$ follows in the same way, using $\|\nabla g\|_{L^1(\mathbb{R}^3)}\leq C\varrho^{-1/3+\gamma}$. We omit the details. For the last bound in (A.3), we can directly use that $\Delta\varphi(x)=\sum_{n\in\mathbb{Z}^3}\Delta\varphi_\infty(x+nL)$ with at most one non-vanishing term in the sum, due to the compact support of V. This implies that $\Delta\varphi=\Delta\varphi_\infty$ on Λ. Using that $\hat{\chi}_> = 1-\hat{\chi}_<$, it then follows that

$$\Delta\widetilde{\varphi}(x) = \Delta\varphi(x) - \int_\Lambda dy\,\Delta\varphi(y)\chi_<(x-y).$$

Hence

$$\|\Delta\widetilde{\varphi}\|_{L^1(\Lambda)} \leq C\|V_\infty\|_{L^1(\mathbb{R}^3)}\big(1+\|\chi_<\|_{L^1(\Lambda)}\big)\leq C,$$

where we used again Assumption 1.1, (1.5), Lemma A.1 and the fact that $\|\chi_<\|_{L^1(\Lambda)}\leq C$ by Lemma A.5 below.

It remains to prove (A.4). We have

$$\varphi(x)-\varphi_\infty(x) = \frac{1}{L^3}\sum_{p\neq 0}\frac{\mathcal{F}(V_\infty f_\infty)(p)}{|p|^2}e^{ip\cdot x} - \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} dp\,\frac{\mathcal{F}(V_\infty f_\infty)(p)}{|p|^2}e^{ip\cdot x}.$$

The function $p\mapsto\mathcal{F}(V_\infty f_\infty)(p)e^{ip\cdot x}$ is square integrable and smooth, uniformly in x for bounded x. Approximating the integral by a Riemann sum, it is then a simple exercise to show that the right-hand side is $O(1/L)$ for large L, uniformly in x for bounded x. We leave the details to the reader. □

Lemma A.4 (Estimates of the L^1, L^2 norms of h_1). *Let $h_1:\Lambda\to\mathbb{R}$ be the periodic function with the Fourier coefficients defined in (4.49). Then $\|h_1\|_{L^1(\Lambda)}\to 0$ and $\|h_2\|_{L^2(\Lambda)}\to 0$ as $L\to\infty$.*

Proof. Using (1.5) and $\hat{\chi}_< + \hat{\chi}_> = 1$, we can write

$$\hat{h}_1(p) = \hat{g}(p)(1-\hat{\chi}_<(p)) \quad \text{with } \hat{g}(p) = \mathcal{F}(V_\infty\varphi_\infty)(p) - \widehat{V\varphi}(p).$$

The function $g : \Lambda \to \mathbb{R}$ with Fourier coefficients $\hat{g}(p)$ is thus the periodization of $V_\infty(\varphi_\infty - \varphi)$, and because of the compact support of V, we have for $x \in \Lambda = [-L/2, L/2]^3$ (and L large enough)

$$g(x) = V_\infty(x)(\varphi_\infty(x) - \varphi(x)).$$

From Assumption 1.1 and A.4, it readily follows that $\|g\|_{L^1(\Lambda)} \to 0$ and $\|g\|_{L^2(\Lambda)} \to 0$ as $L \to \infty$. Now

$$h_1(x) = g(x) - \int_\Lambda dy\, g(x-y)\chi_<(y),$$

hence the same holds for h_1 since $\|\chi_<\|_{L^1(\Lambda)} < C$, as will be shown in Lemma A.5 below. This completes the proof. □

A.2 | Bounds on the Cut-Off Functions

The following Lemma is an easy exercise that we leave to the reader. (Compare with [16, Prop. 4.2].)

Lemma A.5 (L^1 norm of the cut-off functions $\chi_<, \zeta^<$). *Let $\chi_< : \Lambda \to \mathbb{R}$ and $\zeta^< : \Lambda \to \mathbb{R}$ be the cut-off functions with Fourier coefficients introduced in (2.16) and (5.11), respectively. Then*

$$\|\chi_<\|_{L^1(\Lambda)} \leq C, \qquad \|\zeta^<\|_{L^1(\Lambda)} \leq C$$

The next lemma is somewhat more subtle, as it requires uniformity in t.

Lemma A.6 (L^1 norm of the function ζ^t). *Let ζ^t be as in (5.19) for $t \in (0, \infty)$. Then*

$$\|\zeta^t\|_{L^1(\Lambda)} \leq C. \tag{A.6}$$

Proof. We start by writing

$$\zeta^t(x) = \frac{1}{L^3}\sum_k e^{-t|k|^2} e^{ik\cdot x} - \frac{1}{L^3}\sum_k \zeta^<(k) e^{-t|k|^2} e^{ik\cdot x} = \zeta_1^t(x) + \zeta_2^t(x),$$

with $\zeta^<$ as in (5.11). The function ζ_1^t is the periodization of $\zeta_\infty^t(x) := (2\pi)^{-3}\int_{\mathbb{R}^3} dp\, e^{-tp^2} e^{ip\cdot x} = (4\pi t)^{-3/2} e^{-x^2/(4t)}$, hence $\|\zeta_1^t\|_{L^1(\Lambda)} \leq \|\zeta_\infty^t\|_{L^1(\mathbb{R}^3)} = 1$. To estimate the L^1 norm of ζ_2^t it is enough to notice that

$$\zeta_2^t(x) = \int_\Lambda dy\, \zeta_1^t(x-y)\zeta^<(y),$$

which implies that $\|\zeta_2^t\|_{L^1(\Lambda)} \leq \|\zeta_1^t\|_{L^1(\Lambda)}\|\zeta^<\|_{L^1(\Lambda)} \leq C$, using Lemma A.5. This yields (A.6). □

Appendix B: Evaluation of $F(x)$

In this section we explicitly evaluate the integral in (1.10), and show that it equals $a^2\varrho_\uparrow^{\frac{7}{3}}F(\varrho_\downarrow/\varrho_\uparrow)$ with F given in (1.7). In fact, rescaling the integrations over p, r, r' in (1.10) by $k_F^\uparrow = (6\pi^2)^{\frac{1}{3}}\varrho_\uparrow^{1/3}$, we find

$$F(x) = \frac{4}{\pi}(6\pi^2)^{1/3}\int_{\mathbb{R}^3} dp\left(\frac{x}{p^2} - g(x,p)\right) \tag{B.1}$$

and

$$g(x,p) = \frac{9}{8\pi^2}\int_{|k+p|>1>|k|} dk \int_{|q-p|>x^{1/3}>|q|} dq\, \frac{1}{(k+p)^2 - k^2 + (q-p)^2 - q^2}. \tag{B.2}$$

Lemma B.1. *For every $x > 0$, $F(x)$ can be written explicitly as in (1.7).*

Proof. Let us rewrite g as

$$g(x,p) = \frac{9}{8\pi^2}\lim_{\varepsilon\to 0}\int dk\,\chi(k)\int dq\,\chi(x^{-1/3}q)\frac{\chi^c(k+p)\chi^c(x^{-1/3}(q-p))}{(k+p)^2-k^2+(q-p)^2-q^2+i\varepsilon}$$

$$= \frac{9}{8\pi^2}\lim_{\varepsilon\to 0}\Re\int dk\,\chi(k)\int dq\,\chi(x^{-1/3}q)\frac{1-\chi(k+p)-\chi(x^{-1/3}(q-p))}{(k+p)^2-k^2+(q-p)^2-q^2+i\varepsilon}$$

where χ stands for the characteristic function of the unit ball, $\chi^c = 1-\chi$ and we have used that

$$\Re\int dk\,\chi(k)\int dq\,\chi(x^{-1/3}q)\frac{\chi(k+p)\chi(x^{-1/3}(q-p))}{(k+p)^2-k^2+(q-p)^2-q^2+i\varepsilon} = 0$$

because of symmetry. In particular, $g = g_0 - g_1 - g_2$ with

$$g_0(x,p) = \frac{9}{8\pi^2}\lim_{\varepsilon\to 0}\Re\int dk\,\chi(k)\int dq\,\chi(x^{-1/3}q)\frac{1}{(k+p)^2-k^2+(q-p)^2-q^2+i\varepsilon},$$

$$g_1(x,p) = \frac{9}{8\pi^2}\lim_{\varepsilon\to 0}\Re\int dk\,\chi(k)\int dq\,\chi(x^{-1/3}q)\frac{\chi(k+p)}{(k+p)^2-k^2+(q-p)^2-q^2+i\varepsilon},$$

$$g_2(x,p) = \frac{9}{8\pi^2}\lim_{\varepsilon\to 0}\Re\int dk\,\chi(k)\int dq\,\chi(x^{-1/3}q)\frac{\chi(x^{-1/3}(q-p))}{(k+p)^2-k^2+(q-p)^2-q^2+i\varepsilon}.$$

We claim that

$$\int_{\mathbb{R}^3} dp\left(\frac{x}{p^2} - g_0(x,p)\right) = 0$$

which follows from

$$\lim_{\varepsilon\to 0}\Re\int_{\mathbb{R}^3} dp\left(\frac{1}{2p^2} - \frac{1}{(k+p)^2-k^2+(q-p)^2-q^2+i\varepsilon}\right) = 0$$

for fixed k and q. In fact, the Fourier transform of the integrand above is proportional to

$$\frac{1}{|y|}\left(1 - e^{iy(k-q)/2}e^{-\sqrt{-(k-q)^2/4+i\varepsilon}|y|}\right).$$

Taking the real part and $\varepsilon \to 0$, this evaluates to 0 at $y = 0$.

We conclude that

$$F(x) = \frac{4}{\pi}(6\pi^2)^{1/3}\int_{\mathbb{R}^3} dp(g_1(x,p) + g_2(x,p)).$$

By simple scaling, $g_2(x,p) = x^{4/3}g_1(x^{-1}, x^{-1/3}p)$, hence also

$$F(x) = \frac{4}{\pi}(6\pi^2)^{1/3}\left(f(x) + x^{7/3}f(x^{-1})\right) \qquad \text{with} \quad f(x) = \int_{\mathbb{R}^3} dp\, g_1(x,p). \tag{B.3}$$

We have

$$f(x) = \frac{9}{8\pi^2}\lim_{\varepsilon\to 0}\Re\int dp\int dk\,\chi(k)\int dq\,\chi(x^{-1/3}q)\frac{\chi(k+p)}{(k+p)^2-k^2+(q-p)^2-q^2+i\varepsilon}$$

$$= \frac{9}{8\pi^2}\lim_{\varepsilon\to 0}\Re\int dk\,\chi(k)\int dq\,\chi(x^{-1/3}q)\int dp\,\frac{\chi(p)}{p^2-k^2+(q+k-p)^2-q^2+i\varepsilon}dp dq dk.$$

Let us compute

$$f'(x) = \frac{3}{8\pi^2}\lim_{\varepsilon\to 0}\Re\int_{\mathbb{R}^3} dk\,\chi(k)\int_{\mathbb{S}^2} d\omega\int_{\mathbb{R}^3} dp\,\frac{\chi(p)}{p^2-k^2+(x^{1/3}\omega+k-p)^2-x^{2/3}+i\varepsilon}$$

$$= \frac{3}{8\pi^2}x^{4/3}\lim_{\varepsilon\to 0}\Re\int_{\mathbb{R}^3} dk\,\chi(x^{1/3}k)\int_{\mathbb{S}^2} d\omega\int_{\mathbb{R}^3} dp\,\frac{\chi(x^{1/3}p)}{p^2-k^2+(\omega+k-p)^2-1+i\varepsilon}$$

and

$$x^2\left(x^{-4/3}f'(x)\right)' = -\frac{1}{8\pi^2}\lim_{\varepsilon\to 0}\Re\int_{\mathbb{S}^2}d\nu\int_{\mathbb{S}^2}d\omega\int_{\mathbb{R}^3}dp\,\frac{\chi(x^{1/3}p)}{p^2-x^{-2/3}+(\omega+x^{-1/3}\nu-p)^2-1+i\varepsilon}$$
$$-\frac{1}{8\pi^2}\lim_{\varepsilon\to 0}\Re\int_{\mathbb{R}^3}dk\int_{\mathbb{S}^2}d\omega\int_{\mathbb{S}^2}d\nu\,\frac{\chi(x^{1/3}k)}{x^{-2/3}-k^2+(\omega+k-x^{-1/3}\nu)^2-1+i\varepsilon}$$

which can further be rewritten as

$$x^{7/3}\left(x^{-4/3}f'(x)\right)' = -\frac{1}{8\pi^2}\lim_{\varepsilon\to 0}\Re\int_{\mathbb{S}^2}d\nu\int_{\mathbb{S}^2}d\omega\int_{\mathbb{R}^3}dp\,\frac{\chi(p)}{p^2-1+(x^{1/3}\omega+\nu-p)^2-x^{2/3}+i\varepsilon}$$
$$-\frac{1}{8\pi^2}\lim_{\varepsilon\to 0}\Re\int_{\mathbb{S}^2}d\omega\int_{\mathbb{S}^2}d\nu\int_{\mathbb{R}^3}dp\,\frac{\chi(p)}{1-p^2+(x^{1/3}\omega+\nu-p)^2-x^{2/3}+i\varepsilon}.$$

For the integration over p, we use that (for $a^2\geq 4b$)

$$\lim_{\varepsilon\to 0}\Re\int dp\,\frac{\chi(p)}{p^2+ap+b+i\varepsilon} = 2\pi-\frac{\pi}{2}\sqrt{a^2-4b}\ln\left|\frac{b-1-\sqrt{a^2-4b}}{b-1+\sqrt{a^2-4b}}\right|+\frac{\pi}{2}\frac{a^2-2b-2}{|a|}\ln\left|\frac{b+1-|a|}{b+1+|a|}\right|$$

as well as

$$\lim_{\varepsilon\to 0}\Re\int dp\frac{\chi(p)}{ap+b+i\varepsilon} = \frac{2\pi b}{a^2}-\frac{\pi}{|a|^3}\left(a^2-b^2\right)\ln\left|\frac{b-|a|}{b+|a|}\right|.$$

Straightforward manipulations then lead to

$$-8\pi x^{7/3}\left(x^{-4/3}f'(x)\right)' = 8\pi^2\left(2+\frac{1}{x^{1/3}}(x^{2/3}-1)\ln\frac{|1-x^{1/3}|}{1+x^{1/3}}\right).$$

Consequently, we conclude that

$$x^{-4/3}f'(x)-\frac{3}{7}A = \frac{1}{5}\left(9x^{-4/3}-2x^{-2/3}-\left(3x^{-5/3}-5x^{-1}\right)\ln\frac{|1-x^{1/3}|}{1+x^{1/3}}+2\ln\frac{x^{2/3}}{|1-x^{2/3}|}\right)$$

and hence

$$f(x)-Ax^{7/3}-B = \frac{9}{14}x^{1/3}+\frac{99}{70}x-\frac{6}{35}x^{5/3}+\frac{3}{20}\left(\frac{15}{7}-6x^{2/3}+5x^{4/3}\right)\ln\frac{|1-x^{1/3}|}{1+x^{1/3}}+24x^{7/3}\ln\frac{x^{2/3}}{|1-x^{2/3}|}$$

for some constants $A,B\in\mathbb{R}$. Since

$$0=F(0)=\frac{4}{\pi}(6\pi^2)^{1/3}\left(f(0)+\lim_{x\to 0}x^{7/3}f(x^{-1})\right)=\frac{4}{\pi}(6\pi^2)^{1/3}(B+A)$$

necessarily $B=-A$. The value of A is irrelevant as it drops out in the calculation of F using (B.3), and we arrive at the desired formula (1.7). □

Appendix C: Estimate of the Integral in (6.5)

In this section we prove the bound claimed in (6.5).

Lemma C.1.

$$\sup_{0<x\leq 1}\int_{\mathbb{R}^3}dp\int_{|r|<1<|r+p|}dr\int_{|r'|<x<|r'-p|}dr'\frac{1}{\left(|r+p|^2-|r|^2+|r'-p|^2-|r'|^2\right)^2}<\infty.$$

Proof. We start by observing that it suffices to integrate only over $|p|\leq 2$. For $|p|\geq 2$, the integrand is bounded by $C/|p|^4$, hence the resulting integral is finite and bounded uniformly in x.

In the following, let us use the notation $e_k := ||k|^2 - 1|$. Using that $|r' - p| \geq x$ and $x^2 - |r'|^2 \geq x(x - |r'|)$, the integral to consider is bounded above by

$$\mathrm{I} := \int_{|p|<2} dp \int_{|r|<1<|r+p|} dr \int_{|r'|<x<|r'-p|} dr' \frac{1}{e_{r+p}^2 + e_r^2 + x^2(x - |r'|)^2}.$$

Let us split the domain of integration into two domains, depending on whether $e_{r+p} \leq e_r$ or $e_r < e_{r+p}$, and denote the corresponding integrals by I_1 and I_2. In the first case, we have $1 < |r+p|^2 \leq 2 - |r|^2$. Dropping the positive term e_{r+p}^2 in the denominator, we thus get the upper bound

$$\mathrm{I}_1 \leq \frac{4\pi}{3} \int_{|r|<1} dr \int_{|r'|<x} dr' \frac{(2 - |r|^2)^{3/2} - 1}{e_r^2 + x^2(x - |r'|)^2} \leq C \int_{|r|<1} dr \int_{|r'|<x} dr' \frac{e_r}{e_r^2 + x^2(x - |r'|)^2}$$

where we have used that $(2 - |r|^2)^{3/2} - 1 \leq C(1 - |r|^2) = Ce_r$ for $|r| \leq 1$. We further have, for any $y > 0$,

$$\int_{|r'|<x} dr' \frac{y}{y^2 + x^2(x - |r'|)^2} = 4\pi \int_0^x ds \frac{ys^2}{y^2 + x^2(x - s)^2} \leq 4\pi \int_0^\infty ds \frac{yx^2}{y^2 + x^2 s^2} = 2\pi^2 x. \tag{C.1}$$

Hence $\mathrm{I}_1 \leq C$. For the second term I_2 we proceed similarly. If $e_r < e_{r+p}$ we have $(2 - |r+p|^2)_+ \leq |r|^2 \leq 1$, with $(\cdot)_+$ denoting the positive part. We change variables to $q = r + p$. Using $1 < |q| \leq |r| + |p| \leq 3$ and $1 - (2 - |q|^2)_+^{3/2} \leq C(|q|^2 - 1) = Ce_q$ we get

$$\begin{aligned}
\mathrm{I}_2 &\leq \int_{|r|<1} dr \int_{|r'|<x} dr' \int_{1<|q|<3} dq\, \mathbb{1}_{\{(2-|q|^2)_+ \leq |r|^2 \leq 1\}} \frac{1}{e_q^2 + x^2(x - |r'|)^2} \\
&= \frac{4\pi}{3} \int_{1<|q|<3} dq \int_{|r'|<x} dr' \frac{(1 - (2 - |q|^2)_+^{3/2}}{e_q^2 + x^2(x - |r'|)^2} \leq C \int_{|q|<3} dq \int_{|r'|<x} dr' \frac{e_q}{e_q^2 + x^2(x - |r'|)^2}
\end{aligned}$$

which is finite due to (C.1). □