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Schrödinger connection with selfdual nonmetricity vector in 2+1 dimensions

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ABSTRACT

We present a three-dimensional metric affine theory of gravity whose field equations lead to a connection introduced by Schrödinger many decades ago. Although involving nonmetricity, the Schrödinger connection preserves the length of vectors under parallel transport, and appears thus to be more physical than the one proposed by Weyl. By considering solutions with constant scalar curvature, we obtain a self-duality relation for the nonmetricity vector which implies a Proca equation that may also be interpreted in terms of inhomogeneous Maxwell equations emerging from affine geometry.

1. Introduction

In 1918, Weyl proposed a remarkable generalization of Riemannian geometry (see e.g. [1–4]) with an additional symmetry in an attempt of geometrically unifying electromagnetism with gravity [5,6]. In this theory, both the direction and the length of vectors vary under parallel transport. The connection introduced by Weyl involves a nonmetricity tensor whose trace part is known as the Weyl vector. However, Weyl's attempt to identify the trace part of the nonmetricity, associated with stretching and contraction, with the electromagnetic vector potential failed, due to observational inconsistencies [7].

On the other hand, in a series of papers written in the 1940s and collected in [8], with the aim to construct a unified field theory, Schrödinger proposed a symmetric connection which, although involving nonmetricity, preserves the length of vectors under parallel transport. The Schrödinger connection has the form

$$\hat{\Gamma}^{\lambda}{}_{\mu\nu} = \tilde{\Gamma}^{\lambda}{}_{\mu\nu} + g^{\lambda\rho} Z_{\rho\mu\nu} \,, \tag{1}$$

where $\tilde{\Gamma}^{\lambda}{}_{\mu\nu}$ denotes the Levi-Civita connection and $Z_{\mu\nu\rho}$ is a tensor obeying

$$Z_{\lambda\mu\nu} = Z_{\lambda\nu\mu} \,, \quad Z_{(\lambda\mu\nu)} = 0 \,. \tag{2}$$

The generic decomposition of an affine connection $\Gamma^{\lambda}{}_{\mu\nu}$ is given by

$$\Gamma^{\lambda}{}_{\mu\nu} = \tilde{\Gamma}^{\lambda}{}_{\mu\nu} + \underbrace{N^{\lambda}{}_{\mu\nu}}_{\text{distortion}} \,, \tag{3}$$

with

$$N^{\lambda}{}_{\mu\nu} = \underbrace{\frac{1}{2} g^{\rho\lambda}\left(Q_{\mu\nu\rho} + Q_{\nu\rho\mu} - Q_{\rho\mu\nu}\right)}_{\text{deflection}} \underbrace{- g^{\rho\lambda}\left(T_{\rho\mu\nu} + T_{\rho\nu\mu} - T_{\mu\nu\rho}\right)}_{\text{contorsion}} \,, \tag{4}$$

where $Q_{\lambda\mu\nu} \equiv -\nabla_{\lambda} g_{\mu\nu} = -\partial_{\lambda} g_{\mu\nu} + \Gamma^{\rho}{}_{\mu\lambda} g_{\rho\nu} + \Gamma^{\rho}{}_{\nu\lambda} g_{\mu\rho}$ and $T_{\mu\nu}{}^{\lambda} = \Gamma^{\lambda}{}_{[\mu\nu]}$ are the nonmetricity and the torsion tensor respectively [9]. In the case of vanishing torsion and $N_{(\lambda\mu\nu)} = 0$, the connection (3) reduces to

$$\Gamma^{\lambda}{}_{\mu\nu} = \tilde{\Gamma}^{\lambda}{}_{\mu\nu} - g^{\lambda\rho} Q_{\rho\mu\nu} \,, \tag{5}$$

with

$$Q_{(\lambda\mu\nu)} = 0 \,. \tag{6}$$

One immediately sees that (5) coincides with the Schrödinger connection (1) if we identify

$$Z_{\lambda\mu\nu} = -Q_{\lambda\mu\nu} \,. \tag{7}$$

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Any connection respecting (6) preserves the length of (but in general not the angle between) vectors under parallel transport [8,11]. Notice that, on the other hand, one could consider vanishing nonmetricity and then symmetrize the connection (3) in μ, ν to obtain

$$\Gamma^\lambda{}_{(\mu\nu)} := \check{\Gamma}^\lambda{}_{(\mu\nu)} = \tilde{\Gamma}^\lambda{}_{\mu\nu} - 2g^{\lambda\rho}T_{\rho(\mu\nu)} . \tag{8}$$

Then, $\check{\Gamma}^\lambda{}_{(\mu\nu)}$ coincides with (1) under the identification

$$Z_{\lambda\mu\nu} = -2T_{\lambda(\mu\nu)} . \tag{9}$$

Comparing (7) with (9), we see that (1) can be written either in terms of torsion, in terms of nonmetricity fulfilling (6), or in terms of both of them.

The Schrödinger connection seems to have been overlooked in the successive literature, despite its relevant features and the fact that it appears to be more physical than the one proposed by Weyl. In this paper, we present a metric affine theory of gravity in 2+1 spacetime dimensions whose field equations lead to a Schrödinger connection. Intriguingly, by considering solutions with constant scalar curvature, we obtain a self-duality relation [14] for the nonmetricity vector. This implies a Proca equation which may also be interpreted in terms of inhomogeneous Maxwell equations emerging from affine geometry, i.e., from a purely gravitational setup. In this scenario, gauge invariance follows from self-duality and we can properly dub the nonmetricity vector 'photon'.

2. Schrödinger connection with selfdual nonmetricity vector

Before introducing our theory, let us recall the irreducible decomposition of the nonmetricity tensor $Q_{\lambda\mu\nu}$ under the Lorentz group, that reads in three dimensions

$$\begin{aligned} Q_{\lambda\mu\nu} = {} & \frac{2}{5}Q_\lambda g_{\mu\nu} - \frac{1}{5}\tilde{Q}_\lambda g_{\mu\nu} + \frac{3}{5}g_{\lambda(\nu}\tilde{Q}_{\mu)} \\ & - \frac{1}{5}g_{\lambda(\nu}Q_{\mu)} + \Omega_{\lambda\mu\nu} , \end{aligned} \tag{10}$$

where $Q_\lambda \equiv Q_{\lambda\mu}{}^\mu$ and $\tilde{Q}_\lambda \equiv Q^\mu{}_{\mu\lambda}$ are nonmetricity vectors and $\Omega_{\lambda\mu\nu}$ is the traceless part of the nonmetricity.

We propose the action

$$\begin{aligned} S = {} & \frac{1}{2\kappa^2}\int d^3x \left(\sqrt{-g}f(R) + \frac{1}{2\mu}\epsilon^{\mu\nu\rho}Q_\rho \hat{R}_{\nu\mu}\right) \\ & + \int d^3x\, \epsilon^{\mu\nu\rho}\zeta_{\nu\sigma}T_{\rho\mu}{}^\sigma , \end{aligned} \tag{11}$$

where κ denotes the gravitational coupling constant, $f(R)$ is an arbitrary function of the scalar curvature $R = g^{\mu\nu}R_{\mu\nu}(\Gamma)$ (Γ is a general affine connection), $\hat{R}_{\mu\nu} := R^\lambda{}_{\lambda\mu\nu} = \partial_{[\mu}Q_{\nu]}$ denotes the homothetic curvature tensor, μ is a Chern-Simons coupling constant, and $\zeta_{\nu\sigma}$ a Lagrange multiplier. In (11) we also introduced the Levi-Civita symbol $\epsilon^{\mu\nu\rho} = \sqrt{-g}\varepsilon^{\mu\nu\rho}$, where $\varepsilon^{\mu\nu\rho}$ is the Levi-Civita tensor. The action (11) is manifestly diffeomorphism-invariant.

We work in the Palatini formalism, where the metric $g_{\mu\nu}$ and the connection $\Gamma^\lambda{}_{\mu\nu}$ are independent variables. From the variation of (11) with respect to $\zeta_{\mu\nu}$, we get vanishing torsion,

$$T_{\rho\sigma}{}^\nu = 0 . \tag{12}$$

The variation w.r.t. $g^{\mu\nu}$ leads to

$$f'(R)R_{(\mu\nu)} - \frac{1}{2}f(R)g_{\mu\nu} = 0 . \tag{13}$$

Notice that the Chern-Simons term and the piece involving the Lagrange multiplier do not contribute to (13). The trace of (13) yields

$$\frac{f}{2f'} = \frac{R}{3} , \tag{14}$$

which is identically satisfied if we choose

$$f(R) = CR^{3/2} , \tag{15}$$

with C an arbitrary integration constant. With the choice (15), the action (11) becomes invariant under the conformal transformation (see also [11,12])

$$g_{\mu\nu} \mapsto g'_{\mu\nu} = e^{2\Omega}g_{\mu\nu} , \qquad \Gamma^\lambda{}_{\mu\nu} \mapsto \Gamma'^\lambda{}_{\mu\nu} = \Gamma^\lambda{}_{\mu\nu} , \tag{16}$$

where Ω is a scalar function. On the other hand, (14) can also be viewed as an algebraic equation for R admitting generically solutions with constant scalar curvature [11],

$$R = c_k . \tag{17}$$

Plugging (14) into (13), the latter becomes

$$R_{(\mu\nu)} - \frac{R}{3}g_{\mu\nu} = 0 . \tag{18}$$

Observe that, together with (12), (18) would correspond to the Einstein-Weyl equations in three dimensions in the case in which one considers Weyl nonmetricity (see e.g. [12]). Varying (11) w.r.t. $\Gamma^\lambda{}_{\mu\nu}$ and using (12), one gets

$$\begin{aligned} & P_\lambda{}^{\mu\nu} + \delta_\lambda{}^\nu\frac{\partial^\mu f'}{f'} - g^{\mu\nu}\frac{\partial_\lambda f'}{f'} \\ & + \frac{2}{\mu f'}\varepsilon^{\nu\rho\sigma}\delta_\lambda{}^\mu\hat{R}_{\rho\sigma} + \frac{2\kappa^2}{f'}\varepsilon^{\mu\nu\rho}\zeta_{\rho\lambda} = 0 , \end{aligned} \tag{19}$$

where

$$P_\lambda{}^{\mu\nu} = -\frac{\nabla_\lambda\left(\sqrt{-g}g^{\mu\nu}\right)}{\sqrt{-g}} + \frac{\nabla_\sigma\left(\sqrt{-g}g^{\mu\sigma}\right)\delta^\nu{}_\lambda}{\sqrt{-g}} \tag{20}$$

is the Palatini tensor with vanishing torsion. Taking the λ, μ trace of (19) and contracting with the Levi-Civita tensor, we get

$$\zeta_{[\rho\sigma]} = \frac{3}{\kappa^2\mu}\hat{R}_{\rho\sigma} . \tag{21}$$

Plugging (21) into (19), using (10), and taking the λ, ν trace of the resulting equation, one finds

$$\tilde{Q}^\mu - \frac{Q^\mu}{2} + \frac{\partial^\mu f'}{f'} - \frac{4}{\mu}\varepsilon^{\mu\rho\sigma}\hat{R}_{\rho\sigma} = 0 . \tag{22}$$

Then, with (21) and (22), the μ, ν trace of (19) yields

$$\frac{\partial_\lambda f'}{f'} = \frac{1}{6}Q_\lambda + \frac{2}{3\mu f'}\varepsilon^{\alpha\rho\sigma}g_{\alpha\lambda}\hat{R}_{\rho\sigma} . \tag{23}$$

Inserting also (23) into (19) after use of (21) and (22), and taking different contractions with the Levi-Civita tensor, we obtain

$$\zeta_{(\lambda\nu)} = 0 , \tag{24}$$

and vanishing traceless part of the nonmetricity,

$$\Omega_{\lambda\mu\nu} = 0 . \tag{25}$$

Using (21) and (24), one gets

$$\zeta_{\rho\sigma} = \frac{3}{\kappa^2\mu}\hat{R}_{\rho\sigma} . \tag{26}$$

With (22), (23), (25) and (26), (19) becomes

$$\hat{R}_{\rho\sigma}\delta_\lambda{}^{[\mu}\varepsilon^{\nu]\rho\sigma} + \hat{R}_{\lambda\rho}\varepsilon^{\mu\nu\rho} = 0 , \tag{27}$$

Plugging (23) into (22), we find

$$\hat{R}_{\rho\sigma} = \frac{\mu f'}{20}\varepsilon^{\mu\nu\lambda}g_{\rho\mu}g_{\sigma\nu}\left(Q_\lambda - 3\tilde{Q}_\lambda\right), \tag{28}$$

which, used in (23), leads to

$$\frac{\partial_\lambda f'}{f'} = \frac{1}{10}\left(Q_\lambda + 2\tilde{Q}_\lambda\right). \tag{29}$$

Notice also that, exploiting (28), (26) becomes

$$\zeta_{\rho\sigma} = \frac{3f'}{20\kappa^2}\varepsilon^{\mu\nu\lambda}g_{\rho\mu}g_{\sigma\nu}\left(Q_\lambda - 3\tilde{Q}_\lambda\right). \tag{30}$$

Finally, using (28) we can see that (27) is identically satisfied.

Summarizing, one has

$$Q_{\lambda\mu\nu} = \frac{2}{5}Q_\lambda g_{\mu\nu} - \frac{1}{5}\tilde{Q}_\lambda g_{\mu\nu} + \frac{3}{5}g_{\lambda(\nu}\tilde{Q}_{\mu)} - \frac{1}{5}g_{\lambda(\nu}Q_{\mu)}, \tag{31}$$

together with (12), (18), (28) and (29). The final form of the connection, obtained by plugging (12) and (31) into (4), results to be

$$\begin{aligned}\Gamma^\lambda{}_{\mu\nu} = \tilde{\Gamma}^\lambda{}_{\mu\nu} - \frac{3}{10}g_{\mu\nu}Q^\lambda + \frac{2}{5}\delta_{(\mu}{}^\lambda Q_{\nu)} \\ + \frac{2}{5}g_{\mu\nu}\tilde{Q}^\lambda - \frac{1}{5}\delta_{(\mu}{}^\lambda\tilde{Q}_{\nu)}.\end{aligned} \tag{32}$$

Observe that with the choice (15) (with $C=1$), (28) and (29) would lead to the generalized monopole equation

$$\star(d\Sigma + h\Sigma) = dh, \tag{33}$$

where the one-form h and the function Σ are respectively defined by $h_\lambda = -Q_\lambda/6$ and $\Sigma = 3\mu\sqrt{R}/8$, together with

$$\partial_\mu \ln R = \frac{1}{5}\left(Q_\mu + 2\tilde{Q}_\mu\right). \tag{34}$$

Actually, (33) represents a special case of the generalized monopole equation [12]. If Σ were constant (which can always be achieved by a Weyl rescaling (16), under which $\Sigma \mapsto e^{-\Omega}\Sigma$), (33) would boil down to $dh = \star h\Sigma$, which is the self-duality condition [14] in three dimensions. We can thus regard (33) as a conformally invariant generalization of the three-dimensional self-duality condition. We will further elaborate on this point in the sequel. Notice that, in that case, (34) would yield

$$\tilde{Q}_\mu = -\frac{1}{2}Q_\mu. \tag{35}$$

Setting Σ constant above has the same effects as considering the solutions (17) (we discard the trivial case $R=0$). Then, $f(R)$ is constant and we can write

$$f'(R) = C_0, \tag{36}$$

where C_0 is an arbitrary constant. Thus one has $\partial_\lambda f' = 0$, and (29) leads to (35), so that we are left with just one independent nonmetricity vector. Using (17), (18) becomes

$$R_{(\mu\nu)} = \frac{c_k}{3}g_{\mu\nu}. \tag{37}$$

Then, inserting (36) and (35) into (30) and (28), one obtains

$$\zeta_{\rho\sigma} = \frac{3C_0}{8\kappa^2}\varepsilon_{\rho\sigma\tau}Q^\tau, \tag{38}$$

$$\hat{R}_{\rho\sigma} = \frac{C_0\mu}{8}\varepsilon_{\rho\sigma\tau}Q^\tau. \tag{39}$$

Note that (39) can be dualized as

$$-\frac{C_0\mu}{4}Q_\alpha = \varepsilon_{\alpha\rho\sigma}\partial^\rho Q^\sigma, \tag{40}$$

which implies

$$\tilde{\nabla}^\mu Q_\mu = 0, \tag{41}$$

where $\tilde{\nabla}$ denotes the Levi-Civita covariant derivative. Using (35), the nonmetricity tensor (31) becomes

$$Q_{\lambda\mu\nu} = \frac{1}{2}Q_\lambda g_{\mu\nu} - \frac{1}{4}Q_\mu g_{\lambda\nu} - \frac{1}{4}Q_\nu g_{\lambda\mu}, \tag{42}$$

and, in particular, we have

$$Q_{(\lambda\mu\nu)} = 0. \tag{43}$$

Consequently, (32) boils down to

$$\Gamma^\lambda{}_{\mu\nu} = \tilde{\Gamma}^\lambda{}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}Q^\lambda + \frac{1}{2}\delta_{(\nu}{}^\lambda Q_{\mu)} = \tilde{\Gamma}^\lambda_{\mu\nu} - g^{\lambda\rho}Q_{\rho\mu\nu}. \tag{44}$$

(44) corresponds thus to a Schrödinger connection (1) with (7) and nonmetricity given by (42). Moreover, (40) implies that in the present case the nonmetricity vector Q_μ is selfdual [14].

2.1. Self-duality in three dimensions and inhomogeneous Maxwell equations

In what follows, we discuss some interesting consequences of our theory arising from the self-duality relation (40).

Let us first recall that the authors of [14] showed that for space-time dimension $n = 4k-1$, $k = 1,2,3,\ldots$, one may take the 'square root' of the Proca equation for a massive antisymmetric tensor field. The result is a selfdual field; (40) corresponds to the case $k=1$ and can be rewritten as

$$Q_\mu = \frac{1}{2M}\varepsilon_{\mu\nu\rho}\mathcal{F}^{\nu\rho}, \tag{45}$$

where $M = -C_0\mu/4$ is interpreted as a mass parameter that depends, in particular, on the Chern-Simons coupling μ, and we defined the field strength

$$\mathcal{F}_{\mu\nu} = 2\tilde{\nabla}_{[\mu}Q_{\nu]} = 2\partial_{[\mu}Q_{\nu]} = 2\hat{R}_{\mu\nu}. \tag{46}$$

The dual form of (45) reads

$$\mathcal{F}_{\alpha\beta} = M\varepsilon_{\mu\beta\alpha}Q^\mu. \tag{47}$$

One shews that (45) implies (41), together with the covariant Proca equation

$$\tilde{\nabla}^\mu\mathcal{F}_{\mu\nu} - M^2 Q_\nu = 0. \tag{48}$$

Notice that, using (47), (48) can also be written in the form

$$\partial^\mu\mathcal{F}_{\mu\nu} - \frac{1}{2}g^{\alpha\beta}\partial^\mu g_{\alpha\beta}\mathcal{F}_{\mu\nu} - M^2 Q_\nu = 0. \tag{49}$$

Thus, the vector Q_μ could be interpreted as a massive photon gauge field.

Furthermore, (48) corresponds to an inhomogeneous electromagnetic wave equation which follows from the inhomogeneous Maxwell equations. In this context, we find that Q_μ results to be massless and source of itself. In order to see this explicitly, let us consider the following wave equation, implied by (48) and (41):

$$\left(\Box - M^2\right)Q^\mu = 0, \tag{50}$$

where $\Box \equiv \tilde{\nabla}_\alpha\tilde{\nabla}^\alpha$. On the other hand, the inhomogeneous electromagnetic wave equation for Q^μ in the gauge (41) reads

$$\Box Q^{\mu} = \mu_0 J^{\mu} \quad \Leftrightarrow \quad \Box Q^{\mu} = \tilde{\nabla}_{\rho} \mathcal{F}^{\rho\mu} , \tag{51}$$

where μ_0 is the vacuum permeability and J^{μ} the current. Now plug (46) and (47) into (50) to find

$$\Box Q^{\mu} = \frac{M}{2} \varepsilon^{\mu\rho\sigma} \mathcal{F}_{\rho\sigma} = \tilde{\nabla}_{\rho} \left(M \varepsilon^{\mu\rho\sigma} Q_{\sigma} \right) = \tilde{\nabla}_{\rho} \mathcal{F}^{\rho\mu} . \tag{52}$$

We see that (52) coincides with (51), and we can also write

$$J^{\mu} \equiv \frac{1}{\mu_0} \tilde{\nabla}_{\rho} \mathcal{F}^{\rho\mu} = \frac{M}{2\mu_0} \varepsilon^{\mu\rho\sigma} \mathcal{F}_{\rho\sigma} = \frac{M^2}{\mu_0} Q^{\mu} . \tag{53}$$

The source current J^{μ} is covariantly conserved, $\tilde{\nabla}_{\mu} J^{\mu} = 0$. Nevertheless, by this last equation we cannot directly define a globally conserved charge, since we need a local conservation law to do this. On the other hand, considering (49), we deduce that the total current (source current plus self-current)

$$j^{\mu} = J^{\mu} + \frac{1}{2\mu_0} g^{\alpha\beta} \partial_{\rho} g_{\alpha\beta} \mathcal{F}^{\mu\rho} \tag{54}$$

is locally conserved, $\partial_{\mu} j^{\mu} = 0$.

Concluding, (48), which involves a massive photon, can also be interpreted as the inhomogeneous Maxwell equations, with corresponding wave equation, where the photon is source of itself, due to the self-duality relation (40). Gauge invariance of the connection and of the inhomogeneous Maxwell equations follows from self-duality.

We observe that in this new model the metric has no degree of freedom (we recall that in three spacetime dimensions the Weyl tensor vanishes identically). On the other hand, the non-metricity vector Q_{μ} has one dynamical degree of freedom. This is due to the condition $\partial^{\mu} Q_{\mu} = 0$, implied by the Proca equation (48), so that only two of the three components of Q_{μ} are independent, and the self-duality relation (40), for which, at the end, only one mode is propagated (cf. [14] for details on this counting).

3. Discussion

We presented a three-dimensional metric affine theory of gravity whose field equations lead, considering the particular solution with constant scalar curvature, to a connection introduced by Schrödinger in the 1940s. Although involving nonmetricity, the latter preserves the length of vectors under parallel transport. Furthermore, we obtained a self-duality relation for the nonmetricity vector Q_{μ} leading to a Proca equation which may also be interpreted as inhomogeneous Maxwell equation. Gauge invariance follows from self-duality and we can conclude that, in our framework, the inhomogeneous Maxwell equations emerge from affine geometry, i.e., from a purely gravitational setup.

Let us mention that similar results were obtained in [12] (where the authors presented for the first time an action principle for the Einstein-Weyl equations in three dimensions), where the Weyl vector is self-dual. We also remark that the Chern-Simons and Lagrange multiplier terms in (11) break the projective invariance of the connection, which allows for consistent matter couplings (cf. the discussion in [12]).

The model and the results presented in this paper not only appear to be relevant under geometrical and physical perspectives but they also aim to highlight some peculiar and intriguing features of the Schrödinger connection that have been somewhat overlooked in the past literature. Future developments could consist in studying possible applications of our results in the classification of supersymmetric near-horizon geometries along the lines of [15,16], and in the phenomenology related to dark matter and dark energy. It remains to be seen if our model can be extended to higher dimensions.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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