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A Categorical-Algebraic Exploration of Models for Many-Valued Logic

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Introduction

Łukasiewicz logic was introduced in the early 1900s as a non-classical logic system. Unlike classical logic, which only has 0 and 1 as truth values, Łukasiewicz logic is a many-valued logic system that accepts any real number between 0 and 1 as truth value. It is especially helpful to formalize situations where there is uncertainty or imprecision in statements, such as in natural language, where statements may only be partially true or false. The algebraic models of Łukasiewicz logic are the MV-algebras. *MV-algebras* are a class of algebras that generalize Boolean algebras, which are used to model classical logic. In particular, an MV-algebra is an algebraic structure $(A, \oplus, \neg, 0)$ where $\oplus$ is a binary operation which is both associative, commutative, with the constant 0 as neutral element, and $\neg$ is a unary operation, such that the following equalities hold: $\neg\neg x = x$, $x \oplus \neg 0 = \neg 0$, and $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$. The aim of these conditions is to capture particular characteristics of the real unit interval $[0, 1]$, equipped with two operations: the negation $\neg x \coloneqq 1 - x$ and the truncated addition $x \oplus y \coloneqq \min(1, x + y)$. It has been proved in [44] that the variety $\mathbb{MV}$ of MV-algebras is equivalent to the category of lattice-ordered abelian groups with fixed order-unit, denoted by $u\ell\mathbb{A}b$. A *lattice-ordered group* is an algebraic structure of signature $\{+, 0, -, \vee, \wedge\}$ satisfying the axioms of groups, the axioms of lattices, and the axioms related to the distributivity of the group operation over both the lattice operations. A lattice-ordered group is *abelian* if the group operation is commutative. Given a lattice-ordered group G we can define, for every element x of G, $|x| \coloneqq x \vee -x$. An *order-unit* u of G is an element $0 \leq u \in G$ satisfying the following property: for every $x \in G$, there exists a natural number $n \in \mathbb{N}$ such that $|x| \leq nu$. Despite their importance in logic, MV-algebras and lattice-ordered groups have been underexplored from a categorical-algebraic perspective. The goal of this thesis is precisely to investigate the categorical properties of these structures.

The first chapter focuses on the category $\ell\mathbb{G}rp$ of lattice-ordered groups. Being a variety of universal algebras with a unique constant and a group operation, $\ell\mathbb{G}rp$ is a semi-abelian category. Semi-abelian categories were introduced in [39] in order to capture the categorical properties of groups, following the same idea that led to the notion of abelian category, which describes very efficiently the properties of abelian groups and modules over a ring. Semi-abelian categories are pointed, Barr-exact, finitely cocomplete category where the Split Short Five Lemma holds.

However, the effectiveness of the notion of semi-abelian category in expressing the properties of groups is weaker than the notion of abelian category in relation with abelian groups and modules. For this reason, additional conditions have been considered in recent years to get a better approximation of the structural properties of the category of groups; among these, one can mention representability of actions, algebraic coherence, and strong protomodularity. In the first chapter of this thesis, we show that $\ell\mathbb{G}rp$ is strongly protomodular [8], fiber-wise algebraically cartesian closed [15], and its full subcategory $\ell\mathbb{A}b$ of lattice-ordered abelian groups is algebraically coherent [22]. Neither $\ell\mathbb{G}rp$ nor $\ell\mathbb{A}b$ is action accessible [18]; this observation answers an open question presented in [22].

The second chapter of this thesis is dedicated to exploring various categorical properties of the category of MV-algebras. Although the category of MV-algebras is not pointed and, as such, not semi-abelian, it possesses several noteworthy categorical properties. For instance, $\mathbb{MV}$ is a protomodular and arithmetical category, and we provide explicit examples of protomodularity and arithmeticity terms. Additionally, using the categorical equivalence between $\mathbb{MV}$ and $u\ell\mathbb{A}b$, it is proved that in every category of points on $\mathbb{MV}$, subobjects have centralizers. A significant portion of this chapter is focused on the study of idempotent elements in MV-algebras. Given an MV-algebra A, an element $e \in A$ is *idempotent* if it satisfies the identity $e \oplus e = e$. The subset $\mathrm{Idem}(A)$ of idempotent elements of A is a subalgebra in which all elements are idempotent. In other words, $\mathrm{Idem}(A)$ is a Boolean algebra with respect to the inherited operations from A. This assignment establishes an adjunction: the functor $\mathrm{Idem}\colon \mathbb{MV} \to \mathbb{B}\mathrm{oole}$ is the right adjoint of the inclusion functor $i\colon \mathbb{B}\mathrm{oole} \to \mathbb{MV}$. By exploiting the adjunction just introduced and the classical results of Stone Duality, we define the *Pierce structural space* relative to the idempotents of an MV-algebra. The study of this structural space leads to similar results to those obtained for unitary rings and presented in [4]. Thus, this chapter provides a framework for studying the adjunction between MV-algebras and Boolean algebras from the perspective of categorical Galois theory, offering valuable insights for future research in this area.

In the third chapter, we investigate the relationship between the category of MV-algebras and two of its full subcategories: the subcategory $p\mathbb{MV}$ of perfect MV-algebras and the subcategory $s\mathbb{MV}$ of semisimple MV-algebras. For a given MV-algebra A, its *radical*, denoted by $\mathrm{Rad}(A)$, is the intersection of all maximal ideals of A. The radical of A can be characterized as follows: an element a belongs to $\mathrm{Rad}(A)$ if and only if $na \leq \neg a$ for every natural number n, where na denotes the iterated sum of the element a with itself n times. An MV-algebra is *perfect* if it can be expressed as the union of its radical and the negation of it. An MV-algebra is *semisimple* if its radical is trivial, i.e. it consists only of 0. This pair of subcategories of $\mathbb{MV}$ defines a pretorsion theory. The definition of *pretorsion theory* [30] was introduced to generalize the notion of torsion theory to the non-pointed case. In this chapter we also prove that the reflection of semisimple MV-algebras gives rise to an admissible adjunction for the categorical Galois theory.

In particular, we provide an explicit description of central extensions with respect to this Galois structure. These central extensions are exactly the surjective morphisms of MV-algebras whose restriction to the torsion part is either injective or the unique arrow from the initial object to the terminal one. Furthermore, this pretorsion theory induces a stable factorization system $(\mathscr{E}, \mathscr{M})$ on the category $\mathbb{MV}$, where $\mathscr{E}$ consists of all surjective morphisms of MV-algebras whose kernel is contained in the radical of the domain, while $\mathscr{M}$ is given by all morphisms of MV-algebras whose kernel has trivial intersection with the radical of the domain. Finally, by observing that central extensions form a reflective subcategory in the category of regular epimorphisms and that also the adjunction induced by this reflection is admissible for categorical Galois theory, it is possible to study and characterize higher-order central extensions. Using this characterization, we can provide a notion of commutator, in the non-pointed case, between two ideal subalgebras of an MV-algebra, with respect to this Galois structure.

The good properties of the reflection of semisimple MV-algebras have led us to generalize the results obtained in [27] to the non-pointed case, highlighting the connection between pretorsion theories, Galois structures, and stable factorization systems. Specifically, we work in protomodular, Barr-exact and finitely cocomplete categories $\mathbb{C}$, satisfying the following conditions: the initial object $\mathbf{2}$ and the terminal object $\mathbf{1}$ are not isomorphic; every arrow with domain $\mathbf{2}$ and codomain a non-terminal object is a monomorphism; and the terminal object is strict, meaning that every morphism from $\mathbf{1}$ is an isomorphism.
In [26], the authors introduced the notion of a *protoadditive* functor: a functor between pointed protomodular categories is protoadditive if it preserves split short exact sequences. Moreover, the authors of [27] show that a zero-preserving functor between pointed protomodular categories is protoadditive if and only if it preserves pullbacks along split epimorphisms. This characterization allows us to extend this notion to our framework: we define a functor to be protoadditive if it preserves the terminal object, the initial object, and the pullbacks along split epimorphisms.
In the fourth chapter, our attention is devoted to the study of pretorsion theories $(\mathscr{T}, \mathscr{F})$ defined in categories which satisfy the properties described above, such that $\mathscr{T} \cap \mathscr{F} = \{\mathbf{1}, \mathbf{2}\}$. While exploring this setting, we encounter a class of torsion objects that exhibit anomalous behavior. Namely, the class of objects $B \in \mathbb{C}$ such that $F(B) = \mathbf{1}$, where F denotes the reflector onto the subcategory $\mathscr{F}$. In order to handle such objects and obtain results analogous to those presented in [27], we introduce new conditions for a pretorsion theory to satisfy with respect to any objects $A, B \in \mathbb{C}$:

(P1) if $F(B) = \mathbf{1}$, then $F(A \times B) \cong F(A)$;

(P2) if $F(B) = \mathbf{1}$ and $A \neq \mathbf{1}$, then the sequence $\mathbf{2} \times B \xrightarrow{\iota_A \times id_B} A \times B \xrightarrow{\pi_A} A$ is pre-exact, and $\mathbf{2} \times B$ belongs to $\mathscr{T}$.

The conditions described above enable us to define both a Galois structure and a stable factorization system starting from a pretorsion theory.

Specifically, with a pretorsion theory that satisfies (P1) and has a protoadditive reflector onto the subcategory of torsion-free objects, we can construct an admissible adjunction for categorical Galois theory. For this Galois structure, the following fact holds: an effective descent morphism f is a central extension if and only if the domain of the prekernel of f, defined up to isomorphism, is a torsion-free object.

Furthermore, we can define a stable factorization system $(\overline{\mathscr{E}}, \overline{\mathscr{M}})$ on the category $\mathbb{C}$ from a pretorsion theory that satisfies condition (P2) and where, for every arrow $f\colon A \to B$, the composition $T(K[f]) \xrightarrow{\varepsilon_{K[f]}} K[f] \xrightarrow{k} A$ is the prekernel of an arrow with domain A (k represents the prekernel of f, and T is the coreflector on the subcategory $\mathscr{T}$). Specifically, $\overline{\mathscr{E}}$ consists of all precokernels e with $K[e] \in \mathscr{T}$, while $\overline{\mathscr{M}}$ consists of all arrows m with $K[m] \in \mathscr{F}$.

In conclusion, we present some examples of pretorsion theories that satisfy our properties, in addition to our guiding example for MV-algebras. In the dual of the category of M-sets (where M is a fixed monoid), we can consider the pretorsion theory whose torsion objects are the M-sets with at most one fixed point, while the torsion-free objects are the M-sets consisting only of fixed points. In the category $\mathbb{H}$eyt of Heyting algebras, we study the pretorsion theory whose torsion objects are the Heyting algebras in which the negation of each element is either 0 or 1, and the torsion-free objects are the Boolean algebras. Finally, we present a pretorsion theory in the dual of the category of simplicial sets, where the torsion objects are the simplicial sets with at most one vertex, and the torsion-free objects are the simplicial sets consisting only of vertices.

Chapter 1

Categorical-Algebraic Properties of Lattice-ordered Groups

A *lattice-ordered group* is a set endowed with both a group structure and a lattice structure such that the underlying order relation is invariant under translations. In other words, a lattice-ordered group can be defined as an algebraic structure of signature $\{\cdot, e, ^{-1}, \vee, \wedge\}$ satisfying the axioms of groups, the axioms of lattices, and the axioms related to the distributivity of the group product over both the lattice operations. Therefore, the category of lattice-ordered groups (denoted by $\ell\mathbb{G}rp$) can be presented as the variety of models associated with the equational theory just described.

Recently, lattice-ordered groups have emerged in many areas of mathematics. For instance, in the study of many-valued logic (as shown in [44], the category of lattice-ordered abelian groups with a distinguished order-unit is equivalent to the one of MV-algebras, which provides algebraic semantics for Łukasiewicz many-valued propositional logic [21]), in the theory of Bézout domains, in complex intuitionistic fuzzy soft set theory, and in varietal questions in universal algebra.

Although the notion of lattice-ordered groups is as natural as that of rings or partially ordered groups (it suffices to say that examples of lattice-ordered groups include the set of integers $\mathbb{Z}$, the set of rational numbers $\mathbb{Q}$, and the set of real numbers $\mathbb{R}$ with the usual group sum and the usual order structure), there are currently no studies about this variety from a categorical point of view. The purpose of this work is precisely to explore these aspects.

A first observation is that the category of lattice-ordered groups is semi-abelian. In the same spirit of how abelian categories describe the properties of the categories of abelian groups and of modules over a ring, the notion of semi-abelian category is aimed to capture the homological properties of the category of groups. In short, a *semi-abelian category* [39] is a pointed finitely cocomplete category which is Barr-exact and protomodular (i.e. the Split Short Five Lemma holds). Examples of semi-abelian categories include, for instance, groups, rings without unit, loops, Lie algebras, Heyting semilattices, etc. However, the notion of semi-abelian category is not as efficient in capturing the properties of groups as the one of abelian category is with respect to abelian groups and modules. Therefore, additional categorical-algebraic conditions have been introduced over the years to get closer to a characterization of the structural properties of the category of groups; among these, one can mention representability of actions [5], algebraic coherence [22], and strong protomodularity [9]. The first chapter of this thesis is aimed to study which of these properties hold in the category of lattice-ordered groups.

In Section 1.1 we recall some classical facts about lattice-ordered groups and we focus on the notion of semi-direct product in $\ell\mathbb{G}rp$.

In Section 1.2 we study the nature of commutators in $\ell\mathbb{G}rp$ and we show that every subobject admits a centralizer, which coincides with the classical notion of polar; moreover, we prove that $\ell\mathbb{G}rp$ is algebraically cartesian closed.

In Section 1.3 we give an alternative proof of the known fact that $\ell\mathbb{G}rp$ is arithmetical using the observation that the only internal group object is the trivial one.

In Section 1.4 we show that $\ell\mathbb{G}rp$ is strongly protomodular; this property implies that, among other things, the commutativity of internal equivalence relations in the Smith-Pedicchio sense [46] is equivalent to the commutativity in the Huq sense [36] of their associated normal subobjects. Moreover, we prove that in $\ell\mathbb{G}rp$ every internal equivalence relation admits a centralizer and we provide a description of it.

Section 1.5 is devoted to the study of action accessibility, a property related to the existence of centralizers of internal equivalence relations; here we observe that, despite $\ell\mathbb{G}rp$ is not action accessible, there is a construction of centralizers which is very close to the one developed in [18] for the action accessible category of rings without unit.

Section 1.6 is aimed to prove that the category of lattice-ordered groups is fiber-wise algebraically cartesian closed (i.e. each category of points in $\ell\mathbb{G}rp$ is algebraically cartesian closed); in detail, we show that in each category of points in $\ell\mathbb{G}rp$ every subobject admits a centralizer, and we provide a description of it.

In Section 1.7 we study the properties of the Higgins commutator in $\ell\mathbb{G}rp$; in particular, we prove that $\ell\mathbb{G}rp$ satisfies the condition of normality of the Higgins commutators showing that the Huq commutator of a pair of ideals (i.e. kernels of some arrows) is nothing more than the intersection of the two ideals.

Finally, in Section 1.8, we focus our attention on the study of the categorical-algebraic properties of the variety of lattice-ordered abelian groups (denoted by $\ell\mathbb{A}b$). To be precise, we show that $\ell\mathbb{A}b$ is algebraically coherent; this condition implies several algebraic properties (such as, for example, strong protomodularity, normality of the Higgins commutator, and the so-called "Smith is Huq" condition). Furthermore, we observe that the category of lattice-ordered abelian groups provides an example of an algebraically coherent category that is not action accessible (thus solving the Open Problem 6.28 presented in [22]).

1.1 Preliminaries

In this section, we recall the notion of *lattice-ordered group.* Roughly speaking, a lattice-ordered group is a set endowed with a group structure and a lattice structure such that the group operation is distributive with respect to the lattice operations.

Definition 1.1.1. *A* lattice-ordered group *is an algebra* $(X, \cdot, e, ^{-1}, \vee)$ *where:*

LG1) $(X, \cdot, e, ^{-1})$ *is a group,*

LG2) $(X, \vee)$ *is a semilattice (i.e.* $\vee$ *is a binary, associative, commutative and idempotent operation on* X*) and*

LG3) for every $x, y, z \in X$ *the following equalities hold*

$$x \cdot (y \vee z) = (x \cdot y) \vee (x \cdot z) \text{ and}$$
$$(x \vee y) \cdot z = (x \cdot z) \vee (y \cdot z).$$

A morphism *between two lattice-ordered groups* $(X, \cdot, e, ^{-1}, \vee)$ *and* $(Y, \cdot, e, ^{-1}, \vee)$ *is a map* $f\colon X \to Y$ *such that* f *is both a group homomorphism between* $(X, \cdot, e, ^{-1})$ *and* $(Y, \cdot, e, ^{-1})$ *and a semilattice homomorphism between* $(X, \vee)$ *and* $(Y, \vee)$.
The category $\ell\mathbb{G}rp$ *is the category whose objects are the lattice-ordered groups and whose arrows are the morphisms between them.*

Lattice-ordered groups appear in many different fields of mathematics. The set of integers $\mathbb{Z}$, the set of rational numbers $\mathbb{Q}$, and the set of real numbers $\mathbb{R}$ with the usual group sum and the usual order structure are lattice-ordered groups. Moreover, given a totally ordered set Γ we can provide a lattice-ordered group structure on the set of order automorphisms $\mathrm{Aut}(\Gamma)$: for every $f, g \in \mathrm{Aut}(\Gamma)$ the group product is defined as the composition $f \circ g$, and $(f \vee g)(x) \coloneqq \max(f(x), g(x))$ for all $x \in \Gamma$. These and other examples can be found in any textbook on lattice-ordered groups, as for instance [40] and [1].

We will often denote a model of an algebraic theory $(A, \mathscr{O})$ (where $\mathscr{O}$ is the set of internal operations) with the underlying set A. So, for instance, we will indicate a lattice-ordered group $(X, \cdot, e, ^{-1}, \vee)$ simply with X.
Given a lattice-ordered group X we will assume that the product precedes the lattice operation, i.e.

$$x \cdot y \vee z \text{ stands for } (x \cdot y) \vee z.$$

Finally, when we work with group operations we will concatenate the elements to indicate the product between them, i.e.

$$xy \text{ stands for } x \cdot y.$$

In the literature, lattice-ordered groups are usually presented as algebras on the set of operations $\{\cdot, e, ^{-1}, \vee, \wedge\}$ satisfying the group axioms, the lattice axioms and the axioms related to the left and right distributivity of the group operation over both lattice operations. However, in this thesis, we have preferred a presentation that does not directly involve the meet operation in order to facilitate the description of the semi-direct products (topic that will be covered in the final part of this section). In fact, starting from Definition 1.1.1 it is always possible to define, in a unique way, the meet operation. Explicitly, given a lattice-ordered group X we can define

$$x \wedge y \coloneqq (x^{-1} \vee y^{-1})^{-1} \text{ for every } x, y \in X.$$

Clearly, the new binary operation is associative and commutative (it is, moreover, idempotent). We observe that for all $x, y \in X$

$$x \wedge (x \vee y) = (x^{-1} \vee (x \vee y)^{-1})^{-1} = (x^{-1})^{-1} = x;$$

in fact $x \leq x \vee y$ implies $(x \vee y)^{-1} \leq x^{-1}$, and hence $x^{-1} \vee (x \vee y)^{-1} = x^{-1}$.
Furthermore, for every $x, y \in X$, we have

$$x \vee (x \wedge y) = x \vee (x^{-1} \vee y^{-1})^{-1} = x$$

since $x^{-1} \leq x^{-1} \vee y^{-1}$ implies $(x^{-1} \vee y^{-1})^{-1} \leq x$. Therefore, $(X, \vee, \wedge)$ is a lattice.
Now, we have to prove that the group product distributes also over the meet operation; in other words, we want to show that

$$x(y \wedge z) = xy \wedge xz \text{ and } (x \wedge y)z = xz \wedge yz$$

for all $x, y, z \in X$. We have that

$$\begin{aligned} x(y \wedge z) &= x(y^{-1} \vee z^{-1})^{-1} = ((y^{-1} \vee z^{-1})x^{-1})^{-1} \\ &= ((y^{-1}x^{-1}) \vee (z^{-1}x^{-1}))^{-1} = ((xy)^{-1} \vee (xz)^{-1})^{-1} = (xy) \wedge (xz). \end{aligned}$$

The proof of the other equality is similar. Therefore, also $(X, \cdot, e, ^{-1}, \wedge)$ is a lattice-ordered group.
Moreover, it is easy to show that every morphism in $\ell\mathbb{G}rp$ preserves also the meet operation defined above.
Finally, it is a known fact that the lattice $(X, \vee, \wedge)$ is distributive; a proof of this last result can be found e.g. in [40] and [1].

Definition 1.1.2 ([40, 1])**.** *Let X be an object of $\ell\mathbb{G}rp$. For every $x \in X$ we define*

$$x^+ \coloneqq x \vee e,\ x^- \coloneqq x \wedge e \text{ and } |x| \coloneqq x \vee x^{-1};$$

x^+ *is called the* positive part *of* x, x^- *the* negative part *of* x, *and* $|x|$ *the* absolute value *of* x.

The previous definition is useful in order to show that in a lattice-ordered group every element can be seen as the product between a positive element and a negative one. In fact, given an object X of $\ell\mathbb{G}rp$, we define the positive cone of X as

$$P := \{x \in X \mid x \geq e\}.$$

So, for every $x \in X$, it can be shown that

$$x = x^+ x^- \text{ and } |x| = x^+(x^-)^{-1}.$$

Therefore, X is generated by its positive cone (i.e. $X = PP^{-1}$) and, moreover, $|x| \geq e$ holds for every $x \in X$. A proof of these facts can be found, for example, in [40] and [1].
Furthermore, the notion of positive part is extremely useful in order to characterize group homomorphisms between lattice-ordered groups which are, in addition, morphisms of $\ell\mathbb{G}rp$. In fact, the following holds:

Lemma 1.1.3. *Let* X, Y *be two objects of* $\ell\mathbb{G}rp$. *A map* $f\colon X \to Y$ *is a morphism of* $\ell\mathbb{G}rp$ *if and only if* f *preserves the group product and*

$$f(x \vee e) = f(x) \vee e, \text{ for all } x \in X.$$

Proof. One implication is trivial. So, let us suppose that f preserves the group product (i.e. is a group homomorphism) and $f(x \vee e) = f(x) \vee e$ for every x $\in X$. We have to prove that

$$f(x \vee y) = f(x) \vee f(y), \text{ for all } x, y \in X.$$

We have $x \vee y = (xy^{-1} \vee e)y$, hence $f(x \vee y) = f(xy^{-1} \vee e)f(y)$ and, by assumption,

$$f(xy^{-1} \vee e)f(y) = (f(xy^{-1}) \vee e)f(y) = (f(x)f(y)^{-1} \vee e)f(y) = f(x) \vee f(y). \qquad \square$$

Now, we want to provide a description of the ideals (or normal subobject) in the variety $\ell\mathbb{G}rp$. In a category where it makes sense to speak of a kernel of a morphism (for example a pointed finitely complete category), a subobject of X is called an *ideal* if it is the kernel of some morphism. A detailed study of the notion of an ideal in the variety $\ell\mathbb{G}rp$ can be found in [40] and [1].
First of all, we have to recall the definition of a *convex* subset. Given an object X of $\ell\mathbb{G}rp$ and a subset $S \subseteq X$, S is said to be convex if for every $a, b \in S$ and every $x \in X$, if $a \leq x \leq b$ then $x \in S$.
A subobject $A \leq X$ is an ideal if and only if it is normal (in the classical sense) as a subgroup and it is a convex subset.

The aim of the following proposition is to describe the notion of convexity only with terms. This characterization will be crucial for the purpose of working with semi-direct products.

Proposition 1.1.4. *Let X be an object of $\ell\mathbb{G}rp$ and $A \leq X$ a subalgebra. A is convex if and only if for every $a_1, a_2 \in A$ and $x, y \in X$ one has*

$$(a_1x \vee a_2y)(x \vee y)^{-1} \in A.$$

Proof. Let us suppose that A is convex. We consider the following inequalities:

$$((a_1 \wedge a_2)x) \vee ((a_1 \wedge a_2)y) \leq a_1x \vee a_2y \leq ((a_1 \vee a_2)x) \vee ((a_1 \vee a_2)y),$$

hence

$$(a_1 \wedge a_2)(x \vee y) \leq a_1x \vee a_2y \leq (a_1 \vee a_2)(x \vee y),$$

and so

$$(a_1 \wedge a_2) \leq (a_1x \vee a_2y)(x \vee y)^{-1} \leq (a_1 \vee a_2).$$

Thus, since A is convex, we deduce $(a_1x \vee a_2y)(x \vee y)^{-1} \in A$.
Conversely, let us suppose that for every $a_1, a_2 \in A$ and $x, y \in X$ one has $(a_1x \vee a_2y)(x \vee y)^{-1} \in A$. Let us take an element $x \in X$ and two elements $a_1, a_2 \in A$ such that $a_1 \leq x \leq a_2$. In particular, $a_1 \vee x = x$; therefore, by assumption,

$$(a_1e \vee ex)(e \vee x)^{-1} \in A$$

and so

$$x(e \wedge x^{-1}) = x \wedge e = x^- \in A.$$

With a similar argument, from $a_2^{-1} \leq x^{-1} \leq a_1^{-1}$, we can deduce $e \wedge x^{-1} \in A$ and, since A is a subalgebra, we have $x \vee e = x^+ \in A$. Finally, we get $x = x^+x^- \in A$, i.e. A is convex. □

To conclude, we can say that a subobject $A \leq X$ in $\ell\mathbb{G}rp$ is an ideal if and only if for every $a_1, a_2, a \in A$ and $x, y, z \in X$ one has $(a_1x \vee a_2y)(x \vee y)^{-1} \in A$ and $z^{-1}az \in A$.

In the following part of this section, we will deal with the notion of a semi-abelian category. The main idea behind this notion is to capture some of the homological properties of the category of groups; among the examples of semi-abelian category we can find those of groups, rings without unit, Lie algebras, and Heyting semilattices.

Definition 1.1.5 ([39]). *A pointed category (i.e. a category with a zero object) $\mathbb{C}$ is* semi-abelian *if:*

- *it is Barr-exact* [2] *(which means that $\mathbb{C}$ is a regular category in which every internal equivalence relation is a kernel pair);*

- *it has finite coproducts;*
- *it is protomodular* [6] *(i.e. the Split Short Five Lemma holds in* $\mathbb{C}$*).*

In Theorem 1.1 of [17] the authors provided, in the case of a variety $\mathbb{V}$ of universal algebras, a characterization for protomodularity depending on terms. In fact, the authors proved that a variety $\mathbb{V}$ is protomodular if and only if it has 0-ary terms $e_1, \dots, e_n$, binary terms $t_1, \dots, t_n$ and an $(n+1)$-ary term t satisfying the identities

$$t(x, t_1(x, y), \dots, t_n(x, y)) = y \text{ and } t_i(x, x) = e_i$$

for all $i = 1, \dots, n$.

Proposition 1.1.6. *$\ell\mathbb{G}rp$ is a semi-abelian category.*

Proof. Clearly, $\ell\mathbb{G}rp$ is Barr-exact and it has finite coproducts since it is a variety. Moreover, the trivial lattice-ordered group $\{e\}$ is a zero object. Finally, to show that $\ell\mathbb{G}rp$ is protomodular we can take as set of 0-ary terms just $e_1 \coloneqq e$, as set of binary terms just $t_1(x, y) \coloneqq x^{-1}y$ and as (n+1)-ary term (in this case $n = 1$) $t(x, y) \coloneqq xy$. □

As shown in [16], in every semi-abelian category there exist semi-direct products in a categorical sense. In the category of groups, the categorical semi-direct product coincides with the classical one. Now we can describe semi-direct products in the category $\ell\mathbb{G}rp$. In order to do this we will apply, in the next proposition, the results provided in [24].

Proposition 1.1.7. *Let $p\colon A \to B$ be a split epimorphism in $\ell\mathbb{G}rp$ with fixed section $s\colon B \to A$, and $k\colon K \to A$ a kernel of p. Without loss of generality let us suppose that K, B are subalgebras of A and k, s are the inclusions of subalgebras. Then A is isomorphic (as a lattice-ordered group) to the set $K \times B$ endowed with the operations*

- $(k_1, b_1)(k_2, b_2) = (k_1 b_1 k_2 b_1^{-1}, b_1 b_2)$,
- $(k_1, b_1) \vee (k_2, b_2) = ((k_1 b_1 \vee k_2 b_2)(b_1 \vee b_2)^{-1}, b_1 \vee b_2)$,

(which takes the name of semi-direct product *and is indicated as $K \rtimes B$) via the morphism*

$$\begin{aligned} \varphi\colon K \rtimes B &\to A \\ (k, b) &\mapsto kb. \end{aligned}$$

Moreover, considering the following diagram in $\ell\mathbb{G}rp$:

$$\begin{tikzcd} K \arrow[r, "i_K"] \arrow[d, equal] & K \rtimes B \arrow[r, shift left, "p_B"] \arrow[d, "\varphi"] & B \arrow[l, shift left, "i_B"] \arrow[d, equal] \\ K \arrow[r, "k"] & A \arrow[r, shift left, "p"] & B \arrow[l, shift left, "s"] \end{tikzcd}$$

where $i_K(k) = (k, e)$*,* $i_B(b) = (e, b)$ *and* $p_B(k, b) = b$*, we have* $\varphi i_K = k, p\varphi = p_B$ *and* $\varphi i_B = s$*.*

1.2 Centralizers and Algebraic Cartesian Closedness

In this section we study, from a categorical point of view, the commutativity of subobjects in the variety $\ell\mathbb{G}rp$.

In order to introduce the topic, we mention some known results related to the category of groups. Given a group G and two subgroups $A, B \leq G$, the condition that, for every $a \in A$ and $b \in B$, $ab = ba$ can be reformulated in the following equivalent way: there exists a group homomorphism $\varphi\colon A \times B \to G$ making the following diagram commutative:

$$\begin{tikzcd}
A \arrow[dr, "{(id_A,0)}"] \arrow[ddr, bend right, hook] & & B \arrow[dl, "{(0,id_B)}"'] \arrow[ddl, bend left, hook'] \\
& A \times B \arrow[d, "\varphi"] & \\
& G. &
\end{tikzcd}$$

Moreover, it is easy to show that φ must be the group product and, therefore, it is necessarily unique. Hence, with the aim of generalising the notion of commutativity, we must place ourselves in a context in which a morphism φ of this type is unique. This reasoning justifies the following definition:

Definition 1.2.1 ([7])**.** *A pointed category* $\mathbb{C}$ *with finite products is* unital *if, for* X *and* Y *objects of* $\mathbb{C}$*, the pair of morphisms* $(id_X, 0)\colon X \to X \times Y$*,* $(0, id_Y)\colon Y \to X \times Y$ *is jointly extremally epimorphic.*

To be more explicit, a pair of arrows $f\colon A \to B$ and $g\colon C \to B$ of a category $\mathbb{C}$ is said to be jointly extremally epimorphic when for each commutative diagram

$$\begin{tikzcd}
& M \arrow[d, "m"] & \\
A \arrow[ur, "f'"] \arrow[r, "f"'] & B & C \arrow[ul, "g'"'] \arrow[l, "g"]
\end{tikzcd}$$

if m is a monomorphism, then m is an isomorphism.

It has been shown in [3] that every semi-abelian category is unital.

We are ready to mention the generalized notion of commutativity between subobjects.

Definition 1.2.2 ([11])**.** *Let* $\mathbb{C}$ *be a unital category. Two subobjects* $a\colon A \rightarrowtail X$ *and* $b\colon B \rightarrowtail X$ *of* X *are said to* cooperate *(or* commute in the sense of Huq [36]*, and we write* $[a, b] = 0$*) if* *there exists a (necessarily unique) morphism $\varphi\colon A \times B \to X$ (called* cooperator*) such that the following diagram commutes:*

$$\begin{tikzcd}
A \arrow[dr, "(id_A{,}0)"] \arrow[ddr, tail, bend right, "a"'] & & B \arrow[dl, "(0{,}id_B)"'] \arrow[ddl, tail, bend left, "b"] \\
& A \times B \arrow[d, dashed, "\exists\varphi"] & \\
& X. &
\end{tikzcd}$$

Given a subobject $a\colon A \rightarrowtail X$, the centralizer *of a in X, if it exists, is the greatest subobject of X that cooperates with a.*

Now, let us recall the definition of *orthogonal* subobjects of a lattice-ordered group. This concept will be essential in order to study the condition of cooperation.

Definition 1.2.3 ([40])**.** *Let X be an object of $\ell\mathbb{G}rp$. Two elements $a, b \in X$ are called* orthogonal *if*

$$|a| \wedge |b| = e.$$

Two subsets $A, B \subseteq X$ are called orthogonal *(and one writes $A \perp B$) if, for every $a \in A$ and for every $b \in B$, a and b are orthogonal as elements.*

It is a known fact that two orthogonal subobjects of a lattice-ordered group commute as subgroups. More generally, if a and b are orthogonal then $ab = ba$. A proof of this can be found, for instance, in Proposition 2.2.10 of [40].

Proposition 1.2.4. *Let X be an object of $\ell\mathbb{G}rp$ and $A, B \leq X$ two subobjects. Then A and B cooperate if and only if $A \perp B$.*

Proof. ($\Rightarrow$) The cooperator $\varphi\colon A \times B \to X$ is given by $\varphi(a, b) = ab$. In fact, since φ preserves the group operation, we have

$$\varphi(a, b) = \varphi(a, e)\varphi(e, b) = \varphi(i_A(a))\varphi(i_B(b)) = ab.$$

We observe that, for every $a \in A$ and $b \in B$, $(|a|, e) \wedge (e, |b|) = (e, e)$ holds. So, since φ preserves the lattice operations, we get

$$e = \varphi(e, e) = \varphi((|a|, e) \wedge (e, |b|)) = \varphi(|a|, e) \wedge \varphi(e, |b|) = |a| \wedge |b|.$$

($\Leftarrow$) If a cooperator φ exists then it must be the group multiplication because of what we observed at the beginning of the proof. Therefore, we have to show

$$ab = ba \text{ and } (ab)^+ = a^+b^+$$

for all $a \in A$ and $b \in B$. In fact, if $ab = ba$ for every $a \in A$ and $b \in B$, then $\varphi((a,b)(c,d)) = \varphi(ac, bd) = acbd = abcd = \varphi(a,b)\varphi(c,d)$ since, by assumption, $cb = bc$; furthermore, if $(ab)^+ = a^+b^+$ for every $a \in A$ and $b \in B$, then $\varphi((a,b)^+) = \varphi((a,b) \vee (e,e)) = \varphi(a^+, b^+) = a^+b^+ = (ab)^+ = \varphi(a,b)^+ = \varphi(a,b) \vee e$, and thus we can apply Lemma 1.1.3 to say that φ is a morphism of lattice-ordered groups. The first equality holds trivially since $A \perp B$. Let us deal with the second one: $a^+b^+ = (a \vee e)(b \vee e) = ab \vee a \vee b \vee e$ and $(ab)^+ = ab \vee e$; so we have to prove $a \vee b \leq ab \vee e$. Since $|a| \wedge |b| = e$, we have $(a \vee a^{-1}) \wedge (b \vee b^{-1}) = e$ and, by distributivity, we get

$$(a \wedge b) \vee (a \wedge b^{-1}) \vee (a^{-1} \wedge b) \vee (a^{-1} \wedge b^{-1}) = e.$$

Hence, $a^{-1} \wedge b \leq e$ implies $a \vee b^{-1} \geq e$ and, multiplying by b on the right, we obtain $ab \vee e \geq b$; with a similar argument we get $ab \vee e \geq a$. Finally, considering the last two inequalities, we conclude that $ab \vee e \geq a \vee b$. □

The following lemma, related to the properties of orthogonal subobjects, will be of fundamental importance in the next sections.

Lemma 1.2.5. *Let X be an object of $\ell\mathbb{G}rp$ and $A, B \leq X$ two orthogonal subobjects. Then, for every $a \in A$ and $b \in B$, the following equality holds:*

$$(ab) \vee e = (a \vee e)(b \vee e).$$

Proof. Since A and B are orthogonal, then there exists a cooperator $\varphi\colon A \times B \to X$ given by the group product. Therefore $(ab) \vee e = \varphi(a,b) \vee \varphi(e,e) = \varphi((a,b) \vee (e,e)) = \varphi(a \vee e, b \vee e) = (a \vee e)(b \vee e)$ for all $a \in A$ and $b \in B$. □

We, therefore, recall the notion of *polar* of a subset S of a lattice-ordered group (i.e. the set of elements orthogonal to each element of S). We will show that the polar of a subobject is nothing but the centralizer of the subobject. Hence, we will exhibit some properties of centralizers related to being ideals.

Proposition 1.2.6 ([40], Proposition 1.2.6)**.** *Let X be an object of $\ell\mathbb{G}rp$ and $S \subseteq X$ a non-empty subset. Then the set $S^\perp := \{x \in X \mid \text{for each } s \in S\ |x| \wedge |s| = e\}$ (called the* polar *of S) is a convex subalgebra of X.*

Lemma 1.2.7. *Let X be an object of $\ell\mathbb{G}rp$ and $S \subseteq X$ a non-empty subset of X closed under conjugation. Then $S^\perp$ is an ideal of X.*

Proof. First of all we observe that, for all $x, y \in X$, one has

$$|x^{-1}yx| = x^{-1}yx \vee x^{-1}y^{-1}x = x^{-1}(y \vee y^{-1})x = x^{-1}|y|x.$$

We want to show that one has $|x^{-1}yx| \wedge |s| = e$, for every $x \in X$, $y \in S^{\perp}$, and $s \in S$. We observe that $|x^{-1}yx| \wedge |s| = x^{-1}|y|x \wedge |s| = x^{-1}(|y| \wedge x|s|x^{-1})x = x^{-1}(|y| \wedge |xsx^{-1}|)x = x^{-1}(|y| \wedge |\overline{s}|)x = e$, where $xsx^{-1} = \overline{s} \in S$ because S is closed under conjugation, and $|y| \wedge |\overline{s}| = e$ (since $y \in S^{\perp}$). $\square$

Corollary 1.2.8. *Let X be an object of $\ell\mathbb{G}rp$ and $A, B \leq X$ two subobjects. A and B cooperate if and only if $B \subseteq A^{\perp}$. Therefore, $A^{\perp} \leq X$ is the centralizer of $A \leq X$.*

Finally, we recall a property that is strictly related to the existence of centralizers. It is well known that a category $\mathbb{E}$ with finite products is cartesian closed if and only if for every object Y of $\mathbb{E}$ the change-of-base functor $\tau_Y^* \colon \mathbb{E} \to \mathbb{E}/Y$ along the terminal arrow $\tau_Y \colon Y \to \mathbf{1}$ has a right adjoint. For algebraic categories, such adjoints rarely exist, but it turns out to be of interest to consider a variation of this notion; this leads to:

Definition 1.2.9 ([15])**.** *A category $\mathbb{C}$ is* algebraically cartesian closed (a.c.c.) *if for every object X of $\mathbb{C}$ the change-of-base functor $\tau_X^* \colon Pt_{\mathbf{1}}\mathbb{C} \to Pt_X\mathbb{C}$ has a right adjoint, where $\tau_X \colon X \to \mathbf{1}$ is the unique arrow from X to the terminal object.*

In [15] the authors show that the existence of such adjoints is related to the existence of cofree structures for the split epimorphisms $p_Y \colon Y \times X \to Y$ in $Pt_Y\mathbb{E}$ with fixed section (id_Y, u), where $u \colon Y \to X$ can be chosen to be a monomorphism.

Proposition 1.2.10 ([15], Proposition 1.2)**.** *A unital category $\mathbb{C}$ is algebraically cartesian closed if and only if, for every X object of $\mathbb{C}$, each subobject of X has a centralizer.*

Corollary 1.2.11. *The category $\ell\mathbb{G}rp$ is algebraically cartesian closed.*

1.3 Congruence Distributivity and Arithmeticity

It is a widely known fact that in the category $\ell\mathbb{G}rp$ the lattice of congruences on any object is distributive. In this section we provide an alternative proof of this fact based on categorical tools. We recall that a category is a *Mal'tsev category* [19] if it is finitely complete and if every internal reflexive relation is an internal equivalence relation. If the category is regular, this notion is equivalent to the following property: for every object X and for every pair of internal equivalence relations $(s_1, s_2) \colon S \rightarrowtail X \times X$ and $(r_1, r_2) \colon R \rightarrowtail X \times X$ one has $R \circ S = S \circ R$; in detail, $R \circ S \colon \rightarrowtail X \times X$ is defined as the regular image of (p_1, p_3), where (p_1, p_3) is given by the following diagram:

$$
\begin{tikzcd}
R \times_X S \arrow[r, "\pi_S"'] \arrow[d, "\pi_R"] \arrow[dd, bend right, "p_1"'] \arrow[rr, bend left, "p_3"] \arrow[dr, phantom, "\lrcorner", very near start] & S \arrow[r, "s_2"'] \arrow[d, "s_1"] & X \\
R \arrow[r, "r_2"'] \arrow[d, "r_1"] & X & \\
X. & &
\end{tikzcd}
$$

The composite $S \circ R$ can be defined in a similar way. Moreover, if the category is a variety of universal algebras, the property of being a Mal'tsev category is equivalent to the existence of a ternary term $p(x, y, z)$ (called *Mal'tsev term*) satisfying the axioms

$$p(x, x, z) = z \text{ and } p(x, y, y) = x$$

for every object X and for every $x, y, z \in X$. Therefore, if the theory contains a group operation, the associated variety is a Mal'tsev category: in fact, a Mal'tsev ternary term is $p(x, y, z) := xy^{-1}z$.
Then we immediately get the following result:

Corollary 1.3.1. *The category $\ell\mathbb{G}rp$ is a Mal'tsev category.*

If $\mathbb{C}$ is a Barr-exact category with coequalizers then, for every object X of $\mathbb{C}$, the set $\mathrm{Eq}(X)$ of internal equivalence relations on X is a lattice; given two internal equivalence relations $(s_1, s_2) \colon S \rightarrowtail X \times X$ and $(r_1, r_2) \colon R \rightarrowtail X \times X$, the meet $S \wedge R$ is defined as the meet of subobjects of $X \times X$, and the join $(t_1, t_2) \colon S \vee R \rightarrowtail X \times X$ is defined as the kernel pair of $q = \mathrm{coeq}(v_1, v_2)$, where $(v_1, v_2) \colon V \rightarrowtail X \times X$ is the join of S and R as subobjects of $X \times X$ (we recall that the join, as subobjects, of two internal equivalence relations is not, in general, an internal equivalence relation). Thanks to the previous observations, the classical notion of arithmetical variety of universal algebras can be extended to a categorical context as follows:

Definition 1.3.2 ([47]). *A Barr-exact category with coequalizers $\mathbb{C}$ is* arithmetical *if it is a Mal'tsev category and, for any object X of $\mathbb{C}$, the lattice* $\mathrm{Eq}(X)$ *of internal equivalence relations on X is distributive.*

It is a known fact (a proof of this can be found in [3]) that the property of being an arithmetical category is related to the absence of non-trivial *internal group objects* in the category. In fact, the following holds:

Proposition 1.3.3 ([3], Proposition 2.9.9). *Let $\mathbb{C}$ be a semi-abelian category. If in $\mathbb{C}$ the only internal group object is the zero object then $\mathbb{C}$ is arithmetical.*

Proposition 1.3.4. *The only internal group object in $\ell\mathbb{G}rp$ is the zero object.*

Proof. Given an internal group X in $\ell\mathbb{G}rp$, with multiplication $\mu \colon X \times X \to X$ and neutral element $\eta \colon \{*\} \to X$, we want to show that $X \simeq \{e\}$. It is not difficult to see that $\eta(*) = e$ and $\mu(x, y) = xy$ for all $x, y \in X$. Therefore, since μ is a morphism of lattice-ordered groups, for all $a, b, c, d \in X$ the following equality holds

$$\mu((a, b) \vee (c, d)) = \mu(a, b) \vee \mu(c, d).$$

Hence, μ being the group multiplication, we obtain

$$ab \vee ad \vee cb \vee cd = ab \vee cd.$$

We deduce that the following inequality holds:

$$ad \vee cb \leq ab \vee cd.$$

Now, if we consider $a = d$ and $c = b = e$, we get $a^2 \leq a^2 \vee e \leq ae \vee ea = a$; multiplying by a^{-1} we have $a \leq e$ for each $a \in X$ and, by replacing a with a^{-1}, we get $e \leq a$. In other words, we have proved $X = \{e\}$. $\square$

Corollary 1.3.5. *$\ell\mathbb{G}rp$ is an arithmetical category.*

1.4 Strong Protomodularity

Given a category $\mathbb{C}$, we denote by $\mathrm{Pt}(\mathbb{C})$ the category whose objects are the diagrams in $\mathbb{C}$ of the form

$$\begin{tikzcd} A \arrow[r, shift left, "p"] & B \arrow[l, shift left, "s"] \end{tikzcd}$$

where $ps = id_B$, and whose arrows are the pairs (f, g) of arrows of $\mathbb{C}$

$$\begin{tikzcd} A \arrow[r, shift left, "p"] \arrow[d, "f"'] & B \arrow[l, shift left, "s"] \arrow[d, "g"] \\ C \arrow[r, shift left, "q"] & D \arrow[l, shift left, "r"] \end{tikzcd}$$

such that $qf = gp$ and $fs = rg$. We denote by $\pi\colon \mathrm{Pt}(\mathbb{C}) \to \mathbb{C}$ the functor that associates to every split extension (i.e. an object of $\mathrm{Pt}(\mathbb{C})$) (p, s) the codomain of p.

Definition 1.4.1 ([8])**.** *A finitely complete category $\mathbb{C}$ is* strongly protomodular *when all the change-of-base functors of $\pi\colon \mathrm{Pt}(\mathbb{C}) \to \mathbb{C}$ reflect both isomorphisms and normal monomorphisms (in the semi-abelian case a monomorphism is normal if and only if it is the kernel of some arrow).*

In [9] the author shows that, if $\mathbb{C}$ is a pointed protomodular category, there is a characterization of strong protomodularity related to the stability of kernels. Let us consider a diagram in $\mathbb{C}$ of the form

$$\begin{tikzcd} X \arrow[r, "k"] \arrow[d, "m"'] & A \arrow[r, shift right, "p"'] \arrow[d, "f"'] & B \arrow[l, shift right, "s"'] \arrow[d, equal] \\ Y \arrow[r, "l"'] & C \arrow[r, shift right, "q"'] & B \arrow[l, shift right, "r"'] \end{tikzcd}$$

where $k = \ker(p)$, $ps = id_B$, $l = \ker(q)$, $qr = id_B$, m is a normal monomorphism and the right-rightward square, the right-leftward square, and the left square commute. Then $\mathbb{C}$ is strongly protomodular if and only if the composite lm is a normal monomorphism for every diagram of this form.

Proposition 1.4.2. *$\ell\mathbb{G}rp$ is a strongly protomodular category.*

Proof. Let us consider the following commutative diagram (without loss of generality we can assume that the monomorphisms are inclusions):

$$\begin{tikzcd} X \arrow[r, hook] \arrow[d, hook] & A \arrow[r, shift right, "p"'] \arrow[d, "f"'] & B \arrow[l, shift right, "s"'] \arrow[d, equal] \\ Y \arrow[r, hook] & C \arrow[r, shift right, "q"'] & B \arrow[l, shift right, "r"'] \end{tikzcd}$$

where $ps = id_B$, $qr = id_B$, X is the ideal of A determined by $\ker(p)$, Y is the ideal of C determined by $\ker(q)$, X is an ideal of Y and the right-rightward square, the right-leftward square, and the left square commute. We want to show that X is an ideal of C, too.

- X is a normal subgroup of C. This is a known fact about the category of groups (but we give the proof anyway). Let us fix an element $x \in X$ and an element $c \in C$. Then there exist $y \in Y$ and $b \in B$ such that $c = yr(b)$, so $c^{-1}xc = r(b)^{-1}y^{-1}xyr(b) = r(b)^{-1}\overline{x}r(b)$ with $\overline{x} \in X$ (since X is closed under conjugation with the elements of Y). Hence, $f(\overline{x}) = \overline{x}$ and $r(b) = fs(b)$. Thus, $r(b)^{-1}\overline{x}r(b) = f(s(b)^{-1}\overline{x}s(b)) = f(\widetilde{x}) = \widetilde{x}$ with $\widetilde{x} \in X$, since $s(b) \in A$ and X is closed under conjugation with the elements of A.

- X is a convex subset of C. We know that, for every $y \in Y$, if $x_1 \leq y \leq x_2$, with $x_1, x_2 \in X$, then $y \in X$ and, for every $c \in C$, if $y_1 \leq c \leq y_2$, with $y_1, y_2 \in Y$, then $c \in Y$. So, given an element $c \in C$ such that $x_1 \leq c \leq x_2$, with $x_1, x_2 \in X$, we have $c \in Y$ (since $x_1, x_2 \in Y$) and thus $c \in X$. □

In the final part of this section we study, in the case of $\ell\mathbb{G}rp$, the consequences of strong protomodularity relatively to the commutativity, in the Smith-Pedicchio sense, of internal equivalence relations. In particular, we show that every internal equivalence relation admits a centralizer. Let us begin by recalling the necessary notions to deal with this subject.

Definition 1.4.3 ([46], [13])**.** *Let $\mathbb{C}$ be a Mal'tsev category and*

$$\begin{tikzcd} R \arrow[r, shift left=2, "r_1"] \arrow[r, shift right=2, "r_2"'] & X, \arrow[l, "\delta_R"'] \end{tikzcd} \qquad \begin{tikzcd} S \arrow[r, shift left=2, "s_1"] \arrow[r, shift right=2, "s_2"'] & X \arrow[l, "\delta_S"'] \end{tikzcd}$$

a pair of internal equivalence relations on an object X of $\mathbb{C}$. We say that (R, r_1, r_2) and (S, s_1, s_2) commute in the Smith-Pedicchio sense *(and we write $[R, S] = 0$) if, given the following diagram:*

$$\begin{tikzcd} R \times_X S \arrow[r, shift left, "\pi_S"] \arrow[d, shift right, "\pi_R"'] \arrow[dr, phantom, "\lrcorner", very near start] & S \arrow[l, shift left, "\tau_S"] \arrow[d, shift right, "s_1"'] \\ R \arrow[u, shift right, "\tau_R"'] \arrow[r, shift left, "r_2"] & X \arrow[l, shift left, "\delta_R"] \arrow[u, shift right, "\delta_S"'] \end{tikzcd}$$

where $R \times_X S$ is the pullback of r_2 through s_1, $\tau_R = (id_R, \delta_S r_2)$ and $\tau_S = (\delta_R s_1, id_S)$ are induced by the universal property, there exists a unique morphism $p\colon R \times_X S \to X$ (called connector *between R and S) such that $p\tau_S = s_2$ and $p\tau_R = r_1$. The* centralizer *of an internal equivalence relation (R, r_1, r_2) on X, if it exists, is the greatest internal equivalence relation on X which commutes with (R, r_1, r_2).*

It has been shown in Proposition 3.2 of [14] that, in a pointed Mal'tsev category, if two internal equivalence relations (R, r_1, r_2) and (S, s_1, s_2) commute in the Smith-Pedicchio sense, then necessarily their associated normal subobjects j_R and j_S commute in the Huq sense, where $j_R \coloneqq \ker(q_R)$ and $j_S \coloneqq \ker(q_S)$, with $q_R \coloneqq \mathrm{coeq}(r_1, r_2)$ and $q_S \coloneqq \mathrm{coeq}(s_1, s_2)$. Briefly, $[R, S] = 0$ implies $[j_R, j_S] = 0$. The converse is not true in general. We say that a pointed Mal'tsev category satisfies the so-called *Smith is Huq* condition (SH) if $[j_R, j_S] = 0$ implies $[R, S] = 0$. It has been proved in Theorem 6.1 of [14] that in every pointed strongly protomodular category the Smith is Huq condition holds.
Finally, we are ready to prove the following:

Proposition 1.4.4. *In $\ell\mathbb{G}rp$ internal equivalence relations admit centralizers.*

Proof. Since $\ell\mathbb{G}rp$ is a semi-abelian category we have, for every object X, an order-preserving bijection φ between $\mathrm{Eq}(X)$ and the lattice $\mathrm{Ideals}(X)$ of ideals of X, where $\varphi(R) \coloneqq I_R$, with $I_R \coloneqq \{x \in X \mid (x, e) \in R\}$. Given two internal equivalence relations $R \leq X \times X$ and $S \leq X \times X$, we know that $[R, S] = 0$ if and only if $[\varphi(R), \varphi(S)] = 0$ ($\ell\mathbb{G}rp$ is a strongly protomodular category, hence (SH) holds). Moreover, given an internal equivalence relation R on X, we recall from Lemma 1.2.7 that the centralizer $I_R^{\perp} \leq X$ of the ideal I_R associated with R is an ideal. Therefore, since φ is an order-preserving bijection and $\ell\mathbb{G}rp$ satisfies (SH), the centralizer of R in X is $\varphi^{-1}(I_R^{\perp})$. □

1.5 Action Accessibility

To approach the topic covered in this section, let us first review some known properties of the category $\mathbb{G}rp$ of groups. First of all, we recall the general notion of *split extension.*

Definition 1.5.1. *Let $\mathbb{C}$ be a pointed protomodular category. A* split extension *of $\mathbb{C}$ is a diagram of the form*

$$X \xrightarrow{k} A \underset{p}{\overset{s}{\leftrightarrows}} B,$$

where $k = \ker(p)$ and $ps = id_B$.
We denote by $\mathbb{S}pl\mathbb{E}xt_{\mathbb{C}}(X)$ the category whose objects are the split extensions of $\mathbb{C}$ with the same

fixed kernel object X and whose arrows are the pairs (g, f) of arrows in $\mathbb{C}$

$$\begin{tikzcd} X \arrow[r, "k"] \arrow[d, equal] & A \arrow[r, shift right, "p"'] \arrow[d, "g"'] & B \arrow[l, shift right, "s"'] \arrow[d, "f"] \\ X \arrow[r, "l"'] & C \arrow[r, shift right, "q"'] & D \arrow[l, shift right, "r"'] \end{tikzcd}$$

such that $gk = l$,$fp = qg$ and $gs = rf$.

Given a group X, we can define a functor

$$\begin{tikzcd} \mathbb{G}rp^{op} \arrow[rr, "\mathrm{Act}(-,X)"] & & \mathbb{S}et \\ B \arrow[rr, mapsto] \arrow[d, "f"'] & & \{X \to A \leftrightarrows B\}/\sim \\ B' \arrow[rr, mapsto] & & \{X \to A' \leftrightarrows B'\}/\sim \arrow[u, "\mathrm{Act}(f,X)"'] \end{tikzcd}$$

where $\{X \to A \leftrightarrows B\}$ is the set of split extensions with fixed kernel object X and fixed quotient object B; two split extensions $X \to A \leftrightarrows B$ and $X \to \overline{A} \leftrightarrows B$ are equivalent (under the equivalence relation $\sim$) if there exists an arrow $g\colon A \to \overline{A}$ such that $gk = \overline{k}$, $gs = \overline{s}$ and $\overline{p}g = p$

$$\begin{tikzcd} X \arrow[r, "k"] \arrow[d, equal] & A \arrow[r, shift right, "p"'] \arrow[d, "g"'] & B \arrow[l, shift right, "s"'] \arrow[d, equal] \\ X \arrow[r, "\overline{k}"'] & \overline{A} \arrow[r, shift right, "\overline{p}"'] & B \arrow[l, shift right, "\overline{s}"'] \end{tikzcd}$$

(hence, thanks to the Split Short Five Lemma, g is an isomorphism). Finally, $\mathrm{Act}(f, X)$ sends the class of a split extension $X \to A' \leftrightarrows B'$ to the class of the split extension defined via the following diagram, where the right-rightward square is a pullback:

$$\begin{tikzcd} X \arrow[r, "{(k',0)}"] \arrow[d, equal] & A' \times_{B'} B \arrow[r, shift right, "\pi_B"'] \arrow[d, "\pi_{A'}"'] \arrow[dr, phantom, "\lrcorner"] & B \arrow[l, shift right, "{(s'f, id_B)}"'] \arrow[d, "f"] \\ X \arrow[r, "k'"'] & A' \arrow[r, shift right, "p'"'] & B'. \arrow[l, shift right, "s'"'] \end{tikzcd}$$

It is a known fact that, in $\mathbb{G}rp$, there is a one-to-one correspondence between the set $\{X \to A \leftrightarrows B\}/\sim$ and the set of group homomorphisms with domain B and codomain the group $\mathrm{Aut}(X)$ of automorphisms of X. In other words, the functor $\mathrm{Act}(-, X)$ is representable and a representing object is $\mathrm{Aut}(X)$. A pointed protomodular category in which the functor $\mathrm{Act}(-, X)$ is representable for every object X is called *action representable* [5]. However, this condition is extremely strong: there are, in fact, very few examples of action representable categories (for instance, the category of groups and the category of Lie algebras over a commutative ring with unit). It appears therefore legitimate to try to weaken such condition. In order to do this, it is easy to observe that, in $\mathbb{G}rp$, the property of being an action representable category can be restated in the following way: the split extension

$$X \xrightarrow[(id_X,0)]{} X \rtimes \mathrm{Aut}(X) \underset{\pi_{\mathrm{Aut}(X)}}{\overset{(0,id_{\mathrm{Aut}(X)})}{\leftrightarrows}} \mathrm{Aut}(X), \tag{1}$$

corresponding to the action $id_{\mathrm{Aut}(X)} \colon \mathrm{Aut}(X) \to \mathrm{Aut}(X)$, is a terminal object of $\mathbb{S}pl\mathbb{E}xt_{\mathbb{G}rp}(X)$. Therefore, in $\mathbb{G}rp$, for each object of $\mathbb{S}pl\mathbb{E}xt_{\mathbb{G}rp}(X)$ there exists a unique morphism into (1).
In light of this, the authors of [18] have weakened the notion of action representable category in the following way:

Definition 1.5.2 ([18])**.** *An object F of $\mathbb{S}pl\mathbb{E}xt_{\mathbb{C}}(X)$ is said to be* faithful *if for each object E of $\mathbb{S}pl\mathbb{E}xt_{\mathbb{C}}(X)$ there is at most one arrow from E to F.*

Definition 1.5.3 ([18])**.** *Let $\mathbb{C}$ be a pointed protomodular category. An object in $\mathbb{S}pl\mathbb{E}xt_{\mathbb{C}}(X)$ is said to be* accessible *if it admits a morphism into a faithful object. We say that $\mathbb{C}$ is* action accessible *if, for every object X of $\mathbb{C}$, every object in $\mathbb{S}pl\mathbb{E}xt_{\mathbb{C}}(X)$ is accessible.*

Actually, the notion of action accessible category appears as a generalization of the one of action representable category: in fact, if there is a terminal object T of $\mathbb{S}pl\mathbb{E}xt_{\mathbb{C}}(X)$, this object is also faithful and each object of $\mathbb{S}pl\mathbb{E}xt_{\mathbb{C}}(X)$ admits a unique morphism into T. Examples of action accessible categories include, for instance, not necessarily unitary rings (as shown in [18]) and all *categories of interest* in the sense of [45] (as shown in [43]).
One of the interesting properties, among other things, implied by action accessibility is the existence of centralizers of internal equivalence relations (see Theorem 4.1 in [18]). The converse implication, in general, is not true. In fact, in the next part of this section we show that the category $\ell\mathbb{G}rp$ is not action accessible despite the existence of centralizers of internal equivalence relations (as shown in Proposition 1.4.4).

Proposition 1.5.4. *$\ell\mathbb{G}rp$ is not action accessible.*

Proof. Consider the lexicographic product $\mathbb{Z}\overleftarrow{\times}\mathbb{Z}$ of the group of integers $\mathbb{Z}$ (with the usual order) with itself. The underlying set is the product, and in terms of structure the group operations are defined component-wise, while the order is defined as follows: $(a,b) \leq (c,d)$ if and only if $b < d$, or $b = d$ and $a \leq c$. We consider the following split extension

$$\mathbb{Z} \xrightarrow{i_1} \mathbb{Z}\overleftarrow{\times}\mathbb{Z} \underset{p_2}{\overset{i_2}{\leftrightarrows}} \mathbb{Z} \tag{2}$$

where $i_1 = (id_{\mathbb{Z}}, 0)$, $i_2 = (0, id_{\mathbb{Z}})$ and p_2 is the projection on the second component. Now, for every $n \in \mathbb{N}_{>0}$ we can consider the morphism of lattice-ordered groups $f_n \colon \mathbb{Z} \to \mathbb{Z}$ given by

$f_n(x) \coloneqq nx$. This morphism induces the following morphism in $\mathbb{S}pl\mathbb{E}xt_{\ell\mathbb{G}rp}(\mathbb{Z})$:

$$\begin{tikzcd}
\mathbb{Z} \arrow[r, "i_1"] \arrow[d, equal] & \mathbb{Z}\overleftarrow{\times}\mathbb{Z} \arrow[d, "g_n"'] \arrow[r, shift right, "p_2"'] & \mathbb{Z} \arrow[l, shift right, "i_2"'] \arrow[d, "f_n"] \\
\mathbb{Z} \arrow[r, "i_1"'] & \mathbb{Z}\overleftarrow{\times}\mathbb{Z} \arrow[r, shift right, "p_2"'] & \mathbb{Z} \arrow[l, shift right, "i_2"']
\end{tikzcd}$$

where $g_n(x,y) \coloneqq (x, ny)$. Thus we can deduce that (2) is not faithful. So, if $\ell\mathbb{G}rp$ were action accessible then there should exist a faithful object

$$\begin{tikzcd}
X \arrow[r, "k"'] & A \arrow[r, shift right, "p"'] & B \arrow[l, shift right, "s"']
\end{tikzcd}$$

and a morphism

$$\begin{tikzcd}
\mathbb{Z} \arrow[r, "i_1"] \arrow[d, equal] & \mathbb{Z}\overleftarrow{\times}\mathbb{Z} \arrow[d, "g"'] \arrow[r, shift right, "p_2"'] & \mathbb{Z} \arrow[l, shift right, "i_2"'] \arrow[d, "f"] \\
\mathbb{Z} \arrow[r, "k"'] & A \arrow[r, shift right, "p"'] & B. \arrow[l, shift right, "s"']
\end{tikzcd}$$

Then, if we consider the (regular epimorphism, monomorphism)-factorization of (g, f) we get:

$$\begin{tikzcd}
\mathbb{Z} \arrow[r, "i_1"] \arrow[d, equal] & \mathbb{Z}\overleftarrow{\times}\mathbb{Z} \arrow[d, two heads] \arrow[dd, bend right=50, "g"'] \arrow[r, shift right, "p_2"'] & \mathbb{Z} \arrow[l, shift right, "i_2"'] \arrow[d, two heads] \arrow[dd, bend left=50, "f"] \\
\mathbb{Z} \arrow[r, "\overline{k}"'] \arrow[d, equal] & \mathrm{Im}(g) \arrow[d, tail] \arrow[r, shift right, "\overline{p}"'] & \mathrm{Im}(f) \arrow[l, shift right, "\overline{s}"'] \arrow[d, tail] \\
\mathbb{Z} \arrow[r, "k"'] & A \arrow[r, shift right, "p"'] & B. \arrow[l, shift right, "s"']
\end{tikzcd} \tag{3}$$

Therefore, $\mathrm{Im}(f)$ is a quotient (in $\ell\mathbb{G}rp$) of $\mathbb{Z}$. However, $\mathbb{Z}$ has only two ideals: $\{0\}$ and $\mathbb{Z}$. Hence we have two possibilities: $\mathrm{Im}(f) \cong \mathbb{Z}$ or $\mathrm{Im}(f) \cong \{e\}$. If $\mathrm{Im}(f) \cong \mathbb{Z}$ then f is injective and so the split extension

$$\begin{tikzcd}
\mathbb{Z} \arrow[r, "i_1"] & \mathbb{Z}\overleftarrow{\times}\mathbb{Z} \arrow[r, shift right, "p_2"'] & \mathbb{Z} \arrow[l, shift right, "i_2"']
\end{tikzcd}$$

has to be a faithful object, and this is a contradiction. Alternatively, if $\mathrm{Im}(f) \cong \{e\}$, f has to be the trivial morphism, and so $\mathrm{Im}(g) \cong \mathbb{Z}$. Therefore, recalling that the top right-rightward square of (3) is a pullback, we get a contradiction since $\mathbb{Z}\overleftarrow{\times}\mathbb{Z}$ is not isomorphic, as a lattice-ordered group, to $\mathbb{Z} \times \mathbb{Z}$. $\square$

In [18] the authors show that, in the case of the variety $\mathbb{R}ng$ of not necessarily unitary rings, given a split extension there is a procedure, based on centralizers of subobjects, to build a morphism from it into a faithful split extension. This same argument has also been extended in [43] to categories of interest in the sense of [45]. We recall here a sketch of the proof present in [18]. Given an object A of $\mathbb{R}ng$, two subobjects $X, Y \leq A$ cooperate if and only if for every $x \in X$

and for every $y \in Y$

$$xy = 0 = yx.$$

Hence, it can be shown that the centralizer of X in A is the subobject

$$Z_A(X) \coloneqq \{a \in A \mid ax = 0 = xa \text{ for all } x \in X\}.$$

Given an object of $\mathbb{S}pl\mathbb{E}xt_{\mathbb{R}ng}(X)$

$$X \xrightarrow{k} A \underset{s}{\overset{p}{\rightleftarrows}} B, \tag{4}$$

they define $I \coloneqq \{b \in B \mid s(b)k(x) = 0 = k(x)s(b) \text{ for all } x \in X\}$; they prove that I is an ideal of B and $s(I) = Z_A(k(X)) \cap s(B)$ is an ideal of A. Thus, they show that the split extension

$$X \xrightarrow{\overline{k}} A/s(I) \underset{\overline{s}}{\overset{\overline{p}}{\rightleftarrows}} B/I, \tag{5}$$

where the morphisms are induced by the universal property of the quotient, is a faithful object of $\mathbb{S}pl\mathbb{E}xt_{\mathbb{R}ng}(X)$ and the pair $(\pi_{s(I)}, \pi_I)$, obtained by the quotient projections $\pi_{s(I)} \colon A \to A/s(I)$ and $\pi_I \colon B \to B/I$, is a morphism between (4) and (5). Therefore, in the case of rings, there is a canonical way to construct an arrow of $\mathbb{S}pl\mathbb{E}xt_{\mathbb{R}ng}(X)$ into a faithful object making use of the notion of centralizer.

Although the category $\ell\mathbb{G}rp$ is not action accessible, it is possible to emulate the previous construction in this case. This shows that in $\ell\mathbb{G}rp$ centralizers of subobjects have a good behaviour even though the category is not action accessible.

Let us fix a split extension in $\ell\mathbb{G}rp$:

$$X \xrightarrow{k} A \underset{s}{\overset{p}{\rightleftarrows}} B.$$

We want to show that the intersection between $s(B)$ and the centralizer of $k(X)$ in A is an ideal of A. In other words, we need to prove that $k(X)^{\perp} \cap S(B)$ is convex and closed under conjugation in A.

- Convexity: let us consider $s(b_1) \leq a \leq s(b_2)$ where $s(b_1), s(b_2) \in k(X)^{\perp} \cap s(B)$ and $a \in A$. We recall that for all $a \in A$ there exist $x \in k(X)$ and $b \in B$ such that $a = k(x)s(b)$ (see Proposition 1.1.7). Then, applying p to the inequalities, we obtain $b_1 \leq b \leq b_2$ and thus

$$s(b_1) \leq s(b) \leq s(b_2).$$

  Therefore, since $k(X)^{\perp}$ is a convex subobject of A, we get $s(b) \in k(X)^{\perp} \cap s(B)$. Hence, from $s(b_1) \leq a \leq s(b_2)$ multiplying on the right by $s(b^{-1})$, we obtain

$$s(b_1 b^{-1}) \leq k(x) \leq s(b_2 b^{-1}).$$

So, since $s(b_1 b^{-1}), s(b_2 b^{-1}) \in k(X)^{\perp} \cap s(B)$ (because $k(X)^{\perp} \cap s(B)$ is a subalgebra of A and $s(b), s(b_1), s(b_2) \in k(X)^{\perp} \cap s(B)$), we get $k(x) \in k(X)^{\perp}$ ($k(X)^{\perp}$ is a convex subalgebra). Therefore, $k(x) = e$ and then we obtain $a = k(x)s(b) = s(b) \in k(X)^{\perp} \cap s(B)$.

- Closedness under conjugation: let us consider $s(c) \in k(X)^{\perp} \cap s(B)$ (where $c \in B$) and $a = k(x)s(b) \in A$. Then, we have

$$as(c)a^{-1} = k(x)s(b)s(c)s(b^{-1})k(x)^{-1} = k(x)s(d)k(x)^{-1}$$

where $s(d) = s(b)s(c)s(b^{-1}) \in k(X)^{\perp} \cap s(B)$ since $k(X)^{\perp}$ is closed under conjugation and, clearly, $s(d) \in s(B)$. Therefore, we get

$$as(c)a^{-1} = k(x)s(d)k(x)^{-1} = s(d)$$

since $s(d) \in k(X)^{\perp}$ and the elements of $k(X)^{\perp}$ commute with the ones of $k(X)$.

1.6 Fiber-wise Algebraic Cartesian Closedness

In this section we deal with a stronger version of the notion of algebraically cartesian closed category:

Definition 1.6.1 ([15])**.** *A category* $\mathbb{C}$ *is* fiber-wise algebraically cartesian closed *if for every split epimorphism*

$$A \underset{s}{\overset{p}{\rightleftarrows}} B$$

the change-of-base functor

$$p^* \colon Pt_B\mathbb{C} \to Pt_A\mathbb{C}$$

has a right adjoint.

It is not difficult to see that this condition holds for a category $\mathbb{C}$ if and only if every category of points over $\mathbb{C}$ is algebraically cartesian closed. First of all, we observe that the category $Pt_{A \leftrightarrows B}Pt_B\mathbb{C}$ is isomorphic to $Pt_A\mathbb{C}$: an object of $Pt_{A \leftrightarrows B}Pt_B\mathbb{C}$ can be seen as a diagram of type

$$\begin{tikzcd} C \arrow[r, shift right, "q"'] \arrow[d, shift right, "h"'] & B \arrow[l, shift right, "r"'] \arrow[d, equal] \\ A \arrow[r, shift right, "p"'] \arrow[u, shift right, "t"'] & B \arrow[l, shift right, "s"'] \end{tikzcd}$$

where $ps = id_B$, $qr = id_B$, $ht = id_A$, $ph = q$, and $ts = r$; therefore, q and r are uniquely determined by h and t. Hence, each category of points is algebraically cartesian closed if and only if the functor

$$\tau^*\colon Pt_{B=B}Pt_B\mathbb{C} \longrightarrow Pt_{A\leftrightarrows B}Pt_B\mathbb{C}$$

has a right adjoint; thanks to the isomorphism shown above between $Pt_A\mathbb{C}$ and $Pt_{A\leftrightarrows B}Pt_B\mathbb{C}$ and recalling that $\tau = p$, we get that τ^* has a right adjoint if and only if p^* has a right adjoint. Therefore, $\mathbb{C}$ is fiber-wise algebraically cartesian closed if and only if every category of points over $\mathbb{C}$ is algebraically cartesian closed.

Our aim is to show, thanks to the previous observations, that $\ell\mathbb{G}rp$ is fiber-wise algebraically cartesian closed. In order to do this we will prove that, in every category of points over $\ell\mathbb{G}rp$, subobjects have centralizers. As a preliminary remark, we recall that an arrow $(A \leftrightarrows B) \xrightarrow{f} (C \leftrightarrows B)$ of $Pt_B\mathbb{C}$ is a monomorphism if and only if $f\colon A \to C$ is a monomorphism of $\mathbb{C}$.

We are ready to show the existence of centralizers in every category of points over $\ell\mathbb{G}rp$ and to provide an explicit description of them.

Definition 1.6.2. *Let X be an object of $\ell\mathbb{G}rp$ and B a subalgebra of X. A subalgebra L of X is* closed under the action of B *if*

$$blb^{-1} \in L \text{ and } (l_1b_1 \vee l_2b_2)(b_1 \vee b_2)^{-1} \in L$$

for every $l, l_1, l_2 \in L$ and $b, b_1, b_2 \in B$.

Proposition 1.6.3. *Let B be an object of $\ell\mathbb{G}rp$. In the category $Pt_B\ell\mathbb{G}rp$ subobjects have centralizers.*

Proof. Let us consider an object (A, p, s) of $Pt_B\ell\mathbb{G}rp$, i.e. a diagram of the form

$$K \xrightarrow{k} A \underset{p}{\overset{s}{\leftrightarrows}} B$$

where $ps = id_B$ and $k = \ker(p)$. For simplicity, let us suppose that k is the inclusion of $K \leq A$ and s is the inclusion of $B \leq A$. Therefore, by Proposition 1.1.7, we know that A is isomorphic as a lattice-ordered group to $K \rtimes B$, whose operations are defined by

$$(k_1, b_1)(k_2, b_2) = (k_1b_1k_2b_1^{-1}, b_1b_2)$$

and

$$(k_1, b_1) \vee (k_2, b_2) = ((k_1b_1 \vee k_2b_2)(b_1 \vee b_2)^{-1}, b_1 \vee b_2).$$

In other words, the object (A, p, s) is isomorphic to

$$\begin{tikzcd} K \arrow[r, "i_k"] & K \rtimes B \arrow[r, shift right, "p_B"'] & B \arrow[l, shift right, "i_B"'] \end{tikzcd}$$

where $p_B(k, b) = b$, $i_B(b) = (e, b)$ and $i_K(k) = (k, e)$. A subobject (X, q, r) of $(K \rtimes B, p_B, i_B)$ in $Pt_B\ell\mathbb{G}rp$ is a subalgebra $X \leq K \rtimes B$ in $\ell\mathbb{G}rp$ such that, referring to the following diagram,

$$\begin{tikzcd} X \arrow[d, hook] \arrow[dr, shift left, "q"] & \\ K \rtimes B \arrow[r, shift left, "p_B"] & B \arrow[l, shift left, "i_B"] \arrow[ul, shift left, "r"] \end{tikzcd}$$

q is the restriction of p_B to X and $r(b) = (e, b)$ for every $b \in B$ (in particular $\{e\} \times B \leq X$).
Given two subobjects $(X, p_{|X}, r_X)$ and $(Y, p_{|Y}, r_Y)$ of (A, p, s) and the product between them in $Pt_B\ell\mathbb{G}rp$, we need to describe the arrows $i_X : X \to X \times_B Y$ and $i_Y : Y \to X \times_B Y$ induced by the universal property (we recall that the product in $Pt_B\ell\mathbb{G}rp$ is the pullback of $p_{|X}$ along $p_{|Y}$).
Then, if we consider the following diagram

$$\begin{tikzcd} X \arrow[r, "p_{|X}"] \arrow[dr, "i_X"] \arrow[ddr, bend right, "id_X"'] & B \arrow[dr, bend left, "s"] & \\ & X \times_B Y \arrow[r, "\pi_Y"] \arrow[d, "\pi_X"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "p_{|Y}"] \\ & X \arrow[r, "p_{|X}"'] & B \end{tikzcd}$$

we get $i_X(x) = (x, sp(x))$; in a similar way $i_Y : Y \to X \times_B Y$ is given by $i_Y(y) = (sp(y), y)$.
We know that $(x, y) \in X \times_B Y$ if and only if $p(x) = p(y)$. Moreover, given $(x, y) \in X \times_B Y$ we have $(x, y) = (x, sp(x))(sp(y), sp(x))^{-1}(sp(y), y)$ where $(x, sp(x)) \in X \times_B Y$, $(sp(y), y) \in X \times_B Y$ and, since $psp(x) = p(x) = p(y) = psp(y)$, we get $(sp(y), sp(x)) = (sp(x), sp(x)) = (sp(y), sp(y)) \in X \times_B Y$.
Hence, if there exists a cooperator $\varphi : X \times_B Y \to A$, then

$$\varphi(x, y) = \varphi(x, sp(x))\varphi(sp(y), sp(x))^{-1}\varphi(sp(y), y) = xsp(x)^{-1}y = xsp(y)^{-1}y,$$

since

$$\varphi(x, sp(x))\varphi(sp(y), sp(x))^{-1}\varphi(sp(y), y) = \varphi(i_X(x))\varphi(i_X(sp(x)))^{-1}\varphi(i_Y(y))$$

$$\begin{tikzcd} X \arrow[dr, "i_X"] \arrow[ddr, tail, bend right] & & Y \arrow[dl, "i_Y"'] \arrow[ddl, tail, bend left] \\ & X \times_B Y \arrow[d, "\varphi"'] & \\ & A. & \end{tikzcd}$$

Given a subobject (X, q, r) of $(K \times B, p_B, i_B)$ we define

$$\overline{X} \coloneqq \{\overline{x} \in K \mid \text{there exists } b \in B \text{ s.t. } (\overline{x}, b) \in X\}.$$

We want to show that $X = \overline{X} \times B$ as sets. Clearly $X \subseteq \overline{X} \times B$. Conversely, let us take an element $(k, b) \in \overline{X} \times B$. Then $k \in \overline{X}$, and so there exists $b_1 \in B$ such that $(k, b_1) \in X$; but $(e, b), (e, b_1) \in X$, therefore $(k, b) = (k, b_1)(e, b_1)^{-1}(e, b) \in X$. In general, we have a one-to-one correspondence between the subobjects of (A, p, s) and the subalgebras of K closed under the action of B. Hence, under our assumptions, we have $X = \overline{X} \times B$ and $Y = \overline{Y} \times B$. Thus,

$$X \times_B Y = \{((\overline{x}, b_1), (\overline{y}, b_2)) \in X \times Y \mid b_1 = b_2\}$$

and $\varphi\colon X \times_B Y \to K \times B$ is such that

$$\varphi((\overline{x}, b), (\overline{y}, b)) = (\overline{x}, b)(e, b)^{-1}(\overline{y}, b) = (\overline{x}\,\overline{y}, b).$$

Let us show that φ is a group homomorphism if and only if $xy = yx$ for all $x \in \overline{X}$ and $y \in \overline{Y}$. Given an element $((x, b), (y, b)), ((z, c), (w, c)) \in (\overline{X} \times B) \times_B (\overline{Y} \times B)$ one has

$$((x, b), (y, b))((z, c), (w, c)) = ((x, b)(z, c), (y, b)(w, c)) = ((xbzb^{-1}, bc), (ybwb^{-1}, bc)).$$

Therefore,

$$\varphi((x, b), (y, b))\varphi((z, c), (w, c)) = (xy, b)(zw, c) = (xybzwb^{-1}, bc)$$

and, since $((x, b), (y, b))((z, c), (w, c)) = ((xbzb^{-1}, bc), (ybwb^{-1}, bc))$, we get

$$\varphi((xbzb^{-1}, bc), (ybwb^{-1}, bc)) = (xbzb^{-1}ybwb^{-1}, bc).$$

So, φ is a group homomorphism if and only if $bzb^{-1}y = ybzb^{-1}$ for all $z \in \overline{X},\ y \in \overline{Y}$ and $b \in B$. Then, setting $b = e$, we get $zy = yz$ for all $z \in \overline{X}$ and $y \in \overline{Y}$. Moreover, since $\overline{X}$ is closed under the action of B and since the conjugation is a bijection, we get that every element of $\overline{X}$ can be seen as bzb^{-1} for appropriate $b \in B$ and $z \in \overline{X}$; thus, if $zy = yz$ for all $z \in \overline{X}$ and $y \in \overline{Y}$ then $bzb^{-1}y = ybzb^{-1}$ for all $z \in \overline{X},\ y \in \overline{Y}$ and $b \in B$. Now, let us deal with the order structure. We know that φ is a morphism of lattice-ordered groups if and only if φ is a group homomorphism and for all $((x, b), (y, b)) \in (\overline{X} \times B) \times_B (\overline{Y} \times B)$

$$\varphi(((x, b), (y, b)) \vee ((e, e), (e, e))) = \varphi(((x, b), (y, b))) \vee (e, e). \tag{1.1}$$

We know that

$$\begin{aligned}((x,b),(y,b)) \vee ((e,e),(e,e)) &= ((x,b) \vee (e,e),(y,b) \vee (e,e))\\ &= (((xb \vee e)(b \vee e)^{-1}, b \vee e),((yb \vee e)(b \vee e)^{-1}, b \vee e))\\ &= (((xb)^+(b^+)^{-1}, b^+),((yb)^+(b^+)^{-1}, b^+)),\end{aligned}$$

hence one has

$$\varphi(((x,b),(y,b)) \vee ((e,e),(e,e))) = ((xb)^+(b^+)^{-1}(yb)^+(b^+)^{-1}, b^+).$$

Considering the right term of (1.1), we obtain

$$\varphi(((x,b),(y,b))) \vee (e,e) = (xy,b) \vee (e,e) = ((xyb)^+(b^+)^{-1}, b^+).$$

We want to show that φ is a lattice-ordered group morphism if and only if $\overline{X} \perp \overline{Y}$.
If φ is a lattice-ordered group morphism, then

$$(xb)^+(b^+)^{-1}(yb)^+(b^+)^{-1} = (xyb)^+(b^+)^{-1}$$

for each $x \in \overline{X}$, $y \in \overline{Y}$ and $b \in B$; therefore, setting $b = e$, we get

$$(xy)^+ = x^+y^+ \text{ for every } x \in \overline{X} \text{ and } y \in \overline{Y}$$

and so $\overline{X}$ and $\overline{Y}$ are orthogonal (see the proof of Proposition 1.2.4).
Conversely, let us suppose $\overline{X} \perp \overline{Y}$. We want to show that, for all $x \in \overline{X}, y \in \overline{Y}$ and $b \in B$,

$$(xb)^+(b^+)^{-1}(yb)^+(b^+)^{-1} = (xyb)^+(b^+)^{-1}$$

and so we have to prove that $(xb \vee e)(b \vee e)^{-1}(yb \vee e) = xyb \vee e$. We start with the first term:

$$\begin{aligned}(xb \vee e)(b \vee e)^{-1}(yb \vee e) &= (xb(b^{-1} \wedge e) \vee (b^{-1} \wedge e))(yb \vee e)\\ &= xb(b^{-1} \wedge e)(yb \vee e) \vee (b^{-1} \wedge e)(yb \vee e)\\ &= xb(b^{-1} \wedge e)yb \vee xb(b^{-1} \wedge e) \vee (b^{-1} \wedge e)yb \vee (b^{-1} \wedge e).\end{aligned}$$

Now, we know that there exists an element $y_1 \in \overline{Y}$ such that $yb = by_1$ (since $\overline{Y}$ is closed under the action of B and the conjugation is an automorphism of $\overline{Y}$), hence the last term is equal to

$$\begin{aligned}&xb(b^{-1} \wedge e)by_1 \vee xb(b^{-1} \wedge e) \vee (b^{-1} \wedge e)by_1 \vee (b^{-1} \wedge e)\\ &\qquad = x(b \wedge b^2)y_1 \vee x(b \wedge e) \vee (b \wedge e)y_1 \vee (b^{-1} \wedge e).\end{aligned}$$

Moreover, we observe that there exists an element $y_2 \in \overline{Y}$ such that $(b \wedge e)y_1 = y_2(b \wedge e)$, thus

$$\begin{aligned} x(b \wedge e) \vee (b \wedge e)y_1 &= x(b \wedge e) \vee y_2(b \wedge e) = (x \vee y_2)(b \wedge e) \\ &= (xy_2 \vee e)(b \wedge e) = xy_2(b \wedge e) \vee (b \wedge e) = x(b \wedge e)y_1 \vee (b \wedge e) \end{aligned}$$

(since $\overline{X} \perp \overline{Y}$, by Lemma 1.2.5 we know that $x \vee y_2 = xy_2 \vee e$). Then, one has

$$\begin{aligned} &x(b \wedge b^2)y_1 \vee x(b \wedge e) \vee (b \wedge e)y_1 \vee (b^{-1} \wedge e) \\ &\quad = x(b \wedge b^2)y_1 \vee x(b \wedge e)y_1 \vee (b \wedge e) \vee (b^{-1} \wedge e). \end{aligned}$$

We recall that $(b \wedge e) \vee (b^{-1} \wedge e) = (b \vee b^{-1}) \wedge e = |b| \wedge e = e$, so we finally get

$$\begin{aligned} x(b \wedge b^2)y_1 \vee x(b \wedge e)y_1 \vee (b \wedge e) \vee (b^{-1} \wedge e) &= x(b \wedge b^2)y_1 \vee x(b \wedge e)y_1 \vee e \\ &= x[(b \wedge b^2) \vee (b \wedge e)]y_1 \vee e = xby_1 \vee e = xyb \vee e \end{aligned}$$

since $(b \wedge b^2) \vee (b \wedge e) = b(b \wedge e) \vee e(b \wedge e) = (b \vee e)(b \wedge e) = b^+b^- = b$.
To conclude, we have to prove that, if we take a subalgebra $\overline{X} \leq K$ closed under the action of B, then also $\overline{X}^\perp \leq K$ is closed under the action of B (and so the centralizer of $X = \overline{X} \times B$ is $\overline{X}^\perp \times B$ endowed with the semi-direct product structure). Let us consider two elements $y \in \overline{X}^\perp$ and $b \in B$; we want to show that $byb^{-1} \in \overline{X}^\perp$. We know that

$$\begin{aligned} &byb^{-1} \in \overline{X}^\perp \text{ if and only if } |byb^{-1}| \wedge |x| = e \text{ for all } x \in \overline{X} \text{ if and only if} \\ &|y| \wedge |b^{-1}xb| = e \text{ for all } x \in \overline{X} \text{ if and only if } |y| \wedge |x| = e \text{ for all } x \in \overline{X} \end{aligned}$$

because the conjugation is an automorphism of $\overline{X}$; the last assertion holds since $y \in \overline{X}^\perp$. We recall that $\overline{X}^\perp$ is a convex subalgebra of K. For every $y_1, y_2 \in \overline{X}^\perp$ and $b_1, b_2 \in B$ we have $y_1b_1 \leq (y_1 \vee y_2)(b_1 \vee b_2)$ and $y_2b_2 \leq (y_1 \vee y_2)(b_1 \vee b_2)$; we also observe that $(y_1 \wedge y_2)b_1 \leq y_1b_1$ and $(y_1 \wedge y_2)b_2 \leq y_2b_2$. So one has

$$y_1b_1 \vee y_2b_2 \leq (y_1 \vee y_2)(b_1 \vee b_2)$$

and

$$(y_1 \wedge y_2)(b_1 \vee b_2) = (y_1 \wedge y_2)b_1 \vee (y_1 \wedge y_2)b_2 \leq y_1b_1 \vee y_2b_2.$$

Therefore, for all $y_1, y_2 \in \overline{X}^\perp$ and $b_1, b_2 \in B$ we obtain

$$y_1 \wedge y_2 \leq (y_1b_1 \vee y_2b_2)(b_1 \vee b_2)^{-1} \leq y_1 \vee y_2;$$

then, since $\overline{X}^\perp$ is convex in K, we get $(y_1b_1 \vee y_2b_2)(b_1 \vee b_2)^{-1} \in \overline{X}^\perp$ for all $y_1, y_2 \in \overline{X}^\perp$ and $b_1, b_2 \in B$. □

Corollary 1.6.4. *For every object B of $\ell\mathbb{G}rp$, in the category $Pt_B\ell\mathbb{G}rp$ subobjects have centralizers, therefore $Pt_B\ell\mathbb{G}rp$ is algebraically cartesian closed. Hence, the category $\ell\mathbb{G}rp$ is fiber-wise algebraically cartesian closed.*

1.7 Normality of the Higgins Commutator

The aim of this section is to propose a further study regarding the properties of commutators in the category of lattice-ordered groups.
We recall a first notion of categorical commutator strongly linked to the concept of cooperation.

Definition 1.7.1 ([36],[11])**.** *Let $\mathbb{C}$ be a unital category. For a pair of subobjects $a\colon A \rightarrowtail X$ and $b\colon B \rightarrowtail X$ of an object X in $\mathbb{C}$, the* Huq commutator *is the smallest normal subobject $[A,B]_X \rightarrowtail X$ such that the images of a and b cooperate in the quotient $X/[A,B]_X$.*

Then, we remind here the notion of *Higgins commutator*. In a pointed category $\mathbb{C}$ with binary products and coproducts, for each pair of objects H and K we have the following canonical arrows:

$$H \xrightarrow{(id_H,0)} H \times K \xleftarrow{(0,id_K)} K$$

$$H \xleftarrow[{[id_H,0]}]{} H + K \xrightarrow[{[0,id_K]}]{} K;$$

combining them we get a canonical arrow

$$\Sigma = \begin{pmatrix} id_H & 0 \\ 0 & id_K \end{pmatrix} \colon H + K \to H \times K.$$

For instance, in the case of the variety of groups, the morphism Σ associates to each word $h_1k_1h_2k_2\dots h_nk_n$, where $h_i \in H$ and $k_i \in K$ for $i = 1,\dots,n$, the pair $(h_1h_2\dots h_n, k_1k_2\dots k_n) \in H \times K$. It is easy to see that a category with binary products and coproducts is unital if and only if, for every pair of objects H and K, Σ is a strong epimorphism. Hence, again in the case of groups, the kernel of Σ, denoted by $H \diamond K$ and called the *cosmash product* of H and K, can be described as the subgroup of $H + K$ generated by the elements of the form $hkh^{-1}k^{-1}$ with $h \in H$ and $k \in K$. In the light of the above, we are ready to recall the following:

Definition 1.7.2 ([42])**.** *Let $\mathbb{C}$ be a semi-abelian category. Given a pair of subobjects $a\colon A \rightarrowtail X$ and $b\colon B \rightarrowtail X$ of an object X in $\mathbb{C}$, the* Higgins commutator *of A and B is the subobject $[A,B] \rightarrowtail X$ constructed, via the (regular epimorphism, monomorphism)-factorization, as in diagram*

$$\begin{tikzcd} A \diamond B \arrow[r, "k_{A,B}"] \arrow[d, two heads] & A+B \arrow[d, "{[a,b]}"] \\ {[A,B]} \arrow[r, tail] & X, \end{tikzcd}$$

where $k_{A,B}$ is the kernel of $\Sigma = \begin{pmatrix} id_A & 0 \\ 0 & id_B \end{pmatrix} : A + B \to A \times B$.

In general, the Higgins commutator of two normal subobjects is not normal. Therefore, it makes sense to mention the following definition:

Definition 1.7.3 ([23]). *A semi-abelian category $\mathbb{C}$ satisfies the condition of* normality of Higgins commutators (NH) *when, for every pair of normal subobjects $H \rightarrowtail X$, $K \rightarrowtail X$ where X is an object of $\mathbb{C}$, the Higgins commutator $[H, K] \rightarrowtail X$ is a normal subobject of X.*

We have everything we need to prove that that the category of lattice-ordered groups satisfies (NH) (recalling that in $\ell\mathbb{G}rp$ the notion of normal subobject coincide with the one of ideal).

Lemma 1.7.4. *Let $H, K \leq X$ be two convex subalgebras of X in $\ell\mathbb{G}rp$. Then H and K cooperate if and only if $H \cap K = \{e\}$.*

Proof. ($\Rightarrow$) Trivial since $H \perp K$ (thanks to Proposition 1.2.4).
($\Leftarrow$) We want to show that $H \perp K$: let us consider two elements $h \in H$ and $k \in K$; then $e \leq |h| \wedge |k| \leq |h|$ and $e \leq |h| \wedge |k| \leq |k|$. Therefore, since H and K are convex, we have $|h| \wedge |k| \in H \cap K = \{e\}$. $\square$

Notation 1.7.5. *We will write $[H, K]$ for the Higgins commutator of $H \rightarrowtail X$ and $K \rightarrowtail X$, and $[H, K]_Y$ for the Huq commutator of $H \rightarrowtail X$ and $K \rightarrowtail X$ in the subobject Y of X, where H and K are subobjects of Y.*

Proposition 1.7.6. *Let X be an object of $\ell\mathbb{G}rp$ and H, K ideals of X. Then, $[H, K]_X = H \cap K$.*

Proof. Let us prove the inclusion $H \cap K \subseteq [H, K]_X$. Consider the following diagram:

$$\begin{tikzcd} H \arrow[dr, "i_H"] \arrow[ddr, bend right, "q_{|H}"'] & & K \arrow[dl, "i_K"'] \arrow[ddl, bend left, "q_{|K}"] \\ & H \times K \arrow[d, "\varphi"] & \\ & X/[H,K]_X & \end{tikzcd}$$

where $q \colon X \twoheadrightarrow X/[H, K]_X$ is the canonical projection. Then, by Lemma 1.7.4, we know that $q(H) \cap q(K) = \{e\}$. So, since $q(H \cap K) \subseteq q(H) \cap q(K) = \{e\}$ we get $H \cap K \subseteq [H, K]_X$. The other inclusion holds in every semi-abelian category (see Theorem 3.9 in [29]). $\square$

Proposition 1.7.7. *The category $\ell\mathbb{G}rp$ satisfies (NH).*

Proof. Thanks to Theorem 2.8 of [23], it suffices to prove that, given an ideal H of X and an ideal K of Y such that $H, K \leq Y$, then $[H,K]_X = [H,K]_Y$. Thus the statement follows from the previous proposition, since $[H,K]_X = [H,K]_Y = H \cap K$. $\square$

1.8 Algebraic Coherence for $\ell\mathbb{A}b$

In the last part of the chapter we focus on the notion of *algebraically coherent category.* This concept has an important algebraic meaning: an algebraically coherent category satisfies a large set of properties related to the good behaviour of commutators (such as, for example, strong protomodularity); moreover, in the case of the varieties of universal algebra, the property of being fiber-wise algebraically cartesian closed is implied by algebraic coherence.

Definition 1.8.1 ([22])**.** *A category $\mathbb{C}$ with finite limits is* algebraically coherent *if, for every morphism $f\colon X \to Y$ in $\mathbb{C}$, the change-of-base functor*

$$f^*\colon Pt_Y\mathbb{C} \to Pt_X\mathbb{C}$$

is coherent: a functor between categories with finite limits is coherent *if it preserves finite limits and jointly extremally epimorphic pairs.*

Since in the semi-abelian case the split extensions with fixed splitting can be totally described in terms of semi-direct products, the authors in [22] proved the following result:

Proposition 1.8.2 ([22], Theorem 3.21)**.** *Suppose $\mathbb{C}$ is a semi-abelian category. The following are equivalent:*

- $\mathbb{C}$ *is algebraically coherent;*
- *given $K \rightarrowtail X$ and $H \rightarrowtail X$ in $\mathbb{C}$, any action $\xi\colon B\flat X \to X$ which restricts to K and H also restricts to $K \vee H$.*

Let us now try to understand how this result can be interpreted in the category of lattice-ordered groups. We know (see [16]) that, in the semi-abelian case, for each internal action $\xi\colon B\flat X \to X$ there exists a unique (up to isomorphism) split extension

$$X \xrightarrow{k} A \underset{p}{\overset{s}{\leftrightarrows}} B$$

making, in the following diagram, the right-rightward square, the right-leftward square and the left square commute:

$$\begin{tikzcd}
B\flat X \arrow[r, "k_0"] \arrow[d, "\xi"'] & X+B \arrow[d, "{[k,s]}"'] \arrow[r, shift right, "{[0,id_B]}"'] & B \arrow[l, shift right, "\iota_B"'] \arrow[d, equal] \\
X \arrow[r, "k"'] & A \arrow[r, shift right, "p"'] & B. \arrow[l, shift right, "s"']
\end{tikzcd}$$

Therefore, in $\ell\mathbb{G}rp$, ξ restricts to a subalgebra $L \leq X$ if and only if L is closed under the corresponding action of B (in the sense of Definition 1.6.2).

In the next proposition we will deal with the category of *lattice-ordered abelian groups*. A lattice-ordered abelian group is a lattice-ordered group in which the group operation is commutative. The category $\ell\mathbb{A}b$ is the full subcategory of $\ell\mathbb{G}rp$ whose objects are lattice-ordered abelian groups.

Proposition 1.8.3. *$\ell\mathbb{A}b$ is algebraically coherent.*

Proof. Let us consider an object (A, p, s) of $Pt_B\ell\mathbb{A}b$ i.e. a diagram of the form

$$X \xrightarrow{k} A \underset{p}{\overset{s}{\leftrightarrows}} B$$

where $ps = id_B$ and $k = \ker(p)$. For simplicity, let us suppose that k is the inclusion of $X \leq A$ and s is the inclusion of $B \leq A$. Given two subalgebras $K, H \leq X$ closed under the action of B, we want to show that also $K \vee H$ is closed under the action of B.
First of all, let us observe that, given a subalgebra $L \leq X$, the following equality holds for every $l_1, l_2 \in L$ and $b_1, b_2 \in B$:

$$(l_1b_1 \vee l_2b_2)(b_1 \vee b_2)^{-1} = l_2(l_2^{-1}l_1b_1b_2^{-1} \vee e)(b_1b_2^{-1} \vee e)^{-1}.$$

Therefore, L is closed under the action of B if and only if for all $l \in L$ and $b \in B$

$$(lb \vee e)(b \vee e)^{-1} \text{ belongs to } L.$$

Now, in a lattice-ordered abelian group A, the equation

$$xy = (x \vee y)(x \wedge y)$$

holds for all $x, y \in A$; in fact, $x(x \wedge y)^{-1}y = x(x^{-1} \vee y^{-1})y = x \vee y$ and thus $xy = (x \vee y)(x \wedge y)$. Finally, it is easy to see that every element of $K \vee H$ can be written as

$$\bigvee_{i \in I} \bigwedge_{j \in J} k_{i,j}h_{i,j}$$

where I, J are finite sets of indices and $k_{i,j} \in K, h_{i,j} \in H$ for all $i \in I, j \in J$. This statement can be proved by iteratively applying the distributive properties of the lattice operations, the distributivity property of the group product over the lattice operations, and the commutative property of the group product. Therefore, given an element $b \in B$, one has

$$\left(\left(\bigvee_{i\in I}\bigwedge_{j\in J} k_{i,j}h_{i,j}\right)b \vee e\right) = \left(\bigvee_{i\in I}\bigwedge_{j\in J} k_{i,j}h_{i,j}b\right) \vee e$$
$$= \bigvee_{i\in I}\left(\bigwedge_{j\in J} k_{i,j}h_{i,j}b \vee e\right) = \bigvee_{i\in I}\bigwedge_{j\in J}(k_{i,j}h_{i,j}b \vee e)$$

where the first equality holds thanks to the distributivity of the group operation over the lattice operations, the second thanks to the idempotence of the join, and the third thanks to the distributivity of the join over the meet. Therefore

$$\left(\left(\bigvee_{i\in I}\bigwedge_{j\in J} k_{i,j}h_{i,j}\right)b \vee e\right)(b\vee e)^{-1} = \bigvee_{i\in I}\bigwedge_{j\in J}(k_{i,j}h_{i,j}b \vee e)(b\vee e)^{-1}$$

and hence, in order to prove that if K and H are closed under the action of B then also $K \vee H$ is closed under the action of B, it suffices to prove that for every $k \in K$, $h \in H$ and $b \in B$

$$(khb \vee e)(b \vee e)^{-1} \in K \vee H.$$

We need to take care of an intermediate step: we want to show that, for all $k \in K$, $h \in H$ and $b \in B$,

$$((k \vee h)b \vee (k \wedge h)^{-1})(b \vee e)^{-1} \text{ belongs to } K \vee H.$$

To do this we observe that

$$\begin{aligned}((k \vee h)b \vee (k \wedge h)^{-1})(b \vee e)^{-1} &= (kb \vee hb \vee k^{-1} \vee h^{-1})(b \vee e)^{-1}\\ &= (kb \vee k^{-1})(b \vee e)^{-1} \vee (hb \vee h^{-1})(b \vee e)^{-1}\\ &= k^{-1}(k^2b \vee e)(b \vee e)^{-1} \vee h^{-1}(h^2b \vee e)(b \vee e)^{-1}.\end{aligned}$$

So, since K is closed under the action of B, we have $(k^2b \vee e)(b \vee e)^{-1} \in K$ and then we obtain $k^{-1}(k^2b \vee e)(b \vee e)^{-1} \in K$; similarly $h^{-1}(h^2b \vee e)(b \vee e)^{-1} \in H$. Therefore, taking the join of these two terms, we get $((k \vee h)b \vee (k \wedge h)^{-1})(b \vee e)^{-1} \in K \vee H$. To conclude, since $k \wedge h \in K \vee H$, the product

$$(k \wedge h)((k \vee h)b \vee (k \wedge h)^{-1})(b \vee e)^{-1} \text{ belongs to } K \vee H;$$

then we get

$$(khb \vee e)(b \vee e)^{-1} \in H \vee K$$

applying the distributivity property of the group product over the lattice join and recalling that $(k \wedge h)(k \vee h) = kh$. □

This conclusive result answers the Open Problem 6.28 presented in [22]. In fact, the category $\ell\mathbb{A}b$

is algebraically coherent, as just shown, however it is not action accessible: the example provided in 1.5.4 exclusively involves lattice-ordered groups whose group operations are commutative.

Chapter 2

Categorical-Algebraic Properties of MV-Algebras

MV-algebras are a class of algebraic structures used in mathematics to study non-classical logic. They generalize the notion of Boolean algebras, which are used to model classical logic, and can be seen as a mathematical system for reasoning under uncertainty. MV-algebras have applications in various areas of mathematics, such as logic, algebra, topology, and computer science, and can be used to model fuzzy logic, intuitionistic logic, and quantum logic.

More specifically, an *MV-algebra* A is a set equipped with an operation $\oplus$, which is both associative, commutative, and has a neutral element 0, and an operation $\neg$, such that the following equalities hold: $\neg\neg x = x$, $x \oplus \neg 0 = \neg 0$, and $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$. These conditions are intended to capture some properties of the real unit interval $[0, 1]$ equipped with negation $\neg x = 1 - x$ and truncated addition $x \oplus y = \min(1, x + y)$.

MV-algebras are a powerful tool to model Łukasiewicz calculus mathematically. In fact, in Łukasiewicz calculus, which is a type of many-valued logic, the truth values are not limited to just "true" or "false". Instead, they can take on any value from a continuous interval, typically $[0, 1]$. In this sense, MV-algebras provide a natural framework for modeling Łukasiewicz calculus, as they can represent the truth values as elements in the interval $[0, 1]$, and develop an algebraic structure that can naturally manipulate these values.

In this chapter, we will investigate the categorical-algebraic properties of the variety of MV-algebras, denoted by $\mathbb{MV}$. Although $\mathbb{MV}$ is not a semi-abelian category, it possesses many important properties. Recall that a category is semi-abelian [39] if it is a pointed finitely cocomplete category which is Barr-exact and protomodular. However, $\mathbb{MV}$ fails to satisfy the condition of having isomorphic initial and terminal objects; the initial object is the algebra $\{0, 1\}$, while the terminal object is the algebra $\{0 = 1\}$. Nonetheless, being a variety of universal algebra, $\mathbb{MV}$ is complete, cocomplete, and Barr-exact. Furthermore, we will show that $\mathbb{MV}$ is also a protomod-

ular category. To summarize, even though $\mathbb{MV}$ is not a semi-abelian category, it still possesses many important algebraic-categorical properties, making it an interesting object of study in its own right.
The structure of the chapter is organized as follows.
In Section 2.1 we recall some classical facts about MV-algebras and we focus on the notion of ideal of an MV-algebra.
In Section 2.2 we investigate the properties of idempotent elements in an MV-algebra. An element a in an MV-algebra is said to be *idempotent* if the equation $a \oplus a = a$ holds. We will review some known results concerning idempotents and prove some additional properties. The content presented in this part of the chapter will be fundamental for the following sections.
In Section 2.3 we review Stone Duality Theorem for Boolean algebras [49] and establish the notation that will be used in the subsequent parts of the chapter.
In Section 2.4 we deal with extending the construction of the Stone space for Boolean algebras to the case of MV-algebras. In addition, in this section, we will provide an in-depth study of the Pierce spectrum for MV-algebras, obtaining results similar to those known for the case of unitary rings, as presented e.g. in [4].
In Section 2.5 we show that the category of MV-algebras is a protomodular and arithmetical category, by providing explicit descriptions of the protomodularity and arithmeticity terms. Moreover, we prove that subobjects in the category $Pt_B\mathbb{MV} = (\mathbb{MV}/B)\backslash id_B$ (i.e. the coslice over id_B of the slice of $\mathbb{MV}$ over B) have centralizers.

2.1 Preliminaries

In this section, we will recall the fundamental properties of MV-algebras, starting from their definition. We will examine how new operations can be derived from those introduced in the definition. Additionally, we will recall how a partial order can be naturally defined on any MV-algebra. Finally, we will focus on the concept of an ideal in an MV-algebra.

Definition 2.1.1. *An* MV-algebra *is an algebra* $(A, \oplus, \neg, 0)$ *with a binary operation* $\oplus$*, a unary operation* $\neg$ *and a constant* 0 *satisfying the following equations:*

M1) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$*;*

M2) $x \oplus y = y \oplus x$*;*

M3) $x \oplus 0 = x$*;*

M4) $\neg\neg x = x$*;*

M5) $x \oplus \neg 0 = \neg 0$*;*

M6) $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$.

A morphism *between two MV-algebras* $(A, \oplus, \neg, 0)$ *and* $(B, \oplus, \neg, 0)$ *is a map* $f \colon A \to B$ *satisfying the following conditions, for each* $x, y \in A$*:*

H1) $f(0) = 0$*;*

H2) $f(x \oplus y) = f(x) \oplus f(y)$*;*

H3) $f(\neg x) = \neg f(x)$*.*

The category $\mathbb{MV}$ *is the category whose objects are the MV-algebras and whose arrows are the morphisms between them.*

As in the case of $\ell\mathbb{G}rp$, we will indicate an MV-algebra $(A, \oplus, \neg, 0)$ simply with A. Moreover, we will consider the $\neg$ operation more binding than the $\oplus$ operation.

Given an MV-algebra A, we define the constant 1 and the binary operations $\odot$, $\ominus$, $\to$, and d as follows:

- $1 \coloneqq \neg 0$;
- $x \odot y \coloneqq \neg(\neg x \oplus \neg y)$;
- $x \to y \coloneqq \neg x \oplus y$;
- $x \ominus y \coloneqq \neg(\neg x \oplus y) = \neg(x \to y) = x \odot (\neg y)$;
- $d(x, y) \coloneqq (x \ominus y) \oplus (y \ominus x)$ (called *distance*)

We will consider the $\neg$ operation more binding than the $\odot$, $\ominus$, and $\to$ operations.

With respect to the new operations, we get

- $\neg 1 = 0$;
- $x \oplus y = \neg(\neg x \odot \neg y)$;
- $x \oplus 1 = 1$;
- $(x \ominus y) \oplus y = (y \ominus x) \oplus x$;
- $x \to x = \neg x \oplus x = 1$;
- $x = y$ if and only if $d(x, y) = 0$ (see [21], Proposition 1.2.5).

We recall now some known facts that will be useful in the next sections.

Lemma 2.1.2 ([21], Lemma 1.1.3)**.** *Let A be an MV-algebra. For every $x \in A$, $\neg x$ is the unique solution of the simultaneous equations:*
$$\begin{cases} x \oplus \neg x = 1 \\ x \odot \neg x = 0. \end{cases}$$

Lemma 2.1.3 ([21], Lemma 1.1.2)**.** *Let A be an MV-algebra and $x, y \in A$. Then the following conditions are equivalent:*

- $\neg x \oplus y = 1$*;*
- $x \odot \neg y = 0$*;*
- $y = (y \ominus x) \oplus x$*;*
- *there exists an element $z \in A$ such that $y = x \oplus z$.*

Given an MV-algebra A and two elements $x, y \in A$, we write

$$x \leq y$$

if and only if x and y satisfy one of the above equivalent conditions. It can be shown that the relation $\leq$ defines a partial order on A (a proof of this fact can be found in [21]). It is clear that every morphism of MV-algebras preserves the partial order defined above.

Lemma 2.1.4 ([21], Lemma 1.1.4)**.** *In every MV-algebra A the natural order $\leq$ has the following properties:*

- *$x \leq y$ if and only if $\neg y \leq \neg x$;*
- *if $x \leq y$ then, for all $z \in A$, $x \oplus z \leq y \oplus z$ and $x \odot z \leq y \odot z$;*
- *$x \odot y \leq z$ if and only iff $x \leq \neg y \oplus z$.*

Proposition 2.1.5 ([21], Proposition 1.1.5, Proposition 1.1.6, and Proposition 1.5.1)**.** *On every MV-algebra A the natural order determines a lattice structure. Specifically, for every $x, y \in A$, the join $x \vee y$ and the meet $x \wedge y$ are given by*

$$x \vee y \coloneqq (x \ominus y) \oplus y \text{ and } x \wedge y \coloneqq x \odot (\neg x \oplus y).$$

Moreover, the underlying lattice structure on A is distributive.
Finally, the following equations hold:

- $x \odot (y \vee z) = (x \odot y) \vee (x \odot z)$*;*
- $x \oplus (y \wedge z) = (x \oplus y) \wedge (x \oplus z)$*;*

- $x \odot (y \wedge z) = (x \odot y) \wedge (x \odot z)$;
- $x \oplus (y \vee z) = (x \oplus y) \vee (x \oplus z)$.

Given a morphism of MV-algebras $h\colon A \to B$, one can define the *kernel* of h as

$$\ker(h) \coloneqq \{x \in A \,|\, h(x) = 0\}.$$

Kernels of morphisms can be characterized as ideals. A subset $I \subseteq A$ of an MV-algebra A is an *ideal* if and only if for every $x, y \in A$:

- $0 \in I$;
- $x, y \in I$ implies $x \oplus y \in I$;
- $x \in I$ and $y \leq x$ implies $y \in I$.

An ideal I is said to be *proper* if $I \neq A$.
The dual notion of an ideal is the one of a filter. A subset $F \subseteq A$ of an MV-algebra A is a *filter* if and only if for every $x, y \in A$:

- $1 \in F$;
- $x, y \in F$ implies $x \odot y \in F$;
- $x \in F$ and $x \leq y$ implies $y \in F$.

A filter F is said to be *proper* if $F \neq A$. One can easily prove that F is a filter of A if and only if $\neg F \coloneqq \{a \in A \,|\, \neg a \in F\}$ is an ideal of A.
It is known that every kernel is an ideal (we recall that every morphism preserves the order). Given an ideal $I \subseteq A$ we can define a congruence $\sim_I$ on A in the following way: for every $x, y \in A$, $x \sim_I y$ if and only if $d(x, y) \in I$. One can show that the kernel of the quotient projection $\pi\colon A \to A/\sim_I$ is exactly I. This procedure establishes a one-to-one correspondence between kernels of morphisms with domain A and ideals of A. Additionally, it is not difficult to see that a morphism of MV-algebras h is injective if and only if $\ker(h) = \{0\}$.
Finally, given an MV-algebra A and a non-empty subset $S \subseteq A$, the ideal generated by S (i.e. the smallest ideal containing S) exists and it is

$$\langle S \rangle = \{x \in A \,|\, \exists s_1, \dots, s_n \in S \text{ s.t. } x \leq s_1 \oplus \cdots \oplus s_n\}.$$

Lemma 2.1.6. *Let $f\colon A \to B$ be a morphism of MV-algebras and $I \subseteq A$ an ideal. The restriction of f to I is injective if and only if* $\ker(f) \cap I = \{0\}$.

Proof. If the restriction of f is injective and we take an element $a \in \ker(f) \cap I$, we get $f(a) = 0 = f(0)$ and, since $0 \in I$, we obtain $a = 0$. Conversely, if $\ker(f) \cap I = \{0\}$ and we consider two elements $a, b \in I$ such that $f(a) = f(b)$, we get that $f(d(a, b)) = d(f(a), f(b)) = 0$ and so $d(a, b) \in \ker(f) \cap I = \{0\}$ i.e. $a = b$; thus the restriction of f is injective. $\square$

2.2 Properties of Idempotents

In this section, we will examine some properties of idempotent elements in an MV-algebra. Given an MV-algebra A, we denote the set of idempotent elements of A with $\mathrm{Idem}(A)$. We will prove that the assignment defined by $A \mapsto \mathrm{Idem}(A)$ constitutes a functor $\mathrm{Idem}\colon \mathbb{MV} \to \mathbb{B}\mathrm{oole}$, which is right adjoint to the inclusion functor $i\colon \mathbb{B}\mathrm{oole} \hookrightarrow \mathbb{MV}$. Finally, in the second part of this section, we will investigate fundamental properties of idempotent elements that are relevant for the subsequent sections.

Definition 2.2.1 ([21]). *Let A be an MV-algebra. An element $x \in A$ is an* idempotent *(or* Boolean*) element if*

$$x \oplus x = x.$$

The set of idempotent elements of A is called Boolean skeleton *of A and it is denoted by*

$$\mathrm{Idem}(A).$$

Given an MV-algebra $(A, \oplus, \neg, 0)$, it is a known fact that A is a Boolean algebra with respect to its operations if and only if $\oplus$ is an idempotent operation.

Theorem 2.2.2 ([21], Theorem 1.5.3). *For every element x in an MV-algebra A the following conditions are equivalent:*

- $x \in \mathrm{Idem}(A)$;
- $x \vee \neg x = 1$;
- $x \wedge \neg x = 0$;
- $x \odot \neg x = 0$;
- $x \oplus y = x \vee y$, *for all* $y \in A$;
- $x \odot y = x \wedge y$, *for all* $y \in A$.

Therefore, for every MV-algebra A, $\mathrm{Idem}(A)$ is a subalgebra of A. Additionally, the MV-algebra A is a Boolean algebra with respect to its operations if and only if $A = \mathrm{Idem}(A)$.

Remark 2.2.3. *The assignment introduced in Definition 2.2.1 can be extended to a functor*

$$\mathbb{MV} \xrightarrow{\mathrm{Idem}} \mathbb{B}oole$$

$$\begin{array}{ccc} A & \longmapsto & \mathrm{Idem}(A) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \mathrm{Idem}(f)} \\ B & \longmapsto & \mathrm{Idem}(B), \end{array}$$

where $\mathrm{Idem}(f)$ *is the restriction of* f *to* $\mathrm{Idem}(A)$*. In fact, fixed an element* $x \in \mathrm{Idem}(A)$ *one has* $f(x) \oplus f(x) = f(x \oplus x) = f(x)$ *(i.e.* $f(x) \in \mathrm{Idem}(B)$*).*

Proposition 2.2.4. *The inclusion functor* $i : \mathbb{B}\mathrm{oole} \hookrightarrow \mathbb{MV}$ *is both a left and a right adjoint.*

Proof. We start showing that $i \dashv \mathrm{Idem}$. We do this proving the existence of a counit $\varepsilon\colon i\,\mathrm{Idem} \to Id_{\mathbb{MV}}$. For every object A of $\mathbb{MV}$, we define ε_A as the inclusion $\mathrm{Idem}(A) \subseteq A$; clearly the following diagram commutes for all morphisms f, and so ε is a natural transformation:

$$\begin{array}{ccc} i\,\mathrm{Idem}(A) & \overset{\varepsilon_A}{\hookrightarrow} & A \\ {\scriptstyle i\,\mathrm{Idem}(f)}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ i\,\mathrm{Idem}(B) & \underset{\varepsilon_B}{\hookrightarrow} & B. \end{array}$$

Moreover, ε satisfies the universal property of the counit: given a morphism of MV-algebras $g\colon i(B) \to A$, where B is an object of $\mathbb{B}\mathrm{oole}$, we want to prove that there exists a unique morphism of Boolean algebras $\overline{g}\colon B \to \mathrm{Idem}(A)$ such that the following diagram commutes

$$\begin{tikzcd} & i(B) \arrow[dl, "i(\overline{g})"'] \arrow[dr, "g"] & \\ i\,\mathrm{Idem}(A) \arrow[rr, hook, "\varepsilon_A"'] & & A. \end{tikzcd}$$

Since B is a Boolean algebra, $g(b) \in \mathrm{Idem}(A)$ for every $b \in B$ and so $\overline{g}$ is defined as the restriction of g.

Now, we construct the functor

$$\mathbb{MV} \xrightarrow{Q} \mathbb{B}\mathrm{oole}$$

$$\begin{array}{ccc} A & \longmapsto & A/\rho_A \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle \tilde{f}} \\ B & \longmapsto & B/\rho_B, \end{array}$$

where we define, for every MV-algebra A, the congruence $\rho_A \coloneqq \langle\{(x, x \oplus x) \in A \times A \,|\, x \in A\}\rangle \subseteq A \times A$; since each element of A/ρ_A is idempotent, then A/ρ_A is a Boolean algebra. Moreover, for every morphism of MV-algebras $f\colon A \to B$, $\tilde{f}\colon A/\rho_A \to B/\rho_B$ is given by $\tilde{f}([a]) \coloneqq [f(a)]$. We prove that $Q \dashv i$ by showing the existence of a unit $\eta\colon Id_{\mathbb{MV}} \to iQ$. For every object A of $\mathbb{MV}$, we define η_A as the quotient projection on the congruence ρ_A. Easy computations show that following diagram is commutative, so η is a natural transformation:

$$\begin{array}{ccc} A & \xrightarrow{\eta_A} & i(A/\rho_A) \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle i(\tilde{f})} \\ B & \xrightarrow[\eta_B]{} & i(B/\rho_B). \end{array}$$

Moreover, η satisfies the universal property of the unit: consider a morphism of MV-algebras

$h\colon A \to i(B)$, where B is an object of $\mathbb{B}$oole, we want to prove that there exists a unique morphism of Boolean algebras $\hat{h}\colon A/\rho_A \to B$ such that the following diagram commutes

$$\begin{tikzcd} A \arrow[rr, "\eta_A"] \arrow[dr, "h"'] & & A/\rho_A \arrow[dl, "i(\hat{h})"] \\ & i(B). & \end{tikzcd}$$

Since B is a Boolean algebra, we observe that $h(x) = h(x) \oplus h(x) = h(x \oplus x)$ for every $x \in A$; hence, we have $\rho_A \subseteq K[h]$ where $K[h] \coloneqq \{(x, y) \in A \times A \mid h(x) = h(y)\}$. So, by the universal property of the quotient, we can define a unique morphism of MV-algebras (actually, of Boolean algebras) $\hat{h}\colon A/\rho_A \to B$ making the above diagram commutative. $\square$

Proposition 2.2.5 ([21], Proposition 6.4.1)**.** *Given an MV-algebra A, for every element $e \in \mathrm{Idem}(A) \setminus \{0\}$, $(\downarrow e, \oplus, \neg_e, 0)$ is an MV-algebra (where $\downarrow e \coloneqq \{x \in A \mid x \leq e\}$ and $\neg_e x \coloneqq \neg x \wedge e$). Moreover, the map*

$$h_e\colon A \to \downarrow e$$
$$x \mapsto x \wedge e$$

is a morphism of the MV-algebra A onto the MV-algebra $\downarrow e$, with $\ker(h_e) = \downarrow \neg e$.

Given an MV-algebra A and an element $e \in \mathrm{Idem}(A)$, as a consequence of the previous proposition, the following equalities holds for every $x, y \in A$:

- $e \wedge (x \odot y) = (e \wedge x) \odot (e \wedge y)$;
- $e \vee (x \odot y) = (e \vee x) \odot (e \vee y)$;
- $e \wedge (x \oplus y) = (e \wedge x) \oplus (e \wedge y)$;
- $e \vee (x \oplus y) = (e \vee x) \oplus (e \vee y)$.

Proposition 2.2.6 ([21], Lemma 6.4.5)**.** *Let A be an MV-algebra and $e_1, \ldots, e_k$ elements of $\mathrm{Idem}(A)$ such that*

- $e_1 \vee \cdots \vee e_k = 1$,
- $e_i \wedge e_j = 0$ *for* $i \neq j$, $i, j = 1, \ldots k$.

Then there exists an isomorphism of MV-algebras

$$h\colon A \to \prod_{i=1}^{k} \downarrow e_i$$

such that, for every $x \in A$,

$$h(x) \coloneqq (x \wedge e_i)_{i=1}^{k}.$$

Lemma 2.2.7. *Let A be an MV-algebra and $e_1, \dots, e_k$ elements of* $\mathrm{Idem}(A)$ *such that*

- $e_1 \vee \dots \vee e_k = 1$,
- $e_i \wedge e_j = 0$ *for* $i \neq j$, $i, j = 1, \dots k$.

Then, for every $x_1, \dots, x_k \in A$, the following equality holds

$$\neg\left(\bigoplus_{i=1}^{k} x_i \odot e_i\right) = \bigoplus_{i=1}^{k} (\neg x_i \odot e_i).$$

Proof. First of all, we observe that $\neg e_i = \bigoplus_{i \neq j} e_j$, in fact the following equalities hold simultaneously:

$$\begin{cases} e_i \oplus \left(\bigoplus_{j\neq i} e_j\right) = \bigoplus_{i=1}^{k} e_i = 1 \\ e_i \odot \left(\bigoplus_{j\neq i} e_j\right) = \bigoplus_{j\neq i}(e_i \odot e_j) = 0; \end{cases}$$

hence, applying Lemma 2.1.2, we get $\neg e_i = \bigoplus_{i\neq j} e_j$.
We are ready to prove our claim; on the one hand we have

$$\begin{aligned} \left(\bigoplus_{i=1}^{k}(x_i \odot e_i)\right) \oplus \left(\bigoplus_{i=1}^{k}(\neg x_i \odot e_i)\right) &= \bigoplus_{i=1}^{k}(x_i \odot e_i) \oplus (\neg x_i \odot e_i) \\ &= \bigoplus_{i=1}^{k}(x_i \oplus \neg x_i) \odot e_i = \bigoplus_{i=1}^{k} e_i = 1. \end{aligned}$$

On the other hand, since $x_i \odot e_i \leq x_i$ and $x_i \odot e_i \leq e_i$ for $i = 1, \dots k$, we obtain

$$\begin{aligned} \left(\bigoplus_{i=1}^{k}(x_i \odot e_i)\right) \odot \left(\bigoplus_{i=1}^{k}(\neg x_i \odot e_i)\right) &\leq \left(x_i \oplus \bigoplus_{j\neq i} e_j\right) \odot \left(\neg x_i \oplus \bigoplus_{j\neq i} e_j\right) \\ &= (x_i \oplus \neg e_i) \odot (\neg x_i \oplus \neg e_i) \\ &= (x_i \odot \neg x_i) \oplus \neg e_i = \neg e_i; \end{aligned}$$

therefore, for every $i = 1 \dots k$, we get

$$\left(\bigoplus_{i=1}^{k}(x_i \odot e_i)\right) \odot \left(\bigoplus_{i=1}^{k}(\neg x_i \odot e_i)\right) \leq \neg e_i.$$

So, we deduce

$$\left(\bigoplus_{i=1}^{k}(x_i \odot e_i)\right) \odot \left(\bigoplus_{i=1}^{k}(\neg x_i \odot e_i)\right) \leq \bigwedge_{i=1}^{k} \neg e_i = \neg \bigvee_{i=1}^{k} e_i = \neg 1 = 0$$

and, applying again Lemma 2.1.2, the statement follows. $\square$

Lemma 2.2.8. *Let A be an MV-algebra and $e_1, \ldots, e_k$ elements of* $\mathrm{Idem}(A)$ *such that $e_i \odot e_j = 0$ for $i \neq j$, $i, j = 1, \ldots k$. Then, for every $x \in A$, the following equality holds:*

$$x \odot (e_1 \oplus \cdots \oplus e_n) = (x \odot e_1) \oplus \cdots \oplus (x \odot e_n).$$

Proof. Let us consider two idempotent elements $e, f \in \mathrm{Idem}(A)$ such that $e \odot f = 0$. Given an element $x \in A$, since $e \oplus f \in \mathrm{Idem}(A)$, we get

$$x \odot (e \oplus f) = x \wedge (e \vee f) = (x \wedge e) \vee (x \wedge f) = (x \odot e) \vee (x \odot f) \leq (x \odot e) \oplus (x \odot f).$$

Now, we want to show the converse inequality

$$(x \odot e) \oplus (x \odot f) \leq (x \odot e) \vee (x \odot f).$$

First of all, we observe that the equality $e \odot f = 0$ implies the inequalities $e \leq \neg f$ and $f \leq \neg e$. Hence, we obtain

$$(x \odot e) \oplus (x \odot f) \leq (x \odot e) \oplus (x \odot \neg e) \leq e \oplus (x \odot \neg e) = x \vee e,$$
$$(x \odot e) \oplus (x \odot f) \leq (x \odot \neg f) \oplus (x \odot f) \leq (x \odot \neg f) \oplus f = x \vee f.$$

Therefore, we have

$$(x \odot e) \oplus (x \odot f) \leq (x \vee e) \wedge (x \vee f) = x \vee (e \wedge f) = x$$

and, moreover,

$$(x \odot e) \oplus (x \odot f) \leq e \oplus f.$$

Considering the meet of the last two inequalities, we get

$$(x \odot e) \oplus (x \odot f) \leq x \wedge (e \oplus f) = x \odot (e \oplus f)$$

since $e \oplus f \in \mathrm{Idem}(A)$. Finally, the statement follows by induction. □

Lemma 2.2.9. *Let A be an MV-algebra. For every $e \in \mathrm{Idem}(A)$ and $x, y \in A$,*

$$(x \ominus y) \odot e = 0 \textit{ and } (y \ominus x) \odot e = 0 \textit{ implies } x \odot e = y \odot e.$$

Proof. If $e = 0$ the statement follows trivially. Hence, suppose $e \neq 0$. From $(x \ominus y) \odot e = 0$ and $(y \ominus x) \odot e = 0$ we get

$$((x \ominus y) \odot e) \oplus ((y \ominus x) \odot e) = 0$$

and so

$$d(x,y) \wedge e = 0.$$

We know that $h_e\colon A \to\downarrow e$ is a morphism of MV-algebras (see Proposition 2.2.5), where $h_e(z) = z \wedge e = z \odot e$. Therefore, from

$$0 = d(x,y) \odot e = h_e(d(x,y)) = d_e(h_e(x), h_e(y))$$

we get $h_e(x) = h_e(y)$, and so $x \odot e = y \odot e$ (where d_e is the distance operation of the MV-algebra $\downarrow e$). □

2.3 Boolean Algebras and Stone Duality

Stone Duality is a mathematical result that connects two seemingly unrelated areas of mathematics: topology and algebra. It provides a powerful tool for studying the structure and behavior of various mathematical objects, such as Boolean algebras, topological spaces, and lattice-ordered groups. Stone Duality was introduced by Stone [49]. The purpose of this section is to provide a concise overview of Stone Duality. To accomplish this objective, we will present a summarised version of the discussion presented in [4].

In the first part of this section, we will recall the concepts of filters and ideals in Boolean algebras, and explore some of their key properties and applications.

Definition 2.3.1. *Let B be a Boolean algebra. A subset $F \subseteq B$ is a* filter *if*

- $1 \in F$;
- $x, y \in F$ *implies* $x \wedge y \in F$;
- $x \in F$ *and* $x \leq y$ *implies* $y \in F$.

The set of filters of B is denoted by $\mathrm{Filters}(B)$.
A filter F is proper *if* $0 \notin F$. *An* ultrafilter *is a maximal element in the poset of proper filters ordered by inclusion.*

Definition 2.3.2. *Let B be a Boolean algebra. A subset $I \subseteq B$ is an* ideal *if*

- $0 \in I$;
- $x, y \in I$ *implies* $x \vee y \in I$;
- $x \in I$ *and* $y \leq x$ *implies* $y \in I$.

The set of ideals of B is denoted by $\mathrm{Ideals}(B)$.
An ideal I is proper *if* $1 \notin I$. *A* maximal ideal *is a maximal element in the poset of proper ideals ordered by inclusion.*

Proposition 2.3.3. *Consider a filter F of a Boolean algebra B. The following are equivalent:*

- *F is an ultrafilter;*
- *for all $x \in X$, $x \in F$ or $\neg x \in F$;*
- *for all $x, y \in B$, $x \vee y \in F$ implies $x \in F$ or $y \in F$;*
- *there exists a morphism of Boolean algebras $f \colon B \to \mathbf{2} = \{0 < 1\}$ such that $f^{-1}(1) = F$.*

Proposition 2.3.4. *Let B be a Boolean algebra:*

- *every proper filter is contained in an ultrafilter;*
- *every non-zero element of B is contained in an ultrafilter;*
- *for every $x, y \in B$, if $x \nleq y$ then there exists an ultrafilter F such that $x \in F$ and $y \notin F$;*
- *every filter is the intersection of the ultrafilters containing it.*

For every Boolean algebra B, we define the set $\mathrm{Spec}(B)$ as the set of ultrafilters of B and, for every filter H of B, the set

$$O_H \coloneqq \{F \in \mathrm{Spec}(B) \mid H \nsubseteq F\}.$$

Our aim is to recall the definition of the two functors that establish the duality between the category of Boolean algebras and the one of Stone spaces.

Definition 2.3.5. *A topological space (X, τ) is a* Stone space *if (X, τ) is compact, T_2 and has a base of clopens. The category* $\mathbb{S}\mathrm{tone}$ *is the full subcategory of the category of topological spaces which objects are precisely the Stone spaces.*

The first functor, denoted by Spec, maps a Boolean algebra B to the set of ultrafilters on B. More specifically, the set $\mathrm{Spec}(B)$ is equipped with the *Stone topology*

$$\tau \coloneqq \{O_H \subseteq \mathrm{Spec}(B) \mid H \in \mathrm{Filters}(B)\}.$$

Proposition 2.3.6. *Let B be a Boolean algebra. The map*

$$\begin{aligned} O \colon (\mathrm{Filters}(B), \subseteq) &\to (\tau, \subseteq) \\ H &\mapsto O_H \end{aligned}$$

is an isomorphism of partially ordered sets.

As a consequence of this proposition we have the following:

Corollary 2.3.7. *Let B be a Boolean algebra. Then*

- *$O_{\{1\}} = \emptyset$ and $O_B = \mathrm{Spec}(B)$;*

- $O_{H_1 \cap H_2} = O_{H_1} \cap O_{H_2}$;
- $O_{\langle \bigcup_{i\in I} H_i \rangle} = \bigcup_{i\in I} O_{H_i}$, *where, given a subset* $S \subseteq B$, $\langle S \rangle$ *is the filter generated by* S.

Furthermore, since the following lemma holds, the set $\mathrm{Spec}(B)$ equipped with the topology just defined turns out to be a Stone space.

Lemma 2.3.8. *Let* B *be a Boolean algebra. For every* $b \in B$ *the set*

$$O_b \coloneqq O_{\uparrow b} = \{F \in \mathrm{Spec}(B) \mid b \notin F\}$$

is a clopen for τ*. Moreover, the set* $\{O_b \mid b \in B\}$ *is a base for* τ*.*

The behavior of Spec on arrows is contravariant. That is, if $f\colon A \to B$ is a morphism of Boolean algebras, then $\mathrm{Spec}(f)\colon \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is a continuous map between the associated Stone spaces, defined by

$$\mathrm{Spec}(f)(F) \coloneqq f^{-1}(F)$$

for every ultrafilter $F \subseteq B$.

The second functor is the clopen algebra functor Clopen, which assigns to each Stone space X its Boolean algebra $\mathrm{Clopen}(X)$ of clopen subsets (where the Boolean algebra operations are the ones induced by $\mathbf{2}^{\mathrm{Spec}(B)}$), and, to each continuous map $g : X \to Y$ between Stone spaces, it assigns the morphism $\mathrm{Clopen}(g) : \mathrm{Clopen}(Y) \to \mathrm{Clopen}(X)$ given by

$$\mathrm{Clopen}(g)(V) \coloneqq g^{-1}(V)$$

for every clopen subset $V \subseteq Y$.

To conclude, the result of Stone Duality precisely states that the following hold:

Proposition 2.3.9. *Let* B *be a Boolean algebra.* B *is isomorphic as a Boolean algebra to* $\mathrm{Clopen}(\mathrm{Spec}(B)) \coloneqq \{U \subseteq \mathrm{Spec}(B) \mid U$ *is a clopen for* $\tau\}$.

Proposition 2.3.10. *Let* (X, τ) *be a Stone space.* (X, τ) *is homeomorphic to the topological space* $(\mathrm{Spec}(\mathrm{Clopen}(X)), \tau)$.

Theorem 2.3.11. *The category of Boolean algebras is dually equivalent to the category of Stone spaces. The equivalence is given by*

$$\mathbb{B}\mathrm{oole}^{op} \underset{\mathrm{Clopen}}{\overset{\mathrm{Spec}}{\rightleftarrows}} \mathbb{S}\mathrm{tone}\,.$$

Finally, we mention the following results, consequences of Stone Duality, that will be indispensable for tackling the next section.

Lemma 2.3.12. *Let B be a Boolean algebra and $b, c \in B$. For every $a \in B$, we define $U_a \coloneqq O_{\neg a}$. Then, the following hold:*

- *If $b \leq c$ then $O_b \supseteq O_c$, and $U_b \subseteq U_c$;*
- *if $b \neq c$ then $O_b \neq O_c$;*
- *$O_{b \wedge c} = O_b \cap O_c$;*
- *$O_{b \vee c} = O_b \cup O_c$;*
- *$O_1 = \emptyset$, $O_o = B$, $U_1 = B$, and $U_0 = \emptyset$;*
- *$O_{\neg b} = (O_b)^c$.*

Corollary 2.3.13. *Let B be a Boolean algebra. $U \subseteq \mathrm{Spec}(B)$ is a clopen if and only if there exists an element $b \in B$ such that $U = U_b$.*

2.4 Pierce Spectrum for MV-Algebras

In this section we will introduce the *Pierce spectrum* of an MV-algebra based on its idempotent elements, and explore some of its key properties. We will see that the Pierce spectrum allows us to classify MV-algebras in terms of their idempotent elements. This part of the thesis is heavily influenced by the results obtained from the study of the Pierce spectrum for unitary rings, which can be found presented in [4].

We begin by proving some properties related to ideals and idempotent elements.

Definition 2.4.1. *Let A be an MV-algebra. An ideal I of A is* regular *if*

$$I = \langle \mathrm{Idem}(I) \rangle,$$

where $\mathrm{Idem}(I) \coloneqq \{x \in I \,|\, x \oplus x = x\}$.

Lemma 2.4.2. *Let A be an MV-algebra. Given an ideal I of A the following are equivalent:*

i) I is regular;

ii) for every $i \in I$ there exists an element $e \in \mathrm{Idem}(I)$ such that $i = i \odot e$.

Proof. *ii)* $\Rightarrow$ *i)* Since $i = i \odot e \leq e$ we conclude that $I = \langle \mathrm{Idem}(I) \rangle$.
i) $\Rightarrow$ *ii)* Given an element $i \in I$ we know that $i \leq e_1 \oplus e_2 \oplus \cdots \oplus e_n$ with $e_i \in \mathrm{Idem}(I)$ for every $i = 1, \ldots, n$. Furthermore, $e = e_1 \oplus e_2 \oplus \cdots \oplus e_n$ is an element of $\mathrm{Idem}(I)$, therefore there exists an element $z \in A$ such that $i = z \odot e$; so $i \odot e = z \odot e \odot e = z \odot e = i$ (since $e \odot e = e$). $\square$

Proposition 2.4.3. *Let A be an MV-algebra and I and ideal of A.* $\mathrm{Idem}(I)$ *is an ideal of the Boolean algebra* $\mathrm{Idem}(A)$.

Proof.

- $0 \in \mathrm{Idem}(I)$, since $0 \in I$ and $0 \oplus 0 = 0$;
- if $x \in \mathrm{Idem}(I)$ and $y \leq x$ with $y \in \mathrm{Idem}(A)$, then $y \in I \cap \mathrm{Idem}(A) = \mathrm{Idem}(I)$;
- if $x, y \in \mathrm{Idem}(I)$, then $x \vee y = x \oplus y \in I \cap \mathrm{Idem}(A) = \mathrm{Idem}(I)$. □

Definition 2.4.4. *We define the* Pierce Spectrum functor

$$\mathrm{Sp}\colon \mathbb{MV}^{op} \to \mathbb{S}\mathrm{tone}$$

as the composite

$$\mathbb{MV}^{op} \xrightarrow{\mathrm{Idem}^{op}} \mathbb{B}\mathrm{oole}^{op} \xrightarrow{\mathrm{Spec}} \mathbb{S}\mathrm{tone}\,.$$

Lemma 2.4.5. *Let A be an MV-algebra. Every partition in non-empty clopens of a clopen U_e of* $\mathrm{Sp}(A)$ *has the form*

$$U_e = U_{e_1} \cup U_{e_2} \cup \cdots \cup U_{e_n},$$

where

- *each e_i is a non-zero element of* $\mathrm{Idem}(A)$,
- $e_1 \oplus e_2 \oplus \cdots \oplus e_n = e$,
- $e_i \odot e_j = 0$ *for every* $i \neq j$.

Proof. First of all, we observe that U_e can be presented as a finite union of clopens since it is compact (it is a close subset of a compact space). Thanks to Lemma 2.3.12 we get

$$U_e = U_{e_1} \cup U_{e_2} \cup \cdots \cup U_{e_n} = U_{e_1 \vee e_2 \vee \cdots \vee e_n}$$

and so $e = e_1 \vee e_2 \vee \cdots \vee e_n$; we recall that, for every $x \in A$, $f \oplus x = f \vee x$ if $f \in \mathrm{Idem}(A)$. So, we obtain $e_1 \oplus e_2 \oplus \cdots \oplus e_n = e$. In a similar way, from $U_{e_i} \cap U_{e_j} = \emptyset$ (with $i \neq j$) we conclude that $e_i \odot e_j = 0$. □

Lemma 2.4.6.

i) Every MV-Algebra A is a regular ideal of itself;

ii) a finite intersection of regular ideals is a regular ideal;

iii) an arbitrary join of regular ideals (which is computed as the join of ideals) is a regular ideal.

Proof. *i*) $A = \langle\{1\}\rangle$.
ii) Consider two regular ideals I, J of A and fix an element $x \in I \cap J$. We know that there exist $e_1 \in \mathrm{Idem}(I)$ and $e_2 \in \mathrm{Idem}(J)$ such that

$$x = x \odot e_1 = x \odot e_2;$$

therefore we obtain $x = x \odot e_2 = (x \odot e_1) \odot e_2 = x \odot (e_1 \odot e_2)$. Since $e_1 \odot e_2 \in I \cap J$, we deduce that $I \cap J$ is a regular ideal.
iii) Consider a family $\{I_k\}_{k \in K}$ of regular ideals. We show that

$$\bigvee_{k \in K} I_k = \langle S \rangle$$

where $S = \mathrm{Idem}(\bigvee_{k \in K} I_k)$. Clearly, one has $\langle S \rangle \subseteq \bigvee_{k \in K} I_k$. Conversely, if we take an element $x \in \bigvee_{k \in K} I_k$, then there exist $i_{k_i} \in I_{k_i}$, with $k_i \in K$ for $i = 1, \ldots, n$, such that

$$x \leq i_{k_1} \oplus \cdots \oplus i_{k_n}.$$

Since i_{k_i} is an element of the regular ideal I_{k_i}, we know that there exists an element $e_i \in I_{k_i}$ such that $i_{k_i} = i_{k_i} \odot e_i$. Hence, one has

$$x \leq i_{k_1} \oplus \cdots \oplus i_{k_n} = (i_{k_1} \odot e_1) \oplus \cdots \oplus (i_{k_n} \odot e_n) \leq e_1 \oplus \cdots \oplus e_n \in S,$$

and so $x \in \langle S \rangle$. □

Now, we will provide an in-depth analysis about the relationship between the set of idempotent ideals of an MV-algebra A and the set of open subsets of the topological space $\mathrm{Sp}(A)$. To do this, it is necessary to recall the notion of a locale.

Definition 2.4.7. *A* locale *L is a complete lattice such that the equality*

$$a \wedge \left(\bigvee_{i \in I} b_i \right) = \bigvee_{i \in I} (a \wedge b_i)$$

holds for every $a \in L$ and $\{b_i\}_{i \in I} \subseteq L$. A function between two locales is a morphism *of locales if it preserves arbitrary joins and finite meets. An* isomorphism *of locales is a bijective morphism of locales.*

Proposition 2.4.8. *Let A be an MV-algebra. The set of regular ideals of A (denoted by* $\mathrm{RegIdeals}(A)$*), endowed with the intersection as meet and the join of ideals as join, is a locale isomorphic to the one of open subsets of* $(\mathrm{Sp}(A), \tau)$.

Proof. We know, from Proposition 2.3.6, that τ is isomorphic as a locale to $\mathrm{Filters}(\mathrm{Idem}(A))$ which is, in turn, isomorphic as a locale to $\mathrm{Ideals}(\mathrm{Idem}(A))$ (we recall that, given an ideal I of a Boolean algebra B, the set $\neg I = \{x \in B \mid \neg x \in I\}$ is a filter and this assignment establishes a bijection, which preserves and reflects the order, between the set $\mathrm{Ideals}(B)$ and $\mathrm{Filters}(B)$). So, we can restate our claim: the locale $\mathrm{RegIdeals}(A)$ is isomorphic to the locale $\mathrm{Ideals}(\mathrm{Idem}(A))$. We define the map

$$\begin{aligned}\varphi\colon\ \mathrm{RegIdeals}(A) &\to \mathrm{Ideals}(\mathrm{Idem}(A))\\ I &\mapsto \mathrm{Idem}(I) = I \cap \mathrm{Idem}(A).\end{aligned}$$

We prove that φ is an isomorphism of locales.

- φ is injective: if $\varphi(I) = \varphi(J)$, then $I = \langle \varphi(I)\rangle = \langle \varphi(J)\rangle = J$.

- φ is surjective: let us consider an ideal $J \in \mathrm{Ideals}(\mathrm{Idem}(A))$, and define $I \coloneqq \langle J\rangle$. Clearly, one has $J \subseteq \mathrm{Idem}(I)$. Conversely, if we consider an element $e \in \mathrm{Idem}(I) = \langle J\rangle \cap \mathrm{Idem}(A)$, there exist $e_1, \dots, e_n \in J$ such that $e \leq e_1 \oplus \cdots \oplus e_n$ (we recall that the elements of J are idempotent). But $e \leq e_1 \oplus \cdots \oplus e_n = e_1 \vee \cdots \vee e_n$, hence $e \in \mathrm{Idem}(A)$ and $e \in J$; therefore $\mathrm{Idem}(I) \subseteq J$.

- φ preserves and reflects the order: if we consider two elements $I_1 \subseteq I_2$ of $\mathrm{RegIdeals}(A)$ we have $\mathrm{Idem}(I_1) \subseteq \mathrm{Idem}(I_2)$; conversely, if we take $J_1 \subseteq J_2$, then $\langle J_1\rangle \subseteq \langle J_2\rangle$. □

Thanks to the above observation, it is possible to translate the topology of $\mathrm{Sp}(A)$ making use of regular ideals. We have shown, in fact, that the following sets are isomorphic as partially ordered sets

$$\mathrm{Filters}(\mathrm{Idem}(A)) \xrightarrow{\sim} \mathrm{Ideals}(\mathrm{Idem}(A)) \xrightarrow{\sim} \mathrm{RegIdeals}(A).$$

Therefore, we deduce that

$$\mathrm{Sp}(A) \cong \{M \in \mathrm{RegIdeals}(A) \mid M \text{ is a maximal element of } (\mathrm{RegIdeals}(A), \subseteq)\},$$

and so, if we consider the last bijection as a definition of $\mathrm{Sp}(A)$ (and we will do that from here on), we have to translate the topology in the following way:

$$\tau = \{O_I \mid I \in \mathrm{RegIdeals}(A)\}$$

where

$$O_I \coloneqq \{M \in \mathrm{Sp}(A) \mid I \nsubseteq M\}.$$

Definition 2.4.9. *The* structural Pierce space *of an MV-algebra A is*

$$\coprod_{M \in \mathrm{Sp}(A)} A/M$$

endowed with the final topology for which the following maps are continuous

$$s_X^I \colon O_I \to \coprod_{M \in \mathrm{Sp}(A)} A/M$$
$$N \mapsto [x]_N \in A/N$$

for every $I \in \mathrm{RegIdeals}(A)$ and $x \in A$ (where on O_I we are considering the subspace topology induced by $O_I \subseteq \mathrm{Sp}(A)$). In other words, a subset $U \subseteq \coprod_{M \in \mathrm{Sp}(A)} A/M$ is open if and only if $(s_x^I)^{-1}(U)$ is open for every $I \in \mathrm{RegIdeals}(A)$ and $x \in A$.

Notation 2.4.10. *Let A be an MV-algebra. For every $e \in \mathrm{Idem}(A)$ we define $O_e \coloneqq O_{\downarrow e}$ where $\downarrow e \coloneqq \{x \in A \mid x \leq e\}$, therefore*

$$O_e = \{M \in \mathrm{Sp}(A) \mid e \notin M\}.$$

Definition 2.4.11. *A map $f \colon X \to Y$ between two topological spaces (X, τ_X) and (Y, τ_y) is* étale *if for every $x \in X$ there exist two open subsets $U \subseteq X$, and $V \subseteq Y$ such that $x \in U$, $f(x) \in V$ and*

$$f_{|U} \colon U \to V$$

is an homeomorphism.

An étale map is both continuous and open.

Theorem 2.4.12. *Let A be an MV-algebra. The map*

$$p\colon \coprod_{M \in \mathrm{Sp}(A)} A/M \to \mathrm{Sp}(A)$$
$$[x]_N \in A/N \mapsto N$$

is étale.

Proof. First of all, let us show that

$$U_x \coloneqq \{M \in \mathrm{Sp}(A) \mid x \in M\}$$

is an open subset of $\mathrm{Sp}(A)$ for every $x \in A$.
We define

$$\begin{aligned} J &\coloneqq \{e \in \mathrm{Idem}(A) \mid \text{ for every } N \in \mathrm{Sp}(A) \text{ such that } x \notin N \implies e \in N\} \\ &= \bigcap \{\mathrm{Idem}(N) \mid N \in \mathrm{Sp}(A), x \notin N\}, \end{aligned}$$

then J can be seen as an intersection of ideals of $\mathrm{Idem}(A)$ and so there exists a regular ideal

$I \in \mathrm{RegIdeals}(A)$ such that

$$J = \mathrm{Idem}(I).$$

We observe that, for every $N \in \mathrm{Sp}(A)$, the following chain of equivalences holds: $N \in O_I$ if and only if $I \nsubseteq N$ if and only if (since I and N are regular and, therefore, generated by their idempotents) there exists an element $e \in \mathrm{Idem}(A)$ such that $e \in I$ (and so $e \in J$) and $e \notin N$. This last statement implies that $x \in N$: in fact, if we suppose $x \notin N$ then, since $\mathrm{Idem}(N)$ appears in the intersection above and since $e \notin \mathrm{Idem}(A)$, we conclude that $e \notin J$. Conversely, if $x \in N$ then there exists an element $e \in \mathrm{Idem}(A)$ such that $e \in J$ and $e \notin N$. In fact, if $x \in N$ then, since $N \in \mathrm{RegIdeals}(A)$, there exists $e' \in \mathrm{Idem}(N)$ such that $x = x \odot e'$. Now, if we consider $M \in \mathrm{Sp}(A)$, if $x \notin M$ then $e' \notin M$ (in fact $e' \in M$ implies $x \in M$, since $x \leq e'$). But we know that $\mathrm{Idem}(M)$ is maximal as ideal of $\mathrm{Idem}(A)$; therefore $e' \notin M$ implies $\neg e' \in M \subseteq M$. Hence, for every $m \in \mathrm{Sp}(A)$ if $x \notin M$ then $\neg e' \in M$, and so $\neg e' \in J$ and $\neg e' \notin \mathrm{Idem}(N)$ (since $e' \in N$). Then we have $N \in O_I$ if and only if $x \in N$, which is equivalent to saying $N \in U_x$; hence U_x is open. Let us consider s_x^I and s_y^J,

$$\begin{aligned}(s_y^J)^{-1}(s_x^I(O_I)) &= \{M \in O_J \mid s_y^J(M) \in s_x^I(O_I)\} \\ &= \{M \in O_J \mid \exists N \in O_I \text{ and } [y]_M = [x]_N\};\end{aligned}$$

then we conclude that $M = N$ and we have

$$(s_y^J)^{-1}(s_x^I(O_I)) = \{M \in O_I \cap O_J \mid [y]_M = [x]_M\}.$$

Since the equality $[y]_M = [x]_M$ holds if and only if $d(x, y) \in M$, we obtain

$$(s_y^J)^{-1}(s_x^I(O_I)) = O_J \cap O_I \cap U_{d(x,y)}$$

which is a finite intersection of open subsets and, therefore, it is open. Thanks to the definition of final topology we know that s_x^I is open in $\coprod_{M \in \mathrm{Sp}(A)} A/M$ for every $I \in \mathrm{RegIdeals}(A)$ and $x \in A$. We observe that $O_A = \mathrm{Sp}(A)$ and so $s_x^A(\mathrm{Sp}(A))$ is open for every $x \in A$. Moreover, if we consider an element $[x]_M \in \coprod_{M \in \mathrm{Sp}(A)} A/M$ we have $[x]_M \in s_x^A(\mathrm{Sp}(A))$ (in this set we have all the quotient classes of x). So, we get

$$s_x^A(\mathrm{Sp}(A)) \underset{s_x^A}{\overset{p}{\rightleftarrows}} O_A = \mathrm{Sp}(A)$$

where s_x^A and p are continuous. We observe that for every $I \in \mathrm{RegIdeals}(A)$ one has $ps_x^I(N) = p([x]_N) = N$, hence

$$ps_x^I \colon O_I \hookrightarrow \mathrm{Sp}(A)$$

is continuous. Then, if we fix an open subset $O \subseteq \mathrm{Sp}(A)$, we know that $p^{-1}(O)$ is open if and only if $(s_x^I)^{-1}(p^{-1}(O))$ is open for every s_x^I, which means that $(ps_x^I)^{-1}(O)$ is open for every s_x^I; therefore, since ps_x^I is continuous, we deduce that $p^{-1}(O)$ is open, too. Finally, we observe that $ps_x^A(N) = p([x]_N) = N$ for every $N \in \mathrm{Sp}(A)$, and $s_x^A p([x]_N) = s_x^A(N) = [x]_N$ for every $[x]_N \in s_x^A(\mathrm{Sp}(A))$. Therefore, p is étale. $\square$

Theorem 2.4.13. *Every MV-algebra A is isomorphic to the MV-algebra of continuous sections of*

$$\begin{aligned} p\colon \coprod_{M\in \mathrm{Sp}(A)} A/M &\to \mathrm{Sp}(A) \\ [x]_N \in A/N &\mapsto N. \end{aligned}$$

More generally, the MV-algebra of continuous sections defined on O_e, with $e \in \mathrm{Idem}(A)$, is isomorphic as an MV-algebra to $\downarrow e$ (introduced in Proposition 2.2.5).

Proof. Clearly, from the second statement we can deduce the first one putting $e = 1$. So let us prove the second one. Fix an element $e \in \mathrm{Idem}(A)$, for every $x \in A$ we have

$$\begin{aligned} s_x^e := s_x^{\downarrow e}\colon O_e &\to \coprod_{M\in \mathrm{Sp}(A)} A/M \\ N &\mapsto [x]_N. \end{aligned}$$

We define

$$\mathrm{Sec_e}(p) := \{s\colon O_e \to \coprod_{M\in \mathrm{Sp}(A)} A/M \mid s \text{ continuous and } ps = id_{O_e}\}$$

(i.e. the set of continuous sections of p defined on O_e) and

$$\begin{aligned} \varphi_e\colon A &\to \mathrm{Sec_e}(p) \\ x &\mapsto s_x^e; \end{aligned}$$

clearly, $\varphi_e(A)$ inherits the MV-algebra structure from A and, in this setting, φ_e is a morphism of MV-algebras.

We observe that $s_{\neg e}^e = 0$: in fact, if we consider an element $N \in O_e$ we have $e \notin N$ and, by maximality, we get $\neg e \in N$; so $s_{\neg e}^e(N) = [\neg e]_N = 0$. Therefore, the kernel $\ker(\varphi_e)$ of

$$\varphi_e\colon A \to \varphi_e(A)$$

contains $\downarrow \neg e$. So, thanks to the universal property of the quotient, we define ψ_e

$$\begin{tikzcd} A \arrow[r, "\varphi_e"] \arrow[d, two heads, "\pi_{\downarrow\neg e}"'] & \varphi_e(A) \\ A/\downarrow \neg e. \arrow[ur, "\psi_e"'] & \end{tikzcd}$$

We prove that ψ_e is injective: let us consider an element $[x] \in A/\downarrow \neg e$; then $\psi_e([x]) = 0$ implies $s^e_x = 0$ and so $s^e_x(N) = 0$ for every $N \in O_e$. Therefore, $[x]_N = 0$ for every $N \in O_e$; hence, we have $x \in \bigcap_{N \in O_e} = \bigcap_{e \notin N} N = \bigcap_{\downarrow \neg e \subseteq N} N = \downarrow \neg e$ (the last equality holds observing that, in a Boolean algebra, every ideal can be seen as the intersection of the maximal ideals containing it). So we get $[x] = 0$, i.e. ψ_e injective.

To conclude, we have to prove that $\varphi_e(A) = \mathrm{Sec_e}(p)$ (i.e. φ_e is surjective). We consider an element $\sigma \in \mathrm{Sec_e}(p)$; given $M \in O_e$, we know that $\sigma(M) = [x_M]_M \in A/M$. Now, $\sigma^{-1}(s^e_{x_M}(O_e))$ is an open subset of O_e (in the proof of the previous proposition we have seen that the subsets of the form $\sigma^{-1}(s^I_x(O_I))$ are open). Therefore

$$W_M \coloneqq \{N \in O_e \,|\, \sigma(N) = [x_M]_N \in A/N\}$$

is an open subset of O_e. Since $\sigma(M) = [x_M]_M$, we obtain $M \in W_M$. We observe that

$$W_M = \bigcup_{i \in I} O_{e_i}$$

with $e_i \in \mathrm{Idem}(A)$ (we know that the set $\{O_e \,|\, e \in \mathrm{Idem}(A)\}$ is a base for the topology on $\mathrm{Sp}(A)$). Hence, there exists an $e_M \in \{e_i \in \mathrm{Idem}(A) \,|\, i \in I\}$ such that $M \in O_{e_M} \subseteq W_M \subseteq O_e$. So, we have

$$\bigcup_{M \in O_e} O_{e_M}$$

and, since O_e is compact,

$$O_e = \bigcup_{i=1}^{n} O_{e_{M_i}}$$

for an appropriate finite set $\{M_1, \ldots, M_n\} \subseteq O_e$. We define

$$\begin{aligned} O_{e_1} &\coloneqq O_{e_{M_1}}, \\ O_{e_2} &\coloneqq O_{e_{M_2}} \setminus O_{e_1}, \\ &\vdots \\ O_{e_n} &\coloneqq O_{e_{M_n}} \setminus O_{e_{n-1}}, \end{aligned}$$

hence we obtain

$$O_e = O_{e_1} \cup \cdots \cup O_{e_n},$$
$$O_{e_k} \cap O_{e_j} = \emptyset \text{ for every } k \neq j.$$

Moreover, we know that $O_{e_i} \subseteq O_{e_{M_i}} \subseteq W_{M_i}$ and so $\sigma(N) = [x_{M_i}]_N \in A/N$ for every $N \in O_{e_i}$.

We define $x_k \coloneqq x_{M_k}$ for every $k = 1, \dots, n$ and

$$x \coloneqq \bigoplus_{k=1}^{n} (x_k \odot e_k),$$

we prove that $\sigma = s^e_x$. Let us consider an element $M \in O_e$; there exists an $i \in \{1, \dots, n\}$ such that $M \in O_{e_i}$ (i.e. $e_i \notin M$) and for every $j \neq i$, with $j \in \{1, \dots, n\}$, $M \notin O_{e_j}$ (i.e. $e_j \in M$). Then, recalling that $\sigma(M) = [x_i]_M \in A/M$, we get

$$s^e_x(M) = \bigoplus_{k=1}^{n} ([x_k]_M \odot [e_k]_M) = [x_i]_M = \sigma(M),$$

since $[e_k]_M = \delta_{k,i}$. $\square$

Proposition 2.4.14. *Let A be an MV-algebra and $M \in \mathrm{Sp}(A)$, Then we have*

$$\mathrm{Idem}(A/M) = \{[0], [1]\}.$$

Proof. We know that the MV-algebra A/M is the filtered limit of the MV-algebras $A/\langle e_1, \dots, e_n\rangle$ where $e_1, \dots, e_n$ are idempotents of M. If $[x]$ is an idempotent of A/M, then there exists an MV-algebra $A/\langle e_1, \dots, e_n\rangle$ in which x and $x \oplus x$ are identified. Moreover, the ideal $\langle e_1, \dots, e_n\rangle$ is equal to $\downarrow \neg e$, where $\neg e \coloneqq e_1 \oplus \cdots \oplus e_n$ ($\neg e$ is, therefore, an idempotent of A). We recall that the assignment

$$\begin{aligned} h\colon A/\downarrow \neg e &\to \downarrow e \\ [y] &\mapsto y \wedge e \end{aligned}$$

is an isomorphism of MV-algebras. Therefore $x \wedge e$ is an idempotent of $\downarrow e$, and hence of A (since the operation $\oplus$ is defined in the same way). Let us say that $x \wedge e = f \in \mathrm{Idem}(A)$. We observe that $h([f]) = f \wedge e = x \wedge e \wedge e = x \wedge e = f$, so $[x] = [f]$ in $A/\downarrow \neg e$, and then the same equality holds in A/M. Now, recalling that $\mathrm{Idem}(M)$ is a maximal ideal of $\mathrm{Idem}(A)$ and $f \in \mathrm{Idem}(A)$, we encounter two possibilities: if $f \in \mathrm{Idem}(M)$ we get $[x] = [0]$; if $f \notin \mathrm{Idem}(M)$ then, by maximality, $\neg f \in \mathrm{Idem}(M)$ and so $[x] = [1]$. $\square$

Given a Stone space X, an MV-algebra A, and a continuous map $f\colon X \to \mathrm{Sp}(A)$, we define $C((X, f), (\coprod_{M\in\mathrm{Sp}(A)} A/M, p))$ as the set of continuous functions $g\colon X \to \coprod_{M\in\mathrm{Sp}(A)} A/M$ making the following diagram commutative:

$$\begin{tikzcd} X \arrow[rr, "g"] \arrow[dr, "f"'] & & \coprod_{M\in\mathrm{Sp}(A)} A/M \arrow[dl, "p"] \\ & \mathrm{Sp}(A). & \end{tikzcd}$$

The set $C((X, f), (\coprod_{M\in\mathrm{Sp}(A)} A/M, p))$ can be endowed with a structure of MV-algebra, via the structure of MV-algebra of each quotient A/M: given $g, h \in C((X,f),(\coprod_{M\in \mathrm{Sp}(A)} A/M, p))$, $(g\oplus h)(x) \coloneqq g(x)\oplus h(x)$ (we know that $g(x), h(x) \in A/f(x)$ and, therefore, they can be summed in $A/f(x)$); in a similar way we can define 0 and $\neg$.

Theorem 2.4.15. *Let R be an MV-algebra. The right adjoint of the functor*

$$\begin{aligned}\mathrm{Sp}_R \colon \mathbb{MV}^{op}/R &\to \mathbb{S}\mathrm{tone}\,/\,\mathrm{Sp}(R)\\ R \xrightarrow{m_A} A &\mapsto \mathrm{Sp}(A) \xrightarrow{\mathrm{Sp}(m_A)} \mathrm{Sp}(R)\end{aligned}$$

is the functor

$$\begin{aligned}C_R \colon \mathbb{S}\mathrm{tone}\,/\,\mathrm{Sp}(R) &\to \mathbb{MV}^{op}/R\\ X \xrightarrow{f} \mathrm{Sp}(R) &\mapsto R \xrightarrow{m} C((X,f),(\coprod_{M\in\mathrm{Sp}(A)} A/M, p))\end{aligned}$$

where $m(r)\colon X \to \coprod_{M\in\mathrm{Sp}(A)} A/M$ is defined by $m(r)(x) \coloneqq [r]_{f(x)}$. Moreover, this right adjoint is full and faithful.

Proof. Let us first prove that $m(r)$ is a continuous function. Consider an open subset $U \subseteq \coprod_{M\in\mathrm{Sp}(A)} A/M$. Since s_r^R is a continuous function, we can deduce that its pre-image $(s_r^R)^{-1}(U) = \{N \in \mathrm{Sp}(R) \mid [r]_N \in U\}$ is an open subset of $\mathrm{Sp}(R)$. Now, since f is also a continuous function, we can conclude that $f^{-1}((s_r^R)^{-1}(U))$ is an open subset of X. Specifically, we have $f^{-1}((s_r^R)^{-1}(U)) = \{x \in X \mid f(x) \in (s_r^R)^{-1}(U)\} = \{x \in X \mid f(x) = N, [r]_N \in U\} = \{x \in X \mid [x]_{f(x)} \in U\} = (m(r))^{-1}(U)$. Therefore, $m(r)$ is a continuous map, as desired.
If $R = \mathbf{1}$ the result is trivial. So suppose $R \neq \mathbf{1}$ and start from the last statement. An idempotent of the MV-algebra $C((X,f),(\coprod_{M\in\mathrm{Sp}(A)} A/M, p))$ is a continuous map $g\colon X \to \coprod_{M\in\mathrm{Sp}(A)} A/M$ such that $pg = f$ and $g(x) \in R/f(x)$ is an idempotent (and, thanks to Proposition 2.4.14, $g(x)$ is either 0 or 1). Moreover, the subsets $s_0^{\mathrm{Sp}(R)}(\mathrm{Sp}(R)) = \{[0]_M \in R/M \mid M \in \mathrm{Sp}(R)\}$ and $s_1^{\mathrm{Sp}(R)}(\mathrm{Sp}(R)) = \{[1]_M \in R/M \mid M \in \mathrm{Sp}(R)\}$ are open and disjoint (since $1 \notin M$). Hence, the idempotents of $C((X,f),(\coprod_{M\in\mathrm{Sp}(A)} A/M, p))$ are given by the set of continuous maps

$$C((X,f),(\{0,1\}\times\mathrm{Sp}(R), p))$$

where $p\colon \{0,1\}\times\mathrm{Sp}(R) \to \mathrm{Sp}(R)$ is the projection on the second component and $\{0,1\}$ is provided with the discrete topology. Clearly,

$$C((X,f),(\{0,1\}\times\mathrm{Sp}(R), p))$$

is isomorphic, as a Boolean algebra, to $\mathrm{Clopen}(X)$ and, therefore, thanks to the Stone Duality, we get

$$\mathrm{Sp}(C((X,f),(\coprod_{M\in\mathrm{Sp}(A)} A/M, p))) \cong X.$$

Moreover, since $C_R(f) \coloneqq m$ is defined by the equality $m(r)(x) = [r]_{f(x)}$, $\mathrm{Idem}(m)\colon \mathrm{Idem}(R) \to C((X,f),(\{0,1\} \times \mathrm{Sp}(R),p))$ is such that

$$\mathrm{Idem}(m)(e)(x) = \begin{cases} (0, f(x)) & e \in f(x) \\ (1, f(x)) & e \notin f(x). \end{cases}$$

So, we get that

$$X \xrightarrow{f} \mathrm{Sp}(R) \cong \mathrm{Sp}(C((X,f),(\coprod_{M \in \mathrm{Sp}(A)} A/M, p))) \xrightarrow{\mathrm{Sp}(m)} \mathrm{Sp}(R).$$

We want to show the existence of a natural bijection

$$\mathbb{MV}((C((X,f),(\coprod_{M \in \mathrm{Sp}(A)} A/M, p)), m), (A, m_A)) \cong \mathbb{S}\mathrm{tone}((\mathrm{Sp}(A), \alpha), (X, f)),$$

where $\alpha = \mathrm{Sp}(m_A)$. We fix an element φ of

$$\mathbb{MV}(C((X,f),(\coprod_{M \in \mathrm{Sp}(A)} A/M, p)), (A, m_A))$$

i.e. a morphism of MV-algebras $\varphi\colon C((X,f),(\coprod_{M \in \mathrm{Sp}(A)} A/M, p)) \to A$ such that $\varphi m = m_A$. So, for every clopen $U \subseteq X$, we consider the continuous map $\chi_U \colon X \to \coprod_{M \in \mathrm{Sp}(A)} A/M$ defined as

$$\chi_U(x) \coloneqq \begin{cases} [1]_{f(x)} \in R/f(x) & x \in U \\ [0]_{f(x)} \in R/f(x) & x \notin U. \end{cases}$$

Clearly χ_U is an idempotent of $C((X,f),(\coprod_{M \in \mathrm{Sp}(A)} A/M, p))$, and so $\varphi(\chi_U) \in \mathrm{Idem}(A)$. Hence, we can define a morphism of Boolean algebras

$$\begin{aligned} \varphi' \colon \mathrm{Clopen}(X) &\to \mathrm{Idem}(A) \\ U &\mapsto \varphi(\chi_U) \end{aligned}$$

and, thanks to the Stone Duality, we obtain a continuous map $\varphi'' \colon \mathrm{Sp}(A) \to X$; we have to show that $f\varphi'' = \alpha$. In other terms, we have to prove that $\varphi'(f^{-1}(O_e)) = m_A(e)$. We observe that

$$\chi_{f^{-1}(O_e)}(x) = \begin{cases} [1]_{f(x)} \in R/f(x) & x \in f^{-1}(O_e) \\ [0]_{f(x)} \in R/f(x) & x \notin f^{-1}(O_e). \end{cases}$$

If $e \notin f(x)$ then $\neg e \in R/f(x)$, and so $[1]_{f(x)} = [e]_{f(x)}$. Similarly, if $e \in f(x)$ then $[e]_{f(x)} = [0]_{f(x)} \in R/f(x)$. Therefore, $\chi_{f^{-1}(O_e)}(x) = [e]_{f(x)} = m(e)(x)$, so we obtain $\chi_{f^{-1}(O_e)} = m(e)$ and $\varphi(\chi_{f^{-1}(O_e)}) = \varphi(m(e)) = m_A(e)$. This implies $f\varphi'' = \alpha$.

Let us prove that the assignment $\varphi \to \varphi''$ is injective. First of all, we notice that for every $h \in C((X, f), (\coprod_{M\in \mathrm{Sp}(A)} A/M, p))$ the collection

$$h^{-1}(s_r^{\mathrm{Sp}(R)}(\mathrm{Sp}(R)))$$

is an open covering of X. Since X has a base of clopens, we can refine this covering with clopens and, since X is compact, we can extract from it a finite covering $U_1, \dots, U_n \subseteq X$. Now, for every $x \in U_i$ we have an element $r_i \in R$ such that $h(x) = [r_i]_{f(x)}$. So we obtain

$$h = \bigoplus_{i=1}^{n} m_A(r_i) \odot \chi_{U_i}.$$

Hence, if we consider two morphisms of MV-algebras

$$\varphi, \psi \colon C((X, f), (\coprod_{M\in \mathrm{Sp}(A)} A/M, p)) \to A$$

such that $\varphi m = m_A$, we have $\psi m = m_A$ and $\varphi'' = \psi''$. We observe that

$$\begin{aligned} \varphi(h) &= \varphi(\bigoplus_{i=1}^{n} m_A(r_i) \odot \chi_{U_i}) = \bigoplus_{i=1}^{n} \varphi(m_A(r_i)) \odot \varphi(\chi_{U_i}) \\ &= \bigoplus_{i=1}^{n} \psi(m_A(r_i)) \odot \psi(\chi_{U_i}) = \psi(h) \end{aligned}$$

since, by Stone Duality, $\varphi'' = \psi''$ implies $\varphi' = \psi'$, and so $\varphi(\chi_U) = \varphi'(U) = \psi'(U) = \psi(\chi_U)$ for every clopen U of X.

It remains to prove the surjectivity. Let us consider a continuous map $g \colon \mathrm{Sp}(A) \to X$ such that $fg = \alpha$. We have to construct a morphism of MV-algebras φ such that $\varphi'' = g$. Thanks to the Stone Duality, we can prove that $\varphi' = \overline{g}$ where $\overline{g} \colon \mathrm{Clopen}(X) \to \mathrm{Idem}(A)$ is the morphism of Boolean algebras induced, by the Stone Duality, from g. We remind that $\overline{g}(f^{-1}(O_e)) = m_A(e)$. Moreover, we know that every $h \in C((X, f), (\coprod_{M\in \mathrm{Sp}(A)} A/M, p))$ can be written as

$$h = \bigoplus_{i=1}^{n} m_A(r_i) \odot \chi_{U_i},$$

where $\{U_1, \dots, U_n\}$ is a partition of X in clopens. Therefore, we can define

$$\begin{aligned} \varphi \colon C((X, f), (\coprod_{M\in \mathrm{Sp}(A)} A/M, p)) &\to A \\ h &\mapsto \varphi(h) := \bigoplus_{i=1}^{n} m_A(r_i) \odot \overline{g}(U_i). \end{aligned}$$

If we show that φ is well defined (i.e. it does not depend on the partition) and it is a morphism of MV-algebras, then we can conclude that $g = \varphi''$, since $\varphi(\chi_U) = \overline{g}(U)$ for every clopen U of X. Let us consider an other decomposition

$$h = \bigoplus_{j=1}^{m} m_A(s_j) \odot \chi_{V_j};$$

we define $W_{i,j} := U_i \cap V_j$. For every $x \in W_{i,j}$ we have $[r_i]_{f(x)} = h(x) = [s_j]_{f(x)}$; observe that the family $\{W_{i,j}\}_{j=1}^{m}$ is a partition in clopens of U_i, therefore

$$\overline{g}(U_i) = \bigoplus_{j=1}^{m} \overline{g}(W_{i,j}),$$

with $\overline{g}(W_{i,j}) \odot \overline{g}(W_{i,k}) = 0$ for every $j \neq k$. Thanks to Lemma 2.2.8, we obtain

$$m_A(r_i) \odot \overline{g}(U_i) = \bigoplus_{j=1}^{m} m_A(r_i) \odot \overline{g}(W_{i,j})$$

for every $i = 1, \ldots, n$ and

$$m_A(s_j) \odot \overline{g}(V_j) = \bigoplus_{i=1}^{n} m_A(s_j) \odot \overline{g}(W_{i,j})$$

for every $j = 1, \ldots, m$. So, if we show that

$$m_A(r_i) \odot \overline{g}(W_{i,j}) = m_A(s_j) \odot \overline{g}(W_{i,j})$$

we are done. We know that for every $x \in W_{i,j}$ we have $[r_i]_{f(x)} = [s_j]_{f(x)}$, and so $d(r_i, s_j) \in f(x)$. Hence, it suffices to show $m_A(r) \odot \overline{g}(U) = 0$ if $r \in f(x)$ for every $x \in U$ (where U is a clopen of X). In fact, we know that $d(r_i, s_j) \in f(x)$ for every $x \in W_{i,j}$. Therefore, if the previous statement holds, we get $d(r_i, s_j) \odot \overline{g}(W_{i,j}) = 0$ and so, thanks to Lemma 2.2.9, $m_A(r_i) \odot \overline{g}(W_{i,j}) = m_A(s_j) \odot \overline{g}(W_{i,j})$. Let us prove that, if $r \in f(x)$ for every $x \in U$ (where U is a clopen of X), then $m_A(r) \odot \overline{g}(U) = 0$. We observe that $f(U) \subseteq \{M \in \mathrm{Sp}(R) \mid r \in M\} = U_r$. Recalling that U_r is open, we can construct a covering $\{O_{e_i}\}_{i \in I}$ made by clopens of U_r (where $e_i \in \mathrm{Idem}(R)$ for every $i \in I$). Therefore, $\{f^{-1}(O_{e_i})\}_{i \in I}$ is a covering of the closed (and so compact) subset U. Hence, we can extract a finite subcovering and we can modify this subcovering in order to obtain a partition, let us say $\{O_j\}_{j=1}^{n}$. So, we have

$$U \subseteq f^{-1}(O_1) \cup \cdots \cup f^{-1}(O_m),$$

and then

$$\begin{aligned} m_A(r) \odot \overline{g}(U) &= m_A(r) \odot \overline{g}(U \cap (f^{-1}(O_1) \cup \cdots \cup f^{-1}(O_m))) \\ &= m_A(r) \odot \left(\bigoplus_{j=1}^{m} \left(\overline{g}(U) \odot \overline{g}(f^{-1}(O_{e_j})) \right) \right) = \bigoplus_{j=1}^{m} m_A(r) \odot \overline{g}(U) \odot m_A(e_j) \end{aligned}$$

where the last equality holds thanks to Lemma 2.2.8. Finally, let us show that

$$m_A(r) \odot m_A(e_i) = 0$$

for every $i \in I$:

$$\begin{aligned} r \in \bigcap \{M \in \mathrm{Sp}(R) \mid M \in O_{e_i}\} &= \{M \in \mathrm{Sp}(R) \mid e_i \notin M\} \\ &= \{M \in \mathrm{Sp}(R) \mid \neg e_i \in M\} = \downarrow \neg e_i, \end{aligned}$$

then $r \leq \neg e_i$ and so $r \odot e_i = 0$.
It remains to prove that φ is a morphism of MV-algebras. Clearly $\varphi(0) = 0$. Let us consider h_1 and h_2, and decompose them on the same partition (which can be done in light of what has just been shown). Then $h_1 = \bigoplus_{i=1}^{n} m_A(r_i) \odot \chi_{U_i}$ and $h_2 = \bigoplus_{i=1}^{n} m_A(s_i) \odot \chi_{U_i}$. Now, recalling that in every MV-algebra one has $(x \odot e) \oplus (y \odot e) = (x \oplus y) \odot e$ for every idempotent e, we get $h_1 \oplus h_2 = \bigoplus_{i=1}^{n} m_A(r_i \oplus s_i) \odot \chi_{U_i}$ and so $\varphi(h_1 \oplus h_2) = \varphi(h_1) \oplus \varphi(h_2)$. Finally, thanks to Lemma 2.2.7, we know that if $h = \bigoplus_{i=1}^{n} m_A(r_i) \odot \chi_{U_i}$ then $\neg h = \bigoplus_{i=1}^{n} m_A(\neg r_i) \odot \chi_{U_i}$ and so $\varphi(\neg h) = \neg \varphi(h)$. □

In conclusion, our analysis has revealed that the counit of the adjunction, introduced in the previous proposition, is a natural isomorphism. This crucial observation offers interesting possibilities for examining the adjunction between MV-algebras and Boolean algebras through the lens of categorical Galois theory. This result provides a solid foundation for our future work, which aims to explore and advance this idea.

2.5 Protomodularity, Arithmeticity, and Centralizers

In this section, we will study the categorical-algebraic properties of $\mathbb{MV}$. Since $\mathbb{MV}$ is a variety of universal algebras, we know that it is a Barr-exact category. It is known that $\mathbb{MV}$ is also arithmetical and protomodular. We will give an explicit description of the terms of protomodularity and arithmeticity for $\mathbb{MV}$. We will then prove how protomodularity allows us to identify the conditions under which certain commutative squares in $\mathbb{MV}$ are pullbacks. Finally, we will provide a more detailed analysis of the properties of categories $Pt_{\mathbb{MV}}B$, showing in particular that in these categories every subobject has a centralizer.

To determine whether a variety $\mathbb{V}$ is protomodular, we can use Theorem 1.1 of [17]. This theorem states that $\mathbb{V}$ is protomodular if and only if it has 0-ary terms $e_1, \ldots, e_n$, binary terms $t_1, \ldots, t_n$ and an $(n+1)$-ary term t satisfying the identities

$$t(x, t_1(x, y), \ldots, t_n(x, y)) = y \text{ and } t_i(x, x) = e_i$$

for all $i = 1, \ldots, n$.

In the recent work [41], the authors make use of the results obtained for the variety of hoops to prove, among other things, that the variety of MV-algebras is protomodular. Here, we present an alternative proof of this fact by exhibiting different protomodularity terms compared to those introduced in the aforementioned work.

Proposition 2.5.1. *$\mathbb{MV}$ is a protomodular category.*

Proof. We define $t_1(x, y) := x \ominus y$, $t_2(x, y) := x \oplus \neg y$, and $t(x, y, z) := x \oplus (y \odot z)$. Clearly, one has

$$t_1(x, x) = x \ominus x = 0 \text{ and } t_2(x, x) = x \oplus \neg x = 1.$$

Moreover, the following equality holds

$$t(t_1(x, y), t_2(x, y), y) = (x \ominus y) \oplus ((x \oplus \neg y) \odot y) = x;$$

a proof of the last equality can be found in Proposition 1.6.2 of [21]. □

In their work [41], the authors show that $\mathbb{MV}$ is a protomodular category by constructing two binary terms, namely $\alpha_1(x, y)$ and $\alpha_2(x, y)$, as well as a ternary term, $\theta(x, y, z)$. However, we observe that, in their case, the equalities $s_1(x, x) = 1$ and $s_2(x, x) = 1$ hold. Consequently, it is clear that our own protomodularity terms differ from theirs, as anticipated earlier.

Lemma 2.5.2. *Consider a commutative square in $\mathbb{MV}$*

$$\begin{array}{ccc} A & \overset{f}{\twoheadrightarrow} & B \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle k} \\ C & \underset{g}{\twoheadrightarrow} & D, \end{array}$$

where the horizontal arrows are regular epimorphisms. Hence, denoting with $\langle h, f \rangle$ the unique *arrow induced by the universal property of the pullback*

$$\begin{tikzcd}
A \arrow[dr, "\langle h,f\rangle"] \arrow[drr, bend left, "f"] \arrow[ddr, bend right, "h"'] & & \\
& P \arrow[r, two heads, "\pi_B"] \arrow[d, "\pi_C"'] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "k"] \\
& C \arrow[r, two heads, "g"'] & D,
\end{tikzcd}$$

we get:

i) $\langle h, f\rangle$ *is injective if and only the restriction of* h *(considered as map)* $h\colon \ker(f) \to \ker(g)$ *is injective;*

ii) $\langle h, f\rangle$ *is surjective if and only the restriction of* h *(considered as map)* $h\colon \ker(f) \to \ker(g)$ *is surjective.*

Proof. i) ($\Rightarrow$) By assumption $\langle h, f\rangle$ is injective, therefore $\{0\} = \ker(\langle h, f\rangle) = \ker(f) \cap \ker(h)$ and so, thanks to Lemma 2.1.6, we conclude that the restriction of h is injective.
($\Leftarrow$) Consider an element $a \in A$ such that $\langle h, f\rangle(a) = (0, 0)$; then $a \in \ker(f) \cap \ker(h)$ and so $a = 0$, since the restriction of h is injective.
ii) ($\Rightarrow$) Fix an element $c \in \ker(g)$; we know that $(c, 0) \in P$ since $k(0) = 0 = g(c)$; hence, there exists an element $a \in A$ such that $\langle h, f\rangle(a) = (c, 0)$ ($\langle h, f\rangle$ is surjective). Therefore, $a \in \ker(f)$ and $h(a) = c$, i.e. the restriction of h is surjective.
($\Leftarrow$) Given an element $(c, b) \in P$ (i.e. $c \in C$, $b \in B$, and $g(c) = k(b)$) there exists an element $a \in A$ such that $f(a) = b$ (f is surjective). Now, we observe that $gh(a) = kf(a) = k(b) = g(c)$, and so we deduce $c \ominus h(a) \in \ker(g)$ and $h(a) \ominus c \in \ker(g)$. Therefore, since the restriction of h is surjective, there exist $a_1, a_2 \in \ker(f)$ such that $h(a_1) = c \ominus h(a) \in \ker(g)$ and $h(a_2) = h(a) \ominus c \in \ker(g)$ (for our aim, it is most useful to keep in mind that $h(\neg a_2) = c \oplus \neg h(a)$). We recall, from Proposition 1.6.2 of [21], that the following equality holds

$$(c \ominus h(a)) \oplus ((c \oplus \neg h(a)) \odot h(a)) = c.$$

This equality can be reformulated as

$$h(a_1) \oplus (h(\neg a_2) \odot h(a)) = c,$$

so $h(a_1 \oplus (\neg a_2 \odot a)) = c$ and $f(a_1 \oplus (\neg a_2 \odot a)) = f(a_1) \oplus (\neg f(a_2) \odot f(a)) = 0 \oplus (1 \odot f(a)) = b$; we have proved that $\langle h, f\rangle$ is surjective. $\square$

The previous lemma yields the following direct consequence:

Corollary 2.5.3. *A commutative square in* $\mathbb{MV}$

$$\begin{tikzcd} A \arrow[r, two heads, "f"] \arrow[d, "h"'] & B \arrow[d, "k"] \\ C \arrow[r, two heads, "g"'] & D, \end{tikzcd}$$

where the horizontal arrows are regular epimorphism, is a pullback if and only if the restriction of h *(considered as map)* $h\colon \ker(f) \to \ker(g)$ *is bijective.*

It is known that $\mathbb{MV}$ is an arithmetical variety, which means it both congruence distributive and congruence permutable. As we said in the previous chapter, in Theorem 2 of [48] the author proved that a variety $\mathbb{V}$ is arithmetical if and only if there exists a ternary term $r(x, y, z)$ such that

$$r(x, x, z) = z,\ r(x, y, y) = x, \text{ and } r(x, y, x) = x$$

for every object X and for every $x, y, z \in X$. Here, we give a proof that $\mathbb{MV}$ is arithmetical by exhibiting such a term:

Proposition 2.5.4. $\mathbb{MV}$ *is an arithmetical category.*

Proof. We define

$$p(x, y, z) \coloneqq ((x \to y) \to z) \wedge ((z \to y) \to x)$$

and

$$t(x, y, z) \coloneqq (y \to (x \wedge z)) \wedge (x \vee z).$$

We observe that

$$\begin{aligned} p(x, x, z) &= z \wedge ((z \to x) \to x) = z \wedge (\neg(\neg z \oplus x) \oplus x) \\ &= z \wedge (\neg(\neg x \oplus z) \oplus z) = z, \\ p(x, y, y) &= ((x \to y) \to y) \wedge x = (\neg(\neg x \oplus y) \oplus y) \wedge x \\ &= (\neg(\neg y \oplus x) \oplus x) \wedge x = x, \text{ and} \\ p(x, y, x) &= ((x \to y) \to x) \wedge ((x \to y) \to x) \\ &= ((x \to y) \to x) = \neg(\neg x \oplus y) \oplus x \geq x. \end{aligned}$$

Moreover, we have

$$\begin{aligned} t(x, x, z) &= (x \to (x \wedge z)) \wedge (x \vee z) = ((x \to x) \wedge (x \to z)) \wedge (x \vee z) \\ &= (x \to z) \wedge (x \vee z) \geq z, \\ t(x, y, y) &= (y \to (x \wedge y)) \wedge (x \vee y) = (y \to x) \wedge (x \vee y) \geq x, \text{ and} \\ t(x, y, x) &= (y \to x) \wedge x = x. \end{aligned}$$

Therefore, the term

$$r(x, y, z) \coloneqq p(x, y, z) \wedge t(x, y, z)$$

satisfies

$$r(x, x, z) = z,\ r(x, y, y) = x, \text{ and } r(x, y, x) = x. \qquad \square$$

In the final part of this section we study, from a categorical point of view, the commutativity of subobjects in the category $Pt_B\mathbb{MV}$, for every MV-algebra B. In general, given an arbitrary category $\mathbb{C}$, the category $Pt_B\mathbb{C}$ (where B is an object of $\mathbb{C}$) is the nothing more than $(\mathbb{C}/B)\backslash id_B$ (i.e. the coslice over id_B of the slice of $\mathbb{C}$ over B).

In order to show that in $Pt_B\mathbb{MV}$ there are centralizers of subobjects, we need to recall the notion of lattice-ordered abelian group. As we seen in the previous chapter, a *lattice-ordered abelian group* is an algebraic structure of signature $\{+, 0, -, \vee, \wedge\}$ satisfying the axioms of abelian groups, the axioms of lattices, and the axioms related to the distributivity of the group operation over both the lattice operations:

$$x + (y \vee z) = (x + y) \vee (x + z) \text{ and } x + (y \wedge z) = (x + y) \wedge (x + z).$$

We denote by $\ell\mathbb{A}b$ the category whose objects are lattice-ordered abelian groups and whose arrows are maps between lattice-ordered abelian groups which preserves the operations. Given a lattice-ordered abelian group G we can define, for every element x of G, $|x| \coloneqq x \vee -x$. An *order-unit* u of G is an element $0 \leq u \in G$ satisfying the following property: for every $x \in G$, there exists a natural number $n \in \mathbb{N}$ such that $|x| \leq nu$. We denote by $u\ell\mathbb{A}b$ the category whose object are the pairs (G, u), where G is a lattice-ordered abelian group and u is an order-unit of G, and whose arrows are the maps which preserve the operations and the distinguished order-unit. Given an object (G, u) of $u\ell\mathbb{A}b$ we recall from [44] the definition

$$[0, u] \coloneqq \{x \in G \,|\, 0 \leq x \leq u\};$$

and on $[0, u]$ the following new operations are introduced: $x \oplus y \coloneqq (x + y) \wedge u$ and $\neg x \coloneqq u - x$. The structure $([0, u], \oplus, \neg, 0)$ is an MV-Algebra, denoted by $\Gamma(G, u)$; moreover, for every arrow $h\colon (G, u) \to (H, v)$ in $u\ell\mathbb{A}b$, the restriction of h to $[0, u]$ (denoted by $\Gamma(h)$) is a morphism of MV-algebras between $[0, u]$ and $[0, v]$.

Theorem 2.5.5 ([44], Theorem 3.9)**.** *The assignment defined by* Γ *establishes an equivalence of categories between* $u\ell\mathbb{A}b$ *and* $\mathbb{MV}$.

We know that $u\ell\mathbb{A}b$ is complete and cocomplete, since it is equivalent to a variety of universal algebras. We want to describe finite limits in $u\ell\mathbb{A}b$. We prove that they are computed as in $\ell\mathbb{A}b$.

We start dealing with equalizers. Let us consider a diagram in $u\ell\mathbb{A}b$ of the form

$$(X,u) \underset{g}{\overset{f}{\rightrightarrows}} (Y,v);$$

We define $E \coloneqq \{x \in X \mid f(x) = g(x)\}$. We know that E inherits the lattice-ordered abelian group operations from X (limits in $\ell\mathbb{A}b$ are computed as in $\mathbb{S}et$ since $\ell\mathbb{A}b$ is a variety). Moreover, since $f(u) = v = g(u)$, we have $u \in E$. So, given an arrow $k\colon (H,h) \to (X,u)$ of $u\ell\mathbb{A}b$ such that $fk = gk$, then, since E is the equalizer of f and g in $\ell\mathbb{A}b$, k factors through the inclusion of E in X; moreover, $k(h) = u$ and so we can conclude that $(E,u) \hookrightarrow (X,u)$ is the equalizer of f and g in $u\ell\mathbb{A}b$. Let us take a look at the products. We consider two objects (X,u) and (Y,v) of $u\ell\mathbb{A}b$. We prove that $(X \times Y, (u,v))$ is an object of $u\ell\mathbb{A}b$ (where the operations on $X \times Y$ are defined component-wise); in other terms, we have to show that (u,v) is an order-unit. So, fix an element $(x,y) \in X \times Y$. Then, there exist $n_1, n_2 \in \mathbb{N}$ such that $|x| \leq n_1 u$ and $|y| \leq n_2 v$. Thus, taking n as the maximum between n_1 and n_2, we get $|(x,y)| = (|x|,|y|) \leq (nu, nv) = n(u,v)$. Hence, applying similar reasoning to the one seen for equalizers, we obtain that the products in $u\ell\mathbb{A}b$ are computed as in $\ell\mathbb{A}b$.

Proposition 2.5.6. *$Pt_{(B,v)}u\ell\mathbb{A}b$ is a unital category.*

Proof. Since $u\ell\mathbb{A}b$ is arithmetical it is also a Mal'tsev category, and so $Pt_{(B,v)}u\ell\mathbb{A}b$ is unital for every object (B,v). □

Proposition 2.5.7. *Let (B,u) be an object of $u\ell\mathbb{A}b$. In the category $Pt_{(B,u)}u\ell\mathbb{A}b$ subobjects have centralizers.*

Proof. Let us consider an object of $Pt_{(B,u)}u\ell\mathbb{A}b$ $((A,u),p,s)$ and let us suppose, without loss of generality, that $s\colon (B,u) \to (A,u)$ is the inclusion. We observe that (A,p,s) is an object of $Pt_B\ell\mathbb{A}b$. We define $K \coloneqq \{k \in A \mid p(k) = 0\}$ and we observe that K is a subalgebra of A in $\ell\mathbb{A}b$. Hence, applying the results from Proposition 1.6.3, we obtain that A is isomorphic as a lattice-ordered abelian group to $K \rtimes B$, whose operations are defined by

$$(k_1,b_1) + (k_2,b_2) = (k_1 + k_2, b_1 + b_2)$$

and

$$(k_1,b_1) \vee (k_2,b_2) = (((k_1 + b_1) \vee (k_2 + b_2)) - (b_1 \vee b_2), b_1 \vee b_2).$$

In other words, the object (A,p,s) is isomorphic to

$$K \rtimes B \underset{p_B}{\overset{i_B}{\rightleftarrows}} B$$

where $p_B(k,b) = b$ and $i_B(b) = (0,b)$. The isomorphism is given by the arrow of $u\ell\mathbb{A}b$ $\varphi\colon K \rtimes B \to A$, where $\varphi(x,b) \coloneqq x + b$. Clearly φ induces an isomorphism in $u\ell\mathbb{A}b$ between the lattice-ordered abelian groups with order-unit (A, u) and $(K \rtimes B, (0, u))$. Hence, we can apply the same argument of Proposition 1.6.3 to establish the validity of the statement. $\square$

Chapter 3

A Galois Theory for MV-Algebras

Given an MV-algebra A, we define its *radical*, denoted by $\mathrm{Rad}(A)$, as the intersection of all maximal ideals of A. It has been shown that $\mathrm{Rad}(A)$ consists precisely of those elements $a \in A$ that satisfy the inequality $na \leq \neg a$ for every natural number n. This notion of radical has important implications in the study of MV-algebras. In particular, it naturally leads to the definition of two important classes of MV-algebras: the perfect MV-algebras and the semisimple MV-algebras. An MV-algebra A is said to be *perfect* if it can be expressed as the union of its radical $\mathrm{Rad}(A)$ and the set $\neg\,\mathrm{Rad}(A)$, which consists of all elements whose negation belongs to $\mathrm{Rad}(A)$. Interestingly, it has been shown (for a proof of this fact see [21]) that the category of non-trivial perfect MV-algebras is equivalent to that of lattice-ordered abelian groups. An MV-algebra is said to be *semisimple* if its radical is trivial. This notion of semisimplicity plays a crucial role in the study of the structure of MV-algebras, and is intimately related to the notion of simplicity in other areas of algebra.

We denote by $p\mathbb{MV}$ the full subcategory of $\mathbb{MV}$ whose objects are perfect MV-algebras, and by $s\mathbb{MV}$ the full subcategory of $\mathbb{MV}$ whose objects are semisimple MV-algebras. Notably, we observe that $p\mathbb{MV} \cap s\mathbb{MV} = \{\mathbf{1}, \mathbf{2}\}$, and every morphism f from a perfect MV-algebra to a semisimple MV-algebra factors through either $\mathbf{2}$ or $\mathbf{1}$. This consequently gives rise to the question of whether these two categories form a non-pointed version of a torsion theory. The appropriate notion to answer this question is that of a *pretorsion theory*, which was introduced in the recent work [30]. This concept provides a generalization of the classical torsion theory, and it has been successfully applied in various areas of mathematics.

Additionally, we will prove that the subcategory $s\mathbb{MV} \subseteq \mathbb{MV}$ is not only reflective but that the adjunction $S \dashv i$ (with S representing the reflector and i the inclusion) is admissible for categorical Galois theory with respect to the class of all arrows (and also with respect to the class of regular epimorphisms). By studying this Galois structure, we will be able to characterize the trivial, normal, and central extensions related to it.

Finally, it should be noted that the subcategory of the category of regular epimorphisms in $\mathbb{MV}$

whose objects are the central extensions is reflective. The corresponding Galois structure is admissible with respect to double extensions, which opens the way to studying higher-dimensional central extensions.

In Section 3.1 we will review the necessary preliminary concepts required to understand the remaining part of the chapter. Specifically, we will focus on the concepts of pretorsion theories, categorical Galois theory, and factorization systems.

In Section 3.2 we will delve into a detailed study of the adjunction determined by the reflective subcategory $s\mathbb{MV}$ from the perspective of categorical Galois theory. Additionally, we will describe the commutators defined by this Galois structure. Finally, we will investigate some properties of the functor S and show how the Galois structure induces a stable factorization system on $\mathbb{MV}$.

In Section 3.3 we will study the higher-dimensional normal and central extensions relative to the Galois structure determined by the adjunction defined by the subcategory of regular epimorphisms whose objects are the central extensions. This in-depth analysis will allow us to define the commutator of two ideal subalgebras with respect to this Galois structure.

3.1 Pretorsion Theories, Galois Theory, and Factorization Systems

The aim of this section is to provide an introduction to the foundational concepts required for studying pretorsion theories in general categories. Given two full replete subcategories of $\mathbb{C}$ $(\mathscr{T}, \mathscr{F})$, the authors of [30] start by defining a new full subcategory $\mathscr{Z} := \mathscr{T} \cap \mathscr{F}$ of trivial objects in the category $\mathbb{C}$. A morphism is considered to be $\mathscr{Z}$-trivial if it factors through an object of $\mathscr{Z}$. We denote the collection of $\mathscr{Z}$-trivial (or simply trivial) morphisms as $N_{\mathscr{Z}}$. This allows to define $\mathscr{Z}$-prekernels, $\mathscr{Z}$-precokernels, and short $\mathscr{Z}$-pre-exact sequences (or simply prekernels, precokernels, and pre-exact sequences).

Definition 3.1.1 ([30]). *Let $f\colon A \to B$ be a morphism in $\mathbb{C}$. We say that a morphism $k\colon K \to A$ in $\mathbb{C}$ is a* $\mathscr{Z}$-prekernel *of f if the following properties are satisfied:*

- *fk is a morphism of $N_{\mathscr{Z}}$;*
- *whenever $e : E \to A$ is a morphism in $\mathbb{C}$ and fe is in $N_{\mathscr{Z}}$, then there exists a unique morphism $\varphi\colon E \to K$ in $\mathbb{C}$ such that $k\varphi = e$.*

Dually, we have the definition of $\mathscr{Z}$-precokernel.

In [30], the authors show that some properties known for kernels also hold for $\mathscr{Z}$-prekernels. They prove that every $\mathscr{Z}$-prekernel is a monomorphism. Additionally, they show that for every morphism $f\colon A \to B$ in $\mathbb{C}$, the $\mathscr{Z}$-prekernel of f is unique up to unique isomorphism. This means that if $k\colon K \to A$ and $k'\colon K' \to A$ are $\mathscr{Z}$-prekernels of the same arrow f, then there exists a unique isomorphism $\varphi\colon K' \to K$ such that $k\varphi = k'$.

Clearly, the dual of the previous observations hold for $\mathscr{Z}$-precokernels.

Definition 3.1.2 ([30]). *Let $f\colon A \to B$ and $g\colon B \to C$ be morphisms in $\mathbb{C}$. We say that*

$$A \xrightarrow{f} B \xrightarrow{g} C$$

is a short $\mathscr{Z}$-pre-exact sequence *in $\mathbb{C}$ if f is a $\mathscr{Z}$-prekernel of g and g is a $\mathscr{Z}$-precokernel of f.*

A pretorsion theory [31] in a category $\mathbb{C}$ is defined as a pair $(\mathscr{T}, \mathscr{F})$ of full and replete subcategories $\mathscr{T}$ and $\mathscr{F}$ of a category $\mathbb{C}$, which satisfy certain conditions. Specifically, every morphism from an object in $\mathscr{T}$ to an object in $\mathscr{F}$ must be trivial, and, for every object A in $\mathbb{C}$, there must exist a pre-exact sequence with a torsion object in $\mathscr{T}$ as its left endpoint and a torsion-free object in $\mathscr{F}$ as its right endpoint. This broader view allows for more flexibility in the choice of the category $\mathbb{C}$ and the subcategories $\mathscr{T}$ and $\mathscr{F}$. The concept of pretorsion theory can be seen as a generalization of the notion of torsion theory. In fact, when the category $\mathbb{C}$ is pointed, every pretorsion theory such that $\mathscr{T} \cap \mathscr{F}$ reduces to the zero object is, actually, a torsion theory.

Definition 3.1.3 ([31]). *Let $\mathbb{C}$ be an arbitrary category. A* pretorsion theory *$(\mathscr{T}, \mathscr{F})$ in $\mathbb{C}$ consists of two replete (i.e. closed under isomorphism) full subcategories $\mathscr{T}, \mathscr{F}$ of $\mathbb{C}$ satisfying the following two conditions. Set $\mathscr{Z} = \mathscr{T} \cap \mathscr{F}$:*

- $\mathbb{C}(T, F) \subseteq N_{\mathscr{Z}}$ *for every object $T \in \mathscr{T}$, $F \in \mathscr{F}$;*
- *for every object A of $\mathbb{C}$ there is a short $\mathscr{Z}$-pre-exact sequence*

$$T(A) \xrightarrow{\varepsilon_A} A \xrightarrow{\eta_A} F(A)$$

with $T(A) \in \mathscr{T}$ and $F(A) \in \mathscr{F}$. It has been shown that such a $\mathscr{Z}$-pre-exact sequence is unique up to isomorphism.

In [31], the authors prove that, by fixing for each object A a $\mathscr{Z}$-pre-exact sequence as in 3.1.3, every pretorsion theory defines two functors:

$$\begin{tikzcd}
\mathbb{C} \arrow[r, "F"] & \mathscr{F} \\
A \arrow[r, mapsto] \arrow[d, "f"'] & F(A) \arrow[d, "F(f)"] \\
B \arrow[r, mapsto] & F(B)
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{C} \arrow[r, "T"] & \mathscr{T} \\
A \arrow[r, mapsto] \arrow[d, "f"'] & T(A) \arrow[d, "T(f)"] \\
B \arrow[r, mapsto] & T(B)
\end{tikzcd}$$

where $T(f)\colon T(A) \to T(B)$ is the unique morphism such that $f\varepsilon_A = \varepsilon_B T(f)$, and it exists since $\eta_B f \varepsilon_A \in N_{\mathscr{Z}}$; in a similar way, $F(f)\colon F(A) \to F(B)$ is the unique morphism such that $F(f)\eta_A = \eta_B f$, and it exists since $\eta_B f \varepsilon_A \in N_{\mathscr{Z}}$

$$\begin{tikzcd}
T(A) \arrow[r, "\varepsilon_A"] \arrow[d, "\exists! T(f)"'] & A \arrow[r, "\eta_A"] \arrow[d, "f"] & F(A) \arrow[d, "\exists! F(f)"] \\
T(B) \arrow[r, "\varepsilon_B"'] & B \arrow[r, "\eta_B"'] & F(B).
\end{tikzcd}$$

Proposition 3.1.4 ([31], Proposition 3.3)**.** *Let $(\mathscr{T}, \mathscr{F})$ be a pretorsion theory in a category $\mathbb{C}$. Then:*

- *the functor $F\colon \mathbb{C} \to \mathscr{F}$ is a left inverse left adjoint of the inclusion functor $i_{\mathscr{F}}\colon \mathscr{F} \hookrightarrow \mathbb{C}$ and the unit is given by η;*
- *the functor $T\colon \mathbb{C} \to \mathscr{T}$ is a left inverse right adjoint of the inclusion functor $i_{\mathscr{T}}\colon \mathscr{T} \hookrightarrow \mathbb{C}$ and the counit is given by ε.*

In the following pages, we will review several important definitions from categorical Galois theory.

Definition 3.1.5. *Let $\mathbb{C}$ be a category with pullbacks. A class $\mathscr{C}$ of arrows in $\mathbb{C}$ is* admissible *when:*

- *every isomorphism is in $\mathscr{C}$;*
- *$\mathscr{C}$ is closed under composition;*
- *$\mathscr{C}$ is closed under pullbacks, namely for every pullback*

$$\begin{tikzcd} \bullet \arrow[r, "c"] \arrow[d, "d"'] \arrow[dr, phantom, "\lrcorner", very near start] & \bullet \arrow[d, "b"] \\ \bullet \arrow[r, "a"'] & \bullet \end{tikzcd}$$

  if a and b are in $\mathscr{C}$ then c and d are in $\mathscr{C}$.

Definition 3.1.6. *Let $\mathscr{C}$ be an admissible class of morphisms in a category $\mathbb{C}$. For an object C of $\mathbb{C}$, we write $\mathscr{C}/C$ for the following category:*

- *the objects are the pairs (X, f) where $f\colon X \to C$ in $\mathscr{C}$;*
- *the arrows $h\colon (X, f) \to (Y, g)$ are all arrows in $\mathbb{C}$ such that $gh = f$.*

Definition 3.1.7 ([37])**.** *A* relatively admissible adjunction *consists in an adjunction $S \dashv C\colon \mathbb{P} \to \mathbb{A}$ (where $\mathbb{A}$ and $\mathbb{P}$ have pullbacks) and two admissible classes $\mathscr{A} \subseteq \mathbb{A}$, $\mathscr{P} \subseteq \mathbb{P}$ of arrows, such that*

- *$S(\mathscr{A}) \subseteq \mathscr{P}$,*
- *$C(\mathscr{P}) \subseteq \mathscr{A}$,*
- *for every object A of $\mathbb{A}$ the A-component η_A of the unit of the adjunction $S \dashv C$ is in $\mathscr{A}$,*
- *for every object P of $\mathbb{P}$ the P-component ε_P of the counit of the adjunction $S \dashv C$ is in $\mathscr{P}$.*

We denote a relatively admissible adjunction by $(S, C, \mathscr{A}, \mathscr{P})$.

In order to recall the notion of an admissible adjunction, remaining consistent with the notation of previous definitions, we review the definition of the following functors: $S_A\colon \mathscr{A}/A \to \mathscr{P}/S(A)$ and $C_A\colon \mathscr{P}/S(A) \to \mathscr{A}/A$, where A is an object of $\mathbb{A}$. One has $S_A(f\colon B \to A) = S(f)$ and $C_A(g\colon P \to S(A)) = \pi_A$, where the diagram

$$\begin{tikzcd} K \arrow[r, "\pi_A"] \arrow[d, "\pi_{C(P)}"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "\eta_A"] \\ C(P) \arrow[r, "C(g)"'] & CS(A) \end{tikzcd}$$

is a pullback (η_A is the A-component of the unit of the adjunction $S \dashv C$). Moreover, S_A is the left adjoint of C_A.

Definition 3.1.8 ([37]). *A relatively admissible adjunction* $(S, C, \mathscr{A}, \mathscr{P})$ *is* admissible *when the functor* C_A *is full and faithful for every object* A *of* $\mathbb{A}$.

Now, we recall the definition of effective descent morphism, as this concept will be of crucial importance for the detailed analysis of admissible Galois structures.

Definition 3.1.9. *Let* $\mathscr{A}$ *be an admissible class of arrows in a category* $\mathbb{A}$ *with pullbacks. An arrow* $h\colon B \to A$ *is an* effective descent morphism relatively to $\mathscr{A}$ *if:*

- $h \in \mathscr{A}$;
- *the change-of-base functor* $h^*\colon \mathscr{A}/B \to \mathscr{A}/A$ *is monadic.*

It has been shown that, in Barr-exact categories, effective descent morphisms, w.r.t the class of all morphisms, are precisely the regular epimorphisms.

Finally, we observe how the notion of effective descent morphism enables the study of specific classes of arrows with respect to an admissible Galois structure.

Definition 3.1.10 ([37]). *Given a relatively admissible adjunction* $(S, C, \mathscr{A}, \mathscr{P})$

- *a* trivial extension *is an arrow* $f\colon A \to B$ *of* $\mathscr{A}$ *such that the square*

$$\begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[d, "f"'] \arrow[dr, phantom, "\lrcorner", very near start] & PS(A) \arrow[d, "PS(f)"] \\ B \arrow[r, "\eta_B"'] & PS(B) \end{tikzcd}$$

  is a pullback (where η *is the unit of the adjunction* $S \dashv P$*);*

- *a* normal extension *is an arrow* f *of* $\mathscr{A}$ *such that it is an effective descent morphism relatively to* $\mathscr{A}$ *and its kernel pair projections are trivial extensions;*

- *a* central extension *is an arrow f of $\mathscr{A}$ such that there exists an effective descent morphism g relatively to $\mathscr{A}$ and the pullback $g^*(f)$ of f along g is a trivial extension.*

Finally, we are ready to recall the notion of factorization system.

Definition 3.1.11. *A* factorization system *for a category $\mathbb{C}$ is a pair of classes of arrows $(\mathscr{E}, \mathscr{M})$ such that:*

- *for every $e \in \mathscr{E}$ and $m \in \mathscr{M}$ one has $e \downarrow m$, i.e. for every commutative square in $\mathbb{C}$*

$$\begin{tikzcd} A \arrow[r, "e\in\mathscr{E}"] \arrow[d, "g"'] & B \arrow[d, "h"] \arrow[dl, "\exists! d"'] \\ C \arrow[r, "m\in\mathscr{M}"'] & D \end{tikzcd}$$

  there exists a unique arrow $d \colon B \to C$ such that $de = g$ and $md = h$;

- *every arrow f in $\mathbb{C}$ factors as $f = me$, where $m \in \mathscr{M}$ and $e \in \mathscr{E}$.*

A factorization system $(\mathscr{E}, \mathscr{M})$ is stable *if the pullback of every arrow of $\mathscr{E}$ along an arbitrary arrow is, again, an arrow of $\mathscr{E}$.*

3.2 A Galois Theory for MV-Algebras

In this section, we will study the category of MV-algebras from the perspective of categorical Galois theory. Given an MV-algebra A, we denote its radical, i.e. the intersection of its maximal ideals, by $\mathrm{Rad}(A)$. An MV-algebra is said to be *semisimple* if its radical is trivial. It has been shown (for a proof of this fact see [21]) that an MV-algebra is semisimple if and only if it is a subdirect product of subalgebras of the MV-algebra [0,1]. We will also consider perfect MV-algebras, which are defined as follows: an MV-algebra A is said to be *perfect* if $A = \mathrm{Rad}(A) \cup \neg\,\mathrm{Rad}(A)$. We will show that the full subcategory of semisimple MV-algebras and the full subcategory of perfect MV-algebras constitute the torsion-free and torsion parts, respectively, of a pretorsion theory on $\mathbb{MV}$. We will then study the Galois structure defined by the reflector of the subcategory of semisimple MV-algebras.

Definition 3.2.1 ([21])**.** *Given an MV-Algebra A, the* radical *of A is defined as*

$$\mathrm{Rad}(A) := \bigcap \{M \subseteq A \mid \ M \textit{ is a maximal ideal}\}.$$

Let us consider an MV-algebra A. In Proposition 3.6.4 of [21] the authors show that

$$\mathrm{Rad}(A) = \mathrm{Inf}(A) \cup \{0\},$$

where $a \in \mathrm{Inf}(A)$ if and only if $a \neq 0$ and $na \leq \neg a$ for every $n \in \mathbb{N}$. Moreover, in Lemma 7.3.3 of [21], it is proved that

$$\mathrm{Rad}(A) = \bigvee\{J \subseteq A \,|\, J \text{ is a nilpotent ideal of } A\},$$

where an ideal J is said to be nilpotent if, for every $x, y \in J$, one has $x \odot y = 0$ and the join is computed in the poset $\mathrm{Ideals}(A)$ of ideals of A.

Definition 3.2.2 ([21]). *Let A be an MV-algebra. A is* semisimple *if its radical* $\mathrm{Rad}(A)$ *is trivial, i.e.*

$$\mathrm{Rad}(A) = \{0\}.$$

A is perfect *if it can be expressed as the union of its radical* $\mathrm{Rad}(A)$ *and the negation of its radical, i.e.*

$$A = \mathrm{Rad}(A) \cup \neg\,\mathrm{Rad}(A),$$

where $\neg S \coloneqq \{x \in A \,|\, \neg x \in S\}$

Remark 3.2.3. *Given an MV-algebra A, we define $S(A) \coloneqq A/\mathrm{Rad}(A)$. Thanks to Lemma 3.6.6 of [21] we obtain that $S(A)$ is semisimple.*

Remark 3.2.4. *Given an MV-algebra A, we define $P(A) \coloneqq \mathrm{Rad}(A) \cup \neg\,\mathrm{Rad}(A)$. $P(A)$ is a subalgebra of A:*

- $0 \in P(A)$;
- $x \in P(A)$ *implies* $\neg x \in P(A)$;
- *if $x, y \in P(A)$ then $x \oplus y \in P(A)$. To show this, we work on cases: if $x, y \in \mathrm{Rad}(A)$ then $x \oplus y \in \mathrm{Rad}(A)$; if $x \in \mathrm{Rad}(A)$ and $y \in \neg\,\mathrm{Rad}(A)$ then, since $\neg\,\mathrm{Rad}(A)$ is a filter and $y \leq x \oplus y$, we obtain $x \oplus y \in P(A)$; finally, if $x, y \in \neg\,\mathrm{Rad}(A)$, since $\neg\,\mathrm{Rad}(A)$ is a filter and $y \leq x \oplus y$, we get $x \oplus y \in P(A)$.*

Moreover, it is easy to see that $P(A)$ is perfect.

It is straightforward to verify that the argument just presented holds for any arbitrary ideal I of A. In other words, the set $I \cup \neg I$ always forms a subalgebra of A.
If we denote by $s\mathbb{MV}$ the full subcategory of $\mathbb{MV}$ whose objects are the semisimple MV-algebras, we get a functor S described by the following assignment:

$$\mathbb{MV} \xrightarrow{\quad S \quad} s\mathbb{MV}$$

$$\begin{array}{ccc} A & \longmapsto & S(A) = A/\mathrm{Rad}(A) \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle S(f)=\overline{f}} \\ B & \longmapsto & S(B) = B/\mathrm{Rad}(B), \end{array}$$

where $\overline{f}([a]) := [f(a)]$, for every $[a] \in S(A)$. We have to show that $\overline{f}$ is well defined: $a \in \mathrm{Rad}(A)$ if and only if $na \leq \neg a$ for every $n \in \mathbb{N}$, thus $nf(a) = f(na) \leq f(\neg a) = \neg f(a)$ for every $n \in \mathbb{N}$, and so $f(a) \in \mathrm{Rad}(B)$.
Similarly, if we denote by $p\mathbb{MV}$ the full subcategory of $\mathbb{MV}$ whose objects are the perfect MV-algebras, we get a functor P described by the following assignment:

$$\begin{tikzcd}
\mathbb{MV} \arrow[r, "P"] & p\mathbb{MV} \\
A \arrow[r, mapsto] \arrow[d, "f"'] & P(A) = \mathrm{Rad}(A) \cup \neg\,\mathrm{Rad}(A) \arrow[d, "P(f)"] \\
B \arrow[r, mapsto] & P(B) = \mathrm{Rad}(B) \cup \neg\,\mathrm{Rad}(A),
\end{tikzcd}$$

where $P(f)(a) = f(a)$ for every $a \in P(A)$. We have to show that $P(f)$ is well defined: as we saw before, if $a \in \mathrm{Rad}(A)$ then $f(a) \in \mathrm{Rad}(B)$, and so $f \colon A \to B$ restricts to $P(f) \colon P(A) \to P(B)$.

Proposition 3.2.5. *The inclusion functor $i \colon s\mathbb{MV} \hookrightarrow \mathbb{MV}$ is right adjoint to S.*

Proof. We construct the unit of the adjunction $\eta \colon id_{\mathbb{MV}} \to iS$ as the quotient projection $\eta_A \colon A \to A/\mathrm{Rad}(A)$, for every MV-algebra A. Clearly η is a natural transformation. Moreover, η satisfies the universal property of the unit: given a morphism of MV-algebras $g \colon A \to i(B)$, where B is semisimple, we want to prove that there exists a unique morphism of MV-algebras $\tilde{g} \colon S(A) \to B$ such that the following diagram commutes:

$$\begin{tikzcd}
A \arrow[rr, "\eta_A"] \arrow[dr, "g"'] & & iS(A) \arrow[dl, "i(\tilde{g})"] \\
& i(B). &
\end{tikzcd}$$

Since B is semisimple, then if $a \in \mathrm{Rad}(A)$ we have $g(a) = 0$; therefore g induces a unique morphism $\tilde{g}$ such that $\tilde{g}([a]) = g(a)$, for every $a \in A$. $\square$

Proposition 3.2.6. *The inclusion functor $j \colon p\mathbb{MV} \hookrightarrow \mathbb{MV}$ is left adjoint to P.*

Proof. We construct the counit of the adjunction $\varepsilon \colon jP \to id_{\mathbb{MV}}$ as the inclusion $\varepsilon_A \colon j(P(A)) \to A$, for every MV-algebra A. Clearly ε is a natural transformation. Moreover, ε satisfies the universal property of the counit: given a morphism of MV-algebras $h \colon j(B) \to A$, where B is perfect, we want to prove that there exists a unique morphism of MV-algebras $\tilde{h} \colon B \to P(A)$ such that the following diagram commutes

$$\begin{tikzcd}
& j(B) \arrow[dl, "j(\tilde{h})"'] \arrow[dr, "h"] & \\
j(P(A)) \arrow[rr, "\varepsilon_A"'] & & A.
\end{tikzcd}$$

Since B is perfect, then, for every $b \in B$, $b \in \mathrm{Rad}(B)$ or $b \in \neg\,\mathrm{Rad}(B)$ and $h(b) \in \mathrm{Rad}(A)$ or $h(b) \in \neg\,\mathrm{Rad}(A)$, i.e. $h(b) \in P(A)$; therefore h defines a unique morphism $\tilde{h}$ such that $\tilde{h}(b) = h(b)$, for every $b \in B$. □

More specifically, as anticipated at the beginning of this section, the pair of subcategories just introduced allows us to define a pretorsion theory in $\mathbb{MV}$. Therefore, we define the class of trivial objects as

$$\mathscr{Z} := \{\mathbf{1}, \mathbf{2}\},$$

where $\mathbf{1}$ indicates the terminal object of $\mathbb{MV}$ and $\mathbf{2}$ the initial object.

In the remaining part of the chapter, we will assume that the class of zero objects is $\{\mathbf{1}, \mathbf{2}\}$; therefore, we will write prekernel, precokernel, and pre-exact sequence to denote, respectively, $\{\mathbf{1}, \mathbf{2}\}$-prekernel, $\{\mathbf{1}, \mathbf{2}\}$-precokernel, and $\{\mathbf{1}, \mathbf{2}\}$-pre-exact sequence.

Proposition 3.2.7. $(p\mathbb{MV}, s\mathbb{MV})$ *is a pretorsion theory for* $\mathbb{MV}$.

Proof. First of all we observe that

$$p\mathbb{MV} \cap s\mathbb{MV} = \{\mathbf{1}, \mathbf{2}\}.$$

We prove that, given two MV-algebras A, B, with A perfect and B semisimple, $\mathbb{MV}(A, B) \subseteq N_{\mathscr{Z}}$. Let us suppose $B \neq \mathbf{1}$ (this implies $A \neq \mathbf{1}$); then a morphism $f\colon A \to B$ factors in the following way:

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "\chi_{\neg\,\mathrm{Rad}(A)}"'] & & B \\ & \mathbf{2}, \arrow[ur, hook] & \end{tikzcd}$$

where

$$\chi_{\neg\,\mathrm{Rad}(A)}(a) = \begin{cases} 0 & a \in \mathrm{Rad}(A) \\ 1 & a \in \neg\,\mathrm{Rad}(A). \end{cases}$$

In fact, if $a \in A$ then $a \in \mathrm{Rad}(A)$ or $a \in \neg\,\mathrm{Rad}(A)$, since A is perfect, and so $f(a) \in \mathrm{Rad}(B) = \{0\}$ or $f(a) \in \neg\,\mathrm{Rad}(A) = \{1\}$, because B is semisimple. If $B = \mathbf{1}$ we have

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "f"'] & & \mathbf{1} \\ & \mathbf{1}\,. \arrow[ur, equal] & \end{tikzcd}$$

Now, we want to show that, for every MV-algebra B, we have a pre-exact sequence with B in the middle. If $B = \mathbf{1}$ the sequence is given by $B = B = B$. Suppose $B \neq \mathbf{1}$; we prove that

$$P(B) \xhookrightarrow[\varepsilon_B]{} B \xrightarrow[\eta_B]{} S(B)$$

is a pre-exact sequence. Observe that, since $\eta_B(b) = 0$ for every $b \in \mathrm{Rad}(B)$, $\eta_B\varepsilon_B \in N_{\mathscr{Z}}$. We prove that ε_B is a prekernel of η_B. Consider an arrow $\lambda_B \colon C \to B$ such that there exists an arrow $x \colon C \to \mathbf{2}$ making the diagram below commutative

$$\begin{tikzcd} B \arrow[r, two heads, "\eta_B"] & S(B) \\ C \arrow[u, "\lambda_B"] \arrow[r, "x"'] & \mathbf{2}\,. \arrow[u, hook] \end{tikzcd}$$

Notice that if $\eta_B\lambda_B$ factored through $\mathbf{1}$, then we would have $S(B) = \mathbf{1}$; since $S(B) = \mathbf{1}$ implies $1 \in \mathrm{Rad}(B)$, it would follow that $B = \mathbf{1}$. Now, for every $c \in C$, if $x(c) = 0$, then $\lambda_B(c) \in \mathrm{Rad}(B)$ and, if $x(c) = 1$, then $\lambda_B(c) \in \neg\,\mathrm{Rad}(B)$; therefore, λ_B restricts to $\lambda'_B \colon C \to P(B)$ and the following diagram commutes

$$\begin{tikzcd} P(B) \arrow[rr, hook, "\varepsilon_B"] & & B \arrow[rr, two heads, "\eta_B"] & & S(B) \\ & C \arrow[ul, "\lambda'_B"] \arrow[ur, "\lambda_B"'] \arrow[rr, "x"'] & & \mathbf{2}\,. \arrow[ur, hook] & \end{tikzcd}$$

Therefore, ε_B is a prekernel of η_B.
Thus we focus on η_B: we want to show that it is a precokernel of ε_B. Fix an arrow $\theta_B \colon B \to C$ such that $\theta_B\varepsilon_B \in N_{\mathscr{Z}}$. Now, if $\theta_B\varepsilon_B$ factors through $\mathbf{1}$, we can conclude that $C = \mathbf{1}$. Hence, the claim becomes trivial. Then, suppose there exists an arrow $y \colon P(B) \to \mathbf{2}$ making the following diagram commutative:

$$\begin{tikzcd} P(B) \arrow[r, hook, "\varepsilon_B"] \arrow[d, "y"'] & B \arrow[d, "\theta_B"] \\ \mathbf{2} \arrow[r, hook] & C. \end{tikzcd}$$

If $b \in \mathrm{Rad}(B)$, then $\theta_B(b) \in \mathbf{2} \subseteq C$; but, if $\theta_B(b) = 1$, we get $1 \in \mathrm{Rad}(C)$ and so $C = \mathbf{1}$ (we are excluding this case). Hence, θ_B induces a unique morphism $\theta'_B \colon S(B) \to C$ such that the diagram below is commutative

$$\begin{tikzcd} P(B) \arrow[rr, hook, "\varepsilon_B"] \arrow[dr, "y"'] & & B \arrow[rr, two heads, "\eta_B"] \arrow[dr, "\theta_B"] & & S(B) \arrow[dl, "\theta'_B"] \\ & \mathbf{2} \arrow[rr, hook] & & C. & \end{tikzcd}$$

Therefore, η_B is a precokernel of ε_B. □

The purpose of the following part of the section is to examine the adjunction

$$S \dashv i \colon s\mathbb{MV} \hookrightarrow \mathbb{MV}$$

from the perspective of categorical Galois theory.

In the literature, adjunctions studied through the lens of categorical Galois theory are frequently adjunctions between semi-abelian categories. Additionally, in cases where the adjunction is induced by the reflection of a reflective subcategory, it is often required that the subcategory is a Birkhoff subcategory. A full and reflective subcategory $\mathbb{D}$ of a category $\mathbb{C}$ is considered a *Birkhoff* subcategory if it is closed under subobjects and regular images. It is important to mention that our work does not fall under the conditions specified above. Specifically, our category $\mathbb{MV}$ is not semi-abelian (it is not pointed), and the full subcategory $s\mathbb{MV} \hookrightarrow \mathbb{MV}$ is not a Birkhoff subcategory. In fact, let us consider the MV-algebra

$$A \coloneqq \prod_{n\geq 2} L_n$$

where $L_n \coloneqq \{0, \frac{1}{n}, \dots, \frac{n-1}{n}, 1\} \subseteq [0,1]$ and the operations are the ones induced by $[0,1]$. First of all, let us show that A is semisimple. Given an element $x = (x_n)_{n\geq 2} \in A$, with $x \neq 0$, there exists a natural number $n \geq 2$ such that $x_n = \frac{k}{n} \neq 0$, hence $(\neg x)_n = \neg x_n = 1 - \frac{k}{n} = \frac{n-k}{n} \neq 1$. Therefore, there must exist a natural number m such that $mk > n - k$ and so $mx \nleq \neg x$, i.e. $\mathrm{Inf}(A) = \emptyset$ and $\mathrm{Rad}(A) = \{0\}$. Next, we consider the relation $\rho \subseteq A \times A$ defined by $(x, y) \in \rho$ if and only if there exists a natural number $N \geq 2$ such that $x_n = y_n$ for every $n \geq N$, where $x = (x_n)_{n\geq 2}$ and $y = (y_n)_{n\geq 2}$. It is clear that ρ is a congruence. Let $[z] \in A/\rho$ where $z = (z_n)_{n\geq 2}$ and $z_n = \frac{1}{n}$ for every $n \geq 2$. Our goal now is to show that $[z] \in \mathrm{Inf}(A/\rho)$. Fix a natural number $m \in \mathbb{N}$; clearly $\frac{m}{n} \leq \frac{n-1}{n}$ for every $n > m$, that is $mz_n \leq \neg z_n$. We define $y = (y_n)_{n\geq 2} \in A$ such that

$$y_n \coloneqq \begin{cases} 0 \text{ if } n \leq m \\ z_n \text{ if } n > m. \end{cases}$$

Then $m[z] = m[y] \leq \neg[y] = \neg[z]$ and so the radical $\mathrm{Rad}(A/\rho)$ is not trivial, i.e. A/ρ is not semisimple.

Proposition 3.2.8. *For every MV-algebra B the counit of the adjunction*

$$S_B \dashv i_B \colon s\mathbb{MV}/s(B) \to \mathbb{MV}/B$$

is an isomorphism.

Proof. When $B = \mathbf{1}$ the assertion is trivial. Suppose $B \neq \mathbf{1}$; we recall that, for every object $f \colon A \to B$ of $\mathbb{MV}/B$, $S_B(f \colon A \to B) = (\overline{f} \colon S(A) \to S(B))$ and, for every object $\varphi \colon A \to S(B)$, $i_B(\varphi \colon A \to S(B)) = (\varphi' \colon A' \to B)$ is defined by the following pullback:

$$\begin{tikzcd} A' \arrow[r, two heads, "\varphi''"] \arrow[d, "\varphi'"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "\varphi"] \\ B \arrow[r, two heads, "\eta_B"'] & S(B). \end{tikzcd}$$

We want to prove that

$$(S(\varphi')\colon S(A') \twoheadrightarrow S(B)) \cong (\varphi\colon A \twoheadrightarrow S(B)).$$

Consider the following commutative diagram, where φ''' is induced by the universal property of η:

$$\begin{tikzcd} A' \arrow[r, two heads, "\eta_{A'}"] \arrow[dr, two heads, "\varphi''"'] & A'/\operatorname{Rad}(A') \arrow[d, two heads, "\varphi'''"] \\ & A. \end{tikzcd}$$

Clearly φ''' is surjective; let us show that it is also injective. Fix an element $(b,a) \in A'$, where $a \in A$, $b \in B$ and $\varphi(a) = \eta_B(b) = [b]$. If $\varphi'''([b,a]) = 0$, we obtain $\varphi'''\eta_{A'}(b,a) = 0$ and so $\varphi''(b,a) = 0$, i.e. $a = 0$. We want to prove that $(b,0) \in \operatorname{Rad}(A')$: since $b \in \operatorname{Rad}(B)$ we know that $nb \leq \neg b$ for every $n \in \mathbb{N}$; then $n(b,0) = (nb,0) \leq (\neg b, 1) = \neg(b,0)$ for every $n \in \mathbb{N}$ and so $(b,0) \in \operatorname{Rad}(A')$. Therefore $\ker(\varphi''')$ is trivial, i.e. φ''' is injective. It remains to prove that the following diagram is commutative:

$$\begin{tikzcd} A'/\operatorname{Rad}(A') \arrow[d, "\varphi'''"'] \arrow[dr, "S(\varphi')"] & \\ A. \arrow[r, "\varphi"'] & B/\operatorname{Rad}(B) \end{tikzcd}$$

Take an element $(b,a) \in A'$. We know that $\varphi\varphi'''([b,a]) = \varphi(a)$, $\varphi(a) = [b]$, and $S(\varphi')([b,a]) = [b]$. $\square$

We observe that, thanks to the previous proposition, the adjunction $S \dashv i$ is admissible with respect to the class of all arrows in $\mathbb{MV}$ and the class of all arrows in $s\mathbb{MV}$. We will denote this Galois structure as Γ.

Proposition 3.2.9. *Given an arrow $f\colon A \to B$ in $\mathbb{MV}$, the restriction of f (considered as map) $f\colon \operatorname{Rad}(A) \to \operatorname{Rad}(B)$ is injective if and only if $P(f)\colon P(A) \to P(B)$ is injective, or $P(A) = \mathbf{2}$ and $P(B) = \mathbf{1}$.*

Proof. Thanks to Lemma 2.1.6, we know that $f\colon \operatorname{Rad}(A) \to \operatorname{Rad}(B)$ is injective if and only if $\ker(f) \cap \operatorname{Rad}(A) = \{0\}$. Suppose first that $B \neq \mathbf{1}$ (which implies $A \neq \mathbf{1}$). If $a \in \ker(P(f))$, then we have $a \in \operatorname{Rad}(A)$: indeed, if $a \in \neg\operatorname{Rad}(A)$, then $f(a) = 0 \in \neg\operatorname{Rad}(B)$, which contradicts our assumption that $B \neq \mathbf{1}$. Hence, we have $\ker(P(f)) = \ker(f) \cap \operatorname{Rad}(A)$, and so $P(f)$ is injective if and only if $\ker(f) \cap \operatorname{Rad}(A) = \{0\}$. Now suppose that $B = \mathbf{1}$. Then the restriction $f\colon \operatorname{Rad}(A) \to \operatorname{Rad}(B) = \{0\}$ is injective if and only if $\operatorname{Rad}(A) = \{0\}$. Indeed, the condition $\operatorname{Rad}(A) = \{0\}$ is equivalent to either $P(A) = \mathbf{2}$ or $P(A) = \mathbf{1}$. $\square$

Proposition 3.2.10. *Given an arrow $f\colon A \to B$ in $\mathbb{MV}$, the following diagram is a pullback (i.e.*

f is a trivial extension for Γ)

$$\begin{tikzcd} A \arrow[r, two heads, "\eta_A"] \arrow[d, "f"'] & S(A) = A/\operatorname{Rad}(A) \arrow[d, "S(f)=\overline{f}"] \\ B \arrow[r, two heads, "\eta_B"'] & S(B) = B/\operatorname{Rad}(B) \end{tikzcd}$$

if and only if $P(f)\colon P(A) \to P(B)$ is an isomorphism, or $P(A) = \mathbf{2}$ and $P(B) = \mathbf{1}$.

Proof. We apply Lemma 2.5.2 and we get that the above diagram is a pullback if and only if the restriction $f\colon \operatorname{Rad}(A) \to \operatorname{Rad}(B)$ is bijective. Thanks to Proposition 3.2.9, this last statement holds if and only if $P(f)\colon P(A) \to P(B)$ is an isomorphism, or $P(A) = \mathbf{2}$ and $P(B) = \mathbf{1}$. $\square$

Proposition 3.2.11. *Given an effective descent morphism relatively to the class of all arrows (i.e. a regular epimorphism) $f\colon A \twoheadrightarrow B$ in $\mathbb{MV}$, f is a normal extension for Γ if and only if $P(f)\colon P(A) \to P(B)$ is injective, or $P(A) = \mathbf{2}$ and $P(B) = \mathbf{1}$.*

Proof. Consider the kernel pair of f defined by the pullback

$$\begin{tikzcd} \operatorname{Eq}(f) \arrow[r, two heads, "\pi_2"] \arrow[d, two heads, "\pi_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, two heads, "f"] \\ A \arrow[r, two heads, "f"'] & B. \end{tikzcd}$$

We want to prove that $P(\pi_1)$ is an isomorphism if and only if $P(f)$ is injective, or $P(A) = \mathbf{2}$ and $P(B) = \mathbf{1}$. We start by assuming $B \neq \mathbf{1}$. Suppose that $P(f)\colon P(A) \to P(B)$ is injective; we observe that $P(\operatorname{Eq}(f)) = \{(a_1, a_2) \in A \times A \,|\, a_1, a_2 \in \operatorname{Rad}(A) \text{ and } f(a_1) = f(a_2)\} \cup \{(a_1, a_2) \in A \times A \,|\, a_1, a_2 \in \neg\operatorname{Rad}(A) \text{ and } f(a_1) = f(a_2)\}$. Therefore, since $P(f)$ is injective and $\operatorname{Rad}(A) \cap \neg\operatorname{Rad}(A) = \emptyset$ (otherwise we would get $A = \mathbf{1}$), if $(a_1, a_2) \in P(\operatorname{Eq}(f))$ then $f(a_1) = f(a_2)$ and so $a_1 = a_2$. Hence, $P(\operatorname{Eq}(f)) = \{(a, a) \in A \times A \,|\, a \in P(A)\}$ and clearly $P(\pi_1)\colon P(\operatorname{Eq}(f)) \to P(A)$ is an isomorphism. Conversely, if we assume that $P(\pi_1)$ is an isomorphism and we consider an element $a \in P(A)$ such that $P(f)(a) = 0$, then $a \in \operatorname{Rad}(A)$ (otherwise we would obtain $B = \mathbf{1}$), and so $(0, a) \in P(\operatorname{Eq}(f))$ (in fact, $n(0, a) = (0, na) \leq (1, \neg a) = \neg(0, a)$ for every $n \in \mathbb{N}$). So, since $P(\pi_1)$ is an isomorphism and $P(\pi_1)(0, 0) = P(\pi_1)(0, a)$, we get $a = 0$ and then we deduce that $P(f)$ is injective. Finally, let us handle the case $B = \mathbf{1}$. If $A = \mathbf{1}$, then $\operatorname{Eq}(f) = \mathbf{1}$ and so the assertion is trivial. If $A \neq \mathbf{1}$ and $B = \mathbf{1}$ we have two possible situations. If $P(A) = \mathbf{2}$, then $P(\operatorname{Eq}(f)) = \mathbf{2}$ and so $P(\pi_1)$ is an isomorphism and $P(f)\colon \mathbf{2} \to \mathbf{1}$. If $P(A) \neq \mathbf{2}$, hence $P(f)\colon P(A) \to \mathbf{1}$ is not injective and $P(\pi_1)$ is not an isomorphism. In fact, by assumption, we have an element $a \neq 0$ such that $a \in \operatorname{Rad}(A)$; therefore, we observe that there exist two different elements $(0, 0), (0, a) \in P(\operatorname{Eq}(f))$ (since $f(a) \in \operatorname{Rad}(B) = \{0\}$) such that $P(\pi_1)(0, 0) = P(\pi_1)(0, a)$, and so $P(\pi_1)$ is not an isomorphism. $\square$

Proposition 3.2.12. *Given an effective descent morphism relatively to the class of all arrows (i.e. a regular epimorphism) $f\colon A \twoheadrightarrow B$ in $\mathbb{MV}$, f is a central extension for Γ if and only if $P(f)\colon P(A) \to P(B)$ is injective, or $P(A) = \mathbf{2}$ and $P(B) = \mathbf{1}$.*

Proof. Consider the following pullback diagram:

$$\begin{tikzcd} A \times_B C \arrow[r, two heads, "\pi_C"] \arrow[d, two heads, "\pi_A"'] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[d, two heads, "g"] \\ A \arrow[r, two heads, "f"'] & B. \end{tikzcd}$$

Thanks to the previous proposition, if $P(f)$ is injective or $P(A) = \mathbf{2}$ and $P(B) = \mathbf{1}$, then we can choose $g = f$. Conversely, suppose that there exists a regular epimorphism $g\colon C \twoheadrightarrow B$ such that the restriction of π_C (considered as a map) $\pi_C\colon \operatorname{Rad}(A \times_B C) \to \operatorname{Rad}(C)$ is a bijection (i.e. π_C is a trivial extension). We will show that the restriction of f (considered as a map) $f\colon \operatorname{Rad}(A) \to \operatorname{Rad}(B)$ is injective. Thanks to Proposition 3.2.9, this is precisely equivalent to stating that $P(f)$ is injective, or $P(A) = \mathbf{2}$ and $P(B) = \mathbf{1}$. So, it suffices to prove that $\ker(f) \cap \operatorname{Rad}(A) = \{0\}$. To do this, let us fix an element $a \in \ker(f) \cap \operatorname{Rad}(A)$. Then, $f(a) = 0 = g(0)$, and thus $(a, 0)$ belongs to $A \times_B C$. Moreover, we observe that $(a, 0)$ is an element of $\operatorname{Rad}(A \times_B C)$, since $a \in \operatorname{Rad}(A)$. Therefore, recalling that, by assumption, π_C restricted to $\operatorname{Rad}(A \times_B C)$ is an injective map and observing that $\pi_C(a, 0) = \pi_C(0, 0)$, we deduce that $a = 0$. Hence, the restriction of f to $\operatorname{Rad}(A)$ is injective. $\square$

From the two previous propositions, it follows immediately that a regular epimorphism is a central extension if and only if it is a normal extension. It is known that this situation always occurs for Galois structures arising from reflections to a Birkhoff subcategory of a Goursat category, see Theorem 4.8 of [38]. However, our case is not an instance of this general result, since $s\mathbb{MV}$ is not a Birkhoff subcategory of $\mathbb{MV}$.

We provide an example of a central extension which is not trivial. Consider the MV-algebra

$$A \coloneqq \prod_{n \geq 2} L_n$$

and the congruence $\rho \subseteq A \times A$ defined by $(x, y) \in \rho$ if and only if there exists a natural number $N \geq 2$ such that $x_n = y_n$ for every $n \geq N$, where $x = (x_n)_{n \geq 2}$ and $y = (y_n)_{n \geq 2}$. We prove that the quotient projection

$$\pi\colon A \twoheadrightarrow A/\rho$$

is central but not trivial. Recalling that A is semisimple, we immediately get that $P(A) = \mathbf{2}$ and so the map

$$P(\pi)\colon P(A) = \mathbf{2} \to P(A/\rho)$$

is injective (since $P(A/\rho)$ is not trivial). However, the map $P(\pi)$ is not surjective. To see this, let $[z] \in A/\rho$ be fixed, where $z = (z_n)_{n\geq 2}$ and $z_n = \frac{1}{n}$ for every $n \geq 2$. We have already shown that $[z] \in \mathrm{Inf}(A/\rho)$, and it is clear that $[z]$ is not the zero element of A/ρ. Therefore, $[z]$ does not lie in the image of $P(\pi)$, and we conclude that $P(\pi)$ is not surjective. This implies that the morphism π is central but not trivial.

In [28], the authors show that, in the case of an adjunction between semi-abelian categories induced by the reflection of a Birkhoff subcategory, the Galois structure determined by such an adjunction allows for the introduction of a notion of *commutator*. The goal of this part of the section is to present a construction similar to that of [28] for the case of MV-algebras. Specifically, given a regular epimorphism $f\colon A \twoheadrightarrow B$, we seek to identify a subalgebra of A such that f is a central morphism if and only if the subalgebra is trivial (which, in our case, means that it is an element of the class $\mathscr{X}$).

Recalling that every MV-algebra is a distributive lattice with respect to the operations defined in 2.1.5, the result presented in Proposition 3.2.12 can be expressed in a different form: a regular epimorphism $f\colon A \twoheadrightarrow B$ is a central extension if and only if $\ker(f) \subseteq \mathrm{Rad}(A)^{\perp}$. Here, for an MV-algebra A and a non-empty subset $S \subseteq A$, we define the set

$$S^{\perp} \coloneqq \{x \in A \,|\, x \wedge s = 0 \text{ for every } s \in S\}.$$

It can be proved that $S^{\perp}$ is an ideal of A.

Lemma 3.2.13. *Consider a morphism $f\colon A \to B$ in $\mathbb{MV}$. Then, $\ker(f) \cap \mathrm{Rad}(A) = \{0\}$ if and only if $K[f] \cap P(A) \in \mathscr{X}$, where $K[f] \coloneqq A$ if $B = \mathbf{1}$, otherwise $K[f]$ is given by the following pullback:*

$$\begin{tikzcd} K[f] \arrow[r] \arrow[d, "k"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{2} \arrow[d, "\iota_B"] \\ A \arrow[r, "f"'] & B. \end{tikzcd}$$

Proof. We notice that $K[f]$ is precisely given by the union of I with $\neg I$, where $I = \ker(f)$. Therefore, we obtain

$$\mathrm{Rad}(K[f]) = \mathrm{Rad}(A) \cap K[f] = (\mathrm{Rad}(A) \cap \ker(f)) \cup (\mathrm{Rad}(A) \cap \neg\ker(f)).$$

If $B \neq \mathbf{1}$, then $1 \notin \mathrm{Rad}(B)$ and so $\mathrm{Rad}(A) \cap \neg\ker(f) = \emptyset$. Hence $\ker(f) \cap \mathrm{Rad}(A) = \{0\}$ if and only if $\mathrm{Rad}(K[f]) = \{0\}$ (which means $P(K[f]) \in \mathscr{X}$). If $B = \mathbf{1}$, then $\ker(f) = A$ and $K[f] = A$. Hence, the equivalence we need to prove simplifies to: $\mathrm{Rad}(A) = \{0\}$ if and only if $P(A) \in \mathscr{X}$; this equivalence always holds, so the statement is trivial in this case. $\square$

To conclude, based on what has been proved so far, we observe that a regular epimorphism

$f\colon A \twoheadrightarrow B$ is central for Γ if and only if $K[f] \cap P(A) \in \mathscr{X}$. It therefore makes sense to define, for a general regular epimorphism f, the following subalgebra of A:

$$[A, K[f]]_{s\mathbb{MV}} \coloneqq K[f] \cap P(A);$$

$[A, K[f]]_{s\mathbb{MV}}$ has the following property: it belongs to $\mathscr{X}$ if and only if f is central.

In the final part of this section, we will focus on the study of the functor $S\colon \mathbb{MV} \to s\mathbb{MV}$. The authors of [26] introduce the concept of a *protoadditive functor*. A functor F between pointed protomodular categories is protoadditive if it preserves split short exact sequences. Moreover, in [27], the authors show that a functor between pointed protomodular categories that preserves the zero object is protoadditive if and only if it preserves pullbacks along split epimorphisms. This characterization enables the extension of the notion of protoadditivity to the non-pointed case.

Definition 3.2.14. *A functor $S\colon \mathbb{C} \to \mathbb{D}$ between protomodular categories that have both a terminal and an initial object is* protoadditive *if it preserves the terminal object, the initial object, and pullbacks along split epimorphisms.*

Additionally, we will prove how the pretorsion theory in $\mathbb{MV}$ studied in this section induces a stable factorization system. These observations will assist us in introducing the topics covered in the next chapter.

Proposition 3.2.15. *The functor $S\colon \mathbb{MV} \to s\mathbb{MV}$ is protoadditive.*

Proof. It is clear that $S(\mathbf{1}) = \mathbf{1} / \mathbf{1} = \mathbf{1}$ and $S(\mathbf{2}) = \mathbf{2} / \{0\} = \mathbf{2}$. Moreover, since $s\mathbb{MV}$ is a full reflective subcategory of $\mathbb{MV}$, we know that $s\mathbb{MV}$ is closed under the formation of limits. We consider, in $\mathbb{MV}$, the pullback of a split epimorphism along an arbitrary morphism

$$\begin{tikzcd} A \times_B C \arrow[r, "\pi_C"] \arrow[d, "\pi_A"'] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[d, "g"] \\ A \arrow[r, shift left, "p"] & B. \arrow[l, shift left, "s"] \end{tikzcd}$$

We observe that

$$\begin{aligned} \mathrm{Rad}(A \times_B C) &= \{(a, c) \in A \times C \mid a \in \mathrm{Rad}(A),\ c \in \mathrm{Rad}(C),\ p(a) = g(c)\} \\ &= \mathrm{Rad}(A \times C) \cap (A \times_B C). \end{aligned}$$

We now proceed to compute the pullback of $S(p)$ along $S(g)$ in $s\mathbb{MV}$

$$\begin{tikzcd}
S(A\times_B C) \arrow[dr, "\exists!\varphi"] \arrow[drr, bend left, "S(\pi_C)"] \arrow[ddr, bend right, "S(\pi_A)"'] & & \\
& S(A)\times_{S(B)} S(C) \arrow[r, "\pi_{S(C)}"] \arrow[d, "\pi_{S(A)}"'] \arrow[dr, phantom, "\lrcorner", very near start] & S(C) \arrow[d, "S(g)"] \\
& S(A) \arrow[r, shift left, "S(p)"] & S(B). \arrow[l, shift left, "S(s)"]
\end{tikzcd}$$

Using the fact that $S(p)S(\pi_A) = S(p\pi_A) = S(g\pi_C) = S(g)S(\pi_C)$, the universal property of this pullback defines an arrow $\varphi\colon S(A\times_B C) \to S(A)\times_{S(B)}S(C)$ such that $\varphi([a,c]) = ([a],[c])$. We will prove that φ is an isomorphism. It is not difficult to see that φ is injective: if $\varphi([a,c]) = ([0],[0])$, then we have $(a,c) \in A\times_B C$, $a \in \mathrm{Rad}(A)$, and $c \in \mathrm{Rad}(C)$. Thus, $(a,c) \in \mathrm{Rad}(A\times C)\cap A\times_B C$, and so $[a,c] = [0,0]$. Now, let us show that φ is surjective. Let $([a],[c]) \in S(A)\times_{S(B)} S(C)$. Since $S(p)([a]) = S(g)([c])$, we have $[p(a)] = [g(c)]$, and so $[sp(a)] = [sg(c)]$. Our goal is to find $a' \in A$ and $c' \in C$ such that $p(a') = g(c')$ and $[a'] = [a]$, $[c'] = [c]$. First, we observe that $(sp(a)\ominus a, 0) \in A\times_B C$ and $(a\ominus sp(a), 0) \in A\times_B C$. By the definition of φ, we get $\varphi([sp(a)\ominus a, 0]) = ([sp(a)\ominus a],[0])$ and $\varphi([a\ominus sp(a),0]) = ([a\ominus sp(a)],[0])$. Additionally, we obtain $(sg(c),c) \in A\times_B C$ and $\varphi([sg(c),c]) = ([sg(c)],[c]) = ([sp(a)],[c])$ (since $[sp(a)] = [sg(c)]$). Let us define $x \coloneqq [a\ominus sp(a),0]$, $y \coloneqq [sp(a)\ominus a, 0]$, and $z \coloneqq [sg(c),c]$, for simplicity. Recalling that

$$(a\ominus sp(a)) \oplus ((a\oplus \neg sp(a))\odot sp(a)) = a,$$

we get

$$\varphi(x\oplus(\neg y\odot z)) = ([a],[c]).$$

Thus, φ is surjective, and since we have already shown that it is injective, we conclude that φ is an isomorphism. ☐

Lemma 3.2.16. *Given a morphism $f\colon A\to B$ in $\mathbb{MV}$ define*

$$\begin{aligned}\overline{f}\colon A/\theta_f &\to B\\ [a] &\mapsto f(a),\end{aligned}$$

where $\theta_f \coloneqq \ker(f)\cap\mathrm{Rad}(A)$ and $\overline{f}$ is well defined since $\theta_f \subseteq \mathrm{Rad}(f)$. Then we have

$$\ker(\overline{f})\cap\mathrm{Rad}(A/\theta_f) = \{0\}.$$

Proof. Fix an element $[a] \in \ker(\overline{f})\cap\mathrm{Rad}(A/\theta_f)$. Since $f(a) = \overline{f}([a]) = 0$, we immediately get $a \in \ker(f)$. We consider the quotient projection $\pi_f\colon A \twoheadrightarrow A/\theta_f$. Given that $[a] = \pi_f(a) \in$

$\mathrm{Rad}(A/\theta_f) = \bigcap\{M \subseteq A/\theta_f \mid M \text{ is a maximal ideal}\}$ we deduce that a belongs to

$$\bigcap\{\pi_f^{-1}(M) \subseteq A \mid M \text{ is a maximal ideal of } A/\theta_f\}.$$

Thanks to Proposition 1.2.10 of [21], we know that π_f^{-1} defines a bijection, which preserves and reflects the order, between $\{I \subseteq A/\theta_f \mid I \text{ is an ideal}\}$ and $\{J \subseteq A \mid J \text{ is an ideal and } J \supseteq \theta_f\}$. We can observe that the maximal elements of the poset of ideals in A are precisely the maximal elements of the poset $\{J \subseteq A \mid J \text{ is an ideal and } J \supseteq \theta_f\}$, as $\theta_f \subseteq \mathrm{Rad}(A)$ and the radical is contained in every maximal ideal. Thus, we have

$$\bigcap\{\pi_f^{-1}(M) \subseteq A \mid M \text{ is a maximal ideal of } A/\theta_f\} =$$
$$\bigcap\{N \subseteq A \mid N \text{ is a maximal ideal of } A \text{ and } N \supseteq \theta_f\} = \mathrm{Rad}(A).$$

Consequently, if $[a] \in \ker(\overline{f}) \cap \mathrm{Rad}(A/\theta_f)$, then $a \in \ker(f) \cap \mathrm{Rad}(A) = \theta_f$, and therefore $[a] = 0$. $\square$

Proposition 3.2.17. *We define the two following classes of arrows in* $\mathbb{MV}$

$$\mathscr{E} := \{e\colon A \to B \in \mathrm{Arr}(\mathbb{MV}) \mid e \textit{ is surjective and } \ker(e) \subseteq \mathrm{Rad}(A)\} \textit{ and}$$
$$\mathscr{M} := \{m\colon A \to B \in \mathrm{Arr}(\mathbb{MV}) \mid \ker(m) \cap \mathrm{Rad}(A) = \{0\}\}.$$

Then the pair $(\mathscr{E}, \mathscr{M})$ *forms a stable factorization system for* $\mathbb{MV}$.

Proof. We start by considering a commutative square

$$\begin{tikzcd} A \arrow[r, "e\in\mathscr{E}"] \arrow[d, "g"'] & B \arrow[d, "h"] \\ C \arrow[r, "m\in\mathscr{M}"'] & D, \end{tikzcd}$$

and given that e is surjective, we assume $B = A/\ker(e)$. We observe that for every element $a \in \ker(e) \subseteq \mathrm{Rad}(A)$, we have $g(a) \in \mathrm{Rad}(C)$ and, furthermore, $mg(a) = he(a) = h(0) = 0$. Therefore, we get $g(a) \in \mathrm{Rad}(C) \cap \ker(m) = \{0\}$ which implies that the arrow $d\colon B \to C$, where $d([a]) := g(a)$, is well defined. Additionally, we can see that $md([a]) = mg(a) = he(a) = h([a])$. Finally, d is unique since e is an epimorphism. Next, we consider an arbitrary arrow $f\colon A \to B$ in $\mathbb{MV}$ and we construct the factorization

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, two heads, "q"'] & & B \\ & A/\ker(f) \cap \mathrm{Rad}(A), \arrow[ur, "i"'] & \end{tikzcd}$$

where q is the quotient projection and $i([a]) := f(a)$. Clearly, since $\ker(q) = \ker(f) \cap \mathrm{Rad}(A) \subseteq$

Rad(A), we have $q \in \mathscr{E}$. Furthermore, thanks to Lemma 3.2.16, we get

$$\ker(i) \cap \mathrm{Rad}(A/\ker(f) \cap \mathrm{Rad}(A)) = \{0\},$$

and so $i \in \mathscr{M}$. To conclude, we need to show that the factorization system is stable. For this, we consider the pullback

$$\begin{tikzcd} A \times_B C \arrow[r, two heads, "\pi_C"] \arrow[d, "\pi_A"'] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[d, "g"] \\ A \arrow[r, two heads, "e \in \mathscr{E}"'] & B \end{tikzcd}$$

and we observe that π_C is surjective, since it is the pullback of a surjective map. Moreover, $\ker(\pi_C) = \{(a, 0) \in A \times C \mid e(a) = 0\}$, and so every element $(a, 0) \in \ker(\pi_C)$ satisfies $a \in \mathrm{Rad}(A)$. Hence we have $\ker(\pi_C) \subseteq \mathrm{Rad}(A \times_I C)$. $\square$

3.3 A Higher Galois Theory for MV-Algebras

In the final part of this chapter, we will explore the higher-order Galois structure determined by the adjunction $S \dashv i$. Additionally, we will use this analysis to define and examine the commutator of ideal subalgebras in relation with this Galois structure. For an MV-algebra $A \neq \mathbf{1}$, a subalgebra $S \subseteq A$ is said to be *ideal* if there is an ideal I of A such that $S = I \cup \neg I$. If $A = \mathbf{1}$, the only subalgebra is $\mathbf{1}$ itself, and we assume it is also ideal. To express the commutator of ideal subalgebras, we only need to consider regular epimorphisms (i.e. surjective maps) of $\mathbb{MV}$ and $s\mathbb{MV}$, as we will see. In particular, to investigate commutators between ideal subalgebras, we need to use the structure $(S, i, |\operatorname{Ext}\mathbb{MV}|, |\operatorname{Ext} s\mathbb{MV}|)$, where $|\operatorname{Ext} s\mathbb{MV}|$ denotes the class of surjective maps of $s\mathbb{MV}$, and $|\operatorname{Ext}\mathbb{MV}|$ denotes the class of surjective maps of $\mathbb{MV}$. Let us begin by observing that this structure satisfies the necessary conditions to be considered an admissible structure in Galois theory. We know that $s\mathbb{MV}$ is a full and reflective subcategory of $\mathbb{MV}$, and thus closed under limits. Therefore, the set of surjective maps of $\mathbb{MV}$ and $s\mathbb{MV}$ are admissible classes of arrows. Additionally, it is evident that for every surjective map f of $\mathbb{MV}$, $S(f)$ is also surjective. Finally, since every component of the unit of the adjunction $S \dashv i$ is surjective (and the composition of surjective maps is surjective), we can conclude that the Galois structure $(S, i, |\operatorname{Ext}\mathbb{MV}|, |\operatorname{Ext} s\mathbb{MV}|)$ is admissible.

Definition 3.3.1 ([12]). *Let $\mathbb{C}$ be a category with pullbacks. A commutative diagram of regular epimorphism*

$$\begin{tikzcd} A \arrow[r, two heads, "g"] \arrow[d, two heads, "f"'] & C \arrow[d, two heads, "k"] \\ B \arrow[r, two heads, "h"'] & D \end{tikzcd}$$

is a regular pushout *if the morphism $\langle f, g\rangle \colon A \to B \times_D C$ defined by the universal property of* *the pullback*

$$\begin{tikzcd} A \arrow[dr, "\langle f,g \rangle"] \arrow[drr, bend left, "g"] \arrow[ddr, bend right, "f"'] & & \\ & B \times_D C \arrow[r, two heads, "\pi_C"] \arrow[d, two heads, "\pi_B"'] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[d, two heads, "k"] \\ & B \arrow[r, two heads, "h"'] & D \end{tikzcd}$$

is a regular epimorphism.

Applying Lemma 2.5.2 we immediately get the following characterization:

Proposition 3.3.2. *A diagram of regular epimorphism*

$$\begin{tikzcd} A \arrow[r, two heads, "f"] \arrow[d, two heads, "h"'] & B \arrow[d, two heads, "k"] \\ C \arrow[r, two heads, "g"'] & D \end{tikzcd}$$

in the category $\mathbb{MV}$ *is a regular pushout if and only if* $h(\ker(f)) = \ker(g)$.

Let $\operatorname{Ext}\mathbb{MV}$ be the category whose objects are the extensions (i.e. regular epimorphisms) of MV-algebras and whose morphisms are the commutative diagrams between them; let $\operatorname{CExt}_{s\mathbb{MV}}\mathbb{MV}$ be the full subcategory of $\operatorname{Ext}\mathbb{MV}$ determined by the central extensions. We define the functor S_1 as follows:

$$\operatorname{Ext}\mathbb{MV} \xrightarrow{\qquad S_1 \qquad} \operatorname{CExt}_{s\mathbb{MV}}\mathbb{MV}$$

$$\begin{tikzcd} A \arrow[r, two heads, "f"] \arrow[d, "h"'] & B \arrow[d, "k"] & & A/\theta_f \arrow[r, two heads, "\overline{f}"] \arrow[d, "\overline{h}"'] & B \arrow[d, "k"] \\ C \arrow[r, two heads, "g"'] & D & \longmapsto & C/\theta_g \arrow[r, two heads, "\overline{g}"'] & D \end{tikzcd}$$

where $\theta_f \coloneqq \ker(f) \cap \operatorname{Rad}(A)$ and $\theta_g \coloneqq \ker(g) \cap \operatorname{Rad}(C)$. Moreover, we define $\overline{f}([a]) \coloneqq f(a)$, $\overline{g}([c]) \coloneqq g(c)$ and $\overline{h}([a]) \coloneqq [h(a)]$. Since $\theta_f \subseteq \ker(f)$ we immediately get that $\overline{f}$ is well defined (in a similar way one can show that also $\overline{g}$ is well defined). Let us consider an element $a \in \theta_f = \ker(f) \cap \operatorname{Rad}(A)$; then $h(a) \in \theta_g = \ker(g) \cap \operatorname{Rad}(C)$ (from $a \in \operatorname{Rad}(A)$ we get $h(a) \in \operatorname{Rad}(C)$ and from $a \in \ker(f)$ we obtain $h(a) \in \ker(g)$ because $gh = kf$); therefore, $\overline{h}$ is well defined. It remains to prove that $\overline{f}\colon A/\theta_f \twoheadrightarrow B$ and $\overline{g}\colon C/\theta_g \twoheadrightarrow D$ are central extensions. In other words, we have to show that

$$\ker(\overline{f}) \cap \operatorname{Rad}(A/\theta_f) = \{0\} \text{ and } \ker(\overline{g}) \cap \operatorname{Rad}(C/\theta_g) = \{0\}.$$

These equalities are a direct consequence of Lemma 3.2.16.

Being $\mathbb{MV}$ a Mal'tsev variety, it was proved in [35] that the inclusion functor $i_1 \colon \mathrm{CExt}_{s\mathbb{MV}}\,\mathbb{MV} \to \mathrm{Ext}\,\mathbb{MV}$ has a left adjoint, which gives rise to an admissible Galois structure. We provide a detailed proof of this result in our context.

Proposition 3.3.3. *The inclusion functor* $i_1 \colon \mathrm{CExt}_{s\mathbb{MV}}\,\mathbb{MV} \to \mathrm{Ext}\,\mathbb{MV}$ *is right adjoint to* S_1.

Proof. To build the unit of the adjunction, denoted by $\eta^1 \colon \mathrm{Id}_{\mathrm{Ext}\,\mathbb{MV}} \to i_1 S_1$, we need to define η^1_f for each object f in $\mathrm{Ext}\,\mathbb{MV}$. Specifically, we define η^1_f as the pair of arrows (π_f, id_B) in $\mathbb{MV}$, where $\pi_f \colon A \to A/\theta_f$ is the quotient projection. In other terms, η^1_f is given by the commutative square

$$\begin{tikzcd} A \arrow[r, two heads, "f"] \arrow[d, two heads, "\pi_f"'] & B \arrow[d, equal] \\ A/\theta_f \arrow[r, "\overline{f}"'] & B. \end{tikzcd}$$

Consider the commutative square in $\mathbb{MV}$ below:

$$\begin{tikzcd} A \arrow[r, two heads, "f"] \arrow[d, "h"'] & B \arrow[d, "k"] \\ C \arrow[r, two heads, "g"'] & D, \end{tikzcd}$$

where g is an object of $\mathrm{CExt}_{s\mathbb{MV}}\,\mathbb{MV}$ (i.e. $\ker(g) \cap \mathrm{Rad}(C) = \{0\}$). We observe that, if $a \in \theta_f = \ker(f) \cap \mathrm{Rad}(A)$, then $gh(a) = kf(a) = 0$ and $h(a) \in \mathrm{Rad}(C)$. Hence, we can define $\overline{h} \colon A/\theta_f \to C$ such that the two squares in the following diagram are commutative:

$$\begin{tikzcd} A \arrow[r, two heads, "f"] \arrow[d, "\pi_f"'] & B \arrow[d, equal] \\ A/\theta_f \arrow[r, two heads, "\overline{f}"'] \arrow[d, "\overline{h}"'] & B \arrow[d, "k"] \\ C \arrow[r, two heads, "g"'] & D. \end{tikzcd}$$

So, we have proved that η^1 satisfies the universal property of unit of an adjunction. $\square$

In the case of MV-algebras, we define the class of arrows $\mathrm{Ext}^2\,\mathbb{MV}$ of $\mathrm{Ext}\,\mathbb{MV}$ as the class of squares which are regular pushouts. Moreover, we introduce the class of arrows $\mathrm{Ext}\,\mathrm{CExt}_{s\mathbb{MV}}\,\mathbb{MV}$ of $\mathrm{CExt}_{s\mathbb{MV}}\,\mathbb{MV}$ as the squares of $\mathrm{Ext}^2\,\mathbb{MV}$ in which the horizontal arrows are central extensions. The purpose of the following results is to establish the admissibility of the Galois structure

$$(S_1, i_1, \mathrm{Ext}^2\,\mathbb{MV}, \mathrm{Ext}\,\mathrm{CExt}_{s\mathbb{MV}}\,\mathbb{MV}).$$

Furthermore, applying Proposition 3.3 of [12] we get that every regular pushout is a regular epimorphism in $\mathrm{Ext}\,\mathbb{MV}$. Finally, thanks to Theorem 2.1 and Example 3.1 of [25], we conclude that every regular pushout in Ext $\mathbb{MV}$ is an effective descent morphism.

Proposition 3.3.4. $\mathrm{Ext}^2\,\mathbb{MV}$ *is an admissible class of arrows.*

Proof. Every isomorphism is in $\mathrm{Ext}^2\,\mathbb{MV}$: we consider a commutative diagram

$$\begin{tikzcd} A \arrow[r, two heads, "f"] \arrow[d, two heads, "h_1"'] & B \arrow[d, two heads, "h_2"] \\ C \arrow[r, two heads, "g"'] & D \end{tikzcd}$$

where h_1 and h_2 are isomorphisms in $\mathbb{MV}$. If we fix an element $c \in \ker(g)$ we get $h_1^{-1}(c) \in \ker(f)$ and so $c \in h_1(\ker(f))$; this implies $\ker(g) = h_1(\ker(f))$.
$\mathrm{Ext}^2\,\mathbb{MV}$ is closed under composition: to see this, we fix the following diagram, where the two squares are commutative, $h_1(\ker(f)) = \ker(g)$, and $k_1(\ker(g)) = \ker(e)$:

$$\begin{tikzcd} A \arrow[r, two heads, "f"] \arrow[d, two heads, "h_1"'] & B \arrow[d, two heads, "h_2"] \\ C \arrow[r, two heads, "g"'] \arrow[d, two heads, "k_1"'] & D \arrow[d, two heads, "k_2"] \\ E \arrow[r, two heads, "e"'] & F. \end{tikzcd}$$

Then, $k_1h_1(\ker(f)) = k_1(\ker(g)) = \ker(e)$; so, (k_1h_1, k_2h_2) is in $\mathrm{Ext}^2\,\mathbb{MV}$.
$\mathrm{Ext}^2\,\mathbb{MV}$ is closed under pullbacks: consider two arrows (h_1, h_2) and (k_1, k_2) in $\mathrm{Ext}^2\,\mathbb{MV}$

$$\begin{tikzcd} A \arrow[r, two heads, "f"] \arrow[d, two heads, "h_1"'] & B \arrow[d, two heads, "h_2"] \\ C \arrow[r, two heads, "g"'] & D \end{tikzcd} \qquad \begin{tikzcd} E \arrow[r, two heads, "e"] \arrow[d, two heads, "k_1"'] & F \arrow[d, two heads, "k_2"] \\ C \arrow[r, two heads, "g"'] & D, \end{tikzcd}$$

the pullback of (h_1, h_2) along (k_1, k_2) is given by the commutative cube:

$$\begin{tikzcd} & B \times_D F \arrow[rr, two heads, "\pi_F"] \arrow[dd, two heads, "\pi_B"] & & F \arrow[dd, two heads, "k_2"] \\ A \times_C E \arrow[ur, "f \times e"] \arrow[rr, two heads, "\pi_E"] \arrow[dd, "\pi_A"'] & & E \arrow[ur, two heads, "e"] \arrow[dd, "k_1"] & \\ & B \arrow[rr, two heads, "h_2"] & & D \\ A \arrow[ur, two heads, "f"] \arrow[rr, two heads, "h_1"'] & & C, \arrow[ur, two heads, "g"'] & \end{tikzcd}$$

where the front face and the back face are pullbacks in $\mathbb{MV}$. We have to prove that (π_A, π_B) is in $\mathrm{Ext}^2\,\mathbb{MV}$. First of all, we show that $f \times e$ is surjective. Fix an element $(x, y) \in B \times_D F$ (i.e. $x \in B$,

$y \in F$, and $h_2(x) = k_2(y)$). Then, there exist $z \in A$ and $w \in E$ such that $f(z) = x$ and $e(w) = y$. We know that $gh_1(z) = h_2 f(z) = k_2 e(w) = gk_1(w)$. So, $h_1(z) \ominus k_1(w) \in \ker(g) = k_1(\ker(e))$ and $k_1(w) \ominus h_1(z) \in \ker(g) = k_1(\ker(e))$. Therefore, there exist $e_1 \in \ker(e)$ and $e_2 \in \ker(e)$ such that $k_1(e) = h_1(z) \ominus k_1(w)$ and $k_1(e_2) = k_1(w) \ominus h_1(z)$. We define $\overline{w} := e_1 \oplus (\neg e_2 \odot w)$; clearly, we have $e(\overline{w}) = 0 \oplus (1 \odot e(w)) = y$. Moreover, $k_1(\overline{w}) = (h_1(z) \ominus k_1(w)) \oplus ((\neg k_1(w) \oplus h_1(z)) \odot k_1(w)) = h_1(z)$ (the last equality holds thanks to Proposition 1.6.2 of [21]). Hence, we deduce that $(z, \overline{w}) \in A \times_C E$ and $(f \times e)(z, \overline{w}) = (x, y)$, and so we conclude that $f \times e$ is surjective.
It remains to prove that

$$\pi_A(\ker(f \times e)) = \ker(f).$$

Observe that

$$\ker(f \times e) = \{(z, w) \in A \times_C E \mid f(z) = 0, e(w) = 0\}.$$

We recall that $h_1(\ker(f)) = \ker(g) = k_1(\ker(e))$. Therefore, given an element $a \in \ker(f)$, there exists an element $w \in \ker(e)$ such that $k_1(w) = h_1(a)$ i.e. $(a, w) \in A \times_C E$. Finally, we notice that $(f \times e)(a, w) = (0, 0)$, and so $a \in \pi_A(\ker(f \times e))$. This implies $\pi_A(\ker(f \times e)) = \ker(f)$. $\square$

Proposition 3.3.5. $\operatorname{Ext} \mathrm{CExt}_{s\mathbb{MV}} \mathbb{MV}$ *is an admissible class of arrows.*

Proof. Using a similar argument to the one presented in the previous proposition, we can conclude that every isomorphism belongs to $\operatorname{Ext} \mathrm{CExt}_{s\mathbb{MV}} \mathbb{MV}$, and that $\operatorname{Ext} \mathrm{CExt}_{s\mathbb{MV}} \mathbb{MV}$ is closed under composition.
To see that $\operatorname{Ext} \mathrm{CExt}_{s\mathbb{MV}} \mathbb{MV}$ is closed under pullbacks consider two arrows (h_1, h_2) and (k_1, k_2) in $\operatorname{Ext} \mathrm{CExt}_{s\mathbb{MV}} \mathbb{MV}$;

$$\begin{tikzcd}
A \arrow[r, two heads, "f"] \arrow[d, two heads, "h_1"'] & B \arrow[d, two heads, "h_2"] & E \arrow[r, two heads, "e"] \arrow[d, two heads, "k_1"'] & F \arrow[d, two heads, "k_2"] \\
C \arrow[r, two heads, "g"'] & D & C \arrow[r, two heads, "g"'] & D.
\end{tikzcd}$$

We recall that $\ker(f) \cap \mathrm{Rad}(A) = \{0\}$, $\ker(g) \cap \mathrm{Rad}(C) = \{0\}$, and $\ker(e) \cap \mathrm{Rad}(E) = \{0\}$ (since f, g, e are central extensions).
The pullback of (h_1, h_2) along (k_1, k_2) is given by the commutative cube:

$$\begin{tikzcd}
& B \times_D F \arrow[rr, two heads, "\pi_F"] \arrow[dd, two heads, "\pi_B"] \arrow[dr, phantom, "\lrcorner", very near start] & & F \arrow[dd, two heads, "k_2"] \\
A \times_C E \arrow[ur, "f \times e"] \arrow[rr, two heads, "\pi_E"'] \arrow[dd, "\pi_A"'] \arrow[dr, phantom, "\lrcorner", very near start] & & E \arrow[ur, two heads, "e"'] \arrow[dd, "k_1", near start] & \\
& B \arrow[rr, two heads, "h_2"', near start] & & D \\
A \arrow[ur, two heads, "f"'] \arrow[rr, two heads, "h_1"'] & & C, \arrow[ur, two heads, "g"'] &
\end{tikzcd}$$

where the front face and the back are pullbacks in $\mathbb{MV}$. Applying what we have seen in the previous proposition, it remains to prove that

$$\ker(f \times e) \cap \mathrm{Rad}(A \times_C E) = \{0\}.$$

We fix an element $(x, y) \in \ker(f \times e) \cap \mathrm{Rad}(A \times_C E)$; we observe that $x \in \mathrm{Rad}(A)$ and $y \in \mathrm{Rad}(C)$. Moreover, recalling that $(x, y) \in \ker(f \times e)$, we obtain $x \in \ker(f) \cap \mathrm{Rad}(A)$ and $y \in \ker(e) \cap \mathrm{Rad}(E)$. This implies $(x, y) = (0, 0)$, and so we conclude that $f \times e$ is a central extension. □

Proposition 3.3.6. *The data* $(S_1, i_1, \mathrm{Ext}^2\,\mathbb{MV}, \mathrm{Ext}\,\mathrm{CExt}_{s\mathbb{MV}}\,\mathbb{MV})$ *determine a relatively admissible adjunction.*

Proof. To show that $S_1(\mathrm{Ext}^2\,\mathbb{MV}) \subseteq \mathrm{Ext}\,\mathrm{CExt}_{s\mathbb{MV}}\,\mathbb{MV}$, consider an arrow (h, k) in $\mathrm{Ext}^2\,\mathbb{MV}$ (the left square below) and its image under S_1, denoted by $S_1(h, k) = (\overline{h}, k)$ (the right square below):

$$\begin{tikzcd} A \arrow[r, two heads, "f"] \arrow[d, two heads, "h"'] & B \arrow[d, two heads, "k"] \\ C \arrow[r, two heads, "g"'] & D \end{tikzcd} \qquad \begin{tikzcd} A/\theta_f \arrow[r, two heads, "\overline{f}"] \arrow[d, "\overline{h}"'] & B \arrow[d, two heads, "k"] \\ C/\theta_g \arrow[r, two heads, "\overline{g}"'] & D. \end{tikzcd}$$

We aim to prove that $(\overline{h}, k) \in \mathrm{Ext}\,\mathrm{CExt}_{s\mathbb{MV}}\,\mathbb{MV}$. We consider an element $[c] \in \ker(\overline{g})$, which implies that $c \in \ker(g)$. Since $h(\ker(f)) = \ker(g)$, there exists an element $a \in \ker(f)$ such that $h(a) = c$. Thus, we have $[a] \in \ker(\overline{f})$ and $\overline{h}([a]) = [c]$, showing that $\overline{h}(\ker(\overline{f})) = \ker(\overline{g})$. Moreover, since i_1 is the inclusion, we have $i_1(\mathrm{Ext}\,\mathrm{CExt}_{s\mathbb{MV}}\,\mathbb{MV}) \subseteq \mathrm{Ext}^2\,\mathbb{MV}$ trivially. Finally, for every object $f\colon A \twoheadrightarrow B$ of $\mathrm{Ext}\,\mathbb{MV}$, the f-component η^1_f of the unit of the adjunction is defined by the following square:

$$\begin{tikzcd} A \arrow[r, two heads, "f"] \arrow[d, two heads, "\pi_f"'] & B \arrow[d, equal] \\ A/\theta_f \arrow[r, "\overline{f}"'] & B. \end{tikzcd}$$

We want to prove that η^1_f is in $\mathrm{Ext}^2\,\mathbb{MV}$: fix an element $[x] \in \ker(\overline{f})$; then $x \in \ker(f)$ and $\pi_f(x) = [x]$ (i.e. $\pi_f(\ker(f)) = \ker(\overline{f})$). Moreover, since i_1 is the inclusion, the counit ε is a natural isomorphism and, therefore, $\varepsilon_g \in \mathrm{Ext}\,\mathrm{CExt}_{s\mathbb{MV}}\,\mathbb{MV}$, for every object g of $\mathrm{CExt}_{s\mathbb{MV}}\,\mathbb{MV}$. □

Proposition 3.3.7. *For every object* $f\colon A \twoheadrightarrow B$ *of* $\mathrm{Ext}\,\mathbb{MV}$ *the counit of the adjunction*

$$S_{1f} \dashv i_{1f}\colon\ \mathrm{Ext}\,\mathrm{CExt}_{s\mathbb{MV}}\,\mathbb{MV}/S_1(f) \to \mathrm{Ext}^2\,\mathbb{MV}/f$$

is a natural isomorphism.

Proof. An object (h,k) of $\operatorname{Ext}\operatorname{CExt}_{s\mathbb{MV}}\mathbb{MV}/S_1(f)$ is a regular pushout of the following form:

$$\begin{tikzcd}
C \arrow[r, two heads, "g"] \arrow[d, two heads, "h"'] & D \arrow[d, two heads, "k"] \\
A/\theta_f \arrow[r, two heads, "\overline{f}"'] & B.
\end{tikzcd}$$

Hence, $i_{1f}(h,k)$ is defined by the following commutative cube in which the front face and the back face are pullbacks in $\mathbb{MV}$:

$$\begin{tikzcd}
& D \arrow[rr, equal] \arrow[dd, two heads, "k"] \arrow[dr, phantom, "\lrcorner", very near start] & & D \arrow[dd, two heads, "k"] \\
P \arrow[ur, two heads, "gp_C"] \arrow[rr, two heads, "p_C" near start] \arrow[dd, two heads, "p_A"'] \arrow[dr, phantom, "\lrcorner", very near start] & & C \arrow[ur, two heads, "g"'] \arrow[dd, two heads, "h" near start] & \\
& B \arrow[rr, equal] & & B \\
A \arrow[ur, two heads, "f"'] \arrow[rr, two heads, "\pi_f"'] & & A/\theta_f; \arrow[ur, two heads, "\overline{f}"'] &
\end{tikzcd}$$

in fact, we have $i_{1f}(h,k) \coloneqq (p_A, k)$. Therefore, the (h,k)-component $\varepsilon_{(h,k)}$ of the counit of the adjunction $S_{1f} \dashv i_{1f}$ is given by the horizontal arrows of the commutative cube

$$\begin{tikzcd}
& D \arrow[rr, equal] \arrow[dd, two heads, "k"] & & D \arrow[dd, two heads, "k"] \\
P/\theta_{gp_C} \arrow[ur, two heads, "\overline{gp_C}"] \arrow[rr, "\overline{p_C}" near end] \arrow[dd, two heads, "\overline{p_A}"'] & & C \arrow[ur, two heads, "g"] \arrow[dd, two heads, "h" near start] & \\
& B \arrow[rr, equal] & & B \\
A/\theta_f \arrow[ur, two heads, "\overline{f}"] \arrow[rr, equal] & & A/\theta_f. \arrow[ur, dash, "\overline{f}"] &
\end{tikzcd}$$

To show that $\overline{p_C}$ is an isomorphism of MV-algebras, we need to prove both injectivity and surjectivity. First, we note that $\overline{p_C}$ is surjective, since p_C is surjective. To prove injectivity, consider an element $[a,c] \in P/\theta_{gp_C}$ such that $\overline{p_C}([a,c]) = 0$. This implies $c = 0$. Since $(a,c) \in P$ and $c = 0$, we have $[a] = \pi_f(a) = h(0) = 0$. Thus, $a \in \theta_f = \ker(f) \cap \operatorname{Rad}(A)$, which leads to $(a,0) \in \operatorname{Rad}(P)$. Furthermore, $gp_C(a,0) = g(0) = 0$, which means that $(a,0) \in \theta_{gp_C}$, and therefore $[a,0] = 0$ in P/θ_{gp_C}. Hence, $\overline{p_C}$ is injective. Finally, note that $(h,k)(\overline{p_C}, id_D) = (\overline{p_A}, k)$, which completes the proof. $\square$

Corollary 3.3.8. *The Galois structure*

$$\Gamma_1 \coloneqq (S_1, i_1, \operatorname{Ext}^2\mathbb{MV}, \operatorname{Ext}\operatorname{CExt}_{s\mathbb{MV}}\mathbb{MV})$$

is admissible.

We are ready to study the central extensions determined by the structure Γ_1.

Proposition 3.3.9. *Consider an element $(h,k) \in \mathrm{Ext}^2\,\mathbb{MV}$*

$$\begin{tikzcd} A \arrow[r, two heads, "f"] \arrow[d, two heads, "h"'] & B \arrow[d, two heads, "k"] \\ C \arrow[r, two heads, "g"'] & D; \end{tikzcd}$$

(h,k) is a normal extension for Γ_1 if and only if $\langle h,f\rangle \colon A \twoheadrightarrow C\times_D B$ is a normal (or central) extension for Γ.

Proof. Let us consider the pullback of (h,k) along (h,k):

$$\begin{tikzcd} & B\times_D B \arrow[rr, two heads, "\pi_B^1"] \arrow[dd, two heads, "\pi_B^1"] \arrow[drr, phantom, "\lrcorner"] & & B \arrow[dd, two heads, "k"] \\ A\times_C A \arrow[ur, two heads, "f\times f"] \arrow[rr, two heads, "\pi_A^2"'] \arrow[dd, "\pi_A^1"'] \arrow[ddrr, phantom, "\lrcorner"] & & A \arrow[ur, two heads, "f"'] \arrow[dd, two heads, "h"] & \\ & B \arrow[rr, two heads, "k"'] & & D \\ A \arrow[ur, two heads, "f"'] \arrow[rr, two heads, "h"'] & & C. \arrow[ur, two heads, "g"'] & \end{tikzcd}$$

We construct the diagram associated with the naturality of η^1:

$$\begin{tikzcd} & B\times_D B \arrow[rr, equal] \arrow[dd, two heads, "\pi_B^1"] & & B\times_D B \arrow[dd, two heads, "k"] \\ A\times_C A \arrow[ur, two heads, "f\times f"] \arrow[rr, two heads, "\pi_{f\times f}"'] \arrow[dd, "\pi_A^1"'] & & A\times_C A/\theta_{f\times f} \arrow[ur, two heads, "\overline{f\times f}"'] \arrow[dd, "\overline{\pi_A^1}"] & \\ & B \arrow[rr, equal] & & B \\ A \arrow[ur, two heads, "f"'] \arrow[rr, two heads, "\pi_f"'] & & A/\theta_f. \arrow[ur, two heads, "\overline{f}"'] & \end{tikzcd}$$

We have to prove that this cube is a pullback (i.e. the front square is a pullback) if and only if $\langle h,f\rangle$ is central for Γ (i.e. $\ker(\langle h,f\rangle)\cap \mathrm{Rad}(A) = \{0\}$).
We know that the square

$$\begin{tikzcd} A\times_C A \arrow[r, two heads, "\pi_{f\times f}"] \arrow[d, "\pi_A^1"'] & A\times_C A/\theta_{f\times f} \arrow[d, "\overline{\pi_A^1}"] \\ A \arrow[r, two heads, "\pi_f"'] & A/\theta_f \end{tikzcd}$$

is a pullback if and only if the restriction $\pi_A^1\colon \theta_{f\times f} \to \theta_f$ is a bijection (thanks to Lemma 2.5.2). We prove that π_A^1 is always surjective. Let $a \in \theta_f = \ker(f) \cap \mathrm{Rad}(A)$; then $(a, a) \in \ker(f \times f) \cap \mathrm{Rad}(A \times A)$, and $\pi_A^1(a, a) = a$. By Lemma 2.1.6, we know that the restriction of π_A^1 is injective if and only if $\ker(\pi_A^1) \cap \theta_{f\times f} = \{0\}$. Specifically, $\ker(\pi_A^1) \cap \theta_{f\times f} = \ker(\pi_A^1) \cap \ker(f \times f) \cap \mathrm{Rad}(A \times_C A) = \{(0, a) \in A \times A \mid f(a) = 0, h(a) = 0, a \in \mathrm{Rad}(A)\}$. It is clear that $\ker(\pi_A^1) \cap \theta_{f\times f} = \{0\}$ if and only if $\ker(f) \cap \ker(h) \cap \mathrm{Rad}(A) = \{0\}$. The last statement is true if and only if $\langle h, f\rangle\colon A \twoheadrightarrow C \times_D B$ is a normal extension for Γ, observing that $\ker(\langle h, f\rangle) = \ker(h) \cap \ker(f)$. $\square$

Proposition 3.3.10. *Consider an element* $(h, k) \in \mathrm{Ext}^2\, \mathbb{MV}$

$$\begin{tikzcd} A \arrow[r, two heads, "f"] \arrow[d, two heads, "h"'] & B \arrow[d, two heads, "k"] \\ C \arrow[r, two heads, "g"'] & D; \end{tikzcd}$$

(h, k) *is a central extension for* Γ_1 *if and only if* (h, k) *is a normal extension for* Γ_1*.*

Proof. Clearly if (h, k) is normal then it is central. Let us prove that also the other implication is true. We consider two objects (h, k) and (h', k') of $\mathrm{Ext}^2\, \mathbb{MV}$, defined by the following squares:

$$\begin{tikzcd} A \arrow[r, two heads, "f"] \arrow[d, two heads, "h"'] & B \arrow[d, two heads, "k"] \\ C \arrow[r, two heads, "g"'] & D \end{tikzcd} \qquad \begin{tikzcd} A' \arrow[r, two heads, "f'"] \arrow[d, two heads, "h'"'] & B' \arrow[d, two heads, "k'"] \\ C \arrow[r, two heads, "g"'] & D. \end{tikzcd}$$

The pullback of (h, k) along (h', k') is given by the commutative cube

$$\begin{tikzcd} & B \times_D B' \arrow[rr, two heads, "\pi_{B'}"] \arrow[dd, two heads, "\pi_B"] & & B' \arrow[dd, two heads, "k'"] \\ A \times_C A' \arrow[ur, two heads, "f\times f'"] \arrow[rr, two heads, "\pi_{A'}"'] \arrow[dd, "\pi_A"'] & & A' \arrow[ur, two heads, "f'"'] \arrow[dd, "h'"] & \\ & B \arrow[rr, two heads, "k"'] & & D \\ A \arrow[ur, two heads, "f"'] \arrow[rr, two heads, "h"'] & & C. \arrow[ur, two heads, "g"'] & \end{tikzcd}$$

We construct the diagram associated with the naturality of η^1:

$$
\begin{tikzcd}
& B \times_D B' \arrow[rr, equals] \arrow[dd, two heads, "\pi_{B'}"] & & B \times_D B' \arrow[dd, two heads, "\pi_{B'}"] \\
A \times_C A' \arrow[ur, two heads, "f \times f'"] \arrow[rr, two heads, "\pi_{f \times f'}"'] \arrow[dd, "\pi_{A'}"'] & & A \times_C A/\theta_{f \times f} \arrow[ur, two heads, "\overline{f \times f'}"'] \arrow[dd, "\overline{\pi_{A'}}"] & \\
& B' \arrow[rr, equals] & & B' \\
A' \arrow[ur, two heads, "f'"'] \arrow[rr, two heads, "\pi_{f'}"'] & & A/\theta_{f'}. \arrow[ur, two heads, "\overline{f'}"'] &
\end{tikzcd}
$$

Now, if the restriction of $\pi_{A'} \colon \theta_{f \times f'} \to \theta_{f'}$ is bijective, then it is injective and so, thanks to Lemma 2.1.6, we obtain $\ker(\pi_{A'}) \cap \ker(f \times f') \cap \mathrm{Rad}(A \times_C A') = \{0\}$. But $\ker(\pi_{A'}) \cap \ker(f \times f') \cap \mathrm{Rad}(A \times_C A') = \{(a, 0) \in A \times A \mid h(a) = 0, f(a) = 0, a \in \mathrm{Rad}(A)\}$. It is clear that $\ker(\pi_{A'}) \cap \ker(f \times f') \cap \mathrm{Rad}(A \times_C A') = \{0\}$ if and only if $\ker(h) \cap \ker(f) \cap \mathrm{Rad}(A) = \{0\}$. The last statement is true if and only if $\langle h, f \rangle \colon A \twoheadrightarrow C \times_D B$ is a normal extension for Γ (this is implied by the fact that $\ker(\langle h, f \rangle) = \ker(h) \cap \ker(f)$). $\square$

Corollary 3.3.11. *Consider an element* $(h, k) \in \mathrm{Ext}^2\,\mathbb{MV}$

$$
\begin{tikzcd}
A \arrow[r, two heads, "f"] \arrow[d, two heads, "h"'] & B \arrow[d, two heads, "k"] \\
C \arrow[r, two heads, "g"'] & D.
\end{tikzcd}
$$

(h, k) *is a central extension for* Γ_1 *if and only if* $\langle h, f \rangle \colon A \twoheadrightarrow C \times_D B$ *is a central extension for* Γ.

We are now prepared to describe the commutator of two ideal subalgebras with respect to the Galois structure that has just been studied. To accomplish this, we require the following results.

Lemma 3.3.12. *Let* $A \neq \mathbf{1}$ *be an MV-algebra and* $I \subseteq A$ *a proper ideal. Then* $\mathrm{Rad}(A) \cap \neg I = \emptyset$.

Proof. Suppose $x \in \mathrm{Rad}(A) \cap \neg I$. Since $x \in \mathrm{Rad}(A)$, we have $x \leq \neg x$. However, since $\neg x \in I$ by assumption, we also have $x \in I$. Thus, we conclude that $1 = x \oplus \neg x \in I$, which is in contradiction with the fact that I is a proper ideal. Therefore, $\mathrm{Rad}(A) \cap \neg I$ must be empty, as claimed. $\square$

Lemma 3.3.13. *Let A be an MV-algebra. Consider three elements $x, y, z \in A$ such that $x \leq y \oplus z$. Then, there exist $y_1, z_1 \in A$ such that $y_1 \leq y$, $z_1 \leq z$, and $x = y_1 \oplus z_1$.*

Proof. By Theorem 3.9 of [44], we know that $A = [0, u]$ for a certain object (G, u) in $u\ell\mathbb{A}b$, where $u\ell\mathbb{A}b$ denotes the category of lattice-ordered abelian groups with order-unit. We recall that in A the operation $\oplus$ is defined as $a \oplus b = (a + b) \wedge u$, where $+$ denotes the addition in (G, u). Given $x, y, z \in A$ such that $x \leq y \oplus z \leq y + z$, we can apply the Riesz decomposition property (see, for example, Theorem 2.1 in [40]) to obtain two elements $0 \leq y_1 \leq y$ and $0 \leq z_1 \leq z$ such that $x = y_1 + z_1$. Finally, we observe that $y_1 \oplus z_1 = (y_1 + z_1) \wedge u = x \wedge u = x = y_1 + z_1$, and thus the statement holds. $\square$

Proposition 3.3.14. *Let A be an MV-algebra and let I and J be two ideals of A. We define $I \oplus J$ to be the set $\{x \in A \mid \text{there exist } i \in I \text{ and } j \in J \text{ such that } x = i \oplus j\}$. Then, we have $I \vee J = I \oplus J$, where $I \vee J$ denotes the join of I and J computed in* Ideals(A).

Proof. To prove that $I \oplus J$ is an ideal of A, it suffices to show that $0 \in I \oplus J$, $I \oplus J$ is closed under $\oplus$, and $I \oplus J$ is downward-closed. Since $0 = 0 \oplus 0$, we have $0 \in I \oplus J$. Moreover, if $x = i_1 \oplus j_1$ and $y = i_2 \oplus j_2$ with $i_1, i_2 \in I$ and $j_1, j_2 \in J$, then $x \oplus y = (i_1 \oplus i_2) \oplus (j_1 \oplus j_2) \in I \oplus J$, which shows that $I \oplus J$ is closed under $\oplus$. Finally, suppose $z \in I \oplus J$ and $w \in A$ such that $w \leq z$. Then there exist $i \in I$ and $j \in J$ such that $z = i \oplus j$. Applying Lemma 3.3.13, we obtain $w = i_w \oplus j_w$ with $i_w \leq i$ and $j_w \leq j$. Therefore $w \in I \oplus J$, which shows that $I \oplus J$ is downward-closed and completes the proof. $\square$

Consider an MV-algebra A, with $A \neq \mathbf{1}$, and fix I and J two proper ideals. Let $M = I \cup \neg I$ and $N = J \cup \neg J$ be the two ideal subalgebras associated with I and J, respectively. We aim to show that

$$M \vee N = K \cup \neg K,$$

where $M \vee N$ denotes the subalgebra generated by M and N, and $K = I \vee J$ (with this join computed in the poset Ideals(A)). In fact, as K contains both I and J, it is clear that $K \cup \neg K \supseteq M$ and $K \cup \neg K \supseteq N$, which implies $K \cup \neg K \supseteq M \vee N$. Moreover, since $M \vee N$ contains both I and J, we have $M \vee N \supseteq I \oplus J = I \vee J$ (the last equality holds applying the previous proposition). As $M \vee N$ is an MV-algebra, we also have $M \vee N \supseteq K \vee \neg K$. Therefore, the equality we need to show holds.

We distinguish two cases. If $I \vee J = A$, then the diagram

$$\begin{tikzcd} A \arrow[r, two heads, "q_J"] \arrow[d, two heads, "q_I"'] & A/J \arrow[d, two heads] \\ A/I \arrow[r, two heads] & \mathbf{1} \end{tikzcd} \tag{$\clubsuit$}$$

is a regular pushout. To show this, we only need to prove that the restriction $q_I \colon J \to A/I$ is surjective. Let $[a] \in A/I$ be an arbitrary element. Since $I \vee J = A$, there exist $i \in I$ and $j \in J$ such that $a = i \oplus j$. Therefore, $[a] = [i \oplus j] = [j] = q_I(j)$. This implies that q_I is surjective, completing the proof. If $I \vee J \neq A$, then K is a proper ideal of A. Hence, it is easy to see that $K \cap \neg K = \emptyset$. Therefore, we define a morphism of MV-algebras $\chi \colon M \vee N = K \cup \neg K \to \mathbf{2}$ by setting $\chi(x) = 1$ if and only if $x \in \neg K$. Since $I \subseteq K$, this morphism induces a morphism of MV-algebras $\xi_1 \colon (M \vee N)/I \to \mathbf{2}$ defined by $\xi_1([x]) = 1$ if and only if $x \in \neg K$. Similarly, we define a morphism of MV-algebras $\xi_2\colon (M \vee N)/J \to \mathbf{2}$ by $\xi_2([x]) = 1$ if and only if $x \in \neg K$. Let us then show that the diagram

$$\begin{tikzcd} M \vee N \arrow[r, two heads, "q_J"] \arrow[d, two heads, "q_I"'] & (M \vee N)/J \arrow[d, two heads, "\xi_2"] \\ (M \vee N)/I \arrow[r, two heads, "\xi_1"'] & \mathbf{2} \end{tikzcd} \qquad (\spadesuit)$$

is a regular pushout. Consider an element $[x] \in (M \vee N)/I$ such that $[x] \in \ker(\xi_1)$, which means that $x \in K$. Therefore, since $K = I \vee J$, there exist $i \in I$ and $j \in J$ such that $x = i \oplus j$. Now observe that $[x] = [i \oplus j] = [j] = q_I(j)$. Hence, the restriction $p_I\colon J \to \ker(\xi_1)$ is surjective.
To conclude, we note that the vertical arrows of both squares are central extensions for the structure Γ_1 if and only if $\mathrm{Rad}(A) \cap I \cap J = \{0\}$ (where we use the fact that $\mathrm{Rad}(M \vee N) = (M \vee N) \cap \mathrm{Rad}(A)$). Applying Proposition 3.3.12, we can see that this condition is equivalent to requiring that

$$P(A) \cap M \cap N \in \mathscr{X}.$$

Therefore, we can define the commutator between M and N with respect to the adjunction $S_1 \dashv i_1$ as

$$[M, N]_{\mathrm{CExt}_{s\mathbb{MV}}\,\mathbb{MV}} \coloneqq P(A) \cap M \cap N;$$

$[M, N]_{\mathrm{CExt}_{s\mathbb{MV}}\,\mathbb{MV}}$ has the following property: it belongs to $\mathscr{X}$ if and only if the vertical arrows of ($\clubsuit$) or ($\spadesuit$) (depending on the join $I \vee J$) are central extensions for Γ_1. Finally, if at least one of the ideals is not proper (for example, if $I = A$), then the study of centrality reduces to analyzing the behavior of the regular epimorphism $q_J\colon A \twoheadrightarrow A/J$ with respect to the Galois structure Γ. Notably, observing that $N = K[q_J]$ and $M = A$, we have that q_J is central if and only if $P(A) \cap A \cap N = P(A) \cap N \in \mathscr{X}$. Therefore, we can define $[A, N]_{\mathrm{CExt}_{s\mathbb{MV}}\,\mathbb{MV}} \coloneqq [A, N]_{s\mathbb{MV}}$. One possible goal for future work is to further investigate the properties of this commutator.

Chapter 4

Protoadditive Functors in a Multipointed Context and Pretorsion Theories

The goal of this chapter is to explore the relationship between pretorsion theories, Galois structures, and stable factorization systems. In particular, we aim to investigate how pretorsion theories, which generalize torsion theories in non-pointed categories, can be used to define Galois structures and stable factorization systems.
There is a well-known correspondence between semi left exact reflections, that are often associated with torsion-free reflections and factorization systems. Indeed, the pair $(\mathscr{E}, \mathscr{M})$ of classes of morphisms, where $\mathscr{E}$ is the class of morphisms inverted by a semi left exact reflector and $\mathscr{M}$ the class of trivial extensions, is a factorization system [20].
In the previous chapter, we observed how the pretorsion theory $(p\mathbb{MV}, s\mathbb{MV})$ in the category of MV-algebras determines a Galois structure and a stable factorization system. Building on this, we aim to generalize these results to non-pointed categories.
To achieve this goal, we begin by revisiting and deepening our understanding of the concepts of prekernel and precokernel. This will be essential in providing a solid foundation for understanding the contents of the chapter.
We then examine how a pretorsion theory can be used to define a Galois structure in a non-pointed category, and investigate the conditions under which such a structure exists. We also explore the correlation between the reflector associated with the pretorsion theory and the central extensions with respect to the induced Galois structure.
Furthermore, we investigate how pretorsion theories determine a stable factorization system in non-pointed categories. So, we highlight the similarities and differences between the pointed and non-pointed cases, and identify the necessary conditions for the existence of such a system.

The purpose of Section 4.1 is to introduce the working context and examine the properties of prekernels and precokernels. We work, in fact, in categories that are, in a sense, similar to the category of MV-algebras. Thus, we prove that prekernels are always present in these categories. However, we also show that the existence of precokernels cannot be guaranteed. This analysis provides the foundation for the next sections.
Section 4.2 is focusing on a series of technical and specialized results in non-pointed categories. These results are crucial for developing a proof of a variation of the Nine Lemma to the non-pointed case.
Section 4.3 is the centerpiece of this chapter, as we investigate the crucial conditions that a pretorsion theory must meet to determine, on the one hand, an admissible Galois structure with respect to the class of all arrows and, on the other hand, a stable factorization system.
In Section 4.4, we present a series of examples of pretorsion theories that satisfy the properties analyzed in the previous sections. To be specific, we will provide examples of such pretorsion theories in the category of M-sets (with M a fixed monoid), in the variety of Heyting algebras, and in the category of simplicial sets. Our goal is to provide concrete examples of how the concepts and conditions discussed in earlier sections can be applied in practice to develop a deeper understanding of pretorsion theories and their applications.

4.1 Framework Analysis: Properties of Prekernels and Precokernels

4.1.1 Framework and Assumptions

In general, our identifications will be up to isomorphism. Therefore, when this will not cause any confusion, we will say that two objects or two morphisms of a certain category are equal when in fact they are simply isomorphic.
Let us establish the framework for this chapter.

Assumption 4.1.1. *We assume to work in a Barr-exact and protomodular category $\mathbb{C}$ with finite colimits. We define $\mathscr{Z}$ as the full replete subcategory whose objects are (modulo isomorphisms) the initial object $\mathbf{2}$ and the terminal object $\mathbf{1}$. We denote the class of arrows of $\mathbb{C}$ factorizing through an object of $\mathscr{Z}$ as $N_{\mathscr{Z}}$.*
Moreover, we require:

- $\mathbf{2} \neq \mathbf{1}$*;*
- *for every object $A \neq \mathbf{1}$, the unique arrow $\iota_A \colon \mathbf{2} \to A$ is a monomorphism;*
- *for every object A, if there exists an arrow $e \colon \mathbf{1} \to A$ then $A = \mathbf{1}$ and $e = id_{\mathbf{1}}$. Therefore there does not exist an arrow $\mathbf{1} \to \mathbf{2}$.*

Remark 4.1.2. *For every A, if there exists an arrow $\xi\colon A \to \mathbf{2}$ then ξ is a split epimorphism (where a section is given by ι_A).*

Examples of categories satisfying these assumptions include the category $\mathbb{B}\mathrm{oole}$ of Boolean algebras, the category $\mathbb{MV}$ of MV-algebras, the category $\mathbb{H}\mathrm{eyt}$ of Heyting algebras, and the category $\mathbb{S}\mathrm{et}^{\mathrm{op}}$.

Proposition 4.1.3. *Let $\mathbb{E}$ be an elementary topos. The following conditions are equivalent:*

i) for every object $A \neq \mathbf{2}$, the unique arrow $\tau_A\colon A \to \mathbf{1}$ is an epimorphism;

ii) $\mathbb{E}$ is two-valued (i.e. $\mathrm{Sub}(\mathbf{1})$ has exactly two elements).

Proof. *i)* $\Rightarrow$ *ii)* Fix a monomorphism $m\colon V \rightarrowtail \mathbf{1}$. There are two possible cases to consider: either $V = \mathbf{2}$ or m is an epimorphism. In the second situation, since $\mathbb{E}$ is balanced, we conclude that m is an isomorphism and so $V = \mathbf{1}$.
ii) $\Rightarrow$ *i)* We construct the (regular epimorphism, monomorphism)-factorization of τ_A

$$\begin{tikzcd} A \arrow[rr, "\tau_A"] \arrow[dr, two heads, bend right, "e"'] & & \mathbf{1} \\ & I. \arrow[ur, tail, bend right, "m"'] & \end{tikzcd}$$

If $I = \mathbf{2}$, since in an elementary topos the initial object is strict, we can conclude that $A = \mathbf{2}$. If $I \neq \mathbf{2}$, then we know from the assumptions that $I = \mathbf{1}$. This implies $\tau_A = e$, and so τ_A is an epimorphism. □

Based on what we have just proved, we can assert that also categories of the form $\mathbb{E}^{op}$, where $\mathbb{E}$ is a two-valued elementary topos, satisfy the conditions presented in Assumption 4.1.1.

From this point on, we assume to work in a category $\mathbb{C}$ satisfying the conditions presented in Assumption 4.1.1.

As a final result in this subsection, we present the following lemma which will be extremely useful in the subsequent sections:

Lemma 4.1.4. *Consider a regular epimorphism $f\colon A \twoheadrightarrow B$ and a morphism $g\colon B \to C$ such that $gf \in N_{\mathscr{Z}}$. Then $g \in N_{\mathscr{Z}}$.*

Proof. If $C = \mathbf{1}$ the statement is trivial. Therefore, let us assume $C \neq \mathbf{1}$. Thus, there exists a morphism $\alpha\colon A \to \mathbf{2}$ such that the following diagram commutes:

$$\begin{tikzcd} A \arrow[r, two heads, "f"] \arrow[dr, bend right, "\alpha"'] & B \arrow[r, "g"] & C \\ & \mathbf{2}. \arrow[ur, tail, bend right, "\iota_C"'] & \end{tikzcd}$$

So, for the diagonal property of the (regular epimorphism, monomorphism)-factorization, there exists a morphism $d\colon B \to \mathbf{2}$ such that $df = \alpha$ and $g = \iota_C d$, which implies $g \in N_{\mathscr{Z}}$. $\square$

4.1.2 Prekernels and Precokernels

As in the previous chapter, we will consider the class of zero objects $\mathscr{Z} = \{\mathbf{1}, \mathbf{2}\}$ fixed. Therefore, when there is no ambiguity, we will use the terms prekernel, precokernel, and pre-exact sequence to refer, respectively, to $\mathscr{Z}$-prekernel, $\mathscr{Z}$-precokernel, and $\mathscr{Z}$-pre-exact sequence.
As a first step, we will show that every arrow in $\mathbb{C}$ admits a prekernel, and we will provide an explicit description of it. Fix an arrow $f\colon A \to B$, suppose $B \neq \mathbf{1}$, and consider the pullback of f along ι_B:

$$\begin{tikzcd} K[f] \arrow[r, "\chi"] \arrow[d, "k"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{2} \arrow[d, "\iota_B"] \\ A \arrow[r, "f"'] & B. \end{tikzcd}$$

We claim that $k\colon K[f] \to A$ is a prekernel of f. In fact, for every commutative diagram of the following form (we know that if $fe \in N_{\mathscr{Z}}$, then it cannot factor through $\mathbf{1}$, since $B \neq \mathbf{1}$):

$$\begin{tikzcd} & \mathbf{2} \arrow[dr, bend left, "\iota_B"] & \\ E \arrow[ur, bend left, "h"] \arrow[r, "e"] & A \arrow[r, "f"] & B, \end{tikzcd}$$

we have a morphism $\varphi\colon E \to K[f]$ such that $k\varphi = e$ induced by the universal property of the pullback

$$\begin{tikzcd} E \arrow[dr, "\varphi"] \arrow[drr, bend left, "h"] \arrow[ddr, bend right, "e"'] & & \\ & K[f] \arrow[r, "\chi"] \arrow[d, "k"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{2} \arrow[d, "\iota_B"] \\ & A \arrow[r, "f"'] & B. \end{tikzcd}$$

This morphism is unique: suppose there exists a morphism ψ such that $k\psi = e$; since k is a monomorphism (it is the pullback of a monomorphism) we get $\varphi = \psi$.
Furthermore, if $B = \mathbf{1}$, it is easy to see that $\operatorname{preker}(\tau_A\colon A \to \mathbf{1}) = id_A$.

Given an arrow f, we denote the domain of the prekernel of f, defined up to unique isomorphism, as $K[f]$.

Let us now state and prove two lemmas that will help us to understand better how prekernels behave with respect to (regular epimorphism, monomorphism)-factorizations and pullbacks.

Lemma 4.1.5. *Consider a morphism $f\colon A \to B$, with $B \neq \mathbf{1}$, and let $k\colon K[f] \to A$ be the* *prekernel of f. Then, we have $k = \mathrm{preker}(g)$, where g is the regular epimorphism of the (regular epimorphism, monomorphism)-factorization of f.*

Proof. Let $A \overset{g}{\twoheadrightarrow} I \overset{i}{\rightarrowtail} B$ be the (regular epimorphism, monomorphism)-factorization of f. We know that $I \neq \mathbf{1}$, since $B \neq \mathbf{1}$. We consider the following diagram:

$$\begin{tikzcd} K[f] \arrow[r] \arrow[d, "k"'] \arrow[dr, phantom, "(1)"] & \mathbf{2} \arrow[r, equal] \arrow[d, tail, "\iota_I"'] \arrow[dr, phantom, "(2)"] & \mathbf{2} \arrow[d, "\iota_B"] \\ A \arrow[r, two heads, "g"'] & I \arrow[r, tail, "i"'] & B. \end{tikzcd}$$

The square (2) is a pullback since the top arrow is an isomorphism and the bottom arrow is a monomorphism. Recalling that (1)+(2) is a pullback by assumption, we can deduce that (1) is also a pullback. Therefore, we obtain $k = \mathrm{preker}(g)$. □

Lemma 4.1.6. *Consider a pullback diagram:*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "h"'] \arrow[dr, phantom, "(1)"] & B \arrow[d, "l"] \\ C \arrow[r, "g"'] & D, \end{tikzcd}$$

with $D \neq \mathbf{1}$. Then, $K[f]$ is equal to $K[g]$.

Proof. We know that $\mathrm{preker}(f)$ is defined by the pullback

$$\begin{tikzcd} K[f] \arrow[r, "\chi"] \arrow[d, "k"'] \arrow[dr, phantom, "(2)"] & \mathbf{2} \arrow[d, "\iota_B"] \\ A \arrow[r, "f"'] & B. \end{tikzcd}$$

Therefore, considering the commutative diagram

$$\begin{tikzcd} K[f] \arrow[r, "\chi"] \arrow[d, "k"'] \arrow[dr, phantom, "(2)"] & \mathbf{2} \arrow[d, "\iota_B"] \\ A \arrow[r, "f"'] \arrow[d, "h"'] \arrow[dr, phantom, "(1)"] & B \arrow[d, "l"] \\ C \arrow[r, "g"'] & D, \end{tikzcd}$$

since (1) and (2) are pullbacks, we get that (1)+(2) is a pullback diagram and so $K[f]$ is equal to $K[g]$. □

Let us now shift our focus to precokernels. It should be noted that, unlike the case of prekernels, the existence of a precokernel for an arrow in $\mathbb{C}$ cannot be guaranteed, in general. To illustrate this, let us suppose that the two projections $\pi_1, \pi_2 \colon \mathbf{2} \times \mathbf{2} \to \mathbf{2}$ are not equal (which precisely means that $\mathbf{2} \times \mathbf{2}$ and $\mathbf{2}$ are not isomorphic). Under this assumption, the precokernel of $id_{\mathbf{2} \times \mathbf{2}}$ does not exist. Indeed, let us suppose the existence of a precokernel $q \colon \mathbf{2} \times \mathbf{2} \to Q$ of $id_{\mathbf{2} \times \mathbf{2}}$. Therefore, considering the diagram below, there are two morphisms $\varphi_1, \varphi_2 \colon Q \to \mathbf{2}$ such that $\varphi_1 q = \pi_1$ and $\varphi_2 q = \pi_2$:

$$\begin{tikzcd}
 & & \mathbf{2} \\
\mathbf{2} \times \mathbf{2} \arrow[rru, bend left, "\pi_1"] \arrow[rrd, bend right, "\pi_2"'] \arrow[r, "id_{\mathbf{2} \times \mathbf{2}}"] & \mathbf{2} \times \mathbf{2} \arrow[ru, "\pi_1"] \arrow[rd, "\pi_2"'] \arrow[r, "q"] & Q \arrow[u, "\varphi_1"'] \arrow[d, "\varphi_2"] \\
 & & \mathbf{2}\,.
\end{tikzcd}$$

So, we have $\langle \varphi_1, \varphi_2 \rangle q = id_{\mathbf{2} \times \mathbf{2}}$, which implies that q is a split monomorphism. However, in general, a precokernel is an epimorphism, hence q is an isomorphism. Therefore, up to isomorphism, we can assume that $q = id_{\mathbf{2} \times \mathbf{2}}$. But, if $id_{\mathbf{2} \times \mathbf{2}} = \text{precoker}(id_{\mathbf{2} \times \mathbf{2}})$, we obtain a factorization of the following form:

$$\begin{tikzcd}
\mathbf{2} \times \mathbf{2} \arrow[rr, "id_{\mathbf{2} \times \mathbf{2}} = \langle \pi_1, \pi_2 \rangle"] \arrow[rd, bend right, "p"'] & & \mathbf{2} \times \mathbf{2} \\
 & \mathbf{2}\,. \arrow[ru, bend right, "\Delta"'] &
\end{tikzcd}$$

Indeed, $id_{\mathbf{2} \times \mathbf{2}}$ cannot factor through $\mathbf{1}$, since this would imply that $\mathbf{2} \times \mathbf{2} = \mathbf{1}$, and thus $\mathbf{2} = \mathbf{1}$, contradicting our assumption on $\mathbb{C}$. Hence, we have $\langle p, p \rangle = \Delta p = \langle \pi_1, \pi_2 \rangle$, which implies $p = \pi_1 = \pi_2$, in contrast with $\pi_1 \neq \pi_2$.

Let us continue by recalling a known and useful result concerning precokernels.

Proposition 4.1.7 ([31], Lemma 4.4)**.** *If p is a precokernel of some morphism, then p is also the precokernel of its prekernel.*

Now, let us present a list of situations in which we investigate the existence of precokernels. If they exist, we provide an explicit description.

- If there exists a unique arrow $\xi \colon A \to \mathbf{2}$, then $\text{precoker}(f \colon A \to B) = p$, where p is defined by the pushout

$$\begin{tikzcd}
A \arrow[r, "\xi"] \arrow[d, "f"'] & \mathbf{2} \arrow[d, "\iota_P"] \\
B \arrow[r, "p"'] \arrow[ru, phantom, "\ulcorner"] & P.
\end{tikzcd}$$

  Indeed, consider an arrow $g \colon B \to C$ such that $gf \in N_{\mathscr{Z}}$. If $C = \mathbf{1}$, we obtain $\tau_P p = g$.

Otherwise, we must have the following factorization:

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[dr, bend right, "\xi"'] & B \arrow[r, "p"] \arrow[dr, "g"] & P \\
& \mathbf{2} \arrow[r, "\iota_C"'] & C.
\end{tikzcd}$$

Therefore, there exists a unique $\varphi\colon P \to C$ such that $\varphi p = g$ defined by the universal property of the pushout

$$\begin{tikzcd}
A \arrow[r, "\xi"] \arrow[d, "f"'] \arrow[dr, phantom, "\ulcorner", very near end] & \mathbf{2} \arrow[d, "\iota_P"] \arrow[ddr, bend left, "\iota_C"] & \\
B \arrow[r, "p"'] \arrow[drr, bend right, "g"'] & P \arrow[dr, dashed, "\exists!\varphi"] & \\
& & C.
\end{tikzcd}$$

From this, we can conclude $p = \operatorname{precoker}(f)$.

- If there does not exist an arrow $A \to \mathbf{2}$, then $\operatorname{precoker}(f\colon A \to B) = \tau_B$. To see this, consider an arrow g such that $gf \in N_{\mathscr{Z}}$. We must have the following factorization of gf:

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[dr, bend right, "\tau_A"'] & B \arrow[r, "\tau_B"] \arrow[dr, "g"] & \mathbf{1} \arrow[d] \\
& \mathbf{1} \arrow[r] & C.
\end{tikzcd}$$

  So, we can immediately deduce that $C = \mathbf{1}$.

- The arrow $\tau_A\colon A \to \mathbf{1}$ is a precokernel if and only if there does not exist an arrow $A \to \mathbf{2}$. Indeed, if τ_A is a precokernel, then it is also the precokernel of its prekernel, which is id_A. Therefore, if there were an arrow $A \to \mathbf{2}$, there would exist an arrow $\mathbf{1} \to \mathbf{2}$, given by

$$\begin{tikzcd}
A \arrow[r, "id_A"] & A \arrow[r, "\tau_A"] \arrow[dr] & \mathbf{1} \arrow[d] \\
& & \mathbf{2},
\end{tikzcd}$$

  which leads to a contradiction. Conversely, if there is no arrow of the form $A \to \mathbf{2}$, then any arrow g in $N_{\mathscr{Z}}$ with domain A must factor through $\mathbf{1}$, and thus τ_A is the precokernel of id_A:

$$\begin{tikzcd}
A \arrow[r, "id_A"] \arrow[dr, bend right] & A \arrow[r, "\tau_A"] \arrow[dr, dashed, "g=\tau_A"] & \mathbf{1} \arrow[d] \\
& \mathbf{1} \arrow[r] & C = \mathbf{1}\,.
\end{tikzcd}$$

- $f\colon A \to B \in N_{\mathscr{Z}}$ if and only if $\operatorname{precoker}(f) = id_B$ (see Lemma 5.4 in [30]).

Proposition 4.1.8. *Let $f\colon A \to B$ be the precokernel of an arrow $g\colon C \to A$, with $B \neq \mathbf{1}$. We know that gf factors as in the following commutative diagram:*

$$\begin{tikzcd}
C \arrow[r, "g"] \arrow[dr, bend right, "\chi"'] & A \arrow[r, "f"] & B \neq \mathbf{1} \\
& \mathbf{2}\,. \arrow[ur, bend right, "\iota_B"'] &
\end{tikzcd}$$

Then, f can be obtained as the pushout of χ along g.

Proof. We consider the pushout of g along χ and the arrow θ, induced by the universal property, making the following diagram commute:

$$\begin{tikzcd}
C \arrow[r, "g"] \arrow[d, "\chi"'] & A \arrow[d, "q"] \arrow[ddr, bend left, "f"] & \\
\mathbf{2} \arrow[r, "\iota_Q"'] \arrow[drr, bend right, "\iota_B"'] \arrow[ur, phantom, "\ulcorner"] & Q \arrow[dr, dashed, "\exists!\theta"] & \\
& & B.
\end{tikzcd}$$

Then we obtain an arrow γ, making the diagram below commute, determined by the fact that $qg \in N_{\mathscr{Z}}$ and $f = \text{precoker}(g)$:

$$\begin{tikzcd}
& & Q & \\
C \arrow[r, "g"] \arrow[dr, bend right, "\chi"'] & A \arrow[ur, bend left, "q"] \arrow[rr, "f"] & & B \neq \mathbf{1} \arrow[ul, bend right, "\exists!\gamma"'] \\
& \mathbf{2}\,. \arrow[urr, bend right, "\iota_B"'] & &
\end{tikzcd}$$

Finally, we observe that that $\theta\gamma f = f$ and $\gamma\theta q = q$. Since both f and q are epimorphisms (the former is a precokernel and therefore an epimorphism, while the latter is the pushout of the split epimorphism χ), we can deduce that $\theta\gamma = id_B$ and $\gamma\theta = id_Q$. $\square$

Corollary 4.1.9. *Consider a precokernel $f\colon A \to B$, with $B \neq \mathbf{1}$, and its prekernel $k\colon K[f] \to A$. Then the pullback*

$$\begin{tikzcd}
K[f] \arrow[r, "h"] \arrow[d, "k"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{2} \arrow[d, "\iota_B"] \\
A \arrow[r, "f"'] & B
\end{tikzcd}$$

is also a pushout.

Corollary 4.1.10. *Every precokernel $f\colon A \to B$, with $B \neq \mathbf{1}$, is a regular epimorphism.*

Proof. Every precokernel with a codomain different from $\mathbf{1}$ is a pushout of a split epimorphism, and hence a regular epimorphism. $\square$

Proposition 4.1.11. *Consider an arrow $f\colon A \to B$ with prekernel $k\colon K[f] \to A$, and suppose that there exists a unique morphism $\chi\colon K[f] \to \mathbf{2}$. Then we have $k = \operatorname{preker}(q)$, where $q = \operatorname{precoker}(k)$.*

Proof. Let us suppose $B \neq \mathbf{1}$. Then k is defined by the pullback

$$\begin{tikzcd}
K[f] \arrow[r, "\chi"] \arrow[d, "k"'] \arrow[dr, phantom, "\lrcorner", very near start] \arrow[dr, phantom, "(1)"] & \mathbf{2} \arrow[d, "\iota_B"] \\
A \arrow[r, "f"'] & B
\end{tikzcd}$$

Moreover, since there exists a unique morphism $\chi\colon K[f] \to \mathbf{2}$, q is given by the pushout

$$\begin{tikzcd}
K[f] \arrow[r, "\chi"] \arrow[d, "k"'] \arrow[dr, phantom, "(2)"] \arrow[dr, phantom, "\ulcorner", very near end] & \mathbf{2} \arrow[d, "\iota_Q"] \arrow[ddr, bend left, "\iota_B"] & \\
A \arrow[r, "q"'] \arrow[drr, bend right, "f"'] & Q \arrow[dr, dashed, "\exists!\gamma"] & \\
& & B
\end{tikzcd}$$

and, additionally, this pushout induces an arrow $\gamma\colon Q \to B$ such that $\gamma q = f$ (and, of course, $\gamma\iota_Q = \iota_B$). To prove that the square (2) is also a pullback, consider two arrows $\alpha\colon X \to A$ and $\beta\colon X \to \mathbf{2}$ such that $q\alpha = \iota_Q\beta$. Then, we have $f\alpha = \gamma q\alpha = \gamma\iota_Q\beta = \iota_B\beta$. By the universal property of the pullback (1), there exists a unique morphism $\varphi\colon X \to K[f]$ such that $k\varphi = \alpha$ and $\chi\varphi = \beta$. Therefore, the square (2) is also a pullback. If $B = \mathbf{1}$ we have $k = id_A$ and, moreover, the following diagram is both a pullback and a pushout:

$$\begin{tikzcd}
A \arrow[r, "\chi"] \arrow[d, equal] \arrow[dr, phantom, "\lrcorner", very near start] \arrow[dr, phantom, "\ulcorner", very near end] & \mathbf{2} \arrow[d, equal] \\
A \arrow[r, "\chi"'] & \mathbf{2};
\end{tikzcd}$$

hence $\operatorname{precoker}(id_A) = \chi$ and $\operatorname{preker}(\chi) = id_A$. $\square$

4.2 Some Remarks on the Nine Lemma for Pre-Exact Sequences

The Nine Lemma (or 3×3 Lemma) is a fundamental tool in homological algebra that is used to establish exactness of sequences in various contexts. In [10] it is proved that this result holds in any quasi-pointed (the unique arrow from the initial object to the terminal object is a monomorphism), regular, and protomodular category. In the present section, we provide an alternative version of the Nine Lemma that is suitable for our non-pointed case (in which, in general, the morphism $\mathbf{2} \to \mathbf{1}$ is not a monomorphism). To derive this, we adapt to our context some results obtained in [10]. We begin by recalling the following:

Lemma 4.2.1. *Let $\mathbb{C}$ be a regular category. If (1) and (2) are commutative squares, such that (1) and (1)+(2) are pullbacks and f' is a regular epimorphism, then (2) is a pullback:*

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "a"'] \arrow[dr, phantom, "(1)"] & B \arrow[r, "g"] \arrow[d, "b"] \arrow[dr, phantom, "(2)"] & C \arrow[d, "c"] \\
A' \arrow[r, two heads, "f'"'] & B' \arrow[r, "g'"'] & C'.
\end{tikzcd}$$

The purpose of this section is to prove a version of the Nine Lemma that holds in a category satisfying the conditions stated in Assumption 4.1.1. Therefore, from now on, we will assume that $\mathbb{C}$ satisfies these conditions.

We present a set of auxiliary lemmas and propositions that are adaptions of similar known results in the pointed context. These findings will be essential in proving the final theorem within this section.

Proposition 4.2.2. *Consider a commutative diagram as below:*

$$\begin{tikzcd}
K \arrow[r, "k"] \arrow[d, "u"'] & A \arrow[r, two heads, "f"] \arrow[d, "a"] & B \arrow[d, "b"] \\
K' \arrow[r, "k'"'] & A' \arrow[r, two heads, "f'"'] & B',
\end{tikzcd}$$

where f, f' are regular epimorphisms, $k = \mathrm{preker}(f)$, and $k' = \mathrm{preker}(f')$. If u and b are isomorphisms, then a is an isomorphism.

Proof. If $B = \mathbf{1}$, the result is trivial. Therefore, let us assume that $B \neq \mathbf{1}$. Consider the following commutative diagram:

$$\begin{tikzcd}
K \arrow[rr] \arrow[dr, "k"] \arrow[dd, "u"'] & & \mathbf{2} \arrow[dr] \arrow[dd, equal] & \\
& A \arrow[rr, "f"] \arrow[dd, "a"] & & B \arrow[dd, "b"] \\
K' \arrow[rr] \arrow[dr, "k'"'] & & \mathbf{2} \arrow[dr] & \\
& A' \arrow[rr, "f'"'] & & B',
\end{tikzcd}$$

where, by assumption, the top face, the bottom face, and the back face are pullbacks. Then, the diagram defined by the top face and the front face is a pullback. Therefore, we obtain the commutative diagram

$$\begin{tikzcd}
K \arrow[r, "k"] \arrow[d] \arrow[dr, phantom, "(1)"] & A \arrow[r, "a"] \arrow[d, two heads, "f"] \arrow[dr, phantom, "(2)"] & A' \arrow[d, "f'"] \\
\mathbf{2} \arrow[r] & B \arrow[r, "b"'] & B',
\end{tikzcd}$$

where both (1) and (1)+(2) are pullbacks and f is a regular epimorphism. By protomodularity, we conclude that (2) is a pullback, too. We can deduce that a is an isomorphism, as pullback of the isomorphism b. $\square$

Lemma 4.2.3. *Consider a commutative diagram as below:*

$$\begin{tikzcd}
K \arrow[r, "k"] \arrow[d, "u"'] \arrow[dr, phantom, "(1)"] & A \arrow[r, two heads, "f"] \arrow[d, "a"] & B \arrow[d, "b"] \\
K' \arrow[r, "k'"'] & A' \arrow[r, "f'"'] & B',
\end{tikzcd}$$

where f is a regular epimorphism, $k = \mathrm{preker}(f)$, and $k' = \mathrm{preker}(f')$. Furthermore, we require that if $B \neq \mathbf{1}$ then also $B' \neq \mathbf{1}$. Under these assumptions, (1) *is a pullback if and only if b is a monomorphism.*

Proof. Suppose $B \neq \mathbf{1}$ and $B' \neq \mathbf{1}$, and consider the commutative cube

$$\begin{tikzcd}
K \arrow[rr, "k"] \arrow[dr] \arrow[dd, "u"'] & & A \arrow[dr, "f"] \arrow[dd, "a"] & \\
& \mathbf{2} \arrow[rr] \arrow[dd, equal] & & B \arrow[dd, "b"] \\
K' \arrow[rr, "k'"] \arrow[dr] & & A' \arrow[dr, "f'"] & \\
& \mathbf{2} \arrow[rr] & & B',
\end{tikzcd}$$

where the top face and the bottom face are pullbacks. If (1) is a pullback, then the diagram defined by the top face and the front face is a pullback, too. Therefore, in the commutative diagram

$$\begin{tikzcd}
K \arrow[r] \arrow[d, "k"'] \arrow[dr, phantom, "(i)"] & \mathbf{2} \arrow[r, equal] \arrow[d] \arrow[dr, phantom, "(ii)"] & \mathbf{2} \arrow[d] \\
A \arrow[r, two heads, "f"'] & B \arrow[r, "b"'] & B'
\end{tikzcd}$$

we have that (i)+(ii) is a pullback, (i) is a pullback and f is a regular epimorphism. Then, thanks to Lemma 4.2.1, we obtain that (ii) is a pullback, as well. Since in a protomodular category pullbacks reflect monomorphisms, we can conclude that b is a monomorphism.

If we assume that b is a monomorphism, we can deduce that the front face in the cube above is a pullback. Therefore, the diagram defined by the back face and the bottom face forms a pullback.

Since the bottom face is already a pullback, we can conclude that the back face is a pullback, too. In other words, (1) is a pullback.
If $B = \mathbf{1}$, then also $B' = \mathbf{1}$. In this case, b is an isomorphism, $k = id_A$, and $k' = id_{A'}$. So (1) is trivially a pullback. $\square$

It should be noted that the previous result is not true, in general, in the case where $B' = \mathbf{1}$ and $B \neq \mathbf{1}$. This can be illustrated with a counterexample given by the commutative diagram below (in $\mathbb{MV}$):

$$\begin{tikzcd} \mathbf{2} \arrow[r, equal] \arrow[d] \arrow[dr, phantom, "(1)"] & \mathbf{2} \arrow[r, equal] \arrow[d] & \mathbf{2} \arrow[d, "b"] \\ \mathbf{1} \arrow[r, equal] & \mathbf{1} \arrow[r, equal] & \mathbf{1}\,. \end{tikzcd}$$

In fact, (1) is a pullback but b is not a monomorphism.

Lemma 4.2.4. *Consider a commutative diagram as below:*

$$\begin{tikzcd} K \arrow[r, "k"] \arrow[d, two heads, "u"'] & A \arrow[r, two heads, "f"] \arrow[d, "a"'] & B \arrow[d, two heads, "b"] \\ K' \arrow[r, "k'"'] & A' \arrow[r, two heads, "f'"'] & B', \end{tikzcd}$$

where f, f' are regular epimorphisms, $k = \text{preker}(f)$, and $k' = \text{preker}(f')$. If u and b are regular epimorphisms, then a is a regular epimorphism.

Proof. Let a be factored as $a = ip$, where p is a regular epimorphism and i is a monomorphism. Consider the following commutative diagram:

$$\begin{tikzcd} K \arrow[r, "k"] \arrow[d, two heads, "u"'] & A \arrow[r, two heads, "f"] \arrow[d, two heads, "p"'] & B \arrow[d, two heads, "b"'] \\ K' \arrow[r, "j"] \arrow[d, equal] & \mathrm{Im}(a) \arrow[r, "f'i"] \arrow[d, tail, "i"'] & B' \arrow[d, equal] \\ K' \arrow[r, "k'"'] & A' \arrow[r, two heads, "f'"'] & B', \end{tikzcd}$$

where j is determined by the fact that u is a strong epimorphism and i is a monomorphism. If $B' \neq \mathbf{1}$, then $j = \text{preker}(f'i)$; in fact, in the diagram below the two squares are pullbacks:

$$\begin{tikzcd} K' \arrow[r, equal] \arrow[d, "j"'] & K' \arrow[r] \arrow[d, "k'"] & \mathbf{2} \arrow[d] \\ \mathrm{Im}(a) \arrow[r, tail, "i"'] & A' \arrow[r, two heads, "f'"'] & B', \end{tikzcd}$$

and thus the whole rectangle is also a pullback, too. Furthermore, we observe that f' is a regular epimorphism, and so, applying Proposition 4.2.2 to the bottom rectangle of the first diagram in this proof, we can deduce that i is an isomorphism. Hence, a is a regular epimorphism. If $B' = \mathbf{1}$, then k' is an isomorphism. Thus, $k'u = ipk$ is a regular epimorphism, and so i is a regular epimorphism. Since i is also a monomorphism, it must be an isomorphism. Therefore, a is a regular epimorphism. $\square$

Lemma 4.2.5. *Consider a commutative diagram as below:*

$$\begin{tikzcd}
 & A \arrow[r, "f"] \arrow[d, "a"'] & B \arrow[d, "b"] \\
K' \arrow[r, "k'"] \arrow[d, two heads, "u"'] & A' \arrow[r, two heads, "f'"] \arrow[d, two heads, "a'"'] & B' \arrow[d, two heads, "b'"] \\
K'' \arrow[r, "k''"'] & A'' \arrow[r, two heads, "f''"'] & B'',
\end{tikzcd}$$

where a', b', f', f'', u are regular epimorphisms, $a = \mathrm{preker}(a')$, $b = \mathrm{preker}(b')$, $k' = \mathrm{preker}(f')$ and $k'' = \mathrm{preker}(f'')$. Furthermore, suppose $B' \neq \mathbf{1}$, and $B'' \neq \mathbf{1}$. Under these assumptions, f is a regular epimorphism.

Proof. Consider the commutative diagram

$$\begin{tikzcd}
K' \arrow[r, "k'"] \arrow[d, two heads, "u"'] & A' \arrow[r, two heads, "f'"] \arrow[d, "\langle a', f' \rangle"'] & B' \arrow[d, equal] \\
K'' \arrow[r, "\langle k'', \iota_{B'}\alpha \rangle"'] & A'' \times_{B''} B' \arrow[r, two heads, "\pi_{B'}"'] & B'
\end{tikzcd} \qquad (\blacklozenge)$$

where α is the upper arrow in the following pullback square:

$$\begin{tikzcd}
K'' \arrow[r, "\alpha"] \arrow[d, "k''"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{2} \arrow[d, "\iota_{B''}"] \\
A'' \arrow[r, "f''"'] & B''.
\end{tikzcd}$$

In order to prove the commutativity of ($\blacklozenge$), we have to show that $f'k' = \iota_{B'}\alpha u$. We know that $f'k' \in N_{\mathscr{Z}}$, and so we obtain $f'k' = \iota_{B'}\chi$ (where $\chi \colon K' \to \mathbf{2}$). We observe that $b'f'k' = f''k''u = \iota_{B''}\alpha u$; moreover, we notice that $b'f'k' = b'\iota_{B'}\chi = \iota_{B''}\chi$. Since $\iota_{B''}$ is a monomorphism, we deduce $\alpha u = \chi$, and so $\iota_{B'}\alpha u = \iota_{B'}\chi = f'k'$. We prove that $\langle k'', \iota_{B'}\alpha \rangle$ is the prekernel of $\pi_{B'}$.

To see this, we notice that in the diagram

$$\begin{tikzcd}
K'' \arrow[r, "\alpha"] \arrow[d, "{\langle k'', \iota_{B'}\alpha \rangle}"] \arrow[dd, bend right=60, "k''"'] & \mathbf{2} \arrow[d, "\iota_{B'}"] \arrow[dd, bend left=60, "\iota_{B''}"] \\
A'' \times_{B''} B' \arrow[r, two heads, "\pi_{B'}"'] \arrow[d, two heads, "\pi_{A''}"'] & B' \arrow[d, "b'"] \\
A'' \arrow[r, "f''"'] & B''
\end{tikzcd}$$

the rectangle and the bottom square are pullbacks, hence the top square is a pullback, too. If u is a regular epimorphism, then by Lemma 4.2.4 we can deduce that $\langle a', f' \rangle$ is a regular epimorphism. Since $B \neq \mathbf{1}$ and $B' \neq \mathbf{1}$, it follows that $A \neq \mathbf{1}$ and $A' \neq \mathbf{1}$. We can then construct the commutative diagram:

$$\begin{tikzcd}
A \arrow[r, "a"] \arrow[d, "f"'] & A' \arrow[r, two heads, "a'"] \arrow[d, "{\langle a', f' \rangle}"'] & A'' \arrow[d, equal] \\
B \arrow[r, "{\langle \iota_{A''}\beta, b \rangle}"'] & A'' \times_{B''} B' \arrow[r, two heads, "\pi_{A''}"'] & A''
\end{tikzcd} \qquad (\star)$$

where β is the upper arrow in the following pullback square:

$$\begin{tikzcd}
B \arrow[r, "\beta"] \arrow[d, "b"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{2} \arrow[d, "\iota_{B''}"] \\
B' \arrow[r, "b'"'] & B'';
\end{tikzcd}$$

the commutativity of $(\star)$ can be proved using an argument very similar to the one exhibited for ($\blacklozenge$). We observe that $\langle \iota_{A''}\beta, b \rangle$ is the prekernel of $\pi_{A''}$. In fact, we have the rectangle

$$\begin{tikzcd}
B \arrow[r, "{\langle \iota_{A''}\beta, b \rangle}"] \arrow[d, "\beta"'] \arrow[dr, phantom, "(1)"] & A'' \times_{B''} B' \arrow[r, two heads, "\pi_{B'}"] \arrow[d, two heads, "\pi_{A''}"'] \arrow[dr, phantom, "(2)"] & B' \arrow[d, "b'"] \\
\mathbf{2} \arrow[r] & A'' \arrow[r, two heads, "f''"'] & B''
\end{tikzcd}$$

where (1) and (1)+(2) are pullbacks, and so also (2) is a pullback, too. Hence, applying Lemma 4.2.3 to $(\star)$, we obtain that the left square of $(\star)$ is a pullback, since $id_{A''}$ is a monomorphism. So, by regularity, we conclude that f is a regular epimorphism. $\square$

Theorem 4.2.6. *Consider a commutative diagram as below:*

$$\begin{tikzcd}
K \arrow[r, "k"] \arrow[d, "u"'] \arrow[dr, phantom, "(1)"] & A \arrow[r, two heads, "f"] \arrow[d, "a"'] & B \arrow[d, "b"] \\
K' \arrow[r, "k'"] \arrow[d, two heads, "u'"'] & A' \arrow[r, two heads, "f'"] \arrow[d, two heads, "a'"'] & B' \arrow[d, "b'"] \\
K'' \arrow[r, "k''"'] & A'' \arrow[r, two heads, "f''"'] & B'',
\end{tikzcd}$$

where f, f', f'', u', a' are regular epimorphisms and $u = \operatorname{preker}(u')$, $a = \operatorname{preker}(a')$, $k = \operatorname{preker}(f)$, $k' = \operatorname{preker}(f')$, and $k'' = \operatorname{preker}(f'')$. Furthermore, suppose $B' \neq \mathbf{1}$ and $B'' \neq \mathbf{1}$. Then, b' is a regular epimorphism and $b = \operatorname{preker}(b')$.

Proof. We deduce that b' is a regular epimorphism from the fact that $b'f' = f''a'$ and $f''a'$ is regular epimorphism. Since $B'' \neq \mathbf{1}$, we know that A'' and K'' are not terminal either. Using Lemma 4.2.3, we obtain that (1) is a pullback, noting that k'' is a monomorphism. Since (1) is a pullback and $B' \neq \mathbf{1}$, we can apply Lemma 4.2.3 again to conclude that b is a monomorphism. We define $h\colon K[b'] \to B'$ as the prekernel of b'. Then, we consider the following commutative diagram:

$$\begin{tikzcd}
 & & & B \arrow[dl, "l"] \arrow[ddl, bend left, "b"] \\
K \arrow[r, "k"] \arrow[d, "u"'] & A \arrow[r, two heads, "g"] \arrow[d, "a"'] \arrow[urr, bend left, two heads, "f"] & K[b'] \arrow[d, "h"] & \\
K' \arrow[r, "k'"'] & A' \arrow[r, two heads, "f'"'] & B' \arrow[d, "b'"] & \\
 & & B'', &
\end{tikzcd}$$

where g is determined by the fact that $b'f' = f''a'$, and so $b'f'a = f''a'a \in N_{\mathscr{Z}}$. Moreover, observing that $b'bf = f''a'a \in N_{\mathscr{Z}}$ and applying Lemma 4.1.4, we obtain that $b'b \in N_{\mathscr{Z}}$. Hence, there exists a unique arrow l such that $hl = b$. We show that $lf = g$. Recalling that $hlf = bf = f'a = hg$ and that h is a monomorphism, our claim follows. Thanks to Lemma 4.2.5 we observe that g is a regular epimorphism, so l is a regular epimorphism, too. Furthermore, since b is a monomorphism, we get that l is a monomorphism. Thus, l is an isomorphism, which completes the proof. $\square$

It is important to point out that the aforementioned result does not hold, in general, when $B' = \mathbf{1}$

and $B \neq \mathbf{1}$. To see this, consider the commutative diagram in $\mathbb{MV}$

$$\begin{tikzcd}
\mathbf{2} \arrow[r, equal] \arrow[d, equal] & \mathbf{2} \arrow[r, equal] \arrow[d, equal] & \mathbf{2} \arrow[d] \\
\mathbf{2} \arrow[r, equal] \arrow[d, equal] & \mathbf{2} \arrow[r] \arrow[d, equal] & \mathbf{1} \arrow[d, equal] \\
\mathbf{2} \arrow[r, equal] & \mathbf{2} \arrow[r] & \mathbf{1}
\end{tikzcd}$$

and observe that the unique arrow $\mathbf{2} \to \mathbf{1}$ is not the prekernel of $id_{\mathbf{1}}$.

This version of the Nine Lemma is connected to the one explored in [34]. Fixed an ideal of arrows N, the authors recall the notion of a *star* (i.e. an ordered pair of parallel morphisms $(k_1, k_2) \colon K \rightrightarrows A$ such that $k_1 \in N$) and they define the *star-kernel* of an arrow $f \colon A \to B$ as a universal star with respect to the property that $fk_1 = fk_2$. Moreover, they introduce the notion of *star-exact sequence* as a diagram

$$\begin{tikzcd}
K \arrow[r, shift left, "k_1"] \arrow[r, shift right, "k_2"'] & A \arrow[r, two heads, "f"] & B,
\end{tikzcd}$$

where (k_1, k_2) is the star-kernel of f and $f = \mathrm{coeq}(k_1, k_2)$.
In our context, for every regular epimorphism $f \colon A \twoheadrightarrow B$ with $B \neq \mathbf{1}$, the sequence

$$\begin{tikzcd}
K[f] \arrow[r, shift left, "\iota_A t"] \arrow[r, shift right, "k"'] & A \arrow[r, two heads, "f"] & B
\end{tikzcd}$$

is star-exact with respect to the ideal $N_{\mathscr{Z}}$, where the arrows in the diagram above are defined by the following pullback:

$$\begin{tikzcd}
K[f] \arrow[r, two heads, "t"] \arrow[d, tail, "k"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{2} \arrow[d, tail, "\iota_B"] \\
A \arrow[r, two heads] & B.
\end{tikzcd}$$

Hence, the sequences in the diagram of Theorem 4.2 can be considered as star-exact sequences in the sense of [34]. However, our specific version of the Nine Lemma does not directly derive from the results presented in that paper. This is due to the fact that their study assumes the presence of enough trivial objects, which requires closure of the class of zero objects under squares. In contrast, our context lacks this property (for instance, in general, $\mathbf{2} \times \mathbf{2} \notin \mathscr{Z}$).

4.3 Pretorsion Theories, Galois Structures and Factorization Systems

The purpose of the initial part of this section is to present a preliminary investigation about the pretorsion theories $(\mathscr{T}, \mathscr{F})$, such that $\mathscr{T} \cap \mathscr{F} = \mathscr{Z} = \{\mathbf{1}, \mathbf{2}\}$, on a category $\mathbb{C}$ that satisfies the properties presented in Assumption 4.1.1. The exploration aims to provide a foundational understanding of the key concepts and results that will be used in the subsequent sections of this chapter. Let $\mathscr{T}$ and $\mathscr{F}$ be two full replete subcategory of $\mathbb{C}$. We recall that $(\mathscr{T}, \mathscr{F})$ is a pretorsion theory if every arrow that starts from an object in $\mathscr{T}$ and ends in an object in $\mathscr{F}$ factors through an object of $\mathscr{Z}$. Moreover, every object in $\mathbb{C}$ is in the middle of a pre-exact sequence such that the left endpoint is a torsion object, and the right endpoint is a torsion-free object. We denote by F the reflector on the category of torsion-free objects, and by T the coreflector on the category of torsion objects.

Remark 4.3.1. *Since the sequences*

$$\mathbf{2} = \mathbf{2} = \mathbf{2},$$

$$\mathbf{1} = \mathbf{1} = \mathbf{1}$$

are both pre-exact, and $\mathscr{T} \cap \mathscr{F} = \{\mathbf{1}, \mathbf{2}\}$, we can assume $F(\mathbf{2}) = \mathbf{2}$, $F(\mathbf{1}) = \mathbf{1}$, $T(\mathbf{2}) = \mathbf{2}$, and $T(\mathbf{1}) = \mathbf{1}$.

Given an object X in $\mathbb{C}$, the $(\mathscr{T}, \mathscr{F})$-pre-exact sequence associated with X is the unique pre-exact sequence, up to isomorphism,

$$T(X) \longrightarrow X \longrightarrow F(X)$$

such that $T(X) \in \mathscr{T}$ and $F(X) \in \mathscr{F}$.

Proposition 4.3.2. *Consider a pretorsion theory $(\mathscr{T}, \mathscr{F})$ on $\mathbb{C}$. For every object $T \in \mathscr{T}$ there exists at most one arrow $T \to \mathbf{2}$.*

Proof. We fix an object $T \in \mathscr{T}$. First of all, let us show that $F(T) \in \mathscr{Z}$. Consider the $(\mathscr{T}, \mathscr{F})$-pre-exact sequence associated with T:

$$\begin{tikzcd} T_1 \arrow[r, "\varepsilon_T"] & T \arrow[rr, "\eta_T"] \arrow[dr, "\chi"'] & & F_1 \\ & & Z. \arrow[ur] & \end{tikzcd}$$

Observe that $\eta_T \in N_{\mathscr{Z}}$ (since its domain is an object of $\mathscr{T}$ and its codomain an object of $\mathscr{F}$); then η_T factors as above, where $Z \in \mathscr{Z}$. If $Z = \mathbf{1}$, then $F_1 = \mathbf{1}$ and, since $\eta_T = \tau_T$ is a precokernel, we can deduce that there does not exist an arrow $T \to \mathbf{2}$. In fact, suppose we have an arrow $r\colon T \to \mathbf{2}$; then $r\varepsilon_T \in N_{\mathscr{Z}}$. Therefore, there must exist an arrow $\mathbf{1} \to \mathbf{2}$, which leads to a contradiction. If $Z = \mathbf{2}$ the (regular epimorphism, monomorphism)-factorization of η_T is given by:

$$\begin{tikzcd} T \arrow[rr, "\eta_T"] \arrow[dr, bend right, two heads, "\chi"'] & & F_1 \\ & \mathbf{2}; \arrow[ur, bend right, tail, "\iota_{F_1}"'] & \end{tikzcd}$$

applying Lemma 4.1.5, we get $\varepsilon_T = \operatorname{preker}(\chi)$. Hence ε_T is equal to id_T. Moreover, we know that $\eta_T = \operatorname{precoker}(id_T)$ and $\eta_T id_T = \iota_{F_1}\chi$; then η_T is given by the pushout

$$\begin{tikzcd} T \arrow[r, equal] \arrow[d, "\chi"'] & T \arrow[d, "\chi"] \\ \mathbf{2} \arrow[r, equal] \arrow[ur, phantom, "\ulcorner", very near end] & \mathbf{2}\,. \end{tikzcd}$$

So, we obtain that the $(\mathscr{T}, \mathscr{F})$-pre-exact sequence associated with T is

$$\begin{tikzcd} T \arrow[r, equal] & T \arrow[r, "\chi"] & \mathbf{2}\,. \end{tikzcd}$$

In conclusion, if we consider an arrow $r\colon T \to \mathbf{2}$, since $r \in N_{\mathscr{Z}}$, there exists an arrow $\varphi\colon \mathbf{2} \to \mathbf{2}$ such that $\varphi\chi = r$. However, because $\mathbf{2}$ is the initial object, it follows that $\varphi = id_{\mathbf{2}}$, and so $r = \chi$. $\square$

Proposition 4.3.3. *For a pretorsion theory $(\mathscr{T}, \mathscr{F})$ on $\mathbb{C}$, the following statements are equivalent:*

i) for every object $A \in \mathbb{C}$, $F(A) = \mathbf{1}$ if and only if $A = \mathbf{1}$;

ii) for every object $T \in \mathscr{T}$ with $T \neq \mathbf{1}$, there exists a unique arrow $T \to \mathbf{2}$.

Proof. $i) \Rightarrow ii)$ We can apply the same argument as in the previous proof. In fact, the only scenario where we lack an arrow $T \to \mathbf{2}$ is when $F(T) = \mathbf{1}$. This case only occurs when $T = \mathbf{1}$, since, by $i)$, we have $F(T) = \mathbf{1}$ if and only if $T = \mathbf{1}$.
$ii) \Rightarrow i)$ Let $A \in \mathbb{C}$ be a fixed object. If $A = \mathbf{1}$, then the existence of the arrow $\eta_A\colon \mathbf{1} \to F(A)$ implies that $F(A) = \mathbf{1}$. If $F(A) = \mathbf{1}$, then $\eta_A = \tau_A\colon A \to \mathbf{1}$ and $\varepsilon_A = id_A$. Observing that τ_A is a precokernel, we conclude that there does not exist an arrow $A \to \mathbf{2}$. Moreover, as $\varepsilon_A = id_A$, we get $A \in \mathscr{T}$ and so $A = \mathbf{1}$. $\square$

Definition 4.3.4. *We say that a pretorsion theory satisfies* condition (U) *if it fulfills the equivalent conditions outlined in the aforementioned proposition.*

Proposition 4.3.5. *For every object A of $\mathbb{C}$, if there exists an arrow $\chi\colon T(A) \to \mathbf{2}$ then $T(A \times \mathbf{2}) \cong T(A)$.*

Proof. First, we observe that $\mathscr{T}$ has products. This follows from the fact that T is the right adjoint of the inclusion functor $i_{\mathscr{T}}\colon \mathscr{T} \hookrightarrow \mathbb{C}$. For any pair of objects $T_1, T_2 \in \mathscr{T}$, we have $T(i_{\mathscr{T}}(T_1) \times i_{\mathscr{T}}(T_2)) = Ti_{\mathscr{T}}(T_1)\overline{\times}Ti_{\mathscr{T}}(T_2) = T_1\overline{\times}T_2$, where $\overline{\times}$ denotes the product in the subcategory $\mathscr{T}$. Note that the product in $\mathscr{T}$ may differ from the product in the larger category $\mathbb{C}$. So, we need to prove that $T(A)\overline{\times}\,\mathbf{2} \cong T(A)$. Let χ be the unique morphism from $T(A)$ to $\mathbf{2}$. Consider the following commutative diagrams:

$$\begin{tikzcd} T(A) \arrow[r, "\langle id_{T(A)},\chi\rangle"] \arrow[rr, bend right, "id_{T(A)}"'] & T(A)\overline{\times}\,\mathbf{2} \arrow[r, "\pi_{T(A)}"] & T(A) \end{tikzcd}$$

$$\begin{tikzcd} T(A)\overline{\times}\,\mathbf{2} \arrow[r, "\pi_{T(A)}"] \arrow[rr, bend right, "\langle \pi_{T(A)},\chi\pi_{T(A)}\rangle"'] & T(A) \arrow[r, "\langle id_{T(A)},\chi\rangle"] & T(A)\overline{\times}\,\mathbf{2}; \end{tikzcd}$$

recalling that there exists a unique arrow $T(A)\overline{\times}\,\mathbf{2} \to \mathbf{2}$ and, since $\pi_{\mathbf{2}}, \chi\pi_{T(A)}\colon T(A)\overline{\times}\,\mathbf{2} \to \mathbf{2}$, we deduce $\pi_{\mathbf{2}} = \chi\pi_{T(A)}$, so $\langle \pi_{T(A)}, \chi\pi_{T(A)}\rangle = \langle \pi_{T(A)}, \pi_{\mathbf{2}}\rangle = id_{T(A)\overline{\times}\,\mathbf{2}}$. Hence $\langle id_{T(A)}, \chi\rangle$ is an isomorphism. □

Proposition 4.3.6. *Let A be an object of $\mathbb{C}$. If there is no arrow of the form $T(A) \to \mathbf{2}$, then $A \in \mathscr{T}$.*

Proof. We consider the $(\mathscr{T},\mathscr{F})$-pre-exact sequence associated with A and the factorization of $\eta_A\varepsilon_A$:

$$\begin{tikzcd} T(A) \arrow[r, "\varepsilon_A"] \arrow[dr, bend right] & A \arrow[r, "\eta_A"] & F(A) \\ & Z. \arrow[ur, bend right] & \end{tikzcd}$$

We observe that Z must be $\mathbf{1}$, and so $F(A) = \mathbf{1}$. Therefore, we obtain $A = T(A) \in \mathscr{T}$. □

Lemma 4.3.7. *Let $e\colon A \to B$ be a precokernel with $B \neq \mathbf{1}$, such that $K[e] \in \mathscr{T}$. Then, for every arrow $g\colon C \to B$, the pullback $\overline{e}\colon P \to C$ of e along g is a precokernel and $K[\overline{e}] \in \mathscr{T}$.*

Proof. Let us say that the prekernel of e is $k\colon K[e] \to A$ and the prekernel of $\overline{e}$ is $k'\colon K[\overline{e}] \to P$. Thanks to Lemma 4.1.6, we deduce that $K[\overline{e}] = K[e] \in \mathscr{T}$. Furthermore, applying Proposition 4.1.8, we conclude that e is the pushout of a split epimorphism, and so it is a regular epimorphism. Therefore, $\overline{e}$ is a regular epimorphism as pullback of e. We observe that, since k is the prekernel of an arrow with domain different from $\mathbf{1}$ and $K[e] = K[\overline{e}]$, there exists an arrow $t\colon K[\overline{e}] \to \mathbf{2}$. Moreover, this arrow is unique because $K[\overline{e}]$ is in $\mathscr{T}$. In order to prove that $\overline{e}$ is a precokernel, we examine the diagram

$$\begin{tikzcd} K[\overline{e}] \arrow[r, "k'"] \arrow[d, two heads, "t"'] \arrow[dr, phantom, "\lrcorner"] & P \arrow[d, two heads, "\overline{e}"] \\ \mathbf{2} \arrow[r, "\iota_C"'] & C; \end{tikzcd}$$

applying Proposition 14 of [6], we immediately get that the square above is also a pushout. Finally, since there exists a unique arrow $t\colon K[\overline{e}] \to \mathbf{2}$, we conclude $\overline{e} = \text{precoker}(k')$. $\square$

As we can infer from the previous results and as we will see later, the existence of objects $T \neq \mathbf{1} \in \mathscr{T}$ such that $F(T) = \mathbf{1}$ can cause issues. In order to ensure that some of the desired results hold, we need to impose specific constraints on these objects. In particular, we will require the pretorsion theory to satisfy two particular conditions that are related to this class of torsion objects. We will introduce one of them now, while the other will be presented later when needed.

Definition 4.3.8. *We say that a pretorsion theory satisfies* condition (P1) *if, for every pair of objects $A, B \in \mathbb{C}$,*

$$F(B) = \mathbf{1} \ \textit{implies}\ F(A \times B) \cong F(A).$$

Condition (U) clearly implies condition (P1) because, if (U) holds, then $F(B) = \mathbf{1}$ implies $B = \mathbf{1}$. However, the converse is not true in general, as we will show with an example later on.

As mentioned in the introduction, the goal of this chapter is to explore the connection between pretorsion theories, Galois theory, and stable factorization systems in the non-pointed context. Specifically, the results presented here are a variation of those introduced in Section 3 of [27], adapted to the non-pointed case.

Lemma 4.3.9. *Let $(\mathscr{T}, \mathscr{F})$ be a pretorsion theory satisfying condition (U). Consider an object A of $\mathbb{C}$ and the $(\mathscr{T}, \mathscr{F})$-pre-exact sequence associated with A:*

$$T(A) \xrightarrow{\varepsilon_A} A \xrightarrow{\eta_A} F(A).$$

In this sequence, the arrow η_A is a regular epimorphism.

Proof. Consider an object A in the category $\mathbb{C}$. If $F(A) = \mathbf{1}$ then, by our condition (U), we have $A = \mathbf{1}$. Therefore, $\eta_{\mathbf{1}} = id_{\mathbf{1}}$ which is a regular epimorphism. If $A \neq \mathbf{1}$, we can apply Corollary 4.1.10. $\square$

Proposition 4.3.10. *Let $(\mathscr{T}, \mathscr{F})$ be a pretorsion theory satisfying condition (U), and let $\overline{\mathbb{C}}$ represent the class of regular epimorphisms in $\mathbb{C}$ and $\overline{\mathscr{F}}$ represent the class of regular epimorphisms in $\mathscr{F}$. In this context, the adjunction $F \dashv i_{\mathscr{F}}$ is relatively admissible with respect to these classes.*

Proof. The components of the counit and of the unit of the adjunction are, respectively, isomorphisms and regular epimorphisms (see Lemma 4.3.9). Moreover, F is a left adjoint and so preserves colimits: in particular $F(\overline{\mathscr{F}}) \subseteq \overline{\mathbb{C}}$. Finally, we consider the following diagram in $\mathscr{F}$:

$$A \underset{g}{\overset{f}{\rightrightarrows}} B \xrightarrow{q} Q,$$

where $q = \mathrm{coeq}_{\mathscr{F}}(f, g)$. To clarify, we use the symbol $\mathrm{coeq}_{\mathscr{F}}$ to denote the coequalizer computed in $\mathscr{F}$, and we use the symbol $\mathrm{coeq}_{\mathbb{C}}$ to denote the coequalizer computed in $\mathbb{C}$. Consider the coequalizer $q' = \mathrm{coeq}_{\mathbb{C}}(f, g)$ in $\mathbb{C}$ and observe that, since F preserves colimits, we have $F(q') = q$. Thanks to the naturality of the unit η, we obtain $\eta_{Q'} q' = F(q') = q$. This implies that q can be presented in $\mathbb{C}$ as the composite of two regular epimorphisms. Since $\mathbb{C}$ is a Barr-exact category, the composition of two regular epimorphisms is a regular epimorphism, too. Therefore, q is a regular epimorphism in $\mathbb{C}$. □

Proposition 4.3.11. *Let $(\mathscr{T}, \mathscr{F})$ be a pretorsion theory satisfying condition (P1). For every object B in $\mathbb{C}$, the counit of the adjunction $F_B \dashv i_{\mathscr{F}B} : \mathscr{F}/F(B) \to \mathbb{C}/B$ is an isomorphism.*

Proof. First of all, we recall that $i_{\mathscr{F}B}(f\colon A \to F(B)) = (f'\colon A' \to B)$, where f' is defined by the pullback

$$\begin{tikzcd} A' \arrow[r, "f''"] \arrow[d, "f'"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "f"] \\ B \arrow[r, "\eta_B"'] & F(B). \end{tikzcd}$$

So, we have to prove that

$$F_B(f') = (F(f')\colon F(A') \to F(B)) \cong (f\colon A \to F(B)).$$

Suppose $B = \mathbf{1}$; then the following square is a pullback:

$$\begin{tikzcd} A \arrow[r, equal] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d] \\ \mathbf{1} \arrow[r, equal] & \mathbf{1}, \end{tikzcd}$$

and so the assertion holds. Now, if $B \neq \mathbf{1}$ and $F(B) = \mathbf{1}$, we consider the pullback

$$\begin{tikzcd} A \times B \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d] \\ B \arrow[r] & \mathbf{1}\,. \end{tikzcd}$$

Then, since condition (P1) holds by assumption, we have $F(A \times B) \cong F(A)$. Finally, suppose $F(B) \neq \mathbf{1}$ and consider the commutative diagram:

$$\begin{tikzcd} T(A') \arrow[r, "\varepsilon_{A'}"] & A' \arrow[rr, "\eta_{A'}"] \arrow[dr, "f''"'] & & F(A') \arrow[dl, "\exists! f'''"] \\ & & A, & \end{tikzcd}$$

where f''' is induced by the fact that $f''\varepsilon_{A'} \in N_{\mathscr{Z}}$, as its domain is a torsion object and its codomain is torsion-free. We prove that f''' is an isomorphism. Applying Lemma 4.3.7, we immediately get that f'' is a precokernel and $K[f''] = K[\eta_B] = T(B)$. Thus, due to the uniqueness of the $(\mathscr{T}, \mathscr{F})$-pre-exact sequence associated with A', since $K[f''] \in \mathscr{T}$ and $A \in \mathscr{F}$, we conclude that f''' is an isomorphism:

$$\begin{tikzcd} T(A') \arrow[r, "\varepsilon_{A'}"] \arrow[d, "\wr"'] & A' \arrow[r, "\eta_{A'}"] \arrow[d, equal] & F(A') \arrow[d, "\wr"', "f'''"] \\ K[f''] \arrow[r, "k"'] & A' \arrow[r, "f''"'] & A. \end{tikzcd}$$

Finally, we have to show that the following diagram is commutative:

$$\begin{tikzcd} F(A') \arrow[r, "F(f')"] \arrow[d, "f'''"'] & F(B) \arrow[d, equal] \\ A \arrow[r, "f"'] & F(B). \end{tikzcd}$$

We note that $ff'''\eta_{A'} = ff'' = \eta_B f' = F(f')\eta_{A'}$ (where the last equality follows from the fact that η is a natural transformation). Therefore, since $\eta_{A'}$ is an epimorphism (it is a precokernel), we get $ff''' = F(f')$. □

We have just observed that a pretorsion theory $(\mathscr{T}, \mathscr{F})$ satisfying (P1) defines a precokernel-reflective subcategory (i.e. a reflective subcategory with the property that each component of the unit is a precokernel) $\mathscr{F}$ of $\mathbb{C}$. Moreover, the reflector $F\colon \mathbb{C} \to \mathscr{F}$ satisfies $F(\mathbf{1}) = \mathbf{1}$, and $F(f^*(\eta_B))$ is an isomorphism for every object $B \in \mathbb{C}$ and every arrow f in $\mathscr{F}$. Thanks to the next proposition that we are about to state, we can claim the converse, too.

Proposition 4.3.12. *Consider a precokernel-reflective subcategory $\mathscr{F}$ of $\mathbb{C}$ such that the reflector $F\colon \mathbb{C} \to \mathscr{F}$ satisfies $F(\mathbf{1}) = \mathbf{1}$, and $F(f^*(\eta_B))$ is an isomorphism for every object $B \in \mathbb{C}$ and for every arrow f in $\mathscr{F}$. Then, $\mathscr{F}$ is the torsion-free part of a pretorsion theory $(\mathscr{T}, \mathscr{F})$ which satisfies (P1) and such that $\mathscr{F} \cap \mathscr{T} = \mathscr{Z}$.*

Proof. We define the full subcategory of $\mathbb{C}$ whose objects are

$$\mathscr{T} \coloneqq \{T \in \mathbb{C} \,|\, T = K[\eta_X] \text{ for some object } X \in \mathbb{C}\}.$$

To determine $\mathscr{F} \cap \mathscr{T}$, let us consider an object $Z \in \mathscr{F} \cap \mathscr{T}$. Since $Z = K[\eta_X]$, we know that it is defined by the pullback

$$\begin{tikzcd} Z \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Z' \arrow[d, "\chi"] \\ X \arrow[r, "\eta_X"'] & F(X), \end{tikzcd}$$

where $Z' = \mathbf{1}$ if $F(X) = \mathbf{1}$, and $Z' = \mathbf{2}$ if $F(X) \neq \mathbf{1}$. Recalling that $F(\chi^*(\eta_X))$ is an isomorphism and $F(Z) = Z$ (since $Z \in \mathscr{F}$), we obtain $Z = F(Z') = Z'$. Moreover, we have $F(\mathbf{1}) = \mathbf{1}$, by assumption, and $F(\mathbf{2}) = \mathbf{2}$, since F preserves coproducts. So, we get $Z = \mathbf{1}$ or $Z = \mathbf{2}$, i.e.

$\mathscr{F} \cap \mathscr{T} = \mathscr{Z}$. Now, we show that $F(T) \in \mathscr{Z}$ for every $T \in \mathscr{T}$. Since $T = K[\eta_X]$, T is given by a pullback of the form

$$\begin{tikzcd} T \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Z \arrow[d, "\chi"] \\ X \arrow[r, "\eta_X"'] & F(X), \end{tikzcd}$$

where $Z \in \mathscr{Z}$. As before, we can conclude $F(T) = Z \in \mathscr{Z}$. In order to prove that every arrow $f\colon T \to F$ (with $T \in \mathscr{T}$ and $F \in \mathscr{F}$) is an element of $N_{\mathscr{Z}}$, consider the naturality square of η below:

$$\begin{tikzcd} T \arrow[r, "\eta_T"] \arrow[d, "f"'] & F(T) \in \mathscr{Z} \arrow[d, "F(f)"] \\ F \arrow[r, equal] & F. \end{tikzcd}$$

This diagram tells us that $f = F(f)\eta_T \in N_{\mathscr{Z}}$. Furthermore, given an object $X \in \mathbb{C}$, we observe that the $(\mathscr{T}, \mathscr{F})$-pre-exact sequence associated with X is given by

$$T(X) = K[\eta_X] \xrightarrow{\varepsilon_X} X \xrightarrow{\eta_X} F(X),$$

where $\varepsilon_X = \text{preker}(\eta_X)$. Finally, we notice that $(\mathscr{T}, \mathscr{F})$ satisfies (P1). Suppose $F(X) = \mathbf{1}$; then $\eta_X = \tau_X$ and so, applying F to the top arrow of the pullback

$$\begin{tikzcd} X \times Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d] \\ X \arrow[r, "\tau_X"'] & \mathbf{1}, \end{tikzcd}$$

we deduce $F(Y \times X) = F(Y)$. □

Thanks to what has been shown so far, we are finally ready to describe, in terms of the pretorsion theory $(\mathscr{T}, \mathscr{F})$, the normal and central extensions determined by the Galois structure associated with the reflector F.

Theorem 4.3.13. *Consider a pretorsion theory $(\mathscr{T}, \mathscr{F})$ satisfying condition (P1), where F is protoadditive (i.e., $F(\mathbf{1}) = \mathbf{1}$, $F(\mathbf{2}) = \mathbf{2}$, and F preserves the pullback of split epimorphisms along every morphism). Suppose $f\colon A \to B$ is an effective descent morphism (in our case, a regular epimorphism), and let $\Gamma_{\mathscr{F}}$ be the Galois structure associated with the reflector F. Then, the following conditions are equivalent:*

i) f is a normal extension for $\Gamma_{\mathscr{F}}$;

ii) f is a central extension for $\Gamma_{\mathscr{F}}$;

iii) $K[f] \in \mathscr{F}$ (where $K[f]$ denotes the domain, up to isomorphism, of the prekernel of f).

Proof. *i)* $\Rightarrow$ *ii)* This implication is always true.
ii) $\Rightarrow$ *iii)* Let $p\colon E \to B$ be an effective descent morphism such that the morphism $p^*(f)$, defined by the pullback below, is a trivial extension:

$$\begin{tikzcd}
P \arrow[r] \arrow[d, "p^*(f)"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "f"] \\
E \arrow[r, "p"'] & B.
\end{tikzcd}$$

Suppose $B \neq \mathbf{1}$. This implies $E \neq \mathbf{1}$, too. Consider the commutative cube

$$\begin{tikzcd}
& K[p^*(f)] \arrow[rr] \arrow[dl] \arrow[dd] & & K[f] \arrow[dl] \arrow[dd] \\
P \arrow[rr, crossing over] \arrow[dd, "p^*(f)"'] & & A & \\
& \mathbf{2} \arrow[rr, equal] \arrow[dl] & & \mathbf{2} \arrow[dl] \\
E \arrow[rr, "p"'] & & B, \arrow[from=uu, crossing over, "f"] &
\end{tikzcd}$$

in which the front face and the two lateral faces are pullbacks. Therefore, also the back face is a pullback, and hence $K[p^*(f)] \cong K[f]$. By assumption, $p^*(f)$ is a trivial extension, then the square

$$\begin{tikzcd}
P \arrow[r, "\eta_P"] \arrow[d, "p^*(F)"'] \arrow[dr, phantom, "\lrcorner", very near start] & F(P) \arrow[d, "F(p^*(F))"] \\
E \arrow[r, "\eta_E"'] & F(E)
\end{tikzcd}$$

is a pullback. If $F(E) \neq \mathbf{1}$, applying the same argument seen before, we deduce $K[F(p^*(F))] \cong K[p^*(F)] \cong K[f]$. Hence, since $\mathscr{F}$ is closed under limits in $\mathbb{C}$, we obtain $K[F(p^*(F))] \in \mathscr{F}$. Thus $K[f] \in \mathscr{F}$. If $F(E) = \mathbf{1}$, consider the following commutative cube:

$$\begin{tikzcd}
& K[p^*(f)] \arrow[rr] \arrow[dl] \arrow[dd] & & F(P) \arrow[dl, equal] \arrow[dd] \\
P \arrow[rr, crossing over] \arrow[dd, "p^*(f)"'] & & F(P) & \\
& \mathbf{2} \arrow[rr] \arrow[dl] & & \mathbf{1} \arrow[dl, equal] \\
E \arrow[rr] & & \mathbf{1}, \arrow[from=uu, crossing over] &
\end{tikzcd}$$

where the front face and the two lateral faces are pullbacks. Therefore, the back face is a pullback, too. This implies $K[p^*(f)] \cong \mathbf{2} \times F(P) \in \mathscr{F}$ (since $\mathscr{F}$ is closed under products) and so $K[f] \cong K[p^*(f)] \in \mathscr{F}$.

If $B = \mathbf{1}$ and $E = \mathbf{1}$ the statement follows trivially.

Finally, if $B = \mathbf{1}$ and $E \neq \mathbf{1}$, consider the commutative cube

$$\begin{tikzcd}
& \mathbf{2}\times A \arrow[rr, "\pi_A"] \arrow[dl, "\iota_E\times id_A"'] \arrow[dd, "\pi_{\mathbf{2}}"] & & A \arrow[dl, equal] \arrow[dd] \\
E\times A \arrow[rr, "\pi'_A"] \arrow[dd, "\pi'_E"'] & & A \arrow[dd] & \\
& \mathbf{2} \arrow[rr] \arrow[dl, "\iota_E"] & & \mathbf{1} \arrow[dl, equal] \\
E \arrow[rr] & & \mathbf{1}\,. &
\end{tikzcd}$$

Using a similar reasoning to what was proposed in the first part of the proof, we deduce $\mathbf{2}\times A \in \mathscr{F}$. Hence, we have $T(\mathbf{2}\times A) \in \mathscr{Z}$. Now, if there exists an arrow $T(A) \to \mathbf{2}$, applying Proposition 4.3.5, we get $T(A) = T(\mathbf{2}\times A) \in \mathscr{Z}$ and then $A \in \mathscr{F}$. If there does not exist an arrow $T(A) \to \mathbf{2}$, then, as seen Proposition 4.3.6, we have $A \in \mathscr{T}$ and $F(A) = \mathbf{1}$. Hence $f = \eta_A = \tau_A$. So, since f is central, there exists a regular epimorphism $\tau_E \colon E \twoheadrightarrow \mathbf{1}$ such that $\pi_E \colon E \times A \to E$ is trivial (observe that π_E is the pullback of f along τ_E). Given that $F(A) = \mathbf{1}$ and condition (P1) holds, we obtain $F(E \times A) = F(E)$. Moreover, because π_E is trivial, the following diagram is a pullback:

$$\begin{tikzcd}
E\times A \arrow[r, "\eta_{E\times A}"] \arrow[d, "\pi_E"'] \arrow[dr, phantom, "\lrcorner", very near start] & F(E\times A) = F(E) \arrow[d, "F(\pi_E)=id_E"] \\
E \arrow[r, "\eta_E"'] & F(E).
\end{tikzcd}$$

Noting that π_E is an isomorphism, as a pullback of an isomorphism, we can suppose $E \times A = E$ and $\pi_E = id_E$. In conclusion, considering the diagram

$$\begin{tikzcd}
E \arrow[r] \arrow[d, equal] \arrow[dr, phantom, "\lrcorner\ (2)", very near start] & A \arrow[r] \arrow[d, "\eta_A"] \arrow[dr, phantom, "\lrcorner\ (1)", very near start] & \mathbf{1} \arrow[d, equal] \\
E \arrow[r, two heads] & \mathbf{1} \arrow[r, equal] & \mathbf{1},
\end{tikzcd}$$

where (2) is a pullback and (1)+(2) is a pullback, we can apply Lemma 4.2.1 to get that (1) is a pullback. Therefore, η_A is an isomorphism and $A = \mathbf{1}$.

iii) $\Rightarrow$ *i)* We will start by examining the case where $B \neq \mathbf{1}$. As a direct consequence we have $A \neq \mathbf{1}$. Moreover, we can apply a similar argument to the one seen before to the commutative cube

$$
\begin{tikzcd}
& K[\pi_1] \arrow[rr, dash, "\cong"] \arrow[dl] \arrow[dd, shift right] & & K[f] \arrow[dl] \arrow[dd] \\
\mathrm{Eq}(f) \arrow[rr, "\pi_2"] \arrow[dd, shift right, "\pi_1"'] & & A \arrow[dd, "f"] & \\
& \mathbf{2} \arrow[uu, shift right] \arrow[rr, equal] \arrow[dl] & & \mathbf{2} \arrow[dl] \\
A \arrow[uu, shift right, "\Delta"'] \arrow[rr, "f"] & & B, &
\end{tikzcd}
$$

where $(\mathrm{Eq}(f), \pi_1, \pi_2)$ is the kernel pair of f, to deduce $K[f] \cong K[\pi_1]$. Therefore, by assumption, we have $K[\pi_1] \in \mathscr{F}$. Consider the pullback

$$
\begin{tikzcd}
K[\pi_1] \arrow[r] \arrow[d, shift right] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Eq}(f) \arrow[d, shift right, "\pi_1"'] \\
\mathbf{2} \arrow[u, shift right] \arrow[r] & A \arrow[u, shift right, "\Delta"']
\end{tikzcd}
$$

and observe that it is preserved by F, since it is the pullback along a split epimorphism. Hence, also the square

$$
\begin{tikzcd}
F(K[\pi_1]) \arrow[r] \arrow[d, shift right] \arrow[dr, phantom, "\lrcorner", very near start] & F(\mathrm{Eq}(f)) \arrow[d, shift right, "F(\pi_1)"'] \\
\mathbf{2} \arrow[u, shift right] \arrow[r] & F(A) \arrow[u, shift right, "F(\Delta)"']
\end{tikzcd}
$$

is a pullback. Observe that $K[\pi_1] \in \mathscr{F}$ implies $F(K[\pi_1]) = K[\pi_1]$. Suppose $F(A) \neq \mathbf{1}$ and consider the cube

$$
\begin{tikzcd}
& K[\pi_1] \arrow[rr, equal] \arrow[dl] \arrow[dd] & & K[\pi_1] \arrow[dl] \arrow[dd] \\
\mathrm{Eq}(f) \arrow[rr, "\eta_{F(\mathrm{Eq}(f))}"] \arrow[dd, shift right, "\pi_1"'] & & F(\mathrm{Eq}(f)) \arrow[dd] & \\
& \mathbf{2} \arrow[rr, equal] \arrow[dl] & & \mathbf{2} \arrow[dl] \\
A \arrow[uu, shift right] \arrow[rr, "\eta_A"] & & F(A). &
\end{tikzcd}
$$

Since π_1 is a split epimorphism and $\mathbb{C}$ is a protomodular category, we conclude that the front face is a pullback. Thus, π_1 is a trivial extension and so, by definition, f is a normal extension. Let us deal with the case $F(A) = \mathbf{1}$. Analyzing the pullback

$$
\begin{tikzcd}
F(K[\pi_1]) \arrow[r] \arrow[d, shift right] \arrow[dr, phantom, "\lrcorner", very near start] & F(\mathrm{Eq}(f)) \arrow[d, shift right] \\
\mathbf{2} \arrow[u, shift right] \arrow[r] & F(A) = \mathbf{1}, \arrow[u, shift right]
\end{tikzcd}
$$

we deduce $F(\mathrm{Eq}(f)) = \mathbf{1}$ and $K[\pi_1] = F(K[\pi_1]) = \mathbf{2}$. Moreover, considering the commutative cube

$$\begin{tikzcd}
& K[\pi_1] = \mathbf{2} \arrow[rr] \arrow[dl] \arrow[dd] & & \mathbf{1} \arrow[dl, equal] \arrow[dd, equal] \\
\mathrm{Eq}(f) \arrow[rr, "\eta_{F(\mathrm{Eq}(f))}"] \arrow[dd, shift right, "\pi_1"'] & & F(\mathrm{Eq}(f)) = \mathbf{1} \arrow[dd, equal] & \\
& \mathbf{2} \arrow[rr] \arrow[dl] & & \mathbf{1} \arrow[dl, equal] \\
A \arrow[rr, "\eta_A"] \arrow[uu, shift right] & & F(A) = \mathbf{1}, &
\end{tikzcd}$$

once again, we can conclude that the front face is a pullback, and hence f is a normal extension. Finally, if $B = \mathbf{1}$, we obtain $K[f] = A \in \mathscr{F}$. Hence the square

$$\begin{tikzcd}
A \times A \arrow[r, "\eta_{A\times A}"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & F(A \times A) \arrow[d, "F(\pi_1)"] \\
A \arrow[r, "\eta_A"'] & F(A)
\end{tikzcd}$$

is a pullback, since η_A and $\eta_{A\times A}$ are isomorphisms ($A \times A \in \mathscr{F}$). Then, $\pi_1 \colon A \times A \to A$ is a trivial extension, and thus f is normal extension. $\square$

The purpose of the following results is to understand the relationship between pretorsion theories and factorization systems. Specifically, we will see how certain pretorsion theories give rise to stable factorization systems. Moreover, we will show that every stable factorization system $(\mathscr{E}, \mathscr{M})$, such that $\mathbf{2} \to \mathbf{1} \in \mathscr{M}$ and every arrow in $\mathscr{E}$ is a precokernel, induces a pretorsion theory.

Lemma 4.3.14. *For a given pretorsion theory $(\mathscr{T}, \mathscr{F})$, we define two classes of arrows:*

$$\overline{\mathscr{E}} := \{e \text{ precokernel } \mid K[e] \in \mathscr{T}\} \text{ and } \overline{\mathscr{M}} := \{m \mid K[m] \in \mathscr{F}\}.$$

Then, for every $e \in \overline{\mathscr{E}}$ and $m \in \overline{\mathscr{M}}$, we have $e \downarrow m$. Additionally, the unique arrow $\mathbf{2} \to \mathbf{1}$ belongs $\overline{\mathscr{M}}$.

Proof. We recall that $e \downarrow m$ holds if, for every commutative square

$$\begin{tikzcd}
A \arrow[r, "e"] \arrow[d, "a"'] & B \arrow[d, "b"] \arrow[dl, "\exists d"'] \\
C \arrow[r, "m"'] & D,
\end{tikzcd}$$

there exists an arrow $d \colon B \to C$ such that $md = b$ and $de = a$. Let $k(e) \colon K[e] \to A$ be the prekernel of e, and $k(m) \colon K[m] \to C$ the prekernel of m. We consider the following commutative diagram:

$$\begin{tikzcd}
 & & Z \arrow[dr, bend left] & \\
Z' \arrow[dr, bend right] & K[e] \arrow[l, bend right] \arrow[ur, bend left] \arrow[r, "k(e)"] \arrow[d, "\exists!\varphi"'] & A \arrow[r, "e"] \arrow[d, "a"] & B \arrow[d, "b"] \arrow[dl, "\exists!d"'] \\
 & K[m] \arrow[r, "k(m)"'] & C \arrow[r, "m"'] & D,
\end{tikzcd}$$

where $e \in \overline{\mathscr{E}}$, $m \in \overline{\mathscr{M}}$, $ma = be$, and $Z, Z' \in \mathscr{Z}$. Since $mak(e) = bek(e) \in N_{\mathscr{Z}}$, there exists a unique arrow φ such that $ak(e) = k(m)\varphi$. Furthermore, since $K[e] \in \mathscr{T}$ and $K[m] \in \mathscr{F}$, it follows that $\varphi \in N_{\mathscr{Z}}$. Hence, we have $ak(e) = k(m)\varphi \in N_{\mathscr{Z}}$. Additionally, we recall that e is the precokernel of $k(e)$ (by Proposition 4.1.7), and therefore there exists a unique arrow $d\colon B \to C$ such that $de = a$. To conclude, we obtain $mde = ma = be$ and, from the fact that e is an epimorphism, we deduce $md = b$. Furthermore, we can note that $\mathrm{preker}(\mathbf{2} \to \mathbf{1}) = id_{\mathbf{2}}$ and that $\mathbf{2} \in \mathscr{F}$. Therefore, we obtain that $\mathbf{2} \to \mathbf{1}$ is an element of $\overline{\mathscr{M}}$. □

The following definition represents a generalized version, applicable to our non-pointed case, of the original definition presented in [33].

Definition 4.3.15. *We say that a pretorsion theory $(\mathscr{T}, \mathscr{F})$ satisfies* condition (N) *if, for every diagram*

$$T(K[f]) \xrightarrow{\varepsilon_{K[f]}} K[f] \xrightarrow{k} A \xrightarrow{f} B$$

where $k = \mathrm{preker}(f)$, then $k\varepsilon_{K[f]}$ is the prekernel of some arrow.

Definition 4.3.16. *We say that a pretorsion theory $(\mathscr{T}, \mathscr{F})$ satisfies* condition (P2) *if, whenever $F(B) = \mathbf{1}$ and $A \neq \mathbf{1}$, the sequence*

$$\mathbf{2} \times B \xrightarrow{\iota_A \times id_B} A \times B \xrightarrow{\pi_A} A$$

is pre-exact and $\mathbf{2} \times B \in \mathscr{T}$.

Proposition 4.3.17. *If the pretorsion theory $(\mathscr{T}, \mathscr{F})$ satisfies conditions (P2) and (N), then the pair of classes $\overline{\mathscr{E}}$ and $\overline{\mathscr{M}}$, defined in the previous lemma, forms a stable factorization system.*

Proof. We can obtain a factorization of the arrow $\tau_A\colon A \to \mathbf{1}$ as follows:

$$\begin{tikzcd}
A \arrow[rr, "\tau_A"] \arrow[dr, bend right, "\eta_A"'] & & \mathbf{1} \\
 & F(A). \arrow[ur, bend right, "\tau_{F(A)}"'] &
\end{tikzcd}$$

Here, $\mathrm{preker}(\eta_A) = (\varepsilon_A\colon T(A) \to A)$, and, since $T(A) \in \mathscr{T}$, we get $\eta_A \in \overline{\mathscr{E}}$. Similarly, $\mathrm{preker}(\tau_{F(A)}) = id_{F(A)}\colon F(A) \to F(A)$, and, since $F(A) \in \mathscr{F}$, we have $\tau_{F(A)} \in \overline{\mathscr{M}}$. Thus, we can restrict our argument to an arrow $f\colon A \to B$ with $B \neq \mathbf{1}$. By assumption, we know that $k\varepsilon_{K[f]}$ is a prekernel of some arrow. Suppose that $k\varepsilon_{K[f]} = \mathrm{preker}(\tau_A\colon A \to \mathbf{1})$. Then we can assume $k\varepsilon_{K[f]} = id_A$, which implies $A = T(K[f]) \in \mathscr{T}$. Moreover, since $k\varepsilon_{K[f]}$ is an isomorphism, we obtain that k is a regular epimorphism; recalling that k is a monomorphism, as a prekernel, we deduce that k is an isomorphism. Since k is an isomorphism, we can suppose $\mathrm{preker}(f) = id_A$. Additionally, we observe that there exists an arrow $\chi\colon A \to \mathbf{2}$, in fact, given that $B \neq \mathbf{1}$, $K[f]$ is defined by the pullback

$$\begin{tikzcd} K[f] \arrow[r] \arrow[d, "k"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{2} \arrow[d] \\ A \arrow[r, "f"'] & B. \end{tikzcd}$$

So we have an arrow $K[f] \to \mathbf{2}$, and, recalling that $A = T(K[f])$, this implies the existence of $\chi\colon A \to \mathbf{2}$. Thus, A being an object of $\mathscr{T}$, the morphism $\chi\colon A \to \mathbf{2}$ is unique. Moreover, considering that $\mathrm{preker}(f) = id_A$ and $B \neq \mathbf{1}$, the factorization $f = \iota_B\chi$ holds. Finally, the square below is a pushout:

$$\begin{tikzcd} A \arrow[r, "\chi"] \arrow[d, equal] \arrow[dr, phantom, "\ulcorner", very near end] & \mathbf{2} \arrow[d, equal] \\ A \arrow[r, "\chi"'] & \mathbf{2} \end{tikzcd}$$

and so $\chi = \mathrm{precoker}(id_A)$. To sum up, f can be factorized as $f = \iota_B\chi$, where $K[\iota_B] = \mathbf{2} \in \mathscr{F}$ (i.e. $\iota_B \in \overline{\mathscr{M}}$) and χ is a precokernel such that $K[\chi] = A \in \mathscr{T}$ (i.e. $\chi \in \overline{\mathscr{E}}$). We still need to examine the scenario where B is not equal to $\mathbf{1}$, and $k\varepsilon_{K[f]}$ is the prekernel of an arrow $h\colon A \to C$, where C is not equal to $\mathbf{1}$. First of all we construct the precokernel of $k\varepsilon_{K[f]}$. We observe that there exists a unique morphism $\xi\colon T(K[f]) \to \mathbf{2}$, since $T(K[f]) \in \mathscr{T}$. Therefore, the precokernel of $k\varepsilon_{K[f]}$ is given by the pushout

$$\begin{tikzcd} T(K[f]) \arrow[r, "\xi"] \arrow[d, "k\varepsilon_{K[f]}"'] \arrow[dr, phantom, "\ulcorner", very near end] & \mathbf{2} \arrow[d] \\ A \arrow[r, two heads, "q"'] & Q. \end{tikzcd}$$

Notice that q is a regular epimorphism, since it is the pushout of the split epimorphism ξ. Now, we can apply Proposition 4.1.11 to obtain that $k\varepsilon_{K[f]} = \mathrm{preker}(q)$. So, consider the commutative diagram

$$\begin{tikzcd} & & & P \arrow[dr, tail, "j"] & \\ T(K[f]) \arrow[r, "\varepsilon_{K[f]}"] & K[f] \arrow[r, "k"] & A \arrow[ur, two heads, "g"] \arrow[rr, "f"] \arrow[dr, two heads, "q"'] & & B \\ & & & Q \arrow[ur, "m"] \arrow[r, two heads, "e"'] & I, \arrow[u, tail, "i"'] \end{tikzcd}$$

where (g, j) is the (regular epimorphism, monomorphism)-factorization of f, (e, i) the (regular epimorphism, monomorphism)-factorization of m, and m is induced by the universal property of the precokernel q. We claim that $f = mq$ is the $(\overline{\mathscr{E}}, \overline{\mathscr{M}})$-factorization of f. Thanks to Lemma 4.1.5, we get that $k = \mathrm{preker}(g)$ and, by the uniqueness of the (regular epimorphism, monomorphism)-factorization, we can assume $P = I$. Additionally, we recall that, by construction, q is a precokernel and $\mathrm{preker}(q) = k\varepsilon_{K[f]} \colon T(K[f]) \to A$. Hence $q \in \overline{\mathscr{E}}$. So, it remains to prove that $m \in \overline{\mathscr{M}}$ or, equivalently, $K[m] \in \mathscr{F}$. Applying Lemma 4.1.5, we deduce that $K[e] = K[m]$. Therefore, we will show that $K[e] \in \mathscr{F}$. To this end, consider the commutative diagram

$$\begin{tikzcd}
T(K[f]) \arrow[r, "\varepsilon_{K[f]}"] \arrow[d, equal] & K[f] \arrow[r, two heads, "\eta_{K[f]}"] \arrow[d, "k"] & F(K[f]) \arrow[d, "\exists! k'"] \\
T(K[f]) \arrow[r, "k\varepsilon_{K[f]}"] \arrow[d, two heads, "\xi"'] & A \arrow[r, two heads, "q"] \arrow[d, two heads, "g"] & Q \arrow[d, two heads, "e"] \\
\mathbf{2} \arrow[r, "\iota_P"'] & P \arrow[r, equal] & P,
\end{tikzcd}$$

where k' is induced by the universal property of the precokernel $\eta_{K[f]}$. Thanks to Theorem 4.2.6, we conclude that $k' = \mathrm{preker}(e)$ and so $K[e] = F(K[f]) \in \mathscr{F}$.

To complete the proof we have to show that $\overline{\mathscr{E}}$ is pullback stable. For this purpose, fix an arrow $e \colon X \to Y$ of $\overline{\mathscr{E}}$ and suppose $Y \neq \mathbf{1}$. Since $Y \neq \mathbf{1}$, we can apply Lemma 4.3.7 to deduce that $\overline{e} \in \overline{\mathscr{E}}$. In contrast, if the codomain of e is the terminal object, i.e. $e = \tau_X \colon X \to \mathbf{1}$, we obtain $id_X = \mathrm{preker}(e)$ and, moreover, there does not exist an arrow $X \to \mathbf{2}$. But $X = K[e] \in \mathscr{T}$, so $F(X) = \mathbf{1}$. We consider the pullback of e along the arrow $\tau_W \colon W \to \mathbf{1}$

$$\begin{tikzcd}
X \times W \arrow[r, "\pi_W"] \arrow[d, "\pi_X"'] \arrow[dr, phantom, "\lrcorner", very near start] & W \arrow[d, "\tau_W"] \\
X \arrow[r, "e"'] & \mathbf{1}\,.
\end{tikzcd}$$

If $W = \mathbf{1}$, then $\pi_W = e$ and so the pullback of e is a precokernel. If $W \neq \mathbf{1}$, we use condition (P2) to assert that the sequence

$$\mathbf{2} \times X \xrightarrow{\iota_W \times id_X} W \times X \xrightarrow{\pi_W} W$$

is pre-exact. Thus, we have that π_W is the precokernel of $\iota_W \times id_X$ and $\mathbf{2} \times X \in \mathscr{T}$, i.e. $\pi_W \in \overline{\mathscr{E}}$. $\square$

Proposition 4.3.18. *Consider a stable factorization system $(\mathscr{E}, \mathscr{M})$ such that every arrow in $\mathscr{E}$ is a precokernel and $\mathbf{2} \to \mathbf{1} \in \mathscr{M}$. Let $\mathscr{T}$ be the full subcategory of $\mathbb{C}$ whose objects are*

$$\mathscr{T} := \{T \in \mathbb{C} \mid \exists t \colon T \to \mathbf{2}, t \in \mathscr{E}\} \cup \{T \in \mathbb{C} \mid \tau_T \in \mathscr{E}\}$$

and $\mathscr{F}$ *the full subcategory of* $\mathbb{C}$ *whose objects are*

$$\mathscr{F} := \{F \in \mathbb{C} \,|\, \tau_F \in \mathscr{M}\}.$$

Then, the pair $(\mathscr{T}, \mathscr{F})$ *forms a pretorsion theory in* $\mathbb{C}$ *satisfying conditions (P2) and (N), with* $\mathscr{T} \cap \mathscr{F} = \mathscr{Z}$.

Proof. Let us begin with the study of the intersection $\mathscr{T} \cap \mathscr{F}$. It is clear that both $\mathbf{1}$ and $\mathbf{2}$ belong to $\mathscr{T} \cap \mathscr{F}$. Vice versa, fix an object $X \in \mathscr{T} \cap \mathscr{F}$. If there exists an arrow $\chi \colon X \to \mathbf{2}$ belonging to $\mathscr{E}$, then we consider the commutative diagram

$$\begin{tikzcd} X \arrow[r, "\chi \in \mathscr{E}"] \arrow[rr, bend right, "\tau_X \in \mathscr{M}"'] & \mathbf{2} \arrow[r, "\in \mathscr{M}"] & \mathbf{1}, \end{tikzcd}$$

where $\tau_X \in \mathscr{M}$, since $X \in \mathscr{F}$. From the uniqueness of the $(\mathscr{E}, \mathscr{M})$-factorization we deduce that χ is an isomorphism, and so $X = \mathbf{2}$. If there does not exist an arrow $X \to \mathbf{2}$ belonging to $\mathscr{E}$, then we must have $\tau_X \in \mathscr{E}$. Hence, since $X \in \mathscr{F}$, τ_X belongs also to $\mathscr{M}$. Considering the commutative diagram

$$\begin{tikzcd} X \arrow[r, "\tau_X \in \mathscr{E}"] \arrow[d, equal] & \mathbf{1} \arrow[d, equal] \arrow[dl, "\exists ! d"] \\ X \arrow[r, "\tau_X \in \mathscr{M}"'] & \mathbf{1}, \end{tikzcd}$$

we deduce that $X = \mathbf{1}$ (since there must exist an arrow $d \colon \mathbf{1} \to X$).
Let us show that $(\mathscr{T}, \mathscr{F})$ is a pretorsion theory. First of all, we consider an arrow $f \colon T \to F$ from an object $T \in \mathscr{T}$ to an object $F \in \mathscr{F}$. We have to prove that $f \in N_{\mathscr{Z}}$. If there does not exist an arrow $t \colon T \to \mathbf{2}$ belonging to $\mathscr{E}$, then $\tau_T \in \mathscr{E}$. So, from the commutative square

$$\begin{tikzcd} T \arrow[r, "\tau_T \in \mathscr{E}"] \arrow[d, "f"'] & \mathbf{1} \arrow[d, equal] \arrow[dl, "\exists ! d"] \\ F \arrow[r, "\tau_F \in \mathscr{M}"'] & \mathbf{1}, \end{tikzcd}$$

we deduce that there exists an arrow $d \colon \mathbf{1} \to F$, since $\tau_T \downarrow \tau_F$. Thus, $F = \mathbf{1}$ and $f \in N_{\mathscr{Z}}$. If there exists an arrow $t \colon T \to \mathbf{2}$ belonging to $\mathscr{E}$, we consider, as before, the commutative diagram

$$\begin{tikzcd} T \arrow[r, "t \in \mathscr{E}"] \arrow[d, "f"'] & \mathbf{2} \arrow[d] \arrow[dl, "\exists ! d"] \\ F \arrow[r, "\tau_F \in \mathscr{M}"'] & \mathbf{1}; \end{tikzcd}$$

since $t \downarrow \tau_F$, there exists an arrow $d \colon \mathbf{2} \to F$ such that $dt = f$. This implies $f \in N_{\mathscr{Z}}$.

For every object $A \in \mathbb{C}$ we consider the $(\mathscr{E}, \mathscr{M})$-factorization of $\tau_A \colon A \to \mathbf{1}$

$$\begin{tikzcd} A \arrow[rr, "\tau_A"] \arrow[dr, bend right, "e"'] & & \mathbf{1} \\ & I, \arrow[ur, bend right, "m"'] & \end{tikzcd}$$

and we define $(k\colon K[e] \to A) \coloneqq \mathrm{preker}(e)$. Clearly, since $I = K[m]$, we deduce that I is an object of $\mathscr{F}$. We prove that $K[e]$ belongs to $\mathscr{T}$. If $I = \mathbf{1}$, we get $\tau_A \in \mathscr{E}$, and so $K[e] = A \in \mathscr{T}$. If $I \neq \mathbf{1}$, then $K[e]$ is defined by the following pullback:

$$\begin{tikzcd} K[e] \arrow[r, "t\in\mathscr{E}"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{2} \arrow[d] \\ A \arrow[r, "e"'] & I, \end{tikzcd}$$

where $t \in \mathscr{E}$ (we recall that the factorization system is stable). Hence, we have $K[e] \in \mathscr{T}$. In other words, for every object A, the pre-exact sequence

$$K[e] \xrightarrow{k} A \xrightarrow{e} I$$

is such that $K[e] \in \mathscr{T}$ and $I \in \mathscr{F}$. Referring to the diagram above, we define $T(A) \coloneqq K[e]$ and $F(A) \coloneqq I$.
Let us now focus on proving that $(\mathscr{T}, \mathscr{F})$ satisfies condition (N). Consider a morphism $f\colon A \to B$ and assume $B \neq \mathbf{1}$. Given the sequence

$$T(K) \xrightarrow{\varepsilon} K \xrightarrow{k} A \xrightarrow{f} B,$$

we have to verify that $k\varepsilon$ is a prekernel (where $k = \mathrm{preker}(f)$ and $T(K)$ is the torsion part of K). To this end, we study the $(\mathscr{E}, \mathscr{M})$-factorizations of $f\colon A \xrightarrow{e} I \xrightarrow{m} B$ and of $\tau_K\colon K \xrightarrow{e'} I' \xrightarrow{m'} \mathbf{1}$. Generally, for every arrow $\chi\colon K \to \mathbf{2}$ in $\mathbb{C}$, the $\mathscr{E}$ part of the factorization of χ is equal to the $\mathscr{E}$ part of the factorization of τ_K. To prove it, we examine the diagram

$$\begin{tikzcd} & I \arrow[dr, bend left, "m\in\mathscr{M}"] & & \\ K \arrow[ur, bend left, "e\in\mathscr{E}"] \arrow[rr, "\chi"] \arrow[dr, bend right, "e'\in\mathscr{E}"'] & & \mathbf{2} \arrow[r, "z\in\mathscr{M}"] & \mathbf{1} \\ & I', \arrow[urr, bend right, "m'\in\mathscr{M}"'] & & \end{tikzcd}$$

where $\chi = me$ and $\tau_K = m'e'$. Since $\mathscr{M}$ is closed under composition, we obtain $zm \in \mathscr{M}$. Therefore, $zme = m'e'$ and so, from the uniqueness of the $(\mathscr{E}, \mathscr{M})$-factorization, we deduce $e = e'$. Hence, in order to study the torsion part of K, we can focus on the arrow $\chi\colon K \to \mathbf{2}$, whose existence is guaranteed by the fact that K is the prekernel of an arrow with codomain different from the terminal object. Let us put $\chi = m'e'$ with $e' \in \mathscr{E}$ and $m' \in \mathscr{M}$, and consider the commutative diagrams

$$\begin{tikzcd} K \arrow[r, "e'"] \arrow[d, "k"'] & I' \arrow[r, "m'"] \arrow[d, "d"] & \mathbf{2} \arrow[d, "\iota_B"] \\ A \arrow[r, "e"'] & I \arrow[r, "m"'] & B \end{tikzcd} \qquad \begin{tikzcd} K \arrow[r, "e'"] \arrow[d, "ek"'] & I' \arrow[d, "\iota_B m'"] \arrow[dl, "\exists d"'] \\ I \arrow[r, "m"'] & B, \end{tikzcd}$$

where d is induced by the diagonal property of the factorization system (see the square on the right). Thanks to the stability of the factorization system and the uniqueness of the $(\mathscr{E}, \mathscr{M})$-factorization, we conclude that both the squares in the rectangle on the left are pullbacks. Therefore, applying Lemma 4.1.6, we get $k(e) = kk(e')$, where $k(e) = \text{preker}(e)$ and $k(e') = \text{preker}(e')$. By definition of $T(K)$ we have $\varepsilon = k(e')$, and so we deduce that $k\varepsilon = kk(e') = k$ is a prekernel. If $B = \mathbf{1}$, we come to a trivial situation.

Finally, we have to prove that $(\mathscr{T}, \mathscr{F})$ satisfies condition (P2). With this goal, fix an object $B \in \mathbb{C}$ such that $F(B) = \mathbf{1}$. So, by definition, $\tau_B \in \mathscr{E}$. Now, since $\mathscr{E}$ is stable, for every arrow $A \to \mathbf{1}$, with $A \neq \mathbf{1}$, the pullback

$$\begin{tikzcd} A \times B \arrow[r, "\pi_A \in \mathscr{E}"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d] \\ B \arrow[r, "\tau_B \in \mathscr{E}"'] & \mathbf{1} \end{tikzcd}$$

tells us that $\pi_A \in \mathscr{E}$. Hence π_A is a precokernel. Furthermore, consider the prekernel of π_A defined by the pullback

$$\begin{tikzcd} \mathbf{2} \times B \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{2} \arrow[d] \\ A \times B \arrow[r, "\pi_A"'] & A. \end{tikzcd}$$

Then, $\mathbf{2} \times B \in \mathscr{T}$ since it is the domain of the prekernel of an arrow of $\mathscr{E}$. To sum up, we have verified that the sequence

$$\mathbf{2} \times B \xrightarrow{\iota_A \times id_B} A \times B \xrightarrow{\pi_A} A$$

is pre-exact and $\mathbf{2} \times B \in \mathscr{T}$. $\square$

Theorem 4.3.19. *The assignments defined in Proposition 4.3.14 and Proposition 4.3.18 establish a one-to-one correspondence between pretorsion theories $(\mathscr{T}, \mathscr{F})$, satisfying conditions (N) and (P2), and stable factorization systems $(\mathscr{E}, \mathscr{M})$ such that every arrow in $\mathscr{E}$ is a precokernel and $\mathbf{2} \to \mathbf{1} \in \mathscr{M}$.*

Proof. For every pretorsion theory $(\mathscr{T}, \mathscr{F})$ we define

$$\overline{\mathscr{E}} := \{e \text{ precokernel } \mid K[e] \in \mathscr{T}\} \text{ and } \overline{\mathscr{M}} := \{m \mid K[m] \in \mathscr{F}\}.$$

The pretorsion theory induced by this factorization system is given by the two full subcategories whose objects are

$$\overline{\mathscr{T}} := \{T \in \mathbb{C} \mid \exists t\colon T \to \mathbf{2}, t \in \overline{\mathscr{E}}\} \cup \{T \in \mathbb{C} \mid \tau_T \in \overline{\mathscr{E}}\}$$

and

$$\overline{\mathscr{F}} := \{F \in \mathbb{C} \mid \tau_F \in \overline{\mathscr{M}}\}.$$

On the one hand, if we consider an object $T \in \overline{\mathscr{T}}$ such that there exists an arrow $t\colon T \to \mathbf{2} \in \overline{\mathscr{E}}$, then $T \in \mathscr{T}$, since $\operatorname{preker}(t) = id_T$ and so $K[t] = T$. We proceed in a similar way if $T \to \mathbf{1} \in \overline{\mathscr{E}}$. On the other hand, if we take an object $T \in \mathscr{T}$, then either $T \to \mathbf{2}$ or $T \to \mathbf{1}$ must be a precokernel (since $F(T) = \mathbf{2}$ or $F(T) = \mathbf{1}$). So, as before, $T \in \overline{\mathscr{T}}$. In other terms, we have proved $\mathscr{T} = \overline{\mathscr{T}}$. Analogously, we can show that $\mathscr{F} = \overline{\mathscr{F}}$.
For every factorization system $(\mathscr{E}, \mathscr{M})$, satisfying the assumptions of Proposition 4.3.18, we define

$$\overline{\mathscr{T}} = \{T \in \mathbb{C} \mid \exists t\colon T \to \mathbf{2}, t \in \mathscr{E}\} \cup \{T \in \mathbb{C} \mid \tau_T \in \mathscr{E}\}$$

and

$$\overline{\mathscr{F}} = \{F \in \mathbb{C} \mid \tau_F \in \mathscr{M}\}.$$

The factorization system, induced by this pretorsion theory, is defined by the classes of arrows

$$\overline{\mathscr{E}} := \{e \text{ precokernel } \mid K[e] \in \overline{\mathscr{T}}\} \text{ and } \overline{\mathscr{M}} := \{m \mid K[m] \in \overline{\mathscr{F}}\}.$$

Fix a precokernel $e\colon A \to B$, with $B \neq \mathbf{1}$. We prove that $e \in \mathscr{E}$ if and only if $e \in \overline{\mathscr{E}}$. To this end, observe that, since e is a precokernel, the diagram below (which defines the prekernel of e) is both a pullback and a pushout:

$$\begin{tikzcd} K[e] \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{2} \arrow[d] \\ A \arrow[r, "e"'] \arrow[ur, phantom, "\ulcorner", very near end] & B. \end{tikzcd}$$

Due to the closure of $\mathscr{E}$ under both pullbacks and pushouts, we can infer that $e \in \mathscr{E}$ if and only if $K[e] \to \mathbf{2} \in \mathscr{E}$. This equivalence can be further expressed as $K[e] \in \overline{\mathscr{T}}$, which, in turn, is equivalent to $e \in \overline{\mathscr{E}}$. It remains to study the case $B = \mathbf{1}$. If $\tau_A \in \mathscr{E}$, then $A \in \overline{\mathscr{T}}$ and so $\tau_A \in \overline{\mathscr{E}}$. Conversely, if we $\tau_A \in \overline{\mathscr{E}}$, then $A \in \overline{\mathscr{T}}$ and so, since there does not exist an arrow $A \to \mathbf{2}$ (we recall that τ_A is a precokernel), $\tau_A \in \mathscr{E}$. We have shown $\mathscr{E} = \overline{\mathscr{E}}$. Finally, observing that in a factorization system each class is uniquely determined by the other, we conclude $\mathscr{M} = \overline{\mathscr{M}}$. $\square$

For a given pretorsion theory $(\mathscr{T}, \mathscr{F})$, we define

$$\mathscr{E} := \{f \in \mathrm{Arr}(\mathbb{C}) \mid F(f) \text{ is an isomorphism}\} \text{ and}$$
$$\mathscr{E}' := \{g \in \mathscr{E} \mid \text{ every pullback of } g \text{ is an arrow of } \mathscr{E}\}.$$

Lemma 4.3.20. *Let $(\mathscr{T}, \mathscr{F})$ be a torsion theory satisfying condition (P2). Then, the inclusion $\overline{\mathscr{E}} \subseteq \mathscr{E}'$ holds.*

Proof. Since $\overline{\mathscr{E}}$ is closed under pullbacks, showing that $\overline{\mathscr{E}}$ is a subset of $\mathscr{E}$ is sufficient to prove the statement. Let us start with an arrow $e\colon A \to \mathbf{1} \in \overline{\mathscr{E}}$. We observe that $F(A) = \mathbf{1}$, since $K[e] = A \in \mathscr{T}$. Therefore, we conclude $F(e) = id_{\mathbf{1}}$, and so $e \in \mathscr{E}$. Now, consider an arrow $e\colon A \to B$ of $\overline{\mathscr{E}}$, with $B \neq \mathbf{1}$. We know that $e = \mathrm{precoker}(k)$ and $K[e] \in \mathscr{T}$, where $k\colon K[e] \to A$ is the prekernel of e. We notice that $F(K[e]) = \mathbf{2}$. This is because $K[e] \in \mathscr{T}$, which implies $F(K[e]) \in \mathscr{Z}$. Moreover, since there exists an arrow $\chi\colon K[e] \to \mathbf{2}$ $(B \neq \mathbf{1})$, $F(K[e])$ cannot be $\mathbf{1}$. Furthermore, having that χ is unique, we conclude that the diagram below is a pushout:

$$\begin{tikzcd} K[e] \arrow[r, "\chi"] \arrow[d, "k"'] & \mathbf{2} \arrow[d, "\iota_B"] \\ A \arrow[r, "e"'] & B. \end{tikzcd}$$

We can therefore apply F to the above diagram, which yields another pushout (due to the fact that F preserves colimits):

$$\begin{tikzcd} \mathbf{2} \arrow[r, equal] \arrow[d, "F(k)"'] & \mathbf{2} \arrow[d, "\iota_{F(B)}=F(\iota_B)"] \\ F(A) \arrow[r, "F(e)"'] & F(B). \end{tikzcd}$$

In conclusion, as $F(e)$ is the pushout of an isomorphism, it is an isomorphism, too. This implies that $e \in \mathscr{E}$. $\square$

The following result is a generalization of Proposition 4.2 of [31].

Lemma 4.3.21. *Given a sequence in $\mathbb{C}$*

$$F_1 \xrightarrow{f} X \xrightarrow{g} F_2$$

such that $F_1, F_2 \in \mathscr{F}$ and $f = \mathrm{preker}(g)$, then $X \in \mathscr{F}$.

Proof. Consider the $(\mathscr{T}, \mathscr{F})$-pre-exact sequence associated with X

$$\begin{tikzcd} T(X) \arrow[r, "\varepsilon_X"] \arrow[d, "\exists!\varphi"'] & X \arrow[r, "\eta_X"] \arrow[d, equal] & F(X) \\ F_1 \arrow[r, "f"'] & X \arrow[r, "g"'] & F_2, \end{tikzcd}$$

where φ is induced by the universal property of the prekernel f (in fact $g\varepsilon_X \in N_{\mathscr{Z}}$, since it is an arrow with domain a torsion object and codomain a torsion-free object). Then, $\varphi \in N_{\mathscr{Z}}$ because it is an arrow between a torsion object and a torsion-free object. Consequently, $\varepsilon_X \in N_{\mathscr{Z}}$, which implies that η_X is an isomorphism (i.e. $X \in \mathscr{F}$). □

The dual of the previous proposition tells us that, for every sequence

$$T_1 \xrightarrow{f} X \xrightarrow{g} T_2,$$

if $g = \mathrm{precoker}(f)$ and $T_1, T_2 \in \mathscr{T}$, then also $X \in \mathscr{T}$.

Finally, we can provide the analogous of Theorem 3.7 in [27]. We write $\mathrm{EffDes}(\mathbb{C})$ (resp. $\mathrm{NExt}_{\mathrm{F}}(\mathbb{C})$) for the full subcategory of the category of arrows $\mathrm{Arr}(\mathbb{C})$ determined by all effective descent morphisms (in our case regular epimorphisms) in $\mathbb{C}$ (resp. all normal extensions with respect to Γ_F, which is the Galois structure associated with F).

Theorem 4.3.22. *If $(\mathscr{T}, \mathscr{F})$ is a pretorsion theory in $\mathbb{C}$ satisfying our assumptions and such that the functor F is protoadditive, the following properties hold:*

i) $\mathrm{NExt}_{\mathrm{F}}(\mathbb{C})$ *is a reflective subcategory of* $\mathrm{Arr}(\mathbb{C})$*;*

ii) normal extensions are stable under composition;

iii) any effective descent morphism $f \colon A \to B$ factors uniquely (up to isomorphism) as a composite $f = me$, where e is stably in $\mathscr{E}$ and m is a normal extension; moreover, this factorization coincides with the $(\mathscr{E}, \mathscr{M})$-factorization of f.

Proof. *i)* We can follow the same argument that was used in Theorem 3.7 of [27] for the pointed context.
ii) We consider two normal extensions $f \colon A \to B$ and $g \colon B \to C$. It is a known and general fact that gf is an effective descent morphism. We have to prove that gf is normal, i.e. that $K[gf] \in \mathscr{F}$. If $C \neq \mathbf{1}$, we consider the following commutative diagram where the three squares are pullbacks:

$$\begin{tikzcd}
K[\alpha] \arrow[r] \arrow[d, "k'"'] \arrow[dr, phantom, "\lrcorner"] & \mathbf{2} \arrow[d] & \\
K[gf] \arrow[r, "\alpha"'] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & K[g] \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathbf{2} \arrow[d] \\
A \arrow[r, "f"'] & B \arrow[r, "g"'] & C.
\end{tikzcd}$$

Hence $K[\alpha] = K[f] \in \mathscr{F}$, and so we have a sequence

$$K[f] = K[\alpha] \xrightarrow{k'} K[gf] \xrightarrow{\alpha} K[g]$$

where $k' = \text{preker}(\alpha)$, and the objects $K[\alpha], K[g]$ belong to $\mathscr{F}$; therefore we can apply Lemma 4.3.21 to get $K[gf] \in \mathscr{F}$. If $C = \mathbf{1}$, we have $g = \tau_B$, $B = K[g] \in \mathscr{F}$, $K[f] \in \mathscr{F}$, and $K[gf] = K[\tau_A] = A$; again, we can apply Lemma 4.3.21 to the sequence

$$K[f] \xrightarrow{k} A \xrightarrow{f} B,$$

where $k = \text{preker}(f)$. So we get $A \in \mathscr{F}$.
iii) Again, we can follow the same argument that was used in Theorem 3.7 of [27] for the pointed context. □

This final result establishes a foundation for future studies on higher-order central extensions.

4.4 Examples

4.4.1 $\mathbb{M}\mathsf{Set}$ and Fix Points

It is a well-known fact that, for every monoid M, the category $\mathbb{M}\mathrm{Set}$ of sets with a fixed action of M is an elementary topos. Clearly, $\mathbb{M}\mathrm{Set}$ is a two-valued topos ($\mathrm{Sub}(\mathbf{1}) = \{\mathbf{1}, \emptyset\}$). Given an object X of $\mathbb{M}\mathrm{Set}$, we define the set of fix points

$$\mathrm{Fix}(X) \coloneqq \{x \in X \mid mx = x \text{ for every } m \in M\}.$$

We observe that, for every arrow $f \colon \mathbf{1} = \{*\} \to X$ in $\mathbb{M}\mathrm{Set}$ and every $m \in M$, one has $mf(*) = f(m*) = f(*)$. Then, there is a bijection

$$\mathbb{M}\mathrm{Set}(\mathbf{1}, X) \cong \mathrm{Fix}(X).$$

Therefore, the objects of $\mathbb{M}\mathrm{Set}$ for which there exists a unique arrow $\mathbf{1} \to X$ are precisely the ones with exactly one fix point. In the light of what has just been said, we define two full subcategories of $\mathbb{M}\mathrm{Set}$ whose objects are

$$\mathscr{F} \coloneqq \{X \in \mathbb{M}\mathrm{Set} \mid \mathrm{Fix}(X) = X\} \text{ and } \mathscr{T} \coloneqq \{X \in \mathbb{M}\mathrm{Set} \mid |\mathrm{Fix}(X)| \leq 1\}.$$

From this point on, we will consider as class of zero objects $\mathscr{Z} \coloneqq \mathscr{F} \cap \mathscr{T} = \{\mathbf{1}, \emptyset\}$. In order to study prekernels and precokernels in $\mathbb{M}\mathrm{Set}$, we can apply what we have seen in the first section of this chapter. Therefore, given an arrow $f \colon A \to B$ (with $A \neq \emptyset$) in $\mathbb{M}\mathrm{Set}$, $q = \mathrm{precoker}(f)$ is defined by the following pushout diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & B \arrow[d, "q"] \\ \mathbf{1} \arrow[r] \arrow[ur, phantom, "\ulcorner"] & Q[f]. \end{tikzcd}$$

If $A = \emptyset$, we get $\operatorname{precoker}(\emptyset \to B) = id_B$. Additionally, since coproducts in $M\mathbb{S}\mathrm{et}$ are computed as in $\mathbb{S}\mathrm{et}$, we obtain that $Q[f] \cong B/f(A)$, where, given an object X of $M\mathbb{S}\mathrm{et}$ and a subset $Y \subseteq X$ closed under the action of M, we define X/Y as the $M\mathbb{S}\mathrm{et}$ obtained just contracting Y to one point and defining the action of M as the one induced by X. Finally, if there exists a unique arrow $\mathbf{1} \to B$, the prekernel k of an arrow $f\colon A \to B$ is given by the following pullback diagram:

$$\begin{tikzcd} K[f] \arrow[r] \arrow[d, "k"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{1} \arrow[d] \\ A \arrow[r, "f"'] & B. \end{tikzcd}$$

Proposition 4.4.1. *$(\mathscr{F}, \mathscr{T})$ is a pretorsion theory in* $M\mathbb{S}\mathrm{et}$.

Proof. First of all, we consider $F \in \mathscr{F}$ and $T \in \mathscr{T}$. If $\operatorname{Fix}(T) \neq \emptyset$, then every arrow $h\colon F \to T$ factors as

$$\begin{tikzcd} F \arrow[rr, "h"] \arrow[dr, bend right] & & T \\ & \mathbf{1} \arrow[ur, bend right] & \end{tikzcd}$$

since, for every $x \in F$ and every $m \in M$, $mh(x) = h(mx) = h(x) = y$ (where y is the only fix point of T). If $\operatorname{Fix}(T) = \emptyset$, then F must be empty (since every image of a fix element is a fix element), and so $h = \iota_T \in N_{\mathscr{Z}}$. Hence $M\mathbb{S}\mathrm{et}(F, T) \subseteq N_{\mathscr{Z}}$. Now, given an object X of $M\mathbb{S}\mathrm{et}$, we define

$$F(X) \coloneqq \operatorname{Fix}(X),\; T(X) \coloneqq X/\operatorname{Fix}(X),$$

and we have the sequence

$$F(X) \xrightarrow{\varepsilon_X} X \xrightarrow{\eta_X} T(X)$$

where ε_X is the inclusion and η_X the quotient projection. Clearly $\operatorname{Fix}(\operatorname{Fix}(X)) = \operatorname{Fix}(X)$ and $|\operatorname{Fix}(X/\operatorname{Fix}(X))| \leq 1$ (it is exactly 1 when $\operatorname{Fix}(X) \neq \emptyset$, and the unique fix point is $[x]$ with $x \in \operatorname{Fix}(X)$). We prove that the sequence above is pre-exact. If $\operatorname{Fix}(X) \neq \emptyset$, the following square (which we know to be a pushout, thanks to the description of precokernels) is a pullback

$$\begin{tikzcd} \operatorname{Fix}(X) \arrow[r, "\varepsilon_X"] \arrow[d] & X \arrow[d, "\eta_X"] \\ \mathbf{1} \arrow[r] \arrow[ur, phantom, "\ulcorner", very near end] & X/\operatorname{Fix}(X). \end{tikzcd}$$

To show this, we consider an arrow $g\colon Y \to X$ such that $\eta_X g(y) = [x]$, for an arbitrary $x \in \operatorname{Fix}(X)$. Then $g(y) \in \operatorname{Fix}(X)$, and so g restricts to $\operatorname{Fix}(X)$; hence we get $\varepsilon_X = \operatorname{preker}(\eta_X)$ and $\eta_X = \operatorname{precoker}(\varepsilon_X)$. If $\operatorname{Fix}(X) = \emptyset$, we have the following sequence

$$\emptyset \xrightarrow{\iota_X} X \xrightarrow{id_X} X,$$

where $id_X = \operatorname{precoker}(\iota_X)$ (thanks to the description of precokernels), and $\iota_X = \operatorname{preker}(id_X)$ (since every arrow $g \in N_{\mathscr{Z}}$ with codomain X must have $\emptyset$ as domain, otherwise X would have at least one fix point). □

Proposition 4.4.2. *The pretorsion theory* $(\mathscr{T}, \mathscr{F})$ *in* $\mathrm{M}\mathbb{S}\mathrm{et}^{op}$ *satisfies conditions (P1) and (P2).*

Proof. To prove the statement we work, dually, in $\mathrm{M}\mathbb{S}\mathrm{et}$. We recall that both limits and colimits in $\mathrm{M}\mathbb{S}\mathrm{et}$ are computed as in $\mathbb{S}\mathrm{et}$. In general, for every pair A, B of objects of $\mathrm{M}\mathbb{S}\mathrm{et}$ one has $\operatorname{Fix}(A+B) = \operatorname{Fix}(A) + \operatorname{Fix}(B)$; so condition (P1) trivially holds. In order to prove that also condition (P2) holds we have to show that the diagram

$$\begin{tikzcd} A \arrow[r, "\tau_A"] \arrow[d, "i_A"'] & \mathbf{1} \arrow[d, "i_{\mathbf{1}}"] \\ A+B \arrow[r, "\tau_A + id_B"'] & \mathbf{1}+B \end{tikzcd}$$

is both a pullback and a pushout and $\mathbf{1}+B \in \mathscr{T}$, whenever B is such that $\operatorname{Fix}(B) = \emptyset$ and $A \neq \emptyset$. We observe that $|\operatorname{Fix}(\mathbf{1}+B)| = 1 + |\operatorname{Fix}(B)| = 1$, hence $\mathbf{1}+B \in \mathscr{T}$. Moreover, the square is clearly a pushout. In order to show that it is also a pullback, we consider an arrow $a\colon X \to A+B$

$$\begin{tikzcd} X \arrow[drr, bend left, "\tau_X"] \arrow[ddr, bend right, "a"'] & & \\ & A \arrow[r, "\tau_A"] \arrow[d, "i_A"'] & \mathbf{1} \arrow[d, "i_{\mathbf{1}}"] \\ & A+B \arrow[r, "\tau_A + id_B"'] & \mathbf{1}+B \end{tikzcd}$$

such that $(\tau_A + id_B)(a(x)) = *$ for every $x \in X$ (where $*$ is the unique element of $\mathbf{1}$). We observe that $\operatorname{Im}(a) \subseteq i_A(A)$: if there exists an element $x \in X$ such that $a(x) \notin i_A(A)$, then $(\tau_A + id_B)(a(x)) \in i_B(B) \subseteq \mathbf{1}+B$, but $(\tau_A + id_B)(a(x)) = * \in i_{\mathbf{1}}(\mathbf{1})$. This leads to a contradiction because $\mathbf{1}$ and B are disjoint in $\mathbf{1}+B$. Therefore, we get an arrow $\alpha\colon X \to A$ such that $i_A(\alpha(x)) = a(x)$, for every $x \in X$. Since i_A is injective, this arrow α is unique. □

Proposition 4.4.3. *The pretorsion theory* $(\mathscr{T}, \mathscr{F})$ *in* $\mathrm{M}\mathbb{S}\mathrm{et}^{op}$ *satisfies condition (N).*

Proof. To prove the statement we work, dually, in $\mathrm{M}\mathbb{S}\mathrm{et}$. Given an arrow $f\colon A \to B$ we consider

$$A \xrightarrow{f} B \xrightarrow{q} Q[f] = B/f(A) \xrightarrow{\eta_{Q[f]}} T(Q[f]) = (B/f(A))/\operatorname{Fix}(B/f(A))$$

and we have to show that $\eta_{Q[f]}q$ is a precokernel. If $A = \emptyset$, the assertion is trivial. If $A \neq \emptyset$, we have that

$$(B/f(A))/\operatorname{Fix}(B/f(A)) \cong B/(f(A) \cup \operatorname{Fix}(B))$$

where the isomorphism is given by $\overline{\varphi}\colon (B/f(A))/\operatorname{Fix}(B/f(A)) \to B/(f(A)\cup\operatorname{Fix}(B))$ induced by $\varphi\colon B/f(A) \to B/(f(A)\cup\operatorname{Fix}(B))$, where $\varphi([b]) \coloneqq (b)$ ($[b]$ denotes the class of b with respect to the quotient on $f(A)$, while (b) denotes the class of b with respect to the quotient on $f(A)\cup\operatorname{Fix}(B)$). Clearly φ is well defined and surjective. We observe that $\varphi([b]) = \varphi([c])$ implies $b, c \in f(A) \cup \operatorname{Fix}(B)$: if $b, c \in f(A)$, then $[b] = [c]$; if $b \in f(A)$ and $c \in \operatorname{Fix}(B)$, then $[b] \in \operatorname{Fix}(B/f(A))$ (since every element coming from A via f is a fix point of $B/f(A)$) and $[c] \in \operatorname{Fix}(B/f(A))$; if $b, c \in \operatorname{Fix}(B)$, then $[b], [c] \in \operatorname{Fix}(B/f(A))$. Therefore $\overline{\varphi}$ is also injective and $T(Q[f])$ is the precokernel of $f(A) \cup \operatorname{Fix}(B) \hookrightarrow B$. $\square$

We are ready to describe the stable factorization system induced by $(\mathscr{T}, \mathscr{F})$ on $\mathbb{M}\mathrm{Set}^{op}$. In order to simplify the argument, we will work again in $\mathbb{M}\mathrm{Set}$. Recalling what we have studied in the second section, we immediately get that $\overline{\mathscr{E}} = \{e\colon A \to B \in \operatorname{Arr}(\mathbb{M}\mathrm{Set}) \mid e \text{ is a prekernel, } B/e(A) \in \mathscr{T}\}$ and $\overline{\mathscr{M}} = \{m\colon A \to B \in \operatorname{Arr}(\mathbb{M}\mathrm{Set}) \mid B/m(A) \in \mathscr{F}\}$. Hence, if we consider an arrow $f\colon A \to B$, the $(\overline{\mathscr{M}}, \overline{\mathscr{E}})$-factorization of f is given by

$$\begin{tikzcd}
A \arrow[rr, "f"] \arrow[dr, "m"'] & & B \\
& f(A)\cup\operatorname{Fix}(B), \arrow[ur, tail, "e"'] &
\end{tikzcd}$$

where the morphism $m \in \overline{\mathscr{M}}$ is obtained by restricting the codomain of f, while $e \in \overline{\mathscr{E}}$ is the inclusion map. In order to see that e is a precokernel consider the pullback (assume $A \neq \mathbf{1}$, otherwise the statement is trivial)

$$\begin{tikzcd}
f(A)\cup\operatorname{Fix}(B) \arrow[r] \arrow[d, "e"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{1} \arrow[d, "t"] \\
B \arrow[r, "q"'] & B/(f(A)\cup\operatorname{Fix}(B)),
\end{tikzcd}$$

where t is defined and unique since $|\operatorname{Fix}(B/(f(A)\cup\operatorname{Fix}(B)))| = 1$.
We observe that, in general, the functor F, induced by the pretorsion theory $(\mathscr{F}, \mathscr{T})$ in $\mathbb{M}\mathrm{Set}$, does not preserve colimits. For example, if we consider an object A of $\mathbb{M}\mathrm{Set}$ such that $A \neq \emptyset$ and $\operatorname{Fix}(A) = \emptyset$, then $\tau_A\colon A \to \mathbf{1}$ is an epimorphism but $F(\tau_A)\colon \emptyset \to \mathbf{1}$ is not an epimorphism.

Proposition 4.4.4. *The reflector F induced by the pretorsion theory $(\mathscr{T}, \mathscr{F})$ in $\mathbb{M}\mathrm{Set}^{op}$ is protoadditive.*

Proof. Again, we work in $\mathbb{M}\mathrm{Set}$. Clearly $F(\emptyset) = \emptyset$ and $F(\mathbf{1}) = \mathbf{1}$. We prove that F preservers the pushouts of split monomorphisms. Hence, we consider the following pushout:

$$\begin{tikzcd}
X \arrow[r, shift left, "s"] \arrow[d, "f"'] & Y \arrow[l, shift left, "p"] \arrow[d, "i_Y"] \\
Z \arrow[r, "i_Z"'] & P, \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

where $ps = id_X$. We define on Y the following relation: $y \sim w$ if and only if $y = w$ or there exist $x, t \in X$ such that $y = s(x)$, $w = s(t)$, and $f(x) = f(t)$. Let us prove that $\sim$ is an equivalence relation preserved by the action of M. It is clear that $\sim$ is both reflexive and symmetric. We deal with the transitivity: suppose $y \sim w$ and $w \sim r$; clearly if $y = w$ or $w = r$ the property holds, so suppose $y = s(x)$, $w = s(t)$ with $f(x) = f(t)$ and $w = s(\bar{t})$, $r = s(q)$ with $f(\bar{t}) = f(q)$; since s is injective we deduce $t = \bar{t}$, and so $y \sim r$. Finally, if $y \sim w$ and $y \neq w$ then $y = s(x)$, $w = s(t)$ with $f(x) = f(t)$ and, therefore, $my = s(mx)$, $mw = s(mt)$ with $f(mx) = f(mt)$, i.e. $my \sim mw$. We prove that, in the category $\mathbb{S}\mathrm{et}$, the following square is a pushout:

$$\begin{tikzcd} X \arrow[r, "s"] \arrow[d, "f"'] & Y \arrow[d, "i_2"] \\ Z \arrow[r, "i_1"'] & (Z \setminus f(X)) + (Y/\sim), \end{tikzcd}$$

where $i_1(z) = z$ if $z \in Z \setminus f(X)$, $i_1(z) = [s(x)]$ if $z = f(x)$ (if $f(x_1) = f(x_2)$ then, by definition, $s(x_1) \sim s(x_2)$), and $i_2(y) = [y]$. The square is commutative: $i_1 f(x) = [s(x)] = i_2(s(x))$. Now, let us consider two arrows $a \colon Z \to A$ and $b \colon Y \to A$ such that $af = bs$; we define $\varphi \colon (Z \setminus f(X)) + (Y/\sim) \to A$ by putting

$$\varphi(e) = \begin{cases} a(z) & \text{if } e = z \in Z \setminus f(X) \\ b(y) & \text{if } e = [y] \in Y/\sim . \end{cases}$$

φ is well defined: if $[y] = [w]$ and $y \neq w$, then $y = s(x)$, $w = s(t)$ and $f(x) = f(t)$, hence $b(y) = bs(x) = af(x) = af(t) = b(w)$. The uniqueness is guaranteed because i_1, i_2 are jointly epimorphic ($i_1(Z) \supseteq Z \setminus f(X)$ and $i_2(Y) = Y/\sim$). With the action of M induced by i_1 and i_2 we deduce that the square above is a pushout also in $M\mathbb{S}\mathrm{et}$. Now, we observe that $\mathrm{Fix}((Z \setminus f(X)) + (Y/\sim)) = \mathrm{Fix}(Z \setminus f(X)) + \mathrm{Fix}(Y/\sim)$. So, we study the set $\mathrm{Fix}(Y/\sim)$. We consider an element $[y] \in \mathrm{Fix}(Y/\sim)$, such that $y \notin \mathrm{Fix}(Y)$; then, for every $m \in M$ such that $my \neq y$, since $my \sim y$, there exists an element $x \in X$ such that $y = s(x)$ (and, since s is injective, this element x does not depend on m) and an element $x' \in X$ such that $my = s(x')$ with $f(x) = f(x')$. But s is injective, hence, from $s(x') = my = ms(x) = s(mx)$, we deduce $x' = mx$ and then $f(x) = mf(x)$. Moreover, for every element $m \in M$ such that $my = y$, we get $ms(x) = s(x)$ and, applying p, we obtain $mx = x$ and so $mf(x) = x$. In other words, if $y \notin \mathrm{Fix}(Y)$ there exists a unique $x \in X$ such that $y = s(x)$ and $f(x) \in \mathrm{Fix}(Z)$. We are ready to prove that the commutative diagram below is a pushout (i.e. F is protoadditive)

$$\begin{tikzcd} \mathrm{Fix}(X) \arrow[r, "s"] \arrow[d, "f"'] & \mathrm{Fix}(Y) \arrow[d, "i_2"] \\ \mathrm{Fix}(Z) \arrow[r, "i_1"'] & \mathrm{Fix}(Z \setminus f(X)) + \mathrm{Fix}(Y/\sim), \end{tikzcd}$$

where, with a slight abuse of notation, we have used the same symbols to indicate an arrow and its image through F. We consider two arrows in $\mathscr{F}$ (i.e. maps) $a\colon \mathrm{Fix}(Z) \to A$ and $b\colon \mathrm{Fix}(Y) \to A$ such that $af = bs$, and we define $\varphi\colon \mathrm{Fix}(Z \setminus f(X)) + \mathrm{Fix}(Y/\sim) \to A$ by putting

$$\varphi(e) = \begin{cases} a(z) & \text{if } e = z \in \mathrm{Fix}(Z \setminus f(X)) \\ b(y) & \text{if } e = [y], \text{with } y \in \mathrm{Fix}(Y) \\ af(x) & \text{if } e = [y], \text{with } y \notin \mathrm{Fix}(Y), \end{cases}$$

where the third part makes sense because $y = s(x)$ and $f(x) \in \mathrm{Fix}(Z)$. In order to show that φ is well defined we consider $[y], [w] \in \mathrm{Fix}(Y/\sim)$ such that $[y] = [w]$ and we distinguish three cases:

- $y, w \in \mathrm{Fix}(Y)$ (with $y \neq w$, otherwise the assertion easily holds): then there exist $x, t \in X$ such that $y = s(x)$, $w = s(t)$, and $f(x) = f(t)$; hence, applying p, we deduce $x, t \in \mathrm{Fix}(X)$, and then $\varphi([y]) = b(y) = bs(x) = af(x) = af(t) = bs(t) = b(w) = \varphi([w])$.

- $y \in \mathrm{Fix}(Y)$ and $w \notin \mathrm{Fix}(Y)$: then there exist $x, t \in X$ such that $y = s(x)$ (and so, since $y \in \mathrm{Fix}(Y)$, $x \in \mathrm{Fix}(X)$), $w = s(t)$, $f(t) \in \mathrm{Fix}(Z)$, and $f(x) = f(t)$. Since $x \in \mathrm{Fix}(X)$ we can use the commutativity of the square to get $bs(x) = af(x)$, and so $\varphi([y]) = bs(x) = af(x) = af(t) = \varphi([w])$.

- $y, w \notin \mathrm{Fix}(Y)$: then there exist $x, t \in X$ such that $y = s(x)$, $w = s(t)$, $f(x), f(t) \in \mathrm{Fix}(Z)$, and $f(x) = f(t)$; therefore $\varphi([y]) = af(x) = af(t) = \varphi([w])$.

Finally, i_1, i_2 are jointly epimorphic. To show it we deal with the only non-trivial case: we consider an element $[y] \in \mathrm{Fix}(Y/\sim)$ such that $y \notin \mathrm{Fix}(Y)$; therefore there exists $x \in X$ such that $y = s(x)$ and $f(x) \in \mathrm{Fix}(Z)$, and then $i_1(f(x)) = [s(x)] = [y]$. Hence φ is unique and F is protoadditive. $\square$

Thanks to the previous result we can describe the central extensions for $\Gamma_{\mathscr{F}}$, i.e. the Galois structure induced by $(\mathscr{T}, \mathscr{F})$ in $\mathrm{M}\mathbb{S}\mathrm{et}^{op}$. We apply Theorem 4.3.13 and we get that a regular epimorphism (i.e. an effective descent morphism in a Barr-exact context) of $\mathrm{M}\mathbb{S}\mathrm{et}^{op}$ is a central extension if and only if $K[f]$ is an object of $\mathscr{F}$. If we dually translate what has just been said we get that a regular monomorphism f in $\mathrm{M}\mathbb{S}\mathrm{et}$, seen as an arrow of $\mathrm{M}\mathbb{S}\mathrm{et}^{op}$, is a central extension for $\Gamma_{\mathscr{F}}$ if and only if $Q[f] \in \mathscr{F}$ (where $Q[f]$ is the codomain of the precokernel of f in $\mathrm{M}\mathbb{S}\mathrm{et}$). Finally, recalling that limits in $\mathrm{M}\mathbb{S}\mathrm{et}$ are computed as in $\mathbb{S}\mathrm{et}$, we get that a regular epimorphism f of $\mathrm{M}\mathbb{S}\mathrm{et}^{op}$ is a central extension if and only if $f\colon A \to B$, considered as an arrow of $\mathrm{M}\mathbb{S}\mathrm{et}$, is a monomorphism and $B/f(A) \in \mathscr{F}$ (or, equivalently, if $B = \mathrm{Fix}(B) \cup f(A)$).

4.4.2 Double Negation in Heyting Algebras

We recall that a *Heyting algebra* is an algebraic structure $(H, \vee, \wedge, 1, 0, \Rightarrow)$ such that $(H, \vee, \wedge, 1, 0)$ is a bounded lattice and the binary operation $\Rightarrow$ satisfies

$$x \wedge y \leq z \text{ if and only if } x \leq y \Rightarrow z.$$

Given a Heyting algebra H, let $H_{\neg\neg}$ denote the set of regular elements of H. An element $x \in H$ is said to be *regular* if $\neg\neg x = x$ (recalling that, in general, $\neg x := x \Rightarrow 0$). It is a known fact that $(H_{\neg\neg}, \vee_{\neg\neg}, \wedge, 0, 1, \Rightarrow)$ is a Boolean algebra, where

$$x \vee_{\neg\neg} y := \neg(\neg x \wedge \neg y).$$

Therefore, this construction defines a functor

$$\begin{tikzcd} \mathbb{H}\mathrm{eyt} \arrow[r, "F"] & \mathbb{B}\mathrm{oole} \\ H \arrow[r, mapsto] \arrow[d, "f"'] & H_{\neg\neg} \arrow[d, "F(f)"] \\ L \arrow[r, mapsto] & L_{\neg\neg}, \end{tikzcd}$$

where $F(f)$ is simply the restriction of f to $H_{\neg\neg}$. Since it is true that $\neg\neg(\neg\neg x) = \neg\neg x$, we can define the function

$$\begin{aligned} \neg\neg\colon H &\to H_{\neg\neg} \\ x &\mapsto \neg\neg x. \end{aligned}$$

It is known that this map is a surjective morphism of Heyting algebras. Specifically, this morphism is the H-component of the unit of the adjunction $F \dashv i$, where $i\colon \mathbb{B}\mathrm{oole} \to \mathbb{H}\mathrm{eyt}$ is the inclusion functor.

A Heyting algebra H is *pseudo-deterministic* (we thank Mariano Messora for the suggestion of the name) if, for every $x \in H$, either $\neg x = 1$ or $\neg x = 0$. Given a Heyting algebra H, we define $T(H) := \{x \in H \mid \neg x = 0 \text{ or } \neg x = 1\}$. We recall that, in every Heyting algebra, the equations $x \Rightarrow 1 = 1$, $\neg(x \wedge y) = x \Rightarrow \neg y$, $\neg(x \vee y) = \neg x \wedge \neg y$, and $\neg(x \Rightarrow y) = \neg\neg x \wedge \neg y$ hold (a proof of these identities can be found in any book that deals with Heyting algebras and intuitionistic logic). These identities can be used to prove that $T(H)$ is a Heyting algebra whose operations are induced by H. In particular, we get

$$\neg(x \vee y) = \begin{cases} 1 & \text{if } \neg x = 1, \neg y = 1 \\ 0 & \text{otherwise} \end{cases} \qquad \neg(x \wedge y) = \begin{cases} 0 & \text{if } \neg x = 0, \neg y = 0 \\ 1 & \text{otherwise} \end{cases}$$

$$\neg(x \Rightarrow y) = \begin{cases} 1 & \text{if } \neg x = 0, \neg y = 1 \\ 0 & \text{otherwise.} \end{cases}$$

So, $T(H)$ is a pseudo-deterministic Heyting algebra. We denote by $\mathbb{PD}$ the full subcategory whose objects are pseudo-deterministic Heyting algebras. Moreover, the assignment described above establishes a functor

$$\begin{tikzcd} \mathbb{H}\mathsf{eyt} \arrow[r, "T"] & \mathbb{PD} \end{tikzcd}$$

$$\begin{tikzcd} H \arrow[r, mapsto] \arrow[d, "f"'] & T(H) \arrow[d, "T(f)"] \\ L \arrow[r, mapsto] & T(L), \end{tikzcd}$$

where $T(f)$ is given by the restriction of f to $T(H)$. It is easy to observe that the inclusions of $T(H)$ in H are the H-components of the counit of the adjunction $j \dashv T$, where $j \colon \mathbb{PD} \to \mathbb{H}\mathsf{eyt}$ is the inclusion functor.
Since, in every Boolean algebra, $\neg x = 0$ implies $x = 1$ and $\neg x = 1$ implies $x = 0$, we deduce that

$$\mathbb{B}\mathsf{oole} \cap \mathbb{PD} = \{\mathbf{1}, \mathbf{2}\}.$$

From this point on, we will consider as class of zero objects $\mathscr{Z} := \mathbb{B}\mathsf{oole} \cap \mathbb{PD} = \{\mathbf{1}, \mathbf{2}\}$

Proposition 4.4.5. $(\mathbb{PD}, \mathbb{B}\mathsf{oole})$ *is a pretorsion theory for* $\mathbb{H}\mathsf{eyt}$. *Moreover, this pretorsion theory satisfies both conditions (U) and (N).*

Proof. We start by showing that, for any Heyting algebra H, $F(H) = \mathbf{1}$ holds if and only if $H = \mathbf{1}$. It is evident that $H = \mathbf{1}$ implies $F(H) = \mathbf{1}$. Conversely, if $F(H) = \mathbf{1}$, this implies that $\neg\neg 1 = \neg\neg 0$ in H, and as a result, $0 = 1$. Hence, if we can prove that $(\mathbb{PD}, \mathbb{B}\mathsf{oole})$ constitutes a pretorsion theory, it will inevitably fulfill condition (U). Let us consider a morphism of Heyting algebras $f \colon T \to F$, where T is an object of $\mathbb{PD}$ and F is an object of $\mathbb{B}\mathsf{oole}$. Let $x \in T$ be a fixed element. If $\neg x = 0$, then $0 = f(\neg x) = \neg f(x)$. Since F is a Boolean algebra, we deduce that $f(x) = 1$. Similarly, if $\neg x = 1$, we get $f(x) = 0$. Thus, f factors through an object of $\mathscr{Z}$.
We now fix a Heyting algebra $H \neq \mathbf{1}$ and show that the inner commutative square below is both a pullback and a pushout:

$$\begin{tikzcd} K \arrow[drr, bend left, "l"] \arrow[ddr, bend right, "h"'] & & \\ & T(H) \arrow[r, "\chi"] \arrow[d, "\varepsilon_H"'] & \mathbf{2} \arrow[d, "\iota"] \arrow[ddr, bend left, "g"] & \\ & H \arrow[r, "\eta_H"'] \arrow[drr, bend right, "f"'] & F(H) & \\ & & & M, \end{tikzcd}$$

where ε_H is the inclusion, η_H denotes the double negation map, and χ is defined as follow:

$$\chi(x) = \begin{cases} 0 & \text{if } \neg x = 1 \\ 1 & \text{if } \neg x = 0. \end{cases}$$

Simple calculations allow us to verify that χ is a morphism of Heyting algebras. Let us consider two morphisms of Heyting algebras, $h\colon K \to H$ and $l\colon K \to \mathbf{2}$, satisfying the condition $\eta_H h = \iota l$. By observing that $\neg\neg h(k) = \iota l(k)$, for every $k \in K$, and recalling that in any Heyting algebra $\neg\neg\neg x = \neg x$, we can infer that $\neg h(k)$ can only be equal to 0 or 1. Thus, h factors through $T(H)$, and the uniqueness of this factorization is ensured since ε_H is a monomorphism. Therefore, the given square is a pullback. We now fix two morphisms of Heyting algebras, $f\colon H \to M$ and $g\colon \mathbf{2} \to M$, such that $f\varepsilon_H = g\chi$. We observe that the kernel pair of η_H is given by $\mathrm{Eq}(\eta_H) = \{(x, y) \in H \mid \neg\neg x = \neg\neg y\}$. Moreover, since $\neg\neg(x \Rightarrow y) = \neg\neg x \Rightarrow \neg\neg y$, we deduce that, if $(x, y) \in \mathrm{Eq}(\eta_H)$, then $\neg\neg(x \Rightarrow y) = 1$ and $\neg\neg(y \Rightarrow x) = 1$. Hence $\neg(x \Rightarrow y) = 0$ and $\neg(y \Rightarrow x) = 0$, so we see that $x \Rightarrow y$ and $y \Rightarrow x$ are elements of $T(H)$. Clearly, we have $\chi(x \Rightarrow y) = 1$ and $\chi(y \Rightarrow x) = 1$. By commutativity, we deduce that $f(x \Rightarrow y) = 1$ (hence $f(x) \le f(y)$) and $f(y \Rightarrow x) = 1$ (hence $f(y) \le f(x)$). Finally, since η_H is surjective, and therefore the coequalizer of its kernel pair, there exists a unique morphism of Heyting algebras $\overline{f}\colon F(H) \to M$, induced by the universal property of the coequalizer, such that $\overline{f}\eta_H = f$. It is also trivially true that $\overline{f}\chi = g$. Therefore, the square is a pushout. So, given a Heyting algebra H, the associated pre-exact sequence, with a torsion object on the left and a torsion-free object on the right, is the following:

$$T(H) \xrightarrow{\varepsilon_H} H \xrightarrow{\eta_H} F(H).$$

To conclude, we show that this pretorsion theory satisfies condition (N). Let us consider the following diagram:

$$T(K[f]) \xrightarrow{\varepsilon} K[f] \xrightarrow{k} H \xrightarrow{f} L,$$

where $k = \mathrm{preker}(f)$ and $\varepsilon = \varepsilon_{K[f]}$. If $L = \mathbf{1}$, the statement is trivial; hence, let us assume $L \neq \mathbf{1}$. In order to prove that condition (N) holds, we show that the following square is a pullback:

$$\begin{tikzcd}
Y \arrow[rrd, bend left, "b"] \arrow[rdd, bend right, "a"'] & & \\
& T(K[f]) \arrow[r, "\chi"] \arrow[d, "k\varepsilon"'] & \mathbf{2} \arrow[d, "\iota"] \\
& H \arrow[r, two heads, "q"'] & H/(\mathrm{Eq}(\eta_H) \cap \mathrm{Eq}(f)),
\end{tikzcd}$$

where $\mathrm{Eq}(\eta_H) \cap \mathrm{Eq}(f) = \{(x, y) \in H \times H \mid \neg\neg x = \neg\neg y \text{ and } f(x) = f(y)\}$ and q is the quotient projection. Let us consider two arrows $a\colon Y \to H$ and $b\colon Y \to \mathbf{2}$ such that $qa = \iota b$. It can be observed that, for every $y \in Y$, if $b(y) = 0$, then by commutativity, $qa(y) = 0$, which implies that $\neg\neg a(y) = 0$ and $fa(y) = 0$. Recalling that $K[f] = \{x \in H \mid f(x) = 0 \text{ or } f(x) = 1\}$, we can deduce that $a(y) \in T(K[f])$. By a similar reasoning, if $b(y) = 1$, then $a(y) \in T(K[f])$ as well. Thus, we can conclude that a factors through $T(K[f])$, and the uniqueness of this factorization is guaranteed by the fact that $k\varepsilon$ is a composition of monomorphisms, and therefore a monomorphism itself. $\square$

Proposition 4.4.6. *The reflector F is a localization (i.e. it preserves all finite limits). Hence, in particular, F is protoadditive.*

Proof. Clearly, F preserves the terminal object $\mathbf{1}$. So, it suffices to prove that F preserves every pullback. Consider the following pullback in $\mathbb{H}\mathrm{eyt}$:

$$\begin{tikzcd}
H \times_L K \arrow[r, "\pi_K"] \arrow[d, "\pi_H"'] \arrow[dr, phantom, "\lrcorner", very near start] & K \arrow[d, "g"] \\
H \arrow[r, "f"'] & L.
\end{tikzcd}$$

Our goal is to show that $(H \times_L K)_{\neg\neg} = H_{\neg\neg} \times_{L_{\neg\neg}} \times K_{\neg\neg}$. Let $(h, k) \in (H \times_L K)_{\neg\neg}$. Then we have $(h, k) = \neg\neg(h, k)$ and $f(h) = g(k)$. This implies that $\neg\neg h = h$ (i.e. $h \in H_{\neg\neg}$) and $\neg\neg k \in K$ (i.e. $k \in K_{\neg\neg}$). Therefore, $(h, k) \in H_{\neg\neg} \times_{L_{\neg\neg}} \times K_{\neg\neg}$. Conversely, let $(h, k) \in H_{\neg\neg} \times_{L_{\neg\neg}} \times K_{\neg\neg}$. Then we have $\neg\neg h = h$, $\neg\neg k = k$, and $f(h) = g(k)$. It follows that $(h, k) = \neg\neg(h, k)$ and hence $(h, k) \in (H \times_L K)_{\neg\neg}$. Therefore $(H \times_L K)_{\neg\neg} = H_{\neg\neg} \times_{L_{\neg\neg}} \times K_{\neg\neg}$, as required. $\square$

We are now ready to investigate the central extensions pertaining to the Galois structure defined by the adjunction $F \dashv i$. As we have shown, an effective descent morphism $f\colon H \to L$ in $\mathbb{H}\mathrm{eyt}$ (i.e. a surjective map) is a central extension if and only if $K[f]$, which is defined as the set $\{x \in H \mid f(x) = 0 \text{ or } f(x) = 1\}$, is a Boolean algebra. In simpler terms, a surjective map is central if and only if its prekernel is an object of $\mathbb{B}\mathrm{oole}$. Furthermore, since the reflector F is a localization, it follows that an extension is central if and only if it is trivial. This result can also be seen as a consequence of Remark 4.6 in [38].

We now turn our attention to the stable factorization system induced by the pretorsion theory $(\mathbb{PD}, \mathbb{B}\mathrm{oole})$. Thanks to the results obtained in this chapter, we get $\overline{\mathscr{E}} = \{e \in \mathrm{Arr}(\mathbb{H}\mathrm{eyt}) \mid e$ is a precokernel and $K[e] \in \mathbb{PD}\}$ and $\overline{\mathscr{M}} = \{m \in \mathrm{Arr}(\mathbb{H}\mathrm{eyt}) \mid K[m] \in \mathbb{B}\mathrm{oole}\}$. Given an arrow $f\colon H \to L$ in $\mathbb{H}\mathrm{eyt}$, we can construct the following factorization:

$$\begin{tikzcd}
H \arrow[rr, "f"] \arrow[dr, two heads, "e"'] & & L \\
& \overline{H}, \arrow[ur, "\overline{f}"'] &
\end{tikzcd}$$

where $\overline{H} \coloneqq H/(\mathrm{Eq}(f) \cap \mathrm{Eq}(\eta_H))$, e is the quotient projection, and $\overline{f}([x]) \coloneqq f(x)$ for every

$[x] \in \overline{H}$. It is not difficult to see that e is the precokernel of the inclusion of $K[e] = \{x \in H \mid f(x) = 0, \neg x = 1\} \cup \{x \in H \mid f(x) = 1, \neg x = 0\}$, and that $K[e] \in \mathbb{PD}$, being a subset of $T(H)$. Moreover, we observe that $K[\overline{f}] \in \mathbb{B}\text{oole}$. To see this, consider an element $[x] \in K[\overline{f}]$. We need to show that $[x] = \neg\neg[x]$ to prove that $K[\overline{f}]$ is a Boolean algebra. Suppose $\overline{f}([x]) = 0$; the argument can be adapted to the case $\overline{f}([x]) = 1$. To conclude that $[x] = \neg\neg[x]$, we need to prove that $f(x) = f(\neg\neg x)$ and $\neg\neg x = \neg\neg\neg\neg x$. The second equality always holds, while the first is true because $f(x) = 0$. Thus, the proposed factorization is precisely the $(\overline{\mathscr{E}}, \overline{\mathscr{M}})$-factorization.

4.4.3 Contraction of Vertices in $\mathrm{s}\mathbb{S}\mathrm{et}$

We recall that Δ is the category whose objects are the totally ordered sets $[n] \coloneqq \{0, 1, 2, \dots, n\}$, where the order is induced by the usual one of $\mathbb{N}$, and whose morphisms are the order-preserving functions. A *simplicial set* is a functor $X : \Delta^{op} \to \mathbb{S}\mathrm{et}$. The category of simplicial sets, denoted by $\mathrm{s}\mathbb{S}\mathrm{et}$, is defined to have the simplicial sets as objects, and the natural transformations between them as morphisms. For every natural number n and every simplicial set X, we define $X_n \coloneqq X([n])$.

A simplicial set X can be seen as family of sets X_n for each non-negative integer n, and two sets of functions $d_i : X_n \to X_{n-1}$ and $s_i : X_n \to X_{n+1}$ for every $0 \le i \le n$, such that specific conditions are satisfied for each n:

$$d_i d_j = d_{j-1} d_i, \qquad i < j$$

$$s_i s_j = s_j s_{i-1}, \qquad i > j$$

$$d_i s_j = \begin{cases} s_{j-1} d_i, & i < j \\ id, & i = j, j+1 \\ s_j d_{i-1}, & i > j+1. \end{cases}$$

This is the standard way to write the data of a simplicial set following [32]. The elements of X_0 are called the *vertices*. Given that each s_i is an injective map, to simplify the notation, we will assume that these maps are inclusion maps.

Let S denote the terminal object of $\mathrm{s}\mathbb{S}\mathrm{et}$, and V denote the initial object. It is clear that $S = \Delta(-, [0])$, which implies that $S_n = \{*\}$, for every natural number n. Furthermore, $V_n = \emptyset$ for every natural number n. We observe that $\mathrm{s}\mathbb{S}\mathrm{et}$ is a two-valued elementary topos. Indeed, if we consider a simplicial set $X \subseteq S$, then either $X_n = \{*\}$ for all $n \in \mathbb{N}$, or there exists a natural number n such that $X_n = \emptyset$. However, in the latter case, it follows that $X_n = \emptyset$ for all $n \in \mathbb{N}$, and hence $X = V$. Therefore, we have $\mathrm{Sub}(S) = \{S, V\}$, as desired.

We observe that the Yoneda Lemma implies

$$\mathrm{s}\mathbb{S}\mathrm{et}(S,X) = \mathrm{s}\mathbb{S}\mathrm{et}(\Delta(-,[0]),X) = X_0$$

for every simplicial set X. Therefore, the simplicial sets for which there exists a unique arrow $S \to X$ are precisely those such that $|X_0| = 1$. We now define two full subcategories of $\mathrm{s}\mathbb{S}\mathrm{et}$ whose objects are:

$$\mathscr{F} := \{X \in \mathrm{s}\mathbb{S}\mathrm{et} \mid X_n = X_0, X(f) = id_{X_0} \text{ for every } n \in \mathbb{N}, f \in \mathrm{Arr}(\Delta)\}$$

and

$$\mathscr{T} := \{X \in \mathrm{s}\mathbb{S}\mathrm{et} \mid |X_0| \leq 1\}.$$

Given a simplicial set X, we define $F(X)$ as the subobject of X such that $F(X)_n = X_0$ for every $n \in \mathbb{N}$. So, we have $F(X) \in \mathscr{F}$.
From this point on, we will consider as class of zero objects $\mathscr{Z} := \mathscr{F} \cap \mathscr{T} = \{S, V\}$.

Proposition 4.4.7. *$(\mathscr{F}, \mathscr{T})$ is a pretorsion theory in* $\mathrm{s}\mathbb{S}\mathrm{et}$.

Proof. Let us consider an arrow $f\colon P \to T$ in $\mathrm{s}\mathbb{S}\mathrm{et}$, where P is an object of $\mathscr{F}$ and T is an object of $\mathscr{T}$. If $T = V$, then clearly f factors through an object of $\mathscr{Z}$. If $T \neq V$, then $f_0(P_0) = T_0 = \{*\}$. Moreover, by naturality, the following diagram commutes:

$$\begin{tikzcd} P_0 \arrow[r, "f_0"] \arrow[d, equal] & T_0 = \{*\} \arrow[d, hook] \\ P_0 \arrow[r, "f_1"'] & T_1. \end{tikzcd}$$

Thus, $f_1(P_0) = \{*\}$. By iterating this reasoning, we deduce that $f_n(P_0) = \{*\}$ for every natural number n. Therefore, f factors through S. Furthermore, we define the simplicial set morphism $\varepsilon_X\colon F(X) \to X$, where $(\varepsilon_X)_n$ is the inclusion of X_0 in X_n. Let us assume that $X \neq V$ and construct $T(X)$ through the following pushout diagram:

$$\begin{tikzcd} F(X) \arrow[r] \arrow[d, "\varepsilon_X"'] & S \arrow[d, "t"] \\ X \arrow[r, "\eta_X"'] & T(X). \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

Recalling that limits and colimits in $\mathrm{s}\mathbb{S}\mathrm{et}$ are computed, level-wise, as in $\mathbb{S}\mathrm{et}$, we deduce that $T(X)_0 = X_0/X_0 = \{*\}$ and, therefore, $T(X) \in \mathscr{T}$. We now show that the above square is also a pullback. To this end, let $\alpha\colon Y \to X$ be a morphism of simplicial sets such that $\eta_X\alpha = t\tau_Y$:

$$\begin{tikzcd} Y \arrow[drr, bend left, "\tau_Y"] \arrow[dr, bend right, "\alpha"'] & & \\ & F(X) \arrow[r] \arrow[d, "\varepsilon_X"'] & S \arrow[d, "t"] \\ & X \arrow[r, "\eta_X"'] & T(X). \end{tikzcd}$$

Since, for every $n \in \mathbb{N}$ and for every $y \in Y_n$, we have $(\eta_X)_n(\alpha_n(y)) = t_n(*) = *$, it follows that $\alpha_n(y) \in X_0 \subseteq X_n$. Hence, α restricts to an arrow on $F(X)$ and the square is indeed a pullback. Therefore, we have shown that ε_X is the prekernel of η_X, and that η_X is the precokernel of ε_X. Finally, if $X = V$, we clearly have $F(X) = V$, and we define $T(X) \coloneqq V$. $\square$

We observe that the pretorsion theory $(\mathscr{T}, \mathscr{F})$ in $\mathrm{s}\mathbb{S}\mathrm{et}^{op}$ satisfies condition (U). Indeed, if $F(X) = V$ (recall that V is the terminal object in $\mathrm{s}\mathbb{S}\mathrm{et}^{op}$), then $X_0 = \emptyset$, and thus $X = V$. Moreover, we observe that F preserves pullbacks in $\mathrm{s}\mathbb{S}\mathrm{et}^{op}$. To see this, we work dually and we show that F preserves pushouts in $\mathrm{s}\mathbb{S}\mathrm{et}$. Since pushouts in $\mathrm{s}\mathbb{S}\mathrm{et}$ are computed level-wise as in $\mathbb{S}\mathrm{et}$, and $F(X) = X_0$ for every simplicial set X, the claim follows easily. In particular, F is protoadditive, considered as functor on $\mathrm{s}\mathbb{S}\mathrm{et}^{op}$. Finally, we prove that $(\mathscr{T}, \mathscr{F})$ satisfies condition (N). Consider a morphism of simplicial sets $f\colon X \to Y$, where $X \neq V$, and define $Q[f]$ through the following pushout:

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & \mathbf{1} \arrow[d] \\ Y \arrow[r, "q"'] \arrow[ur, phantom, "\ulcorner"] & Q[f]. \end{tikzcd}$$

We need to prove that $\eta_{Q[f]}q$ is a precokernel:

$$X \xrightarrow{f} Y \xrightarrow{q} Q[f] \xrightarrow{\eta_{Q[f]}} T(Q[f]).$$

We observe that $T(Q[f]) = (Y_n/f_n(X_n))/(Y_0/f_0(X_0))$, and applying a similar reasoning to the one seen for the case of $\mathrm{M}\mathbb{S}\mathrm{et}$, we deduce that $Y_n/(Y_0 \cup f_n(X_n)) \cong (Y_n/f_n(X_n))/(Y_0/f_0(X_0))$. Therefore, we conclude that $\eta_{Q[f]}q$ is the precokernel of the inclusion $F(Y) \cup f(X) \hookrightarrow Y$ (where $(F(Y) \cup f(X))_n \coloneqq Y_0 \cup f_n(X_n)$ for every natural number n). Moreover, if $X = V$, then $Q[f] = Y$, $q = id_Y$, and $\eta Q[f] = \eta_Y$. Thus, in this case $\eta_{Q[f]}q = \eta_Y$, which is a precokernel by construction.

We can now describe the stable factorization system induced by $(\mathscr{T}, \mathscr{F})$ on $\mathrm{s}\mathbb{S}\mathrm{et}^{op}$. To simplify the argument, we will follow our discussion in the context of $\mathrm{s}\mathbb{S}\mathrm{et}$. We have $\overline{\mathscr{E}} = \{e\colon X \to Y \in \mathrm{Arr}(\mathrm{s}\mathbb{S}\mathrm{et}) \mid e \text{ is a prekernel, and } Y/e(X) \in \mathscr{T}\}$, and $\overline{\mathscr{M}} = \{m\colon X \to Y \in \mathrm{Arr}(\mathrm{s}\mathbb{S}\mathrm{et}) \mid Y/m(X) \in \mathscr{F}\}$, where $(Y/e(X))_n \coloneqq Y_n/e_n(X_n)$ and $(Y/m(X))_n \coloneqq Y_n/m_n(X_n)$, for every natural number n. Therefore, as previously observed in the case of $\mathrm{M}\mathbb{S}\mathrm{et}$, the $(\overline{\mathscr{M}}, \overline{\mathscr{E}})$-factorization of an arrow

$f\colon X \to Y$ is given by:

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "m"'] & & Y \\ & f(X) \cup F(Y), \arrow[ur, "e"'] & \end{tikzcd}$$

where the morphism $m \in \overline{\mathscr{M}}$ is obtained by restricting the codomain of f, while $e \in \overline{\mathscr{E}}$ is, levelwise, the inclusion map.

Finally, we can characterize the central extensions for $\Gamma_{\mathscr{F}}$, which is the Galois structure induced by $(\mathscr{T}, \mathscr{F})$ in $\mathrm{s}\mathbb{S}\mathrm{et}^{op}$. Theorem 4.3.13 implies that a regular epimorphism of $\mathrm{s}\mathbb{S}\mathrm{et}^{op}$ is a central extension if and only if $K[f]$ is an object of $\mathscr{F}$. In the dual perspective, we can say that a regular monomorphism f in $\mathrm{s}\mathbb{S}\mathrm{et}$, viewed as an arrow of $\mathrm{s}\mathbb{S}\mathrm{et}^{op}$, is a central extension for $\Gamma_{\mathscr{F}}$ if and only if $Q[f]$, which is the codomain of the precokernel of f in $\mathrm{s}\mathbb{S}\mathrm{et}$, belongs to $\mathscr{F}$. Moreover, since every monomorphism in $\mathrm{s}\mathbb{S}\mathrm{et}$ is regular, we can conclude that a regular epimorphism f in $\mathrm{s}\mathbb{S}\mathrm{et}^{op}$ is a central extension if and only if $f\colon X \to Y$, considered as an arrow in $\mathrm{s}\mathbb{S}\mathrm{et}$, is a monomorphism and $Y/f(X) \in \mathscr{F}$, where $Y/f(X)$ is the simplicial set defined by $(Y/f(X))_n := Y_n/f_n(X_n)$, for every natural number n. Equivalently, if $Y_n = Y_0 \cup f_n(X_n)$ for every natural number n.

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