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Multiple Documentational Genesis and Praxeologies: A Networked Approach to Investigate Mathematics Teachers' Documentation Work

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Abstract
This paper investigates the unexpected phenomenon that we call multiple documentational genesis, which occurred during a professional development program for in-service mathematics teachers, guided by researchers in mathematics education. It involves the proliferation of uncoordinated teaching materials created by teachers for their students based on the same initial input provided by researchers. To make sense of this phenomenon, we rely on a theoretical model combining the Documentational Approach to Didactics and the Meta-Didactical Transposition frameworks. We find that the logos component of the meta-didactical praxeology shared among the teachers participating in the program guided their documentation work in a way unforeseen by the researchers. Indeed, the teacher's decision to distance themselves from the researchers' initial input, perceived as non-coherent with their shared logos, led to a disruption in the co-learning partnership that had consistently characterised the program in prior years. This affected not only the sense of community among researchers and teachers but also the collaborative work among teachers themselves. The experimental findings informed new theoretical perspectives in the form of an adjustment of the theoretical model adopted for data analysis to the specific case of teachers participating in a professional development program.

Keywords Documentational approach · Documentational genesis · Meta-didactical praxeologies · Teacher professional development · Educational resources

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Introduction: Navigating Unexpected Experiment Outcomes

In scientific research, our commitment to rigorous experimentation stands as the cornerstone of empirical comprehension. Yet, it is in the unforeseen outcomes that defy our initial expectations where the heart of scientific inquiry truly thrives.

During the experimental journey detailed in this study, conducted within the context of the University of Turin (Italy) teachers' professional development (PD) program SSPM (the acronym stands for Scuole Secondarie Potenziate in Matematica, i.e. Mathematically Enhanced Secondary Schools), an unexpected phenomenon emerged: teachers, in the academic years 2021/2022 and 2022/2023, although prompted by researchers with seemingly similar inputs to design new teaching materials for their students, produced very different outputs.

In 2021/2022, teachers received as input from researchers a collection of digital materials on famous historical mathematical curves. In response, teachers collaborated closely, aligning their efforts to source and integrate additional content. Evidence of this coordinated work was seen in their consistent communication and unified approach to the materials. This collaboration resulted in a coherent output in the form of well-structured and coordinated sequences of teaching materials on these curves, to be used in their classrooms as a consistent path.

Similarly, in 2022/2023, teachers received as input from researchers a collection of slideshows on curricular topics related to contemporary mathematics. In response to this input, teachers sourced additional content, however, their efforts lacked coordination. This uncoordinated approach was evident in the inconsistent communication and varied approaches to the proposed materials. As a result, the output was a proliferation of uncoordinated teaching materials, with each teacher producing outputs that differed significantly in structure, emphasis, and presentation eventually producing a proliferation of uncoordinated teaching materials (Fig. 1).

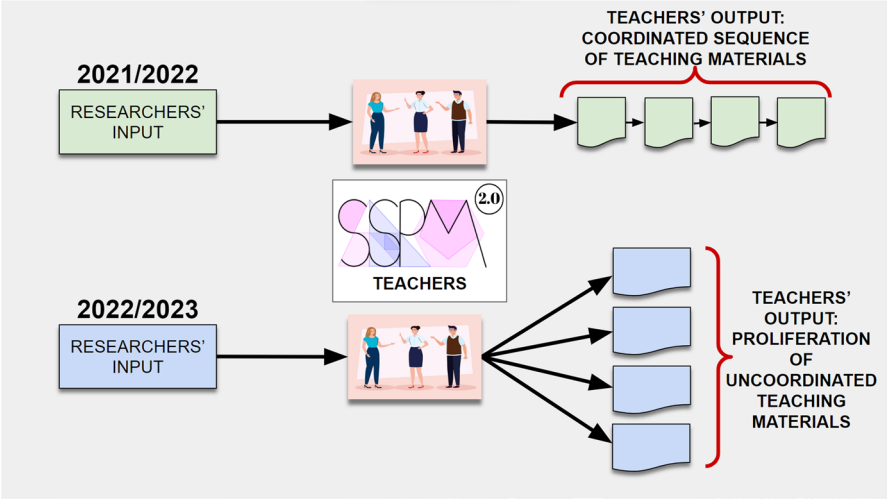

Fig. 1 Outlines of researchers' inputs and teachers' outputs in the different years

This unexpected output is even more surprising considering that, in 2021/2022, teachers spontaneously coordinated even though the whole PD program was delivered online. While, in 2022/2023, the PD program was delivered partially in person, a setting expected to ease coordinated work.

We think that this unexpected phenomenon of proliferation of uncoordinated teaching materials is worth investigating, because it significantly impacted the usability and effectiveness of the teaching materials produced in 2022/2023, as well as the teachers' community and students' learning. Moreover, the results of this investigation can be used in the future to design other PD programs.

In 2021/2022, teachers' coordinated approach encouraged the exchange of ideas and materials, ultimately allowing the entire teaching community to benefit from the shared resources and providing students with a coherent learning trajectory (Simon, 1995; Wilson et al., 2014) on the chosen topic.

Conversely, in 2022/2023, teachers' output resulted in teaching materials tailored to individual preferences, posing challenges for teachers in terms of sharing. Consequently, the sense of community among teachers diminished, and they encountered difficulties in finding common ground for collaboration. Also from the students' perspective, engaging in a single activity on a topic is less beneficial than experiencing an instructional sequence constituting a coherent learning trajectory.

To make sense of this phenomenon we need a fine-grain analysis to identify its potential causes. Since the teachers, the researchers, and the PD program institutional context remained consistent in both years, our focus turns to the inputs provided by the researchers.

Researchers deemed these inputs similar because they were both composed of collections of digital content on specific mathematical topics, expected to foster teachers' curiosity and creativity to design new teaching materials. Nevertheless, they were different from two standpoints: the organisation of the content and the institutional context originating the content.

In 2021/2022, the researchers' input was an unstructured collection of texts, images, videos, GeoGebra files, and links to websites about mathematical curves such as catenary or tractix, gathered in a digital board (Padlet) by the researcher guiding the PD program (the second author), *inside* the Italian institutional context.

In 2022/2023 the researchers' input was a collection of structured and self-standing PowerPoint presentations on curricular topics related to modern mathematics such as geometry or probability, conceived by the Israel project Mathematics News Snapshots for High School (MNS), *outside* the Italian institutional context.

We hypothesise that the peculiar characteristics of the 2022/2023 input might be the origin of the unexpected phenomenon of proliferation of uncoordinated teaching materials obtained as output. The study shall therefore focus on the causal connection between this disrupting input and the subsequent uncoordinated teachers' work.

To pursue our research, we need theoretical tools suitable for (1) investigating teachers transforming existing resources into teaching materials tailored to their local context and students' needs and (2) analysing the reasons supporting teachers' design choices.

We interpret the process of teachers designing teaching materials as an instance of documentational genesis, which can be examined through the lens of the

Documentational Approach to Didactics (DAD) developed by Gueudet and Trouche (2009, 2010, 2012) and their colleagues (Gueudet et al., 2012). Specifically, we name this particular instance as *multiple documentational genesis*, where the term *multiple* signifies the emergence of uncoordinated teaching materials. We consider this *multiple documentational genesis* worth conceptualising and investigating due to its impact on the usability and the didactical value of the final documents.

When teachers engage in the process of documentational genesis, they make various decisions, including selecting resources to base their documentation work on, and determining how to use them with their students. These decisions are guided by underlying general discourses that justify their practices. We conceptualise these justifying discourses as the logos underlying teachers' pedagogical practices. To investigate these aspects, we employ the Meta-Didactical Transposition (MDT) framework (Arzarello et al., 2014). MDT offers a comprehensive approach to understanding teachers' (and researchers') practices and logos when they are engaged in a PD program.

In summary, our study seeks to unravel the complex web of *multiple documentational genesis* and the logos that guides it, using the theoretical and analytical tools provided by the aforementioned frameworks.

Theoretical Frameworks

Documentational Approach to Didactics

The DAD framework (Gueudet & Trouche, 2009, 2010, 2012; Gueudet et al., 2012; Trouche et al., 2019) is suited to examine how teachers use various resources to plan their lessons and support students' learning. The DAD framework draws inspiration from the instrumental approach proposed by Vérillon and Rabardel (1995) in cognitive ergonomics, who introduced the concept of *instrument*, i.e. an artefact associated with a utilization scheme tailored to address a specific activity (Vergnaud, 2009). The process of producing instruments is referred to as instrumental genesis. Similarly, the DAD framework introduces the concept of *document*, i.e. a set of resources associated with a scheme of utilization, tailored to address a specific classroom situation. This core idea is expressed by Gueudet and Trouche (2009) as follows: "Document = Resources + Scheme of utilization" (p. 205). The process of producing documents is referred to as documentational genesis (Gueudet & Trouche, 2009), and it is the result of the active involvement of individual teachers.

In a document, the utilization scheme encompasses both visible and invisible elements. The visible elements are the regularities in the teacher's action for the same class of classroom situations (rules of action) in which the resources are employed. The invisible elements are constituted by the cognitive structure guiding the teacher's action, including the teacher's beliefs and knowledge (operational invariants). The operational invariants act as both driving forces and outcomes of the teacher's actions.

To access visible and invisible components of a document, Gueudet and Trouche (2012) formulated a distinctive methodology for data analysis within the DAD

framework, based on the reflective investigation of teachers' documentation work. This methodology places considerable importance on teachers themselves. Indeed, the active participation of teachers, adopting a reflective perspective, is essential as they have insights into their documentation work and can bring to light concealed resources.

Meta-Didactical Transposition

Meta-Didactical Transposition (MDT) (Arzarello et al., 2014) is based on Chevallard's Anthropological Theory of Didactics (Chevallard, 1985), which conceptualises mathematics teaching as a human activity that occurs within institutions and is characterised by praxeologies. The term *praxeology* is introduced to go beyond the simple praxis: in fact, it comprises two components: know-how (praxis), constituted by a task and a technique to solve it, and know-why (logos), constituted by a technology justifying the technique and a theory to frame the technology (Chevallard, 1985). According to Chevallard (1999), this concept can refer to students' mathematical practices and justifying discourses within the classroom (mathematical praxeologies), or to teachers' practices and justifying discourses used by them to promote mathematical praxeologies in the classroom (didactical praxeologies).

In teachers' PD programs, praxeologies become meta-didactical, as they pertain to researchers' and teachers' practices and justifying discourses to address didactical issues at a meta-level, namely, to reflect on the didactical level.

In these contexts, MDT frames the relationships and reciprocal influences of the two communities—teachers and researchers—modelling the evolution of their meta-didactical praxeologies. Both communities develop new practices (praxis) or justifying discourses (logos) internalising (in the sense of making them part of their praxeologies) components that were initially external (Cusi et al., 2023).

Combining DAD and MDT: A Networked Model

Pocalana and Robutti (2023) introduced a model combining, in the sense of networking (Prediger et al., 2008), the researchers' documental genesis (DAD), which occurred during a teachers' PD program (Pocalana et al., 2023), within their meta-didactical transposition (MDT) for the design and implementation of the PD program itself. In this model, schematised in Fig. 2, the researchers are called didacticians according to Jaworski's (2006) terminology, because, in the context of the PD program, they assume the role of teacher educators. Moreover, since didacticians share a common praxis and logos as a result of their shared institutional and cultural context, the networked model refers to a singular shared meta-didactical praxeology in which the documentation work is embedded.

In the networked model introduced by Pocalana and Robutti (2023), for the first time, DAD and MDT frameworks are combined and interpreted one in light of the other. The documentational genesis is interpreted as part of the techniques adopted by didacticians to accomplish the task of their meta-didactical praxeology, namely, the design and implementation of the PD program. The utilization schemes of the

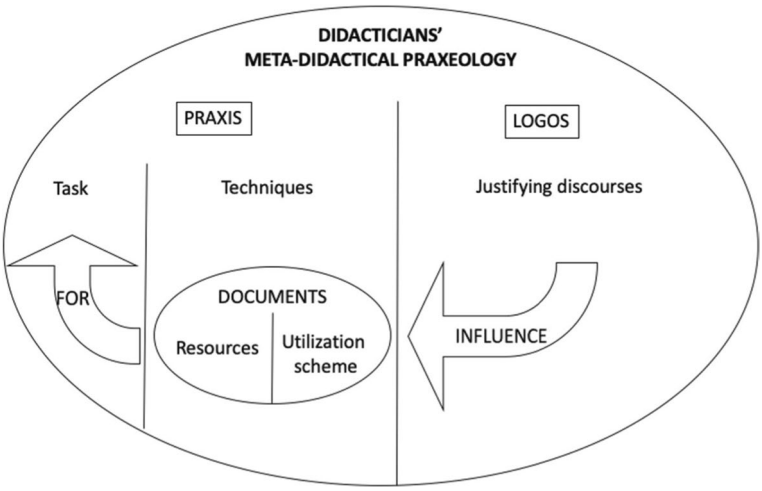

Fig. 2 A networked model for didacticians' work (Pocalana & Robutti, 2023, p. 20)

didacticians' documents encompass both the invariants in their actions and the guiding forces motivating their actions, so they are deeply influenced by the general discourse (logos) justifying the didacticians' practices (praxes) in the context of the PD program.

This DAD and MDT networked model has also been applied to the case of teachers generating documents for their students in the study conducted by Pocalana et al. (2024) and Pocalana and Robutti (2024). So, in light of this model, we formulate the research question guiding the present study:

> RQ: What is the relationship, if any, between the observed phenomenon of *multiple documentational genesis* and teachers' meta-didactical praxeologies?

Methodology

Experimental Environment: The SSPM teachers' PD Program

The experimental environment for this research is the SSPM PD program for secondary school mathematics teachers organised by the University of Turin under the supervision of the third author (Pocalana et al., 2024; Pocalana & Robutti, 2024). The SSPM teacher PD engages mathematics education researchers as teacher educators in programs at two levels: first-level programs for grades 6–8 teachers (Medie 1.0) and grades 9–13 teachers (Liceo 1.0), and second-level programs (Medie 2.0 and Liceo 2.0) for teachers who completed the first-level programs. All programs develop along monthly encounters from September to May, promoting an inquiry-based approach to mathematics education (Laursen & Rasmussen, 2019; Maaß &

Artigue, 2013), hands-on mathematics laboratory teaching (Anichini et al., 2004; Arzarello & Robutti, 2008), and student-centred learning principles (Noyes, 2012).

The primary distinction between 1.0 and 2.0 programs, despite teachers being learners in both, lies in the teachers' interaction with teaching materials (Pocalana et al., 2023). In 1.0 PD program meetings, teachers receive and discuss teaching materials entirely conceived and designed by researchers. In 2.0 meetings, teachers receive input from researchers and collaborate to design original teaching materials for student-centred activities, encouraging exploration and, when possible, hands-on or software-mediated manipulation. This fosters a co-learning partnership (Jaworski, 2003, 2006) between teachers and researchers, promoting a questioning attitude and critical alignment through a process of co-learning inquiry (Pocalana & Robutti, 2022, p. 324). Consequently, 2.0 teachers exhibit extensive experience in documentation work.

The PD program under examination in this paper is the 2022/23 Liceo 2.0 program. It was led by the second author with in-person and online meetings, supervised by the third author. Participants, comprising 32 mathematics teachers with professional experience ranging from 8 to 30 years, have attended PD programs within the SSPM project and other University of Turin initiatives for 5 to 20 years.

SSPM meets MNS

In the 2022/23 Liceo 2.0 PD program, the researchers' input originated from the collaboration between SSPM and the Mathematics News Snapshots (MNS)[1] Project at Technion University in Israel. The MNS Project aims to bridge the gap between teaching and contemporary mathematics research by providing educators with PowerPoint presentations investigating specific mathematics topics, narratives, and recent discoveries (Amit & Movshovitz-Hadar, 2011). These presentations, known as Snapshots, are designed as 30-min lessons for secondary school students, adopting a lecture-driven pedagogical perspective. Currently, Snapshot presentations are available on the MNS website only in Hebrew and English, necessitating translation into other languages to enhance accessibility among teachers and students.

Within the institutional collaboration between the two projects in the two Universities, teachers in the Liceo 2.0 PD program have been enlisted as translators for Snapshot presentations from English into Italian. This collaboration is grounded in the understanding that translating a scientific text requires proficiency in both the source and target languages, coupled with a comprehension of the scientific content to convey it in the target language using the appropriate lexicon. It was emphasised that the final translated presentation should not be extracted for incorporation into other presentations, nor should it be modified to include supplementary materials for personal use. To produce Italian versions faithfully replicating the English originals, SSPM teachers received guidance to preserve the integrity, structure, and graphical elements of the PowerPoint presentation in their translations. Since researchers were

[1] https://mns.org.il/

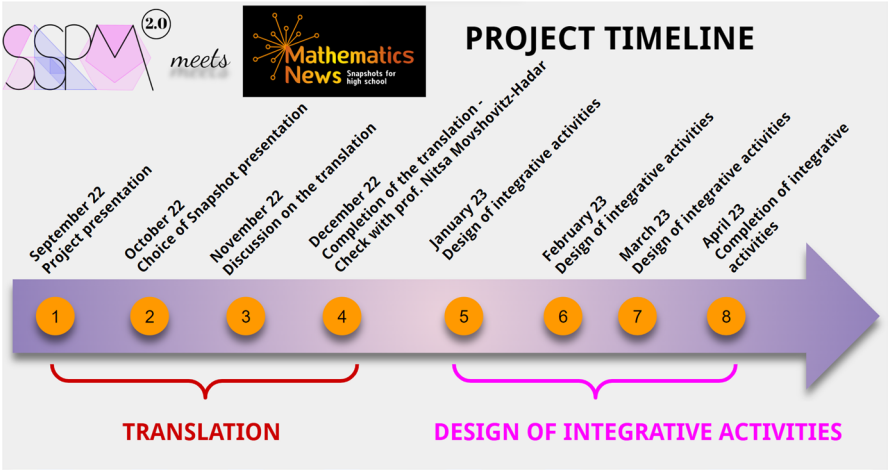

Fig. 3 The timeline of the SSPM Liceo 2.0 program in 2022/23

aware that the translated Snapshot presentations would not be immediately exploitable for hands-on, laboratory, and student-centred activities aligned with the SSPM logos, in the PD program they asked teachers to "design integrative activities", producing new teaching materials to complement the Snapshot presentation. This request prompted teachers' documentational genesis.

An institutional agreement was reached between the MNS and SSPM projects to translate two Snapshot presentations from the 26 available on the MNS website. Researchers facilitated the selection process using a social voting tool[2] allowing each teacher to express up to three votes. Based on the voting results, teachers voluntarily organised into two groups, each focussing on one of the selected presentations.

The timeline of the PD program in Fig. 3 was shared with teachers. Teachers' collaborative work comprised two phases: the first phase – Translation – also including the choice of the Snapshot to be translated and not involving any documentational genesis, and the second phase—Design of integrative activities—where the documentational genesis occurred. Teachers collaborated both synchronously, during monthly meetings, and asynchronously through the Moodle platform provided by the PD program.

Methodological Approach

Pocalana and Robutti's study (2023) has shown how didacticians' documentation work can be conceptualised as part of the praxis component of their shared metadidactical praxeology. In this study, we adopted an analogous combination of DAD and MDT frameworks in a new context: to answer our research question (§ 2.3) regarding the relationship between teachers' *multiple documentational genesis* and

[2] https://bit.ly/triciderSSPM

their meta-didactical praxeologies. More specifically, we analysed teachers' documentation work interpreting it as part of teachers' meta-didactical praxeologies to address the task set by the researchers. We then inferred the logos guiding teachers' praxeologies and, therefore, their documentation work.

In the context of Pocalana and Robutti's study (2023), the subjects under scrutiny were the authors themselves, who served as didacticians in the PD program. So, an introspective attitude was adopted by the authors to recognise their shared praxeology.

In this study, the subjects are teachers, distinct from the authors themselves. Consequently, we faced the challenge of not having direct access to their praxeologies, particularly regarding the motivations and justifications guiding their practice.

Therefore, we first focused on their documentation work, adopting the methodology of *reflective investigation* of teachers' documentation work, developed within the DAD framework (see § 2.1). To do so, we involved the teachers in a reflection on the different phases of their documentational genesis process, to gain a deep insight into their documentation work. As a second step, we inferred their meta-didactical praxeologies for the design of activities for their students.

Data Collection

Coherently with our methodological approach, we collected data from:

- the teaching materials designed by the teachers during the PD program, based on the original MNS resources and on other types of resources retrieved independently;
- oral semi-structured interviews with a sample (n = 6) of teachers, conducted online by the authors.

The oral interviews aimed at eliciting teachers' reflective investigation (Gueudet & Trouche, 2012), to deepen our insight into their documentation work and to gain an understanding of their meta-didactical praxeologies.

Since it was not feasible to interview every teacher, we selected 6 teachers that we considered representative of the Liceo 2.0 group, as they embodied the key characteristics and challenges faced by the whole group. The first author conducted the interviews, with all authors attending via online video calls. Each session lasted about thirty minutes and was recorded and fully transcribed.

Table 1 illustrates the interview questions and research foci:

The first three questions were aimed at shedding light on the first phase of the PD program, involving the selection and translation of the Snapshot presentations for their collaborative work (Fig. 3). Questions number 5 and 6 were aimed at investigating the teachers' documentational genesis process promoting a reflective investigation attitude. In particular, from the responses to question number 5, we sought information about the additional resources that teachers relied on in their documentation work, beyond the initial input provided by the researchers. From the responses to question number 6, we aimed to deepen our insight into the utilization schemes

Table 1 Semi-structured interviews: questions and research foci

Question	Research focus
1. What kind of input has been provided to you by the researchers during last year's Liceo 2.0 PD program?	Praxeologies guiding the selection and translation of Snapshot presentations
2. How did you work on this initial input?	
3. Did you encounter difficulties in dealing with the input provided? If yes, which ones?	
4. Within your group, did you plan just one activity for students, or were there any differentiations? Why?	Documentational genesis - Group internal management
5. In your planning of your activity for students, did you use other resources, in addition to the input provided by the researchers? If so, where did they come from?	Documentational genesis - Additional resources
6. How did you plan to use these resources to design an activity for your students?	Documentational genesis - Schemes of utilization
7. What goals did you set as a teacher in designing your activity?	Documentational genesis + logos of teachers' praxeologies - Educational objectives
8. What educational value do you see in the activity that you designed?	

of the documents designed by the teachers for their students. The final two questions aimed to prompt reflections from the teachers regarding the educational objectives of the activities that they had designed. We particularly relied on the responses to these questions to infer the logos of the teachers' praxeologies related to the design of those activities.

Data Analysis Method

Through the analysis of teachers' produced materials and interview transcripts, we explore the relationship between the observed phenomenon of *multiple documentational genesis* and teachers' meta-didactical praxeologies. The analysis is conducted in two steps:

1) a first step to illuminate the documentational genesis process;
2) a second step to reconstruct the meta-didactical praxeologies guiding the documentational genesis.

For the first step, we enlist the main additional resources that each teacher used in his/her documentation work beyond the initial input provided by the researchers. Following this, we trace the utilization scheme of each document, describing how the teachers plan to work in the classroom with the resources.

For the second step, we use the teachers' narratives to shed light on their meta-didactical praxeologies. From teachers' descriptions of their choice and intended

use of additional resources, we infer a general description of the praxis component of their meta-didactical praxeologies. From the teachers' explanations of the reasons guiding their choices for their documentation work, we infer the general justifying discourses for their praxis, which constitute the logos component of their praxeologies.

Results of Data Analysis

First phase: Choice and Translation of Snapshots

Using the social voting tool proposed by researchers, teachers selected two Snapshot presentations: 'Random Walks' (https://mns.org.il/random-walks/, Fig. 4a) and 'Non-Round Wheels" (https://mns.org.il/non-round-wheels/, Fig. 4b). They then divided into two groups based on these choices. Of the six teachers interviewed, Diana, Simona and Serena are representatives of Group 1 who worked on "Random walks", Amanda, Mario, and Teresa represent Group 2 who worked on "Non-round wheels". All names are pseudonyms, and the interview transcripts have been translated into English by the authors).

In the interviews, all teachers underpinned the novelty of the topic as their main criterion for choosing the Snapshot presentations for translation.

> Simona (Group 1): [...] one topic was an activity already presented during the SSPM PD programs in the past years, some others were simple to approach. In short, the criterion of choice was what we could explore, perhaps not so well known.
> Mario (Group 2): [...] we tried to select some things that interested us, trying not to duplicate things that we already knew or that we had already done elsewhere.

This criterion for the presentation choice can be considered the logos component of a meta-didactical praxeology shared among the teachers (Table 2), in response to the first task assigned by the researchers.

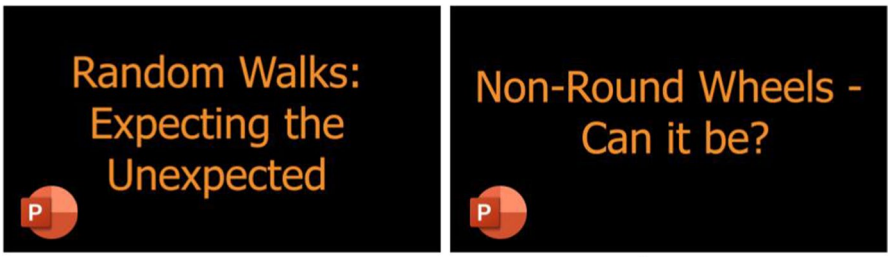

Fig. 4 Cover slides of the two Snapshot presentations chosen by the teachers

Table 2 Shared meta-didactical praxeology for the presentation choice

Praxis	Logos
Task: Choice of the Snapshot presentations to work on *Techniques:* Choose topics that have not yet been covered in SSPM or other PD programs	Favour novelty and curiosity (both for students and for teachers) in the choice of the topic, going beyond the personal comfort zone

Subsequently, each group undertook the translation of their chosen presentation. All interviewees recalled precise and stringent instructions, such as the format and content of the slides had to be fully adherent to the original. This aspect has been perceived as somewhat restrictive by the teachers, who highlighted it several times in the interviews.

> Serena (Group 1): We started translating and maintaining the characteristics and indications given to us: that is, not changing the number of slides, not changing the formatting, not changing... so let's say the attention initially shifted more to this than to the mathematical aspects.
> Teresa (Group 2): In the translation, we were given constraints, that is, it had to be strictly respected what was... what was written, and also the layout, it must be precisely the faithful translation and not embellished or cut. This was exactly the request coming from... from the University that had given us this material.

From these testimonies, we can infer the meta-didactical praxeology that teachers shared and adopted to address the researchers' second task of translating the Snapshot presentations (Table 3).

The production of the translated presentation marked the conclusion of the first phase of the PD program, during which teachers collaborated seamlessly both online and in person. Divided into two independent groups, they adopted a coordinated approach, splitting the work among all participants, discussing translation choices, and ultimately unifying the translations, producing as output translated versions of the two Snapshot presentations that met unanimous agreement from all participants in each group.

Table 3 Shared meta-didactical praxeology for the presentation translation

Praxis	Logos
Task: Translation from English to Italian of the Snapshot presentations from the MNS project	Follow the instructions for the translation, coming from the Israeli researchers leading the MNS project
Techniques: Strictly adhere to the format and content of the original presentation	

Second Phase: Design of Integrative Activities

In the second phase of the PD program, the task assigned to teachers was to design integrative activities. Researchers formulated the task using the term "integrative", considering that teachers could incorporate the presentation of the translated Snapshot slides as resources for their documentational genesis. However, this possibility was promptly rejected by all teachers. Consistent with the logos related to the SSPM project, teachers deemed the Snapshot presentations not fit to be used in their classrooms, as incompatible with the principles of the *mathematics laboratory* approach (Anichini et al., 2004; Arzarello & Robutti, 2008) and the inquiry-based approach in mathematics education (Laursen & Rasmussen, 2019; Maaß & Artigue, 2013).

> Serena (Group 1): Giving a lesson with slides in a classroom is something that isn't right for me. It's not good… It's not part of my being, so it's rare for me to stand there and present slides to a class. So, that was the thing that… the difficulty, one of the biggest difficulties from my point of view.
> Mario (Group 2): Once the translation was done, there was this thing of trying to put it in our comfort zone because objectively those presentations were not for us. [...] We're used to … if we have to do an activity, we do a manipulative activity [...] While here it was very much 'I'll show you and explain it to you by making the slide'.

Teachers shared meta-didactical praxeology regarding the use of Snapshot presentations in their classrooms is described in Table 4.

Multiple Documentational Genesis

When the teachers began to generate documents for their students, they shifted away from the coordinated group approach informing the first phase and opted to work individually or in pairs, choosing partners from the same school. It is in this phase that we observed the proliferation of uncoordinated documents, giving rise to the phenomenon that we named *multiple documentational genesis*. It is important to note that teachers not only worked individually but also did not share their documents with the group until requested to do so by the researchers.

Group 1: Original input "Random walks". We present the documents D1, D2 and D3 produced respectively by Serena, Simona and Diana, representing Group 1 ("Random walks").

Table 4 Shared meta-didactical praxeology for the classroom use of presentations

Praxis	Logos
Task: Use of the translated Snapshot in the classroom *Techniques:* Refuse the presentation of the slides, as they are, to the students	Consider presenting slides in the classroom as not coherent with the SSPM teaching approach

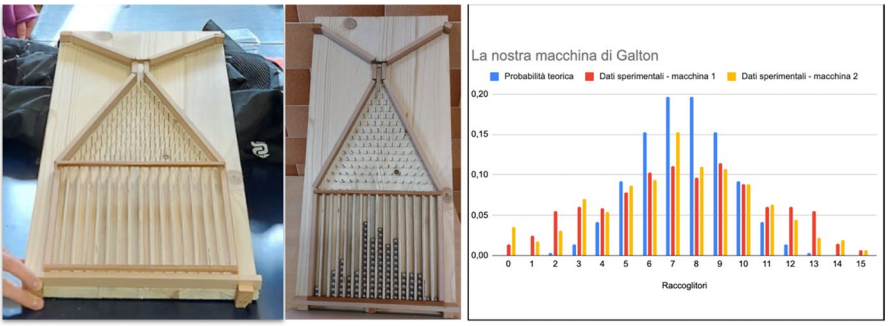

Fig. 5 Students' experimentations with two Galton boards

Table 5 Document D1: Galton Board

Additional resources	Scheme of utilization
- Instructions for building a Galton Board; - plywood panels; - metal balls; - tools for working with plywood	Students and teacher collaborate in: - Figuring out how to build a Galton board made of plywood; - building two concrete models of the Galton board; - conducting an experiment with the two different boards and metal balls to compare the results; - verifying if the empirical results align with the expected results according to the central limit theorem (normal distribution)

Document 1: Galton Board—Teacher: Serena. Document D1 (Table 5) has been created by Serena for a hands-on and experimental activity designed for her 13th-grade students attending a scientific upper-secondary school. The activity is centred on the Galton Board, a machine in which a number of balls perform random walks when dropped vertically through a harrow of pins, causing random lateral deflections in their trajectories. The balls' final configuration provides a practical validation of the central limit theorem and normal distribution.

In the activity, students build and experiment with two different boards (Fig. 5a and 5b) comparing and contrasting experimental and theoretical data (Fig. 5c).

In the interview, describing the idea originating the activity, Serena reports that:

> Serena: The slides were intended for a lecture, and we tried to find activities that could somehow facilitate the transposition, making it a bit more in line with our way of working.
> Serena: We are used to taking a real situation and reading it with the eyes of mathematics, putting our hands on it, trying to see what happens, then trying to see if there is a law behind … it is part of... our way of being a teacher.

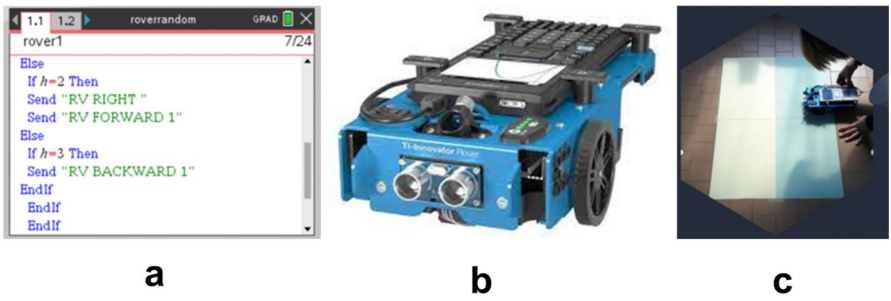

Fig. 6 Programmable graphing calculator controlling a rover with a marker

Table 6 Document D2: Physical random walk

Additional resources	Scheme of utilization
- Programmable graphing calculator; - rover connected and controlled by the graphing calculator; - manual to learn how to program the rover through the graphing calculator	Students and teacher collaborate in: - programming a graphing calculator and a rover controlled by it to obtain a physical representation of a random walk; - breaking down a problem-solving strategy into a series of coding commands

Document D2: Physical random walk with a rover—Teacher: Simona. Document D2 (Table 6) has been created by Simona for an experimental activity based on coding, designed for students attending the 10th grade of a scientific upper-secondary school. Her objective was to provide students with a hands-on experience of a physical random walk performed by a small mobile robot (a rover) connected to a graphic calculator. This activity implied programming a graphing calculator to simulate a random walk (Fig. 6a), connecting the calculator to a rover equipped with a marker (Fig. 6b), and finally visualising the random walk traced by the rover (Fig. 6c).

In the interview, describing the idea originating the activity, Simona reports that:

> Simona: It was natural for each of us to go with what we were doing at that moment, trying to decline it. Some were working on probability and therefore tried to use probability. [...] I decided to use robots and I said, 'okay we could simulate the same thing that was in the slides with the rover'.
> Simona: The use of programming enables one to reflect on partitioning the problem in a sequence of instructions. So, analyse a problem from a mathematical point of view and tell a machine what you do in your brain, in order to make it reach the result following the instructions given by you. [...] And then hypothesise results and try to verify whether these occur or not.

Document 3: Digital random walk—Teacher: Diana. Document D3 (Table 7) has been created by Diana for a coding and experimental activity designed for students attending the 10th grade of a technical upper-secondary school. Diana has a degree

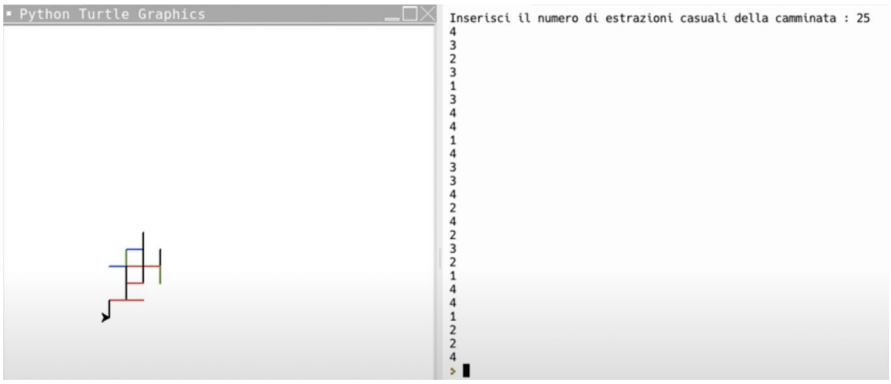

Fig. 7 Python code generating a 2D random walk

Table 7 Document D3: Digital random walk

Main Resources	Scheme of utilization
Python Turtle Graphics	Students and teacher collaborate in: - using Python Turtle Graphic tool to experimentally generate a digital representation of a random walk; - writing a code that allows programming a random walk with the number of random steps as input

in computer science, and she designed an activity (Fig. 7) that merges the Snapshot presentation topic of "Random walks" with her personal technical knowledge. In her activity, she imagined students using Python to program a 2D random walk to be performed in the Cartesian plane.

In the interview, Diana describes her idea as follows:

> Diana: I designed an activity that re-created with Python a random walk by simply throwing the dice and showing the path on the plane, on the Cartesian plane, in 2D only.
> Diana: I have a degree in computer science, so I've always seen mathematics as something useful, serving various purposes.

In the Snapshot presentation, there was an illustration of a planar random walk; however, only the final result was displayed, and there was no accompanying explanation about the generating. This lack initiated Diana's need to find other resources to design a hands-on activity for her students.

Figure 8 presents a visual summary of Group 1 work, highlighting the *multiple documentational genesis*, i.e. the proliferation of uncoordinated documents, emerging after the translation phase.

From the analysis of Group 1's documentational genesis, we can now draw some inferences about teachers' meta-didactical praxeologies. All documents show that teachers rely on laboratory, hands-on experiences to enhance student engagement

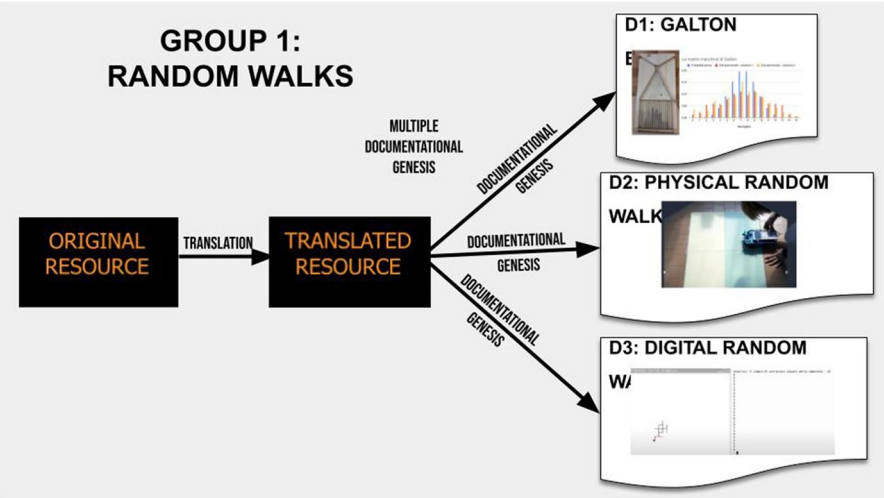

Fig. 8 Teachers' *multiple documentational genesis* for "Random walks"

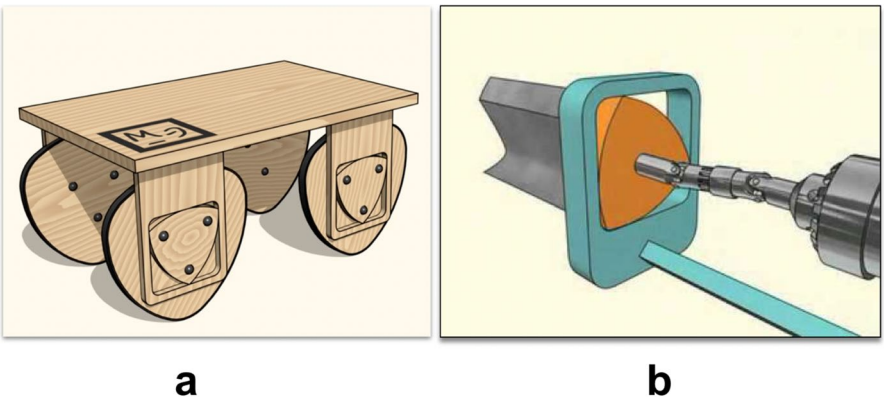

Fig. 9 A square hole drill and a little cart with wheels made with Reuleaux's triangles (Taken from: https://en.etudes.ru/etudes/)

and meaning-making, and to facilitate the interpretation of experimental regularities, which are then linked to mathematical concepts. Teachers employ hands-on and coding activities to bridge the gap between theoretical and practical application, allowing students to visualise phenomena and develop skills in logical sequencing and mathematical reasoning.

Group 2: Original input "Non-round wheels". We present the documents D4, D5 and D6 produced respectively by Mario, Teresa and Amanda, representing Group 2 ("Non-round wheels").

*Document D4: Square hole drill + Reuleaux cart—Teacher: Mario***.** Document D4 (Table 8) has been created by Mario for a hands-on experimental activity designed

Table 8 Document D4: Square hole drill + Reuleaux cart

Additional resources	Scheme of utilization
- Website with instructions to build the Reuleaux cart with proper suspensions - plywood panels; - tools for working with plywood	Students and teacher collaborate in: - building four suspensions composed of Reuleaux triangles moving within an almost perfect square, constructed according to the square hole drill criteria - building a plywood cart with wheels shaped as Reuleaux triangles; - experimentally verifying the cart stability thanks to the wheels' constant width

for students attending the 11th grade of a scientific upper-secondary school, inspired by the Snapshot presentation on Non-round wheels. The activity is centred on the square hole drill (Fig. 9a), which is based on an intriguing phenomenon: when the centre of a Reuleaux triangle[3] is moved along a specific trajectory (made up of four equal patched arcs of ellipses) its vertices trace an almost perfect square, encompassing the area within this geometric shape. Remarkably, the borders of the resulting figure, aside from minor angular sections, will consist of straight segments. This almost perfect square constitutes the ideal suspension for carts equipped with Reuleaux triangle-shaped wheels, which move without rocking (Fig. 9b).

> Mario: I found a website where there were some things that followed what we already had in mind, but it suggested two interesting things about the applications, for example, the square hole made by the drill, and then the little cart with wheels made with Reuleaux's triangle.
> Mario: I thought that one thing that could be interesting to have them do is making the little cart with the wheels made with the Reuleaux triangle.

The square hole drill and the Reuleaux cart were already mentioned in the Snapshot presentations, but Mario wanted to explore additional resources to enable the students to bring these projects to life by designing a hands-on activity.

Document D5:—Leonardo da Vinci's world map—Teacher: Teresa. Document D5 (Table 9) has been created by Teresa for a hands-on activity designed for her 9th-grade students attending the same school as Mario, inspired by the Snapshot presentation on the Non-round wheels. The activity designed by Teresa is centred on Leonardo da Vinci's world map in eight octants in the form of Reuleaux triangles (Fig. 10).

> Teresa: Let's say the idea was to integrate the activity with physical objects, built by us or by the guys with the 3D printer. This was partly to show first-hand, to touch the object (the somewhat concrete aspect of manipulating).

[3] Named after mathematician Franz Reuleaux, a Reuleaux triangle is a shape formed by the intersection of three circles of the same radius, resulting in a curve of constant width.

Fig. 10 Leonardo da Vinci's world map in eight octants in the form of Reuleaux triangles (Taken from: https://en.wikipedia.org/wiki/Reuleaux_triangle)

Table 9 Document D5: Leonardo da Vinci's world map

Additional resources	Scheme of utilization
- Wikipedia - 3D modelling software - 3D printer	Students and teacher collaborate in: - reproducing Leonardo's world map with a 3D modelling software; - 3D printing the model

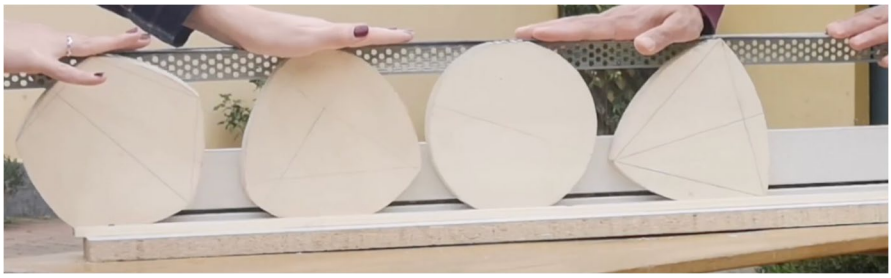

Fig. 11 Reuleaux triangle and other constant-width shapes as wheels (frame taken from the video https://www.youtube.com/channel/UCXWF_gKgyUXVSK3t35SdjFA)

Table 10 Document D6: Cardboard non-round wheels

Additional resources	Scheme of utilization
- Video on the Reuleaux triangle and other constant-width shapes retrieved from a YouTube channel - Cardboard - Tracks for the cardboard constant-width shapes	Students and teacher collaborate in: - building physical models of Reuleaux triangles and other constant-width shapes using cardboard; - verifying that these constant-width shapes always remain contained between two parallel tracks and, when rotated, consistently touch both tracks

Teresa: I think a lot about the little students and sometimes you see that if they touch the objects, then they understand better.

Document D6: Cardboard non-round wheels—Teacher: Amanda. Document D6 (Table 10) is a hands-on and experimental activity imagined by Amanda for 9th or 10th students attending an upper-secondary school focused on humanities). Amanda envisioned engaging her students in constructing a series of non-round wheels by cutting out Reuleaux triangles and other shapes with constant width using cardboard and scissors. The construction activity is followed by an experimental phase, allowing students to verify the geometrical property of the constant width of the non-round shapes by sliding them between two fixed guides (Fig. 11).

Amanda: I thought that students today really need a lot, a lot, a lot of concreteness especially the small ones, right? Let's say especially in grades 9th and 10th. [...] For the activity on the non-round wheels, those shapes would have had to be built. We also watched the video made by Giulia's [Author 3] students on the Reuleaux triangle
Amanda: I thought about cutting out the wheels on a piece of cardboard, and also, putting horizontal guides for the wheel, which always had the same width, making the wheels run and, so, the polygons rotate. Instead of seeing the animation in the slides, let's do it concretely.

Figure 12 presents a visual summary of Group 2 work, highlighting the *multiple documentational genesis*, i.e. the proliferation of uncoordinated documents, emerging after the translation phase.

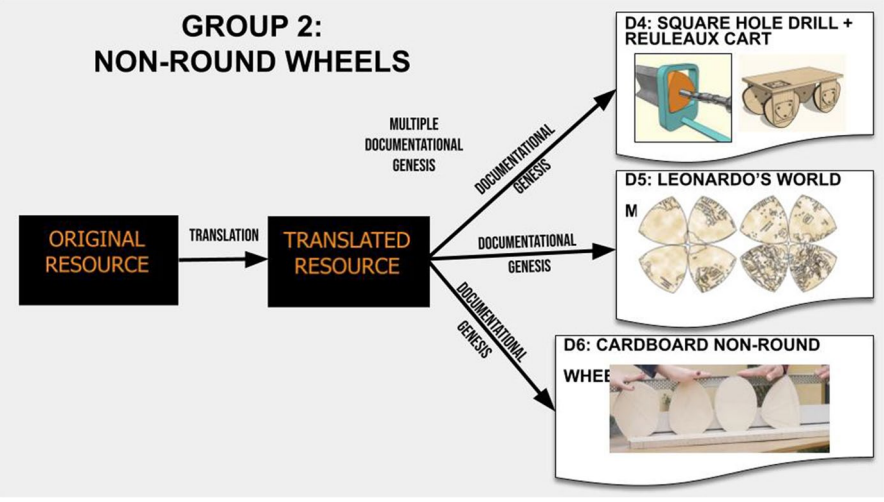

Fig. 12 Teachers' *multiple documentational genesis* for "Non-round wheels"

From the analysis of Group 2's documentational genesis, we can shed light on teachers' meta-didactical praxeologies. All documents demonstrate the use of physical artefacts aligned with mathematical objects, which facilitate empirical verification of theoretical properties and foster meaningful learning. Teachers' choices show that they are guided by the idea that incorporating technological tools to construct physical models of mathematical objects enables students to visually appreciate and directly grasp their characteristics, and hands-on building of these models allows for empirical verification of theoretical properties, promoting deep understanding among students.

Drawing together the analyses of Group 1 and Group 2 output, we observe the phenomenon we named *multiple documentational genesis*, entailing the proliferation of uncoordinated teaching materials produced by teachers resulting in several independent documents elaborated by teachers individually.

Despite the multiplicity of teachers' documentation work, by comparing teachers' meta-didactical praxeologies inferred before we can identify a unique shared meta-didactical praxeology, whose logos guides all documents' schemes of utilization. This passage from a plurality of praxeologies to a *unique shared praxeology,* of which the documentational genesis is part, is the core of the analysis. Specifically, teachers' descriptions of their choices and intended use of additional resources delineate the *shared* praxis. Teachers' explanations of the reasons guiding their choices delineate the *shared* logos.

This shared meta-didactical praxeology is summarised in Table 11.

Findings

From the results of our data analysis, we find that, in the first phase of the PD program (Translation), all observed teachers share the same meta-didactical praxeology to address the task: choice of the Snapshot presentations to work on (Table 2) and translation of the chosen Snapshots (Table 3). For the second phase of the PD program (Design), we can infer that all observed teachers share the same laboratory and student-centred meta-didactical praxeology, which distances them from the classroom use of the translated Snapshot presentations as-is. Indeed, for the documentational genesis, the teachers share a meta-didactical praxeology strictly connected with the SSPM logos, which gives a distinctive, shared imprint on

Table 11 Teachers' shared meta-didactical praxeology for documentational genesis

Praxis	Logos
Task: Design of activities for students, based on Snapshot presentations *Techniques:* Favour - real-world applications of mathematics (D1, D4, D5, D6); - hands-on activities (all documents); - design and creation of physical (D1, D4, D5, D6) or digital artefacts with students (D2, D3, D6)	Favour a teaching approach coherent with - inquiry-based approach; - mathematics laboratory teaching; - student-centred learning principles

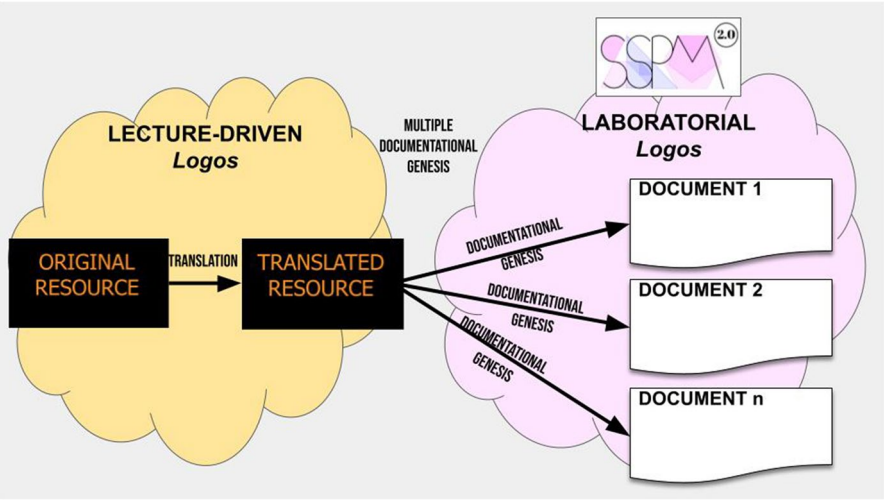

Fig. 13 Interpreting teachers' multiple documentational genesis in light of logos discontinuity

their praxis and, therefore, on their documentation work. This fact can be seen as a strength of the SSPM PD program itself, demonstrating its significant impact on the participating teachers.

We claim that teachers' shared meta-didactical praxeology, particularly the very strong logos related to mathematics laboratory and student-centred approach, is the cause of the observed phenomenon of *multiple documentational genesis*. Indeed, the foreign institutional directive not to modify the translated slides, combined with teachers' reluctance to use these slides as-is with their students, due to their association with a lecture-driven teaching approach, led teachers to create entirely new activities. Although some ideas explored by teachers were present in the Snapshot presentations, they sought additional resources to design new documents, intentionally distancing themselves from the researchers' initial input.

The difference between the coordinated output produced by teachers in 2021/2022 and the uncoordinated output in 2022/2023 can be interpreted through the lens provided by the networking of the DAD and the MDT frameworks, as a result of the teachers' different perceptions of the researchers' inputs. In 2021/2022, teachers perceived the unstructured digital materials on mathematical curves provided by researchers as resources not paired with specific logos-guided utilization schemes. This eased the teachers' documentation work allowing them to produce a coherent output in the form of coordinated sequences of documents guided by the SSPM logos, to be used in their classrooms as a consistent path.

Whereas, in 2022/2023, teachers perceived the structured and self-standing Snapshots presentations, even when translated into Italian, as resources tightly paired with their utilization scheme guided by the MNS lecture-driven logos. To integrate these resources into their laboratory-based teaching approach, teachers had to transition from the foreign MNS logos to the shared SSPM logos. This

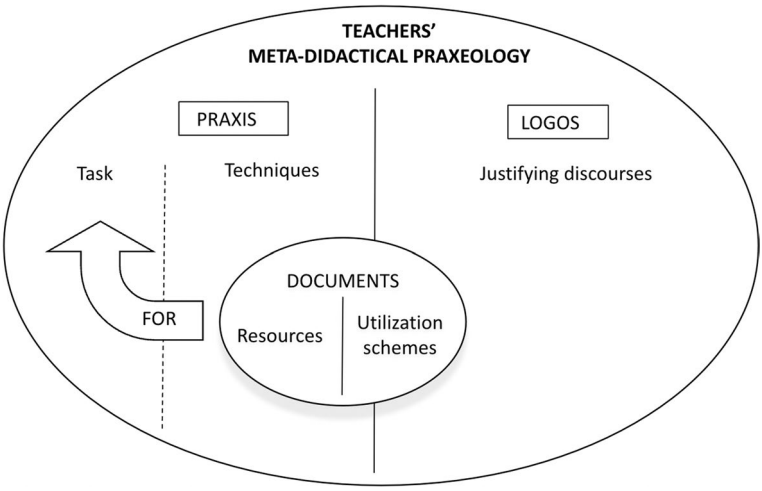

Fig. 14 A new networked model for teachers' work

challenging transition scattered them in various directions, resulting in *multiple documentational genesis.*

In light of this insight, Fig. 13 re-examines Fig. 8 and 12, depicting the observed phenomenon of *multiple documentational genesis*. We now connect this phenomenon to teachers' shared meta-didactical praxeology by immersing the different elements in their respective logos: the original and translated Snapshots on the left are perceived by teachers as guided by the MNS lecture-driven logos, separated from the shared SSPM laboratory logos on the right. This separation is represented by the logos discontinuity at the centre, and the resulting jump gives rise to the *multiple documentational genesis*.

Indeed, this logos discontinuity has been anticipated by researchers, who prompted teachers not only to translate the Snapshot presentations but also to design integrative activities to complement the translated presentation. What had not been foreseen was that teachers would perceive the Snapshot presentations as so distant from the SSPM logos to be deemed altogether unsuitable to bridge the discontinuity, even in their translated form. Furthermore, it was unexpected that teachers would choose to work individually or in pairs, breaking the co-learning partnership typical of the inquiry communities (Jaworski, 2003, 2006) that had always characterised the SSPM 2.0 PD programs (Pocalana & Robutti, 2022; Pocalana et al., 2024; Pocalana & Robutti, 2024).

Theoretical Perspectives

Our experimental findings suggest a modification of the networked model represented in Fig. 2 to address the case of teachers, coherently with recent findings obtained in the case of researchers with the role of teacher educators (didacticians)

by Pocalana and Robutti (in press). In the new model (Fig. 14) that we propose, we aim to schematise the relationship between teachers' documentation work and their meta-didactical praxeology for the design of activities for their students. We confirm that, as in the model used for didacticians, teachers' documentation work is part of their meta-didactical praxeology, but with a difference, in respect to the model in Fig. 2, related to the position of the documents with respect to the praxis and the logos components.

In the new model in Fig. 14, teachers' generation of documents is interpreted not only as part of the praxis, but also as encapsulating elements of the logos component of their meta-didactical praxeology, thus taking into account the hybrid nature of their utilization schemes. This new positioning takes into account that the utilization schemes of the documents encompass justifying discourses guiding teachers' choices. The new model is coherent with the findings by Pocalana and Robutti (in press), who position didacticians' documents, in particular their utilisation schemes, at the intersection between the praxis and the logos components of their meta-didactical praxeology.

Discussion and Conclusion

This work focuses on unravelling the phenomenon we named *multiple documentational genesis*, consisting of the proliferation of uncoordinated teaching materials created by teachers based on the same initial researchers' input, by connecting it with teachers' meta-didactical praxeologies.

Our analysis shows that, when teachers who share a strong common meta-didactical praxeology encounter resources whose utilization scheme in the original context seems to be guided by a logos very different from their shared logos, these resources meet resistance. In our observation, this resistance caused a scattering effect on teachers' documentational genesis, projecting each teacher in a distinct direction and giving rise to the phenomenon of *multiple documentational genesis*.

We interpret this phenomenon as a consequence of the fact that, when teachers encounter a resource, they perceive it as inherently connected to and inseparable from the original logos guiding its utilization scheme in the original context. Therefore, they do not consider changing its utilization scheme to adapt the resource to their logos, e.g. keeping the PowerPoint presentation in its original form (which was mandatory) and intercalating it with hands-on activities to be proposed to students with a laboratory approach. Indeed, in the SSPM institutional context, teachers perceive PowerPoint presentations as representative of a lecture-driven pedagogical approach and, for this reason, they refuse them altogether as non-coherent with their shared logos. Consequently, providing teachers with input not aligned with their shared logos somehow weakened the co-learning partnership (Jaworski, 2003, 2006) among teachers and researchers, as well as among teachers themselves. This fact hindered the teachers' usual collaborative documentation work. So, they ended up conducting their documentation work individually or in pairs, thereby contributing to the emergence of the *multiple documentational genesis*.

The implication of these findings for researchers involved in teacher education is that, to foster coherence and efficacy in teachers' documentational genesis, initial inputs should either be resources as free as possible from pre-existing schemes of utilization, or resources accompanied by utilization schemes aligned with the logos shared in their community. This approach helps prevent *multiple documentational genesis* and fosters collaborative and coordinated documentation work. This is positive for students and for teachers: it offers students a coherent learning trajectory (Simon, 1995; Wilson et al., 2014) and provides teachers with a shared output of their common work enriching their community.

As a limitation of our study, we acknowledge that our sample was constituted by a group of very experienced teachers accustomed to working within a co-learning partnership. In line with the previous study by Pocalana et al. (2023) examining a community of didacticians with a strong shared meta-didactical praxeology, our results illuminate the case of a community of teachers sharing a deeply rooted meta-didactical praxeology characterised by a strong common logos.

Despite this limitation, this study can open the path for further research to extend these findings, ensuring their relevance and applicability across broader teacher populations with varying levels of experience and different PD settings.

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Declarations

All subjects gave their informed consent for inclusion before they participated in the study.

Conflicts of Interest/Competing Interests No potential conflict of interest is reported by the authors.

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