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Crandall–lions viscosity solutions for path-dependent PDEs: The case of heat equation

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We address our interest to the development of a theory of viscosity solutions à la Crandall–Lions for path-dependent partial differential equations (PDEs), namely PDEs in the space of continuous paths $C([0,T];\mathbb{R}^d)$. Path-dependent PDEs can play a central role in the study of certain classes of optimal control problems, as for instance optimal control problems with delay. Typically, they do not admit a smooth solution satisfying the corresponding HJB equation in a classical sense, it is therefore natural to search for a weaker notion of solution. While other notions of generalized solution have been proposed in the literature, the extension of the Crandall–Lions framework to the path-dependent setting is still an open problem. The question of uniqueness of the solutions, which is the most delicate issue, will be based on early ideas from the theory of viscosity solutions and a suitable variant of Ekeland's variational principle. This latter is based on the construction of a smooth gauge-type function, where smooth is meant in the horizontal/vertical (rather than Fréchet) sense. In order to make the presentation more readable, we address the path-dependent heat equation, which in particular simplifies the smoothing of its natural "candidate" solution. Finally, concerning the existence part, we provide a functional Itô formula under general assumptions, extending earlier results in the literature.

Keywords: Path-dependent partial differential equations; viscosity solutions; functional Itô formula

1. Introduction

Path-dependent heat equation refers to the second-order partial differential equation in the space of continuous paths

$$\begin{cases} -\partial_t^H v(t,\boldsymbol{x}) - \dfrac{1}{2}\operatorname{tr}\big[\partial_{\boldsymbol{x}\boldsymbol{x}}^V v(t,\boldsymbol{x})\big] = 0, & (t,\boldsymbol{x}) \in [0,T) \times C\big([0,T];\mathbb{R}^d\big), \\ v(T,\boldsymbol{x}) = \xi(\boldsymbol{x}), & \boldsymbol{x} \in C\big([0,T];\mathbb{R}^d\big). \end{cases} \tag{1.1}$$

Here $C([0,T];\mathbb{R}^d)$ denotes the Banach space of continuous paths $\boldsymbol{x}\colon [0,T] \to \mathbb{R}^d$ equipped with the supremum norm $\|\boldsymbol{x}\|_\infty := \sup_{t\in[0,T]} |\boldsymbol{x}(t)|$, with $|\cdot|$ denoting the Euclidean norm on $\mathbb{R}^d$. The terminal condition $\xi\colon C([0,T];\mathbb{R}^d) \to \mathbb{R}$ is assumed to be continuous and bounded. We refer to equation (1.1) as path-dependent *heat* equation. Similarly as for the usual heat equation, it admits the following Feynman–Kac representation formula in terms of the d-dimensional Brownian motion $\boldsymbol{W} = (\boldsymbol{W}_s)_{s\in[0,T]}$:

$$v(t,\boldsymbol{x}) = \mathbb{E}\big[\xi\big(\boldsymbol{W}^{t,\boldsymbol{x}}\big)\big], \quad \forall (t,\boldsymbol{x}) \in [0,T] \times C\big([0,T];\mathbb{R}^d\big), \tag{1.2}$$

where

$$\boldsymbol{W}_s^{t,\boldsymbol{x}} := \begin{cases} \boldsymbol{x}(s), & s \le t, \\ \boldsymbol{x}(t) + \boldsymbol{W}_s - \boldsymbol{W}_t, & s > t. \end{cases}$$

In the case of the classical heat equation ξ only depends on the terminal value $\boldsymbol{W}_T^{t,\boldsymbol{x}}$.

The peculiarity of equation (1.1) is the presence of the so-called functional or pathwise derivatives $\partial_t^H v$, $\partial_{\boldsymbol{x}\boldsymbol{x}}^V v$, where $\partial_t^H v$ is known as horizontal derivative, while $\partial_{\boldsymbol{x}\boldsymbol{x}}^V v$ is the matrix of second-order vertical derivatives. Those derivatives appeared in [54,55] (under the name of coinvariant derivatives) as building block of the so-called i-smooth analysis, and independently in [1], where they were denoted Clio derivatives; later, they were rediscovered by [32] (from which we borrow terminology and definitions), which adopted a slightly different definition based on the space of càdlàg paths and in addition developed a related stochastic calculus, known as functional Itô calculus, including in particular the so-called functional Itô formula. Differently from the classical Fréchet derivative on $C([0,T];\mathbb{R}^d)$, the distinguished features of the pathwise derivatives are their finite-dimensional nature and the property of being non-anticipative, which follow from the interpretation of t in $\boldsymbol{x}(t)$ as time variable. This means that $v(t,\boldsymbol{x})$ only depends on the values of the path $\boldsymbol{x}$ up to time t; moreover, the horizontal and vertical derivatives at time t are computed keeping the past values frozen, while only the present value of the path (that is $\boldsymbol{x}(t)$) can vary. The related functional Itô calculus was rigorously investigated in [12–14]. [15,18] also gave a contribution in this direction, exploring the relation between pathwise derivatives and Banach space stochastic calculus, built on an appropriate notion of Fréchet type derivative and first conceived in [27], see also [26,28–30].

Partial differential equations in the space of continuous paths (also known as functional or Clio or path-dependent partial differential equations) are mostly motivated by optimal control problems of deterministic and stochastic systems with delay (or path-dependence) in the state variable. Such control systems arise in many fields, as for instance optimal advertising theory [48,49], chemical engineering [45], financial management [39,72], economic growth theory [2], mean field game theory [5], biomedicine [46,80], systemic risk [10]. The underlying deterministic or stochastic controlled differential equations with delay can be studied in two ways: first using a direct approach (see for instance [51,52,54,56,81]), second by lifting them into a suitable infinite-dimensional framework, leading to evolution equations in Hilbert (as in [11,25,42]) or Banach spaces (as in [27,69,70]). The latter methodology turned out to be preferable to address general optimal control problems with delay (see for instance [38,40,41,44,48,50,87]), although such an infinite-dimensional reformulation may require some additional artificial assumptions to be imposed on the original control problem. On the other hand, the direct approach was adopted for special problems where the Hamilton–Jacobi–Bellman equation reduces to a finite-dimensional differential equation, as in [37,57]. This approach can now regain relevance thanks to a well-grounded theory of path-dependent partial differential equations. To this regard, the path-dependent heat equation represents the primary test for such a theory, it indeed requires the main building blocks of the methodology, without overloading the proofs with additional technicalities.

Path-dependent partial differential equations represent a quite recent area of research. Typically, they do not admit a smooth solution satisfying the equation in a classical sense, mainly because of the awkward nature of the underlying space $C([0,T];\mathbb{R}^d)$. This happens also for the path-dependent heat equation, which in particular does not have the smoothing effect characterizing the classical heat equation, except in some specific cases (as shown in [27,31]) with ξ belonging to the class of so-called cylinder or tame functions (therefore depending specifically on a finite number of integrals with respect to the path) or ξ being smoothly Fréchet differentiable. It is indeed quite easy, relying on the probabilistic representation formula (1.2), to see that the function v is not smooth (in the horizontal/vertical sense mentioned above) for terminal conditions of the form

$$\xi(\boldsymbol{x}) = \sup_{0\le t\le T} \boldsymbol{x}(t), \qquad \xi(\boldsymbol{x}) = \boldsymbol{x}(t_0),$$

for some fixed $t_0 \in (0,T)$. For a detailed analysis of the first case above we refer to Section 3.2 in [17], see also Remark 3.8 in [18]. Concerning the second case, see for instance Example 11.1.3 of [88]. It is however worth mentioning that some positive results on smooth solutions were obtained in [18,75]. We also refer to Chapter 9 of [27] and [31], where smooth solutions were investigated using a Fréchet type derivative formulation.

It is therefore natural to search for a weaker notion of solution, as the notion of viscosity solution, commonly used in the standard finite-dimensional case. The theory of viscosity solutions, firstly introduced in [22,23] for first-order equations in finite dimension and later extended to the second-order case in [59–61], provides a well-suited framework guaranteeing the desired existence, uniqueness, and stability properties (for a comprehensive account see [21]). The extension of such a theory to equations in infinite dimension was initiated by [24,62–64,82,85]. One of the structural assumption is that the state space has to be a Hilbert space or, slightly more general, certain Banach space with smooth norm, not including for instance the Banach space $C([0,T];\mathbb{R}^d)$ (notice however that in this paper we do not directly generalize those results to $C([0,T];\mathbb{R}^d)$, as we adopt horizontal/vertical, rather than Fréchet, derivatives on $C([0,T];\mathbb{R}^d)$).

First-order path-dependent partial differential equations were deeply investigated in [68] using a viscosity type notion of solution, which differs from the Crandall–Lions definition as the maximum/minimum condition is formulated on the subset of absolutely continuous paths. Such a modification does not affect existence in the first-order case, however it is particularly convenient for uniqueness, which is indeed established under general conditions. Other notions of generalized solution designed for first-order equations were adopted in [1] as well as in [65–67], where the minimax framework introduced in [83,84] was implemented. We also mention [4], where such a minimax approach was extended to first-order path-dependent Hamilton–Jacobi–Bellman equations in infinite dimension. Concerning the second-order case, a first attempt to extend the Crandall–Lions framework to the path-dependent case was carried out in [73], even though a technical condition on the semi-jets was imposed, namely condition (16) in [73], which narrows down the applicability of such a result. In the literature, this was perceived as an almost insurmountable obstacle, so that the Crandall–Lions definition was not further investigated, while other notions of generalized solution were devised, see [3,9,20,34,58,74,86]. We mention in particular the framework designed in [34] and further investigated in [16,35,36,77–79], where the notion of sub/supersolution adopted differs from the Crandall–Lions definition as the tangency condition is not pointwise but in the sense of expectation with respect to an appropriate class of probability measures. On the other hand, in [20] we introduced the so-called strong-viscosity solution, which is quite similar to the notion of good solution for partial differential equations in finite dimension, that in turn is known to be equivalent to the definition of L^p-viscosity solution, see for instance [53]. We also mention [3], where the authors deal with semilinear path-dependent equations and propose the notion of decoupled mild solution, formulated in terms of generalized transition semigroups; such a notion also adapts to path-dependent equations with integro-differential terms. Finally, we mention [89], which appeared only recently, some time after the present paper was posted on arXiv, but apparently uses the same methodology proposed in this paper to study path-dependent Hamilton–Jacobi–Bellman equations. Unfortunately, it seems that the present version of that paper (v1) contains a relevant gap in the proof of two crucial lemmas (Lemmas 4.2 and 4.3), see for instance, (4.39) in [89].

In the present paper we adopt the natural generalization of the well-known definition of viscosity solution à la Crandall–Lions given in terms of test functions and, under this notion, we establish existence and uniqueness for the path-dependent heat equation (1.1). The uniqueness property is derived, as usual, from the comparison theorem. The proof of this latter, which is the most delicate issue, is known to be quite involved even in the classical finite-dimensional case (see, for instance, [21]), and in its latest form is based on Ishii's lemma. Here we follow instead an earlier approach (see, for instance, Theorem II.1 in [61] or Theorem IV.1 in [62]), which in principle can be applied to any path-dependent equation admitting a "candidate" solution v, for which a probabilistic representation formula holds.

This is the case for equation (1.1), where the candidate solution is given by formula (1.2), but it is also the case for Kolmogorov type equations or, more generally, for Hamilton–Jacobi–Bellman equations. This latter is the class of equations studied in [61] and [62], whose methodology in a nutshell can be described as follows. Let u (resp. w) be a viscosity subsolution (resp. supersolution) of the same path-dependent equation. The desired inequality $u \leq w$ follows if we compare both u and w to the "candidate" solution v, that is if we prove the two inequalities $u \leq v$ and $v \leq w$. Let us consider for instance the first inequality $u \leq v$. In the non-path-dependent and finite-dimensional case (as in [61]), this is proved proceeding as follows: firstly, performing a smoothing of v through its probabilistic representation formula; secondly, taking a local maximum of $u - v_n$ (here it is used the local compactness of the finite-dimensional underlying space), with v_n being a smooth approximation of v; finally, the inequality $u \leq v_n$ is proved proceeding as in the so-called partial comparison theorem (comparison between a viscosity subsolution/supersolution and a smooth supersolution/subsolution), namely exploiting the viscosity subsolution property of u with v_n playing the role of test function. In [62], where such a methodology was extended to the infinite-dimensional case, the existence of a maximum of $u - v_n$ is achieved relying on Ekeland's variational principle, namely exploiting the completeness of the space instead of the missing local compactness.

In this paper, we generalize the methodology sketched above to the path-dependent case. There are however at least two crucial mathematical issues required by such a proof, still not at disposal in the path-dependent framework.

First, given a candidate solution v, it is not a priori obvious how to perform a smooth approximation of v itself starting from its probabilistic representation formula. Here we rely on Lemma 4.1, which in turn exploits the results proved in [18] (Theorem 3.5) and [20] (Theorem 3.12), which are reported and adapted to the present framework in Appendix D of [19] (Lemma D.1 and Lemma D.2, respectively). Notice that such results apply to the case of the path-dependent heat equation (1.1), where there is only the terminal condition ξ in the probabilistic representation formula (1.2) for v. More general results are at disposal in [18] and [20], which cover the case of semilinear path-dependent partial differential equations, characterized by the presence of four coefficients b, σ, F, ξ (see, in particular, Theorem 3.16 in [20] for more details). However, when those other coefficients appear in the path-dependent partial differential equation, we need more information on the sequence $\{v_n\}_n$ approximating v. For instance, we also have to estimate the derivatives of v_n in order to proceed as in [61] or [62]. Since such results are still not at disposal in the path-dependent setting, in order to make the paper more readable and not excessively lengthy, here we address the case of the path-dependent heat equation.

Secondly, concerning the existence of a maximum of $u - v_n$, we rely on a generalized version of Ekeland's variational principle for which we need a smooth gauge-type function with bounded derivatives, as explained below. Our equation is in fact formulated on the non-locally compact space $[0, T] \times C([0, T]; \mathbb{R}^d)$ endowed with the pseudometric

$$d_\infty\big((t, \boldsymbol{x}), \big(t', \boldsymbol{x}'\big)\big) := \big|t - t'\big| + \big\|\boldsymbol{x}(\cdot \wedge t) - \boldsymbol{x}'\big(\cdot \wedge t'\big)\big\|_\infty.$$

Recall that Ekeland's variational principle, in its original form, applied to $([0, T] \times C([0, T]; \mathbb{R}^d), d_\infty)$ states that a perturbation $u(\cdot, \cdot) - v_n(\cdot, \cdot) - \delta d_\infty((\cdot, \cdot), (\bar{t}, \bar{\boldsymbol{x}}))$ of $u(\cdot, \cdot) - v_n(\cdot, \cdot)$ has a strict global maximum, with the perturbation being expressed in terms of the distance d_∞ (the point $(\bar{t}, \bar{\boldsymbol{x}})$ is fixed). As the map $(t, \boldsymbol{x}) \mapsto d_\infty((t, \boldsymbol{x}), (\bar{t}, \bar{\boldsymbol{x}}))$ is not smooth, it cannot be a test function. In order to have a smooth map instead of d_∞, we need a smooth variational principle on $[0, T] \times C([0, T]; \mathbb{R}^d)$. To this end, the starting point is a generalization of the so-called Borwein-Preiss smooth variant of Ekeland's variational principle (see for instance [8]), which works when d_∞ is replaced by a so-called gauge-type function (see Definition 3.1). For the proof of the comparison theorem, we have to construct a gauge-type function which is also smooth and with bounded derivatives, recalling that *smooth* in the present context means in the horizontal/vertical (rather than in the Fréchet) sense. In Section 3 such a gauge-type function is built through a smoothing of d_∞ itself (more precisely, of the part concerning the supremum norm). This latter smoothing is performed by convolution, firstly in the vertical direction, that is in the direction of the map $1_{[t,T]}$ (Lemma 3.1), then in the horizontal direction (Lemma 3.2), the ordering of smoothings being crucial. Notice in particular that the supremum norm is already smooth in the horizontal direction; however, after the vertical smoothing, we lose in general the horizontal regularity because of the presence of the term $1_{[t,T]}$; for this reason, we have also to perform the horizontal smoothing. The resulting smooth gauge-type function with bounded derivatives corresponds to the function ρ_∞ defined in (3.8).

Regarding existence, we prove that the candidate solution v in (1.2) solves in the viscosity sense equation (1.1). We proceed essentially as in the classical non-path-dependent case, relying as usual on Itô's formula, which in the present context corresponds to the functional Itô formula. Such a formula was first stated in [32] and then rigorously proved in [12,13], see also [14,18,43,58,71]. In the present paper, we provide a functional Itô formula under general assumptions (Theorem 2.2). In particular, we do not require any boundedness assumption on the functional $u\colon [0,T]\times C([0,T];\mathbb{R}^d)\to\mathbb{R}$, thus improving (when the semimartingale process is continuous) the results stated in
[12,13].

The paper is organized as follows. Section 2 is devoted to pathwise derivatives and functional Itô calculus. In particular, there is the functional Itô formula (Theorem 2.2) whose complete proof is reported in Appendix A of [19] (notice that [19] coincides with the present paper and contains in addition the Appendices). In Section 3, we prove the smooth variational principle on $[0,T]\times C([0,T];\mathbb{R}^d)$, constructing the smooth gauge-type function with bounded derivatives. In Section 4, we provide the (path-dependent) Crandall–Lions definition of viscosity solution for a general path-dependent partial differential equation. We then study in detail the path-dependent heat equation. In particular, we prove existence showing that the so-called candidate solution v solves in the viscosity sense the path-dependent heat equation (Theorem 4.1). We conclude Section 4 proving the comparison theorem (Theorem 4.2) and uniqueness (Corollary 4.1).

2. Pathwise derivatives and functional Itô calculus

In the present section, we define the pathwise derivatives and state the functional Itô formula under general assumptions.

2.1. Maps on càdlàg paths

Given $T>0$ and $d\in\mathbb{N}^*$, we denote by $D([0,T];\mathbb{R}^d)$ the set of càdlàg functions $\hat{\boldsymbol{x}}\colon [0,T]\to\mathbb{R}^d$. We denote by $\hat{\boldsymbol{x}}(t)$ the value of $\hat{\boldsymbol{x}}$ at $t\in[0,T]$. We also denote by $\mathbf{0}$ the function $\hat{\boldsymbol{x}}\colon [0,T]\to\mathbb{R}^d$ identically equal to zero. We consider on $D([0,T];\mathbb{R}^d)$ the supremum norm $\|\cdot\|_\infty$, namely $\|\hat{\boldsymbol{x}}\|_\infty := \sup_{t\in[0,T]}|\hat{\boldsymbol{x}}(t)|$, where $|\cdot|$ denotes the Euclidean norm on $\mathbb{R}^d$ (we use the same symbol $|\cdot|$ to denote the Euclidean norm on $\mathbb{R}^k$, for any $k\in\mathbb{N}$). We refer to Chapter V in [76] and to Section 15 of Chapter 3 in [6] for a study of the set of càdlàg functions endowed with the uniform metric and a comparison with the Skorokhod space.

We set $\hat{\boldsymbol{\Lambda}} := [0,T]\times D([0,T];\mathbb{R}^d)$ and define $\hat{d}_\infty\colon \hat{\boldsymbol{\Lambda}}\times\hat{\boldsymbol{\Lambda}}\to[0,\infty)$ as

$$\hat{d}_\infty\big((t,\hat{\boldsymbol{x}}),\big(t',\hat{\boldsymbol{x}}'\big)\big) := \big|t-t'\big| + \big\|\hat{\boldsymbol{x}}(\cdot\wedge t)-\hat{\boldsymbol{x}}'\big(\cdot\wedge t'\big)\big\|_\infty.$$

Notice that $\hat{d}_\infty$ is a *pseudometric* on $\hat{\boldsymbol{\Lambda}}$, that is $\hat{d}_\infty$ is not a true metric because one may have $\hat{d}_\infty((t,\hat{\boldsymbol{x}}),(t',\hat{\boldsymbol{x}}'))=0$ even if $(t,\hat{\boldsymbol{x}})\neq(t',\hat{\boldsymbol{x}}')$. We recall that one can construct a true metric space $(\hat{\boldsymbol{\Lambda}}^*,\hat{d}_\infty^*)$, called the metric space induced by the pseudometric space $(\hat{\boldsymbol{\Lambda}},\hat{d}_\infty)$, by means of the equivalence relation which follows from the vanishing of the pseudometric. We also observe that $(\hat{\boldsymbol{\Lambda}},\hat{d}_\infty)$ is a complete pseudometric space. Finally, we denote by $\mathcal{B}(\hat{\boldsymbol{\Lambda}})$ the Borel σ-algebra on $\hat{\boldsymbol{\Lambda}}$ induced by $\hat{d}_\infty$.

Definition 2.1. A map (or functional) $\hat{u}\colon\hat{\boldsymbol{\Lambda}}\to\mathbb{R}$ is said to be *non-anticipative* (on $\hat{\boldsymbol{\Lambda}}$) if it satisfies

$$\hat{u}(t,\hat{\boldsymbol{x}})=\hat{u}\big(t,\hat{\boldsymbol{x}}(\cdot\wedge t)\big),\quad \forall(t,\hat{\boldsymbol{x}})\in\hat{\boldsymbol{\Lambda}}.$$

Remark 2.1. (i) The property of being non-anticipative is crucial and automatically true if the map $\hat{u}\colon\hat{\boldsymbol{\Lambda}}\to\mathbb{R}$ is continuous with respect to $\hat{d}_\infty$.

(ii) More generally, it holds that whenever $\hat{u}\colon\hat{\boldsymbol{\Lambda}}\to\mathbb{R}$ is Borel measurable, namely $\hat{u}$ is measurable with respect to $\mathcal{B}(\hat{\boldsymbol{\Lambda}})$, then $\hat{u}$ is non-anticipative on $\hat{\boldsymbol{\Lambda}}$. As a matter of fact, notice that every open subset B of $\hat{\boldsymbol{\Lambda}}$, endowed with $\hat{d}_\infty$, satisfies the following property: if $(t,\hat{\boldsymbol{x}})\in B$, then $(t,\hat{\boldsymbol{x}}(\cdot\wedge t))\in B$ (this follows from the fact that $\hat{d}_\infty((t,\hat{\boldsymbol{x}}),(t,\hat{\boldsymbol{x}}(\cdot\wedge t)))=0$). As a consequence, by a monotone class argument, the same property holds true for every Borel subset of $\hat{\boldsymbol{\Lambda}}$. Now, let $\hat{u}\colon\hat{\boldsymbol{\Lambda}}\to\mathbb{R}$ be Borel measurable. For every $(t,\hat{\boldsymbol{x}})\in\hat{\boldsymbol{\Lambda}}$, denote

$$B_{\hat{u}(t,\hat{\boldsymbol{x}})}:=\big\{(s,\hat{\boldsymbol{y}})\in\hat{\boldsymbol{\Lambda}}\colon\hat{u}(s,\hat{\boldsymbol{y}})=\hat{u}(t,\hat{\boldsymbol{x}})\big\}.$$

Notice that $B_{\hat{u}(t,\hat{\boldsymbol{x}})}\in\mathcal{B}(\hat{\boldsymbol{\Lambda}})$ and since $(t,\hat{\boldsymbol{x}})\in B_{\hat{u}(t,\hat{\boldsymbol{x}})}$ we deduce that $(t,\hat{\boldsymbol{x}}(\cdot\wedge t))\in B_{\hat{u}(t,\hat{\boldsymbol{x}})}$. This means that $\hat{u}(t,\hat{\boldsymbol{x}}(\cdot\wedge t))=\hat{u}(t,\hat{\boldsymbol{x}})$, namely the map $\hat{u}$ is non-anticipative.

Definition 2.2. We denote by $\boldsymbol{C}(\hat{\boldsymbol{\Lambda}})$ the set of maps $\hat{u}\colon\hat{\boldsymbol{\Lambda}}\to\mathbb{R}$ which are continuous on $\hat{\boldsymbol{\Lambda}}$ with respect to $\hat{d}_\infty$.

Definition 2.3 (Pathwise derivatives). Let $\hat{u}\colon\hat{\boldsymbol{\Lambda}}\to\mathbb{R}$ be non-anticipative.

(i) Given $(t,\hat{\boldsymbol{x}})\in\hat{\boldsymbol{\Lambda}}$, with $t<T$, the *horizontal derivative* of $\hat{u}$ at $(t,\hat{\boldsymbol{x}})$ (if the corresponding limit exists) is defined as

$$\partial_t^H\hat{u}(t,\hat{\boldsymbol{x}}):=\lim_{\delta\to0^+}\frac{\hat{u}(t+\delta,\hat{\boldsymbol{x}}(\cdot\wedge t))-\hat{u}(t,\hat{\boldsymbol{x}})}{\delta}.$$

At $t=T$ the horizontal derivative is defined as

$$\partial_t^H\hat{u}(T,\hat{\boldsymbol{x}}):=\lim_{t\to T^-}\partial_t^H\hat{u}(t,\hat{\boldsymbol{x}}).$$

(ii) Given $(t,\hat{\boldsymbol{x}})\in\hat{\boldsymbol{\Lambda}}$, the *vertical derivatives* of first and second-order of $\hat{u}$ at $(t,\hat{\boldsymbol{x}})$ (if the corresponding limits exist) are defined as

$$\partial_{x_i}^V\hat{u}(t,\hat{\boldsymbol{x}}):=\lim_{h\to0}\frac{\hat{u}(t,\hat{\boldsymbol{x}}+h\mathbf{e}_i1_{[t,T]})-\hat{u}(t,\hat{\boldsymbol{x}})}{h},$$

$$\partial_{x_ix_j}^V\hat{u}(t,\hat{\boldsymbol{x}}):=\partial_{x_j}^V\big(\partial_{x_i}^V\hat{u}\big)(t,\hat{\boldsymbol{x}}),$$

where $\mathbf{e}_1, \dots, \mathbf{e}_d$ is the standard orthonormal basis of $\mathbb{R}^d$.

Finally, we denote $\partial_{\boldsymbol{x}}^V \hat{u} = (\partial_{x_1}^V \hat{u}, \dots, \partial_{x_d}^V \hat{u})$ and $\partial_{\boldsymbol{xx}}^V \hat{u} = (\partial_{x_i x_j}^V \hat{u})_{i,j=1,\dots,d}$.

Definition 2.4. We denote by $\boldsymbol{C}^{1,2}(\hat{\boldsymbol{\Lambda}})$ the set of $\hat{u} \in \boldsymbol{C}(\hat{\boldsymbol{\Lambda}})$ such that $\partial_t^H \hat{u}$, $\partial_{\boldsymbol{x}}^V \hat{u}$, $\partial_{\boldsymbol{xx}}^V \hat{u}$ exist everywhere on $\hat{\boldsymbol{\Lambda}}$ and are continuous.

For later use, we also introduce the following set of maps on càdlàg paths.

Definition 2.5. We denote by $\boldsymbol{C}^{0,2}(\hat{\boldsymbol{\Lambda}})$ the set of $\hat{u} \in \boldsymbol{C}(\hat{\boldsymbol{\Lambda}})$ such that $\partial_{\boldsymbol{x}}^V \hat{u}$, $\partial_{\boldsymbol{xx}}^V \hat{u}$ exist everywhere on $\hat{\boldsymbol{\Lambda}}$ and are continuous.

We can finally state the functional Itô formula for maps on càdlàg paths, whose proof is reported in Appendix A of [19].

Theorem 2.1. *Let* $\hat{u} \in \boldsymbol{C}^{1,2}(\hat{\boldsymbol{\Lambda}})$. *Then, for every* d*-dimensional continuous semimartingale* $\boldsymbol{X} = (\boldsymbol{X}_t)_{t\in[0,T]}$, *where* $\boldsymbol{X} = (X^1, \dots, X^d)$, *defined on some filtered probability space* $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$, *with* $(\mathcal{F}_t)_{t\in[0,T]}$ *satisfying the usual conditions, the following* functional Itô formula *holds*:

$$\hat{u}(t, \boldsymbol{X}) = \hat{u}(0, \boldsymbol{X}) + \int_0^t \partial_t^H \hat{u}(s, \boldsymbol{X})\, ds + \frac{1}{2} \sum_{i,j=1}^d \int_0^t \partial_{x_i x_j}^V \hat{u}(s, \boldsymbol{X})\, d\big[X^i, X^j\big]_s$$

$$+ \sum_{i=1}^d \int_0^t \partial_{x_i}^V \hat{u}(s, \boldsymbol{X})\, dX_s^i, \quad \textit{for all } 0 \le t \le T, \mathbb{P}\textit{-a.s.}$$

Proof. See Appendix A in [19]. □

2.2. Maps on continuous paths

Let $C([0,T]; \mathbb{R}^d)$ denote the set of continuous functions $\boldsymbol{x}\colon [0,T] \to \mathbb{R}^d$. Notice that $C([0,T]; \mathbb{R}^d)$ is a subset of $D([0,T]; \mathbb{R}^d)$. We set $\boldsymbol{\Lambda} := [0,T] \times C([0,T]; \mathbb{R}^d)$ and denote $\boldsymbol{d}_\infty$ the restriction of $\hat{\boldsymbol{d}}_\infty$ to $\boldsymbol{\Lambda} \times \boldsymbol{\Lambda}$. Then, $\boldsymbol{d}_\infty$ is a pseudometric on $\boldsymbol{\Lambda}$ and $(\boldsymbol{\Lambda}, \boldsymbol{d}_\infty)$ is a complete pseudometric space. We denote by $\mathcal{B}(\boldsymbol{\Lambda})$ the Borel σ-algebra on $\boldsymbol{\Lambda}$ induced by $\boldsymbol{d}_\infty$.

Definition 2.6. Let $\hat{u}\colon \hat{\boldsymbol{\Lambda}} \to \mathbb{R}$ be non-anticipative and consider $u\colon \boldsymbol{\Lambda} \to \mathbb{R}$. We say that $\hat{u}$ is *consistent* with u if

$$u(t, \boldsymbol{x}) = \hat{u}(t, \boldsymbol{x}), \quad \forall (t, \boldsymbol{x}) \in \boldsymbol{\Lambda}.$$

The following consistency property is crucial as it implies that, given u admitting two maps $\hat{u}_1$ and $\hat{u}_2$, both being consistent with u, their pathwise derivatives coincide on continuous paths (see also Remark 2.2).

Lemma 2.1. *If* $\hat{u}_1, \hat{u}_2 \in \boldsymbol{C}^{1,2}(\hat{\boldsymbol{\Lambda}})$ *satisfy*

$$\hat{u}_1(t, \boldsymbol{x}) = \hat{u}_2(t, \boldsymbol{x}), \quad \forall (t, \boldsymbol{x}) \in \boldsymbol{\Lambda},$$

then, for all $(t,\boldsymbol{x}) \in \boldsymbol{\Lambda}$,

$$\partial_t^H \hat{u}_1(t,\boldsymbol{x}) = \partial_t^H \hat{u}_2(t,\boldsymbol{x}),$$
$$\partial_{\boldsymbol{x}}^V \hat{u}_1(t,\boldsymbol{x}) = \partial_{\boldsymbol{x}}^V \hat{u}_2(t,\boldsymbol{x}),$$
$$\partial_{\boldsymbol{x}\boldsymbol{x}}^V \hat{u}_1(t,\boldsymbol{x}) = \partial_{\boldsymbol{x}\boldsymbol{x}}^V \hat{u}_2(t,\boldsymbol{x}).$$

Proof. See Appendix B in [19]. □

Thanks to Lemma 2.1 we can now give the following definition (see also Remark 2.2).

Definition 2.7. Let $u\colon \boldsymbol{\Lambda} \to \mathbb{R}$. We say that $u \in \boldsymbol{C}^{1,2}(\boldsymbol{\Lambda})$ if there exists $\hat{u}\colon \hat{\boldsymbol{\Lambda}} \to \mathbb{R}$ consistent with u and satisfying $\hat{u} \in \boldsymbol{C}^{1,2}(\hat{\boldsymbol{\Lambda}})$. Moreover, we define, for all $(t,\boldsymbol{x}) \in \boldsymbol{\Lambda}$,

$$\partial_t^H u(t,\boldsymbol{x}) := \partial_t^H \hat{u}(t,\boldsymbol{x}),$$
$$\partial_{\boldsymbol{x}}^V u(t,\boldsymbol{x}) := \partial_{\boldsymbol{x}}^V \hat{u}(t,\boldsymbol{x}),$$
$$\partial_{\boldsymbol{x}\boldsymbol{x}}^V u(t,\boldsymbol{x}) := \partial_{\boldsymbol{x}\boldsymbol{x}}^V \hat{u}(t,\boldsymbol{x}).$$

Remark 2.2. Notice that, by Lemma 2.1, if $u \in \boldsymbol{C}^{1,2}(\boldsymbol{\Lambda})$ then the definition of the pathwise derivatives of u is independent of the map $\hat{u} \in \boldsymbol{C}^{1,2}(\hat{\boldsymbol{\Lambda}})$ consistent with u.

Theorem 2.2. *Let* $u \in \boldsymbol{C}^{1,2}(\boldsymbol{\Lambda})$. *Then, for every* d*-dimensional continuous semimartingale* $\boldsymbol{X} = (\boldsymbol{X}_t)_{t\in[0,T]}$, *where* $\boldsymbol{X} = (X^1,\dots,X^d)$, *defined on some filtered probability space* $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\in[0,T]},\mathbb{P})$, *with* $(\mathcal{F}_t)_{t\in[0,T]}$ *satisfying the usual conditions, the following* functional Itô formula *holds*:

$$u(t,\boldsymbol{X}) = u(0,\boldsymbol{X}) + \int_0^t \partial_t^H u(s,\boldsymbol{X})\,ds + \frac{1}{2}\sum_{i,j=1}^d \int_0^t \partial_{x_i x_j}^V u(s,\boldsymbol{X})\,d\big[X^i,X^j\big]_s$$
$$+ \sum_{i=1}^d \int_0^t \partial_{x_i}^V u(s,\boldsymbol{X})\,dX_s^i, \quad \textit{for all } 0 \le t \le T,\ \mathbb{P}\textit{-a.s.} \tag{2.1}$$

Proof. Since $u \in \boldsymbol{C}^{1,2}(\boldsymbol{\Lambda})$, by Definition 2.7 there exists a map $\hat{u}\colon \hat{\boldsymbol{\Lambda}} \to \mathbb{R}$ consistent with u and satisfying $\hat{u} \in \boldsymbol{C}^{1,2}(\hat{\boldsymbol{\Lambda}})$. Then, by Theorem 2.1, the following functional Itô formula holds:

$$\hat{u}(t,\boldsymbol{X}) = \hat{u}(0,\boldsymbol{X}) + \int_0^t \partial_t^H \hat{u}(s,\boldsymbol{X})\,ds + \frac{1}{2}\sum_{i,j=1}^d \int_0^t \partial_{x_i x_j}^V \hat{u}(s,\boldsymbol{X})\,d\big[X^i,X^j\big]_s$$
$$+ \sum_{i=1}^d \int_0^t \partial_{x_i}^V \hat{u}(s,\boldsymbol{X})\,dX_s^i, \quad \text{for all } 0 \le t \le T,\ \mathbb{P}\text{-a.s.}$$

The claim follows identifying the pathwise derivatives of $\hat{u}$ with those of u. □

3. Smooth variational principle on $\mathbf{\Lambda}$

The goal of the present section is the proof of a smooth variational principle on $\mathbf{\Lambda}$, which plays a crucial role in the proof of the comparison theorem (Theorem 4.2). To this end, we begin recalling a generalization of the so-called Borwein-Preiss smooth variant ([7]) of Ekeland's variational principle ([33]), corresponding to Theorem 3.1 below. We state it for the case of real-valued (rather than $\mathbb{R} \cup \{+\infty\}$-valued as in [8]) maps on $\mathbf{\Lambda}$. We first recall the definition of gauge-type function for the specific set $\mathbf{\Lambda}$.

Definition 3.1. We say that $\Psi \colon \mathbf{\Lambda} \times \mathbf{\Lambda} \to [0, +\infty)$ is a *gauge-type function* provided that the properties below hold:

(a) Ψ is continuous on $\mathbf{\Lambda} \times \mathbf{\Lambda}$;
(b) $\Psi((t, \boldsymbol{x}), (t, \boldsymbol{x})) = 0$, for every $(t, \boldsymbol{x}) \in \mathbf{\Lambda}$;
(c) for every $\varepsilon > 0$ there exists $\eta > 0$ such that, for all $(t', \boldsymbol{x}'), (t'', \boldsymbol{x}'') \in \mathbf{\Lambda}$, the inequality $\Psi((t', \boldsymbol{x}'), (t'', \boldsymbol{x}'')) \leq \eta$ implies $d_\infty((t', \boldsymbol{x}'), (t'', \boldsymbol{x}'')) < \varepsilon$.

Theorem 3.1. *Let $G \colon \mathbf{\Lambda} \to \mathbb{R}$ be an upper semicontinuous map, bounded from above. Suppose that $\Psi \colon \mathbf{\Lambda} \times \mathbf{\Lambda} \to [0, +\infty)$ is a gauge-type function (according to Definition 3.1) and $\{\delta_n\}_{n \geq 0}$ is a sequence of strictly positive real numbers. For every $\varepsilon > 0$, let $(t_0, \boldsymbol{x}_0) \in \mathbf{\Lambda}$ such that*

$$\sup G - \varepsilon \leq G(t_0, \boldsymbol{x}_0).$$

Then, there exists a sequence $\{(t_n, \boldsymbol{x}_n)\}_{n \geq 1} \subset \mathbf{\Lambda}$ which converges to some $(\bar{t}, \bar{\boldsymbol{x}}) \in \mathbf{\Lambda}$ satisfying the following properties.

(i) $\Psi((\bar{t}, \bar{\boldsymbol{x}}), (t_n, \boldsymbol{x}_n)) \leq \frac{\varepsilon}{2^n \delta_0}$, *for every $n \geq 0$.*
(ii) $G(t_0, \boldsymbol{x}_0) \leq G(\bar{t}, \bar{\boldsymbol{x}}) - \sum_{n=0}^{+\infty} \delta_n \Psi((\bar{t}, \bar{\boldsymbol{x}}), (t_n, \boldsymbol{x}_n))$.
(iii) *For every $(t, \boldsymbol{x}) \neq (\bar{t}, \bar{\boldsymbol{x}})$,*

$$G(t, \boldsymbol{x}) - \sum_{n=0}^{+\infty} \delta_n \Psi\big((t, \boldsymbol{x}), (t_n, \boldsymbol{x}_n)\big) < G(\bar{t}, \bar{\boldsymbol{x}}) - \sum_{n=0}^{+\infty} \delta_n \Psi\big((\bar{t}, \bar{\boldsymbol{x}}), (t_n, \boldsymbol{x}_n)\big).$$

Proof. Theorem 3.1 follows trivially from Theorem 2.5.2 in [8], the only difference being that the latter result is stated on complete *metric* spaces, while here $\mathbf{\Lambda}$ is a complete *pseudometric* space. □

The main ingredient of Theorem 3.1 is the gauge-type function Ψ. In the proof of the comparison theorem we need such a gauge-type function to be also smooth as a map of its first pair, namely $(t, \boldsymbol{x}) \mapsto \Psi((t, \boldsymbol{x}), (t_0, \boldsymbol{x}_0))$, and with bounded derivatives. The most important example of gauge-type function is the pseudometric d_∞ itself, which unfortunately is not smooth enough. The major contribution of the present section is the construction of such a smooth gauge-type function with bounded derivatives, which corresponds to the function ρ_∞ in (3.8). In order to do it, we perform a smoothing of the pseudometric d_∞ itself (more precisely of the part concerning the supremum norm), first in the vertical direction, and then in the horizontal direction. In particular, the next result concerns the smoothing in the vertical direction. The precise form of the mollifier ζ in (3.1) is used to get explicit bounds on $\hat{\kappa}_\infty^{(t_0, \boldsymbol{x}_0)}$ and its derivatives.

Lemma 3.1. *Let $\zeta : \mathbb{R}^d \to \mathbb{R}$ be the probability density function of the standard normal multivariate distribution*

$$\zeta(\mathbf{z}) := \frac{1}{(2\pi)^{\frac{d}{2}}} \mathrm{e}^{-\frac{1}{2}|\mathbf{z}|^2}, \quad \forall \mathbf{z} \in \mathbb{R}^d. \tag{3.1}$$

For every fixed $(t_0, \boldsymbol{x}_0) \in \boldsymbol{\Lambda}$, define the map $\hat{\kappa}_\infty^{(t_0,\boldsymbol{x}_0)} : \hat{\boldsymbol{\Lambda}} \to \mathbb{R}$ as

$$\hat{\kappa}_\infty^{(t_0,\boldsymbol{x}_0)}(t, \hat{\boldsymbol{x}}) := \int_{\mathbb{R}^d} \big\| \hat{\boldsymbol{x}}(\cdot \wedge t) - \boldsymbol{x}_0(\cdot \wedge t_0) - \mathbf{z} 1_{[t,T]} \big\|_\infty \zeta(\mathbf{z})\, d\mathbf{z} - \int_{\mathbb{R}^d} |\mathbf{z}| \zeta(\mathbf{z})\, d\mathbf{z}, \tag{3.2}$$

for all $(t, \hat{\boldsymbol{x}}) \in \hat{\boldsymbol{\Lambda}}$. Moreover, let $\kappa_\infty^{(t_0,\boldsymbol{x}_0)} : \boldsymbol{\Lambda} \to \mathbb{R}$ be given by

$$\kappa_\infty^{(t_0,\boldsymbol{x}_0)}(t, \boldsymbol{x}) := \hat{\kappa}_\infty^{(t_0,\boldsymbol{x}_0)}(t, \boldsymbol{x}),$$

for every $(t, \boldsymbol{x}) \in \boldsymbol{\Lambda}$. Then, the following properties hold.

(1) *For every $(t, \hat{\boldsymbol{x}}) \in \hat{\boldsymbol{\Lambda}}$, the vertical derivatives of first and second-order of $\hat{\kappa}_\infty^{(t_0,\boldsymbol{x}_0)}$ at $(t, \hat{\boldsymbol{x}})$ (namely $\partial^V_{x_i} \hat{\kappa}_\infty^{(t_0,\boldsymbol{x}_0)}(t, \hat{\boldsymbol{x}})$ and $\partial^V_{x_i x_j} \hat{\kappa}_\infty^{(t_0,\boldsymbol{x}_0)}(t, \hat{\boldsymbol{x}})$, for every $i, j = 1, \dots, d$) exist.*

(2) *For every $i, j = 1, \dots, d$, $\partial^V_{x_i} \hat{\kappa}_\infty^{(t_0,\boldsymbol{x}_0)}$ is bounded by the constant 1 and $\partial^V_{x_i x_j} \hat{\kappa}_\infty^{(t_0,\boldsymbol{x}_0)}$ is bounded by the constant $\sqrt{\frac{2}{\pi}}$.*

(3) *$\hat{\kappa}_\infty^{(t_0,\boldsymbol{x}_0)} \geq -C_\zeta$ and $\kappa_\infty^{(t_0,\boldsymbol{x}_0)}(t, \boldsymbol{x}) \geq \|\boldsymbol{x}(\cdot \wedge t) - \boldsymbol{x}_0(\cdot \wedge t_0)\|_\infty - C_\zeta$, for every $(t, \boldsymbol{x}) \in \boldsymbol{\Lambda}$, with*

$$C_\zeta := \int_{\mathbb{R}^d} |\mathbf{z}| \zeta(\mathbf{z})\, d\mathbf{z} = \sqrt{2} \frac{\Gamma(\frac{d}{2} + \frac{1}{2})}{\Gamma(\frac{d}{2})} > 0, \tag{3.3}$$

where $\Gamma(\cdot)$ is the Gamma function.

(4) *For every fixed d, there exists some constant $\alpha_d > 0$ such that*

$$\begin{aligned} &\alpha_d \big(\big\| \boldsymbol{x}(\cdot \wedge t) - \boldsymbol{x}_0(\cdot \wedge t_0) \big\|_\infty^{d+1} \wedge \big\| \boldsymbol{x}(\cdot \wedge t) - \boldsymbol{x}_0(\cdot \wedge t_0) \big\|_\infty \big) \\ &\quad \leq \kappa_\infty^{(t_0,\boldsymbol{x}_0)}(t, \boldsymbol{x}) \leq \big\| \boldsymbol{x}(\cdot \wedge t) - \boldsymbol{x}_0(\cdot \wedge t_0) \big\|_\infty, \end{aligned} \tag{3.4}$$

for all $(t, \boldsymbol{x}) \in \boldsymbol{\Lambda}$. In particular, it holds that $\kappa_\infty^{(t_0,\boldsymbol{x}_0)} \geq 0$.

Proof. See Appendix C in [19], Section C.1. □

We now address the problem of smoothing the map $\hat{\kappa}_\infty^{(t_0,\boldsymbol{x}_0)}$ in the horizontal direction. This is required by the fact that the presence of $1_{[t,T]}$ in the definition of $\hat{\kappa}_\infty^{(t_0,\boldsymbol{x}_0)}$ is an obstruction to horizontal regularity, therefore a further convolution in the time variable t is needed. The latter convolution also provides the continuity on $\hat{\boldsymbol{\Lambda}}$ (notice that the map $(t, \hat{\boldsymbol{x}}) \mapsto \hat{\kappa}_\infty^{(t_0,\boldsymbol{x}_0)}(t, \hat{\boldsymbol{x}})$ is not continuous on $\hat{\boldsymbol{\Lambda}}$, see Remark 3.1).

We perform such a horizontal smoothing to $\hat{\kappa}_\infty^{(t_0,\boldsymbol{x}_0)}/(1 + C_\zeta + \hat{\kappa}_\infty^{(t_0,\boldsymbol{x}_0)})$. We apply it to such a map (rather than to $\hat{\kappa}_\infty^{(t_0,\boldsymbol{x}_0)}$ directly) in order to have bounded derivatives (see item 3 of Lemma 3.2). Moreover, we consider $1 + C_\zeta + \hat{\kappa}_\infty^{(t_0,\boldsymbol{x}_0)}$ (instead of $1 + \hat{\kappa}_\infty^{(t_0,\boldsymbol{x}_0)}$) in order to have a denominator greater than or equal to 1 (this follows from inequality $\hat{\kappa}_\infty^{(t_0,\boldsymbol{x}_0)} \geq -C_\zeta$, see item 3 of Lemma 3.1). The precise form of the mollifier η in (3.5) is used to get explicit bounds on $\hat{\chi}_\infty^{(t_0,\boldsymbol{x}_0)}$ and its derivatives.

Remark 3.1 ([47]). Notice that the map $(t,\hat{\boldsymbol{x}}) \mapsto \hat{\kappa}_\infty^{(t_0,\boldsymbol{x}_0)}(t,\hat{\boldsymbol{x}})$ is not continuous on $\hat{\boldsymbol{\Lambda}}$. As a matter of fact, consider the following example. Take $d=1$, $T=2$, $t_0=0$, $\boldsymbol{x}_0 \equiv 0$, $t=1$, $\hat{\boldsymbol{x}} = 1_{[1,2]}$. Then, it holds that

$$\begin{aligned}\hat{\kappa}_\infty^{(t_0,\boldsymbol{x}_0)}(t,\hat{\boldsymbol{x}}) &= \int_{\mathbb{R}} \big\|\hat{\boldsymbol{x}}(\cdot\wedge t) - \boldsymbol{x}_0(\cdot\wedge t_0) - \mathbf{z}1_{[t,T]}\big\|_\infty \zeta(\mathbf{z})\,d\mathbf{z} - \int_{\mathbb{R}} |\mathbf{z}|\zeta(\mathbf{z})\,d\mathbf{z} \\ &= \int_{\mathbb{R}} |1-\mathbf{z}|\zeta(\mathbf{z})\,d\mathbf{z} - \int_{\mathbb{R}} |\mathbf{z}|\zeta(\mathbf{z})\,d\mathbf{z}.\end{aligned}$$

Now, take $\delta \in (0,1)$, then

$$\begin{aligned}\hat{\kappa}_\infty^{(t_0,\boldsymbol{x}_0)}(t+\delta,\hat{\boldsymbol{x}}) &= \int_{\mathbb{R}} \big\|\hat{\boldsymbol{x}}\big(\cdot\wedge(t+\delta)\big) - \boldsymbol{x}_0(\cdot\wedge t_0) - \mathbf{z}1_{[t+\delta,T]}\big\|_\infty \zeta(\mathbf{z})\,d\mathbf{z} - \int_{\mathbb{R}} |\mathbf{z}|\zeta(\mathbf{z})\,d\mathbf{z} \\ &= \int_{\mathbb{R}} \max\big\{1,|1-\mathbf{z}|\big\}\zeta(\mathbf{z})\,d\mathbf{z} - \int_{\mathbb{R}} |\mathbf{z}|\zeta(\mathbf{z})\,d\mathbf{z}.\end{aligned}$$

In conclusion, we have

$$\begin{aligned}\big|\hat{\kappa}_\infty^{(t_0,\boldsymbol{x}_0)}(t+\delta,\hat{\boldsymbol{x}}) - \hat{\kappa}_\infty^{(t_0,\boldsymbol{x}_0)}(t,\hat{\boldsymbol{x}})\big| &= \int_{\mathbb{R}} \big\{\max\big\{1,|1-\mathbf{z}|\big\} - |1-\mathbf{z}|\big\}\zeta(\mathbf{z})\,d\mathbf{z} \\ &= \int_0^2 \big(1-|1-\mathbf{z}|\big)\zeta(\mathbf{z})\,d\mathbf{z} =: \varepsilon_* > 0,\end{aligned}$$

where ε_* is a constant independent of δ. This proves that $|\hat{\kappa}_\infty^{(t_0,\boldsymbol{x}_0)}(t+\delta,\hat{\boldsymbol{x}}) - \hat{\kappa}_\infty^{(t_0,\boldsymbol{x}_0)}(t,\hat{\boldsymbol{x}})| \nrightarrow 0$ as $\delta \to 0^+$ and shows that $\hat{\kappa}_\infty^{(t_0,\boldsymbol{x}_0)}$ is not continuous on $\hat{\boldsymbol{\Lambda}}$.

Lemma 3.2. *Let $\eta\colon \mathbb{R} \to \mathbb{R}$ be given by*

$$\eta(s) := s\mathrm{e}^{-s}, \quad \forall s \in \mathbb{R}. \tag{3.5}$$

$\forall (t_0,\boldsymbol{x}_0) \in \boldsymbol{\Lambda}$, let $\hat{\kappa}_\infty^{(t_0,\boldsymbol{x}_0)}$ be as in Lemma 3.1 and define the map $\hat{\chi}_\infty^{(t_0,\boldsymbol{x}_0)}\colon \hat{\boldsymbol{\Lambda}} \to \mathbb{R}$ as

$$\hat{\chi}_\infty^{(t_0,\boldsymbol{x}_0)}(t,\hat{\boldsymbol{x}}) := \int_0^{+\infty} \frac{\hat{\kappa}_\infty^{(t_0,\boldsymbol{x}_0)}((t+s)\wedge T, \hat{\boldsymbol{x}}(\cdot\wedge t))}{1 + C_\zeta + \hat{\kappa}_\infty^{(t_0,\boldsymbol{x}_0)}((t+s)\wedge T, \hat{\boldsymbol{x}}(\cdot\wedge t))}\eta(s)\,ds,$$

for all $(t,\hat{\boldsymbol{x}}) \in \hat{\boldsymbol{\Lambda}}$, with C_ζ as in (3.3), where we recall that $1 + C_\zeta + \hat{\kappa}_\infty^{(t_0,\boldsymbol{x}_0)} \geq 1$ (see item 3 of Lemma 3.1). Moreover, let $\chi_\infty^{(t_0,\boldsymbol{x}_0)}\colon \boldsymbol{\Lambda} \to \mathbb{R}$ be given by

$$\chi_\infty^{(t_0,\boldsymbol{x}_0)}(t,\boldsymbol{x}) := \hat{\chi}_\infty^{(t_0,\boldsymbol{x}_0)}(t,\boldsymbol{x}), \quad \forall (t,\boldsymbol{x}) \in \boldsymbol{\Lambda}. \tag{3.6}$$

Then, the following properties hold.

1. *For every $(t,\hat{\boldsymbol{x}}) \in \hat{\boldsymbol{\Lambda}}$, the horizontal and vertical derivatives of first and second-order of $\hat{\chi}_\infty^{(t_0,\boldsymbol{x}_0)}$ at $(t,\hat{\boldsymbol{x}})$ (namely $\partial_t^H \hat{\chi}_\infty^{(t_0,\boldsymbol{x}_0)}(t,\hat{\boldsymbol{x}})$, $\partial_{x_i}^V \hat{\chi}_\infty^{(t_0,\boldsymbol{x}_0)}(t,\hat{\boldsymbol{x}})$ and $\partial_{x_i x_j}^V \hat{\chi}_\infty^{(t_0,\boldsymbol{x}_0)}(t,\hat{\boldsymbol{x}})$, for every $i,j = 1,\dots,d$) exist.*
2. *$\hat{\chi}_\infty^{(t_0,\boldsymbol{x}_0)} \in \boldsymbol{C}^{1,2}(\hat{\boldsymbol{\Lambda}})$ and the map $((t_0,\boldsymbol{x}_0),(t,\hat{\boldsymbol{x}})) \mapsto \hat{\chi}_\infty^{(t_0,\boldsymbol{x}_0)}(t,\hat{\boldsymbol{x}})$ is continuous on $\boldsymbol{\Lambda} \times \hat{\boldsymbol{\Lambda}}$.*
3. *The horizontal derivative of $\hat{\chi}_\infty^{(t_0,\boldsymbol{x}_0)}$ is bounded by the constant $\frac{2}{\mathrm{e}}$; the first-order vertical derivatives of $\hat{\chi}_\infty^{(t_0,\boldsymbol{x}_0)}$ are bounded by the constant $1 + C_\zeta$; the second-order vertical derivatives of $\hat{\chi}_\infty^{(t_0,\boldsymbol{x}_0)}$ are bounded by the constant $(1+C_\zeta)(\sqrt{\frac{2}{\pi}} + 2)$.*

(4) *For every* $(t, \boldsymbol{x}) \in \boldsymbol{\Lambda}$,

$$\alpha_d \frac{\|\boldsymbol{x}(\cdot \wedge t) - \boldsymbol{x}_0(\cdot \wedge t_0)\|_\infty^{d+1} \wedge \|\boldsymbol{x}(\cdot \wedge t) - \boldsymbol{x}_0(\cdot \wedge t_0)\|_\infty}{1 + C_\zeta + \|\boldsymbol{x}(\cdot \wedge t) - \boldsymbol{x}_0(\cdot \wedge t_0)\|_\infty} \leq \chi_\infty^{(t_0, \boldsymbol{x}_0)}(t, \boldsymbol{x}) \leq \big\|\boldsymbol{x}(\cdot \wedge t) - \boldsymbol{x}_0(\cdot \wedge t_0)\big\|_\infty \wedge 1, \tag{3.7}$$

with the same constant α_d *as in* (3.4). *In particular, it holds that* $\chi_\infty^{(t_0, \boldsymbol{x}_0)} \geq 0$.

Proof. See Appendix C in [19], Section C.2. □

In conclusion, by Lemma 3.2 it follows that the map $\rho_\infty \colon \boldsymbol{\Lambda} \times \boldsymbol{\Lambda} \to [0, +\infty)$ given by

$$\rho_\infty\big((t, \boldsymbol{x}), (t_0, \boldsymbol{x}_0)\big) = |t - t_0|^2 + \chi_\infty^{(t_0, \boldsymbol{x}_0)}(t, \boldsymbol{x}), \quad \forall (t, \boldsymbol{x}), (t_0, \boldsymbol{x}_0) \in \boldsymbol{\Lambda}, \tag{3.8}$$

with χ_∞ as in (3.6), is a gauge-type function, which is also smooth as a map of the first pair, namely $(t, \boldsymbol{x}) \mapsto \rho_\infty((t, \boldsymbol{x}), (t_0, \boldsymbol{x}_0))$, and with bounded derivatives.

We now apply Theorem 3.1 to the smooth gauge-type function ρ_∞ with bounded derivatives defined by (3.8), taking $\delta_0 := \delta > 0$ and $\delta_n := \delta/2^n$, for every $n \geq 1$.

Theorem 3.2 (Smooth variational principle on $\boldsymbol{\Lambda}$). *Let* $\delta > 0$ *and* $G \colon \boldsymbol{\Lambda} \to \mathbb{R}$ *be an upper semicontinuous map, bounded from above. For every* $\varepsilon > 0$, *let* $(t_0, \boldsymbol{x}_0) \in \boldsymbol{\Lambda}$ *satisfy*

$$\sup G - \varepsilon \leq G(t_0, \boldsymbol{x}_0).$$

Then, there exists a sequence $\{(t_n, \boldsymbol{x}_n)\}_{n \geq 1} \subset \boldsymbol{\Lambda}$ *which converges to some* $(\bar{t}, \bar{\boldsymbol{x}}) \in \boldsymbol{\Lambda}$ *fulfilling the properties below.*

(i) $\rho_\infty((\bar{t}, \bar{\boldsymbol{x}}), (t_n, \boldsymbol{x}_n)) \leq \frac{\varepsilon}{2^n \delta}$, *for every* $n \geq 0$.
(ii) $G(t_0, \boldsymbol{x}_0) \leq G(\bar{t}, \bar{\boldsymbol{x}}) - \delta \varphi_\varepsilon(t, \boldsymbol{x})$, *where the map* $\varphi_\varepsilon \colon \boldsymbol{\Lambda} \to [0, +\infty)$ *is defined as*

$$\varphi_\varepsilon(t, \boldsymbol{x}) := \sum_{n=0}^{+\infty} \frac{1}{2^n} \rho_\infty\big((t, \boldsymbol{x}), (t_n, \boldsymbol{x}_n)\big), \quad \forall (t, \boldsymbol{x}) \in \boldsymbol{\Lambda}.$$

(iii) *For every* $(t, \boldsymbol{x}) \neq (\bar{t}, \bar{\boldsymbol{x}})$, $G(t, \boldsymbol{x}) - \delta \varphi_\varepsilon(t, \boldsymbol{x}) < G(\bar{t}, \bar{\boldsymbol{x}}) - \delta \varphi_\varepsilon(\bar{t}, \bar{\boldsymbol{x}})$.

Finally, the map φ_ε *satisfies the following properties.*

(1) $\varphi_\varepsilon \in \boldsymbol{C}^{1,2}(\boldsymbol{\Lambda})$ *and is bounded.*
(2) $\partial_t^H \varphi_\varepsilon$ *is bounded by the constant* $2(2T + \frac{2}{\mathrm{e}})$.
(3) *For every* $i, j = 1, \dots, d$, $\partial_{x_i}^V \varphi_\varepsilon$ *is bounded by the constant* $2(1 + C_\zeta)$ *and* $\partial_{x_i x_j}^V \varphi_\varepsilon$ *is bounded by the constant* $2(1 + C_\zeta)(\sqrt{\frac{2}{\pi}} + 2)$.

Proof. Items (i)–(ii)–(iii) follow directly from Theorem 3.1, while items (1)–(2)–(3) follow easily from items (2)–(3)–(4) of Lemma 3.2. □

4. Crandall–Lions (path-dependent) viscosity solutions

4.1. Viscosity solutions

In the present section, we consider the second-order path-dependent partial differential equation

$$\begin{cases} \partial_t^H u(t,\boldsymbol{x}) = F\big(t,\boldsymbol{x},u(t,\boldsymbol{x}),\partial_{\boldsymbol{x}}^V u(t,\boldsymbol{x}),\partial_{\boldsymbol{x}\boldsymbol{x}}^V u(t,\boldsymbol{x})\big), & (t,\boldsymbol{x}) \in [0,T) \times C\big([0,T];\mathbb{R}^d\big), \\ u(T,\boldsymbol{x}) = \xi(\boldsymbol{x}), & \boldsymbol{x} \in C\big([0,T];\mathbb{R}^d\big), \end{cases} \tag{4.1}$$

with $F\colon [0,T] \times C([0,T];\mathbb{R}^d) \times \mathbb{R}^d \times \mathcal{S}(d) \to \mathbb{R}$ and $\xi: C([0,T];\mathbb{R}^d) \to \mathbb{R}$, where $\mathcal{S}(d)$ is the set of symmetric $d \times d$ matrices.

Definition 4.1. We denote by $\boldsymbol{C}^{1,2}_{\text{pol}}(\boldsymbol{\Lambda})$ the set of $\varphi \in \boldsymbol{C}^{1,2}(\boldsymbol{\Lambda})$ such that φ, $\partial_t^H\varphi$, $\partial_{\boldsymbol{x}}^V\varphi$, $\partial_{\boldsymbol{x}\boldsymbol{x}}^V\varphi$ satisfy a polynomial growth condition.

Definition 4.2. We say that an upper semicontinuous map $u\colon \boldsymbol{\Lambda} \to \mathbb{R}$ is a (*path-dependent*) *viscosity subsolution* of equation (4.1) if the following holds.

- $u(T,\boldsymbol{x}) \le \xi(\boldsymbol{x})$, for all $\boldsymbol{x} \in C([0,T];\mathbb{R}^d)$;
- for any $(t,\boldsymbol{x}) \in [0,T) \times C([0,T];\mathbb{R}^d)$ and $\varphi \in \boldsymbol{C}^{1,2}_{\text{pol}}(\boldsymbol{\Lambda})$, satisfying

$$(u-\varphi)(t,\boldsymbol{x}) = \sup_{(t',\boldsymbol{x}')\in\boldsymbol{\Lambda}} (u-\varphi)\big(t',\boldsymbol{x}'\big),$$

  we have

$$-\partial_t^H\varphi(t,\boldsymbol{x}) + F\big(t,\boldsymbol{x},u(t,\boldsymbol{x}),\partial_{\boldsymbol{x}}^V\varphi(t,\boldsymbol{x}),\partial_{\boldsymbol{x}\boldsymbol{x}}^V\varphi(t,\boldsymbol{x})\big) \le 0.$$

We say that a lower semicontinuous map $u\colon \boldsymbol{\Lambda} \to \mathbb{R}$ is a (*path-dependent*) *viscosity supersolution* of equation (4.1) if:

- $u(T,\boldsymbol{x}) \ge \xi(\boldsymbol{x})$, for all $\boldsymbol{x} \in C([0,T];\mathbb{R}^d)$;
- for any $(t,\boldsymbol{x}) \in [0,T) \times C([0,T];\mathbb{R}^d)$ and $\varphi \in \boldsymbol{C}^{1,2}_{\text{pol}}(\boldsymbol{\Lambda})$, satisfying:

$$(u-\varphi)(t,\boldsymbol{x}) = \inf_{(t',\boldsymbol{x}')\in\boldsymbol{\Lambda}} (u-\varphi)\big(t',\boldsymbol{x}'\big),$$

  we have

$$-\partial_t^H\varphi(t,\boldsymbol{x}) + F\big(t,\boldsymbol{x},u(t,\boldsymbol{x}),\partial_{\boldsymbol{x}}^V\varphi(t,\boldsymbol{x}),\partial_{\boldsymbol{x}\boldsymbol{x}}^V\varphi(t,\boldsymbol{x})\big) \ge 0.$$

We say that a continuous map $u\colon \boldsymbol{\Lambda} \to \mathbb{R}$ is a (*path-dependent*) *viscosity solution* of equation (4.1) if u is both a (path-dependent) viscosity subsolution and a (path-dependent) viscosity supersolution of (4.1).

4.2. Path-dependent heat equation

In the present section, we focus on the path-dependent heat equation, namely when $F(t,\boldsymbol{x},r,p,M) = -\frac{1}{2}\text{tr}[M]$

$$\begin{cases} \partial_t^H u(t,\boldsymbol{x}) + \dfrac{1}{2}\text{tr}\big[\partial_{\boldsymbol{x}\boldsymbol{x}}^V u(t,\boldsymbol{x})\big] = 0, & (t,\boldsymbol{x}) \in [0,T) \times C\big([0,T];\mathbb{R}^d\big), \\ u(T,\boldsymbol{x}) = \xi(\boldsymbol{x}), & \boldsymbol{x} \in C\big([0,T];\mathbb{R}^d\big). \end{cases} \tag{4.2}$$

In the sequel, we denote

$$\mathcal{L}u(t,\boldsymbol{x}) := \partial_t^H u(t,\boldsymbol{x}) + \frac{1}{2}\operatorname{tr}\big[\partial_{\boldsymbol{x}\boldsymbol{x}}^V u(t,\boldsymbol{x})\big]. \tag{4.3}$$

On the terminal condition ξ, we impose the assumption

(A) *The function* $\xi : C([0,T];\mathbb{R}^d) \to \mathbb{R}$ *is continuous and bounded.*

4.2.1. *Existence*

The "candidate solution" to equation (4.2) is

$$v(t,\boldsymbol{x}) := \mathbb{E}\big[\xi\big(\boldsymbol{W}^{t,\boldsymbol{x}}\big)\big], \quad \forall (t,\boldsymbol{x}) \in \boldsymbol{\Lambda}, \tag{4.4}$$

where $\boldsymbol{W} = (\boldsymbol{W}_s)_{s\in[0,T]}$ is a d-dimensional Brownian motion on some complete probability space $(\Omega,\mathcal{F},\mathbb{P})$, and the stochastic process $\boldsymbol{W}^{t,\boldsymbol{x}} = (\boldsymbol{W}_s^{t,\boldsymbol{x}})_{s\in[0,T]}$ is given by

$$\boldsymbol{W}_s^{t,\boldsymbol{x}} := \begin{cases} \boldsymbol{x}(s), & s \le t, \\ \boldsymbol{x}(t) + \boldsymbol{W}_s - \boldsymbol{W}_t, & s > t. \end{cases} \tag{4.5}$$

Remark 4.1. The boundedness of ξ in Assumption (A) will be used in the proof of (the comparison) Theorem 4.2. On the other hand, the proof that the function v in (4.4) is continuous and is a viscosity solution of equation (4.2) (see the proof of Theorem 4.1) holds under weaker growth condition on ξ (for instance, ξ having polynomial growth).

Theorem 4.1. *Under Assumption* (A), *the function* v *in* (4.4) *is continuous and bounded*. *Moreover*, v *is a* (*path-dependent*) *viscosity solution of equation* (4.2).

Proof. STEP I. *Continuity of* v. Given $(t,\boldsymbol{x}), (t',\boldsymbol{x}') \in \boldsymbol{\Lambda}$, with $t \le t'$, from (4.5) we have

$$\boldsymbol{W}_s^{t,\boldsymbol{x}} - \boldsymbol{W}_s^{t',\boldsymbol{x}'} = \begin{cases} \boldsymbol{x}(s) - \boldsymbol{x}'(s), & s \le t, \\ \boldsymbol{x}(t) - \boldsymbol{x}'(s) + \boldsymbol{W}_s - \boldsymbol{W}_t, & t < s \le t', \\ \boldsymbol{x}(t) - \boldsymbol{x}'\big(t'\big) + \boldsymbol{W}_{t'} - \boldsymbol{W}_t, & s > t'. \end{cases}$$

Hence,

$$\begin{aligned} \sup_{s\in[0,T]} \big|\boldsymbol{W}_s^{t,\boldsymbol{x}} - \boldsymbol{W}_s^{t',\boldsymbol{x}'}\big| &\le \big\|\boldsymbol{x}(\cdot\wedge t) - \boldsymbol{x}'\big(\cdot\wedge t'\big)\big\|_\infty + \sup_{s\in[t,t']} |\boldsymbol{W}_s - \boldsymbol{W}_t| \\ &\le \big\|\boldsymbol{x}(\cdot\wedge t) - \boldsymbol{x}'\big(\cdot\wedge t'\big)\big\|_\infty + \sum_{i=1}^d \sup_{s\in[t,t']} \big|W_s^i - W_t^i\big|, \end{aligned}$$

where $\boldsymbol{W} = (W^1,\dots,W^d)$ and the second inequality follows from the fact the Euclidean norm on $\mathbb{R}^d$ is estimated by the 1-norm. By the reflection principle, $\sup_{s\in[t,t']}|W_s^i - W_t^i|$ has the same law as $|W_{t'}^i - W_t^i|$, therefore

$$\begin{aligned} \mathbb{E}\Big[\sup_{s\in[0,T]} \big|\boldsymbol{W}_s^{t,\boldsymbol{x}} - \boldsymbol{W}_s^{t',\boldsymbol{x}'}\big|\Big] &\le \big\|\boldsymbol{x}(\cdot\wedge t) - \boldsymbol{x}'\big(\cdot\wedge t'\big)\big\|_\infty + \sum_{i=1}^d \mathbb{E}\big[\big|W_{t'}^i - W_t^i\big|\big] \\ &= \big\|\boldsymbol{x}(\cdot\wedge t) - \boldsymbol{x}'\big(\cdot\wedge t'\big)\big\|_\infty + d\sqrt{\frac{2}{\pi}}\sqrt{\big|t-t'\big|}. \end{aligned}$$

Then, since ξ is bounded and continuous, the continuity of v follows from the above estimate together with the Lebesgue dominated convergence theorem.

STEP II. *v is a viscosity solution of equation* (4.2). For every $t \in [0,T]$, let $\mathbb{F}^t = (\mathcal{F}^t_s)_{s\in[t,T]}$ be the completion of the filtration generated by $(\boldsymbol{W}_s - \boldsymbol{W}_t)_{s\in[t,T]}$. Now, fix $(t,\boldsymbol{x}) \in \boldsymbol{\Lambda}$ and $t' \in [t,T]$. We first prove that

$$v(t,\boldsymbol{x}) = \mathbb{E}\big[v\big(t', \boldsymbol{W}^{t,\boldsymbol{x}}\big)\big]. \tag{4.6}$$

To this end, we begin noticing that by (4.5) we have

$$\boldsymbol{W}^{t,\boldsymbol{x}}_\cdot = \boldsymbol{x}(\cdot\wedge t) + \boldsymbol{W}_{\cdot\vee t} - \boldsymbol{W}_t. \tag{4.7}$$

Therefore,

$$v(t,\boldsymbol{x}) = \mathbb{E}\big[\xi\big(\boldsymbol{x}(\cdot\wedge t) + \boldsymbol{W}_{\cdot\vee t} - \boldsymbol{W}_t\big)\big]. \tag{4.8}$$

Now, notice that, by (4.7),

$$\boldsymbol{W}^{t',\boldsymbol{W}^{t,\boldsymbol{x}}}_\cdot = \boldsymbol{W}^{t,\boldsymbol{x}}_{\cdot\wedge t'} + \boldsymbol{W}_{\cdot\vee t'} - \boldsymbol{W}_{t'} = \boldsymbol{W}^{t,\boldsymbol{x}}_\cdot.$$

This proves the flow property $\boldsymbol{W}^{t,\boldsymbol{x}}_\cdot = \boldsymbol{W}^{t',\boldsymbol{W}^{t,\boldsymbol{x}}}_\cdot$. Then, by the freezing lemma for conditional expectation and formula (4.8), we obtain

$$\begin{aligned} v(t,\boldsymbol{x}) &= \mathbb{E}\big[\xi\big(\boldsymbol{W}^{t,\boldsymbol{x}}\big)\big] \\ &= \mathbb{E}\big[\xi\big(\boldsymbol{W}^{t',\boldsymbol{W}^{t,\boldsymbol{x}}}\big)\big] = \mathbb{E}\big[\xi\big(\boldsymbol{W}^{t,\boldsymbol{x}}_{\cdot\wedge t'} + \boldsymbol{W}_{\cdot\vee t'} - \boldsymbol{W}_{t'}\big)\big] \\ &= \mathbb{E}\big[\mathbb{E}\big[\xi\big(\boldsymbol{W}^{t,\boldsymbol{x}}_{\cdot\wedge t'} + \boldsymbol{W}_{\cdot\vee t'} - \boldsymbol{W}_{t'}\big)|\mathcal{F}^t_{t'}\big]\big] = \mathbb{E}\big[v\big(t', \boldsymbol{W}^{t,\boldsymbol{x}}_{\cdot\wedge t'}\big)\big]. \end{aligned}$$

Finally, recalling that v is non-anticipative, we deduce that $v(t', \boldsymbol{W}^{t,\boldsymbol{x}}_{\cdot\wedge t'}) = v(t', \boldsymbol{W}^{t,\boldsymbol{x}})$, which concludes the proof of formula (4.6).

Let us now prove that v is a viscosity solution of equation (4.2). We only prove the viscosity subsolution property, as the supersolution property can be proved in a similar way. We proceed along the same lines as in the proof of the subsolution property in Theorem 3.66 of [38]. Let $(t,\boldsymbol{x}) \in [0,T) \times C([0,T];\mathbb{R}^d)$ and $\varphi \in \boldsymbol{C}^{1,2}_{\mathrm{pol}}(\boldsymbol{\Lambda})$, satisfying:

$$(v-\varphi)(t,\boldsymbol{x}) = \sup_{(t',\boldsymbol{x}')\in\boldsymbol{\Lambda}} (v-\varphi)\big(t',\boldsymbol{x}'\big).$$

We suppose that $(v-\varphi)(t,\boldsymbol{x}) = 0$ (if this is not the case, we replace φ by $\psi(\cdot,\cdot) := \varphi(\cdot,\cdot) + v(t,\boldsymbol{x}) - \varphi(t,\boldsymbol{x})$). Take

$$\varphi(t,\boldsymbol{x}) = v(t,\boldsymbol{x}) = \mathbb{E}\big[v\big(t+\varepsilon, \boldsymbol{W}^{t,\boldsymbol{x}}\big)\big] \le \mathbb{E}\big[\varphi\big(t+\varepsilon, \boldsymbol{W}^{t,\boldsymbol{x}}\big)\big], \tag{4.9}$$

where the latter inequality follows from the fact that $\sup(v-\varphi) = 0$, so that $v \le \varphi$ on $\boldsymbol{\Lambda}$. Notice that the last expectation in (4.9) is finite, as φ has polynomial growth. Now, by the functional Itô formula (2.1), we have

$$\varphi\big(t+\varepsilon, \boldsymbol{W}^{t,\boldsymbol{x}}\big) = \varphi(t,\boldsymbol{x}) + \int_t^{t+\varepsilon} \mathcal{L}\varphi\big(s, \boldsymbol{W}^{t,\boldsymbol{x}}\big)\,ds + \sum_{i=1}^d \int_t^{t+\varepsilon} \partial^V_{x_i}\varphi\big(s, \boldsymbol{W}^{t,\boldsymbol{x}}\big)\,dW^i_s,$$

where $\mathcal{L}$ was defined in (4.3). Since $\partial_{x_i}^V \varphi$ has polynomial growth, the corresponding stochastic integral is a martingale. Then, plugging the above formula into (4.9) and dividing by ε, we find

$$-\mathbb{E}\left[\frac{1}{\varepsilon}\int_t^{t+\varepsilon} \mathcal{L}\varphi\big(s, \boldsymbol{W}^{t,\boldsymbol{x}}\big)\, ds\right] \le 0.$$

Letting $\varepsilon \to 0^+$, we conclude that

$$-\mathcal{L}\varphi(t, \boldsymbol{x}) \le 0,$$

which proves the viscosity subsolution property. □

4.2.2. *Comparison theorem and uniqueness*

Lemma 4.1. *Suppose that Assumption* (A) *holds. Then, there exists a sequence* $\{\xi_N\}_N$, *with* ξ_N *being a map from* $C([0, T]; \mathbb{R}^d)$ *into* $\mathbb{R}$, *such that the following holds.*

(I) $\{\xi_N\}_N$ *converges pointwise to* ξ *as* $N \to +\infty$.
(II) ξ_N *is bounded uniformly with respect to* N.
(III) *For every* N, *let*

$$v_N(t, \boldsymbol{x}) := \mathbb{E}\big[\xi_N\big(\boldsymbol{W}^{t,\boldsymbol{x}}\big)\big], \forall (t, \boldsymbol{x}) \in \boldsymbol{\Lambda}.$$

Then, $v_N \in \boldsymbol{C}^{1,2}(\boldsymbol{\Lambda})$ *and is a classical* (*smooth*) *solution of equation* (4.2) *with terminal condition* ξ_N.
(IV) v_N *is bounded uniformly with respect to* N.
(V) $\{v_N\}_N$ *converges pointwise to* v *as* $N \to +\infty$.

Proof. Items (I) and (II) follow from Lemma D.2 in [19], while item (III) follows from Lemma D.1 in [19]. Item (IV) is a consequence of item (II). Finally, by items (I)–(II) we can apply the Lebesgue dominated convergence theorem, from which we deduce that item (V) holds. □

Theorem 4.2. *Suppose that Assumption* (A) *holds. Let* $u, w\colon \boldsymbol{\Lambda} \to \mathbb{R}$ *be respectively upper and lower semicontinuous*, *satisfying*

$$\sup u < +\infty, \qquad \inf w > -\infty.$$

Suppose that u (*resp.* w) *is a* (*path-dependent*) *viscosity subsolution* (*resp. supersolution*) *of equation* (4.2). *Then* $u \le w$ *on* $\boldsymbol{\Lambda}$.

Proof. The proof consists in showing that $u \le v$ and $v \le w$ on $\boldsymbol{\Lambda}$ (with v given by (4.4)), from which we immediately deduce the claim. In what follows, we only report the proof of the inequality $u \le v$, as the other inequality (that is $v \le w$) can be deduced from the first one replacing u, v, ξ with $-w$, $-v$, $-\xi$, respectively.

We proceed by contradiction and assume that $\sup(u - v) > 0$. Then, there exists $(t_0, \boldsymbol{x}_0) \in \boldsymbol{\Lambda}$ such that

$$(u - v)(t_0, \boldsymbol{x}_0) > 0.$$

Notice that $t_0 < T$, since $u(T, \cdot) \le \xi(\cdot) = v(T, \cdot)$. We split the rest of the proof into five steps.

STEP I. Let $\{\xi_N\}_N$ and $\{v_N\}_N$ be the sequences given by Lemma 4.1. Then, we notice that there exists $N_0 \in \mathbb{N}$ such that

$$(u - v_{N_0})(t_0, \boldsymbol{x}_0) > 0. \tag{4.10}$$

We also suppose that (possibly enlarging N_0)

$$\left|\xi(\boldsymbol{x}_0) - \xi_{N_0}(\boldsymbol{x}_0)\right| \le \frac{1}{2}(u - v_{N_0})(t_0, \boldsymbol{x}_0). \tag{4.11}$$

STEP II. For every $\lambda > 0$, we set

$$u^\lambda(t, \boldsymbol{x}) := e^{\lambda t} u(t, \boldsymbol{x}), \qquad \xi^\lambda(\boldsymbol{x}) := e^{\lambda T}\xi(\boldsymbol{x}),$$
$$v^\lambda_{N_0}(t, \boldsymbol{x}) := e^{\lambda t} v_{N_0}(t, \boldsymbol{x}), \qquad \xi^\lambda_{N_0}(\boldsymbol{x}) := e^{\lambda T}\xi_{N_0}(\boldsymbol{x}).$$

for all $(t, \boldsymbol{x}) \in \boldsymbol{\Lambda}$. Notice that u^λ is a (path-dependent) viscosity subsolution of the path-dependent partial differential equation

$$\begin{cases} \partial_t^H u^\lambda(t, \boldsymbol{x}) + \frac{1}{2}\operatorname{tr}\big[\partial_{\boldsymbol{x}}^V u^\lambda(t, \boldsymbol{x})\big] = \lambda u^\lambda(t, \boldsymbol{x}), & (t, \boldsymbol{x}) \in [0, T) \times C\big([0, T]; \mathbb{R}^d\big), \\ u^\lambda(T, \boldsymbol{x}) = \xi^\lambda(\boldsymbol{x}), & \boldsymbol{x} \in C\big([0, T]; \mathbb{R}^d\big). \end{cases} \tag{4.12}$$

Similarly, $v^\lambda_{N_0}$ is a classical (smooth) solution of equation (4.12) with ξ^λ replaced by $\xi^\lambda_{N_0}$. We finally notice that by (4.10) we have

$$\big(u^\lambda - v^\lambda_{N_0}\big)(t_0, \boldsymbol{x}_0) > 0.$$

So, in particular,

$$\sup\big(u^\lambda - v^\lambda_{N_0}\big) - \varepsilon = \big(u^\lambda - v^\lambda_{N_0}\big)(t_0, \boldsymbol{x}_0) \le \sup\big(u^\lambda - v^\lambda_{N_0}\big), \tag{4.13}$$

where $\varepsilon := \sup(u^\lambda - v^\lambda_{N_0}) - (u^\lambda - v^\lambda_{N_0})(t_0, \boldsymbol{x}_0)$.

STEP III. Notice that $u^\lambda - v^\lambda_{N_0}$ is upper semicontinuous and bounded from above. Then, by (4.13) and the smooth variational principle (Theorem 3.2) with $G = u^\lambda - v^\lambda_{N_0}$, we deduce that for every $\delta > 0$ there exists a sequence $\{(t_n, \boldsymbol{x}_n)\}_{n \ge 1} \subset \boldsymbol{\Lambda}$ converging to some $(\bar{t}, \bar{\boldsymbol{x}}) \in \boldsymbol{\Lambda}$ (possibly depending on $\varepsilon, \delta, \lambda, N_0$) such that the following holds.

(i) $\rho_\infty((t_n, \boldsymbol{x}_n), (\bar{t}, \bar{\boldsymbol{x}})) \le \frac{\varepsilon}{2^n \delta}$, for every $n \ge 0$, where ρ_∞ is the smooth gauge-type function with bounded derivatives defined by (3.8).

(ii) $(u^\lambda - v^\lambda_{N_0})(t_0, \boldsymbol{x}_0) \le (u^\lambda - (v^\lambda_{N_0} + \delta\varphi_\varepsilon))(\bar{t}, \bar{\boldsymbol{x}})$, where

$$\varphi_\varepsilon(t, \boldsymbol{x}) := \sum_{n=0}^{+\infty} \frac{1}{2^n}\rho_\infty\big((t, \boldsymbol{x}), (t_n, \boldsymbol{x}_n)\big) \quad \forall (t, \boldsymbol{x}) \in \boldsymbol{\Lambda}.$$

(iii) It holds that

$$\big(u^\lambda - \big(v^\lambda_{N_0} + \delta\varphi_\varepsilon\big)\big)(\bar{t}, \bar{\boldsymbol{x}}) = \sup_{(t,\boldsymbol{x}) \in \boldsymbol{\Lambda}} \big(u^\lambda - \big(v^\lambda_{N_0} + \delta\varphi_\varepsilon\big)\big)(t, \boldsymbol{x}). \tag{4.14}$$

We also recall from Theorem 3.2 that φ_ε satisfies the following properties.

(1) $\varphi_\varepsilon \in \boldsymbol{C}^{1,2}(\boldsymbol{\Lambda})$ and is bounded.
(2) $|\partial_t^H \varphi_\varepsilon(t, \boldsymbol{x})| \le 2(2T + \frac{2}{e})$, for every $(t, \boldsymbol{x}) \in [0, T) \times C([0, T]; \mathbb{R}^d)$.
(3) For every $i, j = 1, \dots, d$, $\partial^V_{x_i}\varphi_\varepsilon$ is bounded by the constant $2(1 + C_\zeta)$ and $\partial^V_{x_i x_j}\varphi_\varepsilon$ is bounded by the constant $2(1 + C_\zeta)(\sqrt{\frac{2}{\pi}} + 2)$.

In particular, $\varphi_\varepsilon \in \boldsymbol{C}^{1,2}_{\mathrm{pol}}(\boldsymbol{\Lambda})$.

STEP IV. We prove below that $\bar t < T$. As a matter of fact, by item (ii) of STEP III we have

$$\big(u^\lambda - \big(v^\lambda_{N_0} + \delta\varphi_\varepsilon\big)\big)(\bar t, \bar{\boldsymbol{x}}) \geq \big(u^\lambda - v^\lambda_{N_0}\big)(t_0, \boldsymbol{x}_0). \tag{4.15}$$

On the other hand, if $\bar t = T$ we obtain

$$\big(u^\lambda - \big(v^\lambda_{N_0} + \delta\varphi_\varepsilon\big)\big)(\bar t, \bar{\boldsymbol{x}}) = \mathrm{e}^{\lambda T}\big(\xi(\bar{\boldsymbol{x}}) - \xi_{N_0}(\bar{\boldsymbol{x}})\big) - \delta\varphi_\varepsilon(T, \bar{\boldsymbol{x}}) \leq \mathrm{e}^{\lambda T}\big(\xi(\bar{\boldsymbol{x}}) - \xi_{N_0}(\bar{\boldsymbol{x}})\big), \tag{4.16}$$

where the latter inequality comes from the fact that $\varphi_\varepsilon \geq 0$. Hence, by (4.15) and (4.16) we get

$$\mathrm{e}^{\lambda t_0}(u - v_{N_0})(t_0, \boldsymbol{x}_0) \leq \mathrm{e}^{\lambda T}\big(\xi(\bar{\boldsymbol{x}}) - \xi_{N_0}(\bar{\boldsymbol{x}})\big).$$

Letting $\varepsilon \to 0$, it follows from item (i) above with $n = 0$ and (3.7) that $d_\infty((\bar t, \bar{\boldsymbol{x}}), (t_0, \boldsymbol{x}_0)) \to 0$. Therefore, letting $\varepsilon \to 0$ in the previous inequality, we obtain

$$\mathrm{e}^{\lambda t_0}(u - v_{N_0})(t_0, \boldsymbol{x}_0) \leq \mathrm{e}^{\lambda T}\big(\xi(\boldsymbol{x}_0) - \xi_{N_0}(\boldsymbol{x}_0)\big).$$

By (4.11), we end up with $\mathrm{e}^{\lambda t_0} \leq \frac{1}{2}\mathrm{e}^{\lambda T}$. Letting $\lambda \to 0$, we find a contradiction.

STEP V. Here again $\lambda > 0$ is fixed. By (4.14) and the definition of viscosity subsolution of (4.12) applied to u^λ at the point $(\bar t, \bar{\boldsymbol{x}})$ with test function $v^\lambda_{N_0} + \delta\varphi_\varepsilon$, we obtain

$$-\mathcal{L}\big(v^\lambda_{N_0} + \delta\varphi_\varepsilon\big)(\bar t, \bar{\boldsymbol{x}}) + \lambda u^\lambda(\bar t, \bar{\boldsymbol{x}}) \leq 0.$$

Recalling that $v^\lambda_{N_0}$ is a classical (smooth) solution of equation (4.12) with ξ^λ replaced by $\xi^\lambda_{N_0}$, we find

$$\lambda\big(u^\lambda - v^\lambda_{N_0}\big)(\bar t, \bar{\boldsymbol{x}}) \leq \delta\mathcal{L}\varphi_\varepsilon(\bar t, \bar{\boldsymbol{x}}).$$

By item (ii) in STEP III (namely (4.15)), subtracting from both sides the quantity $\lambda\delta\varphi_\varepsilon(\bar t, \bar{\boldsymbol{x}})$, we obtain

$$\lambda\big(u^\lambda - v^\lambda_{N_0}\big)(t_0, \boldsymbol{x}_0) \leq \lambda\big(u^\lambda - \big(v^\lambda_{N_0} + \delta\varphi_\varepsilon\big)\big)(\bar t, \bar{\boldsymbol{x}}) \leq \delta\mathcal{L}\varphi_\varepsilon(\bar t, \bar{\boldsymbol{x}}) - \lambda\delta\varphi_\varepsilon(\bar t, \bar{\boldsymbol{x}}).$$

Recalling that $\varphi_\varepsilon \geq 0$, we see that

$$\lambda\big(u^\lambda - v^\lambda_{N_0}\big)(t_0, \boldsymbol{x}_0) \leq \lambda\big(u^\lambda - \big(v^\lambda_{N_0} + \delta\varphi_\varepsilon\big)\big)(\bar t, \bar{\boldsymbol{x}}) \leq \delta\mathcal{L}\varphi_\varepsilon(\bar t, \bar{\boldsymbol{x}}).$$

From items (2) and (3) above, it follows that $\mathcal{L}\varphi_\varepsilon(\bar t, \bar{\boldsymbol{x}})$ is bounded by a constant (not depending on ε, δ, λ). Therefore, letting $\delta \to 0^+$, taking into account the notations of STEP II, we have

$$\lambda\mathrm{e}^{\lambda t_0}(u - v_{N_0})(t_0, \boldsymbol{x}_0) = \lambda\big(u^\lambda - v^\lambda_{N_0}\big)(t_0, \boldsymbol{x}_0) \leq 0,$$

which gives a contradiction to (4.10). □

As a direct consequence of the comparison theorem (Theorem 4.2), we obtain the following uniqueness result.

Corollary 4.1. *Under Assumption* (A), *the function* v *in* (4.4) *is the unique* (*path-dependent*) *viscosity solution of equation* (4.2), *where uniqueness holds in the class of all continuous and bounded functions from* $\boldsymbol{\Lambda}$ *to* $\mathbb{R}$.

Proof. By Theorem 4.1, we know that v is continuous and bounded, moreover it is a (path-dependent) viscosity solution of equation (4.2).

Now, let $u : \mathbf{\Lambda} \to \mathbb{R}$ be a continuous and bounded function such that u is a (path-dependent) viscosity solution of (4.2). Then, in particular, u (resp. v) is a (path-dependent) viscosity subsolution (resp. supersolution) of equation (4.2). As a consequence, by the comparison theorem (Theorem 4.2) we deduce that $u \le v$ on $\mathbf{\Lambda}$. Changing the roles of u and v we get the opposite inequality, from which we conclude that $u \equiv v$. □

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