

Existence and regularity for prescribed Lorentzian mean curvature hypersurfaces, and the Born–Infeld model

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Abstract

Given a measure ρ on a domain $\Omega \subset \mathbb{R}^m$, we study spacelike graphs over Ω in Minkowski space with Lorentzian mean curvature ρ and Dirichlet boundary condition on $\partial\Omega$, which solve

$$-\operatorname{div} \left(\frac{Du}{\sqrt{1 - |Du|^2}} \right) = \rho \quad \text{in } \Omega \subset \mathbb{R}^m. \quad (BI)$$

The graph function also represents the electric potential generated by a charge ρ in electrostatic Born-Infeld's theory. Even though there exists a unique minimizer u_ρ of the associated action

$$I_\rho(\psi) \doteq \int_\Omega \left(1 - \sqrt{1 - |D\psi|^2} \right) dx - \langle \rho, \psi \rangle$$

among functions ψ satisfying $|D\psi| \leq 1$, by the lack of smoothness of the Lagrangian density for $|D\psi| = 1$ one cannot guarantee that u_ρ satisfies the Euler-Lagrange equation (BI). A chief difficulty comes from the possible presence of light segments in the graph of u_ρ . In this paper, we investigate the existence of a solution for general ρ . In particular, we give sufficient conditions to guarantee that u_ρ solves (BI) and enjoys log-improved energy and $W_{\text{loc}}^{2,2}$ estimate. Furthermore, we construct examples which suggest a sharp threshold for the regularity of ρ to ensure the solvability of (BI).

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1 Introduction

Spacelike maximal and constant mean curvature hypersurfaces in Lorentzian manifolds, possibly with singularities, and more generally spacelike hypersurfaces M with prescribed Lorentzian mean curvature ρ , play a prominent role in General Relativity. For instance, their use is substantial in connection to positive energy theorems and to the initial value problem for solutions to the Einstein field equation (see [42] and the references therein). Therefore, guaranteeing their existence and grasping their qualitative properties, either in entire space or in a bounded domain with a given boundary condition, helps to understand the mathematics behind Einstein's theory. Once ρ is prescribed, one of the core issues making the existence and regularity problem challenging is the possibility that the constructed hypersurface M ceases to be spacelike somewhere. For example, M may contain a light segment ("go null" in the terminology of [42]), a feature that, as we shall see, can actually occur even for reasonably well-behaved ρ . The appearance of a light segment \overline{xy} forces to properly justify the sentence " M has mean curvature ρ in a neighborhood of \overline{xy} ", and also suggests to allow for possibly

nonsmooth mean curvature functions. Indeed, the behavior of M near a light segment is a problem which is, to the best of our knowledge, mostly open.

The purpose of the present paper is to investigate the prescribed Lorentzian mean curvature problem for general ρ . For simplicity, we shall restrict to hypersurfaces M in the Lorentz-Minkowski ambient space

$$\mathbb{L}^{m+1} \doteq \mathbb{R} \times \mathbb{R}^m \quad \text{with Lorentzian metric} \quad -dx^0 \otimes dx^0 + \sum_{i=1}^m dx^i \otimes dx^i,$$

although most of our results likely allow for extension to more general Lorentzian manifolds. The spacelike condition ensures that M is the multigraph, over some open subset Ω of the totally geodesic slice $\mathbb{R}^m \doteq \{x^0 = 0\}$, of a function u with $|Du| < 1$. Hereafter, we shall only consider single valued graphs. We treat the problem in a bounded domain Ω as well as in the entire \mathbb{R}^m . Following the convention in the literature, we say that the graph of $u \in W^{1,\infty}(\Omega)$ is

- *weakly spacelike* if $|Du| \leq 1$ on Ω ;
- *spacelike* if $|u(x) - u(y)| < |x - y|$ whenever $x, y \in \Omega$, $x \neq y$ and the line segment \overline{xy} is contained in Ω ;
- *strictly spacelike* if $u \in C^1(\Omega)$ and $|Du| < 1$ in Ω .

Given $\phi \in C(\partial\Omega)$, a spacelike hypersurface with Lorentzian mean curvature ρ and boundary (the graph of) ϕ is the graph of a solution $u : \overline{\Omega} \rightarrow \mathbb{R}$ to

$$\begin{cases} -\operatorname{div} \left(\frac{Du}{\sqrt{1 - |Du|^2}} \right) = \rho & \text{on } \Omega \subset \mathbb{R}^m, \\ u = \phi & \text{on } \partial\Omega, \end{cases} \quad (BI)$$

where D and $|\cdot|$ are the connection (gradient) and norm in \mathbb{R}^m . If $\Omega = \mathbb{R}^m$, the boundary assumption in (BI) is replaced by $u(x) \rightarrow 0$ as $|x| \rightarrow \infty$, accomplished by a proper choice of the function space for u .

The name (BI) is an acronym for ‘‘Born-Infeld’’. In fact, our second main motivation for studying (BI) comes from Born-Infeld’s theory for Electromagnetism, proposed by Born and Infeld in [16, 17] to overcome the failure of the principle of finite energy occurring in Maxwell’s model. The reader is referred to the Appendix, where we describe the problem in more detail. In the electrostatic case, according to Born-Infeld’s theory the solution u_ρ to (BI) represents the electric potential generated by a charge ρ , and by setting

$$w_\rho \doteq \frac{1}{\sqrt{1 - |Du_\rho|^2}},$$

the quantity $w_\rho - 1 + \rho u_\rho$ is the energy density of the field generated by u_ρ (with the obvious meaning for ρu_ρ if $\rho \notin L^1$). The role of w_ρ will be essential in our investigation and, hereafter, slightly abusing notation we name w_ρ the *energy density* of u_ρ .

By the above description, (BI) is a nonlinear replacement for Poisson’s equation

$$-\Delta u = \rho \quad (1.1)$$

occurring in Maxwell's model. For $\rho \in L^1(\mathbb{R}^3) \cap L^{6/5}(\mathbb{R}^3)$, solutions to (1.1) (which decay at infinity) do not violate the principle of finite energy since $|Du| \in L^2(\mathbb{R}^3)$, see [27, Remark 1]. However, this is not the case for general sources including point charges, which are natural to describe electrons and motivate the introduction of Born-Infeld's theory. Thus, it is meaningful to consider (BI) for ρ a measure.

It was observed in [5, 6, 11, 42] that a variational approach to (BI) by minimizing the action

$$I_\rho(v) \doteq \int_{\Omega} \left(1 - \sqrt{1 - |Dv|^2}\right) dx - \langle \rho, v \rangle \quad (1.2)$$

($\langle \cdot, \cdot \rangle$ stands for the duality pairing) may not lead to a solution to (BI) because of the lack of smoothness of the Lagrangian density when $|Dv| = 1$. In particular, the possible presence of light segments in the graph of the (unique) minimizer u_ρ is a serious obstacle. Hereafter, with a slight abuse of notation, we call $\overline{xy} \subset \overline{\Omega}$ a *light segment* for u_ρ if $x \neq y$ and

$$|u_\rho(y) - u_\rho(x)| = |x - y|.$$

Since $|Du_\rho| \leq 1$, in such a case u_ρ is linear with slope 1 on \overline{xy} . To the present, the existence and regularity problem for solutions to (BI) was considered only for very restricted classes of measures ρ . In fact, L^∞ sources and juxtaposition of point charges were treated in bounded domains, while for $\Omega = \mathbb{R}^m$ the case $\rho \in L^q(\mathbb{R}^m)$, $q > m$, was recently included. Notice that solvability of (1.1) is known at least for

$$\rho = \sum_{i=1}^k a_i \delta_{x_i} + f \quad \text{with} \quad f \in L^q, \quad q > 1,$$

where δ_x is the Dirac delta at x and $a_i \in \mathbb{R}$. In this paper, we shall see whether similar properties for (BI) hold or not. Among other results in this paper, we here stress the following:

- (i) If $m \geq 3$ and $\rho \in L^2_{\text{loc}}(\mathbb{R}^m) \cap L^p(\mathbb{R}^m)$, $p \in [1, 2m/(m+2)]$, then u_ρ solves (BI) on \mathbb{R}^m (Theorem 1.19);
- (ii) If $m = 2$, Ω is bounded and

$$\rho = \sum_{i=1}^k a_i \delta_{x_i} + f \quad \text{with} \quad f \in L^2_{\text{loc}}(\Omega) \cap L^1(\Omega), \quad x_i \in \Omega,$$

then u_ρ solves (BI) and the graph of u_ρ does not contain light segments. Moreover, we may replace point charges by any singular measure ρ_S supported in Ω with $\mathcal{H}^1(\text{supp } \rho_S) = 0$ (Theorem 1.11);

- (iii) If $m \geq 3$ and $q < m - 1$, there exist $f \in L^q(\mathbb{R}^m)$ and $x, y \in \mathbb{R}^m$ such that the minimizer u_ρ for $\rho = \delta_y - \delta_x + f$ does not solve (BI). Here the graph of u_ρ has a light segment over \overline{xy} (Theorem 1.9 and Corollary 1.10).

Towards these goals, we need to develop new tools. Indeed, in the aforementioned cases studied in the literature, the solvability of (BI) depends on proving that u_ρ is *strictly spacelike* in the domain of consideration. When $\rho \in L^2_{\text{loc}}$ not even spacelikeness is guaranteed, and in fact compactly supported solutions to (BI) on \mathbb{R}^m , with a light segment and source $\rho \in L^2(\mathbb{R}^m)$, do exist in dimension $m \geq 4$ (cf. Theorem 1.7 below). Therefore, a core point of our investigation regards the structure of the set of light segments of u_ρ and its interplay with the solvability of (BI). The existence of a solution will be obtained through new integral bounds ("log"

improvement) for the energy density w_ρ and for the second fundamental form of the graph of u_ρ . Since an efficient way to localize estimates reveals to be one of the major problems to overcome, the case of bounded Ω is particularly challenging.

Notation and agreements.

Hereafter, we write ω_{m-1} for the volume of the unit sphere \mathbb{S}^{m-1} , and indicate with 1_A the characteristic function of a set A . The subscript δ will denote quantities referred to the Euclidean metric on \mathbb{R}^m : d_δ will be the Euclidean distance, $\text{diam}_\delta(E)$ the diameter of a set $E \subset \mathbb{R}^m$ and $|\cdot|_\delta, \mathcal{H}_\delta^k$ the volume and k -dimensional Hausdorff measure in d_δ . Given $x, y \in \mathbb{R}^m$ with $x \neq y$, we let \overline{xy} be the closed segment joining x and y . If $\Omega \subset \mathbb{R}^m$ is an open set, we denote by $\mathcal{M}(\Omega)$ the set of all finite (signed) Borel measures on Ω equipped with the total variation norm $\|\cdot\|_{\mathcal{M}(\Omega)}$. The set $\text{Lip}_c(\Omega)$ will denote the set of Lipschitz functions with compact support in Ω , and we write $\Omega' \Subset \Omega$ when Ω' has compact closure in Ω .

1.1 Known results for bounded domains

After works for maximal hypersurfaces ($\rho = 0$) due to Flaherty [26] and Audounet and Bancel [1], solutions to (BT) in bounded domains Ω and for sources $\rho \in L^\infty(\Omega)$ were studied in depth in the influential work by Bartnik and Simon [5]. To describe the main result therein, for $\phi \in C(\partial\Omega)$, we define

$$\mathcal{Y}_\phi(\Omega) \doteq \left\{ u \in W^{1,\infty}(\Omega) : u \text{ weakly spacelike, } u = \phi \text{ on } \partial\Omega \right\}. \quad (1.3)$$

Remark 1.1. We assumed no regularity of $\partial\Omega$, so the boundary condition has to be intended as in [5]: $u = \phi$ on $\partial\Omega$ iff, for each $x \in \partial\Omega$ and any straight line $\gamma : (0, 1) \rightarrow \Omega$ with $\gamma(0^+) = x$, it holds $u(\gamma(t)) \rightarrow \phi(x)$ as $t \rightarrow 0^+$. In Proposition 3.5 below, we will prove that this definition suffices to guarantee that functions $u \in \mathcal{Y}_\phi(\Omega)$ can be extended continuously on $\partial\Omega$ with value ϕ .

The class of boundary data for which $\mathcal{Y}_\phi(\Omega) \neq \emptyset$ was characterized in [5, p. 149] in terms of the function

$$d_\Omega^-(x, y) \doteq \inf \left\{ \mathcal{H}_\delta^1(\gamma) : \gamma \in \Gamma_{x,y} \right\} \leq +\infty \quad \forall x, y \in \overline{\Omega}, \quad (1.4)$$

where

$$\Gamma_{x,y} = \left\{ \gamma \in C([0, 1], \overline{\Omega}) : \gamma((0, 1)) \subset \Omega, \gamma \text{ piecewise affine and } \gamma(0) = x, \gamma(1) = y \right\},$$

the infimum is defined to be $+\infty$ if $\Gamma_{x,y} = \emptyset$, and γ is called piecewise affine if it consists of finitely many intervals where it is affine. In fact, it is shown in [5, p. 149] that

$$\mathcal{Y}_\phi(\Omega) \neq \emptyset \iff |\phi(x) - \phi(y)| \leq d_\Omega^-(x, y) \quad \forall x, y \in \partial\Omega.$$

Note that the restriction d_Ω of d_Ω^- to $\Omega \times \Omega$ gives the intrinsic metric on Ω . Remarks on the relation between $d_\Omega^-(x, y)$ for $x, y \in \partial\Omega$ and the distance in the metric completion of (Ω, d_Ω) will be given in Subsection 3.2.

Next, we introduce a class of weak solutions to (BT) in bounded domains.

Definition 1.2. Let Ω be a bounded domain in \mathbb{R}^m . For $\rho \in W^{1,\infty}(\Omega)^*$, a *weak solution* to (BT) is a function $u \in \mathcal{Y}_\phi(\Omega)$ such that

- (i) $w \doteq \frac{1}{\sqrt{1 - |Du|^2}} \in L^1_{\text{loc}}(\Omega)$ and
- (ii) $\int_\Omega \frac{Du \cdot D\eta}{\sqrt{1 - |Du|^2}} dx = \langle \rho, \eta \rangle \quad \forall \eta \in \text{Lip}_c(\Omega).$

Given a subdomain $\Omega' \subset \Omega$, we say that u *weakly solves* **(BI)** on Ω' if $w \in L^1_{\text{loc}}(\Omega')$ and (ii) holds for $\eta \in \text{Lip}_c(\Omega')$.

Equation **(BI)** is formally the Euler-Lagrange equation for the functional

$$I_\rho : \mathcal{Y}_\phi(\Omega) \rightarrow \mathbb{R}, \quad I_\rho(v) \doteq \int_{\Omega} \left(1 - \sqrt{1 - |Dv|^2}\right) dx - \langle \rho, v \rangle. \quad (1.5)$$

Although, for ρ lying in a large subset of $W^{1,\infty}(\Omega)^*$, the variational problem for I_ρ admits a unique minimizer u_ρ (cf. Subsection 3.1), the example of a hyperplane with slope 1 and $\rho = 0$ indicates that the requirement $\mathcal{Y}_\phi(\Omega) \neq \emptyset$ does not suffice to guarantee that u_ρ solves **(BI)** (see Ecker [22]). In this respect, note that any solution to **(BI)** is easily seen to coincide with the minimizer u_ρ (cf. Proposition 3.17 below). In [5, Theorem 4.1 and Corollaries 4.2, 4.3], the authors obtained the following striking result:

Theorem 1.3 ([5]). *Let $\Omega \subset \mathbb{R}^m$ be a bounded domain, and let $\phi \in C(\partial\Omega)$. The following properties are equivalent:*

- (i) ϕ admits a spacelike extension on Ω , that is, there exists $\bar{\phi} \in \mathcal{Y}_\phi(\Omega)$ which is spacelike on Ω ;
- (ii) $|\phi(x) - \phi(y)| < d_{\overline{\Omega}}(x, y)$ for every $x, y \in \partial\Omega$, $x \neq y$;
- (iii) for each $\rho \in L^\infty(\Omega)$, there exists $u \in C^1(\Omega) \cap W^{2,2}_{\text{loc}}(\Omega)$, which is strictly spacelike and weakly solves **(BI)**.

We therefore define the set

$$S(\partial\Omega) \doteq \left\{ \phi \in C(\partial\Omega) : \text{any among (i), (ii), (iii) in Theorem 1.3 holds} \right\}.$$

Remark 1.4. No regularity of Ω is assumed in Theorem 1.3. This is quite a contrast with the linear problem $-\Delta u = \rho$ in Ω , $u = \phi$ on $\partial\Omega$, for which we need certain regularity properties of $\partial\Omega$, and comes from the strong restriction $u \in W^{1,\infty}(\Omega)$ for **(BI)**.

Remark 1.5. In a broader setting, the equivalence (i) \Leftrightarrow (ii) was studied in [39, Theorem 1].

Theorem 1.3 does not contain the full generality of the statements in [5]. Indeed, under the only assumption $\mathcal{Y}_\phi(\Omega) \neq \emptyset$ the authors showed that the minimizer u_ρ is strictly spacelike on the complement of the set of light segments

$$K_\phi^\rho \doteq \overline{\bigcup \left\{ \overline{xy} : x, y \in \Omega, x \neq y, \overline{xy} \subset \Omega, |u_\rho(x) - u_\rho(y)| = |x - y| \right\}}, \quad (1.6)$$

hence it solves **(BI)** on $\Omega \setminus K_\phi^\rho$. A key fact proved in [5, Theorem 3.2] is that when $\rho \in L^\infty(\Omega)$, every light segment has to extend up to $\partial\Omega$, a property called there the *anti-peeling theorem*. The proof depends on a comparison argument that is not applicable to more general sources ρ , a case for which the relation between singularities of ρ and properties of light segments, including their existence, is currently unknown.

For the study of hypersurfaces with $\rho \in L^\infty$ in more general ambient Lorentzian manifolds, we suggest to consult the works of Gerhardt [31] and Bartnik [6]. As yet, to the best of our knowledge, more singular ρ on bounded domains were only addressed in a series of works by Klyachin and Miklyukov [39, 40, 37], where the authors treated in depth the case of juxtaposition of point charges. Particularly relevant for us is [40, Theorem 2], that we rephrase as follows:

Theorem 1.6 ([40]). *Let $\Omega \subset \mathbb{R}^m$ be a domain such that (Ω, d_Ω) has compact completion, and let $\phi \in \mathcal{S}(\partial\Omega)$. Fix a k -tuple of points $\mathcal{P} = \{x_1, \dots, x_k\} \subset \Omega$. Then, there exists a constant $M_m(\phi, \mathcal{P})$ such that, for each $a \doteq (a_1, \dots, a_k) \in \mathbb{R}^k$ satisfying $|a| < M_m(\phi, \mathcal{P})$, the minimizer u_ρ with source*

$$\rho = \sum_{j=1}^k a_j \delta_{x_j}$$

solves (BT) and it is strictly spacelike (hence, smooth) on $\Omega \setminus \mathcal{P}$. Furthermore, $M_2(\phi, \mathcal{P}) = +\infty$.

The above result also contains a lower bound for $M_m(\phi, \mathcal{P})$ when $m \geq 3$, which depends on the solution to (BT) with $\rho = 0$, on $\{x_1, \dots, x_k\}$ and on the geometry of Ω .

The case $m = 2$ is rather special and, indeed, maximal surfaces with singularities in \mathbb{L}^3 were also studied from a different point of view by using complex-analytic tools. An Enneper–Weierstrass type representation for maximal surfaces was used by Pryce [49] to provide explicit examples in some special cases, including a maximal surface with two point singularities and, notably, one with a light segment \overline{xy} [49, Example XI]. Regarding the latter, it would be very interesting to determine the singular charge ρ with $\text{supp } \rho \subset \overline{xy}$ generated by such a solution. In [23, 25], the geometric structure of maximal surfaces with singularities was investigated, while [41, 53, 28] described in detail classes of maximal surfaces whose singular set is suitably controlled. Before stating further known results, it should be pointed out that, in the works cited below, the authors consider the equation

$$(1 - |Du|^2)^{3/2} \text{div} \left(\frac{Du}{\sqrt{1 - |Du|^2}} \right) = (1 - |Du|^2)^{3/2} \rho$$

for which the role of light segments may be different. Examples of maximal surfaces in \mathbb{L}^3 whose singular set contains an entire light line were constructed in [29, 55, 3], while an investigation of points at which Du_ρ is light-like can be found in [38, 54, 55]. The behavior near isolated singularities of surfaces with nonconstant, smooth ρ was characterized in [30]. To the best of our knowledge, whether or not the singular sets described in the above mentioned references induce a singular measure in the mean curvature ρ , and which kind of measure, is a problem that has not been considered yet.

1.2 Our contributions for bounded domains

From a variational point of view, even though the minimizer u_ρ for I_ρ in (1.5) may not solve (BT) weakly, if $\phi \in \mathcal{S}(\partial\Omega)$ then u_ρ enjoys nice properties for each reasonably well-behaved source ρ , including signed Radon measures. Inspired by [11], we prove in Proposition 3.13 that the energy density of u_ρ is locally integrable, namely

$$w_\rho = \frac{1}{\sqrt{1 - |Du_\rho|^2}} \in L^1_{\text{loc}}(\Omega), \quad (1.7)$$

and in particular $|Du_\rho| < 1$ a.e. on Ω ; moreover,

$$\int_{\Omega} \frac{Du_\rho \cdot (Du_\rho - D\psi)}{\sqrt{1 - |Du_\rho|^2}} dx \leq \langle \rho, u_\rho - \psi \rangle \quad \forall \psi \in \mathcal{Y}_\phi(\Omega), \quad (1.8)$$

where the integrand in the LHS is shown to belong to $L^1(\Omega)$. As we shall see in Proposition 3.17, u_ρ weakly solves (BI) if and only if equality holds in (1.8), a fact that is not obvious in view of the lack of regularity of $\partial\Omega$ and of ϕ .

Next, we investigate the relation between the integrability of ρ and the possible existence of a light segment in the graph of u_ρ . In Section 6 (Theorem 6.1), we construct the following example.

Theorem 1.7. *For each $m \geq 3$ and $\ell \in \{1, \dots, m-2\}$, there exists a function $u \in C_c^\infty(\mathbb{R}^m)$ with the following properties:*

(i) *the set K of light segments of u is a closed cylinder $\overline{B}^{\ell-1} \times [a, b]$ in a totally geodesic ℓ -plane of \mathbb{R}^m (in particular, if $\ell = 1$ it is a single light segment), and $|Du| < 1$ on $\mathbb{R}^m \setminus K$;*

(ii) *u satisfies*

$$\int_{\mathbb{R}^m} \frac{Du \cdot D\eta}{\sqrt{1 - |Du|^2}} dx = \int_{\mathbb{R}^m} \rho_u \eta dx \quad \forall \eta \in \text{Lip}_c(\mathbb{R}^m),$$

where $\rho_u \in L^q(\mathbb{R}^m)$ for each $q < m - \ell$. In particular, if $\Omega \Subset \mathbb{R}^m$ is a bounded domain containing the support of u , then the restriction of u to Ω weakly solves (BI) with $\phi \equiv 0$ and $\rho = \rho_u$;

(iii) *for each $q < m - \ell$, it holds*

$$w, \quad w|D^2u|, \quad w^2|D^2u(Du, \cdot)|, \quad w^3 D^2u(Du, Du) \in L^q(\mathbb{R}^m),$$

where $w = (1 - |Du|^2)^{-1/2}$ is the energy density of u .

Remark 1.8. Notice that u is smooth, so the presence of a (maximally extended) light segment entirely contained in Ω may not prevent the minimizer to be regular. As pointed out by one of the referees, this also occurs when the light segment extends up to $\partial\Omega$: consider the function $u(y, x) = (1 - |y|^4)x$ in a small ball $\Omega \subset \mathbb{R}^2$ centered at the origin. Then, $u \in C^\infty(\overline{\Omega})$ even though there is a light segment γ joining antipodal points of Ω , and by direct computation, if Ω is small enough, the mean curvature ρ of u on $\Omega \setminus \gamma$ extends smoothly to the entire $\overline{\Omega}$. Furthermore, we may check that u minimizes I_ρ with respect to its own boundary values¹. However, the energy density w_ρ violates (1.7), so the condition $\phi \in \mathcal{S}(\partial\Omega)$ is necessary to get (1.7) even when $|Du_\rho| < 1$ almost everywhere.

Our idea for the construction is as follows. Assuming for simplicity $\ell = 1$ and using coordinates $(y, x_m) \in \mathbb{R}^{m-1} \times \mathbb{R}$, we first perturb the function $(y, x_m) \mapsto x_m$, whose slope is equal to 1, in the y -direction in the following form:

$$U(y, x_m) = (1 - |y|^2) x_m.$$

The core part of the argument is then to devise a suitable cut-off so that U becomes strictly spacelike outside of a small segment of type $\{0\} \times [-t, t]$, it weakly solves (BI) and the mean curvature of the resulting graph has the desired regularity. This is quite subtle and, because of technicalities, the proof is deferred to Section 6. We think that the above regularity bounds for ρ are sharp, see Subsection 1.5 for a list of open problems.

The existence of a weak solution with a light segment allows us to exhibit what, to our knowledge, is the first example of minimizer with spacelike boundary condition that *does not solve (BI)*, even though the corresponding source is rather mild. This is a consequence of the following result, Theorem 4.5 below.

¹The minimizer u_ρ must coincide with u on $\partial\Omega \cup \gamma$ and be strictly spacelike away from γ by [5, Corollary 4.2], hence $u = u_\rho$ follows by standard comparison on each half-ball determined by γ .

Theorem 1.9. Let $\Omega \subset \mathbb{R}^m$ be either a bounded domain or $\Omega = \mathbb{R}^m$. In the first case, let $\phi \in S(\partial\Omega)$. Let u_ρ be a minimizer for I_ρ and assume that u_ρ has a light segment $\overline{xy} \subset \Omega$ with $u_\rho(y) - u_\rho(x) = |y - x|$. Then, for each $\alpha > 0$, u_ρ also minimizes the functional I_{ρ_α} with

$$\rho_\alpha = \rho + \alpha(\delta_y - \delta_x)$$

but it does not solve (BI) weakly for ρ_α .

Indeed, applying Theorem 1.9 to the example in Theorem 1.7 with $\ell = 1$, we infer

Corollary 1.10. Let $m \geq 3$. Then, there exist a smooth domain $\Omega \Subset \mathbb{R}^m$, a function $u \in C_c^\infty(\Omega) \cap \mathcal{Y}_0(\Omega)$, points $x, y \in \Omega$ with $x \neq y$ and a function $\rho_{AC} \in L^q(\Omega)$ for any $q < m - 1$, such that the following properties hold:

(i) \overline{xy} is a light segment for u , and $|Du| < 1$ on $\Omega \setminus \overline{xy}$;

(ii) u minimizes I_ρ with source

$$\rho = \alpha(\delta_y - \delta_x) + \rho_{AC}, \quad \text{for each fixed } \alpha \in \mathbb{R}^+,$$

but it does not solve (BI) weakly.

Observe that Corollary 1.10 makes it impossible to extend Theorem 1.6 (i.e. [40, Theorem 2]) for dimension $m \geq 3$ to more general sources of the type

$$\rho = \sum_{j=1}^k a_j \delta_{x_j} + \rho_{AC} \quad \text{with } \rho_{AC} \in L^q(\Omega), \quad q < m - 1.$$

We next move to results that guarantee the solvability of (BI). To get elliptic estimates, our boundary datum shall be restricted to compact subsets $\mathcal{F} \subset S(\partial\Omega)$ with respect to uniform convergence. Examples of \mathcal{F} include a singleton $\{\phi\}$ and sets of uniformly bounded, c -Lipschitz functions on $\partial\Omega$ with respect to d_δ , with $c < 1$. Under the assumption that the metric space (Ω, d_Ω) has compact completion, a more general example will be studied in Subsection 3.2.

We first consider the 2-dimensional case.

Theorem 1.11. Assume that $\Omega \subset \mathbb{R}^2$ is a bounded domain, and let $\Sigma \Subset \Omega$ be a compact subset satisfying $\mathcal{H}_\delta^1(\Sigma) = 0$. Suppose that $\rho \in \mathcal{M}(\Omega)$ decomposes as

$$\rho = \rho_S + \rho_{AC}, \quad \text{with } \begin{cases} \text{supp } \rho_S \subset \Sigma \\ \rho_{AC} \in L^1(\Omega) \cap L_{loc}^2(\Omega \setminus \Sigma). \end{cases}$$

Then,

- (i) for each $\phi \in S(\partial\Omega)$, the minimizer $u_\rho \in \mathcal{Y}_\phi(\Omega)$ weakly solves (BI) in Ω and does not have light segments;
- (ii) for any given compact set $\mathcal{F} \subset S(\partial\Omega)$, $\mathcal{I}_1, \mathcal{I}_2, \varepsilon > 0$, $q_0 \geq 0$, and any given open set $\Omega' \Subset \Omega \setminus \Sigma$ satisfying

$$\|\rho\|_{\mathcal{M}(\Omega)} \leq \mathcal{I}_1, \quad \|\rho\|_{L^2(\Omega')} \leq \mathcal{I}_2,$$

there exists a constant $C = C(\Omega, \mathcal{F}, q_0, \text{diam}_\delta(\Omega), \mathcal{I}_1, \mathcal{I}_2, \varepsilon, d_\delta(\Omega', \partial\Omega), \Omega')$ such that, for each $\phi \in \mathcal{F}$, it holds

$$\int_{\Omega'_\varepsilon} (1 + \log w_\rho)^{q_0} \left\{ w_\rho |D^2 u_\rho|^2 + w_\rho^3 \left| D^2 u_\rho (Du_\rho, \cdot) \right|^2 + w_\rho^5 [D^2 u_\rho (Du_\rho, Du_\rho)]^2 \right\} dx + \int_{\Omega'_\varepsilon} w_\rho (1 + \log w_\rho)^{q_0+1} dx \leq C,$$

where $\Omega'_\varepsilon \doteq \{x \in \Omega' : d_\delta(x, \partial\Omega') > \varepsilon\}$;

(iii) if $\Omega' \Subset \Omega \setminus \Sigma$ and $\rho \in L^\infty(\Omega')$, then u_ρ is strictly spacelike and $u_\rho \in C_{\text{loc}}^{1,\alpha}(\Omega')$, for some $\alpha > 0$. In particular, if $\rho \in C^\infty(\Omega')$ so is u_ρ .

Remark 1.12. If ρ_S is a sum of Dirac deltas and $\rho_{AC} = 0$, we recover the result by Klyachin-Miklyukov (see Theorem 1.6). However, we stress that our proof is completely different. Indeed, the clever proof in [40] is quite specific to Dirac delta singularities, and it seems difficult to extend to sources whose absolutely continuous part is not in L^∞ .

Remark 1.13. Regarding the second order regularity of u , for general ρ one cannot expect $u_\rho \in W_{\text{loc}}^{2,q}$ for $q \geq 1$, see the beginning of Subsection 4.3.

We briefly overview the strategy of the proof, that relies on several steps. We refer to $\Omega, \mathcal{F}, \text{diam}_\delta(\Omega), \mathcal{I}_1, \mathcal{I}_2, d_\delta(\Omega', \partial\Omega)$ in (ii) as being the *data* of our problem, and fix $\varepsilon > 0$. Hereafter, a constant C will be assumed to depend on the data. We proceed by approximating ρ via convolution to get $\rho_j \rightharpoonup \rho$ weakly in $\mathcal{M}(\Omega)$, let $u_j \in \mathcal{Y}_\phi(\Omega)$ minimize I_{ρ_j} and denote by $w_j \doteq (1 - |Du_j|^2)^{-1/2}$ its energy density. First, we show the following two properties:

($\mathcal{P}0_1$) Proposition 4.10 and Corollary 4.11 (**local second fundamental form estimate**): the squared norm of the second fundamental form Π_j for the graph of u_j over Ω satisfies

$$\int_{\Omega'_{\varepsilon/2}} \|\Pi_j\|^2 w_j^{-1} dx \leq C;$$

($\mathcal{P}0_2$) Lemma 4.4 (**energy estimate**): on Euclidean balls B_r contained in $\Omega_{\varepsilon/2}$,

$$\int_{B_r} w_j dx \leq Cr.$$

Properties ($\mathcal{P}0_1$) and ($\mathcal{P}0_2$) hold in any dimension $m \geq 2$. We stress that, writing Π_j in terms of u_j as in (2.3) in the following, ($\mathcal{P}0_1$) implies bounds on the derivative of the energy density w_j . For the surface case $m = 2$, ($\mathcal{P}0_1$) and ($\mathcal{P}0_2$) imply

($\mathcal{P}1$) Theorem 4.12 (**higher integrability for $m = 2$**):

$$\int_{\Omega'_\varepsilon} w_j \log w_j dx \leq C.$$

The uniform integrability of $\{w_j\}$ granted by ($\mathcal{P}1$) enables us to show

($\mathcal{P}2$) Step 2 in Proof of Theorem 1.11 (**no-light-segment**): u_ρ has no light segments in Ω' (the statement is quantitative in terms of the data).

With the aid of $(\mathcal{P}2)$, we can then refine the integral estimates as follows:

$(\mathcal{P}3)$ Theorem 4.13 (**higher integrability and second fundamental form estimates**): for each $q_0 \geq 0$,

$$\int_{\Omega'_\varepsilon} \left\{ w_j \log w_j + \|\Pi_j\|^2 w_j^{-1} \right\} \log^{q_0} w_j dx \leq C, \quad (1.9)$$

where C also depends on q_0 (and on Ω' in a subtler way). Item (ii) in Theorem 1.11 follows from (1.9), which is technically one of the core parts of the paper. It is important to notice that $(\mathcal{P}3)$ holds in a given dimension m provided that so does $(\mathcal{P}2)$, and in particular, the higher integrability of w_j does not depend on $(\mathcal{P}1)$. To the present, we are able to prove $(\mathcal{P}2)$ only in dimension $m = 2$, and the example in Theorem 1.7 shows the possible failure of $(\mathcal{P}2)$ in dimension $m \geq 4$ when $\rho \in L^2(\Omega')$.

Also, item (iii) in Theorem 1.11 follows from $(\mathcal{P}2)$ by applying arguments in [5]. To prove Item (i) we need one last piece of information. Clearly, $(\mathcal{P}2)$ and the fact that $\mathcal{H}_\delta^1(\Sigma) = 0$ guarantee that u_ρ does not have light segments on the entire Ω . However, the local uniform integrability of $\{w_j\}$ on each $\Omega' \Subset \Omega \setminus \Sigma$ implies

$$\int_{\Omega} w_\rho Du_\rho \cdot D\eta = \langle \rho, \eta \rangle \quad \forall \eta \in \text{Lip}_c(\Omega \setminus \Sigma).$$

To extend the above identity to test functions $\eta \in \text{Lip}_c(\Omega)$, we shall prove the following removable singularity property, which holds in any dimension.

$(\mathcal{P}4)$ Theorem 4.2 (**removable singularity**): if $\{w_j\}$ is locally uniformly integrable on $\Omega \setminus \Sigma$ and $\mathcal{H}_\delta^1(\Sigma) = 0$, then u_ρ solves weakly (BI) .

As we shall see in Remark 4.3, condition $\mathcal{H}_\delta^1(\Sigma) = 0$ cannot be weakened to $\mathcal{H}_\delta^1(\Sigma) < \infty$.

In higher dimensions, the possible failure of $(\mathcal{P}2)$ makes it necessary to investigate the set of light segments K_ϕ^ρ of u_ρ . However, with the aid of the higher integrability Theorem 4.13, we can still get nontrivial information. Namely, for ρ, Σ as in Theorem 1.11 but in dimension $m \geq 3$, the *only* possible obstruction to the solvability of (BI) is when K_ϕ^ρ is non-empty and touches Σ or $\partial\Omega$, as the next Theorem shows.

Theorem 1.14. *Let $m \geq 3$ and $\Omega \subset \mathbb{R}^m$ be a domain, $\Sigma \Subset \Omega$ be compact and $\rho \in \mathcal{M}(\Omega)$ satisfy $\mathcal{H}_\delta^1(\Sigma) = 0$ and*

$$\rho = \rho_S + \rho_{AC}, \quad \text{with} \quad \begin{cases} \text{supp } \rho_S \subset \Sigma, \\ \rho_{AC} \in L^1(\Omega) \cap L^2_{\text{loc}}(\Omega \setminus \Sigma). \end{cases}$$

Given $\phi \in S(\partial\Omega)$, consider the set of light segments K_ϕ^ρ of the minimizer $u_\rho \in \mathcal{Y}_\phi(\Omega)$ defined in (1.6). Then,

(i) u_ρ weakly solves (BI) on $\Omega \setminus K_\phi^\rho$.

Moreover, if $K_\phi^\rho \cap (\partial\Omega \cup \Sigma) = \emptyset$, then u_ρ weakly solves (BI) on the entire Ω .

(ii) For each $\Omega' \Subset \Omega \setminus (\Sigma \cup K_\phi^\rho)$ and $q_0 \geq 0$,

$$\begin{aligned} & \int_{\Omega'} (1 + \log w_\rho)^{q_0} \left\{ w_\rho |D^2 u_\rho|^2 + w_\rho^3 \left| D^2 u_\rho (Du_\rho, \cdot) \right|^2 + w_\rho^5 [D^2 u_\rho (Du_\rho, Du_\rho)]^2 \right\} dx \\ & + \int_{\Omega'} w_\rho (1 + \log w_\rho)^{q_0+1} dx < \infty. \end{aligned}$$

(iii) If $\Omega' \Subset \Omega \setminus (\Sigma \cup K_\phi^\rho)$ and $\rho \in L^\infty(\Omega')$, then u_ρ is strictly spacelike and $u_\rho \in C_{\text{loc}}^{1,\alpha}(\Omega')$, for some $\alpha > 0$. In particular, if $\rho \in C^\infty(\Omega')$ so is u_ρ .

Remark 1.15. Corollary 1.10 shows that, in dimension $m \geq 4$, there exists $\rho_{\text{AC}} \in L^2(\Omega)$ and $\rho_S = \delta_y - \delta_x$ such that $u_\rho \in \mathcal{Y}_0(\Omega)$ does not solve (BI) weakly on the entire Ω . Notice that the support $\Sigma = \{x, y\}$ of ρ_S satisfies $\Sigma \subset K_\phi^\rho$, and therefore condition $K_\phi^\rho \cap \Sigma = \emptyset$ in (i) of Theorem 1.14 cannot be removed.

1.3 Known results for $\Omega = \mathbb{R}^m$

The picture for constant ρ on the entire \mathbb{R}^m is by now well understood. Thanks to Calabi [19], Cheng and Yau [21] and Bartnik (Ecker [22, Theorem F]), we know that if $u : \mathbb{R}^m \rightarrow \mathbb{R}$ minimizes I_0 (i.e. $\rho = 0$) on each open subset $\Omega \Subset \mathbb{R}^m$ with respect to compactly supported variations in Ω , then u is a hyperplane, possibly with slope 1. Note that no growth conditions on u are imposed a-priori. On the contrary, many examples of smooth spacelike graphs with constant $\rho \neq 0$ were constructed in [51, 52].

In view of applications to Born-Infeld's theory, we study I_ρ in \mathbb{R}^m with $m \geq 3$ and for functions decaying at infinity to zero, taking advantage of the different functional settings described by Kiessling in [35] and Bonheure, d'Avenia and Pomponio in [11]. For our purposes, we mildly modify their frameworks and define in Subsection 3.1 a Banach space $\mathcal{Y}(\mathbb{R}^m)$ in such a way that I_ρ is well defined on

$$\mathcal{Y}_0(\mathbb{R}^m) \doteq \left\{ v \in \mathcal{Y}(\mathbb{R}^m) : \|Dv\|_\infty \leq 1 \right\}, \quad (1.10)$$

and so that the latter is closed (and convex) in $\mathcal{Y}(\mathbb{R}^m)$. Our choice does not affect the functional properties of I_ρ shown in [11]: in particular, following [11, Lemma 2.2], I_ρ has a unique minimizer $u_\rho \in \mathcal{Y}_0(\mathbb{R}^m)$ which, by [11, Proposition 2.7] (cf. also Proposition 3.13 herein), satisfies

$$w_\rho \in L_{\text{loc}}^1(\mathbb{R}^m), \quad 0 \leq w_\rho - 1 \leq \frac{|Du_\rho|^2}{\sqrt{1 - |Du_\rho|^2}} \in L^1(\mathbb{R}^m) \quad (1.11)$$

and the variational inequality

$$\int_{\mathbb{R}^m} \frac{Du_\rho \cdot (Du_\rho - D\psi)}{\sqrt{1 - |Du_\rho|^2}} dx \leq \langle \rho, u_\rho - \psi \rangle \quad \forall \psi \in \mathcal{Y}_0(\mathbb{R}^m). \quad (1.12)$$

We then say that u_ρ weakly solves (BI) if

$$\int_{\mathbb{R}^m} \frac{Du_\rho \cdot D\eta}{\sqrt{1 - |Du_\rho|^2}} dx = \langle \rho, \eta \rangle \quad \forall \eta \in \text{Lip}_c(\mathbb{R}^m).$$

In [11], u_ρ was shown to solve (BI) weakly whenever $\rho \in \mathcal{Y}(\mathbb{R}^m)^*$ satisfies any of the following assumptions:

- (i) ρ is radial ([11, Theorem 1.4]);
- (ii) $\rho \in L_{\text{loc}}^\infty(\mathbb{R}^m)$ ([11, Theorem 1.5]). In this case, u_ρ is locally strictly spacelike and thus $u_\rho \in C_{\text{loc}}^{1,\alpha}(\mathbb{R}^m)$ for some $\alpha > 0$, by the regularity theory for quasilinear equations.

A constructive proof of classical solutions for $\rho \in C_0^\alpha(\mathbb{R}^m) \cap L^1(\mathbb{R}^m)$, under a smallness condition on ρ in $C_0^\alpha(\mathbb{R}^m)$, can be found in [20] by Carley and Kiessling. We next discuss the two further cases considered so far.

The case of point charges.

The problem for

$$\rho = \sum_{i=1}^k a_i \delta_{x_i} \tag{1.13}$$

was treated in [10, 11]: in particular, see [10, Theorem 1.2], u_ρ was shown to be locally strictly spacelike (hence, smooth) away from the charges $\{x_i\}$ provided that the points x_i are sufficiently far away depending on the sizes a_i , in the quantitative way recalled in Remark 1.18 below. In this case, u_ρ weakly (indeed, classically) solves (BT) on $\mathbb{R}^m \setminus \{x_1, x_2, \dots, x_k\}$. However, in [10, 11] the authors did not prove equality in (1.12) for test functions which do not vanish at x_i , see [11, Remark 4.4] for more detailed comments.

In [35], Kiessling claimed that for ρ as in (1.13) u_ρ satisfies (BT) without any restriction on the charges a_i , but in [11] it was pointed out that his argument has a subtle flaw. Kiessling subsequently published the erratum [35], where he supplied a proof of his claim using a dual approach to circumvent the obstacle in the direct method pointed out in [11]. From a mathematical point of view it is still desirable to have a proof with a direct use of the functional I_ρ , though.

The case $\rho \in L^q$ for large q .

It is natural to seek a sharp condition on ρ that guarantees both the strict spacelikeness of u_ρ and its local $C^{1,\alpha}$ regularity, for some $\alpha \in (0, 1)$. The evidence coming from the radial case in [11, Section 3], further motivated by the detailed discussion in the Introduction of [13], led Bonheure and Iacopetti to formulate the following

Conjecture ([13, Conjecture 1.4]). *If $m \geq 3$ and $\rho \in \mathcal{Y}^* \cap L_{\text{loc}}^q(\mathbb{R}^m)$ with $q > m$, then u_ρ is strictly spacelike on \mathbb{R}^m and $u_\rho \in C_{\text{loc}}^{1,\alpha}(\mathbb{R}^m)$ for some $\alpha \in (0, 1)$.*

Here, \mathcal{Y}^* is the dual of a functional space \mathcal{Y} where $\mathcal{Y}_0(\mathbb{R}^m)$ embeds as a closed, convex set, a typical case being

$$\rho \in L^p(\mathbb{R}^m) \quad \text{for } p \in [1, 2_*],$$

where, in what follows,

$$2_* \doteq \frac{2m}{m+2}$$

is the conjugate exponent of the Sobolev one 2^* . In fact, in the stated assumptions on ρ , $C_{\text{loc}}^{1,\alpha}$ regularity easily follows from strict spacelikeness by standard theory of quasilinear equations.

To the present, a complete answer to the conjecture is still unknown. After a first partial result in [13], which is in itself remarkable, an almost exhaustive positive answer was given by the combined efforts of Haarala [34] and Bonheure–Iacopetti [14]:

Theorem 1.16 ([34, Theorem 1.3] and [14, Theorems 1.4 and 1.5]). *Assume $m \geq 3$ and $\rho \in L^q(\mathbb{R}^m) \cap L^p(\mathbb{R}^m)$ with $p \in [1, 2_*]$ and $q > m$. Then, u_ρ is strictly spacelike and*

$$u_\rho \in C_{\text{loc}}^{1,1-\frac{m}{q}}(\mathbb{R}^m) \cap W_{\text{loc}}^{2,q}(\mathbb{R}^m).$$

Furthermore, u_ρ weakly solves (BT).

The proof of the theorem is deep, and combines different ingredients that are of independent interest. We emphasize that the global L^q integrability of ρ is fundamental at various stages of the proofs in [34, 14], and hence, the case $\rho \in L_{\text{loc}}^q(\mathbb{R}^m)$ remains an open problem.

1.4 Our contributions for $\Omega = \mathbb{R}^m$

We first address the problem with a superposition of point charges. With the aid of Theorem 4.2 (removable singularity) and Theorem 4.13 (higher integrability), we can refine the results in [10, 11, 35] and prove

Theorem 1.17. *Let ρ be as in (1.13), and let $u_\rho \in \mathcal{Y}(\mathbb{R}^m)$ minimize I_ρ . Then, the following hold.*

- (i) *Light segments of u_ρ cannot cross each other, that is, no point can lie in the interior of two distinct light segments.*
- (ii) *Any maximally extended light segment of u_ρ must be of the type $\overline{x_i x_j}$ for some i, j such that $a_i a_j < 0$. In particular, if $a_i a_j > 0$ for each i, j , then u_ρ does not have light segments.*
- (iii) *If u_ρ does not have any light segment, then it weakly solves (BT) and is smooth, strictly spacelike outside of $\{x_1, \dots, x_k\}$. Furthermore, around x_i , u_ρ is asymptotic to a light cone in the sense of [22], where the cone is future (respectively, past) pointing provided that $a_i < 0$ (respectively, $a_i > 0$).*

Remark 1.18. According to [10, Proof of Theorem 1.2], u_ρ has no light segments whenever

$$\left(\frac{m}{\omega_{m-1}}\right)^{\frac{1}{m-1}} \frac{m-1}{m-2} \left[\left(\sum_{i \in I_-} |a_i|\right)^{\frac{1}{m-1}} + \left(\sum_{i \in I_+} |a_i|\right)^{\frac{1}{m-1}} \right] < \min_{i \neq j} |x_i - x_j|, \quad (1.14)$$

where I_+ (I_-) is the set of indices for which $a_i > 0$ ($a_i < 0$).

Item (i) in Theorem 1.17 is indeed a general fact that holds for every u with $\|Du\|_\infty \leq 1$, see Proposition 3.10. Item (ii) complements [35, Corollary 3.2 and Proposition 3.1], where u_ρ was assumed a priori to satisfy (BT). Notice that our proof is different and essentially suggested by one of the referees. The last part of (iii) in Theorem 1.17 needs some comments, too. In [22], Ecker defined an *isolated singularity* for

$$\operatorname{div} \left(\frac{Du}{\sqrt{1 - |Du|^2}} \right) = 0 \quad \text{on an open set } B$$

as being a point $x_0 \in B$ such that u minimizes I_0 on any $\Omega' \Subset B \setminus \{x_0\}$ (that is, among functions in $\mathcal{Y}_{u_\rho}(\Omega')$), but not on the entire B . He then proves in [22, Theorem 1.5] that an isolated singularity is asymptotic to a future or past pointing light cone centered at x_0 . As a direct application of Ecker's result, in [10, Theorem 3.5] (see also [11, Theorem 1.5]) the authors claim that, for ρ as in (1.13) and $\{x_i\}, \{a_i\}$ matching (1.14), near x_i , u_ρ is asymptotic to a light cone which is upward or downward pointing according to whether $a_i < 0$ or $a_i > 0$. However, without knowing the validity of the Euler-Lagrange equation around x_i , it is not clear to us how to exclude the possibility that u_ρ also minimizes I_0 in a neighborhood of x_i . Our statement that u_ρ solves (BT) in (iii) suffices to guarantee that this does not happen, and therefore to fully justify the conclusions in [11, 10].

Next, we consider the behavior of u_ρ for sources $\rho \in L^2_{\text{loc}}(\mathbb{R}^m)$, and obtain the next

Theorem 1.19. *Let $m \geq 3$ and*

$$\rho \in (L^1(\mathbb{R}^m) + L^p(\mathbb{R}^m)) \cap L^2_{\text{loc}}(\mathbb{R}^m), \quad \text{for some } p \in (1, 2_*].$$

Then, the minimizer u_ρ weakly solves (BI). Moreover, for a given $\mathcal{I} \in \mathbb{R}^+$, there exists a positive constant $\mathcal{I}_0 = \mathcal{I}_0(m, p, \mathcal{I})$ with the following property: if

$$\|\rho\|_{L^1(\mathbb{R}^m) + L^p(\mathbb{R}^m)} \leq \mathcal{I},$$

then for any pair of open sets $\Omega'' \Subset \Omega' \Subset \mathbb{R}^m$ with $d_\delta(\Omega'', \partial\Omega') \geq \mathcal{I}_0$, any $\mathcal{I}_2 > 0$ with

$$\|\rho\|_{L^2(\Omega')} \leq \mathcal{I}_2,$$

and any $q_0 \geq 0$, there exists a constant $C = C(q_0, m, p, \mathcal{I}, \mathcal{I}_0, \mathcal{I}_2, |\Omega'|_\delta)$ such that

$$\begin{aligned} & \int_{\Omega''} (1 + \log w)^{q_0} \left\{ w_\rho |D^2 u_\rho|^2 + w_\rho^3 \left| D^2 u_\rho (Du_\rho, \cdot) \right|^2 + w_\rho^5 [D^2 u_\rho (Du_\rho, Du_\rho)]^2 \right\} dx \\ & + \int_{\Omega''} w_\rho (1 + \log w_\rho)^{q_0+1} dx \leq C. \end{aligned} \quad (1.15)$$

Some comments are in order. First, the example in Theorem 1.7 shows that the set of light segments

$$K^\rho \doteq \overline{\bigcup \left\{ \overline{xy} : x, y \in \mathbb{R}^m, x \neq y, |u_\rho(x) - u_\rho(y)| = |x - y| \right\}} \quad (1.16)$$

may be non-empty, at least if $m \geq 4$. As in the case of bounded domains, the existence of light segments for solutions in \mathbb{R}^3 is unknown. Second, the enhanced second fundamental form estimate (1.15) holds provided that the inequality

$$\int_{\Omega'} \rho^2 \frac{(1 + \log w_\rho)^{q_0+2}}{w_\rho} dx \leq \mathcal{I}_1 \quad (1.17)$$

is satisfied, which is trivially implied by $\rho \in L^2(\Omega')$. Whether (1.17) may be satisfied by less regular sources ρ is an open problem.

We conclude by studying the case when $\rho \in L^2_{\text{loc}}$ away from a small compact set, where the singular part of ρ is supported. Then, Theorem 4.13 guarantees that the sole obstruction to the solvability of (BI) on the entire \mathbb{R}^m may occur when K^ρ intersects the support of the singular measure. As we see in Corollary 1.10 and Remark 1.15, the case can happen for certain $\rho = \rho_{AC} + \rho_S$ where $\rho_{AC} \in L^2_{\text{loc}}$ and $\rho_S = \delta_y - \delta_x$ with $m \geq 4$, the corresponding minimizer does not solve (BI).

Theorem 1.20. *Let $m \geq 3$ and let $\Sigma \Subset \mathbb{R}^m$ be a compact set satisfying $\mathcal{H}^1_\delta(\Sigma) = 0$. Assume that ρ decomposes as*

$$\rho = \rho_S + \rho_2, \quad \text{with} \quad \begin{cases} \rho_S \in \mathcal{M}(\mathbb{R}^m), \text{ supp } \rho_S \subset \Sigma, \\ \rho_2 \in (L^1(\mathbb{R}^m) + L^p(\mathbb{R}^m)) \cap L^2_{\text{loc}}(\mathbb{R}^m \setminus \Sigma), \quad p \in (1, 2_*], \end{cases}$$

Then, the following hold.

- (i) *The minimizer u_ρ weakly solves (BI) on $\mathbb{R}^m \setminus K^\rho$, with K^ρ as in (1.16). Moreover, if $K^\rho \cap \Sigma = \emptyset$, then u_ρ weakly solves (BI) on \mathbb{R}^m .*

- (ii) *For each $\Omega' \Subset \mathbb{R}^m \setminus (\Sigma \cup K^\rho)$ and $q_0 \geq 0$,*

$$\begin{aligned} & \int_{\Omega'} (1 + \log w_\rho)^{q_0} \left\{ w_\rho |D^2 u_\rho|^2 + w_\rho^3 \left| D^2 u_\rho (Du_\rho, \cdot) \right|^2 + w_\rho^5 [D^2 u_\rho (Du_\rho, Du_\rho)]^2 \right\} dx \\ & + \int_{\Omega'} w_\rho (1 + \log w_\rho)^{q_0+1} dx < \infty. \end{aligned}$$

(iii) If $\Omega' \Subset \Omega \setminus (\Sigma \cup K^\rho)$ and $\rho \in L^\infty(\Omega')$, then u_ρ is strictly spacelike and $u_\rho \in C_{\text{loc}}^{1,\alpha}(\Omega')$, for some $\alpha > 0$. In particular, if $\rho \in C^\infty(\Omega')$ so is u_ρ .

1.5 Open problems and outline of the paper

We first address the existence problem for light segments. We think that the regularity of ρ_u in Theorem 1.7 might be sharp, and we are tempted to propose the following

Conjecture 1. *If $\phi \in S(\partial\Omega)$ and $\rho \in L_{\text{loc}}^q(\Omega)$ with $q > m - 1$, then the minimizer u_ρ does not have light segments.*

The case $q = m - 1$, which includes $\rho \in L_{\text{loc}}^2(\Omega)$ when $m = 3$, is particularly subtle.

Question 2. *If $\phi \in S(\partial\Omega)$ and $\rho \in L_{\text{loc}}^{m-1}(\Omega)$, could the minimizer have light segments?*

In view of the techniques developed herein, a negative answer to the above question would be sufficient to extend Theorem 1.11 to dimension $m \geq 3$ and to $\rho_{\text{AC}} \in L_{\text{loc}}^{m-1}(\Omega \setminus \Sigma)$.

Related to the above problems, and in view of Corollary 1.10, we also formulate the following

Question 3. *If $\phi \in S(\partial\Omega)$ and*

$$\rho = \sum_{i=1}^k a_i \delta_{x_i} + \rho_{\text{AC}} \quad \text{with } \rho_{\text{AC}} \in L^q(\Omega), \quad q > m - 1,$$

does the minimizer u_ρ solve (BI) weakly?

An ambitious goal would be to relate the integrability of ρ to the Hausdorff dimension of the set K_ϕ^ρ of light segments. In view of Theorem 1.7 and of its proof, we may expect that the following holds:

Conjecture 4. *If $m \geq 3$, $\phi \in S(\partial\Omega)$ and $\rho \in L^q(\Omega)$ for some $2 \leq q \leq m - 1$, then the Hausdorff dimension of K_ϕ^ρ satisfies $\dim_{\mathcal{H}_\delta}(K_\phi^\rho) \leq m - q$.*

It might be possible that $\dim_{\mathcal{H}_\delta}(K_\phi^\rho) \leq m - q$ could be strengthened to $\mathcal{H}_\delta^{m-q}(K_\phi^\rho) = 0$. If this were true, notice that it would also imply a negative answer to Question 2. If ρ is more singular, we propose the next

Question 5. *For $\rho \in \mathcal{M}(\Omega)$, is it true that $\mathcal{H}_\delta^{m-1}(K_\phi^\rho) = 0$?*

Still about the set of light segments, it would be important to understand the weak limit

$$w_j dx \rightarrow \vartheta \quad \text{in } \mathcal{M}(\Omega'), \quad \Omega' \Subset \Omega :$$

can one characterize the singular part of ϑ , and relate its support to the set K_ϕ^ρ ? Can one characterize the non-negative functional

$$\langle \mathcal{F}, \eta \rangle \doteq \langle \rho, \eta \rangle - \int_{\Omega} \frac{Du_\rho \cdot D\eta}{\sqrt{1 - |Du_\rho|^2}} \quad \eta \in C_c^\infty(\Omega),$$

describing the loss in (1.12)?

Regarding the energy density, we first observe that the integrability of w_ρ in Theorem 1.7 is much higher than the one that we can prove in Theorem 4.13. However, the latter is uniform on a sequence of approximated solutions $\{u_{\rho_j}\}$. We can ask the following

Question 6. Can one prove a local higher integrability $w_\rho \in L^p_{\text{loc}}(\Omega)$, for suitable $p > 1$, under a local higher integrability of ρ , for instance for $\rho \in L^q_{\text{loc}}(\Omega)$ and $q > m - 1$?

Even the case $\rho \in L^q_{\text{loc}}(\mathbb{R}^m)$ and $q > m$ is currently unknown, cf. [34, 14].

Question 7. What about the regularity of u_ρ and w_ρ when $\rho \in L^q$ and $q \in (1, 2)$?

About the higher order regularity for u_ρ , $W^{2,q}$ estimates are unknown apart from the case $q = 2$, considered in the present paper, and $q > m$ treated in [34, 14] for $\Omega = \mathbb{R}^m$. We think that there might be an interpolation result, and therefore propose the following

Question 8. Can one prove that, for $p \in [2, m]$ and $\rho \in L^p_{\text{loc}}$, the minimizer u_ρ satisfies $u_\rho \in W^{2,p}_{\text{loc}}$?

The paper is organized as follows. Section 2 contains some background material from Lorentzian Geometry. Section 3 introduces the functional setting, then moves to discuss the basic properties of u_ρ (convergence under approximation of ρ , integrability), together with various equivalent conditions for the solvability of (BI). In particular, we mention Propositions 3.13 and 3.17, which may have an independent interest. Though preparatory, most of the material in this section did not appear elsewhere in the literature. In Section 4, we develop our main new tools: a removable singularity result, Theorem 1.9, a second fundamental form estimate and a higher integrability result. These are the bulk of the paper, the techniques therein differ from those in the literature and we believe they are applicable beyond the purposes of the present work. Section 5 then contains the proof of our main existence results. In the concluding Section 6, we construct a solution to (BI) with an ℓ -dimensional set of light segments and zero boundary condition.

To a certain extent, each of Sections 2 to 6 can be read independently. In particular, the reader acquainted with Lorentzian Geometry and not focusing on the functional analytic setting may directly skip to Section 4.

A note on constants in elliptic estimates

When constants in our theorems are stated to depend on $\text{diam}_\delta(\Omega)$, $|\Omega'|_\delta$, $d_\delta(\Omega', \partial\Omega)$, in fact they can be bounded uniformly in terms of, respectively, uniform upper bounds for $\text{diam}_\delta(\Omega)$ and $|\Omega'|_\delta$, and lower bounds for $d_\delta(\Omega', \partial\Omega)$. Regarding the dependence of C in Theorem 1.11 from the domain Ω' and from $d_\delta(\Omega', \partial\Omega)$, if $d_\delta(\Omega', \partial(\Omega \setminus \Sigma)) \geq \tau$ and

$$\|\rho\|_{L^2(U_\tau)} \leq \mathcal{I}_2 \quad \text{where } U_\tau = \left\{ x \in \Omega \setminus \Sigma : d_\delta(x, \partial(\Omega \setminus \Sigma)) \geq \tau \right\},$$

then C merely depends on τ . On the other hand, anywhere we write $C = C(\Omega, \dots)$ we mean that we did not investigate the stability of the bounds for sequences of open sets $\{\Omega_j\}$ for which the other data are kept uniformly controlled.

2 Preliminaries from Lorentzian Geometry

In this section, we briefly recall some differential-geometric background that will be used henceforth. Let \mathbb{L}^{m+1} be the Lorentz space with coordinates (x^0, x^1, \dots, x^m) and metric

$$-dx^0 \otimes dx^0 + \sum_{i=1}^m dx^i \otimes dx^i, \quad x \cdot y \doteq -x^0 y^0 + \sum_{i=1}^m x^i y^i, \quad |x|_{\mathbb{L}} \doteq \sqrt{|x \cdot x|}.$$

Given a smooth function $u : \Omega \subset \mathbb{R}^m \rightarrow \mathbb{R}$, consider the graph map

$$F : \Omega \rightarrow \mathbb{L}^{m+1}, \quad F(x) \doteq (u(x), x),$$

and define M to be the manifold $F(\Omega)$ endowed with the metric induced from \mathbb{L}^{m+1} , equivalently, M is Ω endowed with the pull-back metric $g \doteq F^*(\cdot)$. When convenient, g will also be denoted by $\langle \cdot, \cdot \rangle$. Let $\|\cdot\|, \nabla, \Delta_M$ be, respectively, the norm, Levi-Civita connection and Laplace-Beltrami operator associated to g . The Hessian of a function u in the metric g will be denoted by $\nabla^2 u$.

We identify \mathbb{R}^m with the slice $\{x^0 = 0\}$, so $\{x^i\}$ are Cartesian coordinates on \mathbb{R}^m with associated vector fields $\{\partial_i\}$. Given an open set $\Omega \subset \mathbb{R}^m$ and $u \in C^\infty(\Omega)$, we let $u_i \doteq \partial_i u$ and $u_{ij} \doteq (D^2 u)_{ij} = \partial_{ij}^2 u$. By defining

$$X_i \doteq F_* \partial_i = \partial_i + u_i \partial_0,$$

the components of g are written as

$$g_{ij} \doteq X_i \cdot X_j = \delta_{ij} - u_i u_j.$$

Hereafter we assume that g is Riemannian (equivalently, $|Du| < 1$). The inverse metric has components

$$g^{ij} = \delta^{ij} + w^2 u^i u^j, \quad \text{with} \quad w \doteq \frac{1}{\sqrt{1 - |Du|^2}},$$

where $u^i = \delta^{ij} u_j$ are the components of the gradient Du . Then, the volume measure dx_g of g relates to the measure dx on \mathbb{R}^m as follows:

$$dx_g = w^{-1} dx.$$

The future-pointing, unit normal vector to the graph M is given by $\mathbf{n} \doteq w(\partial_0 + u^i \partial_i)$. Note that $\mathbf{n} \cdot \mathbf{n} = -1$ and $w = -\mathbf{n} \cdot \partial_0$. Let superscripts \parallel and \perp denote, respectively, the projection onto TM and TM^\perp with respect to the inner product \cdot in \mathbb{L}^{m+1} . From the chain of identities

$$\langle \partial_0^\parallel, \partial_j \rangle = \partial_0 \cdot F_* \partial_j = -u_j = -\langle \nabla u, \partial_j \rangle,$$

we deduce that

$$\partial_0^\parallel = -\nabla u. \tag{2.1}$$

Denoting by \bar{D} the Levi-Civita connection of \mathbb{L}^{m+1} , we define the second fundamental form of M by

$$\Pi(\partial_i, \partial_j) \doteq \left(\bar{D}_{X_i} X_j \right)^\perp = h_{ij} \mathbf{n}, \quad \text{thus} \quad h_{ij} = -\bar{D}_{X_i} X_j \cdot \mathbf{n} = \bar{D}_{X_i} \mathbf{n} \cdot X_j.$$

From the definition of X_i we obtain $h_{ij} = w u_{ij}$. The (unnormalized) scalar mean curvature $H \doteq g^{ij} h_{ij}$ in direction \mathbf{n} is therefore

$$H = w \Delta u + w^3 D^2 u(Du, Du) = \operatorname{div} \left(\frac{Du}{\sqrt{1 - |Du|^2}} \right),$$

where Δ is the Laplacian on \mathbb{R}^m . Next, since the Christoffel symbols of g are given by $\Gamma_{ij}^k = -w^2 u^k u_{ij}$, we compute the Hessian and Laplacian of a smooth function $\phi : \Omega \rightarrow \mathbb{R}$ in the graph metric g :

$$\begin{aligned} \nabla_{ij}^2 \phi &= \phi_{ij} + w^2 \phi_k u^k u_{ij}; \\ \Delta_M \phi &= g^{ij} \nabla_{ij}^2 \phi = \Delta \phi + w^2 D^2 \phi(Du, Du) + H w D\phi \cdot Du. \end{aligned} \tag{2.2}$$

In addition, the norm of the second fundamental form Π of the graph u is given by

$$\begin{aligned}\|\Pi\|^2 &= g^{ij}g^{kl}h_{ik}h_{jl} = w^2(\delta^{ij} + w^2u^i u^j)u_{ik}(\delta^{kl} + w^2u^k u^l)u_{jl} \\ &= w^2|D^2u|^2 + 2w^4 \left|D^2u(Du, \cdot)\right|^2 + w^6[D^2u(Du, Du)]^2.\end{aligned}\tag{2.3}$$

In particular,

$$\nabla_{ij}^2 u = w^2 u_{ij} = w h_{ij}, \quad \|\nabla^2 u\|^2 = w^2 \|\Pi\|^2, \quad \Delta_M u = Hw \quad \text{on } M.\tag{2.4}$$

Given $o \in \mathbb{R}^m$, we denote by $r_o : \Omega \rightarrow \mathbb{R}$ and $\ell_o : \Omega \rightarrow \mathbb{R}$, respectively, the Euclidean distance from o and the Lorentzian distance from $(u(o), o)$ restricted to the graph of u , that is, we set

$$\begin{aligned}r_o(x) &\doteq |x - o|, \\ l_o(s, x) &\doteq |(s, x) - (u(o), o)|_{\mathbb{L}} = \sqrt{-(s - u(o))^2 + |x - o|^2}, \\ \ell_o(x) &\doteq l_o(u(x), x).\end{aligned}\tag{2.5}$$

We also denote the extrinsic Lorentzian ball centered at o , and more generally the one centered at a subset $A \subset \mathbb{R}^m$, by

$$L_R(o) \doteq \{x \in \Omega : \ell_o(x) < R\}, \quad L_R(A) \doteq \bigcup_{o \in A} L_R(o).\tag{2.6}$$

When it is necessary, we will write ℓ_o^ρ, L_R^ρ to emphasize their dependence on the minimizer $u = u_\rho$ of I_ρ . By (2.2), we get

$$\begin{aligned}\bar{D}l_o^2(u(x), x) &= 2(x^j - o^j)\partial_j + 2(u(x) - u(o))\partial_0; \\ \|\nabla \ell_o(x)\|^2 &= |\bar{D}l_o(u(x), x)|_{\mathbb{L}}^2 + (\bar{D}l_o(u(x), x) \cdot \mathbf{n})^2 \\ &= 1 + \frac{w^2}{\ell_o^2} |Du \cdot (x - o) - (u(x) - u(o))|^2; \\ \Delta_M \ell_o^2(x) &= 2m + 2wH [(x - o) \cdot Du - (u(x) - u(o))] \\ &= 2m + H (\bar{D}l_o^2(u(x), x) \cdot \mathbf{n}).\end{aligned}\tag{2.7}$$

As we shall see in the proof of Theorem 4.13, the construction of cut-off functions based on the Lorentzian distance, instead of those based on the Euclidean one, will be the key to obtain the higher integrability of u_ρ in dimension $m \geq 3$.

3 Basic properties of u_ρ

In this section, we obtain basic properties of the minimizer u_ρ of I_ρ , both for $\Omega \subset \mathbb{R}^m$ a bounded domain ($m \geq 2$) and for $\Omega = \mathbb{R}^m$ ($m \geq 3$).

3.1 Functional setting

We first choose our functional spaces. If $\Omega = \mathbb{R}^m$, our treatment mildly departs from those in [35, 11], and is basically designed to get an explicit description of the sources ρ covered by the method. On the other hand, for bounded Ω , subtleties related to a possibly rough boundary $\partial\Omega$ require extra care in the choice of the functional space, which significantly differs from that in [5].

Definition 3.1. Given $m \geq 2$, we fix $p_1 \in (m, \infty)$ and assume also $p_1 \geq 2^*$ for $m = 3$.

(i) When $m \geq 2$ and $\Omega \subset \mathbb{R}^m$ is a bounded domain, we set

$$\mathcal{Y}(\Omega) \doteq W^{1,p_1}(\Omega) \cap C(\bar{\Omega}), \quad \|v\|_{\mathcal{Y}} \doteq \max \left\{ \|v\|_{W^{1,p_1}(\Omega)}, \|v\|_{C(\bar{\Omega})} \right\};$$

(ii) When $\Omega = \mathbb{R}^m$ and $m \geq 3$, we set

$$\mathcal{Y}(\mathbb{R}^m) \doteq \overline{C_c^\infty(\mathbb{R}^m)}^{\|\cdot\|_{\mathcal{Y}}}, \quad \|v\|_{\mathcal{Y}} \doteq \max \left\{ \|Dv\|_2, \|Dv\|_{p_1} \right\}.$$

Note that, if Ω is bounded and sufficiently regular (Lipschitz is enough), by Morrey's Embedding Theorem $\mathcal{Y}(\Omega) = W^{1,p_1}(\Omega)$ with the equivalent norm $\|\cdot\|_{W^{1,p_1}(\Omega)}$.

Remark 3.2. The case $\Omega = \mathbb{R}^2$ will not be considered in the present paper. We observe that the radially symmetric solution in [17] with a Dirac delta source (cf. (4.9)) has a logarithmic behavior at infinity when $m = 2$, which calls for a different functional setting. For ρ a superposition of point charges, complete classification theorems for entire solutions in \mathbb{R}^2 were obtained by Klyachin [37], and Fernández, López and Souam [25].

The following result can be proved in a similar way as [11, Lemma 2.1], but we give full details for the sake of completeness.

Proposition 3.3. Assume $m \geq 3$ and $\Omega = \mathbb{R}^m$. Then $(\mathcal{Y}(\mathbb{R}^m), \|\cdot\|_{\mathcal{Y}})$ is a reflexive Banach space. Moreover,

$$\mathcal{Y}(\mathbb{R}^m) \hookrightarrow W^{1,q}(\mathbb{R}^m) \quad \forall q \in [2^*, p_1]. \quad (3.1)$$

In particular, $\|\cdot\|_{\mathcal{Y}}$ is equivalent to $\|D \cdot\|_2 + \|\cdot\|_{W^{1,p_1}}$, and $\mathcal{Y}(\mathbb{R}^m) \hookrightarrow C_0(\mathbb{R}^m) \doteq \{u \in C(\mathbb{R}^m) : \lim_{|x| \rightarrow \infty} u(x) = 0\}$ holds.

Proof. First, $\|\cdot\|_{\mathcal{Y}}$ is equivalent to the norm $|u|_{\mathcal{Y}} \doteq \sqrt{\|Du\|_2^2 + \|Du\|_{p_1}^2}$. Hence, to prove the reflexivity of $(\mathcal{Y}(\mathbb{R}^m), \|\cdot\|_{\mathcal{Y}})$ it suffices to show that $(\mathcal{Y}(\mathbb{R}^m), |\cdot|_{\mathcal{Y}})$ is uniformly convex. This easily follows by using the criterion in [18, Exercise 3.29] and the uniform convexity of the norms $\|Du\|_2$ and $\|Du\|_{p_1}$.

To obtain (3.1), let $u \in \mathcal{Y}(\mathbb{R}^m)$. From the choice of p_1 and Hölder's inequality, the next interpolation inequality holds:

$$\|Du\|_q \leq \|u\|_{\mathcal{Y}} \quad \text{for all } q \in [2, p_1]. \quad (3.2)$$

Since $m \in [2, p_1)$ and $q^* \rightarrow \infty$ as $q \rightarrow m^-$, there exists $\hat{q} \in [2, m)$ so that $\hat{q}^* = p_1$. Thus, Sobolev's inequality and (3.2) yield $\|u\|_{p_1} \leq C \|Du\|_{\hat{q}^*} \leq C \|u\|_{\mathcal{Y}}$. Hence, $\mathcal{Y}(\mathbb{R}^m) \hookrightarrow W^{1,p_1}(\mathbb{R}^m)$ holds. In addition, from $\|u\|_{2^*} \leq C \|Du\|_2 \leq \|u\|_{\mathcal{Y}}$, $2 < 2^* \leq p_1$ and (3.2), we see $\mathcal{Y}(\mathbb{R}^m) \hookrightarrow W^{1,2^*}(\mathbb{R}^m)$. Therefore, by the interpolation, (3.1) holds.

The equivalence between $\|\cdot\|_{\mathcal{Y}}$ and $\|D \cdot\|_2 + \|\cdot\|_{W^{1,p_1}}$ is an immediate consequence of (3.1), while $\mathcal{Y}(\mathbb{R}^m) \hookrightarrow C_0(\mathbb{R}^m)$ follows from Morrey's embedding Theorem once we observe that $u \in L^{2^*}(\mathbb{R}^m) \cap C^{0,\alpha}(\mathbb{R}^m)$ implies that u vanishes at infinity. \square

Remark 3.4 (Dual spaces). If $q \in (1, \infty)$ and $\Omega \subset \mathbb{R}^m$ is any domain, then it is well-known that elements in the dual space $W^{1,q}(\Omega)^* = W^{-1,q'}(\Omega)$ can be represented as pairs $(v, V) \in L^{q'}(\Omega) \times [L^{q'}(\Omega)]^m$ where $q' \doteq q/(q-1)$, with the action

$$\langle \rho, \psi \rangle \doteq \int_{\Omega} \psi v dx + \int_{\Omega} D\psi \cdot V dx \quad \forall \psi \in W^{1,q}(\Omega),$$

see for instance [2, Theorem 3.9]. Furthermore, recall that if X_1, X_2 are Banach spaces with $X_1 \cap X_2$ dense in X_1 and X_2 , then $(X_1 \cap X_2)^* = X_1^* + X_2^*$ with the natural norm

$$\|\rho\|_{X_1^* + X_2^*} = \inf \left\{ \|\rho_1\|_{X_1^*} + \|\rho_2\|_{X_2^*} : \rho_j \in X_j^*, \rho = \rho_1 + \rho_2 \right\},$$

see [7, Theorem 2.7.1]. Indeed, inspecting the proof in [7], one deduces that every functional $\rho \in (X_1 \cap X_2)^*$ can be represented as

$$\rho = \rho_1 + \rho_2 \in X_1^* + X_2^*, \quad \text{with} \quad \|\rho_1\|_{X_1^*} + \|\rho_2\|_{X_2^*} \leq \|\rho\|_{(X_1 \cap X_2)^*},$$

the representation being unique (with equality between norms) when $X_1 \cap X_2$ is dense in both X_1 and X_2 . Taking the above observations into account,

- (i) if Ω is a bounded domain, every $\rho \in \mathcal{Y}(\Omega)^*$ can be represented as $\rho = \rho_1 + \rho_2 \in W^{-1,p_1'}(\Omega) + \mathcal{M}(\Omega)$, for some ρ_1, ρ_2 satisfying

$$\|\rho_1\|_{W^{-1,p_1'}} + \|\rho_2\|_{\mathcal{M}} \leq \|\rho\|_{\mathcal{Y}^*}.$$

The representation is unique when $C(\overline{\Omega}) \cap W^{1,p_1}(\Omega)$ is dense in $W^{1,p_1}(\Omega)$, a fact which entails some mild requirement on $\partial\Omega$ such as the segment condition (cf. [2, Theorem 3.22]). However, uniqueness of the representation will not be used in the present work. Notice the continuous inclusion $\mathcal{M}(\Omega) \hookrightarrow \mathcal{Y}(\Omega)^*$.

- (ii) if $\Omega = \mathbb{R}^m$ and $m \geq 3$, then $\mathcal{Y}(\mathbb{R}^m)^* = D^{1,2}(\mathbb{R}^m)^* + W^{-1,p_1'}(\mathbb{R}^m)$, with $D^{1,2}(\mathbb{R}^m)$ being the closure of $C_c^\infty(\mathbb{R}^m)$ with respect to the norm $\|v\|_{D^{1,2}} \doteq \|Dv\|_2$. In particular, because of Proposition 3.3 and Morrey's embedding, $\mathcal{M}(\mathbb{R}^m) \hookrightarrow \mathcal{Y}(\mathbb{R}^m)^*$ and $W^{-1,q'}(\mathbb{R}^m) \hookrightarrow \mathcal{Y}(\mathbb{R}^m)^*$ for each $q \in [2^*, p_1]$. Hence,

$$\mathcal{M}(\mathbb{R}^m) + L^{q'}(\mathbb{R}^m) \hookrightarrow \mathcal{Y}(\mathbb{R}^m)^* \quad \forall q \in [2^*, p_1],$$

where $L^{q'}(\mathbb{R}^m)$ consists of the pairs $(v, 0)$.

Clearly, $\mathcal{Y}_0(\mathbb{R}^m)$ in (1.10) is a closed convex subset of $\mathcal{Y}(\mathbb{R}^m)$. The situation is more subtle for $\mathcal{Y}_\phi(\Omega)$ defined in (1.3), because of the lack of regularity of $\partial\Omega$. However, as the next result shows, the mild sense in which the boundary condition is considered, see Remark 1.1, suffices to guarantee that $\mathcal{Y}_\phi(\Omega) \subset \mathcal{Y}(\Omega)$.

Proposition 3.5. *Let $\Omega \subset \mathbb{R}^m$ be a bounded domain, let $\mathcal{F} \subset C(\partial\Omega)$ be a relatively compact (resp. compact) subset with respect to uniform convergence, and consider*

$$\mathcal{Y}_{\mathcal{F}}(\Omega) \doteq \left\{ v : v \in \mathcal{Y}_\phi(\Omega) \text{ for some } \phi \in \mathcal{F} \right\}.$$

Then $\mathcal{Y}_{\mathcal{F}}(\Omega) \subset C(\overline{\Omega})$ as a relatively compact (resp. compact) subset, where we extend each $v \in \mathcal{Y}_{\mathcal{F}}(\Omega)$ onto $\overline{\Omega}$ by setting $v(x) \doteq \phi(x)$ for $x \in \partial\Omega$.

Proof. First, observe that if $x \in \Omega$ and $\tilde{x} \in \partial\Omega$ is a nearest point to x in the metric d_δ , the boundary condition in Remark 1.1 tested on the segment $tx + (1-t)\tilde{x} \in \Omega$ for any $t \in (0, 1]$ gives, for each $v \in \mathcal{Y}_{\mathcal{F}}(\Omega)$,

$$|v(x) - \phi(\tilde{x})| = \left| v(x) - \lim_{t \rightarrow 0^+} v(tx + (1-t)\tilde{x}) \right| \leq |x - \tilde{x}|. \quad (3.3)$$

The inequality trivially holds also if $x \in \partial\Omega$, by the way v is extended. Whence,

$$\|v\|_{L^\infty(\Omega)} \leq \|\phi\|_{C(\partial\Omega)} + \text{diam}_\delta(\Omega) \leq \sup_{\phi \in \mathcal{F}} \|\phi\|_{C(\partial\Omega)} + \text{diam}_\delta(\Omega) < \infty, \quad (3.4)$$

where the last inequality follows since \mathcal{F} is relatively compact in $C(\partial\Omega)$. This proves the uniform boundedness of $\mathcal{Y}_{\mathcal{F}}(\Omega)$.

Next, we shall show $v \in C(\overline{\Omega})$ for each $v \in \mathcal{Y}_{\mathcal{F}}(\Omega)$, and that $\mathcal{Y}_{\mathcal{F}}(\Omega)$ is uniformly equicontinuous. Let $\varepsilon > 0$ be arbitrary. Since $\widetilde{\mathcal{F}}$ is relatively compact in $C(\partial\Omega)$, \mathcal{F} is uniformly equicontinuous on $\partial\Omega$, hence, there exists $\widetilde{\delta}_\varepsilon > 0$ such that

$$\phi \in \mathcal{F}, x_1, x_2 \in \partial\Omega, |x_1 - x_2| < \widetilde{\delta}_\varepsilon \quad \Rightarrow \quad |\phi(x_1) - \phi(x_2)| < \frac{\varepsilon}{4}.$$

Set

$$\delta_\varepsilon \doteq \frac{1}{4} \min \left\{ \varepsilon, \widetilde{\delta}_\varepsilon \right\} > 0,$$

and pick $x_1, x_2 \in \overline{\Omega}$ with $|x_1 - x_2| < \delta_\varepsilon$. If one among $B_{\delta_\varepsilon}(x_1)$ and $B_{\delta_\varepsilon}(x_2)$ is contained in Ω , property $v \in \mathcal{Y}_\phi(\Omega)$ implies that v is 1-Lipschitz there, whence

$$|x_1 - x_2| < \delta_\varepsilon \quad \Rightarrow \quad |v(x_1) - v(x_2)| \leq |x_1 - x_2| < \delta_\varepsilon < \varepsilon.$$

We therefore assume that $B_{\delta_\varepsilon}(x_j) \cap \partial\Omega \neq \emptyset$ for $j = 1, 2$, and choose $\widetilde{x}_j \in B_{\delta_\varepsilon}(x_j) \cap \partial\Omega$ satisfying $|x_j - \widetilde{x}_j| = d_\delta(x_j, \partial\Omega)$. From $|x_1 - x_2| < \delta_\varepsilon$ and $|x_j - \widetilde{x}_j| < \delta_\varepsilon$ for each j , the triangle inequality implies $|\widetilde{x}_1 - \widetilde{x}_2| < 3\delta_\varepsilon < \widetilde{\delta}_\varepsilon$ and therefore, by using (3.3),

$$\begin{aligned} |v(x_1) - v(x_2)| &\leq |v(x_1) - \phi(\widetilde{x}_1)| + |\phi(\widetilde{x}_1) - \phi(\widetilde{x}_2)| + |\phi(\widetilde{x}_2) - v(x_2)| \\ &\leq |x_1 - \widetilde{x}_1| + \frac{\varepsilon}{4} + |\widetilde{x}_2 - x_2| < 2\delta_\varepsilon + \frac{\varepsilon}{4} \leq \varepsilon. \end{aligned}$$

Hence, $v \in C(\overline{\Omega})$ and $\mathcal{Y}_{\mathcal{F}}(\Omega)$ is uniformly equicontinuous on $\overline{\Omega}$. The relative compactness of $\mathcal{Y}_{\mathcal{F}}(\Omega)$ in $C(\overline{\Omega})$ follows by the Arzelà–Ascoli theorem. If \mathcal{F} is compact, then any limit point of a sequence $\{v_j\} \subset \mathcal{Y}_{\mathcal{F}}(\Omega)$ lies in $\mathcal{Y}_{\mathcal{F}}(\Omega)$ since the limit function is 1-Lipschitz, thus $\mathcal{Y}_{\mathcal{F}}(\Omega)$ is compact in $C(\overline{\Omega})$. \square

Corollary 3.6. *For each bounded domain $\Omega \subset \mathbb{R}^m$ and each $\phi \in C(\partial\Omega)$, $\mathcal{Y}_\phi(\Omega) \subset \mathcal{Y}(\Omega)$ and it is bounded, closed, convex and sequentially weakly compact in $\mathcal{Y}(\Omega)$.*

Proof. By Proposition 3.5, $\mathcal{Y}_\phi(\Omega) \subset C(\overline{\Omega})$ is a compact subset. From $\|Du\|_\infty \leq 1$ for any $u \in \mathcal{Y}_\phi(\Omega)$, we deduce that $\mathcal{Y}_\phi(\Omega) \subset \mathcal{Y}(\Omega)$ is closed, bounded and convex. To prove the sequential weak compactness, let $\{v_j\}$ be a sequence in $\mathcal{Y}_\phi(\Omega)$. Then, up to passing to a subsequence, $v_j \rightarrow v$ weakly in $W^{1,p_1}(\Omega)$ and strongly in $C(\overline{\Omega})$, for some $v \in \mathcal{Y}(\Omega)$. By Remark 3.4, we can represent a given $\rho \in \mathcal{Y}(\Omega)^*$ as $\rho = \rho_1 + \rho_2$ with $\rho_1 \in W^{-1,p'_1}(\Omega)$ and $\rho_2 \in \mathcal{M}(\Omega)$, whence

$$\langle \rho, v_j \rangle = \langle \rho_1, v_j \rangle + \langle \rho_2, v_j \rangle \rightarrow \langle \rho_1, v \rangle + \langle \rho_2, v \rangle = \langle \rho, v \rangle \quad \text{as } j \rightarrow \infty,$$

thus $\{v_j\}$ is weakly convergent. Since $\mathcal{Y}_\phi(\Omega)$ is weakly closed due to the convexity and closedness, we see $v \in \mathcal{Y}_\phi(\Omega)$ and complete the proof. \square

Regarding the minimization problem, for the readers' convenience we reproduce the argument in [11] to show the existence and uniqueness of the minimizer u_ρ in our functional setting. For $\rho \in \mathcal{Y}(\Omega)^*$, we recall that $I_\rho : \mathcal{Y}_\phi(\Omega) \rightarrow \mathbb{R}$ is defined by

$$I_\rho(v) \doteq \int_{\Omega} \left(1 - \sqrt{1 - |Dv|^2}\right) dx - \langle \rho, v \rangle \quad \text{for } v \in \mathcal{Y}_\phi(\Omega).$$

The above discussion guarantees that $\mathcal{Y}_\phi(\Omega)$ is a closed convex subset of $\mathcal{Y}(\Omega)$ (when Ω is bounded, we suppose that $\phi \in C(\partial\Omega)$ is chosen such that $\mathcal{Y}_\phi(\Omega) \neq \emptyset$), and I_ρ is strictly convex since $\overline{B_1(0)} \ni p \mapsto 1 - \sqrt{1 - |p|^2} \in [0, 1]$ is strictly convex. Furthermore, from the inequality $1 - \sqrt{1 - |p|^2} \leq |p|^2$ for $|p| \leq 1$ and using Lebesgue's dominated convergence theorem, I_ρ is continuous on $\mathcal{Y}_\phi(\Omega)$. Combining convexity and continuity, we deduce that I_ρ is weakly lower-semicontinuous. If Ω is a bounded domain, by Corollary 3.6 the set $\mathcal{Y}_\phi(\Omega)$ is bounded and sequentially weakly compact in $\mathcal{Y}(\Omega)$, so the existence of a minimizer is then obvious by the direct method. On the other hand, if $\Omega = \mathbb{R}^m$, then $\|Dv\|_q^q \leq \|Dv\|_2^2$ holds for every $v \in \mathcal{Y}_0(\mathbb{R}^m)$ and $q \in [2, \infty)$ thanks to $\|Dv\|_\infty \leq 1$. Thus, in view of the identity

$$1 - \sqrt{1 - t} = \sum_{j=1}^{\infty} b_j t^j \quad \text{with } b_j \doteq \frac{(2j-2)!}{j!(j-1)!2^{2j-1}}, \quad t \in [0, 1], \quad (3.5)$$

it follows from $3 \leq m < p_1$ that for $v \in \mathcal{Y}_0(\mathbb{R}^m)$,

$$\begin{aligned} \|v\|_{\mathcal{Y}}^2 &\leq \left(\|Dv\|_2^2 + \|Dv\|_{p_1}^2 \right) \leq \left(\|Dv\|_2^2 + \|Dv\|_2^{4/p_1} \right) \\ &\leq 2 \left(\|Dv\|_2^2 + 1 \right) \leq 2 \left[1 + b_1^{-1} \left(I_\rho(v) + \|\rho\|_{\mathcal{Y}^*} \|v\|_{\mathcal{Y}} \right) \right]. \end{aligned} \quad (3.6)$$

Hence, I_ρ is coercive. Since $\mathcal{Y}(\mathbb{R}^m)$ is reflexive, the existence and uniqueness of u_ρ is then a consequence, for instance, of [18, Corollary 3.23].

3.2 Large compact subsets of $\mathcal{S}(\partial\Omega)$

Assume that (Ω, d_Ω) has compact metric completion, that following [40] we denote by Ω_d . We set $\partial\Omega_d = \Omega_d \setminus \Omega$. To stress the difference with d_Ω in (1.4), we write d instead of d_Ω for the metric on Ω_d . We first recall some topological facts. The identity $i : (\Omega, d_\Omega) \rightarrow (\Omega, d_\delta)$ extends by density to a distance non-increasing map $\tilde{i} : (\Omega_d, d) \rightarrow (\overline{\Omega}, d_\delta)$. Since Ω_d is compact and $(\overline{\Omega}, d_\delta)$ is Hausdorff, \tilde{i} is a closed map. From $\tilde{i}(\Omega_d) \supset \Omega$, we deduce that \tilde{i} is also surjective, hence, \tilde{i} is a quotient map. Given $\phi \in C(\partial\Omega)$, let $\tilde{\phi} = \phi \circ \tilde{i} \in C(\partial\Omega_d)$ be its lift.

Let $b \in \mathbb{R}^+$ and $\zeta : \mathbb{R}^+ \rightarrow [0, 1]$, and define the following subset of $\mathcal{S}(\partial\Omega)$:

$$\mathcal{S}_{b,\zeta}(\partial\Omega) \doteq \left\{ \phi \in \mathcal{S}(\partial\Omega) : \|\phi\|_\infty \leq b, \sup_{\substack{x, y \in \partial\Omega_d, \\ d(x, y) = t}} \frac{|\tilde{\phi}(x) - \tilde{\phi}(y)|}{d(x, y)} \leq \zeta(t) \quad \forall t \in \mathbb{R}^+ \right\}, \quad (3.7)$$

where the supremum is defined to be zero if $t > \text{diam}_{d_\Omega}(\Omega)$. We prove that $\mathcal{S}_{b,\zeta}(\partial\Omega)$ is compact in $C(\partial\Omega)$, so let $\{\phi_j\} \subset \mathcal{S}_{b,\zeta}(\partial\Omega)$. By the Arzelá–Ascoli Theorem, $\{\tilde{\phi}_j\}$ is relatively compact in $C(\partial\Omega_d)$ and thus, up to subsequences, $\tilde{\phi}_j \rightarrow \tilde{\phi}$ for some $\tilde{\phi} \in C(\partial\Omega_d)$ which is constant on the fibers of \tilde{i} , and therefore factorizes as $\tilde{\phi} = \phi \circ \tilde{i}$. Since \tilde{i} is a quotient map, $\phi \in C(\partial\Omega)$ (see, for instance, [45, Theorem 22.2]). From $\tilde{\phi}_j \rightarrow \tilde{\phi}$ on $\partial\Omega_d$, we deduce that $\phi_j \rightarrow \phi$ on $\partial\Omega$ and ϕ satisfies the last two conditions in (3.7). To show that $\mathcal{S}_{b,\zeta}(\partial\Omega)$ is compact in $C(\partial\Omega)$, it suffices

to prove that $\phi \in S(\partial\Omega)$. Suppose by contradiction that $\phi \notin S(\partial\Omega)$, and take $x, y \in \partial\Omega$, $x \neq y$ such that $|\phi(x) - \phi(y)| \geq d_{\overline{\Omega}}(x, y)$. Then, being the left-hand side finite, $\Gamma_{xy} \neq \emptyset$ and we can lift the interior of any path $\gamma \in \Gamma_{xy}$ to a path $\tilde{\gamma} : (0, 1) \rightarrow \Omega_d$ of the same length of γ , with $\tilde{\gamma}((0, 1)) \subset \Omega$. Choose paths $\gamma_\varepsilon \in \Gamma_{x,y}$ with $\mathcal{H}_\delta^1(\gamma_\varepsilon) \downarrow d_{\overline{\Omega}}(x, y)$ as $\varepsilon \downarrow 0$. It is easy to check that $\tilde{\gamma}_\varepsilon(0^+) \doteq \tilde{x}_\varepsilon \in \tilde{i}^{-1}(x)$ and $\tilde{\gamma}_\varepsilon(1^-) \doteq \tilde{y}_\varepsilon \in \tilde{i}^{-1}(y)$. Since the fibers $\tilde{i}^{-1}(x)$ and $\tilde{i}^{-1}(y)$ are compact, up to subsequences $\tilde{x}_{\varepsilon_k} \rightarrow \tilde{x} \in \tilde{i}^{-1}(x)$ and $\tilde{y}_{\varepsilon_k} \rightarrow \tilde{y} \in \tilde{i}^{-1}(y)$. By $x \neq y$, we have $0 < d(\tilde{x}, \tilde{y}) = \lim_{k \rightarrow \infty} d(\tilde{x}_{\varepsilon_k}, \tilde{y}_{\varepsilon_k}) \leq d_{\overline{\Omega}}(x, y)$. However, from the last property in (3.7) for $\tilde{\phi}_j$, we get the following contradiction:

$$\begin{aligned} d(\tilde{x}, \tilde{y}) \leq d_{\overline{\Omega}}(x, y) &\leq |\phi(x) - \phi(y)| = |\tilde{\phi}(\tilde{x}) - \tilde{\phi}(\tilde{y})| = \lim_{j \rightarrow \infty} |\tilde{\phi}_j(\tilde{x}) - \tilde{\phi}_j(\tilde{y})| \\ &\leq \zeta(d(\tilde{x}, \tilde{y})) d(\tilde{x}, \tilde{y}) < d(\tilde{x}, \tilde{y}). \end{aligned}$$

3.3 Three useful lemmas

We shall introduce three lemmas that will be repeatedly used. First, we search for an efficient way to approximate $\psi \in \mathcal{Y}(\Omega)$ by a sequence in $\mathcal{Y}(\Omega)$ having suitably prescribed boundary values.

Lemma 3.7 (Approximation). *Let $\Omega \subset \mathbb{R}^m$ be a domain, let $u, \psi : \Omega \rightarrow \mathbb{R}$ and for $\varepsilon > 0$ define*

$$\psi_\varepsilon^u \doteq \max\{u, \psi - \varepsilon\} + \min\{u, \psi + \varepsilon\} - u = \begin{cases} u & \text{if } |\psi - u| < \varepsilon, \\ \psi + \varepsilon & \text{if } u \geq \psi + \varepsilon, \\ \psi - \varepsilon & \text{if } u \leq \psi - \varepsilon. \end{cases}$$

Consider a sequence $\{\varepsilon_j\} \subset \mathbb{R}^+$, $\varepsilon_j \rightarrow 0$ and functions $u_j : \Omega \rightarrow \mathbb{R}$, and define $\psi_j \doteq \psi_{\varepsilon_j}^{u_j}$.

- (i) Assume that $m \geq 2$, Ω is bounded, $\phi_j, \phi \in C(\partial\Omega)$, $u_j \in \mathcal{Y}_{\phi_j}(\Omega)$ and $u, \psi \in \mathcal{Y}_\phi(\Omega)$:
- (a) if $\|\phi_j - \phi\|_{C(\partial\Omega)} < \varepsilon_j$, then $\psi_j \in \mathcal{Y}_{\phi_j}(\Omega)$ and $\psi_j \equiv u_j$ on $\Omega \setminus \Omega_j$ for some set $\Omega_j \Subset \Omega$. Moreover, if for some $\Omega' \Subset \Omega$ it holds $\psi \equiv u$ and $|u_j - u| < \varepsilon_j$ on $\Omega \setminus \Omega'$, then $\psi_j \equiv u_j$ on $\Omega \setminus \Omega'$;
 - (b) if $u_j \rightarrow u$ in $\mathcal{Y}(\Omega)$ as $j \rightarrow \infty$, then $\psi_j \rightarrow \psi$ in $W^{1,q}(\Omega) \cap C(\overline{\Omega})$ for each $q \in [1, \infty)$.
- (ii) If $m \geq 3$, $\Omega = \mathbb{R}^m$ and $u, u_j, \psi \in \mathcal{Y}_0(\mathbb{R}^m)$ satisfy $u_j \rightarrow u$ in $\mathcal{Y}(\mathbb{R}^m)$, then $\{\psi_j\} \subset \mathcal{Y}_0(\mathbb{R}^m)$ and (a) holds. Furthermore, (b) holds with $q \in [2^*, \infty)$, and $\|D\psi_j - D\psi\|_q \rightarrow 0$ for all $q \in [2, \infty)$.

Proof. (i) By Proposition 3.5, $u, u_j, \psi \in C(\overline{\Omega})$. By construction,

$$\psi_j \in C(\overline{\Omega}), \quad \|\psi_j - \psi\|_\infty \leq \varepsilon_j \rightarrow 0, \quad \Omega_j \doteq \{ |u_j - \psi| \geq \varepsilon_j \} \Subset \Omega. \quad (3.8)$$

Next, the identity

$$D\psi_j = \begin{cases} D\psi & \text{a.e. on } |\psi - u_j| \geq \varepsilon_j, \\ Du_j & \text{a.e. on } |\psi - u_j| < \varepsilon_j \end{cases} \quad (3.9)$$

implies that $|D\psi_j| \leq 1$ a.e. on Ω . Since $\psi_j = u_j$ on $\Omega \setminus \Omega_j$ and $u_j \in \mathcal{Y}_{\phi_j}(\Omega)$, we infer that $\psi_j \in \mathcal{Y}_{\phi_j}(\Omega)$. Thus, (a) holds. As for (b), if $u_j \rightarrow u$ in $\mathcal{Y}(\Omega)$, then we get $u_j \rightarrow u$ in $C(\overline{\Omega})$.

Thus, fix $\{\delta_j\}$ such that $\delta_j \rightarrow 0$ and $\|u_j - u\|_\infty < \delta_j$. Taking a subsequence $\{j_k\}$, we have $Du_{j_k}(x) \rightarrow Du(x)$ a.e. Ω . Then as $k \rightarrow \infty$, a.e. Ω ,

$$\begin{aligned} |D\psi_{j_k} - D\psi| &= |Du_{j_k} - D\psi| \cdot \mathbb{1}_{\{|\psi - u_{j_k}| < \varepsilon_{j_k}\}} \leq |Du_{j_k} - D\psi| \cdot \mathbb{1}_{\{|\psi - u| < \varepsilon_{j_k} + \delta_{j_k}\}} \\ &\rightarrow |Du - D\psi| \cdot \mathbb{1}_{\{|\psi - u| = 0\}} = 0, \end{aligned} \quad (3.10)$$

where we used Stampacchia's theorem (see [24, Theorem 4.4]). Since the limit is unique, $D\psi_j \rightarrow D\psi$ a.e. on Ω . Thus, the dominated convergence theorem with $\|D\psi_j\|_\infty \leq 1$ yields $\|D\psi_j - D\psi\|_q \rightarrow 0$ for each $q \in [1, \infty)$. From (3.8), (b) also holds.

(ii) When $\Omega = \mathbb{R}^m$, from (3.9) it is easily seen that $\psi_j \in \mathcal{Y}_0(\mathbb{R}^m)$. In addition, by Proposition 3.3, $\|u_j - u\|_\infty \rightarrow 0$ and $\mathcal{Y}_0(\mathbb{R}^m) \hookrightarrow C_0(\mathbb{R}^m)$. Hence, we may apply the same argument as above to prove (a) in this case. As for (b), setting $f_k \doteq |Du_{j_k} - D\psi|$, $g_k \doteq \mathbb{1}_{\{|\psi - u| < \varepsilon_{j_k} + \delta_{j_k}\}}$ and $f = |Du - D\psi|$, we deduce from (3.10) that

$$\|D\psi_{j_k} - D\psi\|_2 \leq \|f_k g_k\|_2 \leq \|(f_k - f)g_k\|_2 + \|f g_k\|_2 \leq \|(f_k - f)\|_2 + \|f g_k\|_2 \rightarrow 0$$

as $k \rightarrow \infty$, where we used $u_{j_k} \rightarrow u$ in $\mathcal{Y}(\mathbb{R}^m)$, $f g_k \rightarrow 0$ a.e. \mathbb{R}^m and the dominated convergence theorem. The bound $\|D\psi_{j_k} - D\psi\|_\infty \leq 2$ then implies $\|D\psi_{j_k} - D\psi\|_q \rightarrow 0$ for all $q \in [2, \infty)$. Since the limit is unique, $\|D\psi_j - D\psi\|_q \rightarrow 0$ for all $q \in [2, \infty)$. From $\|\psi_j - \psi\|_\infty \rightarrow 0$ and Sobolev's inequality, it follows that $\|\psi_j - \psi\|_{W^{1,q}} \rightarrow 0$ for all $q \in [2^*, \infty)$ and (b) also holds. \square

In the above lemma, assume that Ω is bounded. If $\phi_j \rightarrow \phi$ in $C(\partial\Omega)$ and $u_j \rightarrow u$ weakly in $\mathcal{Y}(\Omega)$, then the strong convergence $\psi_j \rightarrow \psi$ in $\mathcal{Y}(\Omega)$ may fail. However, for future use, we need to guarantee the existence of $\psi_j \in \mathcal{Y}_{\phi_j}(\Omega)$ and $\psi \in \mathcal{Y}_\phi(\Omega)$ such that $\psi_j \rightarrow \psi$ strongly in $\mathcal{Y}(\Omega)$. We propose a construction that depends on the following compactness result for Lorentzian balls.

Lemma 3.8. *Let $\Omega \subset \mathbb{R}^m$ be a domain, let $\mathcal{G} \subset W^{1,\infty}(\Omega)$ be compact in $C(K)$ for each compact set $K \subset \Omega$, and assume that $\|Du\|_\infty \leq 1$ on Ω for each $u \in \mathcal{G}$. Fix an open subset $\Omega' \Subset \Omega$ and $\tilde{\varepsilon} > 0$. Then, the following are equivalent:*

- (a) *For each $\Omega'' \Subset \Omega'$ with $d_\delta(\Omega'', \partial\Omega') \geq \tilde{\varepsilon}$, every $u \in \mathcal{G}$ does not have a light segment $\overline{xy} \subset \overline{\Omega'} \setminus \Omega''$ with $x \in \partial\Omega''$, $y \in \partial\Omega'$.*
- (b) *There exists $R = R(\mathcal{G}, \tilde{\varepsilon}, \Omega') > 0$ such that $L_R^u(\Omega'') \Subset \Omega'$ for each $u \in \mathcal{G}$ and each subset $\Omega'' \Subset \Omega'$ satisfying $d_\delta(\Omega'', \partial\Omega') \geq \tilde{\varepsilon}$, where L_R^u is the Lorentzian ball of radius R associated to the graph of u .*

Furthermore, the following are equivalent:

- (a') *Every $u \in \mathcal{G}$ does not have light segments in Ω' .*
- (b') *For each $\varepsilon > 0$, there exists $R = R(\mathcal{G}, \varepsilon, \Omega') > 0$ such that for each pair of open subsets $\Omega_1 \Subset \Omega_2 \subset \Omega'$ with $d_\delta(\Omega_1, \partial\Omega_2) \geq \varepsilon$, it holds $L_R^u(\Omega_1) \Subset \Omega_2$ for each $u \in \mathcal{G}$.*

Proof. (b) \Rightarrow (a) and (b') \Rightarrow (a') are obvious. The proofs of (a) \Rightarrow (b) and (a') \Rightarrow (b') are analogous, so we only prove (a') \Rightarrow (b'). Assume by contradiction the existence of $\varepsilon > 0$, $\Omega_1^{(j)} \Subset \Omega_2^{(j)}$ with $d_\delta(\Omega_1^{(j)}, \partial\Omega_2^{(j)}) \geq \varepsilon$, $u_j \in \mathcal{G}$, points $z_j \in \partial\Omega_1^{(j)}$ and $p_j \in \partial\Omega_2^{(j)}$ such that

$$\overline{z_j p_j} \subset \overline{\Omega_2^{(j)}} \subset \overline{\Omega'}, \quad \mathcal{H}_\delta^1(\overline{z_j p_j}) \geq \varepsilon, \quad |z_j - p_j| - |u_j(z_j) - u_j(p_j)| \leq \frac{1}{j}. \quad (3.11)$$

Since \mathcal{G} is compact in $C(\overline{\Omega'})$, up to subsequences, $u_j \rightarrow u \in \mathcal{G}$ in $C(\overline{\Omega'})$, $z_j \rightarrow z \in \overline{\Omega'}$ and $p_j \rightarrow p \in \overline{\Omega'}$. Passing to the limit in (3.11), u has a light segment \overline{zp} of length $\geq \varepsilon$. Noticing that $B_\varepsilon(z_j) \subset \Omega'$ for each j , we get $B_\varepsilon(z) \subset \Omega'$ and thus part of \overline{zp} lies in Ω' , a contradiction. \square

Lemma 3.9. *Let $\Omega \subset \mathbb{R}^m$ be a bounded domain, $\phi_j, \phi \in S(\partial\Omega)$ satisfy $\phi_j \rightarrow \phi$ in $C(\partial\Omega)$ as $j \rightarrow \infty$. Then, there exist $\psi_j \in \mathcal{Y}_{\phi_j}(\Omega)$ and $\psi \in \mathcal{Y}_\phi(\Omega)$ such that $\psi_j \rightarrow \psi$ strongly in $\mathcal{Y}(\Omega)$.*

Proof. By Theorem 1.3, we can find ψ_j (respectively, ψ) of class $C^\infty(\Omega)$ solving (BI) with $\rho = 0$ and boundary value ϕ_j (resp. ϕ). The comparison principle and $\phi_j \rightarrow \phi$ in $C(\partial\Omega)$ imply $\psi_j \rightarrow \psi$ in $C(\overline{\Omega})$. For each compact set $K \subset \Omega$, we apply Lemma 3.8 with $\mathcal{G} := \{\psi_j\}_j \cup \{\psi\}$ to obtain $R > 0$ such that $L_R^{\psi_j}(K) \Subset \Omega$ for all $j \geq 1$. The monotonicity formula [5, Lemma 2.1] and the argument in [5, Proof of Theorem 4.1] imply that there exists $\theta > 0$ such that

$$\sup_{j \geq 1} \sup_K |D\psi_j| \leq 1 - \theta$$

and $\{\psi_j\}$ is bounded in $W_{\text{loc}}^{2,2}(\Omega)$. Hence, up to a subsequence, $D\psi_{j_k} \rightarrow D\psi$ a.e. Ω . The dominated convergence theorem then guarantees $\|D\psi_{j_k} - D\psi\|_p \rightarrow 0$ for each $p \in [1, \infty)$, whence $\psi_{j_k} \rightarrow \psi$ strongly in $\mathcal{Y}(\Omega)$. Since ψ is independent of the choices of subsequences, we obtain $\psi_j \rightarrow \psi$ strongly in $\mathcal{Y}(\Omega)$. \square

We conclude this subsection with a simple observation, which however implies an important property of light segments.

Proposition 3.10. *Let $m \geq 2$, $\Omega \subset \mathbb{R}^m$ be any nonempty domain, and let $u \in W^{1,\infty}(\Omega)$ with $|Du| \leq 1$ almost everywhere. Then, light segments in the graph of u cannot cross each other, that is, no point of Ω can lie in the interior of two distinct light segments.*

Proof. The conclusion directly follows from [56, Lemma 3.5], according to which, if $\overline{xy} \subset \Omega$ is a light segment for u , then u is differentiable at any interior point of \overline{xy} with gradient being the unit vector along \overline{xy} pointing towards the increasing direction of u . \square

3.4 Convergence of minimizers

Our proof of the solvability of (BI) depends on an approximation procedure, smoothing ρ by convolution. Thus, it entails a convergence result for minimizers.

Proposition 3.11. *Let $\{\rho_j\} \subset \mathcal{Y}(\Omega)^*$, and consider the following assumptions:*

- (i) $\Omega \subset \mathbb{R}^m$ is a bounded domain with $m \geq 2$ and $\{\phi_j\} \subset S(\partial\Omega)$ satisfies $\phi_j \rightarrow \phi \in S(\partial\Omega)$ strongly in $C(\partial\Omega)$. Assume that $\rho_j = \mu_j + f_j$, where $\mu_j \in \mathcal{M}(\Omega)$, $f_j \in \mathcal{Y}(\Omega)^*$, and that

$$\mu_j \rightarrow \mu \text{ weakly in } \mathcal{M}(\Omega), \quad f_j \rightarrow f \text{ strongly in } \mathcal{Y}(\Omega)^*. \quad (3.12)$$

- (ii) $\Omega = \mathbb{R}^m$ with $m \geq 3$, $\rho_j = \mu_j + f_j$ where μ_j and f_j satisfy (3.12). Assume also that, for each $\varepsilon > 0$, there exists $R_\varepsilon > 0$ such that

$$|\mu_j| \left(\mathbb{R}^m \setminus B_{R_\varepsilon} \right) < \varepsilon \text{ for each } j \geq 1. \quad (3.13)$$

Under either (i) or (ii), by setting $\rho \doteq \mu + f$, up to a subsequence, $u_{\rho_j} \rightarrow u_\rho$ strongly in $W^{1,q}(\Omega) \cap C(\bar{\Omega})$, respectively, for every $q \in [1, \infty)$ if Ω is a bounded domain, and for every $q \in [2^*, \infty)$ if $\Omega = \mathbb{R}^m$. Furthermore, $\|Du_{\rho_j} - Du_\rho\|_q \rightarrow 0$ for every $q \in [2, \infty)$ when $\Omega = \mathbb{R}^m$. In particular,

$$u_{\rho_j} \rightarrow u_\rho \quad \text{in } \mathcal{Y}(\Omega), \quad \langle \rho_j, u_{\rho_j} \rangle \rightarrow \langle \rho, u_\rho \rangle \quad \text{as } j \rightarrow \infty.$$

Proof. For the ease of notation, we write u_j instead of u_{ρ_j} . We first suppose that Ω is bounded. Notice that by $\phi \in S(\partial\Omega)$, $\mathcal{Y}_\phi \neq \emptyset$ and the minimizer u_ρ exists. On the other hand, due to Proposition 3.5 and $u_j \in \mathcal{Y}_{\phi_j}(\Omega)$, $\{u_j\}$ is relatively compact in $C(\bar{\Omega})$ and hence it is bounded in $W^{1,q}(\Omega)$ for any $q \in [1, \infty]$. Up to a subsequence, $u_j \rightharpoonup u$ weakly in $W^{1,q}(\Omega)$ for each fixed $q \in (1, \infty)$, and strongly in $C(\bar{\Omega})$, which yields $u \in \mathcal{Y}_\phi(\Omega)$.

From (3.5) we get

$$\begin{aligned} \int_{\Omega} \left(1 - \sqrt{1 - |Du|^2}\right) dx &= \sum_{k=1}^{\infty} b_k \|Du\|_{2k}^{2k} \leq \sum_{k=1}^{\infty} b_k \liminf_{j \rightarrow \infty} \|Du_j\|_{2k}^{2k} \\ &\leq \lim_{n \rightarrow \infty} \liminf_{j \rightarrow \infty} \sum_{k=1}^n b_k \|Du_j\|_{2k}^{2k} \\ &\leq \liminf_{j \rightarrow \infty} \int_{\Omega} \left(1 - \sqrt{1 - |Du_j|^2}\right) dx. \end{aligned} \quad (3.14)$$

Since $u_j \rightharpoonup u$ weakly in $\mathcal{Y}(\Omega)$ and strongly in $C(\bar{\Omega})$, our assumptions on $\{\mu_j\}$ and $\{f_j\}$ give

$$\lim_{j \rightarrow \infty} \langle \rho_j, u_j \rangle = \lim_{j \rightarrow \infty} (\langle \mu_j, u_j \rangle + \langle f_j, u_j \rangle) = \langle \mu, u \rangle + \langle f, u \rangle = \langle \rho, u \rangle. \quad (3.15)$$

Hence, by (3.14) and the minimality of u_ρ and u_j , we obtain

$$I_\rho(u_\rho) \leq I_\rho(u) \leq \liminf_{j \rightarrow \infty} I_{\rho_j}(u_j). \quad (3.16)$$

By Lemma 3.9, choose $\psi_j \in \mathcal{Y}_{\phi_j}(\Omega)$ and $\psi \in \mathcal{Y}_\phi(\Omega)$ such that $\psi_j \rightarrow \psi$ in $\mathcal{Y}(\Omega)$ as $j \rightarrow \infty$. Having fixed a sequence $\varepsilon_j \rightarrow 0$ with $\varepsilon_j > |\phi_j - \phi|$ on $\partial\Omega$, we consider the approximation $\bar{u}_j \doteq (u_\rho)_{\varepsilon_j}^{\psi_j} \in \mathcal{Y}_{\phi_j}(\Omega)$ of u_ρ given by Lemma 3.7, which therefore satisfies $\bar{u}_j \rightarrow u_\rho$ in $\mathcal{Y}(\Omega)$. Weak convergence $\rho_j \rightharpoonup \rho$ and continuity of I_ρ imply that $I_{\rho_j}(\bar{u}_j) \rightarrow I_\rho(u_\rho)$. Hence, by the minimality of u_j we obtain

$$\limsup_{j \rightarrow \infty} I_{\rho_j}(u_j) \leq \limsup_{j \rightarrow \infty} I_{\rho_j}(\bar{u}_j) = I_\rho(u_\rho). \quad (3.17)$$

From (3.16), we infer that $I_\rho(u_\rho) = I_\rho(u) = \lim_{j \rightarrow \infty} I_{\rho_j}(u_j)$, so $u = u_\rho$ by the uniqueness of minimizer.

Finally, we shall show $u_j \rightarrow u$ strongly in $W^{1,q}(\Omega)$ for any $q \in [1, \infty)$. To this end, notice that $I_{\rho_j}(u_j) \rightarrow I_\rho(u_\rho)$ and (3.15) with $u = u_\rho$ imply

$$\int_{\Omega} \left(1 - \sqrt{1 - |Du_j|^2}\right) dx \rightarrow \int_{\Omega} \left(1 - \sqrt{1 - |Du_\rho|^2}\right) dx.$$

If there exists $k_0 > 0$ such that

$$\varepsilon_0 \doteq \liminf_{j \rightarrow \infty} \|Du_j\|_{2k_0}^{2k_0} - \|Du_\rho\|_{2k_0}^{2k_0} > 0,$$

then by (3.5) we can choose $h_0 > k_0$ so large that

$$\int_{\Omega} \left(1 - \sqrt{1 - |Du_{\rho}|^2}\right) dx - \sum_{k=1}^{h_0} b_k \|Du_{\rho}\|_{2k}^{2k} < \frac{b_{k_0} \varepsilon_0}{2},$$

and therefore deduce the following contradiction:

$$\begin{aligned} \int_{\Omega} \left(1 - \sqrt{1 - |Du_{\rho}|^2}\right) dx &< \frac{b_{k_0} \varepsilon_0}{2} + \sum_{k=1}^{h_0} b_k \|Du_{\rho}\|_{2k}^{2k} \\ &\leq \liminf_{j \rightarrow \infty} \sum_{k=1}^{h_0} b_k \|Du_j\|_{2k}^{2k} - \frac{b_{k_0} \varepsilon_0}{2} \\ &\leq \liminf_{j \rightarrow \infty} \int_{\Omega} \left(1 - \sqrt{1 - |Du_j|^2}\right) dx - \frac{b_{k_0} \varepsilon_0}{2} \\ &= \int_{\Omega} \left(1 - \sqrt{1 - |Du_{\rho}|^2}\right) dx - \frac{b_{k_0} \varepsilon_0}{2}. \end{aligned}$$

Thus, $\|Du_j\|_{2k} \rightarrow \|Du_{\rho}\|_{2k}$ for each $k \geq 1$. The uniform convexity of $L^{2k}(\Omega)$ and $\|u_j - u_{\rho}\|_{\infty} \rightarrow 0$ imply that $Du_j \rightarrow Du_{\rho}$ in $L^{2k}(\Omega)$, hence $u_j \rightarrow u_{\rho}$ in $W^{1,2k}(\Omega)$ for any $k \geq 1$. By Hölder's inequality, $u_j \rightarrow u_{\rho}$ strongly in $W^{1,q}(\Omega)$ for each $q \in [1, \infty)$ and we complete the proof for the case Ω is a bounded domain.

When $\Omega = \mathbb{R}^m$ with $m \geq 3$, first observe that by our assumptions $\{\rho_j\}$ is uniformly bounded in $\mathcal{Y}(\Omega)^*$. Hence, from $I_{\rho_j}(u_j) \leq I_{\rho_j}(0) = 0$ and the coercivity estimate (3.6) for $v = u_j$, we deduce that $\{u_j\}$ is uniformly bounded in $\mathcal{Y}(\mathbb{R}^m)$. By Proposition 3.3 and $\|Du_j\|_{\infty} \leq 1$, $\{u_j\}$ is bounded in $W^{1,q}(\mathbb{R}^m)$ for each $q \in [2^*, \infty)$, hence in $L^{\infty}(\mathbb{R}^m)$. Up to a subsequence, $u_j \rightharpoonup u$ weakly in $W^{1,q}(\mathbb{R}^m)$ for each $q \in [2^*, \infty)$, $u_j \rightarrow u$ in $C_{\text{loc}}(\mathbb{R}^m)$, and $u_j \rightharpoonup u$ weakly in $\mathcal{Y}(\mathbb{R}^m)$ by the reflexivity of $\mathcal{Y}(\mathbb{R}^m)$. Since each u_j is 1-Lipschitz, so is u and $u \in \mathcal{Y}_0(\mathbb{R}^m)$. Coupling condition (3.13) for $\{\mu_j\}$ with the convergence $u_j \rightarrow u$ in $C_{\text{loc}}(\mathbb{R}^m)$ and the uniform boundedness of $\{u_j\}$, we deduce that $\langle \mu_j, u_j \rangle \rightarrow \langle \mu, u \rangle$, hence (3.15) holds. Then, arguing as above with $\bar{u}_j \doteq u_{\rho}$, we may verify $u = u_{\rho}$ and $Du_j \rightarrow Du_{\rho}$ strongly in $L^q(\mathbb{R}^m)$ for each $q \in [2, \infty)$. Hence, $u_j \rightarrow u_{\rho}$ strongly in $W^{1,q}(\mathbb{R}^m)$ for every $q \in [2^*, \infty)$, in particular $u_j \rightarrow u_{\rho}$ in $\mathcal{Y}(\mathbb{R}^m)$. Concluding, $u_j \rightarrow u_{\rho}$ in $C(\mathbb{R}^m)$ follows from Proposition 3.3. \square

Remark 3.12. In the proof of Proposition 3.11 (i), condition $\phi_j, \phi \in \mathcal{S}(\partial\Omega)$ is only used to construct a good competitor \bar{u}_j to obtain (3.17). When $\phi_j \equiv \phi$ for each j , we may simply choose $u_j = u_{\rho}$ and weaken the condition $\phi \in \mathcal{S}(\partial\Omega)$ to $\mathcal{Y}_{\phi}(\Omega) \neq \emptyset$.

3.5 Local integrability of w and the Euler-Lagrange inequality

In this subsection, assuming $\phi \in \mathcal{S}(\partial\Omega)$ if Ω is bounded, we show that the minimizer u_{ρ} is not too degenerate and solves an Euler-Lagrange inequality. The argument in the proof is inspired by [11, Proposition 2.6], in particular, case (ii) in the following is essentially contained therein.

Proposition 3.13. *Let $\Omega \subset \mathbb{R}^m$ be a domain.*

- (i) *Assume that $m \geq 2$ and that Ω is bounded. For any given compact subset $\mathcal{F} \subset \mathcal{S}(\partial\Omega)$, and any $\varepsilon, I_1 > 0$, there exists a constant $C = C(\Omega, \mathcal{F}, m, p_1, I_1, \text{diam}_{\delta}(\Omega), \varepsilon)$ such that if*

$$\phi \in \mathcal{F}, \quad \rho \in \mathcal{Y}(\Omega)^* \text{ with } \|\rho\|_{\mathcal{Y}^*} \leq I_1,$$

then for each open subset $\Omega' \Subset \Omega$ with $d_\delta(\Omega', \partial\Omega) \geq \varepsilon$ the minimizer u_ρ satisfies

$$\int_{\Omega'} \frac{dx}{\sqrt{1 - |Du_\rho|^2}} \leq C. \quad (3.18)$$

In particular, $|Du_\rho| < 1$ a.e. on Ω . Moreover, for each $\psi \in \mathcal{Y}_\phi(\Omega)$,

$$\frac{Du_\rho \cdot (Du_\rho - D\psi)}{\sqrt{1 - |Du_\rho|^2}} \in L^1(\Omega), \quad (3.19)$$

$$\sqrt{1 - |D\psi|^2} - \sqrt{1 - |Du_\rho|^2} \leq \frac{Du_\rho \cdot (Du_\rho - D\psi)}{\sqrt{1 - |Du_\rho|^2}} \quad \text{a.e. on } \Omega \quad (3.20)$$

and

$$\int_{\Omega} \left(\sqrt{1 - |D\psi|^2} - \sqrt{1 - |Du_\rho|^2} \right) dx \leq \int_{\Omega} \frac{Du_\rho \cdot (Du_\rho - D\psi)}{\sqrt{1 - |Du_\rho|^2}} dx \leq \langle \rho, u_\rho - \psi \rangle. \quad (3.21)$$

(ii) Assume that $m \geq 3$ and that $\Omega = \mathbb{R}^m$. For any given $I_1 > 0$ and $\Omega' \Subset \mathbb{R}^m$, there exists a constant $C' = C'(m, p_1, I_1, |\Omega'|_\delta) > 0$ such that if $\|\rho\|_{\mathcal{Y}^*} \leq I_1$, then (3.18) holds with C' . Furthermore, (3.19)–(3.21) hold for each $\psi \in \mathcal{Y}_0(\mathbb{R}^m)$.

Remark 3.14. Notice that choosing $\Omega = \mathbb{R}^m$ and $\psi = 0$ in (3.19) we infer the integrability condition in (1.11) mentioned in the Introduction.

Proof of Proposition 3.13. (i) We first prove (3.18). Fix $\Omega' \Subset \Omega$ with $d_\delta(\Omega', \partial\Omega) \geq \varepsilon$. Given $\psi \in \mathcal{Y}_\phi(\Omega)$, observe that $u_t \doteq (1-t)u_\rho + t\psi \in \mathcal{Y}_\phi(\Omega)$ for every $t \in (0, 1]$. Thus, $I_\rho(u_\rho) \leq I_\rho(u_t)$, and rearranging we get

$$\frac{1}{t} \int_{\Omega} \left(\sqrt{1 - |Du_t|^2} - \sqrt{1 - |Du_\rho|^2} \right) dx \leq \langle \rho, u_\rho - \psi \rangle \quad \forall t \in (0, 1]. \quad (3.22)$$

Next, the concavity of the map $p \mapsto \sqrt{1 - |p|^2}$ on $\overline{B_1(0)}$ implies that

$$\sqrt{1 - |Du_t|^2} \geq (1-t)\sqrt{1 - |Du_\rho|^2} + t\sqrt{1 - |D\psi|^2} \quad \text{a.e. on } \Omega, \forall t \in (0, 1],$$

which yields

$$\sqrt{1 - |D\psi|^2} - \sqrt{1 - |Du_\rho|^2} \leq \frac{1}{t} \left\{ \sqrt{1 - |Du_t|^2} - \sqrt{1 - |Du_\rho|^2} \right\} \quad \text{a.e. on } \Omega. \quad (3.23)$$

Let $\mathcal{S} \subset \mathcal{Y}(\Omega)$ be the set of minimizers of I_0 (i.e. with $\rho = 0$) whose boundary value lies in \mathcal{F} . For given $\phi \in \mathcal{F}$ we denote by $\bar{\phi} \in \mathcal{S}$ the corresponding minimizer. The compactness of \mathcal{F} and Propositions 3.5 and 3.11 guarantee that \mathcal{S} is compact in $C(\overline{\Omega})$. By Theorem 1.3, every $u \in \mathcal{S}$ does not have light segments in Ω , thus applying the first part of Lemma 3.8 for $\Omega_\varepsilon \Subset \Omega_{\varepsilon/2}$ we obtain $R = R(\Omega, \mathcal{F}, \varepsilon) > 0$ such that $L_R^u(\Omega_\varepsilon) \Subset \Omega_{\varepsilon/2}$ for each $u \in \mathcal{S}$. From the monotonicity formula [5, Lemma 2.1], we infer the existence of $\theta = \theta(\Omega, \mathcal{F}, \varepsilon)$ such that

$$\sup_{\Omega'} |D\bar{\phi}| \leq 1 - 4\theta. \quad (3.24)$$

We take $\psi = \bar{\phi}$, and note that on the set of full measure $V \subset \Omega'$ of points where u_ρ is differentiable it holds $|Du_t| < 1$ for every $t \in (0, 1]$. We set

$$K \doteq \left\{ x \in \Omega : 1 - \theta < |Du_\rho(x)| \right\},$$

and split the domain of integration Ω in (3.22) into the sets $\Omega \setminus \Omega'$, $V \cap K$ and $V \cap K^c$. We use (3.23) on $\Omega \setminus \Omega'$ and the identity

$$\begin{aligned} & \frac{1}{t} \left\{ \sqrt{1 - |Du_t|^2} - \sqrt{1 - |Du_\rho|^2} \right\} \\ &= \frac{2Du_\rho \cdot (Du_\rho - D\psi) - t|Du_\rho - D\psi|^2}{\sqrt{1 - |Du_t|^2} + \sqrt{1 - |Du_\rho|^2}} \quad \text{a.e. on } \Omega \cap \{|D\psi| + |Du_\rho| < 2\} \end{aligned} \quad (3.25)$$

to deduce that

$$\begin{aligned} \langle \rho, u_\rho - \bar{\phi} \rangle &\geq \int_{\Omega \setminus \Omega'} \left(\sqrt{1 - |D\bar{\phi}|^2} - \sqrt{1 - |Du_\rho|^2} \right) dx \\ &\quad + \int_{V \cap K} \frac{2Du_\rho \cdot (Du_\rho - D\bar{\phi}) - t|Du_\rho - D\bar{\phi}|^2}{\sqrt{1 - |Du_t|^2} + \sqrt{1 - |Du_\rho|^2}} dx \\ &\quad + \int_{V \cap K^c} \frac{2Du_\rho \cdot (Du_\rho - D\bar{\phi}) - t|Du_\rho - D\bar{\phi}|^2}{\sqrt{1 - |Du_t|^2} + \sqrt{1 - |Du_\rho|^2}} dx. \end{aligned} \quad (3.26)$$

Recalling (3.24), we restrict to t small enough so that $4t < \theta^2$. By the definition of K , the next inequality holds on $\Omega' \cap K$:

$$2Du_\rho \cdot (Du_\rho - D\bar{\phi}) - t|Du_\rho - D\bar{\phi}|^2 \geq 2[(1 - \theta)^2 - (1 - 4\theta)] - 4t > 4\theta > 0. \quad (3.27)$$

Remark also that the last term in the right-hand side of (3.26) is bounded uniformly with respect to $t \in (0, 1)$. Thus, letting $t \rightarrow 0$ in (3.26) and using (3.27), Fatou's lemma and the dominated convergence theorem, we infer

$$\begin{aligned} \langle \rho, u_\rho - \bar{\phi} \rangle &\geq \int_{\Omega \setminus \Omega'} \left(\sqrt{1 - |D\bar{\phi}|^2} - \sqrt{1 - |Du_\rho|^2} \right) dx \\ &\quad + \int_{V \cap K} \frac{2\theta}{\sqrt{1 - |Du_\rho|^2}} dx + \int_{V \cap K^c} \frac{Du_\rho \cdot (Du_\rho - D\bar{\phi})}{\sqrt{1 - |Du_\rho|^2}} dx. \end{aligned} \quad (3.28)$$

From

$$\left| \int_{\Omega \setminus \Omega'} \sqrt{1 - |D\bar{\phi}|^2} - \sqrt{1 - |Du_\rho|^2} dx \right| \leq |\Omega \setminus \Omega'|_\delta \quad (3.29)$$

and the following straightforward estimate on $\Omega' \cap K^c$:

$$\int_{\Omega' \cap K^c} \left| \frac{Du_\rho \cdot (Du_\rho - D\bar{\phi})}{\sqrt{1 - |Du_\rho|^2}} \right| dx \leq \int_{\Omega' \cap K^c} \frac{2dx}{\sqrt{2\theta - \theta^2}} \leq \frac{2|\Omega'|_\delta}{\sqrt{2\theta - \theta^2}},$$

it follows from (3.28) and $|\Omega' \setminus V| = 0$ that

$$\int_{\Omega' \cap K} \frac{2\theta}{\sqrt{1 - |Du_\rho|^2}} dx \leq |\Omega \setminus \Omega'|_\delta + \langle \rho, u_\rho - \bar{\phi} \rangle + \frac{2|\Omega'|_\delta}{\sqrt{2\theta - \theta^2}}.$$

Therefore,

$$\begin{aligned} \int_{\Omega'} \frac{dx}{\sqrt{1 - |Du_\rho|^2}} &= \int_{\Omega' \cap K} \frac{dx}{\sqrt{1 - |Du_\rho|^2}} + \int_{\Omega' \cap K^c} \frac{dx}{\sqrt{1 - |Du_\rho|^2}} \\ &\leq \frac{1}{2\theta} \left(|\Omega \setminus \Omega'|_\delta + \|\rho\|_{Y^*} \|u_\rho - \bar{\phi}\|_Y + \frac{2|\Omega'|_\delta}{\sqrt{2\theta - \theta^2}} \right) + \frac{|\Omega'|_\delta}{\sqrt{2\theta - \theta^2}}. \end{aligned} \quad (3.30)$$

For $\psi \in \mathcal{Y}_\phi(\Omega)$, (3.4) and simple estimates for the W^{1,p_1} norm give

$$\|u_\rho - \bar{\phi}\|_Y \leq 4 \left(\sup_{\phi \in \mathcal{F}} \|\phi\|_{C(\partial\Omega)} + \text{diam}_\delta(\Omega) + |\Omega|_\delta^{\frac{1}{p_1}} \right).$$

Hence, (3.18) holds by (3.30). Notice that, by (3.18) and the arbitrariness of Ω' , $|Du_\rho| < 1$ a.e. on Ω .

Next, we shall prove (3.19)–(3.21). Let $\psi \in \mathcal{Y}_\phi(\Omega)$ and consider as above $u_t \doteq (1-t)u_\rho + t\psi \in \mathcal{Y}_\phi(\Omega)$ for $t \in (0, 1)$. By combining $|Du_\rho| < 1$ a.e. Ω , (3.25) and (3.23), for each $t \in (0, 1)$,

$$\sqrt{1 - |D\psi|^2} - \sqrt{1 - |Du_\rho|^2} \leq \frac{2Du_\rho \cdot (Du_\rho - D\psi) - t|Du_\rho - D\psi|^2}{\sqrt{1 - |Du_t|^2} + \sqrt{1 - |Du_\rho|^2}} \quad \text{a.e. on } \Omega. \quad (3.31)$$

Thus letting $t \rightarrow 0$ on the set $\{|Du_\rho| < 1\}$, we deduce (3.20).

On the other hand, from (3.22) and (3.25), it follows that

$$\int_{\Omega} \frac{2Du_\rho \cdot (Du_\rho - D\psi) - t|Du_\rho - D\psi|^2}{\sqrt{1 - |Du_t|^2} + \sqrt{1 - |Du_\rho|^2}} dx \leq \langle \rho, u_\rho - \psi \rangle.$$

Using a variant of Fatou's lemma as $t \rightarrow 0$ and (3.31), we therefore deduce

$$\int_{\Omega} \left(\sqrt{1 - |D\psi|^2} - \sqrt{1 - |Du_\rho|^2} \right) dx \leq \int_{\Omega} \frac{Du_\rho \cdot (Du_\rho - D\psi)}{\sqrt{1 - |Du_\rho|^2}} dx \leq \langle \rho, u_\rho - \psi \rangle,$$

which proves (3.21). Taking (3.20) into account, both the negative and the positive part of

$$\frac{Du_\rho \cdot (Du_\rho - D\psi)}{\sqrt{1 - |Du_\rho|^2}}$$

are integrable, and (3.19) holds.

(ii) We first observe that (3.6), $I_\rho(u_\rho) \leq I_\rho(0) = 0$ and $\|\rho\|_{Y^*} \leq I_1$ imply that $\|u_\rho\|_Y \leq C_1(m, I_1)$. One can therefore perform the same computations in (3.22)–(3.28) with $\Omega = \mathbb{R}^m$, $\bar{\phi} = 0$, $\theta = 1/8$ and replacing (3.29) with

$$0 \leq \int_{\mathbb{R}^m \setminus \Omega'} \left(1 - \sqrt{1 - |Du_\rho|^2} \right) dx \leq I_\rho(u_\rho) + \langle \rho, u_\rho \rangle \leq I_1 C_1.$$

Inequality (3.30) becomes

$$\int_{\Omega'} \frac{dx}{\sqrt{1 - |Du_\rho|^2}} \leq 4(2\mathcal{I}_1 C_1 + C_2 |\Omega'|_\delta) + C_2 |\Omega'|_\delta,$$

for some absolute constant C_2 . The rest of the proof follows verbatim, taking into account that $1 - \sqrt{1 - |p|^2} \leq |p|^2$ on $\overline{B_1(0)}$ and thus $\sqrt{1 - |D\psi|^2} - \sqrt{1 - |Du_\rho|^2} = (1 - \sqrt{1 - |Du_\rho|^2}) - (1 - \sqrt{1 - |D\psi|^2}) \in L^1(\mathbb{R}^m)$. This completes the proof. \square

Remark 3.15. Inequality (3.20) has a nice geometric interpretation, holding more generally for pairs of Lipschitz functions u, ψ with $|Du| < 1, |D\psi| \leq 1$ a.e. on Ω . Indeed, if we consider the normal vectors $\mathbf{n}'_u \doteq Du + \partial_0, \mathbf{n}'_\psi = D\psi + \partial_0$ (respectively, timelike and causal a.e. on Ω), the reversed Cauchy-Schwarz inequality $-\mathbf{n}'_u \cdot \mathbf{n}'_\psi \geq |\mathbf{n}'_u|_\perp |\mathbf{n}'_\psi|_\perp$ is equivalent to

$$\frac{\mathbf{n}'_u}{|\mathbf{n}'_u|_\perp} \cdot (\mathbf{n}'_u - \mathbf{n}'_\psi) \geq |\mathbf{n}'_\psi|_\perp - |\mathbf{n}'_u|_\perp,$$

that can be rewritten as (3.20) with u replacing u_ρ .

3.6 Global minimizers VS solutions to (BI)

In this subsection, we describe in detail the interplay between solutions of (BI) and global minimizers of I_ρ , stating some useful equivalent characterizations of the solvability of (BI) that, perhaps surprisingly, hold without assuming any regularity of $\partial\Omega$.

Definition 3.16. We say that $u \in \mathcal{Y}_\phi(\Omega)$ is a *local minimizer* for I_ρ if $I_\rho(u) \leq I_\rho(\psi)$ for every $\psi \in \mathcal{Y}_\phi(\Omega)$ with $\{u \neq \psi\} \Subset \Omega$. Similarly, for $\Omega = \mathbb{R}^m$, we say that $u \in \mathcal{Y}_0(\mathbb{R}^m)$ is a *local minimizer* for I_ρ if $I_\rho(u) \leq I_\rho(\psi)$ for every $\psi \in \mathcal{Y}_0(\mathbb{R}^m)$ with $\{u \neq \psi\} \Subset \mathbb{R}^m$.

Proposition 3.17 (Minimizers VS solutions to (BI)). *Let $m \geq 2$, Ω be a bounded domain, $\phi \in S(\partial\Omega)$ and u a local minimizer. Then, $u = u_\rho$. Furthermore, the following are equivalent:*

(i) u is a weak solution to (BI), that is,

$$\frac{1}{\sqrt{1 - |Du|^2}} \in L^1_{\text{loc}}(\Omega), \quad \int_{\Omega} \frac{Du \cdot D\eta}{\sqrt{1 - |Du|^2}} dx = \langle \rho, \eta \rangle \quad \forall \eta \in \text{Lip}_c(\Omega); \quad (3.32)$$

(ii) $u = u_\rho$ and

$$\int_{\Omega} \frac{Du \cdot (Du - D\psi)}{\sqrt{1 - |Du|^2}} dx = \langle \rho, u - \psi \rangle \quad \forall \psi \in \mathcal{Y}_\phi(\Omega) \text{ strictly spacelike};$$

(iii) $u = u_\rho$ and

$$\int_{\Omega} \frac{Du \cdot (Du - D\psi)}{\sqrt{1 - |Du|^2}} dx = \langle \rho, u - \psi \rangle \quad \forall \psi \in \mathcal{Y}_\phi(\Omega) \text{ with } \{\psi \neq u\} \Subset \Omega; \quad (3.33)$$

(iv) $u = u_\rho$ and

$$\int_{\Omega} \frac{Du \cdot (Du - D\psi)}{\sqrt{1 - |Du|^2}} dx = \langle \rho, u - \psi \rangle \quad \forall \psi \in \mathcal{Y}_\phi(\Omega).$$

In particular, if u is a classical solution to (BT), then u satisfies any of (i)–(iv).

The same assertions hold true for $m \geq 3$ and $\Omega = \mathbb{R}^m$.

Proof. Since the case $\Omega = \mathbb{R}^m$ may be proved similarly, we only deal with bounded domains. Let Ω be a bounded domain and u a local minimizer. For $\psi \in \mathcal{Y}_\phi(\Omega)$ and $\varepsilon_j \downarrow 0$, consider the approximation $\psi_j \doteq \psi_{\varepsilon_j}^u$ constructed in Lemma 3.7, that satisfies $\{\psi_j \neq u\} \Subset \Omega$. We first notice $I_\rho(u) \leq I_\rho(\psi_j)$. Since I_ρ is continuous on $\mathcal{Y}_\phi(\Omega)$ as observed in Subsection 3.1, Lemma 3.7 implies $I_\rho(\psi_j) \rightarrow I_\rho(\psi)$ and thus $I_\rho(u) \leq I_\rho(\psi)$. Whence, $u = u_\rho$. Also, if u is a classical solution to (BT), then an integration by parts shows that (3.32) holds.

We next prove that (iv) \Rightarrow (ii) \Rightarrow (i) \Rightarrow (iii) \Rightarrow (iv).

(iv) \Rightarrow (ii) is obvious.

(ii) \Rightarrow (i).

Since $u = u_\rho$, the integrability $(1 - |Du|^2)^{-1/2} \in L^1_{\text{loc}}(\Omega)$ follows by Proposition 3.13. By density and the dominated convergence theorem, it is enough to prove (i) for $\eta \in C^1_c(\Omega)$. Fix an open set Ω' satisfying $\{\eta \neq 0\} \Subset \Omega' \Subset \Omega$, and choose a strictly spacelike extension $\bar{\phi}$ of ϕ , for instance the solution to (BT) for $\rho = 0$ as in Theorem 1.3. Since $\sup_{\Omega'} |D\bar{\phi}| < 1$, for $|t|$ small enough, the function $\psi \doteq \bar{\phi} + t\eta \in \mathcal{Y}_\phi(\Omega)$ is strictly spacelike and thus

$$\int_{\Omega} \frac{Du \cdot (Du - D\bar{\phi} - tD\eta)}{\sqrt{1 - |Du|^2}} dx = \langle \rho, u - \bar{\phi} - t\eta \rangle.$$

Differentiating at $t = 0$ gives (3.32).

(i) \Rightarrow (iii).

Identity (3.33) follows immediately from (3.32) since $u - \psi \in \text{Lip}_c(\Omega)$. To show that (3.33) implies $u = u_\rho$, first observe that $|Du| < 1$ a.e on Ω , in view of the first property in (3.32). Let $\psi \in \mathcal{Y}_\phi(\Omega)$ with $\{\psi \neq u\} \Subset \Omega$. Apply Remark 3.15 and (3.33) to deduce

$$\int_{\Omega} \left(\sqrt{1 - |D\psi|^2} - \sqrt{1 - |Du|^2} \right) dx \leq \int_{\Omega} \frac{Du \cdot (Du - D\psi)}{\sqrt{1 - |Du|^2}} dx = \langle \rho, u - \psi \rangle,$$

which can be rewritten as $I_\rho(u) \leq I_\rho(\psi)$. Hence, u is a local minimizer and thus it coincides with u_ρ .

(iii) \Rightarrow (iv).

We recall (3.21), argue by contradiction and suppose that there exist $\psi \in \mathcal{Y}_\phi(\Omega)$ and $\delta > 0$ such that

$$\int_{\Omega} \frac{Du \cdot (Du - D\psi)}{\sqrt{1 - |Du|^2}} dx \leq \langle \rho, u - \psi \rangle - \delta. \quad (3.34)$$

Select $\Omega' \Subset \Omega$ satisfying

$$\int_{\Omega \setminus \Omega'} \left| \frac{Du \cdot (Du - D\psi)}{\sqrt{1 - |Du|^2}} \right| dx < \frac{\delta}{4}, \quad (3.35)$$

which is possible by (3.19). Fix a sequence $\varepsilon_j \downarrow 0$ and consider the approximation ψ_j for ψ constructed in Lemma 3.7 by choosing $u_j = u$ for each j . By construction, $\psi_j \equiv u$ on $\Omega \setminus \Omega_j$ for some $\Omega_j \Subset \Omega$, and, without loss of generality, we can assume that $\Omega' \subset \Omega_j$ as well as $D\psi_j \rightarrow D\psi$ a.e. Ω . From $\psi_j \rightarrow \psi$ strongly in $\mathcal{Y}(\Omega)$, we get

$$\langle \rho, u - \psi_j \rangle \rightarrow \langle \rho, u - \psi \rangle \quad \text{as } j \rightarrow \infty. \quad (3.36)$$

Also, by (3.18) in Proposition 3.13 and the dominated convergence theorem,

$$\int_{\Omega'} \frac{Du \cdot (Du - D\psi_j)}{\sqrt{1 - |Du|^2}} dx \rightarrow \int_{\Omega'} \frac{Du \cdot (Du - D\psi)}{\sqrt{1 - |Du|^2}} dx \quad \text{as } j \rightarrow \infty. \quad (3.37)$$

By the definition of ψ_j ,

$$Du - D\psi_j = (Du - D\psi) \cdot \mathbb{1}_{V_j}, \quad \text{where } V_j \doteq \{|u - \psi| \geq \varepsilon_j\}, \quad (3.38)$$

hence from (3.34) and (3.36), we infer

$$\begin{aligned} \langle \rho, u - \psi_j \rangle - \delta &\geq \int_{\Omega} \frac{Du \cdot (Du - D\psi)}{\sqrt{1 - |Du|^2}} dx - o_j(1) \\ &= \int_{\Omega \setminus \Omega'} \frac{Du \cdot (Du - D\psi)}{\sqrt{1 - |Du|^2}} dx + \int_{\Omega'} \frac{Du \cdot (Du - D\psi_j)}{\sqrt{1 - |Du|^2}} dx - o_j(1) \quad \text{by (3.37)} \\ &\geq -\frac{\delta}{4} + \int_{\Omega'} \frac{Du \cdot (Du - D\psi_j)}{\sqrt{1 - |Du|^2}} dx - o_j(1) \quad \text{by (3.35)} \\ &= -\frac{\delta}{4} + \int_{\Omega_j} \frac{Du \cdot (Du - D\psi_j)}{\sqrt{1 - |Du|^2}} dx - \int_{\Omega_j \setminus \Omega'} \frac{Du \cdot (Du - D\psi_j)}{\sqrt{1 - |Du|^2}} dx - o_j(1) \\ &= -\frac{\delta}{4} + \langle \rho, u - \psi_j \rangle - \int_{(\Omega_j \setminus \Omega') \cap V_j} \frac{Du \cdot (Du - D\psi)}{\sqrt{1 - |Du|^2}} dx - o_j(1) \\ &\quad \text{by (3.33) and (3.38)} \\ &\geq -\frac{\delta}{2} + \langle \rho, u - \psi_j \rangle - o_j(1) \quad \text{by (3.35),} \end{aligned}$$

a contradiction if j is large enough. \square

Remark 3.18. For $\Omega = \mathbb{R}^m$, a different proof of (iii) \Rightarrow (iv) was given in [11, Theorem 6.4].

4 Main tools

In this section, we prove the tool results needed to obtain our existence/non-existence theorems detailed in the Introduction. The main achievements herein are Theorem 4.2 on removable singularities, Theorem 4.5 about the nonsolvability of (BI), the L^2 -estimates for the second fundamental form in Proposition 4.10 and Corollary 4.11, and Theorem 4.13 guaranteeing the higher integrability of w_ρ . To reach our goals, we first need to regularize ρ and u_ρ , a device which will also be necessary in Section 5.

4.1 Setup for our strategy

According to Remark 3.4, defining $p = q'$ it holds

$$\mathcal{M}(\Omega) + W^{-1,p}(\Omega) \subset \mathcal{Y}(\Omega)^* \quad \text{for each } \begin{cases} p \in [p'_1, \infty) & \text{if } \Omega \text{ is bounded,} \\ p \in [p'_1, 2_*] & \text{if } \Omega = \mathbb{R}^m. \end{cases}$$

We shall hereafter restrict to

$$\rho \in \mathcal{M}(\Omega) + L^p(\Omega) \quad \text{for } p \in (1, 2_*],$$

where $L^p(\Omega) \subset W^{-1,p}(\Omega)$ is the set of pairs $(v, 0)$ as in Remark 3.4.

Since $2_* = 1$ when $m = 2$, hereafter the space $L^p(\Omega)$ is tacitly assumed to be empty when $p \in (1, 2_*]$ and $m = 2$.

Notice that $\mathcal{M}(\Omega) + L^p(\Omega) \hookrightarrow \mathcal{Y}(\Omega)^*$ provided that p_1 is sufficiently large. For instance, we may (and henceforth do) choose

$$p_1 = 3 \quad \text{if } m = 2, \quad p_1 = \max\{2^*, m\} + p' \quad \text{if } m \geq 3. \quad (4.1)$$

By a standard mollifying argument (see [48, Chapter 2]) and Young's inequality, for given

$$\rho = \mu + f \in \mathcal{M}(\Omega) + L^p(\Omega)$$

we can find sequences of functions $g_j, f_j \in C^\infty(\bar{\Omega})$ such that, setting $\mu_j \doteq g_j dx$ and recalling $p = q'$,

$$\begin{aligned} \|\mu_j\|_{\mathcal{M}(\Omega)} &\leq \|\mu\|_{\mathcal{M}(\Omega)}, & \|f_j\|_{L^p(\Omega)} &\leq \|f\|_{L^p(\Omega)} \\ \mu_j &\rightarrow \mu \text{ weakly in } \mathcal{M}(\Omega), & f_j &\rightarrow f \text{ strongly in } L^p(\Omega) \text{ (hence, in } \mathcal{Y}(\Omega)^*). \end{aligned}$$

Define $\rho_j \doteq \mu_j + f_j$. When $\Omega = \mathbb{R}^m$, the construction via convolution also guarantees, for each $\varepsilon > 0$, the existence of $R_\varepsilon > 0$ such that (3.13) holds for $\{\mu_j\}$. Moreover, up to replacing ρ, f by $\rho \mathbb{1}_{B_j}$ and $f \mathbb{1}_{B_j}$ and using a diagonal argument, we can assume that $g_j, f_j \in C_c^\infty(\mathbb{R}^m)$.

Fix $\phi \in \mathcal{S}(\partial\Omega)$ if Ω is bounded, and denote the minimizer of I_{ρ_j} by u_j . Because of Theorem 1.3 or [11, Theorem 1.5 and Remark 3.4], respectively if Ω is bounded or if $\Omega = \mathbb{R}^m$, u_j is a smooth solution to (BT) with Lorentzian mean curvature $H_j \doteq -(g_j + f_j)$ (thus, u_j minimizes I_{ρ_j} with $\rho_j = -H_j dx$). Write $w_j \doteq (1 - |Du_j|^2)^{-1/2}$. Proposition 3.11 yields $u_j \rightarrow u_\rho$ strongly in $\mathcal{Y}(\Omega)$ and $\langle \rho_j, u_j \rangle \rightarrow \langle \rho, u_\rho \rangle$. Therefore, using Proposition 3.17, to show that u_ρ weakly solves (BT) it is enough to prove that

$$\lim_{j \rightarrow \infty} \int_{\Omega} w_j Du_j \cdot D\eta \, dx = \int_{\Omega} w_\rho Du_\rho \cdot D\eta \, dx \quad \forall \eta \in \text{Lip}_c(\Omega). \quad (4.2)$$

Since $\|Du_j\|_\infty \leq 1$ and we may assume $Du_j \rightarrow Du_\rho$ a.e. on Ω , identity (4.2) follows from Vitali's convergence theorem (see [57, Theorem 3.1.9]) provided that $\{w_j\}$ is locally uniformly integrable in the following sense.

Definition 4.1. Let $\Omega \subset \mathbb{R}^m$ be an open subset. We say that a subset $\mathcal{W} \subset L^1_{\text{loc}}(\Omega)$ is *locally uniformly integrable on Ω* if, for each $\Omega' \Subset \Omega$ and $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon, \Omega')$ such that

$$A \subset \Omega' \text{ measurable, } |A| < \delta \quad \implies \quad \int_A |w| dx < \varepsilon \quad \forall w \in \mathcal{W}.$$

By de la Vallée-Poussin's Theorem (see, for instance, [57, Theorem 3.1.10]), \mathcal{W} is locally uniformly integrable if and only if there exists a compact exhaustion $\{\Omega_k\}_{k=1}^\infty$ of Ω , that is, $\Omega_k \Subset \Omega$, $\Omega_k \uparrow \Omega$, and increasing convex functions $f_k : \mathbb{R}_0^+ \rightarrow \mathbb{R}_0^+$ such that

$$\lim_{t \rightarrow \infty} \frac{f_k(t)}{t} = +\infty, \quad \sup_{w \in \mathcal{W}} \int_{\Omega_k} f_k(|w|) dx < \infty \quad \forall k.$$

The purpose of the next subsections is to obtain a local uniform integrability for $\{w_j\}$. We begin by studying the behavior of u_ρ in regions where ρ is singular.

4.2 Removable and unremovable singularities

To our knowledge, the only removable singularity theorem for the prescribed Lorentzian mean curvature equation is the one in [44]. The theorem considers maximal graphs u that are smooth and strictly spacelike in a domain $\Omega' \setminus E$, where $E \Subset \Omega'$ is compact. Under the assumption that the p -capacity of E is zero for some $p \in (1, m]$, and that

$$\int_{\Omega' \setminus E} w^{\frac{p}{p-1}} dx < \infty, \quad (4.3)$$

then u can be smoothly extended to a spacelike maximal solution on Ω' . In particular, by the known relation between Hausdorff measure and capacity (cf. [24]), compact subsets E with $\mathcal{H}_\delta^{m-p}(E) = 0$ are removable for maximal graphs satisfying (4.3). However, the proof seems not easy to extend to more general measures $\rho \neq 0$, and currently we are unable to prove an a-priori estimate yielding (4.3). Therefore, we take a different approach. Our contribution is the following result, which applies to any measure and only needs a local uniform integrability for the sequence of energy densities $\{w_j\}$.

Theorem 4.2 (Removable singularity). *Assume $\Omega \subset \mathbb{R}^m$ is either a bounded domain with $m \geq 2$ or \mathbb{R}^m with $m \geq 3$. Let*

$$\rho \in \mathcal{M}(\Omega) + L^p(\Omega), \quad p \in (1, 2_*],$$

and, if Ω is bounded, let $\phi \in S(\partial\Omega)$. Choose $\{p_1, \rho_j, u_j, w_j\}$ as in Subsection 4.1. Suppose that $E \Subset \Omega$ is a compact set with $\mathcal{H}_\delta^1(E) = 0$. Then, for every open subset $\Omega' \subset \Omega$,

$$\begin{aligned} \{w_j\} \text{ is locally uniformly} & \implies \{w_j\} \text{ is locally uniformly integrable on } \Omega', \text{ and} \\ \text{integrable on } \Omega' \setminus E & \int_{\Omega'} \frac{Du_\rho \cdot D\eta}{\sqrt{1 - |Du_\rho|^2}} = \langle \rho, \eta \rangle \quad \forall \eta \in \text{Lip}_c(\Omega'). \end{aligned}$$

In particular, if $\{w_j\}$ is locally uniformly integrable on $\Omega \setminus E$, then u_ρ weakly solves (BI).

Remark 4.3. The above requirements on E cannot be weakened to $\mathcal{H}_\delta^1(E) < \infty$. Indeed, consider the example in Corollary 1.10, and set $E = \overline{xy}$. Since $u = u_\rho$ has no light segments in $\Omega \setminus \overline{xy}$, the energies $\{w_j\}$ are locally uniformly integrable there. This can be shown by combining Lemma 3.8 with [5, Lemma 2.1], proceeding as in [5, Proof of Theorem 4.1]. However, u_ρ does not solve (BI), so E is not removable. As a related example, one can see the nice [38, Example 2].

The result is a consequence of the next lemma, which estimates the growth of w on balls centered at a given point.

Lemma 4.4. *Let $\Omega \subset \mathbb{R}^m$ be an open set, $H \in C^\infty(\Omega)$ and $u \in C^\infty(\Omega)$ be a strictly spacelike solution of*

$$-\text{div} \left(\frac{Du}{\sqrt{1 - |Du|^2}} \right) = \rho \doteq -H dx \quad \text{on } \Omega.$$

For any given $y \in \Omega$, define

$$J_y(s) \doteq \int_{B_s(y)} \frac{dx}{\sqrt{1 - |Du|^2}}, \quad 0 < s < d_\delta(y, \partial\Omega).$$

Then, for each $0 < s < t < d_\delta(y, \partial\Omega)$, it holds

$$J_y(s) \leq s \left[\frac{J_y(t)}{t} + |\rho|(B_t(y)) \right]. \quad (4.4)$$

Proof. Let $\varphi \in \text{Lip}_c(\Omega)$. Up to a translation, we may assume $u(y) = 0$. Let M be the graph of u . Recalling (2.4), we first test $\Delta_M u = Hw$ against $u\varphi$ and integrate by parts to deduce

$$\int \varphi \|\nabla u\|^2 dx_g = - \int u\varphi Hw dx_g - \int \langle u\nabla u, \nabla \varphi \rangle dx_g.$$

We set $o = y$ in (2.5) and write $\ell(x) = \ell_y(x)$. Multiplying the equation $\Delta_M \ell^2 = 2m + H\bar{D}\ell^2 \cdot \mathbf{n}$ in (2.7) by φ and integrating by parts we get

$$2m \int \varphi dx_g = -2 \int \ell \langle \nabla \ell, \nabla \varphi \rangle dx_g - \int \varphi H\bar{D}\ell^2 \cdot \mathbf{n} dx_g.$$

Noting that $\ell^2(x) = r^2(x) - u^2(x)$ and $u(y) = 0$, and using the identities

$$\ell \nabla \ell = r\nabla r - u\nabla u, \quad w^2 = 1 + \|\nabla u\|^2, \quad \bar{D}\ell^2 \cdot \mathbf{n} = 2w[r(Du, Dr) - u],$$

we infer

$$\begin{aligned} m \int \varphi w^2 dx_g &= m \int \varphi dx_g + m \int \varphi \|\nabla u\|^2 dx_g \\ &= - \int \ell \langle \nabla \ell, \nabla \varphi \rangle dx_g - \int \varphi Hw[r(Du, Dr) - u] dx_g \\ &\quad - m \int u\varphi Hw dx_g - m \int \langle u\nabla u, \nabla \varphi \rangle dx_g \\ &= - \int \langle r\nabla r + (m-1)u\nabla u, \nabla \varphi \rangle dx_g - \int \varphi Hw[r(Du, Dr) + (m-1)u] dx_g. \end{aligned} \quad (4.5)$$

First, since $\|\nabla \varphi\| \leq w|D\varphi|$, $|(Du, Dr)| \leq 1$ and $|u| \leq r$ due to $\|Du\|_\infty \leq 1$, we get

$$\begin{aligned} \langle r\nabla r + (m-1)u\nabla u, \nabla \varphi \rangle &\leq \|r\nabla r + (m-1)u\nabla u\| \|\nabla \varphi\| \\ &\leq mr \max\{\|\nabla r\|, \|\nabla u\|\} \|\nabla \varphi\| \leq mr|D\varphi|w^2. \end{aligned}$$

Setting

$$T_\rho(\varphi) \doteq -\frac{1}{m} \int \varphi Hw[r(Du, Dr) + (m-1)u] dx_g,$$

we deduce from (4.5) the following inequality:

$$\int \varphi w^2 dx_g \leq \int |D\varphi|rw^2 dx_g + T_\rho(\varphi). \quad (4.6)$$

Let $0 < s < t < d_\delta(y, \partial\Omega)$ and consider, for $\varepsilon > 0$ small enough,

$$\varphi(x) \doteq \left(\min \left\{ 1, \frac{s + \varepsilon - r(x)}{\varepsilon} \right\} \right)_+ \in \text{Lip}_c(B_t(y)) \subset \text{Lip}_c(\Omega).$$

From $|u| \leq r$, $|(Du, Dr)| \leq 1$ on the support of φ , $|\varphi| \leq 1$ and $dx_g = w^{-1}dx$, and using the coarea formula, we get

$$|T_\rho(\varphi)| \leq \int_{B_{s+\varepsilon}(y)} r|H|w dx_g = \int_0^{s+\varepsilon} \sigma \left[\int_{\partial B_\sigma(y)} |H| d\mathcal{H}_\delta^{m-1} \right] d\sigma.$$

Letting $\varepsilon \rightarrow 0$ and observing that

$$\int |D\varphi|rw^2 dx_g = \int |D\varphi|rw dx \rightarrow s \int_{\partial B_s(y)} w d\mathcal{H}_\delta^{m-1}$$

for a.e. s , from (4.6), we obtain

$$\int_{B_s(y)} w dx \leq s \int_{\partial B_s(y)} w d\mathcal{H}_\delta^{m-1} + \int_0^s \left[\sigma \int_{\partial B_\sigma(y)} |H| d\mathcal{H}_\delta^{m-1} \right] d\sigma \quad \text{for a.e. } s \in [0, t].$$

By the coarea formula, the above inequality can also be rewritten as

$$-\frac{d}{ds} \frac{J_y(s)}{s} \leq \frac{1}{s^2} \int_0^s \sigma f_y(\sigma) d\sigma \quad \text{for a.e. } s \in (0, t],$$

where

$$f_y(\sigma) = \int_{\partial B_\sigma(y)} |H| d\mathcal{H}_\delta^{m-1}.$$

Integrating on $[s, t]$ and using Tonelli's Theorem, we deduce

$$\begin{aligned} -\frac{J_y(t)}{t} + \frac{J_y(s)}{s} &\leq \int_s^t \frac{1}{\tau^2} \left\{ \int_0^\tau \sigma f_y(\sigma) d\sigma \right\} d\tau \\ &= \int_0^t \sigma f_y(\sigma) \left\{ \int_{\max\{s, \sigma\}}^t \frac{d\tau}{\tau^2} \right\} d\sigma \\ &\leq \int_0^t \sigma f_y(\sigma) \left[-\frac{1}{\tau} \right]_\sigma^t d\sigma \leq \int_0^t \sigma f_y(\sigma) \frac{1}{\sigma} d\sigma \\ &= \int_0^t f_y(\sigma) d\sigma = \int_{B_t(y)} |H| dx = |\rho|(B_t(y)), \end{aligned}$$

which proves (4.4). □

Using Lemma 4.4 and a covering argument, we shall prove Theorem 4.2:

Proof of Theorem 4.2. Write $\rho = \mu + f$ with $\mu \in \mathcal{M}(\Omega)$ and $f \in L^p(\Omega)$. Referring to Subsection 4.1, for $m = 2$ the term f does not appear, and our choice of p_1 imply that $\rho \in \mathcal{Y}(\Omega)^*$. Let μ_j, f_j be as therein, thus $\mu_j \rightharpoonup \mu$ weakly in $\mathcal{M}(\Omega)$ and $f_j \rightarrow f$ strongly in $L^p(\Omega)$. Choose $0 < R_0 \leq d_\delta(E, \partial\Omega)/20$. The relative compactness of $B_{10R_0}(E)$ implies that $\rho_j = \mu_j + f_j dx \rightharpoonup \rho$ weakly in $\mathcal{M}(B_{10R_0}(E))$, so in particular there exists a constant $C_{\mathcal{M}}$ such that

$$\|\rho_j\|_{\mathcal{M}(B_{10R_0}(E))} \leq C_{\mathcal{M}} \quad \text{for each } j \geq 1. \quad (4.7)$$

Write $\rho_j = -H_j dx$. By Proposition 3.13, there exists a constant $C(R_0)$, depending on $\phi, R_0, \|\rho\|_{\mathcal{Y}^*}$ such that

$$\int_{B_{4R_0}(E)} w_j dx \leq C(R_0). \quad (4.8)$$

For $x \in B_{R_0}(E)$ and $s \in (0, R_0]$, set

$$J_{x,j}(s) \doteq \int_{B_s(x)} w_j dx.$$

Note that (4.8) implies $J_{x,j}(R_0) \leq C(R_0)$ for all $j \geq 1$ and $x \in B_{R_0}(E)$, hence Lemma 4.4 and (4.7), (4.8) ensure that for all $x \in B_{R_0}(E)$, $j \geq 1$ and $s \in (0, R_0)$,

$$J_{x,j}(s) \leq s \left[\frac{C(R_0)}{R_0} + |\rho_j|(B_{R_0}(x)) \right] \leq C_1 s,$$

for some $C_1(R_0, C(R_0), C_{\mathcal{M}})$. By our assumption $\mathcal{H}_\delta^1(E) = 0$ and since E is compact, for given $\tau > 0$ we can cover E with finitely many balls $\{B_k\}_{k=1}^N$, $B_k = B_{r_k}(x_k)$ satisfying $r_k < R_0$ and $\sum_k r_k \leq \tau$. We can also assume that $x_k \in B_{R_0}(E)$ for each k . Therefore, for each fixed $\varepsilon > 0$ we can take $\tau > 0$ small enough to satisfy

$$\int_{\bigcup_{k=1}^N B_k} w_j dx \leq \sum_{k=1}^N J_{x_k,j}(r_k) \leq C_1 \sum_{k=1}^N r_k \leq C_1 \tau < \frac{\varepsilon}{2}.$$

Let $\Omega'' \Subset \Omega'$ be a relatively compact subset. By defining $U \doteq \bigcup_{k=1}^N B_k$, our assumption yields that $\{w_j\}$ is uniformly integrable on $\Omega'' \setminus U$. Thus, there exists $\delta > 0$ such that $A \subset \Omega'' \setminus U$ and $|A| < \delta$ imply $\int_A w_j dx < \varepsilon/2$. Then, for each subset $A \subset \Omega''$ with $|A| < \delta$,

$$\int_A w_j dx \leq \int_{A \cap U} w_j dx + \int_{A \setminus U} w_j dx < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

which means that $\{w_j\}$ is uniformly integrable on Ω'' . In particular, (4.2) holds for every fixed $\eta \in \text{Lip}_c(\Omega')$ by Vitali's Theorem. \square

We next consider singularities which cannot be removed. While the examples in Section 6 show that solutions to (BI) may possess light segments when $\rho \in L^q(\Omega)$ and $q < m - 1$, we shall now prove that such solutions exhibit, in a sense, a ‘‘borderline’’ behavior.

Theorem 4.5. *Let $\Omega \subset \mathbb{R}^m$ be either a bounded domain with $m \geq 2$ and $\phi \in S(\partial\Omega)$, or $\Omega = \mathbb{R}^m$ with $m \geq 3$. Let $\rho \in \mathcal{Y}(\Omega)^*$, and assume that the minimizer u_ρ has a light segment $\overline{xy} \subset \Omega$ with $u_\rho(y) - u_\rho(x) = |y - x|$. Then, for each $\alpha > 0$, u_ρ also minimizes the functional I_{ρ_α} with*

$$\rho_\alpha = \rho + \alpha(\delta_y - \delta_x),$$

but it does not solve (BI) weakly for ρ_α .

Proof. For simplicity, in this proof we remove the subscript ρ and denote by $I \doteq I_\rho$ and $u \doteq u_\rho$. We also write $I_\alpha \doteq I_{\rho_\alpha}$ and denote its minimizer by u_α . We argue by contradiction and assume that $u_\alpha \neq u$ for some $\alpha > 0$. By uniqueness of the minimizer, we infer

$$I(u) = I_\alpha(u) + \alpha[u(y) - u(x)] > I_\alpha(u_\alpha) + \alpha[u(y) - u(x)],$$

which implies

$$u(y) - u(x) < \frac{I(u) - I_\alpha(u_\alpha)}{\alpha}.$$

Similarly,

$$I_\alpha(u_\alpha) = I(u_\alpha) - \alpha[u_\alpha(y) - u_\alpha(x)] > I(u) - \alpha[u_\alpha(y) - u_\alpha(x)],$$

thus,

$$u_\alpha(y) - u_\alpha(x) > \frac{I(u) - I_\alpha(u_\alpha)}{\alpha}.$$

Therefore, $u_\alpha(y) - u_\alpha(x) > u(y) - u(x) = |y - x|$, contradicting the fact that $u_\alpha \in \mathcal{Y}_\phi(\Omega)$.

We have therefore proved that $u = u_\alpha$ for each $\alpha > 0$. By Theorem 1.3, pick a strictly spacelike extension $\bar{\phi}$ of ϕ , so that, in particular, $|y - x| - \bar{\phi}(y) + \bar{\phi}(x) > 0$. Since u minimizes I , we see from Proposition 3.13 that

$$\begin{aligned} \int_{\Omega} \frac{Du \cdot (Du - D\bar{\phi})}{\sqrt{1 - |Du|^2}} dx &\leq \langle \rho, u - \bar{\phi} \rangle = \langle \rho_\alpha, u - \bar{\phi} \rangle - \alpha \langle (\delta_y - \delta_x), u - \bar{\phi} \rangle \\ &= \langle \rho_\alpha, u - \bar{\phi} \rangle - \alpha [|y - x| - \bar{\phi}(y) + \bar{\phi}(x)] \\ &< \langle \rho_\alpha, u - \bar{\phi} \rangle. \end{aligned}$$

Therefore, due to Proposition 3.17, u does not solve (BI) for ρ_α . \square

Remark 4.6. When Ω is bounded, the condition $\phi \in S(\partial\Omega)$ only serves to guarantee that u_ρ does not solve (BI) with charge ρ_α . In other words, the fact that u_ρ minimizes I_{ρ_α} only requires $\mathcal{Y}_\phi(\Omega) \neq \emptyset$.

4.3 Local second fundamental form estimate

We first observe that $W_{\text{loc}}^{2,q}$ estimates, for $q \geq 1$, are not to be expected for general ρ . An easy counterexample can be produced building on the explicit expression of u_ρ for $\rho = \omega_{m-1} \delta_0$ in the unit ball with zero boundary values:

$$u_\rho(x) = \int_{|x|}^1 \frac{dt}{\sqrt{t^{2m-2} + 1}} \quad \text{on } B_1(0) \subset \mathbb{R}^m. \quad (4.9)$$

Fix $R \in (0, 1)$ and let $s \in (0, \|u\|_\infty)$, be the constant value of u on $\partial B_R(0)$. Then, the function $u_s \doteq \min\{u, s\}$ solves

$$\begin{cases} \operatorname{div} \left(\frac{Du_s}{\sqrt{1 - |Du_s|^2}} \right) = -R^{1-m} \mathcal{H}_\delta^{m-1} \llcorner \partial B_R(0) & \text{on } B_1(0), \\ u_s = 0 & \text{on } \partial B_1(0). \end{cases}$$

Clearly, $u_s \notin W_{\text{loc}}^{2,q}$ for any $q \geq 1$. Note however that, by explicit computation, $u \in W^{2,q}(B_1(0))$ for each $q \in [1, m)$.

It is reasonable to guess that $u_\rho \in W_{\text{loc}}^{2,2}(\Omega)$ provided that $\rho \in L^2(\Omega)$. Indeed, a stronger estimate holds. First, observe that integrating (2.3) on a domain Ω' we get

$$\int_{M'} \|\Pi\|^2 dx_g = \int_{\Omega'} w \left\{ |D^2u|^2 + 2w^2 \left| D^2u(Du, \cdot) \right|^2 + w^4 [D^2u(Du, Du)]^2 \right\} dx, \quad (4.10)$$

where M' denotes the graph of $u = u_\rho$ over Ω' . In this subsection, we prove local second fundamental form estimates for the graph of u_ρ in regions Ω' where $\rho \in L^2$. We first treat the case of smooth $\rho = -H dx$ with $H \in C^\infty(\bar{\Omega})$, and then argue by approximation. Let u be the smooth solution to (BI), with subscript ρ suppressed. The search for estimates on $\|\Pi\|^2$ is natural in view of the Jacobi equation

$$\Delta_M w = - \left\langle \nabla H, \partial_0^\parallel \right\rangle + \|\Pi\|^2 w, \quad (4.11)$$

see [4, p.519], which will be our starting point. First, we need to decompose $\|\Pi\|^2$ into convenient pieces. Recall that, by (2.4), $\|\Pi\|^2 = w^{-2} \|\nabla^2 u\|^2$. We rewrite $\|\nabla^2 u\|^2$ as follows:

Lemma 4.7. Assume $du(x) \neq 0$ at $x \in M$ and set $v \doteq \nabla u / \|\nabla u\|$ in a neighborhood of x . Denote by A the traceless second fundamental form of the level set $\{u = u(x)\}$ in the direction $-v$ and write $u_{vv} \doteq \nabla^2 u(v, v)$. Then

$$\begin{aligned} \|\nabla^2 u\|^2 &= \|\nabla u\|^2 \|A\|^2 + \frac{1}{m-1} (H^2 w^2 - 2Hw u_{vv}) \\ &\quad + \frac{m}{m-1} \|\nabla \|\nabla u\|\|^2 + \frac{m-2}{m-1} \|\nabla^\top \|\nabla u\|\|^2, \end{aligned} \quad (4.12)$$

where ∇^\top stands for the component of ∇ tangent to the level set $\{u = u(x)\}$.

Proof. Consider an orthonormal frame $\{v, e_\alpha\}$, $2 \leq \alpha \leq m$ on M . We denote by u_{ij} the components of $\nabla^2 u$ in the above frame. Then,

$$\langle \nabla \|\nabla u\|, e_\alpha \rangle = u_{\alpha v}, \quad \langle \nabla \|\nabla u\|, v \rangle = u_{vv},$$

thus

$$\|\nabla^2 u\|^2 = \sum_{\alpha, \beta=2}^m u_{\alpha\beta}^2 + 2\|\nabla^\top \|\nabla u\|\|^2 + u_{vv}^2. \quad (4.13)$$

Next, it follows from the definition of A that

$$\|\nabla u\| A_{\alpha\beta} = u_{\alpha\beta} - \frac{\sum_{\gamma=2}^m u_{\gamma\gamma}}{m-1} \delta_{\alpha\beta}.$$

Splitting the norm of the matrix $[u_{\alpha\beta}]$ into its trace and traceless parts, and recalling (2.4), we get

$$\begin{aligned} \sum_{\alpha, \beta=2}^m u_{\alpha\beta}^2 &= \|\nabla u\|^2 \|A\|^2 + \frac{1}{m-1} \left(\sum_{\alpha=2}^m u_{\alpha\alpha} \right)^2 = \|\nabla u\|^2 \|A\|^2 + \frac{(\Delta_M u - u_{vv})^2}{m-1} \\ &= \|\nabla u\|^2 \|A\|^2 + \frac{1}{m-1} (H^2 w^2 - 2Hw u_{vv} + u_{vv}^2). \end{aligned}$$

Inserting this into (4.13) and noting that $\|\nabla \|\nabla u\|\|^2 = \|\nabla^\top \|\nabla u\|\|^2 + u_{vv}^2$, we obtain (4.12). \square

Remark 4.8. When $H = 0$, we obtain the classical refined Kato inequality for harmonic functions

$$\|\nabla^2 u\|^2 \geq \frac{m}{m-1} \|\nabla \|\nabla u\|\|^2.$$

It is convenient to rewrite the equations in terms of the hyperbolic angle

$$\beta \doteq \operatorname{arcch} w = \log \left(w + \sqrt{w^2 - 1} \right).$$

Note that $w \mapsto \beta$ is a diffeomorphism on $\{du \neq 0\}$. The identities

$$w = \operatorname{ch} \beta, \quad \|\nabla u\| = \sqrt{w^2 - 1} = \operatorname{sh} \beta, \quad u_{vv} = \langle \nabla \|\nabla u\|, v \rangle = \operatorname{ch} \beta \langle \nabla \beta, v \rangle,$$

(4.12) and the fact that $\Pi = w^{-1} \nabla^2 u = 0$ a.e. on the set $\{du = 0\}$ due to Stampacchia's theorem allow us to rewrite $\|\Pi\|^2 = w^{-2} \|\nabla^2 u\|^2$ as

$$\|\Pi\|^2 = \left[\frac{\operatorname{sh}^2 \beta}{\operatorname{ch}^2 \beta} \|A\|^2 + \frac{H^2}{m-1} - \frac{2H \langle \nabla \beta, v \rangle}{m-1} + \frac{m \|\nabla \beta\|^2}{m-1} + \frac{m-2}{m-1} \|\nabla^\top \beta\|^2 \right] \cdot \mathbb{1}_{\{du \neq 0\}} \quad (4.14)$$

a.e. on Ω . We therefore deduce that, for some constant $C = C(m) > 0$,

$$\|\Pi\|^2 \leq C(m) \left[\frac{\text{sh}^2 \beta}{\text{ch}^2 \beta} \|A\|^2 + \|\nabla \beta\|^2 + H^2 \right] \cdot \mathbb{1}_{\{du \neq 0\}} \quad (4.15)$$

and that, for every $M' \Subset M$,

$$\int_{M'} \|\Pi\|^2 dx_g \leq C \iff \int_{M' \cap \{du \neq 0\}} \left[\frac{\text{sh}^2 \beta}{\text{ch}^2 \beta} \|A\|^2 + \|\nabla \beta\|^2 + H^2 \right] dx_g \leq C',$$

where C and C' might be different, but with the same qualitative dependence on the data of our problem (BT). We point out that in (4.14) the coefficient of $\|\nabla \beta\|^2$ is strictly larger than 1 and this plays a crucial role in our analysis. Indeed, it guarantees that the coefficient of $\|\nabla \beta\|^2$ in inequality (4.20) below is positive for sufficiently small $\varepsilon > 0$, allowing us to bound the integral of $\|\nabla \beta\|^2$ from above.

We next rewrite the Jacobi equation in a way that is more suited to our purposes. We begin with the following

Lemma 4.9. *Define*

$$Y \doteq \frac{\nabla w - H \nabla u}{w} \quad \text{on } M. \quad (4.16)$$

Then,

$$\text{div}_M Y = \|\Pi\|^2 - H^2 - \left\langle Y, \frac{\nabla w}{w} \right\rangle. \quad (4.17)$$

Proof. We first claim that

$$\Delta_M w = \left(\|\Pi\|^2 - H^2 \right) w + \text{div}_M (H \nabla u) \quad \text{on } M. \quad (4.18)$$

Indeed, the identity follows from (4.11) once we recall that $\partial_0^\parallel = -\nabla u$ and that

$$\langle \nabla H, \nabla u \rangle = \text{div}_M (H \nabla u) - H \Delta_M u = \text{div}_M (H \nabla u) - H^2 w.$$

From (4.18) we therefore obtain

$$\Delta_M \log w = \|\Pi\|^2 - H^2 - \frac{\|\nabla w\|^2}{w^2} + \text{div}_M \left(\frac{H \nabla u}{w} \right) + H \left\langle \frac{\nabla u}{w}, \frac{\nabla w}{w} \right\rangle,$$

which is (4.17) up to rearranging terms. \square

By (4.14), $\nabla u = \text{sh } \beta v$ and $\nabla w/w = \text{sh } \beta \nabla \beta / \text{ch } \beta$, we rewrite the vector field Y as

$$Y = \frac{\text{sh } \beta}{\text{ch } \beta} (\nabla \beta - H v) \quad (4.19)$$

and $\text{div}_M Y$ as

$$\begin{aligned} \text{div}_M Y = & \left[\frac{\text{sh}^2 \beta}{\text{ch}^2 \beta} \|A\|^2 - \frac{m-2}{m-1} H^2 - \frac{2}{m-1} H \langle \nabla \beta, v \rangle \right. \\ & \left. + \frac{m}{m-1} \|\nabla \beta\|^2 + \frac{m-2}{m-1} \|\nabla^\top \beta\|^2 - \frac{\text{sh } \beta}{\text{ch } \beta} \langle Y, \nabla \beta \rangle \right] \cdot \mathbb{1}_{\{du \neq 0\}} \end{aligned}$$

a.e. on Ω . By (4.19) with $0 \leq \text{sh } \beta / \text{ch } \beta \leq 1$ and Cauchy-Schwarz's and Young's inequalities, we have

$$\begin{aligned} \left| \frac{\text{sh } \beta}{\text{ch } \beta} \langle Y, \nabla \beta \rangle \right| &\leq \|\nabla \beta - H\nu\| \|\nabla \beta\| \leq \|\nabla \beta\|^2 + |H| \|\nabla \beta\| \leq (1 + \varepsilon) \|\nabla \beta\|^2 + \frac{4}{\varepsilon} H^2, \\ |H \langle \nabla \beta, \nu \rangle| &\leq |H| \|\nabla \beta\| \leq \frac{1}{2\varepsilon} |H|^2 + \frac{\varepsilon}{2} \|\nabla \beta\|^2. \end{aligned}$$

Thus there exist constants $C_m, C_{m,\varepsilon}$ such that

$$\text{div}_M Y \geq \left[\frac{\text{sh}^2 \beta}{\text{ch}^2 \beta} \|A\|^2 - C_{m,\varepsilon} H^2 + \left\{ \frac{1}{m-1} - \frac{C_{m,\varepsilon}}{2} \right\} \|\nabla \beta\|^2 \right] \cdot \mathbb{1}_{\{du \neq 0\}} \quad (4.20)$$

a.e. on Ω . We notice from the smoothness of Y, H and from estimate (4.20) that the function $\|\nabla \beta\|^2 \mathbb{1}_{\{du \neq 0\}}$ is integrable on the graph of u .

Proposition 4.10. *There exists a constant $C = C_m > 0$ such that, for every $\varphi \in \text{Lip}_c(\Omega)$,*

$$\int_M \varphi^2 \|\Pi\|^2 dx_g \leq C_m \left(\int_M \|\nabla \varphi\|^2 dx_g + \int_M \varphi^2 H^2 dx_g \right). \quad (4.21)$$

Proof. We test (4.20) with the function φ^2 to obtain

$$\begin{aligned} &\int_{\{du \neq 0\}} \left[\frac{\text{sh}^2 \beta}{\text{ch}^2 \beta} \|A\|^2 + \left\{ \frac{1}{m-1} - \frac{C_{m,\varepsilon}}{2} \right\} \|\nabla \beta\|^2 \right] \varphi^2 dx_g \\ &\leq \int \varphi^2 \text{div}_M Y dx_g + C_{m,\varepsilon} \int H^2 \varphi^2 dx_g \\ &= -2 \int \varphi \langle \nabla \varphi, Y \rangle dx_g + C_{m,\varepsilon} \int H^2 \varphi^2 dx_g. \end{aligned} \quad (4.22)$$

Since, from its very definition, $Y = 0$ on $\{du = 0\}$, and since $0 \leq \text{sh } \beta / \text{ch } \beta \leq 1$, using Cauchy-Schwarz's and Young's inequalities we see from (4.19) that

$$\begin{aligned} |\varphi \langle \nabla \varphi, Y \rangle| &\leq \{ |\varphi \langle \nabla \varphi, \nabla \beta \rangle| + |\varphi H \langle \nabla \varphi, \nu \rangle| \} \mathbb{1}_{\{du \neq 0\}} \\ &\leq \frac{1}{2\varepsilon} \|\nabla \varphi\|^2 + \frac{\varepsilon}{2} \varphi^2 \|\nabla \beta\|^2 \mathbb{1}_{\{du \neq 0\}} + \frac{1}{2} \varphi^2 H^2 + \frac{1}{2} \|\nabla \varphi\|^2. \end{aligned}$$

Recalling that $\|\nabla \beta\|^2 \mathbb{1}_{\{du \neq 0\}}$ is integrable, it follows from (4.22) that

$$\begin{aligned} &\int_{\{du \neq 0\}} \left[\frac{\text{sh}^2 \beta}{\text{ch}^2 \beta} \|A\|^2 + \left\{ \frac{1}{m-1} - \frac{C_{m,\varepsilon}}{2} - \varepsilon \right\} \|\nabla \beta\|^2 \right] \varphi^2 dx_g \\ &\leq C_{m,\varepsilon} \int H^2 \varphi^2 dx_g + C_\varepsilon \int \|\nabla \varphi\|^2 dx_g. \end{aligned}$$

Choosing a small $\varepsilon > 0$ and taking (4.15) into account, we readily deduce (4.21) and complete the proof. \square

Using (4.10), (4.21) and the approximation in Subsection 4.1, we prove the following result. We recall that, for $m = 2$, the space $L^p(\Omega)$ below is meant to be empty.

Corollary 4.11. *Let $\Omega \subset \mathbb{R}^m$ be a domain. Assume that either*

- $m \geq 2$, Ω is bounded, $\mathcal{F} \subset S(\partial\Omega)$ is a compact subset, and $\phi \in \mathcal{F}$;

- $m \geq 3$, $\Omega = \mathbb{R}^m$.

Fix $\mathcal{I}_1, \mathcal{I}_2 \in \mathbb{R}^+$, $\Omega' \Subset \Omega$ and, for $\varepsilon > 0$, define $\Omega'_\varepsilon \doteq \{x \in \Omega' : d_\delta(x, \partial\Omega') > \varepsilon\}$. Let $p \in (1, 2_*]$. Then, there exists a constant

$$C = \begin{cases} C(\Omega, \mathcal{F}, m, \text{diam}_\delta(\Omega), p, \mathcal{I}_1, \mathcal{I}_2, \varepsilon, d_\delta(\Omega', \partial\Omega)) & \text{if } \Omega \text{ is bounded,} \\ C(m, p, \mathcal{I}_1, \mathcal{I}_2, \varepsilon, |\Omega'|_\delta) & \text{if } \Omega = \mathbb{R}^m \end{cases} \quad (4.23)$$

such that for each $\rho \in \mathcal{M}(\Omega) + L^p(\Omega)$ satisfying

$$\|\rho\|_{\mathcal{M}(\Omega) + L^p(\Omega)} \leq \mathcal{I}_1, \quad \|\rho\|_{L^2(\Omega')} \leq \mathcal{I}_2,$$

it holds

$$\int_{\Omega'_\varepsilon} \left\{ w_\rho |D^2 u_\rho|^2 + w_\rho^3 |D^2 u_\rho(Du_\rho, \cdot)|^2 + w_\rho^5 [D^2 u_\rho(Du_\rho, Du_\rho)]^2 \right\} dx \leq C. \quad (4.24)$$

In particular,

$$\begin{aligned} \int_{\Omega'_\varepsilon} \frac{1}{w_\rho} \left\{ |D \log w_\rho|^2 + |Dw_\rho \cdot Du_\rho|^2 \right\} dx &\leq C, \\ \int_{\Omega'_\varepsilon} \left\{ |D \log w_\rho| + |Dw_\rho \cdot Du_\rho| \right\} dx &\leq C. \end{aligned} \quad (4.25)$$

Proof. We choose p_1 as in (4.1) to guarantee that $\rho \in \mathcal{Y}(\Omega)^*$, and referring to Subsection 4.1, we approximate ρ through convolution obtaining $\{\rho_j\}$ with $\rho_j = -H_j dx$ and $H_j \in C^\infty(\bar{\Omega})$ (resp. $H_j \in C_c^\infty(\mathbb{R}^m)$). Let u_j be the smooth solution to (BI) with source ρ_j , and write $w_j \doteq (1 - |Du_j|^2)^{-1/2}$. For the ease of notation, in the next arguments we write u, w instead of u_ρ, w_ρ . Proposition 3.11 yields $u_j \rightarrow u$ strongly in $W^{1,q}(\Omega)$, for each $q \in [1, \infty)$ if Ω is bounded and each $q \in [2^*, \infty)$ if $\Omega = \mathbb{R}^m$. We fix $\varphi \in C_c^1(\Omega')$ so that $\varphi \equiv 1$ on Ω'_ε and $|D\varphi(x)| \leq 2/\varepsilon$ for each $x \in \Omega$. From

$$\|\nabla\varphi\|^2 = |D\varphi|^2 + w_j^2 (Du_j \cdot D\varphi)^2 \leq (1 + w_j^2 |Du_j|^2) |D\varphi|^2 = w_j^2 |D\varphi|^2,$$

(4.10) and Proposition 4.10 with u_j , it follows that

$$\begin{aligned} \int_{\Omega} \varphi^2 w_j \left\{ |D^2 u_j|^2 + 2w_j^2 |D^2 u_j(Du_j, \cdot)|^2 + w_j^4 [D^2 u_j(Du_j, Du_j)]^2 \right\} dx \\ \leq C_m \int_{\Omega} \left\{ w_j |D\varphi|^2 + \varphi^2 \rho_j^2 w_j^{-1} \right\} dx. \end{aligned}$$

Combining this estimate with $w_j \geq 1$, the properties of φ and Proposition 3.13, we find a constant C as in (4.23) such that

$$\sup_{j \geq 1} \int_{\Omega'_\varepsilon} w_j \left\{ |D^2 u_j|^2 + 2w_j^2 |D^2 u_j(Du_j, \cdot)|^2 + w_j^4 [D^2 u_j(Du_j, Du_j)]^2 \right\} dx \leq C. \quad (4.26)$$

In particular, $\{u_j\}$ is bounded in $W^{2,2}(\Omega'_\varepsilon)$ and we may suppose that $u_j \rightharpoonup u$ weakly in $W^{2,2}(\Omega'_\varepsilon)$. From the $W^{1,q}$ convergence we may also suppose that $u_j(x) \rightarrow u(x)$, $Du_j(x) \rightarrow Du(x)$ and $w_j(x) \rightarrow w(x)$ for a.e. $x \in \Omega'_\varepsilon$.

Fix $N > 1$ and set

$$w_{N,j}(x) \doteq \min\{w_j(x), N\}, \quad w_N(x) \doteq \min\{w(x), N\}.$$

By (4.26), we have

$$\sup_{j \geq 1, N > 1} \int_{\Omega'_\varepsilon} w_{N,j} \left\{ \left| D^2 u_j \right|^2 + 2w_{N,j}^2 \left| D^2 u_j (Du_j, \cdot) \right|^2 + w_{N,j}^4 \left[D^2 u_j (Du_j, Du_j) \right]^2 \right\} dx \leq C. \quad (4.27)$$

From $w_j \rightarrow w$, $Du_j \rightarrow Du$ a.e. on Ω , $w_{N,j} \leq N$ and $|Du_j| \leq 1$, it follows that for every $1 \leq i_1, i_2 \leq m$ and $q \in [1, \infty)$,

$$\begin{aligned} & \left\| w_{N,j} - w_N \right\|_{L^q(\Omega'_\varepsilon)} + \left\| w_{N,j}^{3/2} (u_j)_{i_1} - w_N^{3/2} u_{i_1} \right\|_{L^q(\Omega'_\varepsilon)} \\ & \quad + \left\| w_{N,j}^{5/2} (u_j)_{i_1} (u_j)_{i_2} - w_N^{5/2} u_{i_1} u_{i_2} \right\|_{L^q(\Omega'_\varepsilon)} \rightarrow 0. \end{aligned}$$

Since $u_j \rightarrow u$ weakly in $W^{2,2}(\Omega'_\varepsilon)$, for any $\psi \in L^\infty(\Omega'_\varepsilon)$, we see

$$\begin{aligned} & \int_{\Omega'_\varepsilon} w_{N,j}^{1/2} (u_j)_{i_1, i_2} \psi \, dx \rightarrow \int_{\Omega'_\varepsilon} w_N^{1/2} u_{i_1, i_2} \psi \, dx, \\ & \int_{\Omega'_\varepsilon} w_{N,j}^{3/2} (u_j)_{i_1, i_2} (u_j)_{i_3} \psi \, dx \rightarrow \int_{\Omega'_\varepsilon} w_N^{3/2} u_{i_1, i_2} u_{i_3} \psi \, dx, \\ & \int_{\Omega'_\varepsilon} w_{N,j}^{5/2} (u_j)_{i_1, i_2} (u_j)_{i_3} (u_j)_{i_4} \psi \, dx \rightarrow \int_{\Omega'_\varepsilon} w_N^{5/2} u_{i_1, i_2} u_{i_3} u_{i_4} \psi \, dx. \end{aligned}$$

Thus, the density of $L^\infty(\Omega'_\varepsilon)$ in $L^2(\Omega'_\varepsilon)$ yields

$$\begin{aligned} & w_{N,j}^{1/2} D^2 u_j \rightharpoonup w_N^{1/2} D^2 u, \quad w_{N,j}^{3/2} D^2 u_j (Du_j, \cdot) \rightharpoonup w_N^{3/2} D^2 u (Du, \cdot), \\ & w_{N,j}^{5/2} D^2 u_j (Du_j, Du_j) \rightharpoonup w_N^{5/2} D^2 u (Du, Du) \end{aligned}$$

weakly in $L^2(\Omega'_\varepsilon)$. Hence, by (4.27) and the lower semicontinuity of the norm, we obtain

$$\sup_{N > 1} \int_{\Omega'_\varepsilon} w_N \left\{ \left| D^2 u \right|^2 + 2w_N^2 \left| D^2 u (Du, \cdot) \right|^2 + w_N^4 \left[D^2 u (Du, Du) \right]^2 \right\} dx \leq C.$$

By letting $N \rightarrow \infty$ and using the monotone convergence theorem, (4.24) holds.

The first in (4.25) readily follows from

$$|D \log w|^2 = w^4 \left| D^2 u (Du, \cdot) \right|^2, \quad Dw \cdot Du = w^3 D^2 u (Du, Du)$$

a.e. on Ω . On the other hand, the second in (4.25) is derived from Hölder's inequality and Proposition 3.13:

$$\begin{aligned} & \int_{\Omega'_\varepsilon} \{|D \log w| + |Dw \cdot Du|\} \, dx \\ & \leq \left(\int_{\Omega'_\varepsilon} w \, dx \right)^{1/2} \left(\int_{\Omega'_\varepsilon} \frac{1}{w} \{|D \log w|^2 + |Dw \cdot Du|^2\} \, dx \right)^{1/2} \leq C. \end{aligned}$$

This concludes the proof. \square

4.4 Higher regularity (log-improvements)

The main purpose of this subsection is to obtain a local higher integrability for the energy density of u_ρ of the type

$$\int_{\Omega''} w_\rho \log(1 + w_\rho) dx \leq C,$$

where $\Omega'' \Subset \Omega' \Subset \Omega$ and we assume $\rho \in L^2(\Omega')$. We shall split the discussion according to whether $m = 2$ or not. Indeed, in dimension 2 we can rely on the trace theorem and Corollary 4.11 to ensure higher integrability with no additional assumption. As we shall see, the situation is more intricate in dimension $m \geq 3$. For $m = 2$, we prove:

Theorem 4.12. *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain, let $\mathcal{F} \subset S(\partial\Omega)$ be compact and $\phi \in \mathcal{F}$. Fix $\Omega' \Subset \Omega$ and for $\varepsilon > 0$, define $\Omega'_\varepsilon \doteq \{x \in \Omega' : d_\delta(x, \partial\Omega') > \varepsilon\}$. Let $\rho \in \mathcal{M}(\Omega)$ satisfy*

$$\|\rho\|_{\mathcal{M}(\Omega)} \leq \mathcal{I}_1, \quad \|\rho\|_{L^2(\Omega')} \leq \mathcal{I}_2$$

for some constants $\mathcal{I}_1, \mathcal{I}_2$. Then, there exists $C = C(\Omega, \mathcal{F}, \text{diam}_\delta(\Omega), \mathcal{I}_1, \mathcal{I}_2, \varepsilon, d_\delta(\Omega', \partial\Omega))$ such that the energy density $w_\rho = (1 - |Du_\rho|^2)^{-1/2}$ satisfies

$$\int_{\Omega'_\varepsilon} w_\rho \log(1 + w_\rho) dx \leq C. \quad (4.28)$$

In particular, u_ρ weakly solves (BI) on Ω' .

Proof. We fix p_1 as in (4.1) and, as in the proof of Corollary 4.11, we find $\rho_j \doteq -H_j dx$ satisfying $H_j \in C^\infty(\bar{\Omega})$ and

$$\sup_{j \geq 1} \|\rho_j\|_{\mathcal{M}(\Omega)} \leq \mathcal{I}_1, \quad \sup_{j \geq 1} \|\rho_j\|_{L^2(\Omega')} \leq \mathcal{I}_2.$$

Denote by u_j the minimizer of I_{ρ_j} and by $w_j = (1 - |Du_j|^2)^{-1/2}$. We recall that, for each Radon measure μ on \mathbb{R}^m , the following trace inequality holds for some constant $C = C(m)$, see [43, Corollary 1.1.2]:

$$\int \varphi d\mu \leq C \left[\sup_{x \in \mathbb{R}^m, r > 0} \frac{\mu(B_r(x))}{r^{m-1}} \right] \int |D\varphi| dx \quad \forall \varphi \in C_c^\infty(\mathbb{R}^m). \quad (4.29)$$

By Proposition 3.13,

$$\int_{\Omega'} w_j dx \leq C_1(\Omega, \mathcal{F}, \text{diam}_\delta(\Omega), \mathcal{I}_1, d_\delta(\Omega', \partial\Omega)),$$

while, by Corollary 4.11,

$$\int_{\Omega'_{\varepsilon/2}} |D \log w_j| dx \leq C_2(\Omega, \mathcal{F}, \text{diam}_\delta(\Omega), \mathcal{I}_1, \mathcal{I}_2, \varepsilon, d_\delta(\Omega', \partial\Omega)).$$

Hereafter, C_j will denote a constant depending on the same data as C_2 . We consider the measure $\mu \doteq w_j dx \llcorner \Omega'_\varepsilon$ and set $\varphi \doteq \psi \log(1 + w_j)$ for a cut-off function ψ satisfying $\psi \equiv 1$ on $\Omega'_{3\varepsilon/4}$ and $\text{supp } \psi \subset \Omega'_{\varepsilon/2}$. By (4.4), for each $x \in \Omega'_{\varepsilon/4}$ and $r < \varepsilon/8$,

$$\mu(B_r(x)) = \int_{B_r(x) \cap \Omega'_\varepsilon} w_j dx \leq r \left[\frac{8}{\varepsilon} \int_{B_{\varepsilon/8}(x)} w dx + C(\mathcal{I}_1) \right] \leq C_3 r.$$

On the other hand, if $x \in \Omega'_{\varepsilon/4}$ and $r \geq \varepsilon/8$, then

$$\mu(B_r(x)) \leq \int_{\Omega'} w_j dx \leq C_1 \leq C_4 r.$$

When $x \notin \Omega'_{\varepsilon/4}$ and $r < \varepsilon/8$, we clearly have $\mu(B_r(x)) = 0$. Hence, $\mu(B_r(x)) \leq C_5 r$ for each $x \in \mathbb{R}^2$, $r > 0$. Our dimensional restriction, (4.29) and (4.25) imply

$$\begin{aligned} \int_{\Omega'_\varepsilon} w_j \log(1 + w_j) dx &\leq C_6 \int_{\mathbb{R}^2} \left| D(\psi \log(1 + w_j)) \right| dx \\ &\leq C_6 \int_{\Omega'_{\varepsilon/2}} \left[\log(1 + w_j) |D\psi| + \psi |D \log w_j| \right] dx \leq C_7. \end{aligned}$$

Now (4.28) follows by letting $j \rightarrow \infty$ and using Fatou's lemma. Finally, the fact that u_ρ weakly solves (BT) on Ω' follows from (4.28) and the discussion in Subsection 4.1. \square

We remark that Theorem 4.12 cannot be extended to dimension $m \geq 4$. Otherwise, the entire proof of Theorem 1.11 in Subsection 5.2 would work for dimension $m \geq 4$, which contradicts the example in Remark 1.15 (cf. Theorem 4.5). In dimension $m = 3$, proving that $\{w_j\}$ is locally uniformly integrable on a subdomain where ρ is of class L^2 is an open problem, which seems challenging.

Nevertheless, when $\rho \in L^2(\Omega')$ we can prove a higher integrability for w_ρ on Ω'_ε which holds in any dimension, provided we further assume that the Lorentzian ball $L_R^\rho(\Omega'_\varepsilon)$ defined in (2.6) satisfies

$$L_R^\rho(\Omega'_\varepsilon) \Subset \Omega' \quad \text{for some } R > 0. \quad (4.30)$$

The requirement assures that Lorentzian balls of radius R centered at every point $o \in \Omega'_\varepsilon$ are relatively compact in Ω' , and so it enables to localize by means of cut-off functions depending on the Lorentzian distance ℓ_o from o . Such cut-off functions are better behaved than those based on the Euclidean distance r_o , because $\Delta_M \ell_o$ can be controlled more efficiently than $\Delta_M r_o$. More precisely, if $u \in \mathcal{Y}_\phi(\Omega)$ and $\phi \in S(\partial\Omega)$, then from (2.7) we get

$$\|\nabla \ell_o^2\|^2 \leq 4\ell_o^2 + 16w^2 |x - o|, \quad |\Delta_M \ell_o^2| \leq 2m + 4wH |x - o|. \quad (4.31)$$

On the other hand, computing the gradient and Laplacian of r_o and using (2.2), we get

$$|\Delta_M r_o^2| \leq C(1 + w^2 + |H|w).$$

As we will discuss in Remark 4.14 below, the advantage of using ℓ_o instead of r_o is exactly the absence of the addendum w^2 in the upper bound (4.31) for $|\Delta_M \ell_o^2|$. The necessity to differentiate the cut-off function to second order will be apparent in the proof of the next theorem, the second main result of this subsection. Under assumption (4.30), inequality (4.35) below provides both a log-improvement for the energy density and allows to improve, by a factor of $\log^q w$, the conclusion (4.24) in Corollary 4.11.

To introduce the statement, we first recall that, by Proposition 3.13, given $\Omega' \Subset \Omega$ and \mathcal{I}_1 such that $\rho = -H dx$ and $\|\rho\|_{\mathcal{M}(\Omega)} \leq \mathcal{I}_1$, the identity $dx_g = w^{-1} dx$ implies

$$\int_M |H|w dx_g \leq \mathcal{I}_1, \quad \int_{M'} w^2 dx_g \leq C,$$

where M' is the graph over Ω' and C is a constant depending on the data of our problem.

Theorem 4.13. Let $\Omega \subset \mathbb{R}^m$ be either

- a bounded domain, $m \geq 2$, $\mathcal{F} \subset S(\partial\Omega)$ is compact and $\phi \in \mathcal{F}$, or
- $\Omega = \mathbb{R}^m$ and $m \geq 3$.

Let

$$H \in C^\infty(\bar{\Omega}) \text{ if } \Omega \text{ is bounded,} \quad H \in C_c^\infty(\mathbb{R}^m) \text{ if } \Omega = \mathbb{R}^m,$$

define the measure $\rho = -H dx$, and let $u \in \mathcal{Y}_\phi(\Omega)$ be the minimizer of I_ρ . Assume that

$$\|u\|_{L^\infty(\Omega)} \leq I_0, \quad \|\rho\|_{\mathcal{M}(\Omega)+L^p(\Omega)} \leq I_1, \quad (4.32)$$

for some constants $I_0, I_1 > 0$ and $p \in (1, 2_*]$. Suppose that there exist two open subsets $\Omega'' \Subset \Omega' \Subset \Omega$ such that

$$\int_{\Omega'} H^2 \frac{(1 + \log w)^{q_0+2}}{w} dx \leq I_{2,q_0}, \quad (4.33)$$

for some $q_0 \in \mathbb{N} \cup \{0\}$ and $I_{2,q_0} \in \mathbb{R}^+$, and that for some $R > 0$ it holds

$$L_R^\rho(\Omega'') \Subset \Omega'.$$

Then, there exists a constant

$$C = \begin{cases} C(\Omega, \mathcal{F}, m, \text{diam}_\delta(\Omega), I_0, I_1, q_0, I_{2,q_0}, d_\delta(\Omega', \partial\Omega), R) & \text{if } \Omega \text{ is bounded,} \\ C(m, p, I_0, I_1, q_0, I_{2,q_0}, |\Omega'|_\delta, R) & \text{if } \Omega = \mathbb{R}^m \end{cases} \quad (4.34)$$

such that

$$\int_{\Omega''} \frac{(1 + \log w)^{q_0}}{w} \{ \|\Pi\|^2 + w^2 \log w \} dx \leq C. \quad (4.35)$$

Proof. By Theorem 1.3 or [11, Theorem 1.5 and Remark 3.4], we know that u is smooth and strictly spacelike. In particular, $L_s^\rho(\Omega'') \Subset L_t^\rho(\Omega'')$ if $0 \leq s < t$. Define p_1 as in (4.1). We proceed by induction on $q \in \{0, \dots, q_0\}$. Set for convenience

$$\bar{R} \doteq \frac{R}{q_0 + 1},$$

and define the sequence

$$\Omega'' \doteq \Omega_{q_0+1} \Subset \Omega_{q_0} \Subset \dots \Subset \Omega_1 \Subset \Omega_0 \Subset \Omega', \quad \Omega_q \doteq L_{(q_0+1-q)\bar{R}}^\rho(\Omega'') \text{ for } q \geq 0.$$

Let M_q be the graph of u over Ω_q . By rephrasing (4.33) in terms of the graph metric and the hyperbolic angle β , there exists a constant \bar{I}_{2,q_0} only depending on I_{2,q_0} such that

$$\int_{M_0} H^2 (1 + \beta)^{q_0+2} \leq \bar{I}_{2,q_0},$$

where, hereafter in the proof, integration on subsets of the graph of u will always be performed with respect to the graph measure dx_g , that will be omitted as far as no confusion arises. Hence,

$$\int_{M_0} H^2 (1 + \beta)^{q+2} \leq \bar{I}_{2,q_0} \quad \text{for each } q \in \{0, 1, \dots, q_0\}. \quad (4.36)$$

As a starting point, observe that Proposition 3.13 and (4.32) imply the existence of

$$\bar{\mathcal{I}}_{1,0} = \begin{cases} \bar{\mathcal{I}}_{1,0}(\Omega, \mathcal{F}, m, \text{diam}_\delta(\Omega), p, \mathcal{I}_0, \mathcal{I}_1, d_\delta(\Omega', \partial\Omega)) & \text{if } \Omega \text{ is bounded,} \\ \bar{\mathcal{I}}_{1,0}(m, p, \mathcal{I}_0, \mathcal{I}_1, |\Omega'|_\delta) & \text{if } \Omega = \mathbb{R}^m, \end{cases}$$

such that

$$\int_{M_0} |H| \text{ch } \beta + \int_{M_0} \text{ch}^2 \beta \leq \bar{\mathcal{I}}_{1,0}. \quad (\mathcal{A}_0)$$

We shall prove the following inductive step:

if there exists

$$\mathcal{J}_{1,q} = \begin{cases} \mathcal{J}_1(\Omega, \mathcal{F}, m, \text{diam}_\delta(\Omega), p, \mathcal{I}_0, \mathcal{I}_1, d_\delta(\Omega', \partial\Omega), q_0, q, R) & \text{if } \Omega \text{ is bounded,} \\ \mathcal{J}_1(m, p, \mathcal{I}_0, \mathcal{I}_1, |\Omega'|_\delta, q_0, q, R) & \text{if } \Omega = \mathbb{R}^m, \end{cases}$$

such that

$$\int_{M_q} |H|(1+\beta)^q \text{ch } \beta + \int_{M_q} (1+\beta)^q \text{ch}^2 \beta \leq \mathcal{J}_{1,q}, \quad (\mathcal{A}_q)$$

then there exists

$$\mathcal{J}_{2,q} = \begin{cases} \mathcal{J}_2(\Omega, \mathcal{F}, m, \text{diam}_\delta(\Omega), p, \mathcal{I}_0, \mathcal{I}_1, d_\delta(\Omega', \partial\Omega), q_0, q, \mathcal{J}_{1,q}, R) & \text{if } \Omega \text{ is bounded,} \\ \mathcal{J}_2(m, p, \mathcal{I}_0, \mathcal{I}_1, |\Omega'|_\delta, q_0, q, \mathcal{J}_{1,q}, R) & \text{if } \Omega = \mathbb{R}^m, \end{cases}$$

such that

$$\int_{M_{q+1}} (1+\beta)^q \|\Pi\|^2 + \int_{M_{q+1}} (1+\beta)^{q+1} \text{ch}^2 \beta \leq \mathcal{J}_{2,q}. \quad (\mathcal{B}_q)$$

In view of (4.15) and (4.36), to obtain (\mathcal{B}_q) from (\mathcal{A}_q) it is enough to show that

$$\int_{M_{q+1} \cap \{du \neq 0\}} (1+\beta)^q \left[\frac{\text{sh}^2 \beta}{\text{ch}^2 \beta} \|A\|^2 + \|\nabla \beta\|^2 + \beta \text{sh}^2 \beta \right] \leq \mathcal{J}_{2,q},$$

with $\mathcal{J}_{2,q}$ possibly different, but depending on the same data. We first show that $(\mathcal{B}_q) \Rightarrow (\mathcal{A}_{q+1})$ for each $0 \leq q \leq q_0 - 1$: by (4.36) and Young's inequality,

$$\begin{aligned} \int_{M_{q+1}} |H|(1+\beta)^{q+1} \text{ch } \beta &\leq \int_{M_{q+1}} H^2(1+\beta)^{q+2} + \int_{M_{q+1}} (1+\beta)^q \text{ch}^2 \beta \\ &\leq \bar{\mathcal{I}}_{2,q_0} + \mathcal{J}_{2,q}, \end{aligned}$$

hence (\mathcal{A}_{q+1}) holds with $\mathcal{J}_{1,q+1} \doteq \bar{\mathcal{I}}_{2,q_0} + 2\mathcal{J}_{2,q}$.

Since we verified (\mathcal{A}_0) , if the implication $(\mathcal{A}_q) \Rightarrow (\mathcal{B}_q)$ is proved, then the induction hypothesis implies (\mathcal{B}_{q_0}) , which is equivalent to (4.35).

With the above preparation, it suffices to prove that $(\mathcal{A}_q) \Rightarrow (\mathcal{B}_q)$. For small $t > 0$, we consider a smooth approximation $\beta_t \in C^\infty(\Omega)$ of β defined by

$$\text{ch } \beta_t \doteq \sqrt{w^2 + t} \quad \Leftrightarrow \quad \beta_t = \log \left(\sqrt{w^2 + t} + \sqrt{w^2 + t - 1} \right).$$

Note that

$$\begin{aligned} \beta &\leq \beta_t \leq \beta + 1 \quad \text{for small enough } t, & \nabla \beta_t &= 0 \quad \text{a.e. on } \{du = 0\}, \\ \beta_t &\downarrow \beta, \quad \|\nabla \beta_t\| \uparrow \|\nabla \beta\| \cdot \mathbb{1}_{\{du \neq 0\}} \quad \text{as } t \downarrow 0, & \langle \nabla \beta_t, \nabla \beta \rangle \mathbb{1}_{\{du \neq 0\}} &\geq 0. \end{aligned} \quad (4.37)$$

Define also

$$\bar{u} \doteq u - \|u\|_\infty \leq 0. \quad (4.38)$$

We consider the smooth vector field $Y + \beta_t \nabla e^{\bar{u}}$, where Y is defined in (4.16), and compute its divergence. For $\varepsilon \in (0, 1)$ to be specified later, we use (4.20) to deduce that for some positive constants C_m and $C_{m,\varepsilon}$ depending, respectively, on m and on (m, ε) ,

$$\begin{aligned} \operatorname{div}_M (Y + \beta_t \nabla e^{\bar{u}}) &\geq \left[\frac{\operatorname{sh}^2 \beta}{\operatorname{ch}^2 \beta} \|A\|^2 - C_{m,\varepsilon} H^2 + \left\{ \frac{1}{m-1} - C_m \varepsilon \right\} \|\nabla \beta\|^2 \right] \cdot \mathbb{1}_{\{du \neq 0\}} \\ &\quad + e^{\bar{u}} \langle \nabla \beta_t, \nabla u \rangle + \beta_t e^{\bar{u}} H \operatorname{ch} \beta + \beta_t e^{\bar{u}} \operatorname{sh}^2 \beta. \end{aligned} \quad (4.39)$$

The reason for adding the field $\beta_t \nabla e^{\bar{u}}$ is the appearance of the last term in the above inequality, $\beta_t e^{\bar{u}} \operatorname{sh}^2 \beta$, which is of the order of $w^2 \log w$ for large w . Its control will give us the log-improvement for the energy density we search for. Hereafter, $C_m, C_{m,\varepsilon}$ as well as the constants $C_q, C_{q,\varepsilon}$, may vary from line to line.

We integrate (4.39) against the test function

$$\psi = \varphi^2 (1 + \beta_t)^q, \quad \varphi \in \operatorname{Lip}_c(\Omega_q), \quad \varphi^2 \in W^{2,\infty}(\Omega_q). \quad (4.40)$$

By

$$\nabla \psi = (1 + \beta_t)^q \nabla \varphi^2 + q \varphi^2 (1 + \beta_t)^{q-1} \nabla \beta_t,$$

we see that

$$\begin{aligned} &\int_{\{du \neq 0\}} \varphi^2 (1 + \beta_t)^q \left[\frac{\operatorname{sh}^2 \beta}{\operatorname{ch}^2 \beta} \|A\|^2 - C_{m,\varepsilon} H^2 + \left\{ \frac{1}{m-1} - C_m \varepsilon \right\} \|\nabla \beta\|^2 \right] \\ &\quad + \int_M \varphi^2 (1 + \beta_t)^q e^{\bar{u}} \langle \nabla \beta_t, \nabla u \rangle + \int_M \varphi^2 (1 + \beta_t)^q \beta_t e^{\bar{u}} H \operatorname{ch} \beta \\ &\quad + \int_M \varphi^2 (1 + \beta_t)^q e^{\bar{u}} \beta_t \operatorname{sh}^2 \beta \\ &\leq - \int_M (1 + \beta_t)^q \langle \nabla \varphi^2, Y + \beta_t \nabla e^{\bar{u}} \rangle - q \int_M \varphi^2 (1 + \beta_t)^{q-1} \langle \nabla \beta_t, Y + \beta_t \nabla e^{\bar{u}} \rangle. \end{aligned}$$

Rearranging the terms and using Cauchy-Schwarz's inequality together with (4.37), we obtain

$$\begin{aligned} &\int_{\{du \neq 0\}} \varphi^2 (1 + \beta_t)^q \left[\frac{\operatorname{sh}^2 \beta}{\operatorname{ch}^2 \beta} \|A\|^2 + \left\{ \frac{1}{m-1} - C_m \varepsilon \right\} \|\nabla \beta\|^2 \right] \\ &\quad + \int_M \varphi^2 (1 + \beta_t)^q e^{\bar{u}} \beta_t \operatorname{sh}^2 \beta \\ &\leq - \int_M (1 + \beta_t)^q \langle \nabla \varphi^2, Y + \beta_t \nabla e^{\bar{u}} \rangle - q \int_M \varphi^2 (1 + \beta_t)^{q-1} \langle \nabla \beta_t, Y + \beta_t \nabla e^{\bar{u}} \rangle \\ &\quad + \int_{\{du \neq 0\}} \varphi^2 (1 + \beta_t)^q e^{\bar{u}} \|\nabla \beta\| \operatorname{sh} \beta + \int_M \varphi^2 (1 + \beta_t)^{q+1} e^{\bar{u}} |H| \operatorname{ch} \beta \\ &\quad + C_{m,\varepsilon} \int_M \varphi^2 (1 + \beta_t)^q H^2. \end{aligned}$$

From $\bar{u} \leq 0$ (see (4.38)) and

$$\varphi^2 (1 + \beta_t)^q e^{\bar{u}} \|\nabla \beta\| \operatorname{sh} \beta \leq \varepsilon \varphi^2 (1 + \beta_t)^q \|\nabla \beta\|^2 + \varepsilon^{-1} \varphi^2 (1 + \beta_t)^q \operatorname{sh}^2 \beta,$$

we infer

$$\begin{aligned}
& \int_{\{du \neq 0\}} \varphi^2(1 + \beta_t)^q \left[\frac{\text{sh}^2 \beta}{\text{ch}^2 \beta} \|A\|^2 + \left\{ \frac{1}{m-1} - C_m \varepsilon \right\} \|\nabla \beta\|^2 \right] \\
& \quad + \int_M \varphi^2(1 + \beta_t)^q e^{\bar{u}} \beta_t \text{sh}^2 \beta \\
& \leq - \int_M (1 + \beta_t)^q \langle \nabla \varphi^2, Y + \beta_t \nabla e^{\bar{u}} \rangle - q \int_M \varphi^2(1 + \beta_t)^{q-1} \langle \nabla \beta_t, Y + \beta_t \nabla e^{\bar{u}} \rangle \quad (4.41) \\
& \quad + \varepsilon^{-1} \int_M \varphi^2(1 + \beta_t)^q \text{sh}^2 \beta + \int_M \varphi^2(1 + \beta_t)^{q+1} |H| \text{ch} \beta \\
& \quad + C_{m,\varepsilon} \int_M \varphi^2(1 + \beta_t)^q H^2.
\end{aligned}$$

Because of (\mathcal{A}_q) , (4.36) and the first in (4.37),

$$\begin{aligned}
& \int_M \varphi^2(1 + \beta_t)^q \text{sh}^2 \beta \leq C_q \|\varphi\|_\infty^2 \mathcal{J}_{1,q}, \\
& \int_M \varphi^2(1 + \beta_t)^{q+1} |H| \text{ch} \beta \leq \frac{\|\varphi\|_\infty^2}{2} \left\{ \int_{M_q} (1 + \beta_t)^{q+2} H^2 + \int_{M_q} (1 + \beta_t)^q \text{ch}^2 \beta \right\} \quad (4.42) \\
& \leq C_q \|\varphi\|_\infty^2 \left[\bar{\mathcal{I}}_{2,q_0} + \mathcal{J}_{1,q} \right].
\end{aligned}$$

Notice that due to (4.19),

$$\|\nabla \varphi\|^2 \leq \omega^2 |D\varphi|^2 = \text{ch}^2 \beta |D\varphi|^2, \quad \|Y\|^2 \cdot \mathbb{1}_{\{du \neq 0\}} \leq 2 \left[\|\nabla \beta\|^2 + H^2 \right] \cdot \mathbb{1}_{\{du \neq 0\}}.$$

Using $Y = 0$ a.e. on $\{du = 0\}$, Young's inequality and assumption (\mathcal{A}_q) , we infer

$$\begin{aligned}
& - \int_M (1 + \beta_t)^q \langle \nabla \varphi^2, Y \rangle \\
& \leq \varepsilon \int_{\{du \neq 0\}} \varphi^2(1 + \beta_t)^q \left[\|\nabla \beta\|^2 + H^2 \right] + \frac{4}{\varepsilon} \int_{\{du \neq 0\}} (1 + \beta_t)^q \|\nabla \varphi\|^2 \\
& \leq \varepsilon \int_{\{du \neq 0\}} \varphi^2(1 + \beta_t)^q \left[\|\nabla \beta\|^2 + H^2 \right] + 4\varepsilon^{-1} \|D\varphi\|_\infty^2 \int_{M_q} (1 + \beta_t)^q \text{ch}^2 \beta \\
& \leq \varepsilon \int_{\{du \neq 0\}} \varphi^2(1 + \beta_t)^q \left[\|\nabla \beta\|^2 + H^2 \right] + C_{q,\varepsilon} \|D\varphi\|_\infty^2 \mathcal{J}_{1,q}.
\end{aligned}$$

Moreover, from (4.19), $\bar{u} \leq 0$, (4.37) and $Y + \beta_t \nabla e^{\bar{u}} = 0$ a.e. on $\{du = 0\}$ it follows that

$$\begin{aligned}
& - q \int_M \varphi^2(1 + \beta_t)^{q-1} \langle \nabla \beta_t, Y + \beta_t \nabla e^{\bar{u}} \rangle \\
& \leq - q \int_{\{du \neq 0\}} \varphi^2(1 + \beta_t)^{q-1} \left\langle \nabla \beta_t, -\frac{\text{sh} \beta}{\text{ch} \beta} H \nu + \beta_t \nabla e^{\bar{u}} \right\rangle \\
& \leq q \int_{\{du \neq 0\}} \varphi^2(1 + \beta_t)^{q-1} \|\nabla \beta\| \|H\| + q \int_{\{du \neq 0\}} \varphi^2(1 + \beta_t)^q \text{ch} \beta \|\nabla \beta\| \\
& \leq 2\varepsilon \int_{\{du \neq 0\}} \varphi^2(1 + \beta_t)^q \|\nabla \beta\|^2 + \frac{q^2}{\varepsilon} \int_M \varphi^2(1 + \beta_t)^{q-2} H^2 + \frac{q^2}{\varepsilon} \int_M \varphi^2(1 + \beta_t)^q \text{ch}^2 \beta \\
& \leq 2\varepsilon \int_{\{du \neq 0\}} \varphi^2(1 + \beta_t)^q \|\nabla \beta\|^2 + \varepsilon^{-1} C_q \|\varphi\|_\infty^2 \left[\bar{\mathcal{I}}_{2,q_0} + \mathcal{J}_{1,q} \right].
\end{aligned}$$

Plugging these inequalities into (4.41), we get

$$\begin{aligned}
& \int_{\{du \neq 0\}} \varphi^2 (1 + \beta_t)^q \left[\frac{\text{sh}^2 \beta}{\text{ch}^2 \beta} \|A\|^2 + \left\{ \frac{1}{m-1} - C_m \varepsilon \right\} \|\nabla \beta\|^2 \right] \\
& \quad + \int_M \varphi^2 (1 + \beta_t)^q e^{\bar{u}} \beta_t \text{sh}^2 \beta \\
& \leq - \int_M (1 + \beta_t)^q \langle \nabla \varphi^2, \beta_t \nabla e^{\bar{u}} \rangle + C_{m,q,\varepsilon} \|\varphi\|_{W^{1,\infty}}^2 \left[\bar{\mathcal{I}}_{2,q_0} + \mathcal{J}_{1,q} \right].
\end{aligned} \tag{4.43}$$

We next examine the term

$$K \doteq - \int_M (1 + \beta_t)^q \langle \nabla \varphi^2, \beta_t \nabla e^{\bar{u}} \rangle.$$

For $U \in \Omega_q$, we choose φ satisfying (4.40) and

$$\varphi = 0 \quad \text{on } \partial U. \tag{4.44}$$

Hereafter, we will denote by C_j a constant depending on the same quantities as (4.34). Since $\nabla \beta_t = 0$ a.e. on $\{du = 0\}$, we compute

$$\begin{aligned}
K &= - \int_M (1 + \beta_t)^q \beta_t \langle \nabla \varphi^2, \nabla (e^{\bar{u}} - 1) \rangle \\
&= - \int_M \langle \nabla \varphi^2, \nabla [(1 + \beta_t)^q \beta_t (e^{\bar{u}} - 1)] \rangle + \int_{\{du \neq 0\}} (e^{\bar{u}} - 1) \langle \nabla \varphi^2, \nabla [(1 + \beta_t)^q \beta_t] \rangle.
\end{aligned} \tag{4.45}$$

The last integral can be easily estimated by using (4.32), (4.37) and the definition of \bar{u} :

$$\begin{aligned}
& \left| \int_{\{du \neq 0\}} (e^{\bar{u}} - 1) \langle \nabla \varphi^2, \nabla [(1 + \beta_t)^q \beta_t] \rangle \right| \\
& \leq \varepsilon \int_{\{du \neq 0\}} \varphi^2 (1 + \beta_t)^q \|\nabla \beta\|^2 + 4\varepsilon^{-1} (1 + q)^2 \|e^{\bar{u}} - 1\|_{L^\infty(\Omega_q)}^2 \int_M (1 + \beta_t)^q \|\nabla \varphi\|^2 \\
& \leq \varepsilon \int_{\{du \neq 0\}} \varphi^2 (1 + \beta_t)^q \|\nabla \beta\|^2 + \varepsilon^{-1} C_1 \|D\varphi\|_\infty^2 \mathcal{J}_{1,q}.
\end{aligned} \tag{4.46}$$

On the other hand, since $\varphi^2 \in W^{2,\infty}(\Omega_q)$ with $\text{supp } \varphi \in \Omega_q$, we get

$$\begin{aligned}
- \int_M \langle \nabla \varphi^2, \nabla [(1 + \beta_t)^q \beta_t (e^{\bar{u}} - 1)] \rangle &= \int_M (1 + \beta_t)^q \beta_t (e^{\bar{u}} - 1) \Delta_M \varphi^2 \\
&= \int_M (1 + \beta_t)^q \beta_t (1 - e^{\bar{u}}) (-\Delta_M \varphi^2).
\end{aligned} \tag{4.47}$$

We set $U = L_{\bar{R}}(o)$ where $o \in \Omega_{q+1}$. Then $U \in \Omega_q$ and since u is smooth with $\|Du\|_\infty < 1$, $\partial L_{\bar{R}}(o)$ is smooth. We also set

$$\varphi(x) \doteq (\bar{R}^2 - \ell_o^2(x))_+.$$

It is easily seen that (4.40) and (4.44) are satisfied. Moreover, by (2.7) and

$$-\Delta_M \ell_o^4 = -2\|\nabla \ell_o^2\|^2 - 2\ell_o^2 \Delta_M \ell_o^2 \leq -2\ell_o^2 \Delta_M \ell_o^2, \tag{4.48}$$

it follows that on U ,

$$\begin{aligned} -\Delta_M \varphi^2 &= -\Delta_M (\bar{R}^4 - 2\bar{R}^2 \ell_o^2 + \ell_o^4) \leq 2(\bar{R}^2 - \ell_o^2) \Delta_M \ell_o^2 \\ &\leq 4\bar{R}^2 (m+2 |H| \operatorname{ch} \beta |x - o|) \\ &\leq C_2 (1 + |H| \operatorname{ch} \beta). \end{aligned} \quad (4.49)$$

Remark also that

$$\|\varphi\|_{W^{1,\infty}} \leq C_3.$$

From (\mathcal{A}_q) , (4.47), (4.49), $0 \leq 1 - e^{\bar{u}} \leq 1$, $\beta \leq \operatorname{ch}^2 \beta$, (4.46) and (4.42), we deduce

$$\begin{aligned} K &\leq C_2 \int_{M_q} (1 + \beta_t)^q \beta_t (1 + |H| \operatorname{ch} \beta) \\ &\quad + C_1 \varepsilon^{-1} \|D\varphi\|_\infty^2 \mathcal{J}_{1,q} + \varepsilon \int_{\{du \neq 0\}} \varphi^2 (1 + \beta_t)^q \|\nabla \beta\|^2 \\ &\leq C_3 \varepsilon^{-1} [\bar{\mathcal{I}}_{2,q_0} + \mathcal{J}_{1,q}] + \varepsilon \int_{\{du \neq 0\}} \varphi^2 (1 + \beta_t)^q \|\nabla \beta\|^2. \end{aligned} \quad (4.50)$$

Since $\varphi \geq \bar{R}^2/2$ on $L_{\bar{R}/2}(o)$, it follows from (4.43) and (4.50) that

$$\begin{aligned} \int_{L_{\bar{R}/2}(o)} (1 + \beta_t)^q \left[\frac{\operatorname{sh}^2 \beta}{\operatorname{ch}^2 \beta} \|A\|^2 + \left\{ \frac{1}{m-1} - C_m \varepsilon \right\} \|\nabla \beta\|^2 \right] \cdot \mathbb{1}_{\{du \neq 0\}} \\ + \int_{L_{\bar{R}/2}(o)} e^{\bar{u}} (1 + \beta_t)^q \beta_t \operatorname{sh}^2 \beta \leq C_4 C_{m,q,\varepsilon} [\mathcal{J}_{1,q} + \bar{\mathcal{I}}_{2,q_0}]. \end{aligned}$$

Choosing $\varepsilon = [2C_m(m-1)]^{-1}$, noting that $e^{\bar{u}} \geq e^{-2I_0}$ and letting $t \rightarrow 0$, we deduce

$$\int_{L_{\bar{R}/2}(o)} (1 + \beta)^q \left[\frac{\operatorname{sh}^2 \beta}{\operatorname{ch}^2 \beta} \|A\|^2 + \|\nabla \beta\|^2 + \beta \operatorname{sh}^2 \beta \right] \cdot \mathbb{1}_{\{du \neq 0\}} \leq C_5. \quad (4.51)$$

Consider a maximal set of disjoint Euclidean balls $\{B_{\bar{R}/4}(o_1), \dots, B_{\bar{R}/4}(o_s)\}$ with $o_i \in \Omega_{q+1}$. Since $B_{\bar{R}/4}(o_i) \subset L_{\bar{R}/4}(o_i) \Subset \Omega_q \Subset \Omega'$, we get

$$s \leq \left\lceil \frac{|\Omega'_\delta|}{\omega_m (\bar{R}/4)^m} \right\rceil \doteq \tau(m, R, q_0, |\Omega'_\delta|).$$

Using that $\{B_{\bar{R}/2}(o_j)\}$ covers Ω_{q+1} and $B_{\bar{R}/2}(o_j) \subset L_{\bar{R}/2}(o_j) \Subset \Omega_q$, summing up (4.51) we conclude

$$\int_{M_{q+1}} (1 + \beta)^q \left[\frac{\operatorname{sh}^2 \beta}{\operatorname{ch}^2 \beta} \|A\|^2 + \|\nabla \beta\|^2 + \beta \operatorname{sh}^2 \beta \right] \cdot \mathbb{1}_{\{du \neq 0\}} \leq C_5 \tau,$$

which proves (\mathcal{B}_q) . \square

Remark 4.14. We comment on the choice of φ in the above proof. For a general cut-off function φ , in view of (2.2), one could just obtain the bound

$$|\Delta_M \varphi^2| \leq m \|D^2 \varphi^2\|_\infty (1 + \operatorname{ch}^2 \beta) + \|D\varphi^2\|_\infty |H| \operatorname{ch} \beta,$$

which inserted into (4.47) would make necessary to estimate a term of the type

$$\int_U (1 + \beta_t)^q \beta_t \operatorname{ch}^2 \beta. \quad (4.52)$$

Such a term cannot be absorbed into the last addendum on the left-hand side of (4.43). This is the main reason why we use the extrinsic Lorentzian distance. Furthermore, the translation performed in the first line of (4.45) and the choice of \bar{u} in (4.38) are crucial to make sure that the coefficient which multiplies $-\Delta_M \varphi^2$ in (4.47) is non-negative. Hence, an upper estimate for $-\Delta_M \varphi^2$ is sufficient and we can get rid of the term $\|\nabla \ell_o\|$ in (4.48), that would have lead, again, to the appearance of an integral of the type (4.52).

5 Proofs of the main theorems

5.1 Proof of Theorem 1.17

Item (i) directly follows from Proposition 3.10. To prove (ii) we consider a maximally extended light segment $\bar{x}y$, which we know to be bounded because $u_\rho \rightarrow 0$ at infinity by Proposition 3.3. Due to maximality and to the anti-peeling theorem [5, Theorem 3.2] applied to $\Omega \in \mathbb{R}^m \setminus \{x_1, \dots, x_k\}$, $\bar{x}y$ must start and end at points in $\{x_1, \dots, x_k\}$, say $\bar{x}y = \bar{x}_i x_j$. It remains to prove $a_i a_j < 0$. To this end, we argue by contradiction and suppose $a_i a_j > 0$. By changing the indices i and j and considering $-u_\rho$ if necessary, we may suppose $a_j > 0$ and $u_\rho(x_i) - u_\rho(x_j) = |x_i - x_j|$. According to Theorem 4.5 and Remark 4.6, for any $\alpha > 0$ the function u_ρ is also a minimizer for I_{ρ_α} with $\rho_\alpha \doteq \rho + \alpha(\delta_{x_i} - \delta_{x_j})$. Now we choose $\alpha = a_j$ and deduce that u_ρ minimizes $I_{\rho_{a_j}}$. Since ρ_{a_j} is of type (1.13) and has no charge at x_j , the anti-peeling theorem implies that the light segment $\bar{x}_i x_j$ can be extended beyond x_j , contradicting the maximality of $\bar{x}_i x_j$.

We prove (iii). Consider the approximation $\{\rho_j, H_j, u_j, w_j\}$ in Subsection 4.1 and fix $\Omega' \in \mathbb{R}^m \setminus \{x_1, \dots, x_k\}$ with smooth boundary. Then

$$\sup_{j \geq 1} \|H_j\|_{L^\infty(\Omega')} < \infty. \quad (5.1)$$

By Proposition 3.11, $u_j \rightarrow u_\rho$ in $L^\infty(\mathbb{R}^m)$ and $\mathcal{G} \doteq \{u_\rho\} \cup \{u_j : j \in \mathbb{N}\}$ is compact in $C(\mathbb{R}^m)$. Thus, for given $\Omega'' \Subset \Omega'$, by Lemma 3.8 and the assumption that u_ρ has no light-segments, there exists $R > 0$ independent of j such that the Lorentzian ball $L_R^{\rho_j}(\Omega'') \Subset \Omega'$ for all $j \geq 1$. By (5.1), we can apply Theorem 4.13 to deduce

$$\sup_{j \geq 1} \left\| w_j \log(1 + w_j) \right\|_{L^1(\Omega'')} < \infty.$$

Thus, the sequence $\{w_j\}$ is locally uniformly integrable on Ω' . By the arbitrariness of Ω' , $\{w_j\}$ is locally uniformly integrable on $\Omega \setminus \{x_1, \dots, x_k\}$; hence, Theorem 4.2 with $E = \{x_i\}_{i=1}^k$ implies

$$\int_{\mathbb{R}^m} w_\rho D u_\rho \cdot D \eta \, dx = \langle \rho, \eta \rangle = \sum_{i=1}^k a_i \eta(x_i) \quad \forall \eta \in \text{Lip}_c(\mathbb{R}^m). \quad (5.2)$$

Therefore, u_ρ weakly solves (BT). Since u_ρ does not possess light segments, for each ball $\tilde{B} \Subset \mathbb{R}^m \setminus \{x_1, \dots, x_k\}$ we have

$$|u_\rho(x) - u_\rho(y)| < |x - y| = d_{\tilde{B}}(x, y) \quad \forall x, y \in \partial \tilde{B} \text{ with } x \neq y.$$

Moreover, u_ρ minimizes I_0 on \tilde{B} with respect to its own boundary values, thus u_ρ coincides with the solution given in Bartnik-Simon's [5], which is smooth and strictly spacelike. Hence, u_ρ is smooth and strictly spacelike away from x_1, \dots, x_k . We next prove that u_ρ has an isolated

singularity at each x_i , in the sense of Ecker [22]. Fix $B \doteq B_r(x_i)$ with $x_j \notin \overline{B}$ for $j \neq i$, and choose $\eta \in \text{Lip}_c(B)$ with $\eta = -a_i$ in a neighborhood of x_i . Suppose by contradiction that u_ρ minimizes I_0 in B , that is,

$$I_0(u_\rho) = \inf \left\{ I_0(v) : v \in \mathcal{Y}_{u_\rho}(B) \right\}, \quad I_0(v) \doteq \int_B \left(1 - \sqrt{1 - |Dv|^2} \right) dx. \quad (5.3)$$

Since $D\eta = 0$ around x_i and u_ρ is strictly spacelike away from x_1, \dots, x_k , for $t > 0$ small enough it holds $u_\rho + t\eta \in \mathcal{Y}_{u_\rho}(B)$. Using Proposition 3.13 and comparing to (5.2), we get

$$0 \geq \int_B w_\rho Du_\rho \cdot (Du_\rho - D(u_\rho + t\eta)) dx = -t \int_B w_\rho Du_\rho \cdot D\eta dx = t|a_i|^2 > 0,$$

which is a contradiction.

To conclude, [22, Theorem 1.5] ensures that u_ρ is asymptotic to a light cone C near x_i , and we can therefore apply the argument in [10, Theorem 3.5] to deduce that C is upward or downward pointing respectively when $a_i < 0$ or $a_i > 0$. \square

5.2 Proof of Theorem 1.11

Let $\Sigma \Subset \Omega$ and $\rho \in \mathcal{M}(\Omega)$ satisfy the assumptions in Theorem 1.11. Fix $\mathcal{F}, \mathcal{I}_1, \mathcal{I}_2, \Omega'$ and ε as in (ii):

$$\phi \in \mathcal{F}, \quad \|\rho\|_{\mathcal{M}(\Omega)} \leq \mathcal{I}_1, \quad \|\rho\|_{L^2(\Omega')} \leq \mathcal{I}_2. \quad (5.4)$$

We also choose $p_1 = 3$ for $\mathcal{Y}(\Omega)$ (any $p_1 > 2$ works). We split the proof into several steps.

Step 1: for each ϕ, ρ satisfying (5.4), and for each $\varepsilon > 0$, there exists

$$C_1(\Omega, \mathcal{F}, \text{diam}_\delta(\Omega), \mathcal{I}_1, \mathcal{I}_2, \varepsilon, d_\delta(\Omega', \partial\Omega))$$

such that

$$\int_{\Omega'_\varepsilon} w_\rho \log(1 + w_\rho) dx \leq C_1, \quad \Omega'_\varepsilon \doteq \{x \in \Omega' : d_\delta(x, \partial\Omega') > \varepsilon\}.$$

Proof of Step 1. This directly follows from Theorem 4.12 and (5.4). \square

The higher integrability allows to prove the next no-light-segment property.

Step 2: The minimizer u_ρ does not have light segments in Ω' .

Proof of Step 2. Assume by contradiction that $\overline{xy} \subset \Omega'$ is a light segment for u_ρ . Up to renaming, $u_\rho(y) - u_\rho(x) = |y - x|$. Define

$$\tilde{\rho} \doteq \rho + \delta_y - \delta_x.$$

By Theorem 4.5, u_ρ also minimizes $I_{\tilde{\rho}}$: $u_\rho = u_{\tilde{\rho}}$. To reach our desired contradiction, we tweak the argument in Theorem 4.5 used to show that u_ρ does not solve (BT). Let $\{\varphi_j\}$ be a mollifier and define $\rho_j = \varphi_j * \rho$ and $\tilde{\rho}_j = \varphi_j * \tilde{\rho}$. Call $u_j, \tilde{u}_j \in \mathcal{Y}_{\phi}(\Omega)$, respectively, the minimizers of I_{ρ_j} and $I_{\tilde{\rho}_j}$, and denote by w_j and \tilde{w}_j , respectively, their energy densities. In view of Proposition 3.11 and $u_\rho = u_{\tilde{\rho}}$, as $j \rightarrow \infty$, we have $u_j \rightarrow u_\rho$ and $\tilde{u}_j \rightarrow u_\rho$ in $C(\overline{\Omega})$. Notice that, by the properties of convolutions (see [48, Proof of Proposition 2.7]),

$$\|\rho_j\|_{\mathcal{M}(\Omega)} \leq \|\rho\|_{\mathcal{M}(\Omega)} \leq \mathcal{I}_1, \quad \|\tilde{\rho}_j\|_{\mathcal{M}(\Omega)} \leq \|\tilde{\rho}\|_{\mathcal{M}(\Omega)} \leq \mathcal{I}_1 + 2$$

and for each $\Omega'' \in \Omega' \setminus \{x, y\}$, j large enough and ε small enough,

$$\|\rho_j\|_{L^2(\Omega''_{\varepsilon/4})} + \|\tilde{\rho}_j\|_{L^2(\Omega''_{\varepsilon/4})} \leq \|\rho\|_{L^2(\Omega'')} + \|\tilde{\rho}\|_{L^2(\Omega'')} \leq 2\mathcal{I}_2 + 2.$$

Hence, we can apply Theorem 4.12 on $\Omega'' \in \Omega' \setminus \{x, y\}$ to both u_j and to \tilde{u}_j to deduce that $\{w_j\}$ and $\{\tilde{w}_j\}$ are locally uniformly integrable on $\Omega' \setminus \{x, y\}$. Then, Theorem 4.2 with $E = \{x, y\}$ guarantees that

$$\int w_\rho Du_\rho \cdot D\eta \, dx = \langle \rho, \eta \rangle, \quad \int w_\rho Du_\rho \cdot D\eta \, dx = \langle \tilde{\rho}, \eta \rangle \quad \forall \eta \in \text{Lip}_c(\Omega').$$

However, choosing η such that $\eta(y) \neq \eta(x)$, we deduce

$$\langle \tilde{\rho}, \eta \rangle = \langle \rho, \eta \rangle + \eta(y) - \eta(x) \neq \langle \rho, \eta \rangle,$$

giving the desired contradiction. \square

Hereafter, we denote with $\{\rho_j, u_j, w_j\}$ the approximation described in Subsection 4.1. With the aid of Step 2 and $\rho \in L^2(\Omega')$, an application of Lemma 3.8, Corollary 4.11 and Theorem 4.13 gives the next improved higher integrability and second fundamental form estimates for u_ρ , which conclude the proof of Theorem 1.11 (ii).

Step 3: *Higher integrability, Theorem 1.11 (ii): for each $\varepsilon > 0$, $q_0 > 0$, there exists a constant*

$$C = C(\Omega, \mathcal{F}, \text{diam}_\delta(\Omega), \mathcal{I}_1, \mathcal{I}_2, \varepsilon, \Omega', q_0) > 0$$

such that for each ρ and ρ satisfying (5.4),

$$\begin{aligned} & \int_{\Omega'_\varepsilon} (1 + \log w_\rho)^{q_0} \left\{ w_\rho |D^2 u_\rho|^2 + w_\rho^3 \left| D^2 u_\rho (Du_\rho, \cdot) \right|^2 + w_\rho^5 [D^2 u_\rho (Du_\rho, Du_\rho)]^2 \right\} dx \\ & + \int_{\Omega'_\varepsilon} w_\rho (1 + \log w_\rho)^{q_0+1} dx \leq C. \end{aligned}$$

Proof of Step 3. Let $\mathcal{S} \subset \mathcal{Y}(\Omega)$ be the set of minimizers u_ρ whose boundary value ϕ and source ρ satisfy (5.4). Because of the compactness of \mathcal{F} and of Propositions 3.5 and 3.11, taking into account the lower semicontinuity of $\|\cdot\|_{L^2(\Omega')}$ and $\|\cdot\|_{\mathcal{M}(\Omega)}$ under weak convergence, we deduce that \mathcal{S} is compact in $C(\bar{\Omega})$. Applying the second part of Lemma 3.8, for $\varepsilon > 0$ we infer the existence of

$$R = R(\Omega, \mathcal{F}, \text{diam}_\delta(\Omega), \mathcal{I}_1, \mathcal{I}_2, \varepsilon, \Omega').$$

such that $L_R^{\rho_j}(\Omega'_\varepsilon) \Subset L_R^{\rho_j}(\Omega')$ for each $u \in \mathcal{S}$. Theorem 4.13 with $\Omega'' = \Omega'_\varepsilon$ ensures that (4.35) holds for u_j uniformly in j . The corresponding inequality for the pointwise limit u_ρ , which is a rewriting of our desired estimate, then follows by the same method as that in Corollary 4.11. \square

Step 4: *Weak solvability and no light segments, Theorem 1.11 (i).*

Proof of Step 4. Applying Step 1 to the mollified sources ρ_j , we deduce that $\{w_j\}$ are locally uniformly integrable in $\Omega \setminus \Sigma$. Using $\mathcal{H}_\delta^1(\Sigma) = 0$, Theorem 4.2 implies that the limit u_ρ is a weak solution to (BT) on Ω . On the other hand, by Step 2, u_ρ does not have light segments in any set $\Omega'' \Subset \Omega \setminus \Sigma$, hence in $\Omega \setminus \Sigma$. Since $\mathcal{H}_\delta^1(\Sigma) = 0$, there are no light segments on the entire Ω . \square

Step 5: Regularity for $\rho \in L^\infty$, Theorem 1.11 (iii).

Proof of Step 5. Let $\rho \in L^\infty(\Omega')$, and fix a domain $\Omega'' \Subset \Omega'$. Due to Step 2, every point $x \in \Omega''$ has positive Lorentzian distance from $\partial\Omega'$, with a uniform bound depending on the data of our problem. We can therefore use the local gradient estimate in [5, Lemma 2.1] as in [5, Proof of Theorem 4.1] to deduce an L^∞ -estimate for w_ρ and a $W^{2,2}$ -estimate for u_ρ in Ω'' . From Theorem 1.11 (i) and (ii), $u_\rho \in W_{\text{loc}}^{2,2}(\Omega')$ is a strong solution to

$$-\sum_{i=1}^m \partial_i (a_i(Du_\rho)) = \rho \quad \text{in } \Omega'', \text{ where } a_i(p) \doteq (1 - |p|^2)^{-1/2} p_i : B_1(0) \rightarrow \mathbb{R}.$$

By differentiating formally the equation in x_k , we see that $(u_\rho)_k \in W^{1,2}(\Omega'')$ is a weak solution to

$$-\sum_{i=1}^m \partial_i \sum_{n=1}^m \frac{\partial a_i}{\partial p_n} (Du_\rho)(u_\rho)_{nk} = \sum_{i=1}^m \partial_i (\rho \delta_{ki}) \quad \text{in } \Omega''.$$

Since $(\partial a_i / \partial p_n)$ is bounded and uniformly elliptic on Ω'' due to the L^∞ -bound of w_ρ , applying [33, Theorem 8.22 or Corollary 8.24], we see that $(u_\rho)_k \in C_{\text{loc}}^\alpha(\Omega'')$ for some α , hence, $u_\rho \in C_{\text{loc}}^{1,\alpha}(\Omega'')$. By bootstrapping, $u_\rho \in C^\infty(\Omega')$ whenever $\rho \in C^\infty(\Omega')$. \square

By Steps 1–5, we complete the proof of Theorem 1.11. \square

Remark 5.1. Referring to the approximations $\{u_j\}$ of u_ρ in Subsection 4.1, because of Theorem 4.13, Lemma 3.8 and the argument in Step 2 above, we deduce that the uniform integrability of $\{w_j \log w_j\}$ on a subdomain Ω' where $\rho \in L^2$ is equivalent to the nonexistence of light segments for u_ρ on Ω' .

5.3 Proof of Theorem 1.14

The proof is similar to the one of Theorem 1.11. We consider the approximation $\{\rho_j, H_j, u_j, w_j\}$ in Subsection 4.1. Fix $\Omega' \Subset \Omega \setminus (\Sigma \cup K_\phi^\rho)$ and a small $\varepsilon > 0$. Then,

$$\|\rho_j\|_{L^2(\Omega'_\varepsilon)} \leq \|\rho\|_{L^2(\Omega')} \quad \text{for } j \text{ large enough.}$$

Let $\Omega'' \Subset \Omega'_\varepsilon$. From the definition of K_ϕ^ρ and Proposition 3.11, the first part of Lemma 3.8 applied to $\mathcal{E} \doteq \{u_j\}_j \cup \{u\}$ guarantees the existence of R such that $L_R^{\rho_j}(\Omega'') \Subset \Omega'$ for each j , and therefore, by Theorem 4.13 we deduce that, for each $q_0 \in \mathbb{R}^+$,

$$\sup_j \int_{\Omega''} \left\{ w_j (1 + \log w_j) + \|\Pi_j\|^2 w_j^{-1} \right\} (1 + \log w_j)^{q_0} dx < \infty.$$

Hence, Theorem 1.14 (ii) holds by the same argument as the one in Corollary 4.11. In the case $\rho \in L^\infty(\Omega')$, from $L_R^{\rho_j}(\Omega'') \Subset \Omega'$ and $\|\rho_j\|_{L^\infty(\Omega'')} \leq \|\rho\|_{L^\infty(\Omega')}$ for large enough j we can proceed as in the proof of Step 5 in Theorem 1.11 to get $w_\rho \in L^\infty(\Omega'')$ and then $u_\rho \in C_{\text{loc}}^{1,\alpha}(\Omega')$, which proves Theorem 1.14 (iii).

Summarizing, in our assumptions $\{w_j\}$ is locally uniformly integrable on $\Omega \setminus (\Sigma \cup K_\phi^\rho)$. Theorem 4.2 ensures that u_ρ satisfies (BT) on $\Omega \setminus K_\phi^\rho$. Moreover, if $K_\phi^\rho \cap (\partial\Omega \cup \Sigma) = \emptyset$, then we can choose open sets Ω'', Ω' such that $K_\phi^\rho \subset \Omega'' \Subset \Omega' \Subset \Omega \setminus \Sigma$. By the definition of K_ϕ^ρ and applying Lemma 3.8, we get the existence of R such that $L_R^{\rho_j}(\Omega'') \Subset \Omega'$ for each j , and therefore a uniform integrability of $\{w_j\}$ on Ω'' by Theorem 4.13. Hence, $\{w_j\}$ is locally uniformly integrable on the entire $\Omega \setminus \Sigma$, and u_ρ solves (BT) on Ω by Theorem 4.2. Thus, Theorem 1.14 (i) holds and this completes the proof. \square

5.4 Proof of Theorems 1.19 and 1.20

We begin with the following proposition:

Proposition 5.2. *Let $m \geq 3$ and $\mathcal{I} > 0$ be given. Then there exists a constant $\mathcal{J} = \mathcal{J}(m, \mathcal{I}, p_1) > 0$ such that for any $\rho \in \mathcal{Y}(\mathbb{R}^m)^*$ with $\|\rho\|_{\mathcal{Y}^*} \leq \mathcal{I}$, the minimizer u_ρ satisfies*

$$\|u_\rho\|_\infty \leq \mathcal{J}. \quad (5.5)$$

Moreover, $L_\varepsilon^\rho(\Omega'') \Subset \Omega'$ holds provided $\varepsilon > 0$ and $\Omega'' \subset \Omega' \subset \mathbb{R}^m$ satisfy

$$d_\delta(\Omega'', \mathbb{R}^m \setminus \Omega') \geq 2\mathcal{J} + \varepsilon. \quad (5.6)$$

Proof. Remark that the minimizer u_ρ satisfies $I_\rho(u_\rho) \leq I_\rho(0) = 0$. Recalling (3.6) and noting that $b_1 = 1/2$ in (3.5), we see that for each $\rho \in \mathcal{Y}(\mathbb{R}^m)^*$ with $\|\rho\|_{\mathcal{Y}^*} \leq \mathcal{I}$,

$$\|u_\rho\|_{\mathcal{Y}}^2 \leq 4 [1 + 2\|\rho\|_{\mathcal{Y}^*} \|u_\rho\|_{\mathcal{Y}}] \leq 4 + 8\mathcal{I} \|u_\rho\|_{\mathcal{Y}}.$$

Hence, minimizers are uniformly bounded in $\mathcal{Y}(\mathbb{R}^m)$ when $\|\rho\|_{\mathcal{Y}^*} \leq \mathcal{I}$ and by virtue of Proposition 3.3, (5.5) holds.

Let $\Omega'' \subset \Omega'$ satisfy (5.6). Notice that (5.5) implies that for each $x, o \in \mathbb{R}^m$ and each $\rho \in \mathcal{Y}(\mathbb{R}^m)^*$ with $\|\rho\|_{\mathcal{Y}^*} \leq \mathcal{I}$,

$$(\ell_o^\rho)^2(x) = r_o^2(x) - |u_\rho(x) - u_\rho(o)|^2 \geq r_o^2(x) - 4\mathcal{J}^2.$$

Hence, for any $x \in \mathbb{R}^m \setminus \Omega'$ and $o \in \Omega''$,

$$(\ell_o^\rho(x))^2 \geq 4\mathcal{J}\varepsilon + \varepsilon^2,$$

which implies $L_\varepsilon^\rho(\Omega'') \Subset \Omega'$. \square

Proof of Theorem 1.19. Define p_1 as in (4.1) for $m \geq 3$, and choose $\{\rho_j, u_j, w_j\}$ as in Subsection 4.1. Under the assumptions of Theorem 1.19, in view of Proposition 5.2, there exists $\mathcal{J} = \mathcal{J}(m, \mathcal{I}, p)$ such that $\|u_j\|_\infty \leq \mathcal{J}$ and $L_\varepsilon^{\rho_j}(\Omega'') \Subset \Omega'$ for any $\varepsilon > 0$ with $d_\delta(\Omega'', \mathbb{R}^m \setminus \Omega') \geq 2\mathcal{J} + \varepsilon$. Then the local uniform higher integrability of $\{w_j\}$ and the fact that u_ρ solves (BI) directly follow from Theorems 4.2 and 4.13. \square

Proof of Theorem 1.20. The proof follow verbatim that of Theorem 1.14, with the help of the L^∞ estimates in Proposition 5.2, and is left to the reader. \square

6 Weak solutions with light segments

In this section we construct the example in Theorem 1.7. First, for $\ell \in \{1, \dots, m-2\}$ we write $x \in \mathbb{R}^m$ as

$$x = (y, z, x_m), \quad \text{with } y \in \mathbb{R}^{m-\ell}, \quad z \in \mathbb{R}^{\ell-1}.$$

If $\ell = 1$, then the variable z can be omitted, which allows for some computational simplifications. The idea is to consider the function

$$U(x) = (1 - \varepsilon^{2\kappa} |y|^{2\kappa}) x_m \quad (6.1)$$

for $\kappa \geq 1$. Notice that the set $\{|y| = 0\}$ is an ℓ -dimensional subspace made up of light segments, but U does not satisfy a spacelike boundary condition in any bounded smooth domain Ω containing the origin. For this reason, for $\varepsilon > 0$ we fix cut-off functions ϑ_ε , ζ_ε and A_ε as follows:

- $\vartheta_\varepsilon \in C_c^\infty(\mathbb{R})$ is defined by $\vartheta_\varepsilon(t) \doteq \vartheta_1(\varepsilon t)$ and $\vartheta_1(t)$ satisfies

$$\begin{aligned} \vartheta_1(t) &\in C_c^\infty(\mathbb{R}), \quad \vartheta_1'(t) \leq 0 \quad \text{for } t \geq 0, \quad \text{supp } \vartheta_1 \subset [-2, 2], \\ \vartheta_1(t) &\equiv 1 \text{ for } 0 \leq t \leq 1, \quad \vartheta_1(t) = 1 - \frac{e^2}{2} \exp\left(-\frac{1}{t-1}\right) \text{ for } 1 < t \leq \frac{3}{2}. \end{aligned} \quad (6.2)$$

- $\zeta_\varepsilon \in C_c^\infty(\mathbb{R})$ satisfies

$$\zeta_\varepsilon \equiv 1 \quad \text{on } \left[-\frac{1}{2\varepsilon}, \frac{1}{2\varepsilon}\right], \quad \zeta_\varepsilon \equiv 0 \quad \text{on } \mathbb{R} \setminus \left(-\frac{1}{\varepsilon}, \frac{1}{\varepsilon}\right), \quad \|\zeta_\varepsilon'\|_{L^\infty(\mathbb{R})} \leq 4\varepsilon. \quad (6.3)$$

- Having chosen a function $a_\varepsilon \in C_c^\infty(\mathbb{R})$ with

$$\begin{aligned} a_\varepsilon(-t) &= a_\varepsilon(t), \quad a_\varepsilon(t) = \begin{cases} 1 & \text{if } t \in [0, \varepsilon], \\ 0 & \text{if } t \in [2\varepsilon, \infty), \end{cases} \\ a_\varepsilon'(t) &< 0 \quad \text{if } t \in (\varepsilon, 2\varepsilon), \quad a_\varepsilon(t) = 1 - d_\varepsilon \exp\left(-\frac{1}{t-\varepsilon}\right) \quad \text{if } t \in \left(\varepsilon, \frac{3\varepsilon}{2}\right], \end{aligned} \quad (6.4)$$

where $d_\varepsilon > 0$ is chosen so that $a_\varepsilon(3\varepsilon/2) = 1/2$, A_ε is defined by

$$A_\varepsilon(t) \doteq \int_0^t a_\varepsilon(s) \, ds \in C^\infty(\mathbb{R}). \quad (6.5)$$

For $\kappa \geq 1$, we then define $U_\varepsilon(y, z, x_m)$ by

$$U_\varepsilon(y, z, x_m) \doteq (1 - \varepsilon^{2\kappa} |y|^{2\kappa}) \zeta_\varepsilon(|y|) \vartheta_\varepsilon(|z|) \zeta_\varepsilon(x_m) A_\varepsilon(x_m),$$

If $\ell = 1$, $\vartheta_\varepsilon(|z|)$ is replaced by 1. Notice that $U_\varepsilon \in C_c^2(\mathbb{R}^m)$ and $U_\varepsilon \in C_c^\infty(\mathbb{R}^m)$ if $\kappa \in \mathbb{N}$. Remark that

$$U_\varepsilon(0, z, x_m) = x_m \quad \text{if } |z| \leq \frac{1}{\varepsilon} \text{ and } |x_m| \leq \varepsilon,$$

and a direct computation shows that $|DU_\varepsilon| < 1$ on the complement of the above set, see below. Hence, the union of the light segments of U_ε is the ℓ -dimensional compact cylinder

$$(0, z, x_m) \in \{0\} \times \overline{B}_{1/\varepsilon}^{\ell-1} \times [-\varepsilon, \varepsilon].$$

We hereafter denote with W_ε , ρ_{U_ε} and Π_{U_ε} the energy density, the mean curvature and the second fundamental form of the graph of U_ε . Theorem 1.7 follows from the next one applied with $\kappa = 1$:

Theorem 6.1. *Assume $m \geq 3$, $1 \leq \ell \leq m - 2$ and $\kappa \in [1, m - \ell)$. Then*

$$W_\varepsilon \in L_{\text{loc}}^q(\mathbb{R}^m) \quad \text{and} \quad \rho_{U_\varepsilon}, \|\Pi_{U_\varepsilon}\| \in L^q(\mathbb{R}^m) \quad \text{for all } q < \frac{m-\ell}{\kappa}, \quad (6.6)$$

and U_ε satisfies

$$\int_{\mathbb{R}^m} \frac{DU_\varepsilon \cdot D\eta}{\sqrt{1 - |DU_\varepsilon|^2}} \, dx = \int_{\mathbb{R}^m} \rho_{U_\varepsilon} \eta \, dx \quad \text{for each } \eta \in C_c^\infty(\mathbb{R}^m).$$

Proof. For (6.6), since $|\rho_{U_\varepsilon}| \leq C \|\Pi_{U_\varepsilon}\|$ it is enough to estimate W_ε and $\|\Pi_{U_\varepsilon}\|$.

For computational reasons, with a slight abuse of notation we write U_ε as a function of the triple (r, s, x_m) , with $r = |y|$, and $s = |z|$:

$$U_\varepsilon(r, s, x_m) = u_\varepsilon(r, x_m) \vartheta_\varepsilon(s),$$

where we set

$$u_\varepsilon(r, x_m) = (1 - \varepsilon^{2\kappa} r^{2\kappa}) \zeta_\varepsilon(r) \zeta_\varepsilon(x_m) A_\varepsilon(x_m).$$

It is readily checked that for a function $u(r, s, x_m)$ it holds

$$Du = u_r \frac{y}{|y|} + u_s \frac{z}{|z|} + u_m e_m \quad (6.7)$$

and

$$D^2u = \begin{pmatrix} u_{rr} \frac{y}{|y|} \otimes \frac{y}{|y|} + \frac{u_r}{r} \left(I_{m-\ell} - \frac{y}{|y|} \otimes \frac{y}{|y|} \right) & u_{rs} \frac{y}{|y|} \otimes \frac{z}{|z|} & u_{rm} \frac{y}{|y|} \\ u_{rs} \frac{z}{|z|} \otimes \frac{y}{|y|} & u_{ss} \frac{z}{|z|} \otimes \frac{z}{|z|} + \frac{u_s}{s} \left(I_{\ell-1} - \frac{z}{|z|} \otimes \frac{z}{|z|} \right) & u_{sm} \frac{z}{|z|} \\ u_{rm} \frac{y^T}{|y|} & u_{sm} \frac{z^T}{|z|} & u_{mm} \end{pmatrix}, \quad (6.8)$$

where I_k is the identity matrix of size k . Since the matrix

$$u_{rr} \frac{y}{|y|} \otimes \frac{y}{|y|} + \frac{u_r}{r} \left(I_{m-\ell} - \frac{y}{|y|} \otimes \frac{y}{|y|} \right)$$

has eigenvalues u_{rr} and u_r/r with multiplicities 1 and $m - \ell - 1$ respectively, we see that

$$\left| D^2u \right|^2 = u_{rr}^2 + (m - \ell - 1) \frac{u_r^2}{r^2} + u_{ss}^2 + (\ell - 1) \frac{u_s^2}{s^2} + u_{mm}^2 + 2u_{rs}^2 + 2u_{rm}^2 + 2u_{sm}^2. \quad (6.9)$$

Also, from (6.7) and (6.8) it follows that

$$\begin{aligned} D^2u(Du, \cdot) &= [u_{rr}u_r + u_{rs}u_s + u_{rm}u_m] \frac{y}{|y|} + [u_{rs}u_r + u_{ss}u_s + u_{sm}u_m] \frac{z}{|z|} \\ &\quad + [u_{rm}u_r + u_{sm}u_s + u_{mm}u_m] e_m \end{aligned} \quad (6.10)$$

and that

$$D^2u(Du, Du) = u_{rr}u_r^2 + 2u_{rs}u_ru_s + 2u_{rm}u_ru_m + u_{ss}u_s^2 + 2u_{sm}u_su_m + u_{mm}u_m^2. \quad (6.11)$$

For $u(|y|, x_m)$, (6.7)–(6.11) also hold with $\ell = 1$ and $u_s, u_{rs}, u_{ss}, u_{ms} = 0$.

Computing the gradient of U_ε , we obtain

$$|DU_\varepsilon|^2 = [(u_\varepsilon)_r]^2 + [(u_\varepsilon)_m]^2 \vartheta_\varepsilon^2 + u_\varepsilon^2 (\vartheta'_\varepsilon)^2 = |Du_\varepsilon|^2 \vartheta_\varepsilon^2 + u_\varepsilon^2 (\vartheta'_\varepsilon)^2, \quad (6.12)$$

and moreover

$$\begin{aligned} (u_\varepsilon)_r &= [\zeta'_\varepsilon(r) (1 - \varepsilon^{2\kappa} r^{2\kappa}) - \zeta_\varepsilon(r) 2\kappa \varepsilon^{2\kappa} r^{2\kappa-1}] \zeta_\varepsilon(x_m) A_\varepsilon(x_m), \\ (u_\varepsilon)_m &= \zeta_\varepsilon(r) (1 - \varepsilon^{2\kappa} r^{2\kappa}) [\zeta'_\varepsilon(x_m) A_\varepsilon(x_m) + \zeta_\varepsilon(x_m) a_\varepsilon(x_m)]. \end{aligned} \quad (6.13)$$

Hereafter, C and C_ε will denote constants whose value may change from line to line, with C_ε possibly depending on ε . From (6.4), we see that

$$|A_\varepsilon(x_m)| \leq 2\varepsilon \quad \text{for all } x_m \in \mathbb{R}. \quad (6.14)$$

Hence, using also (6.2), notice that

$$|u_\varepsilon(r, x_m)| \leq 2\varepsilon, \quad 0 \leq \vartheta_\varepsilon(s) \leq 1, \quad |\vartheta'_\varepsilon(s)| \leq C\varepsilon. \quad (6.15)$$

We first consider the region

$$\Omega_0 \doteq \left\{ |x_m| \geq \frac{3\varepsilon}{2} \right\} \subset \mathbb{R}^m.$$

Since $a_\varepsilon(x_m) \leq 1/2$ and $|\xi_\varepsilon| \leq 1$ due to (6.3), if $\varepsilon > 0$ is small, then (6.14), (6.3) and (6.15) give

$$\begin{aligned} W_\varepsilon^{-2} &= 1 - |DU_\varepsilon|^2 \geq 1 - |Du_\varepsilon|^2 \vartheta_\varepsilon^2 - C\varepsilon^2 \\ &\geq 1 - C\varepsilon^2 - (a_\varepsilon(x_m))^2 \geq 1/2. \end{aligned} \quad (6.16)$$

Since $U_\varepsilon \in C_c^2(\mathbb{R}^m)$, we get $W_\varepsilon \leq \sqrt{2}$ and $\|\Pi_{U_\varepsilon}\| \leq C$ on Ω_0 . Similarly, we study the region

$$\Omega_1 \doteq \left\{ |x_m| \leq \frac{3\varepsilon}{2}, |y| \geq \frac{1}{2\varepsilon} \right\}.$$

For $\delta_\kappa \doteq 2^{-2\kappa} > 0$ and $|y| \geq 1/(2\varepsilon)$,

$$0 \leq \zeta_\varepsilon(r) (1 - \varepsilon^{2\kappa} |y|^{2\kappa}) \leq 1 - \delta_\kappa.$$

Thus, by (6.3), (6.14), (6.13) and $0 \leq a(x_m) \leq 1$, if ε is small enough, then for some constant $\gamma_\kappa > 0$,

$$\begin{aligned} W_\varepsilon^{-2} &\geq 1 - |Du_\varepsilon|^2 \vartheta_\varepsilon^2 - C\varepsilon^2 \geq 1 - |Du_\varepsilon|^2 - C\varepsilon^2 \\ &\geq 1 - C\varepsilon^2 - (1 - \delta_\kappa)^2 [C\varepsilon^2 + 1] \geq \gamma_\kappa^2 > 0. \end{aligned}$$

Therefore, W_ε and thus $\|\Pi_{U_\varepsilon}\|$ are bounded on Ω_1 , too. Summarizing,

$$W_\varepsilon \leq C, \quad \|\Pi_{U_\varepsilon}\| \leq C \quad \text{on} \quad \left\{ |x_m| \geq \frac{3\varepsilon}{2} \right\} \cup \left\{ |y| \geq \frac{1}{2\varepsilon} \right\}. \quad (6.17)$$

Next, we shall check the integrability of W_ε and Π_{U_ε} on

$$\begin{aligned} \Omega_2 &\doteq \left\{ |y| < \frac{1}{2\varepsilon} \right\} \cup \left\{ |z| < \frac{1}{\varepsilon}, |x_m| \leq \varepsilon \right\}, \\ \Omega_3 &\doteq \left\{ |y| < \frac{1}{2\varepsilon} \right\} \cup \left\{ |z| < \frac{1}{\varepsilon}, \varepsilon \leq |x_m| \leq \frac{3\varepsilon}{2} \right\}. \end{aligned}$$

By (6.2), we have $U_\varepsilon(r, s, x_m) = u_\varepsilon(r, x_m) = (1 - \varepsilon^{2\kappa} r^{2\kappa}) A_\varepsilon(x_m)$ in a neighborhood of $\Omega_2 \cup \Omega_3$. In particular,

$$\begin{aligned} (u_\varepsilon)_r &= -2\kappa \varepsilon^{2\kappa} r^{2\kappa-1} A_\varepsilon(x_m), & (u_\varepsilon)_m &= (1 - \varepsilon^{2\kappa} r^{2\kappa}) a_\varepsilon(x_m), \\ (u_\varepsilon)_{rr} &= -2\kappa(2\kappa - 1) \varepsilon^{2\kappa} r^{2\kappa-2} A_\varepsilon(x_m), & (u_\varepsilon)_{rm} &= -2\kappa \varepsilon^{2\kappa} r^{2\kappa-1} a_\varepsilon(x_m), \\ (u_\varepsilon)_{mm} &= (1 - \varepsilon^{2\kappa} r^{2\kappa}) a'_\varepsilon(x_m). \end{aligned} \quad (6.18)$$

The bounds $1/2 \leq a_\varepsilon(x_m) \leq 1$ following from (6.4) lead to

$$\begin{aligned} W_\varepsilon^{-2} &= 1 - |Du_\varepsilon(x)|^2 = 1 - 4\kappa^2 \varepsilon^{4\kappa} r^{4\kappa-2} A_\varepsilon^2(x_m) - (1 - \varepsilon^{2\kappa} r^{2\kappa})^2 a_\varepsilon^2(x_m) \\ &= (1 - a_\varepsilon(x_m)) (1 + a_\varepsilon(x_m)) \\ &\quad + \varepsilon^{2\kappa} r^{2\kappa} \left[(2 - \varepsilon^{2\kappa} r^{2\kappa}) a_\varepsilon^2(x_m) - 4\kappa^2 \varepsilon^{2\kappa} r^{2\kappa-2} A_\varepsilon^2(x_m) \right] \\ &\geq 1 - a_\varepsilon(x_m) + \varepsilon^{2\kappa} r^{2\kappa} \left[\frac{1}{4} - 16\kappa^2 \varepsilon^4 \right]. \end{aligned}$$

Thus, for sufficiently small $\varepsilon > 0$, we get

$$W_\varepsilon \leq C_\varepsilon (1 - a_\varepsilon(x_m) + r^{2\kappa})^{-\frac{1}{2}} \quad \text{on } \Omega_2 \cup \Omega_3. \quad (6.19)$$

In particular, using $0 \leq a_\varepsilon(x_m) \leq 1$ we deduce

$$W_\varepsilon \leq C_\varepsilon r^{-\kappa} \in L^q(\Omega_2 \cup \Omega_3) \quad \text{for each } q < \frac{m-\ell}{\kappa}.$$

Regarding Π_{U_ε} , since $\kappa \geq 1$ and U_ε has bounded support, it follows from (6.9), (6.10), (6.11) and (6.18) that for $u = U_\varepsilon (= u_\varepsilon)$

$$\begin{aligned} |D^2 U_\varepsilon| &\leq C \left\{ |u_{rr}| + \left| \frac{u_r}{r} \right| + |u_{rm}| + |u_{mm}| \right\} \leq C_\varepsilon (r^{2\kappa-2} + r^{2\kappa-1} + |a'_\varepsilon(x_m)|) \\ &\leq C_\varepsilon (r^{2\kappa-2} + |a'_\varepsilon(x_m)|), \\ |D^2 U_\varepsilon(DU_\varepsilon, \cdot)| &\leq |u_{rr}u_r + u_{rm}u_m| + |u_{rm}u_r + u_{mm}u_m| \leq C_\varepsilon (r^{4\kappa-3} + r^{2\kappa-1} + |a'_\varepsilon(x_m)|) \\ &\leq C_\varepsilon (r^{2\kappa-1} + |a'_\varepsilon(x_m)|), \\ |D^2 U_\varepsilon(DU_\varepsilon, DU_\varepsilon)| &\leq |u_{rr}u_r^2 + 2u_{rm}u_ru_m + u_{mm}u_m^2| \leq C_\varepsilon (r^{6\kappa-4} + r^{4\kappa-2} + |a'_\varepsilon(x_m)|). \end{aligned}$$

By using (6.19), (2.3) and $W_\varepsilon \geq 1$, we deduce

$$\begin{aligned} \|\Pi_{U_\varepsilon}\| &\leq W_\varepsilon |D^2 U_\varepsilon| + 2W_\varepsilon^2 |D^2 U_\varepsilon(DU_\varepsilon, \cdot)| + W_\varepsilon^3 |D^2 U_\varepsilon(DU_\varepsilon, DU_\varepsilon)| \\ &\leq C_\varepsilon [r^{\kappa-2} + r^{-1} + W_\varepsilon^3 |a'_\varepsilon(x_m)|] \leq C_\varepsilon (r^{-1} + W_\varepsilon^3 |a'_\varepsilon(x_m)|). \end{aligned} \quad (6.20)$$

Whence, to prove that $\|\Pi_{U_\varepsilon}\| \in L^q(\Omega_2 \cup \Omega_3)$ for $q < (m-\ell)/\kappa$, taking into account (6.19) and that $a'_\varepsilon = 0$ on $[0, \varepsilon]$ it suffices to show

$$(1 - a_\varepsilon(x_m) + |y|^{2\kappa})^{-\frac{3}{2}} |a'_\varepsilon(x_m)| \in L^q \left(\left\{ |y| < \frac{1}{2\varepsilon}, |z| < \frac{1}{\varepsilon}, \varepsilon \leq |x_m| \leq \frac{3\varepsilon}{2} \right\} \right) \quad (6.21)$$

for each $q < \frac{m-\ell}{\kappa}$. Notice that it is enough to check it for $\frac{m-\ell}{3\kappa} < q < \frac{m-\ell}{\kappa}$ and for $\varepsilon \leq x_m \leq 3\varepsilon/2$, since a_ε is even. Due to (6.19) and since we can reduce to integrate in the variables (y, x_m) , using polar coordinates we get

$$\begin{aligned} &\int_\varepsilon^{\frac{3\varepsilon}{2}} dx_m \int_{|y| \leq 1/(2\varepsilon)} (1 - a_\varepsilon(x_m) + |y|^{2\kappa})^{-\frac{3q}{2}} |a'_\varepsilon(x_m)|^q dy \\ &\leq C_\varepsilon \int_\varepsilon^{\frac{3\varepsilon}{2}} dx_m \int_0^{\frac{1}{2\varepsilon}} |a'_\varepsilon(x_m)|^q (1 - a_\varepsilon(x_m) + r^{2\kappa})^{-\frac{3q}{2}} r^{m-\ell-1} dr \\ &\leq C_\varepsilon \int_\varepsilon^{\frac{3\varepsilon}{2}} dx_m \int_0^{(1-a_\varepsilon(x_m))^{1/(2\kappa)}} |a'_\varepsilon(x_m)|^q (1 - a_\varepsilon(x_m))^{-\frac{3q}{2}} r^{m-\ell-1} dr \\ &\quad + C_\varepsilon \int_\varepsilon^{\frac{3\varepsilon}{2}} dx_m \int_{(1-a_\varepsilon(x_m))^{1/(2\kappa)}}^{\frac{1}{2\varepsilon}} |a'_\varepsilon(x_m)|^q r^{-3q\kappa+m-\ell-1} dr \\ &\leq C_\varepsilon \int_\varepsilon^{\frac{3\varepsilon}{2}} |a'_\varepsilon(x_m)|^q (1 - a_\varepsilon(x_m))^{-\frac{3q}{2} + \frac{m-\ell}{2\kappa}} dx_m. \end{aligned} \quad (6.22)$$

Recalling $a_\varepsilon(x_m) = 1 - d_\varepsilon \exp\left(-\frac{1}{x_m-\varepsilon}\right)$ in (6.4), we have

$$|a'_\varepsilon(x_m)|^q (1 - a_\varepsilon(x_m))^{\frac{m-\ell-3q\kappa}{2\kappa}} \leq C_\varepsilon (x_m - \varepsilon)^{-2q} \exp\left(\frac{\kappa q - (m-\ell)}{2\kappa(x_m - \varepsilon)}\right).$$

Hence, if $\frac{m-\ell}{3\kappa} < q < \frac{m-\ell}{\kappa}$, then

$$\int_{\varepsilon}^{\frac{3\varepsilon}{2}} |a'_\varepsilon(x_m)|^q (1 - a_\varepsilon(x_m))^{-\frac{3q}{2} + \frac{m-\ell}{2\kappa}} dx_m < \infty.$$

Thus, $\|\Pi_{U_\varepsilon}\| \in L^q(\Omega_2 \cup \Omega_3)$ holds for each $q < (m-\ell)/\kappa$, as required.

If $\ell = 1$, that is, if the variable z is missing, we have therefore concluded the desired integrability properties of W_ε and $\|\Pi_{U_\varepsilon}\|$, since so far we only used that $0 \leq \vartheta_\varepsilon \leq 1$. The reader may therefore skip to the end of the proof, where we check that U_ε is a weak solution. To conclude for $\ell \geq 2$, we shall check the integrability of $\|\Pi_{U_\varepsilon}\|$ on $\Omega_4 \cup \Omega_5$, where

$$\begin{aligned} \Omega_4 &\doteq \left\{ |y| \leq \frac{1}{2\varepsilon}, \frac{1}{\varepsilon} < |z| \leq \frac{3}{2\varepsilon}, |x_m| \leq \frac{3\varepsilon}{2} \right\}, \\ \Omega_5 &\doteq \left\{ |y| \leq \frac{1}{2\varepsilon}, \frac{3}{2\varepsilon} \leq |z| \leq \frac{2}{\varepsilon}, |x_m| \leq \frac{3\varepsilon}{2} \right\}. \end{aligned}$$

This is achieved by similar estimates, though computationally more demanding.

We first prove that $|DU_\varepsilon| < 1$ on $\Omega_4 \cup \Omega_5$. Since $U_\varepsilon(r, s, x_m) = (1 - \varepsilon^{2\kappa} r^{2\kappa}) \vartheta_\varepsilon(s) A_\varepsilon(x_m)$ on $\Omega_4 \cup \Omega_5$,

$$\begin{aligned} (U_\varepsilon)_r &= -2\kappa \varepsilon^{2\kappa} r^{2\kappa-1} \vartheta_\varepsilon A_\varepsilon, & (U_\varepsilon)_s &= (1 - \varepsilon^{2\kappa} r^{2\kappa}) \vartheta'_\varepsilon A_\varepsilon, & (U_\varepsilon)_m &= (1 - \varepsilon^{2\kappa} r^{2\kappa}) \vartheta_\varepsilon a_\varepsilon, \\ (U_\varepsilon)_{rr} &= -2\kappa(2\kappa - 1) \varepsilon^{2\kappa} r^{2\kappa-2} \vartheta_\varepsilon A_\varepsilon, & (U_\varepsilon)_{rs} &= -2\kappa \varepsilon^{2\kappa} r^{2\kappa-1} \vartheta'_\varepsilon A_\varepsilon, \\ (U_\varepsilon)_{rm} &= -2\kappa \varepsilon^{2\kappa} r^{2\kappa-1} \vartheta_\varepsilon a'_\varepsilon, & (U_\varepsilon)_{ss} &= (1 - \varepsilon^{2\kappa} r^{2\kappa}) \vartheta''_\varepsilon A_\varepsilon, \\ (U_\varepsilon)_{sm} &= (1 - \varepsilon^{2\kappa} r^{2\kappa}) \vartheta'_\varepsilon a'_\varepsilon, & (U_\varepsilon)_{mm} &= (1 - \varepsilon^{2\kappa} r^{2\kappa}) \vartheta_\varepsilon a'_\varepsilon. \end{aligned} \tag{6.23}$$

Thus,

$$\begin{aligned} W_\varepsilon^{-2} &= 1 - |DU_\varepsilon|^2 \\ &= 1 - 4\kappa^2 \varepsilon^{4\kappa} r^{4\kappa-2} \vartheta_\varepsilon^2 A_\varepsilon^2 - (1 - 2\varepsilon^{2\kappa} r^{2\kappa} + \varepsilon^{4\kappa} r^{4\kappa}) [(\vartheta'_\varepsilon)^2 A_\varepsilon^2 + \vartheta_\varepsilon^2 a_\varepsilon^2] \\ &= 1 - (\vartheta'_\varepsilon)^2 A_\varepsilon^2 - \vartheta_\varepsilon^2 a_\varepsilon^2 + \varepsilon^{2\kappa} r^{2\kappa} [(2 - \varepsilon^{2\kappa} r^{2\kappa}) \{(\vartheta'_\varepsilon)^2 A_\varepsilon^2 + \vartheta_\varepsilon^2 a_\varepsilon^2\} - 4\kappa^2 \varepsilon^{2\kappa} r^{2\kappa-2} \vartheta_\varepsilon^2 A_\varepsilon^2]. \end{aligned}$$

By

$$|A_\varepsilon(x_m)| \leq 2\varepsilon, \quad \frac{1}{2} \leq a_\varepsilon(x_m) \leq 1, \quad \varepsilon r = |\varepsilon y| \leq \frac{1}{2} \quad \text{for each } (y, z, x_m) \in \Omega_4 \cup \Omega_5,$$

if $\varepsilon > 0$ is sufficiently small, then

$$(2 - \varepsilon^{2\kappa} r^{2\kappa}) \vartheta_\varepsilon^2 a_\varepsilon^2 - 4\kappa^2 \varepsilon^{2\kappa} r^{2\kappa-2} \vartheta_\varepsilon^2 A_\varepsilon^2 \geq \frac{1}{8} \vartheta_\varepsilon^2.$$

Therefore, for every $(y, z, x_m) \in \Omega_4 \cup \Omega_5$,

$$W_\varepsilon^{-2} \geq 1 - (\vartheta'_\varepsilon(|z|))^2 A_\varepsilon^2(x_m) - \vartheta_\varepsilon^2(|z|) a_\varepsilon^2(x_m) + \frac{1}{8} \varepsilon^{2\kappa} |y|^{2\kappa} \vartheta_\varepsilon^2(|z|). \tag{6.24}$$

When $(y, z, x_m) \in \Omega_5$, by $3/2 \leq \varepsilon|z| \leq 2$ and (6.2), we see that

$$(\vartheta'_\varepsilon(|z|))^2 \leq C\varepsilon^2, \quad \vartheta_\varepsilon^2(|z|) \leq \vartheta_\varepsilon^2\left(\frac{3}{2\varepsilon}\right) = \frac{1}{4},$$

which implies that if ε is sufficiently small, then for all $(y, z, x_m) \in \Omega_5$,

$$W_\varepsilon^{-2} = 1 - |DU_\varepsilon|^2 \geq 1 - C\varepsilon^4 - \frac{1}{4} \geq \frac{1}{2}.$$

Hence,

$$W_\varepsilon, \left\| \Pi_{U_\varepsilon} \right\| \in L^\infty(\Omega_5). \quad (6.25)$$

On the other hand, when $(y, z, x_m) \in \Omega_4$, we have $\vartheta_\varepsilon(|z|) \geq 1/2$, and (6.24) yields

$$W_\varepsilon^{-2} \geq 1 - 4\varepsilon^2(\vartheta'_\varepsilon(|z|))^2 - \vartheta_\varepsilon^2(|z|)a_\varepsilon^2(x_m) + \frac{\varepsilon^{2\kappa}|y|^{2\kappa}}{32}.$$

Thus, to show $|DU_\varepsilon| < 1$, it suffices to prove

$$4\varepsilon^2 (\vartheta'_\varepsilon(s))^2 + \vartheta_\varepsilon^2(s) = 4\varepsilon^4 (\vartheta'_1(\varepsilon s))^2 + \vartheta_1^2(\varepsilon s) < 1 \quad \text{for each } \frac{1}{\varepsilon} < s \leq \frac{3}{2\varepsilon}. \quad (6.26)$$

To this end, from (6.2) and

$$\vartheta'_1(t) = -\frac{e^2}{2}(t-1)^{-2} \exp(-(t-1)^{-1}),$$

it follows that for $1 < t \leq \frac{3}{2}$

$$\begin{aligned} & 4\varepsilon^4 (\vartheta'_1(t))^2 + \vartheta_1^2(t) \\ &= \varepsilon^4 e^4 (t-1)^{-4} \exp(-2(t-1)^{-1}) + \left[1 - \frac{e^2}{2} \exp(-(t-1)^{-1}) \right]^2 \\ &= 1 - e^2 \left[1 - \frac{e^2}{4} \exp(-(t-1)^{-1}) - \varepsilon^4 e^2 (t-1)^{-4} \exp(-(t-1)^{-1}) \right] \exp(-(t-1)^{-1}). \end{aligned}$$

Since

$$1 - \frac{e^2}{4} \exp(-(t-1)^{-1}) \geq 1 - \frac{e^2}{4} e^{-2} = \frac{3}{4} \quad \text{for every } 1 < t \leq \frac{3}{2},$$

for sufficiently small $\varepsilon > 0$,

$$4\varepsilon^4 (\vartheta'_1(t))^2 + \vartheta_1^2(t) \leq 1 - \frac{e^2}{2} \exp(-(t-1)^{-1}) < 1. \quad (6.27)$$

Hence, $|DU_\varepsilon| < 1$ on Ω_4 . In addition, by $1 - 4\varepsilon^2(\vartheta'_\varepsilon(|z|))^2 - \vartheta_\varepsilon^2(|z|)a_\varepsilon^2(x_m) \geq 0$, we have

$$\begin{aligned} W_\varepsilon(y, z, x_m) &\leq C_\varepsilon \left[1 - 4\varepsilon^2 (\vartheta'_\varepsilon(|z|))^2 - \vartheta_\varepsilon^2(|z|)a_\varepsilon^2(x_m) + |y|^{2\kappa} \right]^{-1/2} \quad \text{on } \Omega_4. \\ &\leq C_\varepsilon |y|^{-\kappa} \end{aligned} \quad (6.28)$$

Thus, $W_\varepsilon \in L^q(\Omega_4)$ for $q < \frac{m-\ell}{\kappa}$. To show $\|\Pi_{U_\varepsilon}\| \in L^q(\Omega_4)$, by $\kappa \geq 1$, (6.9), (6.10), (6.11) and (6.23) we deduce that, for $(y, z, x_m) \in \Omega_4$,

$$\begin{aligned} |D^2 U_\varepsilon| &\leq C_\varepsilon \{ |y|^{2\kappa-2} + |\vartheta''_\varepsilon(|z|)| + |\vartheta'_\varepsilon(|z|)| + |a'_\varepsilon(x_m)| \}, \\ |D^2 U_\varepsilon(DU_\varepsilon, \cdot)| &\leq C_\varepsilon \{ |y|^{2\kappa-1} + |\vartheta'_\varepsilon(|z|)| + |a'_\varepsilon(x_m)| \}, \\ |D^2 U_\varepsilon(DU_\varepsilon, DU_\varepsilon)| &\leq C_\varepsilon \{ |y|^{4\kappa-2} + (\vartheta'_\varepsilon(|z|))^2 + |a'_\varepsilon(x_m)| \}. \end{aligned} \quad (6.29)$$

Due to (6.28), we verify that for all $q < (m-\ell)/\kappa$

$$W_\varepsilon(y, z, x_m)|y|^{2\kappa-2} + W_\varepsilon^2(y, z, x_m)|y|^{2\kappa-1} + W_\varepsilon^3(y, z, x_m)|y|^{4\kappa-2} \leq C_\varepsilon |y|^{-1} \in L^q(\Omega_4). \quad (6.30)$$

On the other hand, by (6.26) and (6.27), we notice that

$$1 - 4\varepsilon^2 (\vartheta'_\varepsilon(|z|))^2 - \vartheta_\varepsilon^2(|z|) \geq \frac{e^2}{2} \exp(-(\varepsilon|z| - 1)^{-1}),$$

which yields

$$W_\varepsilon(y, z, x_m) \leq C_\varepsilon \exp\left(\frac{1}{2}(\varepsilon|z| - 1)^{-1}\right) \quad \text{on } \Omega_4.$$

From (6.2),

$$|\vartheta''_\varepsilon(|z|)| + |\vartheta'_\varepsilon(|z|)| \leq C_\varepsilon (\varepsilon|z| - 1)^{-4} \exp(-(\varepsilon|z| - 1)^{-1}).$$

Hence,

$$\begin{aligned} & W_\varepsilon(y, z, x_m) \{|\vartheta''_\varepsilon(|z|)| + |\vartheta'_\varepsilon(|z|)|\} + W_\varepsilon^3(y, z, x_m) (\vartheta'_\varepsilon(|z|))^2 \\ & \leq C_\varepsilon (\varepsilon|z| - 1)^{-4} \exp\left(-\frac{1}{2}(\varepsilon|z| - 1)^{-1}\right) \in L^\infty(\Omega_4). \end{aligned} \quad (6.31)$$

Moreover,

$$\begin{aligned} & W_\varepsilon^2(y, z, x_m) |\vartheta'_\varepsilon(|z|)| \\ & = W_\varepsilon^{2-\kappa^{-1}}(y, z, x_m) W_\varepsilon^{\kappa^{-1}}(y, z, x_m) |\vartheta'_\varepsilon(|z|)| \\ & \leq C_\varepsilon \exp\left(\frac{2-\kappa^{-1}}{2}(\varepsilon|z| - 1)^{-1}\right) (C_\varepsilon |y|^{-\kappa})^{\kappa^{-1}} (\varepsilon|z| - 1)^{-2} \exp(-(\varepsilon|z| - 1)^{-1}) \\ & = C_\varepsilon (\varepsilon|z| - 1)^{-2} \exp\left(-\frac{1}{2\kappa}(\varepsilon|z| - 1)^{-1}\right) |y|^{-1} \in L^q(\Omega_4) \quad \text{if } q < \frac{m-\ell}{\kappa}. \end{aligned} \quad (6.32)$$

By (6.29), (6.30), (6.31), (6.32) and $W_\varepsilon \geq 1$, to show $\|\Pi_{U_\varepsilon}\| \in L^q(\Omega_4)$ for $q < (m-\ell)/\kappa$, it remains to prove

$$W_\varepsilon^3(x, y, z_m) |a'_\varepsilon(x_m)| \in L^q(\Omega_4) \quad \text{for each } q < \frac{m-\ell}{\kappa}. \quad (6.33)$$

Since $a'_\varepsilon(x_m) = 0$ for $|x_m| \leq \frac{\varepsilon}{2}$ and a_ε is even, we may suppose $\frac{\varepsilon}{2} < x_m \leq \frac{3\varepsilon}{2}$. In this case, from (6.4) and (6.2), notice that

$$\begin{aligned} & 1 - 4\varepsilon^2 (\vartheta'_\varepsilon(|z|))^2 - \vartheta_\varepsilon^2(|z|) a_\varepsilon^2(x_m) \\ & = [1 + \vartheta_\varepsilon(|z|) a_\varepsilon(x_m)] [1 - \vartheta_\varepsilon(|z|) a_\varepsilon(x_m)] - 4\varepsilon^4 (\vartheta'_1(\varepsilon|z|))^2 \\ & \geq 1 - \vartheta_\varepsilon(|z|) a_\varepsilon(x_m) - 4\varepsilon^4 (\vartheta'_1(\varepsilon|z|))^2 \\ & \geq 1 - \left[1 - \frac{e^2}{2} \exp(-(\varepsilon|z| - 1)^{-1})\right] \left[1 - d_\varepsilon \exp(-(x_m - \varepsilon)^{-1})\right] - 4\varepsilon^4 (\vartheta'_1(\varepsilon|z|))^2 \\ & \geq c_0 \left\{ \exp(-(\varepsilon|z| - 1)^{-1}) + \exp(-(x_m - \varepsilon)^{-1}) \right\} \doteq c_0 R^2(|z|, x_m). \end{aligned}$$

Thus, by (6.28),

$$W_\varepsilon(y, z, x_m) \leq C_\varepsilon \left\{ R^2(|z|, x_m) + |y|^{2\kappa} \right\}^{-\frac{1}{2}}.$$

Then we proceed as in (6.22) and for $\frac{m-\ell}{3\kappa} < q < \frac{m-\ell}{\kappa}$, we obtain

$$\begin{aligned}
& \int_{\varepsilon}^{\frac{3\varepsilon}{2}} dx_m \int_{\frac{1}{\varepsilon} < |z| < \frac{3}{2\varepsilon}} dz \int_{|y| \leq \frac{1}{2\varepsilon}} (W_{\varepsilon}^3(y, z, x_m) |a'_{\varepsilon}(x_m)|)^q dy \\
& \leq C_{\varepsilon} \int_{\varepsilon}^{\frac{3\varepsilon}{2}} dx_m \int_{\frac{1}{\varepsilon} < |z| < \frac{3}{2\varepsilon}} dz \int_{|y| \leq R^{1/\kappa}(|z|, x_m)} R^{-3q}(|z|, x_m) |a'_{\varepsilon}(x_m)|^q dy \\
& \quad + C_{\varepsilon} \int_{\varepsilon}^{\frac{3\varepsilon}{2}} dx_m \int_{\frac{1}{\varepsilon} < |z| < \frac{3}{2\varepsilon}} dz \int_{R^{1/\kappa}(|z|, x_m) \leq |y| \leq \frac{1}{2\varepsilon}} |y|^{-3\kappa q} |a'_{\varepsilon}(x_m)|^q dy \\
& \leq C_{\varepsilon} \int_{\varepsilon}^{\frac{3\varepsilon}{2}} dx_m \int_{\frac{1}{\varepsilon} < |z| < \frac{3}{2\varepsilon}} R^{-3q + \frac{m-\ell}{\kappa}}(|z|, x_m) |a'_{\varepsilon}(x_m)|^q dz \\
& \leq C_{\varepsilon} \int_0^{\frac{\varepsilon}{2}} dt \int_1^{\frac{3}{2}} \left\{ \exp\left(-\frac{1}{s-1}\right) + \exp\left(-\frac{1}{t}\right) \right\}^{\frac{m-\ell-3\kappa q}{2\kappa}} t^{-2q} \exp\left(-\frac{q}{t}\right) ds \\
& = C_{\varepsilon} \int_0^{\frac{\varepsilon}{2}} dt \int_0^{\frac{1}{2}} \left\{ \exp\left(-\frac{1}{s}\right) + \exp\left(-\frac{1}{t}\right) \right\}^{\frac{m-\ell-3\kappa q}{2\kappa}} t^{-2q} \exp\left(-\frac{q}{t}\right) ds \\
& \leq C_{\varepsilon} \int_0^{\frac{\varepsilon}{2}} dt \int_0^t \exp\left(\frac{3\kappa q - m + \ell}{2\kappa t}\right) t^{-2q} \exp\left(-\frac{q}{t}\right) ds \\
& \quad + C_{\varepsilon} \int_0^{\frac{\varepsilon}{2}} dt \int_t^{\frac{1}{2}} \exp\left(\frac{3\kappa q - m + \ell}{2\kappa s}\right) t^{-2q} \exp\left(-\frac{q}{t}\right) ds \\
& \leq C_{\varepsilon} \int_0^{\frac{\varepsilon}{2}} t^{-2q} \exp\left(\frac{\kappa q - m + \ell}{2\kappa t}\right) dt < \infty.
\end{aligned}$$

Hence, (6.33) holds and (6.6) follows.

Finally, we prove that u is a weak solution. Let $\eta \in \text{Lip}_c(\Omega)$. First, observe that our estimates guarantee that

$$W_{\varepsilon}(y, z, x_m) \leq C_{\varepsilon} |y|^{-\kappa} \quad \text{for each } (y, z, x_m) \in \mathbb{R}^m.$$

Hence, $W_{\varepsilon} \in L^1(\mathbb{R}^m)$. From $\rho_{U_{\varepsilon}} \in L^q(\mathbb{R}^m)$ and the dominated convergence theorem, it follows that

$$\int_{\mathbb{R}^m} \rho_{U_{\varepsilon}} \eta dx = \lim_{\tau \rightarrow 0} \int_{\{|y| > \tau\}} \rho_{U_{\varepsilon}} \eta dx = - \lim_{\tau \rightarrow 0} \int_{\{|y| > \tau\}} \text{div}(W_{\varepsilon} DU_{\varepsilon}) \eta dx. \quad (6.34)$$

Integration by parts gives

$$- \int_{\{|y| > \tau\}} \text{div}(W_{\varepsilon} DU_{\varepsilon}) \eta dx = \int_{\{|y| = \tau\}} W_{\varepsilon} \eta DU_{\varepsilon} \cdot \frac{y}{|y|} d\mathcal{H}_{\delta}^{m-1} + \int_{\{|y| > \tau\}} W_{\varepsilon} DU_{\varepsilon} \cdot D\eta dx. \quad (6.35)$$

By (6.13) and

$$\left| DU_{\varepsilon} \cdot \frac{y}{|y|} \right| = |(U_{\varepsilon})_r| = |(u_{\varepsilon})_r \vartheta_{\varepsilon}| \leq C \tau^{2\kappa-1} \quad \text{if } |y| = \tau,$$

it follows from the estimate for W_{ε} and the assumption $\ell \leq m-2$ that

$$\limsup_{\tau \rightarrow 0} \int_{\{|y| = \tau\}} \left| W_{\varepsilon} \eta DU_{\varepsilon} \cdot \frac{y}{|y|} \right| d\mathcal{H}_{\delta}^{m-1} \leq \lim_{\tau \rightarrow 0} C \tau^{-\kappa} \tau^{2\kappa-1} \tau^{m-\ell-1} = 0.$$

Finally, since $W_\varepsilon \in L^1$, it follows from (6.34) and (6.35) that

$$\int_{\mathbb{R}^m} \rho_{U_\varepsilon} \eta \, dx = \int_{\mathbb{R}^m} W_\varepsilon DU_\varepsilon \cdot D\eta \, dx,$$

and we complete the proof. \square

A The Born-Infeld model

We here recall the Born-Infeld model of electromagnetism [16, 17]. Concise but informative introductions can be found in [11, 12], see also [58, 35, 36, 8] for a thorough account of the physical literature. We remark that the Born-Infeld model also proved to be relevant in the theory of superstrings and membranes, see [32, 58] and the references therein.

As outlined in the Introduction, one of the main concerns of the theory was to overcome the failure of the principle of finite energy occurring in Maxwell's model, which we now describe. In a spacetime (N^4, g) with metric $g = g_{ab}dy^a \otimes dy^b$ of signature $(-, +, +, +)$ ($g_{00} < 0$), the electromagnetic field is described as a closed 2-form $F = \frac{1}{2}F_{ab}dy^a \wedge dy^b$ which, according to the Lagrangian formulation of Maxwell's theory due to Schwarzschild (cf. [50] and [46, p.88]), in the absence of charges and currents, is required to be stationary for the action

$$\mathcal{L}_{\text{MS}} \doteq \int_{N^4} \mathsf{L}_{\text{MS}} \sqrt{-|g|} dy \quad \text{with} \quad \mathsf{L}_{\text{MS}} \doteq -\frac{F^{ab}F_{ab}}{4},$$

where $|g|$ is the determinant of g and $F^{ab} \doteq g^{ac}g^{bd}F_{cd}$. The presence of a vector field J describing charges and currents is taken into account by adding the action

$$\mathcal{L}_J \doteq \int_{N^4} \mathsf{L}_J \sqrt{-|g|} dy, \quad \mathsf{L}_J = J^a \Phi_a,$$

where we assumed that F is globally exact and we set $F = d\Phi$. By its very definition, the energy-impulse tensor T associated to $\mathcal{L}_{\text{MS}} + \mathcal{L}_J$ has components

$$T_{ab} = \frac{-2}{\sqrt{-|g|}} \frac{\partial((\mathsf{L}_{\text{MS}} + \mathsf{L}_J)\sqrt{-|g|})}{\partial g^{ab}} = F_{ac}F_{bp}g^{cp} - \frac{1}{4}F^{cp}F_{cp}g_{ab} + J^c \Phi_c g_{ab}$$

and in particular T_{00} describes the energy density. In Minkowski space \mathbb{L}^4 , by writing in Cartesian coordinates $\{x^a\}$ the electromagnetic tensor in terms of the electric and magnetic fields $\mathbf{E} = E_j dx^j$ and $\mathbf{B} = B_j dx^j$ as

$$F = \sum_{j=1}^3 E_j dx^j \wedge dx^0 + B_1 dx^2 \wedge dx^3 + B_2 dx^3 \wedge dx^1 + B_3 dx^1 \wedge dx^2,$$

the vector potential as $\Phi = -\varphi dx^0 + \mathbf{A} = -\varphi dx^0 + A_j dx^j$ and $J = \rho \partial_{x^0} + \mathbf{J} = \rho \partial_{x^0} + J^j \partial_{x^j}$, the Maxwell Lagrangian and energy densities become

$$\mathsf{L}_{\text{MS}} + \mathsf{L}_J = \frac{1}{2}(|\mathbf{E}|^2 - |\mathbf{B}|^2) - \rho\varphi + \mathbf{A}(\mathbf{J}), \quad T_{00} = \frac{1}{2}(|\mathbf{E}|^2 + |\mathbf{B}|^2) + \rho\varphi - \mathbf{A}(\mathbf{J}).$$

Restricting to the electrostatic case with no current density ($\mathbf{B} = 0$, \mathbf{E} independent of x^0 , $\mathbf{J} = 0$), from $\mathbf{E} = -d\varphi$ the potential φ turns out to be stationary for the reduced action

$$J_\rho(v) \doteq \frac{1}{2} \int_{\mathbb{R}^3} |Dv|^2 dx - \langle \rho, v \rangle,$$

where $\langle \rho, v \rangle$ is the duality pairing given, for smooth ρ , by integration. However, for $\rho = \delta_{x_0}$ the Dirac delta centered at a point x_0 , the Newtonian potential $\bar{u}_\rho = \text{const} \cdot |x - x_0|^{2-m}$ solving the Euler-Lagrange equation $-\Delta \bar{u}_\rho = \rho$ for J_ρ has infinite energy on punctured balls centered at x_0 :

$$\int_{B_R \setminus B_\varepsilon} T_{00} dx = \frac{1}{2} \int_{B_R \setminus B_\varepsilon} |D\bar{u}_\rho|^2 dx \rightarrow \infty \quad \text{as } \varepsilon \rightarrow 0,$$

a fact of serious physical concern (cf. [17]). The problem also persists for certain sources $\rho \in L^1(\mathbb{R}^m)$, see [27, 11]. To avoid it, Born and Infeld in [16, 17] proposed an alternative Lagrangian density \mathcal{L}_{BI} , defined by

$$\mathcal{L}_{\text{BI}} \sqrt{-|g|} \doteq \sqrt{-|g|} - \sqrt{-|g + F|}$$

(we follow the convention in [58], which changes signs in \mathcal{L}_{BI} with respect to [17]). Indeed, the authors also indicated a further Lagrangian, already suggested in [15], see [17, (2.27) and (2.28)]. However, later works in [9, 47] pointed out a distinctive feature of \mathcal{L}_{BI} , that of generating an electrodynamics free from the phenomenon of birefringence (cf. [36, 8]). Computing the determinants in the expression of \mathcal{L}_{BI} , Born and Infeld obtained

$$\mathcal{L}_{\text{BI}} = 1 - \sqrt{1 + \mathbb{F} - \mathbb{G}^2}, \quad \text{where } \mathbb{F} \doteq \frac{F^{ab} F_{ab}}{2}, \quad \mathbb{G} \doteq \frac{F_{ab} (\star F)^{ab}}{4}$$

and \star is the Hodge dual, so \mathcal{L}_{BI} is asymptotic to \mathcal{L}_{MS} for small F and flat g . In Minkowski space and Cartesian coordinates $\{x^a\}$, \mathcal{L}_{BI} becomes

$$\mathcal{L}_{\text{BI}} = 1 - \sqrt{1 - |\mathbf{E}|^2 + |\mathbf{B}|^2 - (\mathbf{B} \cdot \mathbf{E})^2}. \quad (\text{A.1})$$

Here, we set the maximal field strength to be 1 for convenience. The energy-impulse tensor associated to $\mathcal{L}_{\text{BI}} + \mathcal{L}_J$, and its component T_{00} in Cartesian coordinates, are thus

$$\begin{aligned} T_{ab} &= \mathcal{L}_{\text{BI}} g_{ab} + \frac{F_{ac} F_{bp} g^{cp} - \mathbb{G}^2 g_{ab}}{\sqrt{1 + \mathbb{F} - \mathbb{G}^2}} + J^c \Phi_c g_{ab}, \\ T_{00} &= \frac{1 + |\mathbf{B}|^2}{\sqrt{1 - |\mathbf{E}|^2 + |\mathbf{B}|^2 - (\mathbf{B} \cdot \mathbf{E})^2}} - 1 + \rho\varphi - \mathbf{A}(\mathbf{J}). \end{aligned}$$

In the electrostatic case, the potential u_ρ generated by ρ is therefore required to minimize the action I_ρ in (1.2) on $\Omega = \mathbb{R}^3$ among weakly spacelike functions with a suitable decay at infinity, and its energy density is given by

$$T_{00} = \frac{1}{\sqrt{1 - |Du_\rho|^2}} - 1 + \rho u_\rho = w_\rho - 1 + \rho u_\rho,$$

making apparent the link between w_ρ and T_{00} mentioned in the Introduction. Whence, the integrability (1.11) proved in [11, Proposition 2.7] can be rephrased as the remarkable property

$$T_{00} - \rho u_\rho \in L^1(\mathbb{R}^3) \quad (\text{A.2})$$

for ρ in a large class of distributions including any finite measure on \mathbb{R}^3 . The case $\rho = \delta_{x_0}$ was previously considered by Born and Infeld in [17] to support the consistency of their theory. For solutions in bounded domains, Proposition 3.13 guarantees the same desirable property: $T_{00} - \rho u_\rho \in L^1_{\text{loc}}(\Omega)$, provided that the boundary datum ϕ is not too degenerate.

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