UNIVERSITÀ DEGLI STUDI DI MILANO

FACOLTÀ DI SCIENZE E TECNOLOGIE

Dipartimento di Matematica Federico Enriques

Corso di Dottorato di Ricerca in Matematica

XXXIII ciclo

Tesi di Dottorato di Ricerca

Weight Structures and Motivic Comparisons of p-adic Cohomology Theories for Rigid Analytic Spaces

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MAT 02

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A. A. 2024/2025

“古人之观于天地、山川、草木、虫鱼、鸟兽，往往有得，
以其求思之深而无不在也。夫夷以近，则游者众；险以远，则至者少。
而世之奇伟、瑰怪，非常之观，而人之所罕至焉。”
——王安石《游褒禅山记》

The ancients, in their contemplation of the heavens and the earth,
the mountains and rivers, the trees and living creatures,
often attained deep insight—for their seeking was profound
and touched all things. The world’s most wondrous sights
lie in perilous and distant places, where few have ever set foot.

Abstract

In this thesis, we study weight structures and p-adic cohomology theories for smooth rigid analytic spaces in the motivic context. The work consists of two independent components.

The first part concerns about weight structures, a powerful tool for studying motives introduced by Bondarko. More precisely, we construct a bounded monoidal weight structure on compact rigid analytic motives over a complete non-archimedean field K, using Galois descent. This extends the weight structure on rigid analytic motives with good reduction studied in [BGV25]. In particular, the full subcategory of compact rigid analytic motives over K admits a symmetric monoidal functor, called the weight complex functor, landing in a stable ∞-category of bounded chain complexes.

The second part establishes a comparison between two p-adic cohomology theories for smooth rigid analytic spaces over $\mathbb{C}_p$, the completed algebraic closure of $\mathbb{Q}_p$. Specifically, we prove that, for adic étale motives over $\mathbb{C}_p$, the vector bundles on the Fargues–Fontaine curve arising from their Hyodo–Kato cohomology coincide with their de Rham–Fargues–Fontaine cohomology. The latter provides an overconvergent refinement of crystalline vector bundles, albeit constructed on the generic fiber. This equivalence is formulated in the setting of symmetric monoidal ∞-categories and respects the full motivic structure. Moreover, we enrich both realizations with Galois actions, obtaining $G_{\breve{\mathbb{Q}}_p}$-equivariant solid quasi-coherent sheaves on the Fargues–Fontaine curve; in this equivariant setting, the comparison equivalence becomes canonical. In addition, we show that the Fargues–Fontaine cohomology defined via the décalage functor is also motivic and agrees with the de Rham-Fargues–Fontaine cohomology through their mutual comparisons with Hyodo–Kato cohomology.

Combining these two aspects, we construct two spectral sequences converging to the Hyodo–Kato cohomology of smooth quasi-compact rigid analytic spaces over $\mathbb{Q}_p$ (without reduction assumptions) and to the de Rham–Fargues–Fontaine cohomology of such spaces over $\mathbb{C}_p$. In particular, for a smooth quasi-compact rigid analytic space over $\mathbb{Q}_p$ (resp. over $\mathbb{C}_p$) and each $i \geq 0$, its i-th Hyodo–Kato cohomology (resp. de Rham–Fargues–Fontaine cohomology) admits a finite increasing filtration. For the Hyodo–Kato case, this filtration is the weight filtration in the sense of Deligne; for the de Rham–Fargues–Fontaine case, this is a new type of filtration, distinct from the Harder–Narasimhan filtration.

Acknowledgments

First and foremost, I would like to express my sincere gratitude to my supervisor, Alberto Vezzani, for introducing me to the elegant world of motivic realizations for rigid analytic spaces. I am deeply thankful for his guidance, patience, and constant encouragement throughout my doctoral studies. Our discussions have been invaluable to me: I have always come away from our meetings with a clearer understanding and a renewed sense of inspiration. I am particularly grateful for the way he always manages to illustrate abstract ideas with well-chosen examples, revealing the underlying essence behind technical constructions. This thesis has greatly benefited from his previous work and recent research results. Of course, none of these insights would have been possible without the magic of motives. Beyond mathematics, I am indebted to him for his kindness and his genuine concern for both my academic growth and personal well-being, which I have found profoundly human and encouraging.

I would also like to express my sincere gratitude to the anonymous referees of this thesis for the time and attention they devoted to reading it, and for their thoughtful and encouraging comments. I am also very thankful to many mathematicians for their valuable feedback on parts of this work. In particular, I would like to thank Veronika Ertl for helpful suggestions on the exposition and for comments that improved the clarity of the writing, Arthur-César Le Bras for stimulating discussions during my short visit, which helped clarify parts of the exposition concerning the Fargues–Fontaine curve, and Xu Shen for insightful questions that made me realize that the new filtration on de Rham–Fargues–Fontaine cohomology discussed in this thesis is essentially different from the Harder–Narasimhan filtration.

I am grateful to the members of the arithmetic geometry group in Milan for their friendliness and for the pleasant and stimulating atmosphere they have created. I also thank all the members of the Department of Mathematics for their help and for maintaining a welcoming environment—although, ever since I arrived here, I have never actually seen the department building, as it has been under construction the whole time. I am especially thankful to Stefania Lorenza Leonardi, the doctoral secretary, for her kind assistance and for always helping with every practical matter, and to my fellow PhD students and colleagues for their support and kindness throughout these years.

I am deeply grateful to many friends, both in Milan and elsewhere, for making these years enjoyable and for their companionship beyond mathematics. In particular, I would like to thank Krishna Kumar Madhavan Vijayalakshmi for countless conversations about the "survival guide", especially about the permesso di soggiorno. I am also grateful to Ziqi

Liu for our many casual discussions and for being a reliable lunch companion on my visits to the department. I wish to thank Shi He, Xiangsheng Wei and Ruichuan Zhang, friends I first met in Beijing and have remained close to ever since, for their continued friendship and encouragement over the years. I am also thankful to Shengkai Li, Jie Yao, and Zeke Liu for keeping in touch and for our lively group chats, which have often brightened my days. In particular, I am deeply thankful to Zeke for the warm hospitality whenever I was in Zürich, always providing comfortable accommodation—even though his own address seemed to change every time.

Finally, I would like to express my deepest gratitude to my parents and to my grandparents for their unconditional love, support, and encouragement throughout my life. I am especially grateful for the respect they have always shown for my choices, which has given me the freedom and confidence to pursue my own path.

I am also profoundly grateful to my girlfriend, Chang Qin. I am deeply grateful for her unfailing love, understanding, and encouragement, and for standing by me through the challenges of studying and living abroad. I am also thankful for our shared experiences in discovering and enjoying the culture and everyday life abroad, which have greatly enriched my years overseas.

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Chapter I

Introduction

We begin by introducing the background relevant to the topics addressed in this thesis, which consists of two independent components.

Throughout the introduction, we fix a complete non-archimedean field K over $\mathbb{Q}_p$, with a perfect residue field k. Let $\bar{K}$ be an algebraic closure of K, and let $C = \hat{\bar{K}}$ be its completion, whose residue field is denoted by $\bar{k}$. These choices determine two rings of Witt vectors, $K_0 = \operatorname{Frac} W(k)$ and $\breve{K} = \operatorname{Frac} W(\bar{k})$.

§ 1.1 Rigid Analytic Motives

Throughout this thesis, we adopt the language of motives—that is, we study motives associated to geometric objects rather than the geometric objects themselves. The motivation is straightforward: the (∞-)categories of motives often exhibit better formal properties than the corresponding category of geometric objects. In particular, a key advantage is that the category of motives is equipped with Grothendieck's six-functor formalism (see [Ayo07a; Ayo07b; AGV22]).

The construction of the ∞-category of rigid analytic motives over K begins with the étale site $(\mathbf{RigSm}_K, \text{ét})$, where $\mathbf{RigSm}_K$ is the category of smooth rigid analytic spaces (see §3.1.1 for a discussion of various models for rigid analytic geometry) over K. The ∞-category of rigid analytic motives, denoted by $\mathbf{RigDA}(K)$, is defined by inverting $\mathbb{B}^1$-homotopies and Tate objects in the ∞-category of étale sheaves valued in derived $\mathbb{Q}$-modules over $\mathbf{RigSm}_K$:

$$\mathbf{RigDA}(K) := \mathbf{Shv}_{\text{ét}}(\mathbf{RigSm}_K, \mathcal{D}(\mathbb{Q}))\left[(\mathbb{B}^1\text{-homotopy}, T_K)^{-1}\right]$$

where T_K is the Tate object, encoding the first cohomology of the rigid analytic torus $\mathbb{T}^1_K = \operatorname{Spa}(K\langle X^{\pm 1}\rangle)$. As shown in [AGV22], this ∞-category is stable, compactly generated, and carries a symmetric monoidal ∞-category structure. In other words, it is an object in $\mathbf{CAlg}(\mathcal{P}\mathbf{r}^{\text{st}}_{\omega})$.

Similarly, one can construct the ∞-category of algebraic motives over k and of formal motives over $\mathcal{O}_K$. The main difference lies in the type of homotopy to be inverted: for these two categories, one inverts $\mathbb{A}^1$-homotopy instead of $\mathbb{B}^1$-homotopy. Let $\mathbf{DA}(k)$ and $\mathbf{FDA}(\mathcal{O}_K)$

denote these two categories, respectively. More generally, the same construction applies over arbitrary bases: given a rigid analytic space S (resp. a scheme X, a formal scheme $\mathfrak{X}$), we obtain the ∞-categories of rigid analytic motives over S (resp. algebraic motives over X, formal motives over $\mathfrak{X}$).

Rigid Analytic Motives with Good Reduction

We next continue to focus on the base given by K. There are two remarkable relationships among algebraic motives and rigid analytic motives, studied in [AGV22; BGV25]; see also §3.1.3 and §3.1.4.

Given a formal scheme $\mathfrak{X}$ over $\mathcal{O}_K$, we can associate it to a scheme over the residue field k, given by its ***special fiber*** $\mathfrak{X}_\sigma$; on the other hand, we can take the ***adic generic fiber*** $\mathfrak{X}_\eta$ of $\mathfrak{X}$, which is a rigid analytic space over K. The progress here induces two functors

$$\mathbf{Sm}_k \xleftarrow{(-)_\sigma} \mathbf{FSm}_{\mathcal{O}_K} \xrightarrow{(-)_\eta} \mathbf{RigSm}_K.$$

By the constructions of motives, we have two functors between categories of motives:

$$\mathbf{DA}(k) \xleftarrow{(-)_\sigma} \mathbf{FDA}(\mathcal{O}_K) \xrightarrow{(-)_\eta} \mathbf{RigDA}(K). \tag{1.1.1}$$

A theorem of Ayoub ([AGV22, Theorem 3.1.10] or [Ayo15, Corollaires 1.4.24, 1.4.29]) shows that the first functor is an equivalence of ∞-categories. Intuitively, it allows us to lift motives over residue field k to a formal model over $\mathcal{O}_K$, and this lifting is canonical (up to homotopy). Taking the adic generic fiber of this lifting yields a symmetric monoidal functor

$$\xi\colon \mathbf{DA}(k) \to \mathbf{RigDA}(K) \tag{1.1.2}$$

called the ***Monsky–Washnitzer functor***.

Modules in algebraic motives The cohomological motive $M^{\mathrm{coh}}(\mathbb{G}_m)$ of the multiplicative group $\mathbb{G}_m$ defines a commutative algebra object in $\mathbf{DA}(k)$. Taking modules over this commutative algebra, the resulting category $\mathbf{Mod}_{M^{\mathrm{coh}}(\mathbb{G}_m)}(\mathbf{DA}(k))$ embeds into the category of rigid analytic motives over K. In other words, a refinement of the Monsky–Washnitzer functor (1.1.2) gives a fully faithful functor

$$\bar{\xi}\colon \mathbf{Mod}_{M^{\mathrm{coh}}(\mathbb{G}_m)}(\mathbf{DA}(k)) \hookrightarrow \mathbf{RigDA}(K).$$

The essential image of this functor is often referred to as the ∞-category of ***rigid analytic motives with good reductions***, denoted by $\mathbf{RigDA}_{\mathrm{gr}}(K)$. As a result, we have an equivalence

$$\mathbf{Mod}_{M^{\mathrm{coh}}(\mathbb{G}_m)}(\mathbf{DA}(k)) \simeq \mathbf{RigDA}_{\mathrm{gr}}(K) \tag{1.1.3}$$

of ∞-categories.

This full subcategory also admits an equivalent description: it is a full subcategory of $\mathbf{RigDA}(K)$ generated under small colimits by motives $M(\mathfrak{X}_\eta^{\mathrm{ad}})$ associated to the adic generic fibers of smooth formal scheme $\mathfrak{X}$ over $\mathcal{O}_K$. However, contrary to what its name might suggest, this category contains not only motives associated to rigid analytic space over K with good reduction but also those with semistable reduction.

Algebraic motives with monodromy operators Besides realizing rigid analytic motives with good reduction as an algebraic motive over k equipped with a $M^{\mathrm{coh}}(\mathbb{G}_m)$-action, there is another approach that interprets them as algebraic motives with an additional structure. Specifically, let $\mathbf{DA}_N(k)$ be the ∞-category of algebraic motives with monodromy operators. Informally, objects in $\mathbf{DA}_N(k)$ are pairs $(M, N\colon M \to M(-1))$ consisting of an algebraic motive M over k and a morphism N from M to its (-1)-Tate twist. When M is compact, then the operator N is nilpotent. After choosing a pseudo-uniformizer $\varpi \in \mathcal{O}_K$, there is an identification

$$\mathbf{DA}_N(k) \simeq \mathbf{RigDA}_{\mathrm{gr}}(K). \tag{1.1.4}$$

Different choices of pseudo-uniformizer may affect the resulting monodromy operator. Typically, we take the canonical choice $\varpi = p$.

These two perspectives, from algebraic motives to rigid analytic motives, allow us to study the latter using methods from algebraic geometry.

Rigid Analytic Motives has Potentially Good Reduction

As shown in [AGV22, Theorem 3.3.3], the full subcategory $\mathbf{RigDA}_{\mathrm{gr}}(K)$ provides a tool to study the entire category $\mathbf{RigDA}(K)$. More precisely, every compact rigid analytic motive becomes of good reduction after a finite extension, by [Ayo15, Theorem 2.5.34] or [AGV22, Proposition 3.7.17]. In particular, the full subcategory of compact analytic motives can be described as a "union" of rigid analytic motives with good reductions up to finite Galois extensions: there is an equivalence in $\mathbf{CAlg}(\mathcal{P}\mathbf{r}_\omega^{\mathrm{L}})$

$$\mathbf{RigDA}(K) \simeq \operatorname{colim} \mathbf{RigDA}_{L\text{-}\mathrm{gr}}(K) \tag{1.1.5}$$

where L runs through finite Galois extensions of K and $\mathbf{RigDA}_{L\text{-}\mathrm{gr}}(K)$ is the full subcategory of $\mathbf{RigDA}(K)$ spanned by those rigid analytic motives whose base change to L lies in $\mathbf{RigDA}_{\mathrm{gr}}(L)$. By Galois descent, the full subcategory $\mathbf{RigDA}_{L\text{-}\mathrm{gr}}(K)$ is equivalent to the ∞-category of $\mathrm{Gal}(L/K)$-equivariant rigid analytic motive over L, denoted by $\mathbf{RigDA}_{\mathrm{gr}}(L)^{\mathrm{h}\,\mathrm{Gal}(L/K)}$:

Proposition 1.1.1 (Proposition 3.2.7)**.** *The canonical base change functor induces an equivalence*

$$\mathbf{RigDA}_{L\text{-}\mathrm{gr}}(K) \xrightarrow{\simeq} \mathbf{RigDA}_{\mathrm{gr}}(L)^{\mathrm{h}\,\mathrm{Gal}(L/K)}$$

in $\mathbf{CAlg}(\mathcal{P}\mathbf{r}^{\mathrm{L}}_{\omega})$.

More precisely, the category $\mathbf{RigDA}_{\mathrm{gr}}(L)^{\mathrm{h}\,\mathrm{Gal}(L/K)}$ is obtained by taking the homotopy fixed points of the $\mathrm{Gal}(L/K)$-action on $\mathbf{RigDA}_{\mathrm{gr}}(L)$. It is a compactly generated ∞-category with explicit description for compact generators:

Proposition 1.1.2 (Proposition 3.2.10, Proposition 3.2.11)**.** *The ∞-category $\mathbf{RigDA}_{\mathrm{gr}}(L)^{\mathrm{h}\,\mathrm{Gal}(L/K)}$ is compactly generated with a set of compact generators of the form $\mathrm{Nm}_L\iota_L\xi_L(M(X))$, where X runs through proper smooth varieties over the residue field k_L of L, and the functor Nm_L is the left adjoint of the forgetful functor $\iota_L\colon \mathbf{RigDA}_{\mathrm{gr}}(L)^{\mathrm{h}\,\mathrm{Gal}(L/K)} \to \mathbf{RigDA}_{\mathrm{gr}}(L)$. Moreover, the underlying motive of $\mathrm{Nm}_L\iota_L\xi_L(M(X))$ is given by*

$$\iota_L\mathrm{Nm}_L(\xi_L M(X)) \simeq \bigoplus_{e_{L/K}} \xi_L M(X),$$

where $e_{L/K}$ is the ramification index of L/K.

Using the identification (1.1.5) provided by Proposition 1.1.1, we obtain the following explicit version of [AGV22, Theorem 3.3.3]:

Proposition 1.1.3 (Proposition 3.2.13)**.** *We have an equivalence in $\mathbf{CAlg}(\mathcal{P}\mathbf{r}^{\mathrm{L}}_{\omega})$*

$$\mathbf{RigDA}(K) \simeq \operatorname{colim} \mathbf{RigDA}_{\mathrm{gr}}(L)^{\mathrm{h}\,\mathrm{Gal}(L/K)}$$

where L runs through finite Galois extensions of K.

The Inertial Galois Enrichment of Rigid Analytic Motives

The equivalences (1.1.3) and (1.1.5) also hold in more general settings. For instance, they remain valid after replacing K by C. In particular, there is a full subcategory $\mathbf{RigDA}_{\mathrm{gr}}(C)$ consisting of rigid analytic motives with good reduction over C. Since C is algebraically closed, the equivalence (1.1.5) implies that the inclusion functor $\mathbf{RigDA}_{\mathrm{gr}}(C) \xrightarrow{\simeq} \mathbf{RigDA}(C)$ is an equivalence.

On the other hand, this field shares the same residue field $\bar{k}$ with $\breve{K} := \operatorname{Frac} W(\bar{k})$. In light of (1.1.3), there is an equivalence $\mathbf{RigDA}_{\mathrm{gr}}(\breve{K}) \simeq \mathbf{RigDA}_{\mathrm{gr}}(C)$. Furthermore, these two categories can be identified in a more canonical way: the inclusion functor $\mathbf{RigDA}_{\mathrm{gr}}(\breve{K}) \xrightarrow{\simeq} \mathbf{RigDA}_{\mathrm{gr}}(C)$ is an equivalence, thanks to [BKV25, Proposition 3.23 (2)].

These observations show that the base change functor

$$\mathbf{RigDA}_{\mathrm{gr}}(\breve{K}) \xrightarrow{\simeq} \mathbf{RigDA}(C) \tag{1.1.6}$$

is an equivalence. The intuition of this equivalence is that every rigid analytic motive M over C admits a unique (up to isomorphism) model $\bar{M} \in \mathbf{RigDA}_{\mathrm{gr}}(\breve{K})$. Thus, the motive

M carries a canonical action by the Galois group $G_{\breve{K}} = \mathrm{Gal}(C/\breve{K})$. In other words, under the equivalence (1.1.6), rigid analytic motives over C naturally acquire a $G_{\breve{K}}$-enrichment:

$$\alpha\colon \mathbf{RigDA}(C) \simeq \mathbf{RigDA}_{\mathrm{gr}}(\breve{K}) \to \mathbf{RigDA}(C)^{\mathrm{h}\,G_{\breve{K}}}, \tag{1.1.7}$$

where the last functor is induced by the base change. The $G_{\breve{K}}$-enrichment enables us to equip the coefficients of the motivic realization on $\mathbf{RigDA}(C)$ with a $G_{\breve{K}}$-action.

§ 1.2 Fargues–Fontaine Cohomology Theory

A central development in the geometrization of p-adic Hodge theory and the local Langlands correspondence is the introduction of the ***Fargues–Fontaine curve***. In this thesis, however, we focus primarily on the role of the Fargues–Fontaine curve in p-adic Hodge theory, especially in the cohomological comparisons between various p-adic realizations of rigid analytic varieties.

We first recall the geometrization of certain results in [BMS18] in terms of the Fargues–Fontaine curve. Let $\mathbf{FF} := \mathbf{FF}_{C^\flat,\mathbb{Q}_p}$ be the absolute (adic) Fargues–Fontaine curve. In [BMS18], Bhatt-Morrow-Scholze introduced and studied several p-adic cohomology theories: let $\mathfrak{X}$ be a proper smooth formal scheme over the ring of integers $\mathcal{O}_C$ of C, then we have the following cohomology:

- the p-adic étale cohomology $\mathrm{R}\boldsymbol{\Gamma}_{\text{ét}}(\mathfrak{X}_C, \mathbb{Z}_p)$ of the rigid analytic generic fiber $\mathfrak{X}_C$ of $\mathfrak{X}$;
- the crystalline cohomology $\mathrm{R}\boldsymbol{\Gamma}_{\mathrm{crys}}(\mathfrak{X}_{\bar{k}}/W(\bar{k}))$ of the special fiber $\mathfrak{X}_{\bar{k}}$ of $\mathfrak{X}$;
- a new cohomology theory $\mathrm{R}\boldsymbol{\Gamma}_{\mathrm{crys}}(\mathfrak{X}_C/B_{\mathrm{dR}}^+)$ for the rigid analytic generic fiber;
- a new cohomology theory $\mathrm{R}\boldsymbol{\Gamma}_{A_{\mathrm{inf}}}(\mathfrak{X})$.

Let us consider them on the Fargues–Fontaine curve. Let $i \geq 0$. Then the p-adic étale cohomology $H^i_{\text{ét}}(\mathfrak{X}_C, \mathbb{Q}_p)$ yields a trivial semistable vector bundle $\mathcal{E}_0 := H^i_{\text{ét}}(\mathfrak{X}_C, \mathbb{Q}_p) \otimes \mathcal{O}_{\mathbf{FF}}$ of slope zero. As shown in [BMS18, Theorem 1.7], the crystalline B_{dR}^+-cohomology $H^i_{\mathrm{crys}}(\mathfrak{X}_C/B_{\mathrm{dR}}^+)$ is a finite free B_{dR}^+-module and compares with $H^i_{\text{ét}}(\mathfrak{X}_C, \mathbb{Q}_p)$ after base change to B_{dR}. In other words, $H^i_{\mathrm{crys}}(\mathfrak{X}_C/\mathbb{B}_{\mathrm{dR}}^+)$ is a B_{dR}^+-lattice inside $H^i_{\text{ét}}(\mathfrak{X}_C, \mathbb{Q}_p) \otimes B_{\mathrm{dR}}$. From the perspective of B-pairs, this comparison yields a modification of the trivial vector bundle $\mathcal{E}_0$ at the distinguished point x_C of $\mathbf{FF}$. More precisely, this defines a vector bundle $\mathcal{E}_1$ on the Fargues–Fontaine curve, together with an isomorphism

$$\iota\colon \mathcal{E}_0|_{\mathbf{FF}\setminus\{x_C\}} \simeq \mathcal{E}_1|_{\mathbf{FF}\setminus\{x_C\}}.$$

The modification of $\mathcal{E}_1$ at x_C is exactly given by the crystalline B_{dR}^+-lattice $H^i_{\mathrm{crys}}(\mathfrak{X}_C/B_{\mathrm{dR}}^+)$.

By Fargues' theorem (see, for example, [SW20, Theorem 14.1.1]), this modification corresponds to a φ-module over A_{inf}, which is in fact given by $H^i_{A_{\mathrm{inf}}}(\mathfrak{X})$ defined in [BMS18].

The comparison with the crystalline cohomology $H^i_{\text{crys}}(\mathfrak{X}_{\bar{k}}/W(\bar{k}))$ in [BMS18, Theorem 1.8] shows that the modification $\mathcal{E}_1$ is isomorphic to $\mathcal{E}(H^i_{\text{crys}}(\mathfrak{X}_{\bar{k}}/W(\bar{k}))[1/p], \varphi)$, where the functor $\mathcal{E}$ sends φ-modules over $\breve{K}$ to vector bundles on the Fargues–Fontaine curve as defined by Fargues and Fontaine. Therefore, the crystalline cohomology can be reconstructed from cohomological data on the rigid generic fibers. Taken together with the work of Colmez-Nizioł [CN17; Niz19], these results motivate the following conjecture:

Conjecture ([Far19, Conjecture 1.13], [Sch19, Conjecture 6.4])**.** There exists a motivic cohomology[1]

$$\mathrm{R}\boldsymbol{\Gamma}_{\mathrm{FF}} \colon \mathbf{RigDA}(C)_\omega \to \mathbf{Perf}(\mathbf{FF})$$

sending smooth quasi-compact rigid analytic varieties X over C to perfect complexes on the Fargues–Fontaine curve; in other words, each cohomology $\mathcal{H}^i_{\mathbf{FF}}(X)$ is a vector bundle on the Fargues–Fontaine curve. It is required to satisfy the following properties:

(1) the pullback $x_C^* \mathrm{R}\boldsymbol{\Gamma}_{\mathbf{FF}}(X)$ to the distinguished point is isomorphic to the overconvergent de Rham cohomology $\mathrm{R}\boldsymbol{\Gamma}^\dagger_{\mathrm{dR}}(X)$;

(2) the complete stalk of $\mathrm{R}\boldsymbol{\Gamma}_{\mathrm{FF}}(X)$ at x_C recovers the overconvergent B^+_{dR}-cohomology $\mathrm{R}\boldsymbol{\Gamma}^\dagger_{\mathrm{crys}}(X/B^+_{\mathrm{dR}})$;

(3) if X admits a semistable (weak) formal model $\mathfrak{X}$, then $\mathrm{R}\boldsymbol{\Gamma}_{\mathrm{FF}}(X)$ agrees with the vector bundle associated to the log-rigid cohomology of the special fiber $\mathfrak{X}_\sigma$ in the sense of [Gro05]; see next part for a more precise discussion.

From now on, we refer to such cohomology as a ***Fargues–Fontaine cohomology***. Next, we will see there are two versions of this cohomology, constructed via different approaches. Before proceeding further, it is important to note that such cohomology can extend to the entire category of motives by using Clausen and Scholze's condensed mathematics. This yields a motivic realization functor on the whole category

$$\mathrm{R}\boldsymbol{\Gamma}_{\mathrm{FF}} \colon \mathbf{RigDA}(C) \to \mathbf{QCoh}(\mathbf{FF})$$

taking values in solid quasi-coherent sheaves, in the sense of [And21], on the Fargues–Fontaine curve.

Motivic Hyodo–Kato Cohomology

The part (3) of the conjecture concerns a comparison with log-rigid cohomology on the special fiber. More precisely, we need to require that X has a dagger structure $X^\dagger$ admitting a semistable weak formal model $\mathfrak{X}$. Taking the pullback to the special fiber

[1] Throughout this thesis, all the motivic cohomology is covariant by taking duals. For further details, see §4.2.

$\mathfrak{X}_\sigma$, the log structure on $\mathfrak{X}$ defined by its special fiber, we obtain a log scheme over $\bar{k}^0 = (\mathrm{Spec}(k), \mathbb{N} \to k : 1 \mapsto 0)$. Thus, part (3) requires that the log-rigid cohomology of the special fiber $\mathfrak{X}_\sigma$ gives rise to the same perfect complex as the Fargues–Fontaine cohomology $\mathrm{R}\boldsymbol{\Gamma}_{\mathrm{FF}}(X)$.

However, Colmez and Nizioł, adapting a construction of Beilinson for algebraic varieties in [Bei13], defined an (overconvergent)[1] Hyodo–Kato cohomology theory for (dagger) rigid analytic spaces. In the case of semistable reduction, their local-global compatibility result shows that the Hyodo–Kato cohomology agrees with the log-rigid cohomology on the special fiber.

On the other hand, as we are studying these cohomology theories in the motivic framework, an important result of Vezzani [Vez18] shows that every rigid analytic motive has a dagger structure. Consequently, we may restate part (3) of the conjecture as follows:

(3') the Fargues–Fontaine cohomology agrees with the overconvergent Hyodo–Kato cohomology.

Since the Fargues–Fontaine cohomology is required to be motivic, it is natural to require a comparison with the motivic overconvergent Hyodo–Kato cohomology.

The motivic Hyodo–Kato cohomology theory is constructed in [BKV25; BGV25] using two different approaches: in [BKV25], the authors show that the Colmez-Nizioł's Hyodo–Kato cohomology has the rig-étale descent, $\mathbb{A}^1$-invariance, and maps the Tate object to an invertible object via the Hyodo–Kato isomorphism; and in [BGV25], the authors give a log-free method to construct the Hyodo–Kato cohomology theory from the rigid cohomology theory defined on the special fiber. This thesis mainly treats the motivic Hyodo–Kato cohomology by the log-free approach: starting from the ***rigid realization functor***

$$\mathrm{R}\boldsymbol{\Gamma}_{\mathrm{rig}} \colon \mathbf{DA}(\bar{k}) \to \mathcal{D}_{\varphi}(\breve{K}), \tag{1.2.1}$$

we add the monodromy operators on both sides to obtain a new realization functor

$$\widehat{\mathrm{R}\Gamma}_{\mathrm{rig}} \colon \mathbf{DA}_N(\bar{k}) \to \mathcal{D}_{(\varphi,N)}(\breve{K}).$$

As we introduced earlier, the source category can be identified, after choosing a pseudo-uniformizer $\varpi \in \mathcal{O}_C$, with $\mathbf{RigDA}_{\mathrm{gr}}(C) \simeq \mathbf{RigDA}(C)$ via the equivalences (1.1.4) and Proposition 1.1.3. This yields a cohomology realization taking values of (φ, N)-modules:

$$\mathrm{R}\boldsymbol{\Gamma}^{\varpi}_{\mathrm{HK}} \colon \mathbf{RigDA}(C) \to \mathcal{D}_{(\varphi,N)}(\breve{K}), \tag{1.2.2}$$

referred to as ***ϖ-Hyodo–Kato realization functor***. We usually take $\varpi = p$ as the canonical choice. In the case, we simply refer to it as the ***Hyodo–Kato realization functor***, denoted by $\mathrm{R}\boldsymbol{\Gamma}_{\mathrm{HK}}$, which agrees with the Colmez-Nizioł's Hyodo–Kato cohomology theory as shown in [BGV25, §4.8].

[1]The cohomology theories involved in the thesis are overconvergent since our category $\mathbf{RigDA}(K)$ serves as a model for mixed motives that allows us to study non-proper spaces.

Remark 1.2.1. As observed in [BGV25, Remark 4.18] and [BKV25, Remark 3.5], different choices for pseudo-uniformizer affect only the behaviors of the monodromy operator on the Hyodo–Kato cohomology.

The Fargues–Fontaine Cohomology via the Motivic Method

In [LBV23], Le Bras and Vezzani construct a motivic realization[1], called the ***de Rham–Fargues–Fontaine cohomology***. We still denote it by

$$\mathrm{R\Gamma}_{\mathrm{FF}}\colon \mathbf{RigDA}(C) \to \mathbf{QCoh}(\mathbf{FF}) \tag{1.2.3}$$

The cohomology of compact motives under this realization is given by vector bundles, as shown in [LBV23, Theorem 4.46]. Its comparison with the overconvergent de Rham cohomology follows directly from its construction, and in loc. cit., the authors also show that it can be compared with the overconvergent B^+_{dR}-cohomology. In other words, $\mathrm{R\Gamma}_{\mathrm{FF}}$ satisfies (1) and (2) of the conjecture.

One of the main results of this thesis is to prove that this realization satisfies (3') of the conjecture:

Theorem 1.2.2 (Theorem 4.4.1)**.** *There is a monoidal equivalence between the de Rham–Fargues–Fontaine realization $\mathrm{R\Gamma}_{\mathrm{FF}}$ in (1.2.3) and $\mathcal{E}_N \circ \mathrm{R\Gamma}_{\mathrm{HK}}$, where $\mathcal{E}_N\colon \mathcal{D}_{(\varphi,N)}(\breve{K}) \to \mathbf{QCoh}(\mathbf{FF})$ is Fargues–Fontaine's functor sending (φ, N)-modules to vector bundles on the Fargues–Fontaine curve.*

Therefore, the de Rham–Fargues–Fontaine cohomology confirms the conjecture of Scholze and Fargues. We next sketch the idea of proof of Theorem 1.2.2.

By the construction of the motivic Hyodo–Kato cohomology theory introduced above, the Hyodo–Kato cohomology is naturally compared with the rigid cohomology via the Monsky–Washnitzer functor: there is a canonical monoidal equivalence

$$\mathrm{R\Gamma}_{\mathrm{HK}} \circ \xi \simeq \mathrm{R\Gamma}_{\mathrm{rig}}\colon \mathbf{DA}(\bar{k}) \to \mathcal{D}_{\varphi}(\breve{K}) \tag{1.2.4}$$

in $\mathbf{CAlg}(\mathcal{P}\mathbf{r}^{\mathrm{st}}_{\omega})$. If Theorem 1.2.2 holds, the de Rham–Fargues–Fontaine cohomology can compute the rigid cohomology as well. Therefore, we first prove the comparison of the de Rham–Fargues–Fontaine cohomology and rigid cohomology using the standard Frobenius trick on the Fargues–Fontaine curve and the semi-separatedness property of motives:

Proposition 1.2.3 (Proposition 4.2.9 and [LBV23, Proposition 5.11])**.** *There is a canonical monoidal equivalence*

$$\mathrm{R\Gamma}_{\mathrm{FF}} \circ \xi \simeq \mathcal{E} \circ \mathrm{R\Gamma}_{\mathrm{rig}}$$

in $\mathbf{CAlg}(\mathcal{P}\mathbf{r}^{\mathrm{st}}_{\omega})$.

[1] More precisely, they define a relative version of this realization functor.

Therefore, these two realization functors are not merely objects in $\mathbf{CAlg}(\mathcal{P}\mathbf{r}^{\mathrm{st}}_{\omega})$, but also in the more refined category $\mathbf{CAlg}(\mathcal{P}\mathbf{r}^{\mathrm{st}}_{\omega})_{\mathbf{DA}(\bar{k})/-}$ via their compatibility with rigid cohomology. In other words, we have two objects in the mapping space

$$\mathrm{Map}_{\mathbf{CAlg}(\mathcal{P}\mathbf{r}^{\mathrm{st}}_{\omega})_{\mathbf{DA}(\bar{k})/-}}\left(\mathbf{RigDA}(C), \mathbf{QCoh}(\mathbf{FF})\right).$$

To prove Theorem 1.2.2, it suffices to show that the space above is connected, i.e., it has exactly one connected component. Using the monadicity (1.1.3) of $\mathbf{RigDA}_{\mathrm{gr}}(C) \simeq \mathbf{RigDA}(C)$, this is reduced to extension groups of vector bundles on the Fargues–Fontaine curve; see Proposition 2.3.3. More precisely, one has

$$\pi_0\mathrm{Map}_{\mathbf{CAlg}(\mathcal{P}\mathbf{r}^{\mathrm{st}}_{\omega})_{\mathbf{DA}(\bar{k})/-}}\left(\mathbf{RigDA}(C), \mathbf{QCoh}(\mathbf{FF})\right) \simeq \mathrm{Ext}^1(\mathcal{O}_{\mathbf{FF}}(-1), \mathcal{O}_{\mathbf{FF}}) \simeq 0.$$

This establishes a monoidal equivalence between $\mathrm{R}\Gamma_{\mathrm{FF}}$ and $\mathcal{E}_N \circ \mathrm{R}\Gamma_{\mathrm{HK}}$. However, there exists a big space of choices for such an equivalence, since

$$\pi_1\mathrm{Map}_{\mathbf{CAlg}(\mathcal{P}\mathbf{r}^{\mathrm{st}}_{\omega})_{\mathbf{DA}(\bar{k})/-}}\left(\mathbf{RigDA}(C), \mathbf{QCoh}(\mathbf{FF})\right) \simeq \mathrm{Hom}(\mathcal{O}_{\mathbf{FF}}(-1), \mathcal{O}_{\mathbf{FF}}) \simeq B^{\varphi=p},$$

where B is the ring of analytic functions on $\mathcal{Y}_{(0,\infty)} = \mathrm{Spa}(A_{\mathrm{inf}}, A_{\mathrm{inf}}) \setminus (V(p[p^\flat]))$. This shows that there is no canonical comparison between these two kinds of realization functors.

To obtain a canonical comparison, we can keep track of Galois action. More precisely, let $\breve{K} = \mathrm{Frac}\, W(\bar{k})$, where $\bar{k}$ is the residue field of C. Then, we can enrich both realization functors above to take values in the ∞-category of $G_{\breve{K}}$-equivariant solid quasi-coherent sheaves on $\mathbf{FF}$.

The functor $\mathcal{E}^{\mathrm{ari}}$, which sends the Hyodo–Kato realization $\mathrm{R}\Gamma_{\mathrm{HK}}$ to a $G_{\breve{K}}$-equivariant vector bundle on the Fargues–Fontaine curve, is already known from [FF18]. Let's explain how to enrich the de Rham–Fargues–Fontaine cohomology with $G_{\breve{K}}$-action. To this end, we use the inertial Galois-enrichment of motives, namely, the functor (1.1.7), $\alpha\colon \mathbf{RigDA}(C) \to \mathbf{RigDA}(C)^{\mathrm{h}G_{\breve{K}}}$. Composing the $G_{\breve{K}}$-equivariant de Rham–Fargues–Fontaine cohomology with this enrichment, we get the Galois-refinement of $\mathrm{R}\Gamma_{\mathrm{HK}}$:

$$\mathrm{R}\Gamma^{\mathrm{ari}}_{\mathrm{FF}}\colon \mathbf{RigDA}(C) \to \mathbf{QCoh}(\mathbf{F})^{\mathrm{h}G_{\breve{K}}} \tag{1.2.5}$$

where $\mathbf{QCoh}(\mathbf{FF})^{\mathrm{h}G_{\breve{K}}}$ is the ∞-category of $G_{\breve{K}}$-equivariant solid quasi-coherent sheaves on $\mathbf{FF}$.

With these enhancements, we obtain the canonical comparison natural isomorphism:

Theorem 1.2.4 (Theorem 4.4.10)**.** *There is a unique natural isomorphism* $\mathrm{R}\Gamma^{\mathrm{ari}}_{\mathrm{FF}} \simeq \mathcal{E}^{\mathrm{ari}}_N \circ \mathrm{R}\Gamma_{\mathrm{HK}}$ *in* $\mathbf{CAlg}(\mathcal{P}\mathbf{r}^{\mathrm{st}}_{\omega})$ *such that it is compatible with the natural isomorphisms between the rigid realization functor and the de Rham–Fargues–Fontaine realization, as well as with the Hyodo–Kato realization.*

The Fargues–Fontaine Cohomology via the Décalage Functor

There is another version of Fargues–Fontaine cohomology, denoted by $\widetilde{\mathrm{R}\Gamma}_{\mathbf{FF}}(X)$, defined for smooth dagger varieties X over C using the décalage functor, as introduced in [LB18; Bos23b]. Relying on Bosco's comparison between B-cohomology and Hyodo–Kato cohomology, we show that it is also motivic:

Proposition 1.2.5 (Proposition 4.4.13)**.** *The cohomology theory* $\widetilde{\mathrm{R}\Gamma}_{\mathbf{FF}}$ *satisfies rig-étale descent,* $\mathbb{A}^1$*-invariance, and sends the Tate object to an invertible object. Hence, it defines a motivic realization*

$$\widetilde{\mathrm{R}\Gamma}_{\mathbf{FF}} \colon \mathbf{RigDA}(C) \to \mathbf{QCoh}(\mathbf{FF}).$$

Remark 1.2.6. Le Bras has shown in [LB18] that this version of Fargues–Fontaine cohomology is defined on effective motives. In this thesis, we provide an alternative proof demonstrating that it also extends to the T-stabilization, which enables us to apply the module description (1.1.3).

We establish a motivic comparison between these two versions of Fargues–Fontaine cohomology theories by their comparisons with the Hyodo–Kato cohomology:

Proposition 1.2.7 (Proposition 4.4.14)**.** *There is a (non-canonical) monoidal equivalence* $\mathrm{R}\Gamma_{\mathbf{FF}} \simeq \widetilde{\mathrm{R}\Gamma}_{\mathbf{FF}}$ *in* $\mathbf{CAlg}(\mathcal{P}\mathbf{r}^{\mathrm{st}}_{\omega})$.

§ 1.3 Weight Structures and Weight Filtrations

The second topic of this thesis concerns weight structures on triangulated categories, introduced by Bondarko in [Bon10]. They can be viewed as a dual notion of t-structures (see Remark 2.2.4). Subsequently, Sosnilo developed the theory in the context of stable ∞-categories in [Sos19; Sos22].

More precisely, a ***weight structure*** on a stable ∞-category $\mathcal{C}$ consists of a pair $(\mathcal{C}_{w\geq 0}, \mathcal{C}_{w\leq 0})$ of retract-closed, shift-closed full subcategories of $\mathcal{C}$, which satisfy conditions dual to those in the definition of t-structures (see Definition 2.2.3). As in the theory of t-structures, one can define full subcategories $\mathcal{C}_{[a,b]}$ for any pairs of integers $a \leq b$. In particular, $\mathcal{C}_{[0,0]}$ is called the ***heart*** of w. If $\mathcal{C}$ is the union of all of these full subcategories, we say w is ***bounded***. Here is a basic example of a bounded weight structure:

Example 1.3.1. Let $\mathcal{A}$ be an additive category, and denote by $\mathcal{K}^{\mathrm{b}}(\mathcal{A})$ the ∞-category of bounded chain complexes over $\mathcal{A}$. Then $\mathcal{K}^{\mathrm{b}}(\mathcal{A})_{w\geq 0}$ is defined as the full subcategory consisting of those chain complexes concentrated in non-negative degrees; $\mathcal{K}^{\mathrm{b}}(\mathcal{A})_{w\leq 0}$ is defined similarly. It is straightforward to verify that this construction defines a bounded weight structure.

A more substantial example, studied in [Bon10; Héb10; Bon14], is the ***Chow weight structure*** on the stable ∞-category $\mathbf{DA}(k)$ of motives over a field k. One of the main motivations for introducing weight structures is precisely their application to the study of motives. Since the construction of motivic t-structures is very difficult and remains largely open, weight structures provide a more accessible framework for organizing and analyzing motives.

This naturally leads to the question: does there exist a weight structure on the stable ∞-category of rigid analytic motives?

As observed in [BGV25] and [BKV25, Appendix A], starting from the Chow weight structure on the algebraic motives, one can construct a good weight structure on the full subcategory $\mathbf{RigDA}_{\mathrm{gr}}(K)$ of $\mathbf{RigDA}(K)$. Even though the weight structure is defined only on this subcategory, it allows us to give the "motivic Hyodo–Kato isomorphism" ([BGV25, Theorem 4.53]) and get a weight filtration on the Hyodo–Kato cohomology, as a formal result. These indicate that weight structures on rigid analytic motives provide a powerful tool for connecting motivic ideas with p-adic cohomology theories.

A second objective of this thesis is to extend the weight structure on the full subcategory $\mathbf{RigDA}_{\mathrm{gr}}(K)$ to the whole category. Furthermore, this extended weight structure remains compatible with the symmetric monoidal structure on $\mathbf{RigDA}(K)$ in the sense of [Aok20].

Theorem 1.3.2 (Theorem 3.3.10)**.** *Let K be a complete non-archimedean field with perfect residue field k. There exists a bounded weight structure w on $\mathbf{RigDA}(K)_\omega$ that is compatible with monoidal structure and extends the one on $\mathbf{RigDA}_{\mathrm{gr}}(K)_\omega$ mentioned above. Moreover, it extends to its Ind-completion $\mathbf{RigDA}(K)$ such that $\mathbf{RigDA}(K)_{w\geq 0}$ is closed under small colimits.*

To construct the weight structure in Theorem 1.3.2, we apply Galois descent to relate compact analytic motives over K with analytic motives with good reduction. In other words, we use Proposition 1.1.3: every compact analytic motive over K becomes one with good reduction after a field extension.

It then suffices to construct a weight structure on $\mathbf{RigDA}_{\mathrm{gr}}(L)^{\mathrm{h\,Gal}(L/K)}$ and glue it to obtain one on $\mathbf{RigDA}(K)$. The former category, interpreted as the homotopy fixed points under the Galois action, arises as the limit of $\mathbf{RigDA}_{\mathrm{gr}}(L)$ along this Galois action. Although in general it is difficult to equip a limit of stable ∞-categories with a weight structure, the situation here is special: the translation functors involved are autoequivalences. As shown in Proposition 1.1.2, this leads to an explicit formula for compact generators, which forms a negative class. This yields a bounded weight structure on the compact objects (Proposition 2.2.17), which can then be extended to the entire category as usual (discussed in §2.2.3).

Weight Complex Functors

As an application of Theorem 1.3.2, we can construct a p-adic analytic generalization of Rapoport-Zink's weight spectral sequence together with weight filtration in [RZ82]. The similar arguments also yield a new filtration on the vector bundles on the Fargues–Fontaine curve.

The key ingredient that allows us to obtain such filtrations from (bounded) weight structures is the weight complex functor studied in [Bon10; Sos19]: let $\mathcal{C}$ be a stable symmetric monoidal ∞-category equipped with a bounded weight structure w, then there is a symmetric monoidal functor, called ***weight complex functor***

$$W_{\bullet}\colon \mathcal{C} \to \mathcal{K}^{\mathrm{b}}(\mathrm{h}\,\mathcal{H}_w),$$

where $\mathcal{H}_w$ is the heart of this weight structure. Moreover, equipped with the bounded weight structure given in Example 1.3.1, this functor is weight-exact; namely, it restricts to the canonical functor $\mathcal{H}_w \to \mathrm{h}\,\mathcal{H}_w$.

Weight Filtrations on the Hyodo–Kato Cohomology

Now we explain how to construct a weight filtration on Hyodo–Kato cohomology using the weight complex functor. For this purpose, we give a refinement of Hyodo–Kato cohomology. As before, one has the Hyodo–Kato cohomology over K

$$\mathrm{R\Gamma}_{\mathrm{HK},K}\colon \mathbf{RigDA}_{\mathrm{gr}}(K) \to \mathcal{D}_{(\varphi,N)}(K_0)$$

where $K_0 = \operatorname{Frac} W(k)$, obtaining by adding the monodromy operators on the rigid cohomology $\mathrm{R\Gamma}_{\mathrm{rig}}\colon \mathbf{DA}(k) \to \mathcal{D}_{\varphi}(K_0)$. We can extend it to the entire category $\mathbf{RigDA}(K)$ using Proposition 1.1.3: there is a functor in $\mathbf{CAlg}(\mathcal{P}\mathbf{r}^{\mathrm{st}}_{\omega})$

$$\mathrm{R\Gamma}^{\mathrm{ari}}_{\mathrm{HK},K}\colon \mathbf{RigDA}(K) \to \mathcal{D}_{(\varphi,N,G_K)}(\breve{K}),$$

where $\mathcal{D}_{(\varphi,N,G_K)}(\breve{K})$ is the derived ∞-category of (φ, N, G_K)-modules over $\breve{K}$, in the sense of Fontaine. After forgetting the monodromy operator, it factors through the weight complex functor

Proposition 1.3.3 (Lemma 4.3.2)**.** *Assume k is a finite field. Let $\mathrm{R\Gamma}^{(\varphi,G_K)}_{\mathrm{HK}}$ be the composite functor*

$$\mathbf{RigDA}(K) \to \mathcal{D}_{(\varphi,N,G_K)}(\breve{K}) \to \mathcal{D}_{(\varphi,G_K)}(\breve{K}),$$

i.e., the (φ, G_K)-part of the Hyodo–Kato cohomology. Here $\mathcal{D}_{(\varphi,G_K)}(\breve{K})$ is the derived ∞-category of (φ, G_K)-modules over $\breve{K}$ in the sense of Fontaine. Then its restriction to compact part factors through the weight complex functor with respect to the weight structure defined in Theorem 1.3.2.

Thus, every compact motive M in $\mathbf{RigDA}(K)$ is associated to a bounded chain complex $W_\bullet M$. We can filter it by naive truncations. Then the spectral sequence associated to this filtered chain complex gives the following spectral sequence:

Proposition 1.3.4 (Weight Spectral Sequence, Proposition 4.3.3)**.** *Assume k is a finite field. Let M be a compact motive in $\mathbf{RigDA}(K)$. There is a convergent (homological) spectral sequence of (φ, G_K)-modules*

$$E^1_{pq} = H_q\mathbf{R\Gamma}_{\mathrm{HK}}(W_pM) \Rightarrow H_{p+q}\mathbf{R\Gamma}_{\mathrm{HK}}(M),$$

where $W_\bullet M$ is the weight complex of M. Moreover, this spectral sequence degenerates at the second page.

The filtration induced by this spectral sequence is called ***weight filtration*** in the sense of Deligne. In fact, every motive associated to a smooth quasi-compact rigid analytic space is compact. Therefore, we have get the following weight filtration on the Hyodo–Kato cohomology of these rigid analytic spaces:

Corollary 1.3.5 (Corollary 4.3.5)**.** *Assume k is a finite filed. Let X be a smooth quasi-compact rigid analytic space over K. Then, for each $i \geq 0$, there is a finite increasing filtration on the i-th (overconvergent) Hyodo–Kato cohomology $H^i_{\mathrm{HK}}(X)$ of X satisfying:*

- *it is stable under G_K-action;*
- *the k-th graded piece is pure of weight $i+k$, i.e., the Frobenius eigenvalues with complex norm $|k|^{(i+k)/2}$;*
- *the monodromy operator induces a map*

$$\mathrm{gr}^W_k H^i_{\mathrm{HK}}(X) \to \mathrm{gr}^W_{k-2} H^i_{\mathrm{HK}}(X).$$

Remark 1.3.6. If X has semi-stable reduction, then the weight filtration above has already been given in [BGV25]. However, we do not assume any formal model of X here. Moreover, M can come from varieties without "smoothness" and "properness". For example, let X be locally of finite K-scheme. Then the analytification of $M(X) \in \mathbf{DA}(K)$ is compact by [LBV23, Remark 6.6].

A New Filtration for Vector Bundles over the Fargues–Fontaine Curve

By the comparison result between de Rham–Fargues–Fontaine cohomology and Hyodo–Kato cohomology, namely Theorem 1.2.2, we can deduce that the de Rham–Fargues–Fontaine realization functor factors through the weight complex functor as well. This yields a filtration on the de Rham–Fargues–Fontaine cohomology of compact motives with similar arguments. By contrast, there is no notion of weight for vector bundles on the

Fargues–Fontaine curve; instead, we can talk about slopes. We therefore have the following filtration on the de Rham–Fargues–Fontaine cohomology in terms of slopes:

Proposition 1.3.7 (Corollary 4.4.6)**.** *Assume k is a finite field. Let X be a smooth quasi-compact rigid analytic space over C. Then, for each $i \geq 0$, the vector bundle $\mathcal{H}^i_{\mathrm{FF}}(X)$ admits a finite increasing filtration whose j-th graded piece is of slope $(i+j)/2$.*

This provides a new filtration on vector bundles on the Fargues–Fontaine curve, which is different from the Harder-Narasimhan filtration.

§ 1.4 Organization

This thesis is organized as follows.

- Chapter 2 recollects aspects of ∞-categories used in this thesis. In §2.1, we review how to compute limits and colimits of presentable ∞-categories and their variants and then focus on a special type of limit—namely, homotopy fixed points. In §2.2, we review the basic theory of weight structures, focusing on the construction of bounded weight structures and the associated weight complex functor (§2.2.2), as well as on the extension of such bounded structures (defined on the subcategory of compact objects) to the entire category (§2.2.3). In the final section, we compute mapping spaces from module categories; see Proposition 2.3.3.

- Chapter 3 begins in §3.1 with an introduction of the basic theory of motives used in this thesis. In §3.2, we study Galois-equivariant rigid analytic motives: in §3.2.1, we define these motives, and in the remaining subsections we prove Proposition 1.1.1, 1.1.2 and 1.1.3. We review weight structures on motives in §3.3: in §3.3.1, we recall the Chow weight structure on algebraic motives and explain how to induce the weight structure on $\mathbf{RigDA}_{\mathrm{gr}}(K)$; then, using results from §3.2.1, we extend the weight structure to the entire category in 3.3.2.

- Chapter 4 focuses on comparisons of realization functors. In §4.1, we recall the derived ∞-category of (φ, N, G_K)-modules and solid quasi-coherent sheaves on analytic spaces (in particular, on the Fargues–Fontaine curve), which provide the coefficients for the realization functors. In §4.2, we review the realization functors involved in this thesis and prove comparison results with the rigid cohomology. In §4.3, we extend the Hyodo–Kato realization functor to the entire category and construct the corresponding weight filtration. In the final section, we prove the main comparison theorems: the monoidal comparison between de Rham–Fargues–Fontaine realization and Hyodo–Kato realization, and the uniqueness of Galois-enhanced comparison.

Notation and Conventions

Set Theory

Since presentable ∞-categories are heavily used in this thesis, set-theoretic issues inevitably arise. For convenience, however, we will not dwell on set theory and do not treat these matters in full detail[1]. To this end, we assume the axiom of Grothendieck universes and fix two (uncountable) Grothendieck universes $\mathcal{U}$ and $\mathcal{V}$ with $\mathcal{U} \in \mathcal{V}$. We will refer to a mathematical object as small if it belongs to $\mathcal{U}$; in particular, presentable ∞-categories will always be understood with respect to the universe $\mathcal{U}$. Finally, we denote the countable ordinal number by ω.

∞-Categories

We will freely use the language of ∞-categories in the model developed by Lurie, and we mostly follow notations in [HTT; HA]. More precisely, we adopt the following conventions:

- Let $\mathbf{\Delta}$ denote the simplex category, whose objects are linearly ordered sets of the form $[n] = \{0 < 1 < \cdots < n\}$ and whose morphisms are non-decreasing functions. Let $\mathcal{S}\mathbf{et}$ be the ordinary category of $\mathcal{V}$-small sets. The category of $\mathcal{V}$-small simplicial sets, denoted by $\mathbf{s}\mathcal{S}\mathbf{et}$, is then the presheaf category $\mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathcal{S}\mathbf{et})$.

- If $\mathcal{C}$ is a $\mathcal{V}$-small category, its nerve is denoted by $\mathrm{N}_\bullet(\mathcal{C}) \in \mathbf{s}\mathcal{S}\mathbf{et}$. If $\mathcal{C}_\bullet$ is a $\mathcal{V}$-small simplicial category, its homotopy coherent nerve is denoted by $\mathrm{N}^{\mathrm{hc}}_\bullet(\mathcal{C}_\bullet) \in \mathbf{s}\mathcal{S}\mathbf{et}$.

- Unless otherwise stated, all ∞-categories are assumed to be $\mathcal{V}$-small. If $\mathcal{C}$ is an ∞-category, we will write "subcategory" to mean "sub-∞-category". The full subcategory of (ω-)compact objects in $\mathcal{C}$ is denoted by $\mathcal{C}_\omega$. The homotopy category of $\mathcal{C}$ is denoted by $\mathrm{h}\,\mathcal{C}$, which is an ordinary category. For objects X, Y in $\mathcal{C}$, the mapping space is denoted by $\mathrm{Map}_{\mathcal{C}}(X, Y)$, or simply by $\mathrm{Map}(X, Y)$ when $\mathcal{C}$ is clear from context. The corresponding mapping spectrum is denoted by $\mathrm{map}_{\mathcal{C}}(X, Y)$ or simply by $\mathrm{map}(X, Y)$. Finally, we will abbreviate $\pi_0\mathrm{Map}_{\mathcal{C}}(X, Y) \cong \mathrm{Hom}_{\mathrm{h}\,\mathcal{C}}(X, Y)$ by $\mathrm{Hom}_{\mathcal{C}}(X, Y)$.

- Let $\mathcal{C}$ be a stable ∞-category. We write the suspension and the loop functors as $[1]$ and $[-1]$, respectively, instead of Σ and Ω. This agrees with the shift functor on its homotopy category. Throughout, we adopt the homological indexing convention for homological algebras. For every $n \geq 0$ and $X, Y \in \mathcal{C}$, we set $\mathrm{Ext}^n_{\mathcal{C}}(X, Y) := \pi_{-n}\mathrm{map}(X, Y)$.

- We denote by $\mathcal{C}\mathbf{at}_\infty$ the ∞-category of ($\mathcal{U}$-)small ∞-categories and by $\mathcal{C}\mathbf{AT}_\infty$ the ∞-category of locally small ∞-categories. More precisely, objects of $\mathcal{C}\mathbf{at}_\infty$ are $\mathcal{V}$-

[1]Those seeking a more precise treatment of set-theoretic matters may refer to [HTT; HA; Kerodon].

small ∞-categories whose mapping spaces are $\mathcal{U}$-small. In particular, we only use $\mathcal{U}$-small spaces, and we denote the ∞-category of $\mathcal{U}$-small spaces (a.k.a., anima, or ∞-groupoids) by $\mathcal{S}$. The stabilization of $\mathcal{S}$, the ∞-category of spectra, is denoted by $\mathcal{S}\mathbf{p}$.

- For presentable ∞-categories, we denote by $\mathcal{P}\mathbf{r}^{\mathrm{L}}$ (resp. $\mathcal{P}\mathbf{r}^{\mathrm{R}}$) the (non-full) ∞-subcategory of $\mathcal{C}\mathbf{AT}_\infty$ spanned by presentable ∞-categories and left adjoint functors (resp. right adjoint functors). Variants include: the (non-full) subcategory of $\mathcal{P}\mathbf{r}^{\mathrm{L}}$ (resp. $\mathcal{P}\mathbf{r}^{\mathrm{R}}$) spanned by compactly generated ∞-categories and compact-preserving (resp. filtered colimit-preserving and limit-preserving) functors is denoted by $\mathcal{P}\mathbf{r}^{\mathrm{L}}_\omega$ (resp. $\mathcal{P}\mathbf{r}^{\mathrm{R}}_\omega$); and the full subcategory of $\mathcal{P}\mathbf{r}^{\mathrm{L}}$ (resp. $\mathcal{P}\mathbf{r}^{\mathrm{L}}_\omega$) spanned by those which are also stable ∞-categories is denoted by $\mathcal{P}\mathbf{r}^{\mathrm{st}}$ (resp. $\mathcal{P}\mathbf{r}^{\mathrm{st}}_\omega$). Note that we will not use the full subcategory of $\mathcal{P}\mathbf{r}^{\mathrm{R}}$ spanned by stable ∞-categories.

- For symmetric monoidal ∞-categories: by a monoidal structure on an ∞-category, we always mean a symmetric monoidal structure. If an ∞-category $\mathcal{C}$ carries symmetric monoidal structure $\mathcal{C}^\otimes$, then, by an abuse of notation, we will sometimes also write $\mathcal{C}$ for $\mathcal{C}^\otimes$ when the context is clear. In this case, we denoted by $\mathbf{CAlg}(\mathcal{C})$ the ∞-category of commutative algebra objects of $\mathcal{C}^\otimes$ and by $\mathbf{Mod}_A(\mathcal{C})$ the ∞-category of A-modules in $\mathcal{C}$ for $A \in \mathbf{CAlg}(\mathcal{C})$. Given another symmetric monoidal ∞-category $\mathcal{D}$, we write $\mathbf{Fun}^\otimes(\mathcal{C}, \mathcal{D})$ for the ∞-category of symmetric monoidal functors from $\mathcal{C}$ to $\mathcal{D}$.

Geometric Objects

All rings are assumed to be $\mathcal{U}$-small, commutative and unitary. All the schemes, formal schemes and adic spaces are also assumed to be $\mathcal{U}$-small. For motives, we work with the Morel–Voevodsky ∞-category of étale motives with rational coefficients.

Chapter II

Preliminaries on ∞-Categories

In this chapter, we review the necessary aspects of ∞-category theory that will be used in the sequel. We begin in §2.1 by recalling how to compute limits and colimits of presentable ∞-categories; many of these results will be used throughout the thesis. In §2.2, we review the theory of weight structures on stable ∞-categories developed by Bondarko and Sosnilo. Finally, we describe the mapping spaces of monoidal functors from module categories, which will play a crucial role in comparing the realization functors in §4.4.

§ 2.1 Limits and Colimits of ∞-Categories

The purpose of this section is to review limits and colimits of compactly generated ∞-categories as well as monoidal analogues. In §2.1.2, we then focus on a specific type of limit of ∞-categories, namely group actions on ∞-categories and their homotopy fixed points, following [HA, §6.1.6] and [NS18, Chapter I].

§ 2.1.1 Limits and Colimits of Presentable ∞-Categories

Presentable ∞-Categories

We begin by recalling how to compute the limits/colimits of presentable ∞-categories.

Proposition 2.1.1 ([HTT, Proposition 5.5.3.13, Theorem 5.5.3.18])**.** *The ∞-categories $\mathcal{P}\mathbf{r}^{\mathrm{L}}$ and $\mathcal{P}\mathbf{r}^{\mathrm{L}}$ admit small limits. Moreover, the inclusions*

$$\mathcal{P}\mathbf{r}^{\mathrm{L}} \hookrightarrow \mathcal{C}\mathbf{AT}_{\infty}, \quad \mathcal{P}\mathbf{r}^{\mathrm{R}} \hookrightarrow \mathcal{C}\mathbf{AT}_{\infty}$$

preserve small limits.

Remark 2.1.2. Under the equivalence $\mathcal{P}\mathbf{r}^{\mathrm{L}} \simeq (\mathcal{P}\mathbf{r}^{\mathrm{R}})^{\mathrm{op}}$ ([HTT, Corollary 5.5.3.4]), we know that both $\mathcal{P}\mathbf{r}^{\mathrm{L}}$ and $\mathcal{P}\mathbf{r}^{\mathrm{R}}$ admit small colimits from the proposition above.

Compactly Generated ∞-Categories

Most of the ∞-categories in this thesis are compactly generated, so it is also important to understand how their limits or colimits are computed.

Proposition 2.1.3 ([HTT, Proposition 5.5.7.6 and Its Proof]). *The ∞-category $\mathcal{P}\mathbf{r}^{\mathrm{R}}_{\omega}$ admits small limits and the inclusion functors $\mathcal{P}\mathbf{r}^{\mathrm{R}}_{\omega} \hookrightarrow \mathcal{P}\mathbf{r}^{\mathrm{R}} \hookrightarrow \mathbf{CAT}_{\infty}$ preserve small limits.*

Moreover, let $p\colon K \to \mathcal{P}\mathbf{r}^{\mathrm{R}}_{\omega}$ be a diagram of compactly generated ∞-categories $\{\mathcal{C}_{\alpha}\}$ with a limit $\mathcal{C}$ in $\mathcal{P}\mathbf{r}^{\mathrm{R}}$, then $\mathcal{C}$ is also compactly generated by images of $\mathcal{C}_{\alpha,\omega}$ under F_{α}, where $F_{\alpha}\colon \mathcal{C}_{\alpha} \to \mathcal{C}$ is the left adjoint of the canonical functor $G_{\alpha}\colon \mathcal{C} \to \mathcal{C}_{\alpha}$.

Remark 2.1.4. Under the equivalence $\mathcal{P}\mathbf{r}^{\mathrm{L}}_{\omega} \simeq (\mathcal{P}\mathbf{r}^{\mathrm{R}}_{\omega})^{\mathrm{op}}$ ([HTT, Proposition 5.5.7.2 and Corollary 5.5.3.4]), we know that $\mathcal{P}\mathbf{r}^{\mathrm{L}}_{\omega}$ admits small colimits and the inclusion functor $\mathcal{P}\mathbf{r}^{\mathrm{L}}_{\omega} \hookrightarrow \mathcal{P}\mathbf{r}^{\mathrm{L}}$ preserves small colimits (see also [HA, Lemma 5.3.2.9]).

Next, we study limits in $\mathcal{P}\mathbf{r}^{\mathrm{L}}_{\omega}$. We begin with some notational preliminaries. Given a compactly generated ∞-category $\mathcal{C}$, its full subcategory of compact objects $\mathcal{C}_{\omega}$ is a small ∞-category, i.e., $\mathcal{C}_{\omega} \in \mathcal{C}\mathbf{at}_{\infty}$. Moreover, this full subcategory lies in $\mathcal{C}\mathbf{at}^{\mathrm{fcolim,idem}}_{\infty}$, the (non-full) subcategory of $\mathcal{C}\mathbf{at}_{\infty}$ spanned by idempotent complete small ∞-categories that admit finite colimits and functors preserving finite colimits.

Before stating the limit result of $\mathcal{P}\mathbf{r}^{\mathrm{L}}_{\omega}$, we recall a useful categorical lemma:

Lemma 2.1.5 ([Kerodon, Variant 04JX, Corollary 04JW]). *Let $\mathcal{C}$ be an ∞-category and $\mathcal{D}$ a reflexive subcategory of $\mathcal{C}$. Given a small diagram $u\colon K \to \mathcal{D}$ in $\mathcal{D}$, then:*

(1) *The limit of u in $\mathcal{D}$ exists if and only if the limit of u in $\mathcal{C}$ exists. In the case, the inclusion functor preserves limits.*

(2) *If u admits colimit in $\mathcal{C}$, then u admits colimit in $\mathcal{D}$.*

Proposition 2.1.6. *The ∞-category $\mathcal{C}\mathbf{at}^{\mathrm{fcolim,idem}}_{\infty}$ admits small limits and filtered colimits. Moreover, the inclusion functor $\mathcal{C}\mathbf{at}^{\mathrm{fcolim,idem}}_{\infty} \hookrightarrow \mathcal{C}\mathbf{at}_{\infty}$ preserves small limits and filtered colimits.*

Proof. The filtered colimit case is easy since $\mathcal{C}\mathbf{at}^{\mathrm{fcolim,idem}}_{\infty}$ is closed under filtered colimits in $\mathcal{C}\mathbf{at}_{\infty}$ by [HTT, Corollary 4.4.5.21]. For the limit case, let $\mathcal{C} = \lim \mathcal{C}_i$ be a small limit in $\mathcal{C}\mathbf{at}_{\infty}$, where each $\mathcal{C}_i$ lies in $\mathcal{C}\mathbf{at}^{\mathrm{fcolim,idem}}_{\infty}$. Since the full subcategory of idempotent complete ∞-categories is a reflexive subcategory of $\mathcal{C}\mathbf{at}_{\infty}$ by [HTT, Proposition 5.4.2.16], we know that $\mathcal{C}$ is idempotent complete from the previous lemma. Since we can rewrite small limits by pullbacks and small products ([HTT, Proposition 4.4.2.6] or [Kerodon, Proposition 03UL, Remark 03UM]), we conclude that $\mathcal{C}$ admits finite colimits from [HTT, Lemma 5.4.5.5] and the fact that small colimits of the product of ∞-categories are computed pointwise. ■

Corollary 2.1.7. *The functor $\mathcal{C} \mapsto \mathbf{Ind}(\mathcal{C})$ and $\mathcal{C} \mapsto \mathcal{C}_{\omega}$ induce an equivalence of ∞-categories between $\mathcal{C}\mathbf{at}^{\mathrm{fcolim,idem}}_{\infty}$ and $\mathcal{P}\mathbf{r}^{\mathrm{L}}_{\omega}$. In particular, $\mathcal{P}\mathbf{r}^{\mathrm{L}}_{\omega}$ admits small limits.*

Proof. The equivalence is just [HA, Lemma 5.3.2.9] or [HTT, Proposition 5.5.7.8]. Therefore, we conclude that $\mathcal{P}\mathbf{r}^{\mathrm{L}}_{\omega}$ admits small limits from Proposition 2.1.6. ■

We remark that the inclusion $\mathcal{P}\mathbf{r}^{\mathrm{L}}_{\omega} \hookrightarrow \mathcal{P}\mathbf{r}^{\mathrm{L}}$ does not preserve small limits in general unless the result limits in $\mathcal{P}\mathbf{r}^{\mathrm{L}}$ are already compactly generated:

Proposition 2.1.8. *Let $u\colon \mathcal{I} \to \mathcal{P}\mathbf{r}^{\mathrm{L}}_{\omega}$ be a small diagram in $\mathcal{P}\mathbf{r}^{\mathrm{L}}_{\omega}$. Let $\mathcal{C}$ (resp. $\mathcal{C}'$) be the limit of this diagram in $\mathcal{P}\mathbf{r}^{\mathrm{L}}_{\omega}$ (resp. $\mathcal{P}\mathbf{r}^{\mathrm{L}}$). Then the obvious functor $F\colon \mathcal{C} \to \mathcal{C}'$ in $\mathcal{P}\mathbf{r}^{\mathrm{L}}$ is fully faithful, and its essential image is the Ind-completion of $\mathcal{C}'_{\omega}$.*

Proof. Using [HTT, Lemma 5.4.5.7], we can restrict F to a functor $\mathcal{C}_{\omega} \to \mathcal{C}'_{\omega}$. It is fully faithful as $\mathcal{C}_{\omega} \to \mathcal{C}'$ is the limit of fully faithful functors $\mathcal{C}_{i,\omega} \hookrightarrow \mathcal{C}_i$. This restriction of F is also essentially surjective by [HTT, Lemma 5.4.5.5]. Therefore, their Ind-completion gives the equivalence $F\colon \mathcal{C} \xrightarrow{\simeq} \mathbf{Ind}\left(\mathcal{C}'_{\omega}\right) \subseteq \mathcal{C}'$; here we use the equivalence $\mathcal{C} \simeq \mathbf{Ind}\left(\mathcal{C}_{\omega}\right)$ since $\mathcal{C}$ is compactly generated. ∎

Presentable Stable ∞-Categories

In addition, there are two other useful variants of $\mathcal{P}\mathbf{r}^{\mathrm{L}}$: let $\mathcal{P}\mathbf{r}^{\mathrm{st}}$ (resp. $\mathcal{P}\mathbf{r}^{\mathrm{st}}_{\omega}$) denote the full subcategory of $\mathcal{P}\mathbf{r}^{\mathrm{L}}$ (resp. $\mathcal{P}\mathbf{r}^{\mathrm{L}}_{\omega}$) spanned by those ∞-categories that are also stable.

In order to understand limits in $\mathcal{P}\mathbf{r}^{\mathrm{st}}_{\omega}$, we let $\mathcal{C}\mathbf{at}^{\mathrm{ex}}_{\infty}$ be the subcategory of $\mathcal{C}\mathbf{at}_{\infty}$ spanned by stable ∞-categories and exact functors, and let $\mathcal{C}\mathbf{at}^{\mathrm{perf}}_{\infty}$ be the full subcategory of $\mathcal{C}\mathbf{at}^{\mathrm{ex}}_{\infty}$ spanned by those stable ∞-categories that are idempotent-complete.

Remark 2.1.9. By our definition, functors in $\mathcal{P}\mathbf{r}^{\mathrm{st}}$ are left adjoint functors. One could use $\mathcal{P}\mathbf{r}^{\mathrm{R}}$ to define another version of $\mathcal{P}\mathbf{r}^{\mathrm{st}}$, but since most functors in this thesis are left adjoints, we only work with the $\mathcal{P}\mathbf{r}^{\mathrm{L}}$ version.

Lemma 2.1.10. *The equivalence in 2.1.6 restricts to an equivalence between $\mathcal{P}\mathbf{r}^{\mathrm{st}}_{\omega}$ and $\mathcal{C}\mathbf{at}^{\mathrm{perf}}_{\infty}$.*

Proof. It suffices to show, for any $\mathcal{C} \in \mathcal{P}\mathbf{r}^{\mathrm{st}}_{\omega}$, the full subcategory $\mathcal{C}_{\omega}$ is stable. This is clear since $\mathcal{C}_{\omega} \hookrightarrow \mathcal{C}$ preserves finite colimits and $\mathcal{C}_{\omega}$ is closed under the shift functor. ∎

Lemma 2.1.11. *The ∞-category $\mathcal{C}\mathbf{at}^{\mathrm{perf}}_{\infty}$ admits small limits and filtered colimits. Moreover, the inclusions $\mathcal{C}\mathbf{at}^{\mathrm{perf}}_{\infty} \hookrightarrow \mathcal{C}\mathbf{at}^{\mathrm{ex}}_{\infty} \hookrightarrow \mathcal{C}\mathbf{at}_{\infty}$ preserve small limits and filtered colimits.*

Proof. Note that $\mathcal{C}\mathbf{at}^{\mathrm{perf}}_{\infty}$ is a reflexive subcategory of $\mathcal{C}\mathbf{at}^{\mathrm{ex}}_{\infty}$ ([BGT13, Lemma 2.20]), and the latter admits small limits and filtered colimits, which are computed in $\mathcal{C}\mathbf{at}_{\infty}$ ([HA, Theorem 1.1.4.4, Proposition 1.1.4.6]). Therefore, we use Lemma 2.1.5: small limits in $\mathcal{C}\mathbf{at}^{\mathrm{perf}}_{\infty}$ exist and are computed in $\mathcal{C}\mathbf{at}^{\mathrm{ex}}_{\infty}$; and filtered colimits in $\mathcal{C}\mathbf{at}^{\mathrm{perf}}_{\infty}$ exist and are also computed in $\mathcal{C}\mathbf{at}^{\mathrm{ex}}_{\infty}$ because filtered colimits of idempotent complete ∞-categories are idempotent complete by [HTT, Corollary 4.4.5.21]. ∎

Proposition 2.1.12.

(1) *The ∞-category $\mathcal{P}\mathbf{r}^{\mathrm{st}}$ admits small limits and small colimits. Moreover, the inclusion $\mathcal{P}\mathbf{r}^{\mathrm{st}} \to \mathcal{P}\mathbf{r}^{\mathrm{L}}$ preserves small limits and colimits.*

(2) *The ∞-category $\mathcal{P}\mathbf{r}^{\mathrm{st}}_{\omega}$ admits small limits and filtered colimits. Moreover,*

- *the inclusion functor* $\mathcal{P}\mathbf{r}^{\mathrm{st}}_{\omega} \hookrightarrow \mathcal{P}\mathbf{r}^{\mathrm{L}}_{\omega}$ *preserves small limits and filtered colimits;*
- *the inclusion functor* $\mathcal{P}\mathbf{r}^{\mathrm{st}}_{\omega} \hookrightarrow \mathcal{P}\mathbf{r}^{\mathrm{st}}$ *preserves filtered colimits.*

Proof. (1) This follows from Proposition 2.1.1 and [HA, Theorem 1.1.4.4].

(2) Under the equivalence in Lemma 2.1.10, it suffices to show the existence of limits and filtered colimits in $\mathcal{C}\mathbf{at}^{\mathrm{perf}}_{\infty}$. This has been done in Lemma 2.1.11. For the last assertion, the first part is obvious; for the inclusion $\mathcal{P}\mathbf{r}^{\mathrm{st}}_{\omega} \hookrightarrow \mathcal{P}\mathbf{r}^{\mathrm{st}}$, it is enough to observe that filtered colimits in both of these two categories are computed in $\mathcal{P}\mathbf{r}^{\mathrm{L}}$, as explained in (1) and Remark 2.1.4.

■

As we saw, most limits of special kinds of ∞-categories ultimately reduce to limits in $\mathcal{C}\mathbf{AT}_{\infty}$. Let us recall an explicit formula to compute (small) limits of ∞-categories in a concrete way.

Proposition 2.1.13 (Mapping Spaces of Limits)**.** *Let $\mathcal{F}\colon I \to \mathcal{C}\mathbf{AT}_{\infty}$ be a diagram of ∞-categories indexed by a simplicial set I. For each vertex i of I, we let $\mathcal{C}_i$ denote the image of i under $\mathcal{F}$ and put $\mathcal{C} = \lim_{\mathcal{F}} \mathcal{C}_i$ with canonical functors $p_i\colon \mathcal{C} \to \mathcal{C}_i$. Then, given any objects X, Y in $\mathcal{C}$ with $X_i = p_i(X)$, $Y_i = p_i(Y)$, then we have a homotopy equivalence*

$$\mathrm{Map}_{\mathcal{C}}(X, Y) \simeq \lim \mathrm{Map}_{\mathcal{C}_i}(X_i, Y_i).$$

Proof. For each $i \in I$, we have a pullback diagram

$$\begin{tikzcd} \mathrm{Map}_{\mathcal{C}_i}(X_i, Y_i) \arrow[r] \arrow[d] & \mathrm{Fun}(\Delta^1, \mathcal{C}_i) \arrow[d] \\ \Delta^0 \arrow[r, "{(X_i, Y_i)}"] & \mathrm{Fun}(\partial\Delta^1, \mathcal{C}_i) \end{tikzcd}$$

in the (ordinary) category of simplicial sets $\mathbf{sSet}$. The exponentiation of isofibrations ([Kerodon, Corollary 01F3]) shows the right vertical map is an isofibration. This implies that this diagram is a categorical pullback square, i.e., $\mathrm{Map}_{\mathcal{C}_i}(X_i, Y_i)$ is equivalent to the homotopy fiber product (see [Kerodon, Proposition 033P]). After taking the homotopy coherent nerve, we get a pullback square in $\mathcal{C}\mathbf{AT}_{\infty}$, where the commutative diagram above represents the boundary of this pullback square. The limit of these squares yields a pullback square in $\mathcal{C}\mathbf{AT}_{\infty}$ whose boundary is given by

$$\begin{tikzcd} \lim \mathrm{Map}_{\mathcal{C}_i}(X_i, Y_i) \arrow[r] \arrow[d] & \mathbf{Fun}(\Delta^1, \mathcal{C}) \arrow[d] \\ \Delta^0 \arrow[r, "{(X, Y)}"] & \mathbf{Fun}(\partial\Delta^1, \mathcal{C}) \end{tikzcd}.$$

This follows from the facts that limits of functors are computed levelwise and that limits commute with limits. This gives the desired homotopy equivalence of spaces. ■

Symmetric Monoidal Categories

The ∞-category $\mathcal{P}\mathbf{r}^{\mathrm{L}}$ carries a symmetric monoidal structure (see [HA, Proposition 4.8.1.15]). More precisely, the tensor product operator is given by ([HA, Proposition 4.8.1.17])

$$\mathcal{C} \otimes \mathcal{D} \simeq \mathrm{RFun}(\mathcal{C}^{\mathrm{op}}, \mathcal{D}),$$

where RFun is the ∞-category of functors admitting left adjoints. The subcategory $\mathcal{P}\mathbf{r}^{\mathrm{L}}_{\omega}$ inherits the symmetric monoidal structure via the natural inclusion $\mathcal{P}\mathbf{r}^{\mathrm{L}}_{\omega} \to \mathcal{P}\mathbf{r}^{\mathrm{L}}$, by [HA, Lemma 5.3.2.11]. Moreover, the subcategories $\mathcal{P}\mathbf{r}^{\mathrm{st}}$ and $\mathcal{P}\mathbf{r}^{\mathrm{st}}_{\omega}$ also inherit the symmetric monoidal structure ([HA, §4.8.2], especially [HA, Proposition 4.8.2.7]). Hence, we may consider their commutative algebra objects.

Proposition 2.1.14. *Let $\mathcal{C} \in \{\mathcal{P}\mathbf{r}^{\mathrm{L}}, \mathcal{P}\mathbf{r}^{\mathrm{L}}_{\omega}, \mathcal{P}\mathbf{r}^{\mathrm{st}}, \mathcal{P}\mathbf{r}^{\mathrm{st}}_{\omega}\}$ with the forgetful functor $\theta\colon \mathbf{CAlg}(\mathcal{C}) \to \mathcal{C}$.*

(1) *The forgetful functor $\theta\colon \mathbf{CAlg}(\mathcal{C}) \to \mathcal{C}$ is conservative.*

(2) *The ∞-category $\mathbf{CAlg}(\mathcal{C})$ admits small limits. Moreover, the forgetful functor $\theta\colon \mathbf{CAlg}(\mathcal{C}) \to \mathcal{C}$ preserves small limits.*

(3) *If $\mathcal{C} \in \{\mathcal{P}\mathbf{r}^{\mathrm{L}}, \mathcal{P}\mathbf{r}^{\mathrm{L}}_{\omega}, \mathcal{P}\mathbf{r}^{\mathrm{st}}\}$, then $\mathbf{CAlg}(\mathcal{C})$ admits small colimits. Moreover, the forgetful functor only preserves sifted colimits*[1].

(4) *The ∞-category $\mathbf{CAlg}(\mathcal{P}\mathbf{r}^{\mathrm{st}}_{\omega})$ admits filtered colimits and the forgetful functor preserves them.*

Proof. The conservativity follows from [HA, Lemma 3.2.2.6]. For part (2), it follows from [HA, Corollary 3.2.2.5] and our previous discussions on limits of these presentable ∞-categories.

For parts (3) and (4), we apply [HA, Corollary 3.2.3.2, Corollary 3.2.3.3]. Recall that the tensor products of $\mathcal{P}\mathbf{r}^{\mathrm{L}}$ preserve small colimits separately in each variable (see [HA, Remark 4.8.1.24 & Lemma 5.3.2.11]); the same then holds for $\mathcal{P}\mathbf{r}^{\mathrm{L}}_{\omega}$, $\mathcal{P}\mathbf{r}^{\mathrm{st}}$ and $\mathcal{P}\mathbf{r}^{\mathrm{st}}_{\omega}$. Therefore, these two parts follow from our previous discussions on colimits. ■

§ 2.1.2 Homotopy Fixed Points

Notation 2.1.15. Given a (multiplicative) group G, we let BG denote the groupoid defined by G. To be precise, BG has exactly one object $*$ with $\mathrm{Hom}(*, *) = G$, and the composition law is given by the multiplication of G. The ∞-groupoid $B_{\bullet}G$ is obtained by taking the nerve of it.

[1] See Remark 2.3.2 for why it fails to preserve all colimits.

Definition 2.1.16. Let G be a group and $\mathcal{C}$ an ∞-category with an object X.

(1) A ***G-action*** on X is a functor $B_\bullet G \to \mathcal{C}$ sending $*$ to X. We let $\mathcal{C}^{BG} := \mathrm{Fun}(B_\bullet G, \mathcal{C})$ denote the ***∞-category of G-objects*** in $\mathcal{C}$.

(2) Assume $\mathcal{C}$ admits all limits indexed by $B_\bullet G$. The ***homotopy fixed point*** functor is given by

$$(-)^{\mathrm{h}\,G} \colon \mathcal{C}^{BG} \to \mathcal{C}$$
$$(\rho \colon B_\bullet G \to \mathcal{C}) \mapsto \lim \rho.$$

Remark 2.1.17. (1) In the following, we take $\mathcal{C} = \mathbf{CAT}_\infty$, $\mathcal{P}\mathbf{r}^{\mathrm{L}}$ or $\mathcal{P}\mathbf{r}^{\mathrm{R}}$ in the Definition 2.1.16. This, in particular, defines group actions on ∞-categories and the corresponding ∞-category of homotopy fixed points. In practice, we will consider the case where $X = \mathbf{RigDA}(K)$ is the ∞-category of rigid analytic motives (see §3.1.2) over a complete non-archimedean field K, which serves as the primary example in this paper.

(2) Although $B_\bullet G$ is the nerve of a 1-category, giving a G-action on an object $X \in \mathcal{C}$ is not simply given by a group homomorphism $G \to \mathrm{Aut}_{\mathrm{h}\,\mathcal{C}}(X)$, which is only equivalent to a morphism $\mathrm{sk}_2(B_\bullet G) \to \mathcal{C}$. For us, we are interested in a special case $\mathcal{C} = \mathbf{CAT}_\infty$; therefore, using Grothendieck's construction, a functor $B_\bullet G \to \mathbf{CAT}_\infty$ is given by (up to isomorphism) taking the simplicial nerve of a functor from BG to the simplicial category of ∞-categories, see [Kerodon, Corollary 0387]. Therefore, for an ∞-category $\mathcal{C}$ with a G-action, we often write it as a pair $\tilde{\mathcal{C}} = (\mathcal{C}, \rho)$, where ρ represents the G-action on $\mathcal{C}$.

From now on, we focus on group actions on ∞-categories and their homotopy fixed points. Proposition 2.1.13 is a good tool to understand mapping spaces of homotopy fixed points. As a preparation step, we first recall how to compute homotopy groups of homotopy fixed points of G-spectra (i.e., a G-object in the ∞-category $\mathcal{S}\mathbf{p}$ of spectra).

Proposition 2.1.18 (Homotopy Fixed Points Spectral Sequence)**.** *Given a finite group G, let E be a G-spectrum. Then there is a convergent spectral sequence*

$$E^2_{p,q} \simeq H^{-p}(G, \pi_q(E)) \Rightarrow \pi_{p+q}(E^{\mathrm{h}\,G}).$$

Proof. This is a special case of the Bousfield-Kan spectral sequence in [BK72]; more precisely, we apply the Bousfield-Kan spectral sequence to the totalization of the cosimplicial space $\mathrm{Map}(E_\bullet G, X)^{\mathrm{h}\,G}$; see also [GJ09, Proposition 7.7]. ∎

Corollary 2.1.19. *Let $\mathcal{C}$ be an ∞-category with an action by a finite group G. Given a pair of objects $\bar{M}$, $\bar{N}$ in $\mathcal{C}^{\mathrm{h}\,G}$, we let M, N denote their underlying objects in $\mathcal{C}$. Then there is a homotopy equivalence of spaces*

$$\mathrm{Map}_{\mathcal{C}^{\mathrm{h}\,G}}(\bar{M}, \bar{N}) \simeq \mathrm{Map}_{\mathcal{C}}(M, N)^{\mathrm{h}\,G}.$$

Moreover, if $\mathcal{C}$ is $\mathbb{Q}$-linear, then, for each integer $i \geq 0$, we have an isomorphism

$$\pi_i \mathbf{Map}_{\mathcal{C}^{\mathrm{h}\,G}}(\bar{M}, \bar{N}) \simeq \left(\pi_i \mathbf{Map}_{\mathcal{C}}(M, N)\right)^G.$$

Proof. The first assertion follows from Proposition 2.1.13. Under the homotopy equivalence in the first assertion, we apply Proposition 2.1.18 to $E = \mathbf{Map}_{\mathcal{C}}(M, N)$: there is a convergent spectral sequence

$$E^2_{p,q} = H^{-p}\left(G, \pi_q \mathbf{Map}_{\mathcal{C}}(M, N)\right) \Rightarrow \pi_{p+q} \mathbf{Map}_{\mathcal{C}^{\mathrm{h}\,G}}(\bar{M}, \bar{N}).$$

Now assume $\mathcal{C}$ is $\mathbb{Q}$-linear; then each term $E^2_{p,q}$ on the second page is a $\mathbb{Q}$-vector space. The vanishing of higher Galois cohomologies of $\mathbb{Q}$-vector spaces ([SP, Lemma 0DV3]) implies that the spectral sequence collapses on the second page with non-trivial terms $E^2_{0,i} = \left(\pi_i \mathbf{Map}_{\mathcal{C}}(M, N)\right)^G$. Therefore, the convergence of the spectral sequence gives isomorphisms. ■

It is easy to deduce, from Corollary 2.1.19, that taking homotopy fixed points preserves full faithfulness of functors. To clarify it, we introduce the following notion:

Definition 2.1.20. Let G be a group and let $\mathrm{ev}\colon \mathcal{C}\mathbf{AT}_\infty^{BG} \to \mathcal{C}\mathbf{AT}_\infty$ denote the evaluation functor at the point, i.e., induced by applying the functor $\mathbf{Fun}(-, \mathcal{C}\mathbf{AT}_\infty)$ to the obvious map $\Delta^0 \to B_\bullet G$. Given a G-∞-category $\tilde{\mathcal{C}} = (\mathcal{C}, \rho)$ (see Remark 2.1.17 (2)), a ***G*-*subcategory*** of $\tilde{\mathcal{C}}$ is a G-∞-category $\tilde{D} = (\mathcal{D}, \tau)$ together with a functor $\tilde{\iota}\colon \tilde{\mathcal{D}} \to \tilde{\mathcal{C}}$ such that $\iota := \mathrm{ev}(\tilde{\iota})$ is a fully faithful functor.

Remark 2.1.21. In Definition 2.1.20, the G-action on $\mathcal{D}$ is unique (up to homotopy) because the functor $\tilde{\iota}\colon \tilde{D} \to \tilde{C}$ is a Cartesian morphism with respect to the evaluation functor ev. Indeed, for every G-∞-category $\tilde{E}$, the fully faithful functor $\iota\colon \mathcal{D} \to \mathcal{C}$ induces a fully faithful functor (see [GHN17, Lemma 5.2] for a stronger form)

$$\mathbf{Map}(\mathcal{E}, \mathcal{D}) \to \mathbf{Map}(\mathcal{E}, \mathcal{C}).$$

Therefore, the commutative diagram of Kan complexes

$$\begin{array}{ccc} \mathbf{Map}(\tilde{\mathcal{E}}, \tilde{\mathcal{D}}) & \longrightarrow & \mathbf{Map}(\tilde{\mathcal{E}}, \tilde{\mathcal{C}}) \\ \downarrow & & \downarrow \\ \mathbf{Map}(\mathcal{E}, \mathcal{D}) & \hookrightarrow & \mathbf{Map}(\mathcal{E}, \mathcal{C}) \end{array}$$

is a homotopy pullback square.

Corollary 2.1.22. *Let $\mathcal{C}$ be an ∞-category equipped with an action by a finite group G. Assume $\mathcal{D}$ is a G-subcategory of $\mathcal{C}$. Then the functor*

$$\mathcal{D}^{\mathrm{h}\,G} \to \mathcal{C}^{\mathrm{h}\,G}$$

obtained by applying $(-)^{\mathrm{h}\,G}$ to the canonical inclusion, is fully faithful.

Proof. Consider the commutative diagram (up to homotopy)

$$\begin{tikzcd} \mathcal{D}^{\mathrm{h}G} \arrow[r] \arrow[d] & \mathcal{C}^{\mathrm{h}G} \arrow[d] \\ \mathcal{D} \arrow[r, hook] & \mathcal{C} \end{tikzcd}$$

where vertical morphisms are forgetful functors. Let $F\colon \mathcal{D}^{\mathrm{h}G} \to \mathcal{C}^{\mathrm{h}G}$ denote the induced functor and let $\bar{M}$ and $\bar{N}$ be two objects in $\mathcal{D}^{\mathrm{h}G}$ with underlying objects M, N in $\mathcal{D}$. The commutative diagram above shows the underlying objects of $F(\bar{M})$ and $F(\bar{N})$ in $\mathcal{C}$ are $\iota(M)$ and $\iota(N)$, respectively; here $\iota\colon \mathcal{D} \hookrightarrow \mathcal{C}$ is the inclusion. We deduce from Corollary 2.1.19:

$$\mathbf{Map}_{\mathcal{D}^{\mathrm{h}G}}(\bar{M}, \bar{N}) \simeq \mathbf{Map}_{\mathcal{D}}(M, N)^{\mathrm{h}G} \simeq \mathbf{Map}_{\mathcal{C}}(\iota(M), \iota(N))^{\mathrm{h}G} \simeq \mathbf{Map}_{\mathcal{C}^{\mathrm{h}G}}(F(\bar{M}), F(\bar{N})).$$

■

We end the subsection with a useful and well-known fact about $B_\bullet G$.

Lemma 2.1.23. *Given a group G, we have a homotopy equivalence in $\mathcal{S}$,*

$$B_\bullet G \simeq |G_\bullet|,$$

where $G_\bullet$ is the simplicial object in $\mathcal{S}$ given by

$$\mathbf{N}_\bullet(\boldsymbol{\Delta}^{\mathrm{op}}) \xrightarrow{B_\bullet G} \mathbf{N}_\bullet(\mathbf{Set}) \hookrightarrow \mathcal{S}.$$

Proof. It follows from the homotopy equivalence $B_\bullet G \simeq \mathrm{Sing}_\bullet(|B_\bullet G|)$ and the fact that the latter is just computed by the given geometric realization; for instance, see [Kerodon, Variant 04QS]. ■

§ 2.2 Weight Structures

This section provides a review of the basic theory of weight structures on stable ∞-categories and relevant constructions (in §2.2.3) used in defining the weight structure on rigid analytic motives.

Notation 2.2.1. Let $\mathcal{C}$ be a stable ∞-category (more generally, an additive ∞-category) and $\mathcal{N}$ a full subcategory of $\mathcal{C}$. We define

$$\begin{aligned} {}^{\perp}\mathcal{N} &:= \{\, X \in \mathcal{C} \,|\, \mathrm{Hom}_{\mathcal{C}}(X, N) = 0,\ \forall N \in \mathcal{N} \,\} \\ \mathcal{N}^{\perp} &:= \{\, X \in \mathcal{C} \,|\, \mathrm{Hom}_{\mathcal{C}}(N, X) = 0,\ \forall N \in \mathcal{N} \,\}. \end{aligned}$$

We will call ${}^{\perp}\mathcal{N}$ (resp. $\mathcal{N}^{\perp}$) the ***left orthogonal complement*** (resp. ***right orthogonal complement***) of $\mathcal{N}$ in $\mathcal{C}$.

Since π_0 preserves arbitrary products of spaces[1], we have the following closure properties of orthogonal complements.

Lemma 2.2.2. *Let $\mathcal{C}$ be a stable ∞-category and $\mathcal{N}$ a full subcategory of $\mathcal{C}$.*

(1) *The left orthogonal complement ${}^{\perp}\mathcal{N}$ of $\mathcal{N}$ in $\mathcal{C}$ is closed under small coproducts (if exist) and extensions.*

(2) *The right orthogonal complement $\mathcal{N}^{\perp}$ of $\mathcal{N}$ in $\mathcal{C}$ is closed under small products (if exist) and extensions.*

§ 2.2.1 Basic Properties of Weight Structures

Definition 2.2.3 **([Bon10, Definition 1.1.1], [Sos22, Definition 3.1.1]).** A ***weight structure*** on a stable ∞-category $\mathcal{C}$ consists of a pair of retract-closed full subcategories $w = (\mathcal{C}_{w\leq 0}, \mathcal{C}_{w\geq 0})$ satisfying the following properties:

Semi-invariance $\mathcal{C}_{w\geq 0}[1] \subseteq \mathcal{C}_{w\geq 0}$, $\mathcal{C}_{w\leq 0}[-1] \subseteq \mathcal{C}_{w\leq 0}$; and we write, for any integers $n \in \mathbb{Z}$,

$$\mathcal{C}_{w\geq n} := \mathcal{C}_{w\geq 0}[n], \qquad \mathcal{C}_{\leq n} := \mathcal{C}_{w\leq 0}[n].$$

Weak Orthogonality If $X \in \mathcal{C}_{w\leq 0}$ and $Y \in \mathcal{C}_{w\geq 1}$, then $\mathrm{Hom}_{\mathcal{C}}(X, Y) \simeq 0$.

Weight Decomposition For any object $X \in \mathcal{C}$, we have a cofiber sequence

$$X_{\leq 0} \to X \to X_{\geq 1},$$

where $X_{\leq 0} \in \mathcal{C}_{w\leq 0}$ and $X_{\geq 1} \in \mathcal{C}_{w\geq 1}$. This is called a ***weight decomposition*** of X.

In the case, we also say $\mathcal{C}$ is a ***weighted stable ∞-category***.

Remark 2.2.4 (Dual to t-Structures)**.** In [Pau08], weight structures were independently studied under the name “co-t-structure”. In fact, the axioms of a weight structure are dual to those of a t-structure on a stable ∞-category (or a triangulated category) $\mathcal{C}$: apart from the Semi-invariance, the role of the pair $(\mathcal{C}_{t\geq 0}, \mathcal{C}_{t\leq -1})$ in the axioms of a t-structure corresponds to the pair $(\mathcal{C}_{w\leq 0}, \mathcal{C}_{w\geq 1})$ for a weight structure.

Remark 2.2.5 (Homotopy Categories)**.** It is clear that weight structures are essentially determined by weight structures on the homotopy categories (which are triangulated categories) in the original sense of Bondarko, see [Bon10]. But, for our purpose, it is much more convenient to state it in terms of stable ∞-categories.

[1] This holds because products of spaces can be computed as Kan complexes, where two points lie in the same connected component whenever they are connected by exactly an edge. However, this fails for simplicial sets; see [Kerodon, Warning 00GY] for a counterexample.

Remark 2.2.6 ((Co)homological Convention)**.** In this thesis, we use the "homological convention" for weight structures, whereas in [Bon10], the "cohomological convention" was used. In the latter convention, the roles of $\mathcal{C}_{w\geq 0}$ and $\mathcal{C}_{w\leq 0}$ are interchanged, that is, the pair $(\mathcal{C}^{w\leq 0}, \mathcal{C}^{w\geq 0}) = (\mathcal{C}_{w\geq 0}, \mathcal{C}_{w\leq 0})$ is for the cohomological convention.

Definition 2.2.7. Let $\mathcal{C}$ be a stable ∞-category equipped with a weight structure $w = (\mathcal{C}_{w\leq 0}, \mathcal{C}_{w\geq 0})$.

(1) For every pair (a, b) of integers with $a \leq b$, we let $\mathcal{C}_{[a,b]}$ denote the full subcategory of $\mathcal{C}$ generated by $\mathcal{C}_{w\geq a} \cap \mathcal{C}_{w\leq b}$. In particular, if $a = b = n$, we write it simply by $\mathcal{C}_{w=n}$.

(2) We let $\mathcal{C}^{\mathrm{b}}$ denote the full subcategory of $\mathcal{C}$ generated by those objects X satisfying $X \in \mathcal{C}_{[a,b]}$ for some integers $a \leq b$. The weight structure w on $\mathcal{C}$ is said to be ***bounded*** if $\mathcal{C} = \mathcal{C}^{\mathrm{b}}$.

(3) We will write $\mathcal{C}_{w=0}$ by $\mathcal{C}^{\heartsuit_w}$ and call it the ***heart*** of the weight structure w.

Example 2.2.8. Let $\mathcal{A}$ be an additive category. The stable ∞-category $\mathcal{K}^{\mathrm{b}}(\mathcal{A})$ of bounded chain complexes has the canonical bounded weight structure:

$$\mathcal{K}^{\mathrm{b}}(\mathcal{A})_{w\geq 0} = \left\{ M \in \mathcal{K}^{\mathrm{b}}(\mathcal{A}) \,\middle|\, M \text{ is isomorphic to a complex whose all negative degrees vanish} \right\}$$
$$\mathcal{K}^{\mathrm{b}}(\mathcal{A})_{w\leq 0} = \left\{ M \in \mathcal{K}^{\mathrm{b}}(\mathcal{A}) \,\middle|\, M \text{ is isomorphic to a complex whose all positive degrees vanish} \right\}$$

The heart of this weight structure is exactly $\mathbf{N}_\bullet(\mathcal{A})$.

Remark 2.2.9 (n-Weight Decomposition)**.** Let $\mathcal{C}$ be a stable ∞-category with a weight structure w and $n \in \mathbb{Z}$. For every $X \in \mathcal{C}$, we have a weight decomposition

$$X[-n]_{\leq 0} \to X[-n] \to X[-n]_{\geq 1}$$

for $X[-n]$. We then take $X_{\leq n} = X[-n]_{\leq 0}[n]$ and $X_{\geq n+1} = X[-n]_{\geq 1}[n]$ and get a cofiber sequence

$$X_{\leq n} \to X \to X_{\geq n+1}$$

with $X_{\leq n} \in \mathcal{C}_{w\leq n}$ and $X_{\geq n+1} \in \mathcal{C}_{w\geq n+1}$. We will call any such decomposition for X the ***n-weight decomposition*** of X.

Proposition 2.2.10 ([Bon10, Proposition 1.3.3])**.** *Let $\mathcal{C}$ be a stable ∞-category with a weight structure w. Let n be an integer.*

(1) *We have $\mathcal{C}_{w\geq n} = \mathcal{C}_{w\leq n-1}^{\perp}$ and $\mathcal{C}_{w\leq n} = {}^{\perp}\mathcal{C}_{w\geq n+1}$.*

(2) *The full subcategories $\mathcal{C}_{w\geq n}$, $\mathcal{C}_{w\leq n}$ and $\mathcal{C}_{w=n}$ are stable under extensions.*

(3) *The full subcategory $\mathcal{C}_{w\geq n}$ is stable under small products (if exists) and cofibers in $\mathcal{C}$.*

(4) *The full subcategory* $\mathcal{C}_{w\leq n}$ *is stable under small coproduct (if exists) and fibers in* $\mathcal{C}$.

(5) *If* $X \to W \to Y$ *is a cofiber sequence in* $\mathcal{C}$ *with* $X, Y \in \mathcal{C}_{w=n}$*, then it is split, i.e.* $W \simeq X \oplus Y$.

Proof. We only need to prove (1) since the other parts immediately follows from (1) and Lemma 2.2.2. Let's prove the first equality in (1), and the proof for the second one is similar. It suffices to prove for $n = 0$. The inclusion $\mathcal{C}_{w\geq 0} \subseteq \mathcal{C}_{w\leq -1}^{\perp}$ is from the Weak Orthogonality. For the converse, let $X \in \mathcal{C}_{w\leq -1}^{\perp}$. We choose a (-1)-weight decomposition for X:

$$X_{\leq -1} \to X \to X_{\geq 0}.$$

By the choice of X, we know the second morphism is an isomorphism; hence X lies in $\mathcal{C}_{w\geq 0}$. ■

Like t-structures, we also keep track of how weights vary under functors.

Definition 2.2.11 ([Bon14, Definition 1.2.1])**.** Let $(\mathcal{C}, w_{\mathcal{C}})$ and $(\mathcal{D}, w_{\mathcal{D}})$ be weighted stable ∞-categories and let $F\colon \mathcal{C} \to \mathcal{D}$ be an exact functor of the underlying stable ∞-categories. We say

(1) F is ***left weight-exact*** if $F(\mathcal{C}_{w\leq 0}) \subseteq \mathcal{D}_{w\leq 0}$.

(2) F is ***right weight-exact*** if $F(\mathcal{C}_{w\geq 0}) \subseteq \mathcal{D}_{w\geq 0}$.

(3) F is ***weight-exact*** if it is both left and right weight-exact.

Remark 2.2.12. In Definition 2.2.11, if $w_{\mathcal{C}}$ is a bounded weight structure on $\mathcal{C}$, then the weight-exactness of F can be checked using $F(\mathcal{C}^{\heartsuit_w})$. To be precise, if $w_{\mathcal{C}}$ is bounded, then F is weight-exact if and only if the essential image of $\mathcal{C}^{\heartsuit_w}$ under F lies in $\mathcal{D}^{\heartsuit_w}$ by [Bon14, Remark 1.2.3 (10)]; see also Proposition 2.2.17.

There is a simple way to compare two weight structures on a stable ∞-category.

Lemma 2.2.13 ([Bon10])**.** *Let* $\mathcal{C}$ *be a stable* ∞*-category with two weight structures* w *and* v.

(1) *If either* $\mathcal{C}_{w\leq 0} = \mathcal{C}_{v\leq 0}$ *or* $\mathcal{C}_{w\geq 0} = \mathcal{C}_{v\geq 0}$*, then* $w = v$.

(2) *If the identity functor* $\mathrm{Id}_{\mathcal{C}}\colon (\mathcal{C}, w) \to (\mathcal{C}, v)$ *is weight-exact, then* $w = v$.

Proof. (1) This is an immediately result of Proposition 2.2.10 (1).

(2) By the assumption, we have $\mathcal{C}_{w\leq 0} \subseteq \mathcal{C}_{v\leq 0}$. Using Proposition 2.2.10 (1), we can know that $\mathcal{C}_{v\leq -1} \subseteq \mathcal{C}_{w\leq -1}$, hence $\mathcal{C}_{v\leq 0} \subseteq \mathcal{C}_{w\leq 0}$. Using the assumption again, we know this inclusion is an equality. Therefore, we conclude from (1).

■

Definition 2.2.14 ([Aok20, Definition 4.1]). Let $\mathcal{C}^{\otimes}$ be a stable symmetric monoidal ∞-category. We say a weight structure $w = (\mathcal{C}_{w\geq 0}, \mathcal{C}_{w\leq 0})$ on the underlying ∞-category $\mathcal{C}$ is ***compatible*** with the symmetric monoidal structure if two full subcategories $\mathcal{C}_{w\geq 0}$ and $\mathcal{C}_{w\leq 0}$ are stable under the tensor product functor. In the case, we also say that w is a ***monoidal weight structure*** on $\mathcal{C}^{\otimes}$.

Remark 2.2.15. If a weight structure on the underlying ∞-category of a stable symmetric monoidal ∞-category $\mathcal{C}^{\otimes}$ is compatible with the monoidal structure, then, by [HA, Proposition 2.2.1.1], the symmetric monoidal structure can be restricted to full subcategories $\mathcal{C}_{w\geq 0}$ and $\mathcal{C}_{w\leq 0}$, $\mathcal{C}^{\heartsuit_w}$. We will use the obvious notations $\mathcal{C}^{\otimes}_{w\geq 0}$, $\mathcal{C}^{\otimes}_{w\leq 0}$, $(\mathcal{C}^{\heartsuit_w})^{\otimes}$ to denote the monoidal restrictions.

§ 2.2.2 Bounded Weight Structures and Weight Complex Functors

We focus on bounded weight structures in the subsection. We recall how these weight structures are recovered by their hearts.

Definition 2.2.16 ([Bon10, Definition 4.5.1], [BS18, Definition 1.2.2]). Let $\mathcal{C}$ be a stable ∞-category. A full subcategory $\mathcal{N}$ of $\mathcal{C}$ is called ***negative*** if the mapping space $\mathrm{Map}_{\mathcal{C}}(X, Y)$ is connected for all $X, Y \in \mathcal{N}$.

Proposition 2.2.17. *Let $\mathcal{C}$ be a stable ∞-category.*

(1) *If $\mathcal{C}$ has a bounded weight structure w, then its heart $\mathcal{C}^{\heartsuit_w}$ is additive, idempotent complete, negative and it generates $\mathcal{C}$ under finite colimits and negative shift.*

(2) *Conversely, if $\mathcal{N}$ be a full subcategory of $\mathcal{C}$ satisfying*

- *the full subcategory $\mathcal{N}$ is negative;*
- *the full subcategory $\mathcal{N}$ generated $\mathcal{C}$ under finite colimits, negative shifts and retracts.*

then

$$\mathcal{C}_{w\geq 0} = \{\textit{retracts of finite colimits of objects in } \mathcal{N}\}$$
$$\mathcal{C}_{w\leq 0} = \{\textit{retracts of finite limits of objects in } \mathcal{N}\}$$

define the unique bounded weight structure on $\mathcal{C}$ whose heart contains $\mathcal{N}$. Moreover, the heart is the idempotent completion of $\mathcal{N}$.

Proof. The first assertion follows from [Bon10, Corollary 1.5.7] and Definitions. The second assertion follows from [Bon10, Theorem 4.3.2 (II)]. ■

Corollary 2.2.18. *Let $\mathcal{C}^{\otimes} \in \mathbf{CAlg}(\mathcal{P}\mathbf{r}^{\mathrm{st}})$. Assume its underlying ∞-category $\mathcal{C}$ is equipped with a bounded weight structure w. Then the weight structure w is compatible with the monoidal structure if and only if $\mathcal{C}^{\heartsuit_w}$ contains the tensor unit and is stable under tensor products.*

Proof. The "only if" part is obvious (see Remark 2.2.15). Conversely, since $\mathcal{C}^{\otimes} \in \mathbf{CAlg}(\mathcal{P}\mathbf{r}^{\mathrm{L}})$, the tensor products preserve small colimits separately in each variable; in particular, they are exact (see [HA, Proposition 1.1.4.1]). So we can conclude from Proposition 2.2.17. ■

Notation 2.2.19. Let $\mathcal{C}$, $\mathcal{D}$ be two additive ∞-categories.

(1) We let $\mathbf{Fun}_{\mathrm{add}}(\mathcal{C}, \mathcal{D})$ denote the full subcategory of $\mathbf{Fun}(\mathcal{C}, \mathcal{D})$ generated by additive functors. If, moreover, $\mathcal{C}$, $\mathcal{D}$ admit symmetric monoidal structures, we let $\mathbf{Fun}^{\otimes}_{\mathrm{add}}(\mathcal{C}, \mathcal{D})$ denote the full subcategory of $\mathbf{Fun}^{\otimes}(\mathcal{C}, \mathcal{D})$ generated by those monoidal functors whose underlying functors are additive.

(2) Suppose $\mathcal{C}$, $\mathcal{D}$ are stable and equipped with weight structures. We let $\mathbf{Fun}_{\text{w-ex}}(\mathcal{C}, \mathcal{D})$ denote the full subcategory of $\mathbf{Fun}(\mathcal{C}, \mathcal{D})$ generated by exact and weight-exact functors. If, moreover, $\mathcal{C}$ and $\mathcal{D}$ admit symmetric monoidal structures compatible with their weight structures, we let $\mathbf{Fun}^{\otimes}_{\text{w-ex}}(\mathcal{C}, \mathcal{D})$ denote the full subcategory $\mathbf{Fun}^{\otimes}_{\text{w-ex}}(\mathcal{C}, \mathcal{D})$ generated by those functors whose underlying functors lie in $\mathbf{Fun}_{\text{w-ex}}(\mathcal{C}, \mathcal{D})$.

Proposition 2.2.20. *Let $\mathcal{C}$ and $\mathcal{D}$ be two bounded weighted stable ∞-categories. Then the restriction functor*

$$\mathbf{Fun}_{\text{w-ex}}(\mathcal{C}, \mathcal{D}) \xrightarrow{\simeq} \mathbf{Fun}_{\mathrm{add}}\left(\mathcal{C}^{\heartsuit_w}, \mathcal{D}^{\heartsuit_w}\right)$$

is an equivalence of ∞-categories. Moreover, if $\mathcal{C}$, $\mathcal{D}$ admit symmetric monoidal structures compatible with their weight structures, then the commutative diagram

$$\begin{tikzcd}
\mathbf{Fun}^{\otimes}_{\text{w-ex}}(\mathcal{C}, \mathcal{D}) \arrow[r] \arrow[d] & \mathbf{Fun}^{\otimes}_{\mathrm{add}}\left(\mathcal{C}^{\heartsuit_w}, \mathcal{D}^{\heartsuit_w}\right) \arrow[d] \\
\mathbf{Fun}_{\text{w-ex}}(\mathcal{C}, \mathcal{D}) \arrow[r, "\simeq"] & \mathbf{Fun}_{\mathrm{add}}\left(\mathcal{C}^{\heartsuit_w}, \mathcal{D}^{\heartsuit_w}\right)
\end{tikzcd}$$

is a Cartesian diagram of sets at 0-simplexes.

Proof. The first assertion is [Sos19, Proposition 3.3], and the monoidal assertion follows from [Aok20, Theorem 4.3]. ■

A very important result of Proposition 2.2.17 is that every object in a bounded weighted stable ∞-category can be assigned to a chain complex.

Construction 2.2.21 (Weight Complex Functor[Bon10; Sos19])**.** Let $\mathcal{C}$ be a bounded weighted stable ∞-category. From Proposition 2.2.17, we know $\mathrm{h}\,\mathcal{C}^{\heartsuit_w}$ is an additive category. We have seen, in Example 2.2.8, that $\mathrm{N}_\bullet(\mathrm{h}\,\mathcal{C}^{\heartsuit_w})$ is the heart of the bounded weighted stable ∞-category $\mathcal{K}^{\mathrm{b}}(\mathrm{h}\,\mathcal{C}^{\heartsuit_w})$. Using Proposition 2.2.20, the unit $\mathcal{C}^{\heartsuit_w} \to \mathrm{N}_\bullet(\mathrm{h}\,\mathcal{C}^{\heartsuit_w})$ of the homotopy-nerve adjunction can be extended to a weight-exact functor

$$W_\bullet \colon \mathcal{C} \to \mathcal{K}^{\mathrm{b}}(\mathrm{h}\,\mathcal{C}^{\heartsuit_w}).$$

We call it the ***weight complex functor*** of $\mathcal{C}$. Moreover, if $\mathcal{C}$ admits a symmetric monoidal structure compatible with the weight structure, then the weight complex functor has the symmetric monoidal refinement by the last assertion of the previous proposition due to Ko Aoki.

§ 2.2.3 Constructions of Weight Structures

As we have seen in Proposition 2.2.17, constructing bounded weight structures is not difficult. Now we recall constructions of general weight structures (not necessarily bounded).

Weight Structures From Negative Classes

If we only assume the negative condition in Proposition 2.2.17, it also allows us to construct weight structures of general stable ∞-categories. However, the weight structure may fail to be bounded.

Proposition 2.2.22 ([BS19, Theorem 2.2.1])**.** *Let $\mathcal{C}$ be a stable ∞-category admitting small colimits. Assume $\mathcal{N}$ is a full subcategory of $\mathcal{C}$ that generates $\mathcal{C}$ under small colimits. Let $\mathcal{C}_{w\geq 0}$ and $\mathcal{C}_{w\leq 0}$ be the full subcategories generated by $(\mathcal{N}[i])_{i\geq 0}$ and $(\mathcal{N}[i])_{i\leq 0}$, respectively, under small coproducts and extensions. If $\mathcal{N} \subseteq {}^{\perp}\mathcal{C}_{w\geq 0}$ holds, then $w = (\mathcal{C}_{w\geq 0}, \mathcal{C}_{w\leq 0})$ is the unique weight structure w on $\mathcal{C}$ satisfying*

- *the heart of w contains $\mathcal{N}$; more precisely, the heart of w is the idempotent completion of the full subcategory generated by (possibly infinite) direct sums of $\mathcal{N}$;*
- *the full subcategory $\mathcal{C}_{w\geq 0}$ is closed under small coproducts (or equivalently small colimits).*

Proof. The Semi-invariance is obvious. For the Weak Orthogonality, it suffices to show ${}^{\perp}\mathcal{C}_{w\geq 1}$ is closed under extension, small coproducts and contains $\mathcal{N}[i]$ for all $i \leq 0$. The closure properties have been shown in Lemma 2.2.2 and the third condition is guaranteed by the assumption.

It remains to prove the existences of Weight Decomposition. We let $\mathcal{W}$ be the full subcategory of $\mathcal{C}$ spanned by those X such that, for every $n \in \mathbb{Z}$, there is a cofiber sequence

$$X_{\leq n} \to X_{\geq n} \to X$$

where $X_{\leq n} \in \mathcal{C}_{w \leq n}$ and $X_{\geq n} \in \mathcal{C}_{w \geq n}$. We need to prove $\mathcal{N} \subseteq \mathcal{W}$ and $\mathcal{W}$ is closed under small colimits so that $\mathcal{W} = \mathcal{C}$. Let $X \in \mathcal{N}$. If $n \leq 0$, we have the cofiber sequence $X[-n] \oplus X[-1] \to X[-n] \to X$; if $n > 0$, we can take the trivial decomposition $0 \to X \to X$. This proves $\mathcal{N}$ is contained in $\mathcal{W}$. Next we prove $\mathcal{W}$ is closed under small colimits, or equivalently, closed under small coproducts and cofibers. Since $\mathcal{C}_{w \geq 0}$ and $\mathcal{C}_{w \leq 0}$ are already closed under small coproducts, it suffices to treat cofibers. Let $X \to Y$ be a morphism in $\mathcal{W}$, and choose decompositions (with obvious notations):

$$X_{\leq n-1} \to X_{\geq n-1} \to X$$
$$Y_{\leq n} \to Y_{\geq n} \to Y.$$

Let Z be the cofiber of $X \to Y$. Using the orthogonality between $\mathcal{C}_{w \leq n-1}$ and $\mathcal{C}_{w \geq n}$, we know that there is a unique (up to homotopy) morphism $X_{\leq n-1} \to Y_{\leq n}$ such that the following diagram

$$\begin{tikzcd}
X_{\geq n-1} \arrow[r] \arrow[d] & Y_{\geq n} \arrow[r] \arrow[d] & Q \arrow[d] \\
X_{\leq n-1} \arrow[r] \arrow[d] & Y_{\leq n} \arrow[r] \arrow[d] & P \arrow[d] \\
X \arrow[r] & Y \arrow[r] & Z
\end{tikzcd}$$

with exact rows and columns, is commutative. Since $\mathcal{C}_{w \geq n}$ and $\mathcal{C}_{w \leq n}$ are extension-closed, we know that $P \in \mathcal{C}_{w \leq n}$ and $Q \in \mathcal{C}_{w \geq n}$, which provide a desired decomposition for Z.

For the uniqueness, if v is the other weight structure on $\mathcal{C}$ satisfying the two conditions. In the case, we know that $\mathcal{C}_{v \geq 0}$, $\mathcal{C}_{v \leq 0}$ are closed under small coproducts, extensions (see also Proposition 2.2.10), and they contain $\mathcal{N}$. The construction of w show that we have inclusions $\mathcal{C}_{w \geq 0} \subseteq \mathcal{C}_{v \geq 0}$ and $\mathcal{C}_{w \leq 0} \subseteq \mathcal{C}_{v \leq 0}$; therefore, we conclude $w = v$ from Lemma 2.2.13. For the left part, we refer readers to the proof of [BS19, Theorem 2.2.1]. ■

Weight Structures From Compact Objects

Proposition 2.2.17 provides a convenient method for constructing bounded weight structures on the "stable envelopes" of good additive ∞-categories. Typically, such stable envelopes arise as the compact part of stable compactly generated ∞-categories. We now explain how to extend these weight structures to the entire categories. To this end, we begin with a general construction from compact objects.

Proposition 2.2.23. *Let $\mathcal{C}$ be a stable ∞-category admitting small colimits. Let S be a set[1] of compact objects of $\mathcal{C}$. Then there is a unique weight structure w on $\mathcal{C}$ with*

$$\mathcal{C}_{w\geq 0} = \left\{ X \in \mathcal{C} \,\middle|\, \mathrm{Hom}_{\mathcal{C}}(T[-n], X) \simeq 0 \text{ for all } T \in S,\ n > 0 \right\}.$$

Proof. This is [Bon10, Proof of Theorem 4.3.2, Theorem 4.5.2] or [Pau12, Theorem 5]. ∎

We summarize by putting these constructions on a compactly generated stable ∞-category.

Proposition 2.2.24. *Let $\mathcal{C}$ be a compactly generated stable ∞-category. Assume the full subcategory $\mathcal{C}_\omega$ of compact objects has a weight structure w. Then the weight structures obtained by taking $\mathcal{N} = \mathcal{C}_\omega^{\heartsuit_w}$ in Proposition 2.2.22 or $S = \mathcal{C}_\omega^{\heartsuit_w}$ in Proposition 2.2.23 are the same ones that extend w. Moreover, this is the unique weight structure on $\mathcal{C}$ extending w on $\mathcal{C}_\omega$ such that $\mathcal{C}_{w\geq 0}$ is closed under small colimits.*

Proof. Since objects in $\mathcal{C}_\omega^{\heartsuit_w}$ are compact and $\mathcal{C}$ is compactly generated, we see that Proposition 2.2.22 does give a weight structure W_s extending w. We let W_c denote the one from Proposition 2.2.23. The negativity of $\mathcal{C}_\omega^{\heartsuit_w}$ shows the heart of W_c contains $\mathcal{C}_\omega^{\heartsuit_w}$, and its construction shows $\mathcal{C}_{W_c\geq 0}$ is closed under small coproducts. Now using the uniqueness in Proposition 2.2.22, we know that $W_c = W_s$. The last assertion also follows from the uniqueness in Proposition 2.2.22. ∎

Corollary 2.2.25. *Let $\mathcal{C}$ be a compactly generated stable ∞-category. Assume there is a small negative full subcategory $\mathcal{N}$ consisting of compact objects, and it generates $\mathcal{C}$ under small colimits and negative shifts. Then:*

(1) *There is a unique weight structure w on $\mathcal{C}$ satisfying*

- *the heart of w contains $\mathcal{N}$;*
- *$\mathcal{C}_{w\geq 0}$ is closed under small colimits.*

(2) *The weight structure w in (1) satisfies*

  (a) *the heart of w is the idempotent completion of the full subcategory generated by $\mathcal{N}$ under directed sums;*

  (b) *the weight structure w restricts to the weight structure on $\mathcal{C}_\omega$ whose heart contains $\mathcal{N}$;*

  (c) *given $X \in \mathcal{C}$, then $X \in \mathcal{C}_{w\geq 0}$ if and only if $X \in (\mathcal{N}[i])^\perp$ for all $i < 0$;*

  (d) *given $X \in \mathcal{C}_\omega$, then $X \in \mathcal{C}_{w\leq 0}$ if and only if $X \in {}^\perp\mathcal{N}[i]$ for all $i > 0$.*

[1] It can be empty. But in that case, the weight structure is trivial, i.e., $\mathcal{C}_{w\geq 0} = \mathcal{C}$.

(3) *Assume $\mathcal{C}$ admits a symmetric monoidal structure whose tensor unit is in $\mathcal{C}^{\heartsuit_w}$. If for any pair (X, Y) of objects in $\mathcal{N}$, the tensor product $X \otimes Y$ is still in the heart, then w is compatible with the monoidal structure.*

Proof. (1) By our assumption, $\mathcal{N}[-1]^{\perp}$ contains all $\mathcal{N}[i]$ for $i \geq 0$ and $\mathcal{N}[-1]^{\perp}$ is closed under extensions and small coproducts (because of compactness of objects in $\mathcal{N}$); thus $\mathcal{N}$ satisfies the assumption in Proposition 2.2.22, we apply it to get a desired weight structure on $\mathcal{C}$.

(2) The part (a) is clear. For part (b), applying Proposition 2.2.17 to $\mathcal{N}$, we get the unique bounded weight structure on $\mathcal{C}_\omega$ whose heart contains $\mathcal{N}$. Then we can extend it to $\mathcal{C}$ using Proposition 2.2.24. We let w' denote this new weight structure on $\mathcal{C}$. So we need to show $w = w'$. This clearly follows from the uniqueness of w in (1). It remains to prove (c) and (d). There are obvious inclusions

$$\mathcal{C}_{w\geq 0} \subseteq (\mathcal{N}[-i])^{\perp}, \quad \mathcal{C}_{w\leq 0} \subseteq {}^{\perp}\mathcal{N}[i],$$

for all $i > 0$. We need to prove the converse directions. For (c), it is clear because ${}^{\perp}\{X\}$ is closed under small coproducts and extensions, and then the assumption on X implies $\mathcal{C}_{w\leq -1} \subseteq {}^{\perp}\{X\}$; equivalently, this means $X \in \mathcal{C}_{w\geq 0}$ by Proposition 2.2.10. The similar argument works to prove (d) when X is compact.

(3) We need to prove that $\mathcal{C}_{w\leq 0}$ (resp. $\mathcal{C}_{w\geq 0}$) is closed under the tensor product. Their proofs are similar. Let us prove for $\mathcal{C}_{w\leq 0}$ (for $\mathcal{C}_{w\geq 0}$, one needs to notice that $\mathcal{C}_{w\geq 0}$ is stable under small coproducts). We define $\mathcal{D}$ as the full subcategory of $\mathcal{C}$ spanned by those $X \in \mathcal{C}$ such that $\mathcal{C}_{w\leq 0} \otimes X \subseteq \mathcal{C}_{w\leq 0}$. Then it suffices to prove $\mathcal{C}_{w\leq 0} \subseteq \mathcal{D}$. For this, we need to show $\mathcal{D}$ is closed under coproducts, extensions and contains $\mathcal{N}[i]$ for all $i \leq 0$. As $\mathcal{C}_{w\leq 0}$ is closed under coproduct and extensions, it remains to show $\mathcal{N}[i] \subseteq \mathcal{D}$ for all $i \leq 0$. For this, we define a full subcategory $\mathcal{D}'$ of $\mathcal{C}$ in a way similar to $\mathcal{D}$, that is, $X \in \mathcal{D}'$ if $X \otimes \mathcal{N}[i] \subseteq \mathcal{C}_{w\leq 0}$ for $i \leq 0$. By our condition in the statement, we have inclusions $\mathcal{N}[j] \subseteq \mathcal{D}'$ for all $j \leq 0$. It is easy to see that $\mathcal{D}'$ is closed under small coproducts and extensions as well. This implies $\mathcal{C}_{w\leq 0} \subseteq \mathcal{D}'$, as desired.

∎

Corollary 2.2.26. *Let $F\colon \mathcal{C}_1 \to \mathcal{C}_2$ be a functor in $\mathcal{P}\mathbf{r}^{\mathrm{st}}_{\omega}$. Assume, for $i = 1, 2$, there is a small full subcategory $\mathcal{N}_i$ satisfying the conditions in Corollary 2.2.25. Let w_i be the weight structure obtained in Corollary 2.2.25 for $i = 1, 2$. Then the following are equivalent:*

(1) *F is weight-exact;*

(2) *after restricting to compact objects, $F\colon \mathcal{C}_{1,\omega} \to \mathcal{C}_{2,\omega}$ is weight-exact;*

(3) *F sends $\mathcal{N}_1$ into the heart of w_2.*

Proof. We prove the non-obvious direction (3) $\Rightarrow$ (1). In fact, this follows from the fact that F preserves small colimits and extensions. ■

Weight Structures on Modules

Given a symmetric monoidal ∞-category $\mathcal{C}^{\otimes}$ and a commutative algebra $A \in \mathcal{C}$, we have the ∞-category $\mathbf{Mod}_A(\mathcal{C})$ of A-modules in $\mathcal{C}$. There is a canonical forgetful functor $\mathbf{Mod}_A(\mathcal{C}) \to \mathcal{C}$ that admits a left adjoint ([HA, Corollary 4.2.4.8])

$$\mathrm{Free}\colon \mathcal{C} \to \mathbf{Mod}_A(\mathcal{C})$$

called the ***free module functor***. In fact, $\mathbf{Mod}_A(\mathcal{C})$ also admits a symmetric monoidal structure given by relative tensor products ([HA, §4.4]) such that the free module functor is symmetric monoidal.

The next proposition is essentially proved in [Sos22, Lemma 3.4.1] and [BGV25, Proposition 4.22].

Proposition 2.2.27. *Let $\mathcal{C}^{\otimes}$ be a compactly generated stable symmetric monoidal ∞-category with the compact tensor unit. Assume the underlying ∞-category $\mathcal{C}$ admits a weight structure w that restricts to a bounded weight structure on $\mathcal{C}_\omega$. Let $A \in \mathbf{CAlg}(\mathcal{C})$ satisfy $A \otimes -\colon \mathcal{C} \to \mathcal{C}$ is right weight-exact (e.g, w is compatible with the monoidal structure on $\mathcal{C}$ and $A \in \mathcal{C}_{w\geq 0}$). Then there is a unique weight structure W on $\mathbf{Mod}_A(\mathcal{C})$ satisfying the following conditions:*

(1) $\mathbf{Mod}_A(\mathcal{C})_{W\geq 0}$ *is closed under colimits.*

(2) *the free module functor* $\mathrm{Free}\colon \mathcal{C} \to \mathbf{Mod}_A(\mathcal{C})$ *is weight-exact.*

(3) *it restricts to a bounded weight structure on the compact part and the restriction of the free functor on compact objects of $\mathcal{C}$ is also weight-exact.*

Moreover, if w is compatible with the symmetric monoidal structure of $\mathcal{C}$, then W is compatible with the symmetric monoidal structure of $\mathbf{Mod}_A(\mathcal{C})$.

Proof. We use Corollary 2.2.25. Note that $\mathbf{Mod}_A(\mathcal{C})$ is compactly generated as $\mathcal{C}$ is ([HA, Lemma 5.3.2.12]). We need to take $\mathcal{N}$ as the full subcategory of $\mathbf{Mod}_A(\mathcal{C})$ spanned by all free modules associated to objects in $\mathcal{C}_\omega^{\heartsuit_w}$. Since $\mathcal{C}_\omega^{\heartsuit_w}$ generates $\mathcal{C}$ under colimits and shifts, and $\mathbf{Mod}_A(\mathcal{C})$ is generated by free modules under colimits, this shows that $\mathbf{Mod}_A(\mathcal{C})$ is generated by $\mathcal{N}$ under colimits and shifts. To use Corollary 2.2.25, it remains to show $\mathcal{N}$ is negative. Let $M, N \in \mathcal{C}_\omega^{\heartsuit_w}$. Then, for any integer $i \in \mathbb{Z}$, we have

$$\mathrm{Map}_{\mathbf{Mod}_A(\mathcal{C})}(\mathrm{Free}(M), \mathrm{Free}(N)[i]) \simeq \mathrm{Map}_{\mathcal{C}}(M, A \otimes N[i]).$$

Since $A \otimes -$ is right weight exact, we know that $A \otimes N[i]$ is in $\mathcal{C}_{w>0}$ for any $i > 0$. In particular, for any $i > 0$, we have

$$\mathrm{Hom}_{\mathbf{Mod}_A(\mathcal{C})}(\mathrm{Free}(M), \mathrm{Free}(N)[i]) \simeq \mathrm{Hom}_{\mathcal{C}}(M, A \otimes N[i]) \simeq 0,$$

where the last isomorphism holds as w is a weight structure on $\mathcal{C}_\omega$.

For the last assertion, we need to check that, for any $X, Y \in \mathcal{C}_\omega^{\heartsuit_w}$, the tensor product $\mathrm{Free}(X) \otimes_A \mathrm{Free}(Y)$ is in the heart of W. The tensor product is just the free module associated to $X \otimes Y$ (which is in $\mathcal{C}_\omega^{\heartsuit_w}$ by the assumption). Therefore, we conclude from the weight-exactness of the free module functor. ■

§ 2.3 Functors From Module Categories

We recall two facts of commutative algebras in symmetric monoidal ∞-categories from [HA]. They will be used to establish the comparison of p-adic cohomology theories in §4.4.

§ 2.3.1 Free Commutative Algebras

Let $\mathcal{C}$ be a symmetric monoidal ∞-category. The forgetful functor $\theta\colon \mathbf{CAlg}(\mathcal{C}) \to \mathcal{C}$ admits a left adjoint (e.g., see [HA, Proposition 3.1.3.13])

$$S\colon \mathcal{C} \to \mathbf{CAlg}(\mathcal{C})$$

called ***free commutative algebra functor***. Moreover, for each $X \in \mathcal{C}$, we have

$$\theta(S(X)) = \coprod_{n \geq 0} \mathrm{Sym}^n(X).$$

Example 2.3.1. There is a motivic example for free algebras: let K be a complete non-archimedean field with perfect residue field. There is a commutative algebra $\chi\mathbb{1}$ in $\mathbf{DA}(k)$ (see §3.1.3), which is a free commutative algebra since. More precisely, there is an isomorphism $\chi\mathbb{1} \simeq S(\mathbb{1}(-1)[-1])$ in $\mathbf{CAlg}(\mathbf{DA}(k))$; see [Ayo24, Theorem 3.13] or [BGV25, Proposition 2.63].

Remark 2.3.2. In Proposition 2.1.14 (3), although $\mathbf{CAlg}(\mathcal{C})$ admits all small colimits, the forgetful functor cannot preserve all of them. The point is finite coproducts in $\mathbf{CAlg}(\mathcal{C})$ are not simply computed in $\mathcal{C}$. Informally, there is no obvious way to equip a tensor product on $\mathcal{A}_1 \coprod \mathcal{A}_2$ where $\mathcal{A}_1$ and $\mathcal{A}_2$ are symmetric monoidal categories. However, finite coproducts of free commutative algebras can be computed in $\mathcal{C}$. Therefore, the existence of finite products of general commutative algebras $\mathcal{A}_i$ follows by rewriting each $\mathcal{A}_i$ as a geometric realization of free commutative algebras in $\mathbf{CAlg}(\mathcal{C})$.

§ 2.3.2 Base Change of Algebras

Let $F\colon \mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ be a functor in $\mathbf{CAlg}(\mathcal{P}\mathbf{r}^{\mathrm{st}}_{\omega})$. We next use the identifications (from [HA, Proposition 3.4.1.3])

$$\mathbf{CAlg}(\mathcal{P}\mathbf{r}^{\mathrm{st}}_{\omega})_{\mathcal{C}^{\otimes}/-} \simeq \mathbf{CAlg}(\mathbf{Mod}_{\mathcal{C}^{\otimes}}(\mathcal{P}\mathbf{r}^{\mathrm{st}}_{\omega})), \quad \mathbf{CAlg}(\mathcal{P}\mathbf{r}^{\mathrm{st}}_{\omega})_{\mathcal{D}^{\otimes}/-} \simeq \mathbf{CAlg}(\mathbf{Mod}_{\mathcal{D}^{\otimes}}(\mathcal{P}\mathbf{r}^{\mathrm{st}}_{\omega})).$$

There is a natural adjunction studied in [HA, §4.5.3]:

$$F^{*}\colon \mathbf{CAlg}(\mathcal{P}\mathbf{r}^{\mathrm{st}}_{\omega})_{\mathcal{C}^{\otimes}/-} \rightleftarrows \mathbf{CAlg}(\mathcal{P}\mathbf{r}^{\mathrm{st}}_{\omega})_{\mathcal{D}^{\otimes}/-}\colon F_{*}.$$

More precisely, F_* is induced by composing with F, and F^* sends $\mathcal{C}^{\otimes} \to \mathcal{E}^{\otimes}$ to $\mathcal{D}^{\otimes} \otimes_{\mathcal{C}^{\otimes}} \mathcal{E}^{\otimes}$.

Under our assumptions, we can identify $\mathbf{Mod}_A(\mathcal{C})$ and $\mathcal{D}$ as objects in $\mathbf{CAlg}(\mathcal{P}\mathbf{r}^{\mathrm{st}}_{\omega})_{\mathcal{C}/-}$ via the free module functor ([HA, §4.2.4]) $\mathrm{Free}\colon \mathcal{C} \to \mathbf{Mod}_A(\mathcal{C})$ and the monoidal functor $F\colon \mathcal{C} \to \mathcal{D}$, respectively.

Proposition 2.3.3. *Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor in $\mathbf{CAlg}(\mathcal{P}\mathbf{r}^{\mathrm{st}}_{\omega})$ and $A \in \mathbf{CAlg}(\mathcal{C})$. Then there is a homotopy equivalence*

$$\mathrm{Map}_{\mathbf{CAlg}(\mathcal{P}\mathbf{r}^{\mathrm{st}}_{\omega})_{\mathcal{C}/-}}(\mathbf{Mod}_A(\mathcal{C}), \mathcal{D}) \simeq \mathrm{Map}_{\mathbf{CAlg}(\mathcal{D})}(FA, \mathbb{1}_{\mathcal{D}}).$$

Furthermore, if $A = S(t)$ is a free algebra in $\mathcal{C}$, then we have a homotopy equivalence

$$\mathrm{Map}_{\mathbf{CAlg}(\mathcal{P}\mathbf{r}^{\mathrm{st}}_{\omega})_{\mathcal{C}/-}}(\mathbf{Mod}_A(\mathcal{C}), \mathcal{D}) \simeq \mathrm{Map}_{\mathcal{D}}(F(t), \mathbb{1}_{\mathcal{D}}).$$

Proof. In fact, here $\mathcal{D} = F_*\mathcal{D}$ in $\mathbf{CAlg}(\mathcal{P}\mathbf{r}^{\mathrm{st}}_{\omega})_{\mathcal{C}/-}$; thus, we have

$$\begin{aligned}\mathrm{Map}_{\mathbf{CAlg}(\mathcal{P}\mathbf{r}^{\mathrm{st}}_{\omega})_{\mathcal{C}/-}}(\mathbf{Mod}_A(\mathcal{C}), \mathcal{D}) &\simeq \mathrm{Map}_{\mathbf{CAlg}(\mathcal{P}\mathbf{r}^{\mathrm{st}}_{\omega})_{\mathcal{D}/-}}(F^*\mathbf{Mod}_A(\mathcal{C}), \mathcal{D}) \\ &\simeq \mathrm{Map}_{\mathbf{CAlg}(\mathcal{P}\mathbf{r}^{\mathrm{st}}_{\omega})_{\mathcal{D}/-}}(\mathbf{Mod}_{FA}(\mathcal{D}), \mathcal{D})\end{aligned}$$

here the last homotopy equivalence follows from

$$F^*\mathbf{Mod}_A(\mathcal{C}) \simeq \mathcal{D} \otimes_{\mathcal{C}} \mathbf{Mod}_A(\mathcal{C}) \simeq \mathbf{Mod}_{FA}(\mathcal{D})$$

by [HA, Theorem 4.8.4.6]. Therefore, we conclude the first assertion from the full faithfulness of the functor

$$\begin{aligned}\mathbf{CAlg}(\mathcal{D}) &\hookrightarrow \mathbf{CAlg}(\mathcal{P}\mathbf{r}^{\mathrm{st}}_{\omega})_{\mathcal{D}/-} \\ A &\mapsto \mathbf{Mod}_A(\mathcal{D});\end{aligned}$$

see [HA, Corollary 4.8.5.21]. For the last assertion, we use the universal property of free algebras:

$$\begin{aligned}\mathrm{Map}_{\mathbf{CAlg}(\mathcal{D})}(FA, \mathbb{1}_{\mathcal{D}}) &\simeq \mathrm{Map}_{\mathbf{CAlg}(\mathcal{C})}(A, G(\mathbb{1}_{\mathcal{D}})) \\ &\simeq \mathrm{Map}_{\mathcal{C}}(t, G(\mathbb{1}_{\mathcal{D}})) \\ &\simeq \mathrm{Map}_{\mathcal{D}}(F(t), \mathbb{1}_{\mathcal{D}})\end{aligned}$$

where G is the right adjoint of F. ∎

Chapter III

A Weight Structure on Rigid Analytic Motives

The goal of this chapter is to construct a weight structure, as recalled in §2.2, on the stable ∞-category of rigid analytic motives over a complete non-archimedean field. This extends the weight structure defined in [BGV25, §4.3]. We begin in §3.1 by recalling the theory of rigid analytic motives and their relationship with algebraic motives over the residue field. These results will also be useful for the discussion of motivic realization functors in the next chapter. Then we study Galois actions on the ∞-categories of rigid analytic motives in §3.2, which is key to constructing the weight structure on rigid analytic motives in §3.3.

§ 3.1 Étale Motives

In this section, we will review the necessary theory of (étale) motives. Since our work involves both rigid analytic spaces and adic spaces, we begin in §3.1.1 to clarify the ambient categories of our geometric objects. We then recall the construction of motives in §3.1.2. Finally, in §3.1.3 and §3.1.4, we explain two perspectives of rigid analytic motives in terms of algebraic motives.

§ 3.1.1 Non-Archimedean Analytic Geometry

We clarify our setup for non-archimedean analytic spaces in this subsection. As is known, there are several approaches to non-archimedean analytic spaces, including Tate's rigid analytic varieties [Tat71; BGR84], Raynaud's formal models [Ray74; BL93a; BL93b; Abb11; FK18], Berkovich spaces [Ber90; Ber93], Huber's adic spaces [Hub93; Hub94; Hub96], and more recently, Clausen-Scholze's analytic stacks [CS19b; CS19a].

In this thesis, we primarily work with Huber's theory of adic spaces. However, we also make use of the six-functor formalism for rigid analytic motives developed in [AGV22], which is formulated using the formal model approach. Therefore, we briefly explain the context in which our geometric objects lie and also recall how different models relate to each other.

Formal Models and Rigid Analytic Spaces

Throughout this thesis, by a formal scheme, we mean an adic formal scheme with finite ideal type, i.e., a ringed space that is locally isomorphic to $\mathrm{Spf}(A)$, where A is an adic ring with an ideal of definition I of finite type; and morphisms between formal schemes are adic morphisms of formal schemes, i.e., morphisms that are locally induced by adic morphisms of adic rings. Now let **FSch** denote the category of formal schemes, and let $\mathbf{FSch}^{\mathrm{qcqs}}$ denote the full subcategory spanned by quasi-compact and quasi-separated formal schemes.

• **The Raynaud's Generic Fiber.** We define the category $\mathbf{RigSpc}^{\mathrm{qcqs}}$ of ***quasi-compact quasi-separated rigid analytic spaces*** as the localization of $\mathbf{FSch}^{\mathrm{qcqs}}$ with respect to the class of admissible formal blowing-ups, and the canonical functor will be denoted by

$$(-)^{\mathrm{rig}}\colon \mathbf{FSch}^{\mathrm{qcqs}} \to \mathbf{RigSpc}^{\mathrm{qcqs}}.$$

We can extend $\mathbf{RigSpc}^{\mathrm{qcqs}}$ along open immersions (defined by open immersions of formal models) to a category **RigSpc** (c.f [FK18, §II.2.2.(c)]), referred to as the category of ***rigid analytic spaces***, and then extend the functor above to a functor

$$(-)^{\mathrm{rig}}\colon \mathbf{FSch} \to \mathbf{RigSpc}. \tag{3.1.1}$$

We will refer to this functor as the ***Raynaud's generic fiber functor***. Given a rigid analytic space X, if there is a formal scheme $\mathfrak{X} \in \mathbf{FSch}$, we will call $\mathfrak{X}$ a ***formal model*** of X. Clearly, every quasi-compact and quasi-separated rigid analytic space always admits a formal model.

• **The Special Fiber.** Every scheme can be regarded as a formal scheme whose ideal of definition is (0). Therefore, we have full subcategories $\mathbf{Sch} \subseteq \mathbf{FSch}$ and $\mathbf{Sch}^{\mathrm{qcqs}} \subseteq \mathbf{FSch}^{\mathrm{qcqs}}$. The inclusion functor from the category of reduced schemes to **FSch** admits a right adjoint, which we will denote by $\mathfrak{X} \mapsto \mathfrak{X}_\sigma$ and refer to as the ***special fiber functor***.

• **The Comparison With Tate's Construction.** Finally, let's explain how this construction relates to the Tate's approach. For this, let R be a complete valuation ring of height 1 whose fraction field is denoted by K. We restrict to the category $\mathbf{FSch}_R^{\mathrm{lft}}$ of formal R-scheme that are locally of (topologically) finite type[1]. For every affine formal R-scheme $\mathrm{Spf}(A)$ in $\mathbf{FSch}_R^{\mathrm{lft}}$, we have an affinoid K-Tate algebra $A \otimes_R K$ (because A is of finite type and K can be regarded as a localization of R). Therefore, this construction can be globalized into a functor

$$\begin{aligned} \mathbf{FSch}_R^{\mathrm{lft}} &\to \mathbf{RigVar}_K \\ \mathrm{Spf}(A) &\mapsto \mathrm{Sp}(A \otimes_R K) \end{aligned}$$

[1] A more general construction is given by Berthelot in [Ber86], see also [dJ95, §7].

where $\mathbf{RigVar}_K$ is the category of ***rigid analytic varieties*** over K in the sense of Tate ([Tat71; BGR84]). In fact, this functors factor through the restriction of the Raynaud generic fiber functor and induces an equivalence between $\mathbf{RigVar}_K^{\mathrm{qcqs}}$ and $\mathbf{RigSpc}_R^{\mathrm{lft,qcqs}}$ ([Ray74; BL93a; Bos14]).

More generally, given a rigid analytic space S that is locally of finite type over $\mathrm{Spf}(R)^{\mathrm{rig}}$, there is a rigid analytic variety S_0 over K and a functor

$$(-)_0\colon \mathbf{RigSpc}_S^{\mathrm{lft}} \to \mathbf{RigVar}_{S_0}$$

which restricts to an equivalence between $\mathbf{RigSpc}_S^{\mathrm{lft,qs}}$ and $\mathbf{RigVar}_{S_0}^{\mathrm{qs}}$ ([FK18, Theorem II.B.2.5]).

Adic Spaces

Notation 3.1.1. Let A be a topological ring. We will let A° denote the subring of power-bounded elements and let $A^{\circ\circ} \subseteq A^\circ$ denote the ideal of topologically nilpotent elements.

Recall that a ***Huber ring*** is a topological ring A with an open subring A_0 that is an adic ring (whose ideal of definition is of finite type), and the subring A_0 is called the ***ring of definition***. A ***Huber pair*** (A, A^+) consists of a topological ring A and an integrally closed open subring A^+ contained in A°, which is called the ***ring of integral elements***.

From now on, all Huber pairs are assumed to be ***sheafy*** and ***complete*** so that we have an ***affinoid adic space*** $\mathrm{Spa}(A, A^+)$ and we can recover the Huber pair (A, A^+) from its structure sheaves; and, for brevity, we write $\mathrm{Spa}(A) := \mathrm{Spa}(A, A^\circ)$ for any (complete and sheafy) Huber ring A.

There are several useful Huber pairs:

Definition 3.1.2. Let (A, A^+) is a Huber pair.

(1) We say A is ***Tate*** if $A^{\circ\circ} \cap A^\times \neq \varnothing$. In this case, any element in this intersection is called a ***pseudo-uniformizer***.

(2) We say A is ***analytic*** if the ideal generated by $(A^\triangleright)^{\circ\circ}$ is the unit ideal. In particular, any Tate Huber pair is analytic.

(3) We say A is ***uniform*** if A° is a ring of definition, i.e. A° is bounded.

We also say the Huber pair (A, A^+) is ***Tate/analytic/uniform*** if A is.

Remark 3.1.3 (Structure of (Uniform) Tate Huber Pair)**.** Let (A, A^+) be a Tate Huber pair, and let ϖ be a pseudo-uniformizer of A. We can find a ring of definition A_0 of A containing ϖ (otherwise, we replace ϖ by a suitable power of it). Then we have $A = A_0[\varpi^{-1}]$ and A_0 is ϖ-adic. In fact, as ϖ is a unit, each $\varpi^n A_0$ is open in A_0. On the other hand, A_0 is open bounded and ϖ is topologically nilpotent; thus, $\varpi^n A_0$ is a fundamental neighborhood of 0. The last assertion is clear: for any $a \in A$, we have $\lim_{n\to\infty} a\varpi^n = 0$, which implies $a\varpi^k \in A_0$ for $k \gg 0$; equivalently, this means $a \in A_0[\varpi^{-1}]$. In particular, if A is uniform, we have $A = A^+[\varpi^{-1}]$.

Usually, we only consider analytic adic spaces that are locally of the form Spa(A) where A is analytic because we focus on analytic geometry. Next we let **AnAdic** denote the category of analytic adic spaces, i.e., those adic spaces that are locally isomorphic to Spa(A), where A is either analytic or Tate ([SW20, Proposition 4.3.1]). We next recall how to relate analytic adic spaces with rigid analytic spaces and rigid analytic varieties from [Hub94; Hub96].

• **The Comparison With RigSpc.** In [Hub94], Huber assigned every locally noetherian formal scheme $\mathfrak{X}$ to an adic space $\mathfrak{X}^{\mathrm{ad}}$ [Hub94, Theorem 2.2, Proposition 4.1] and see [Zav23, Corollary 1.2] for a more general case. In the affine case, it is given by Spf$(A) \mapsto$ Spa(A, A). This is a fully faithful functor ([Hub94, Proposition 4.2]). Now we let $\mathfrak{S}$ be a locally noetherian formal scheme and let $\mathfrak{S}^{\mathrm{ad}}_{\mathrm{an}}$ denote the open subspace of $\mathfrak{S}^{\mathrm{ad}}_{\mathrm{an}}$ by removing non-analytic points. Then we have a functor

$$\begin{aligned} (-)^{\mathrm{ad}}_{\eta} \colon \mathbf{FSch}_{\mathfrak{S}} &\to \mathbf{AnAdic}_{\mathfrak{S}^{\mathrm{ad}}_{\mathrm{an}}} \\ \mathfrak{X} &\mapsto \mathfrak{X}^{\mathrm{ad}}_{\mathrm{an}}. \end{aligned} \tag{3.1.2}$$

referred to as the ***adic generic fiber functor***. This functor factors through $\mathbf{RigSpc}_{\mathfrak{S}^{\mathrm{rig}}}$, which will also be denoted by

$$(-)^{\mathrm{ad}}_{\eta} \colon \mathbf{RigSpc}_{\mathfrak{S}^{\mathrm{rig}}} \to \mathbf{AnAdic}_{\mathfrak{S}^{\mathrm{ad}}_{\eta}}. \tag{3.1.3}$$

By restricting, this functor induces an equivalence between $\mathbf{RigSpc}^{\mathrm{lft,qs}}_{\mathfrak{S}^{\mathrm{rig}}}$ and $\mathbf{AnAdic}^{\mathrm{lft,qs}}_{\mathfrak{S}^{\mathrm{ad}}_{\eta}}$ (see [Hub96, (1.1.12)], [FK18, Theorem II.A.5.2]).

However, we won't work with (analytic) adic spaces with finite conditions. Fortunately, there is a good embedding of some "good" analytic spaces into the category of rigid analytic spaces. For this, recall that an adic space X is ***uniform*** if for all open affinoid $U = $ Spa$(A) \subseteq X$, the Huber pair A is uniform. This is equivalent to saying X has an open cover by affinoid $U = $ Spa(A) with A ***stably uniform*** (in the sense of [BV18]).

Proposition 3.1.4. *There is a fully faithful functor from the category* $\mathbf{AnAdic}^{\mathrm{uniform}}$ *of uniform analytic adic spaces to the category* **RigSpc** *which is compatible with open immersions. Specifically, in the affinoid case, the functor is given by* Spa$(A, A^+) \mapsto$ Spf$(A^+)^{\mathrm{rig}}$. *Moreover, this embedding is a section of the adic generic fiber functor* (3.1.3).

Proof. The embedding is given in [AGV22, Corollary 1.2.7]. For the last assertion, it suffices to show that the morphism Spa$(A, A^+) \xrightarrow{\simeq}$ Spa$(A^+)_{\mathrm{an}}$ is an isomorphism. Indeed, this morphism is well-defined as $(A^+, A^+) \to (A, A^+)$ is adic and the latter is analytic. For convenience, we may assume A is uniform and Tate (otherwise, we can shrink $\mathrm{Spa}(A, A^+)$). An analytic point $x \in \mathrm{Spa}(A^+)_{\mathrm{an}}$ is equivalent to giving a morphism $(\kappa(x), \kappa(x)^+) \to (A^+, A^+)$ of Huber pairs, where $\kappa(x)$ is a non-archimedean field. By Remark 3.1.3, this is equivalent to a morphism $(\kappa(x), \kappa(x)^+) \to (A, A^+)$ of Huber pairs. The latter lies in $\mathrm{Spa}(A, A^+)_{\mathrm{an}} = \mathrm{Spa}(A, A^+)$. ∎

• **The Comparison With RigVar.** Now let K be a complete non-archimedean field with valuation ring $\mathcal{O}_K$. Then we can consider rigid analytic varieties over K. Indeed, the assignment $\mathrm{Sp}(A) \mapsto \mathrm{Spa}(A, A^\circ)$ extends to a fully faithful functor

$$\mathbf{RigVar}_K \to \mathbf{AnAdic}_K,$$

where $\mathbf{AnAdic}_K$ is the full subcategory of $\mathbf{AnAdic}$ spanned by those defined over $\mathrm{Spa}(K, \mathcal{O}_K)$. It also induces the continuous functor (in the sense of [SP, Definition 00WV]) between associated sites with respect to analytic topologies. By restricting, we have an equivalence between the category $\mathbf{RigVar}_K^{\mathrm{qs}}$ of quasi-separated rigid analytic varieties over K and the category $\mathbf{AnAdic}_K^{\mathrm{lft,qs}}$ of quasi-separated adic spaces locally of finite type over $\mathrm{Spa}(K, \mathcal{O}_K)$. For details of these, we refer to [Hub94, Proposition 4.5].

Finally, let's recall the relation between the adic generic fiber functor and the Raynaud's generic fiber functor. For this, we need to start from the category $\mathbf{FSch}_{\mathcal{O}_K}^{\mathrm{lft}}$. In fact, there is a commutative diagram (up to an isomorphism) by [Hub94, Remark 4.6]

$$\begin{tikzcd} \mathbf{FSch}_{\mathcal{O}_K}^{\mathrm{lft}} \arrow[rr, "\mathrm{Spf}(A)\mapsto \mathrm{Sp}(A\otimes_{\mathcal{O}_K} K)"] \arrow[dr, "(-)_\eta^{\mathrm{ad}}"'] & & \mathbf{RigVar}_K \arrow[dl, hook] \\ & \mathbf{AnAdic}_K & \end{tikzcd} \quad . \tag{3.1.4}$$

§ 3.1.2 ∞-Categories of Étale Motives

We give a quick recall from [Ayo07b; Ayo14b; Ayo15; AGV22] the constructions of ∞-categories of algebraic, formal and analytic étale motives (with $\mathbb{Q}$-coefficients).

The ∞-Categories of Algebraic and Formal Motives

Let $\mathfrak{S}$ be a formal scheme. Let $\mathbf{FSm}/\mathfrak{S}$ denote the category of formal schemes that are smooth over $\mathfrak{S}$ ([AGV22, §1.3]). We will only consider the étale topology on it. For any $\mathfrak{X} \in \mathbf{FSm}/\mathfrak{S}$, we let $\mathbb{A}^1_{\mathfrak{X}} = \mathrm{Spf}(\mathcal{O}_{\mathfrak{X}}\langle T\rangle)$ denote the relative formal affine line over $\mathfrak{X}$.

Now we define $\mathbf{FDA}_{\text{ét}}^{\mathrm{eff}}(\mathfrak{S})$ as the full subcategory of $\mathbf{Shv}_{\text{ét}}(\mathbf{FSm}/\mathfrak{S}, \mathcal{D}(\mathbb{Q}))$, the ∞-category of étale sheaves valued in the derived ∞-category $\mathcal{D}(\mathbb{Q})$, generated by local objects with respect to the collection of morphisms $\mathbb{Q}_{\mathfrak{S}}(\mathbb{A}^1_{\mathfrak{X}}) \to \mathbb{Q}_{\mathfrak{S}}(\mathfrak{X})$, for any $\mathfrak{X} \in \mathbf{FSm}/\mathfrak{S}$, induced by Yoneda's embedding and their desuspensions. The inclusion functor admits a left adjoint

$$\mathrm{L}_{\mathbb{A}^1}\colon \mathbf{Shv}_{\text{ét}}(\mathbf{FSm}/\mathfrak{S}, \mathcal{D}(\mathbb{Q})) \to \mathbf{FDA}^{\text{eff}}_{\text{ét}}(\mathfrak{S}), \tag{3.1.5}$$

called the $\mathbb{A}^1$***-localization functor***. Let $T_{\mathfrak{S}}$ (or simply T if $\mathfrak{S}$ is clear from the context) be the image of the cofiber of the split inclusion $\mathbb{Q}(\mathfrak{S}) \to \mathbb{Q}(\mathbb{A}^1_{\mathfrak{S}} \setminus 0_{\mathfrak{S}})$ under $\mathrm{L}_{\mathbb{A}^1}$. Then we put (in the sense of [Rob15, Definition 2.6])

$$\mathbf{FDA}_{\text{ét}}(\mathfrak{S}) := \mathbf{FDA}^{\text{eff}}_{\text{ét}}(\mathfrak{S})[T_{\mathfrak{S}}^{-1}].$$

We call this the ∞***-category of (étale) motives over*** $\mathfrak{S}$, and simply write $\mathbf{FDA}(\mathfrak{S})$ for it in the next. Given a smooth $\mathfrak{S}$-scheme $\mathfrak{X}$, we let $M(\mathfrak{X})$ denote the image of $\mathfrak{X}$ in $\mathbf{FDA}(\mathfrak{S})$.

As a localization, $\mathbf{FDA}^{\text{eff}}_{\text{ét}}(\mathfrak{S})$ admits a unique symmetric monoidal structure such that $\mathrm{L}_{\mathbb{A}^1}$ is symmetric monoidal. In particular, $\mathbf{FDA}(\mathfrak{S})$ indeed has a symmetric monoidal structure. Moreover, $\mathbf{FDA}(\mathfrak{S})$ is an ∞-category in $\mathbf{CAlg}(\mathcal{P}\mathbf{r}^{\mathrm{L}}_{\omega})$ underlying a six-functor formalism for good $\mathfrak{S}$ in the sense of [AGV22, Definition 2.4.14] (e.g., $\mathfrak{S} = \mathrm{Spf}(\mathcal{O}_K)$ for some complete valuation ring $\mathcal{O}_K$ of height 1). For more details, we refer to [AGV22, §3].

We will let $\mathbb{1}_{\mathfrak{S}}$ (or simply $\mathbb{1}$) denote the monoidal unit of $\mathbf{FDA}(\mathfrak{S})$. For any integer $n \in \mathbb{N}$, we denote by $\mathbb{1}(n)$ the image of $T^{\otimes n}[-n]$ in $\mathbf{FDA}(\mathfrak{S})$ and let $\mathbb{1}(-n)$ denote its monoidal inverse. They are called ***Tate twist***.

Remark 3.1.5. (1) In [AGV22], authors use the notation $\mathbf{FSH}_{\text{ét}}(\mathfrak{S}, \mathbb{Q})$ instead of $\mathbf{FDA}(\mathfrak{S})$. In fact, they work with more general coefficients (connective ring spectra), and, under their convention, they write **FDA** when the ring spectrum is the Eilenberg-Mac Lane spectrum of an ordinary ring. In other words, we have $\mathbf{FDA}(\mathfrak{S}) = \mathbf{FSH}_{\text{ét}}(\mathfrak{S}, H\mathbb{Q})$ (the former is defined here).

(2) In the construction above, we only consider the étale sheaves on $\mathbf{FSm}/\mathfrak{S}$. One can replace $\mathbf{Shv}_{\text{ét}}(\mathbf{FSm}/\mathfrak{S}, \mathbb{Q})$ by the ∞-category of hypersheaves to define a new category $\mathbf{FDA}^{\wedge}(\mathfrak{S})$. In fact, for good $\mathfrak{S}$ (including what we will use in the following), there is an equivalence $\mathbf{FDA}(\mathfrak{S}) \simeq \mathbf{FDA}^{\wedge}(\mathfrak{S})$ of ∞-categories (see [AGV22, Proposition 3.2.2]).

The ∞-category of formal motives can specialize to ∞-categories of algebraic motives: if $\mathfrak{S} = S$ is a scheme, then we write $\mathbf{DA}(S)$ instead of $\mathbf{FDA}(S)$. Indeed, this is the same as starting from étale sheaves on the category $\mathbf{Sm}/S$ of smooth S-schemes since our formal schemes and morphisms between them are assumed to be adic.

Proposition 3.1.6 ([AGV22, Theorem 3.1.10], [Ayo15, Corollaire 1.4.29])**.** *Let $\mathfrak{S}$ be a formal scheme and let $\mathfrak{S}_\sigma$ be its special fiber. Then the special fiber functor induces an equivalence*

$$\mathbf{FDA}(\mathfrak{S}) \xrightarrow{\simeq} \mathbf{DA}(\mathfrak{S}_\sigma).$$

Remark 3.1.7 (Compact Generation,[AGV22, Remark 2.4.23, Proposition 3.2.3])**.** As we mentioned above, only for good $\mathfrak{S}$, the ∞-category $\mathbf{FDA}(\mathfrak{S})$ is compactly generated in which $\mathbf{DA}(\mathfrak{S}_\sigma)$ is also compactly generated by the equivalence above. In fact, as our motives have $\mathbb{Q}$-coefficients, we only need to assume $\mathfrak{S}_\sigma$ is locally of finite Krull dimension. In the case, a set of compact generators of $\mathbf{DA}(\mathfrak{S}_\sigma)$ (resp. $\mathbf{FDA}(\mathfrak{S})$) is given by motives, up to negative shift and negative Tate twist, associated to those quasi-compact and quasi-separated smooth $\mathfrak{S}_\sigma$-schemes (resp. formal $\mathfrak{S}$-schemes).

Notation 3.1.8. For brevity, we denote $\mathbf{FDA}(\mathrm{Spf}(R))$ (resp. $\mathbf{DA}(\mathrm{Spec}(R))$) by $\mathbf{FDA}(R)$ (resp. $\mathbf{DA}(R)$) for any adic ring (resp. discrete ring) R.

The ∞-Categories of Rigid Analytic Motives

We can define the ∞-category of rigid analytic motives using a similar recipe. Let S be a rigid analytic space. We let $\mathbf{RigSm}/S$ denote the category of smooth S-rigid analytic spaces ([AGV22, Definition 1.3.13]). For any $X \in \mathbf{RigSm}/S$, if X admits a formal model $\mathfrak{X}$, we set $\mathbb{B}^n_X := (\mathbb{A}^n_{\mathfrak{X}})^{\mathrm{rig}}$, which is independent of the choice of the formal model and can be defined via gluing along open immersions in general. There is an open rigid analytic subspace $\mathbb{T}^1_X$ of $\mathbb{B}^1_X$: it is locally given by $\mathrm{Spf}(\mathcal{O}_{\mathfrak{X}}\langle T^{\pm 1}\rangle) \subseteq \mathrm{Spf}(\mathcal{O}_{\mathfrak{X}}\langle T\rangle)$.

We can construct $\mathbf{RigDA}(S)$ similarly. Firstly, we let $\mathbf{RigDA}^{\mathrm{eff}}_{\text{ét}}(S)$ be the full subcategory of $\mathbf{Shv}_{\text{ét}}(\mathbf{RigSm}/S, \mathcal{D}(\mathbb{Q}))$ from the same construction as above, with $\mathbb{A}^1_S$ replaced by $\mathbb{B}^1_S$. Then we have the $\mathbb{B}^1$***-localization functor***

$$\mathrm{L}_{\mathbb{B}^1} : \mathbf{Shv}_{\text{ét}}(\mathbf{RigSm}/S, \mathcal{D}(\mathbb{Q})) \to \mathbf{RigDA}^{\mathrm{eff}}_{\text{ét}}(S). \tag{3.1.6}$$

Secondly, we need to invert the Tate object T_S as above; here this object is defined as the cofiber of the split inclusion $\mathbb{Q}_S(S) \to \mathbb{Q}_S(\mathbb{T}^1_S)$ in $\mathbf{RigDA}^{\mathrm{eff}}_{\text{ét}}(S)$. Thus we get the ***$\infty$-category of rigid analytic (étale) motives*** over S

$$\mathbf{RigDA}(S) := \mathbf{RigDA}^{\mathrm{eff}}_{\text{ét}}(S)[T_S^{-1}].$$

As in the algebraic case, we have the ***Tate twist*** of rigid analytic motives. Recall from [AGV22, §2], if we assume now S is locally of finite Krull dimension in the sense of [AGV22, Notation 1.1.11, Remark 1.1.12], the ∞-category $\mathbf{RigDA}(S)$ is indeed in $\mathbf{CAlg}(\mathcal{P}\mathbf{r}^{\mathrm{L}}_\omega)$. Moreover, a set of compact generators is given by motives, up to negative shift and negative Tate twist, associated to those quasi-compact and quasi-separated smooth S-rigid analytic spaces by [AGV22, Proposition 2.4.22]. It underlies the six-functor formalism developed in [AGV22].

Remark 3.1.9. As in Remark 3.1.5, we write $\mathbf{RigDA}(S)$ in place of $\mathbf{RigSH}$ as used in [AGV22] since our chosen coefficient is $H\mathbb{Q}$, and for a suitable S, we also have an equivalence $\mathbf{RigDA}(S) \simeq \mathbf{RigDA}^\wedge(S)$ of ∞-categories for good S where the latter is defined via hypersheaves, see [AGV22, Proposition 2.4.19]. Furthermore, we will also use $\mathbf{RigDA}(S)$ to refer to both the symmetric monoidal ∞-category $\mathbf{RigDA}(S)^{\otimes}$ (as in [AGV22, Definition 2.1.15]) and its underlying ∞-category.

Remark 3.1.10. Let S be a rigid analytic space. An effective rigid analytic motive over S can be explained as an étale (hyper)sheaf $\mathcal{F}$ on $\mathbf{RigSm}/S$ such that, for every $X \in \mathbf{RigSm}$, the canonical morphism $\mathcal{F}(\mathbb{Q}_S(X)) \to \mathcal{F}(\mathbb{Q}_X(\mathbb{B}^1_X))$ is an isomorphism of spectra. As we saw, we define rigid analytic motives by inverting the Tate object T_S in the sense of [Rob15, Definition 2.6]. By [Rob15, Corollary 2.22], the underlying presentable ∞-category $\mathbf{RigDA}(S)$ is the colimit indexed by $\mathbb{N}$ in $\mathcal{P}\mathbf{r}^{\mathrm{L}}$:

$$\mathbf{RigDA}(S) = \operatorname{colim}\left(\mathbf{RigDA}^{\mathrm{eff}}(S) \xrightarrow{-\otimes T_S} \mathbf{RigDA}^{\mathrm{eff}}(S) \xrightarrow{-\otimes T_S} \mathbf{RigDA}^{\mathrm{eff}}(S) \to \cdots).\right)$$

Under the equivalence $\mathcal{P}\mathbf{r}^{\mathrm{R}} \simeq (\mathcal{P}\mathbf{r}^{\mathrm{L}})^{\mathrm{op}}$, we have a limit in $\mathcal{C}\mathbf{AT}_{\infty}$ (see Proposition 2.1.1):

$$\mathbf{RigDA}(S) = \lim\left(\cdots \xrightarrow{\underline{\mathrm{Hom}}(T_S,-)} \mathbf{RigDA}^{\mathrm{eff}}(S) \xrightarrow{\underline{\mathrm{Hom}}(T_S,-)} \mathbf{RigDA}^{\mathrm{eff}}(S)\right)$$

where $\underline{\mathrm{Hom}}(T_S, -)$ is the right adjoint of $-\otimes T_S$. Therefore, a rigid analytic motive over S is a T-spectrum, i.e., given by $E = (E_n)_{n\in\mathbb{N}}$ with equivalences $E_n \simeq \underline{\mathrm{Hom}}(T_S, E_{n+1})$ where each E_n is an effective rigid analytic motive over S. In particular, we have a functor $E = (E_n) \mapsto E_0$, which is often denoted by

$$\Omega^{\infty}_{T_S} \colon \mathbf{RigDA}(S) \to \mathbf{RigDA}^{\mathrm{eff}}(S).$$

It is the right adjoint of the canonical projection $\Sigma^{\infty}_{T_S} \colon \mathbf{RigDA}^{\mathrm{eff}}(S) \to \mathbf{RigDA}(S)$. So for every $X \in \mathbf{RigSm}/S$, the associated rigid analytic motive of X over S is $\Sigma^{\infty}_{T} \circ \mathbb{L}_{\mathbb{B}^1}(\mathbb{Q}_S(X))$, where $\mathbb{Q}_S(-)$ is the Yoneda's embedding, which is usually denoted by $M_S(X)$ or simply $M(X)$ if S is clear. More generally, one can define the rigid analytic motive associated to non-smooth spaces: let $f \colon X \to S$ be a locally of finite type morphism ([AGV22, Corollary 4.3.18]), then we define $M_S(X) := f_! f^! \mathbb{1}_X$. This coincides with the smooth case by the six-functor formalism.

Remark 3.1.11 (Rational Coefficients)**.** Throughout this thesis, we work exclusively with $\mathbb{Q}$-coefficients for motives. This choice is motivated by the fact that many assumptions in [AGV22] are automatically satisfied in the $\mathbb{Q}$-linear setting, and it also allows us to apply Corollary 2.1.19.

Note that $\mathbf{RigDA}(S)$ is $\mathbb{Q}$-linear: the ∞-category $\mathbf{RigDA}^{\mathrm{eff}}(S)$ is $\mathbb{Q}$-linear since it is a full subcategory of $\mathbf{Shv}_{\mathrm{\acute{e}t}}(\mathbf{RigSm}/S, \mathcal{D}(\mathbb{Q}))$; under the identification (see [HA, Proposition 3.4.1.3]):

$$\mathbf{CAlg}(\mathcal{P}\mathbf{r}^{\mathrm{L}})_{\mathbf{Mod}_{H\mathbb{Q}}/-} \simeq \mathbf{CAlg}(\mathbf{Mod}_{\mathbf{Mod}_{H\mathbb{Q}}}(\mathcal{P}\mathbf{r}^{\mathrm{L}})),$$

the canonical functor $\mathbf{RigDA}^{\mathbf{eff}}(S) \to \mathbf{RigDA}(S)$ in $\mathbf{CAlg}(\mathcal{P}\mathbf{r}^{\mathrm{L}})$ implies $\mathbf{RigDA}(S)$ is also $\mathbb{Q}$-linear. Equivalently, this can be verified via the $\mathbb{Q}$-linear DG perspective; see [SAG, Remark D.1.2.3.].

Remark 3.1.12 (Adic Perspective)**.** In most cases, we are interested in analytic adic spaces over $\mathrm{Spa}(\mathbb{Z}_p)$. Thus, one may restrict to a suitable collection of analytic adic spaces over $\mathrm{Spa}(\mathbb{Z}_p)$, as done in [Vez19a; LBV23]. In the case, for any such adic space S over $\mathrm{Spa}(\mathbb{Z}_p)$, we have

$$\mathbb{B}^1_S \simeq S \times_{\mathrm{Spa}(\mathbb{Z}_p)} \mathrm{Spa}(\mathbb{Z}_p\langle T\rangle), \quad \mathbb{T}^1_S \simeq S \times_{\mathrm{Spa}(\mathbb{Z}_p)} \mathrm{Spa}(\mathbb{Z}_p\langle T^{\pm 1}\rangle).$$

So $\mathbb{B}^1_S$ is the adic closed unit disk over S and $\mathbb{T}^1_S$ is the adic unit circle over S. We remark that the space $\mathbb{B}^1_S$ is just a subspace of the adic affine line over S. For instance, we look at the special case $S = \mathrm{Spa}(K, \mathcal{O}_K)$, where K is a non-archimedean field over $\mathbb{Q}_p$. Then we indeed have

$$\mathbb{B}^1_K \simeq \mathrm{Spa}(K\langle T\rangle, \mathcal{O}_K\langle T\rangle) \simeq \left(\mathbb{A}^1_{\mathrm{Spf}(\mathcal{O}_K)}\right)^{\mathrm{rig}}$$

by (3.1.4) and the adic affine line is the increasing union of closed discs $|T| \leq |\varpi|^{-n}$:

$$\mathbb{A}^{1,\mathrm{an}}_K = \bigcup_{n\geq 0} \mathrm{Spa}(K\langle \varpi^n T\rangle) \supset \mathbb{B}^1_K$$

where $\varpi \in \mathcal{O}_K$ is a pseudo-uniformizer. Thus, in the analytic setting, the appropriate notion of motives involves $\mathbb{B}^1$-invariance, in contrast to $\mathbb{A}^1$-invariance in the algebraic setting.

Finally, it is worth noting that the étale topoi over a uniform adic space constructed from the adic and formal perspectives are, in fact, equivalent.

§ 3.1.3 Rigid Analytic Motives with Good Model

Rigid analytic geometry is usually connected to algebraic geometry via the analytification functor. However, for motives, there is a new and deep perspective, developed in [Ayo20] and [AGV22, §3].

Let S be a rigid analytic space that admits a formal model $\mathfrak{S}$, i.e., Raynaud's generic fiber (3.1.1) of $\mathfrak{S}$ is isomorphic to S. The special fiber of $\mathfrak{S}$ is denoted by $\mathfrak{S}_\sigma$. Then we can define the ***Monsky–Washnitzer functor*** studied in [Vez18]

$$\xi\colon \mathbf{DA}(\mathcal{S}_\sigma) \simeq \mathbf{FDA}(\mathcal{S}) \xrightarrow{(-)_\eta} \mathbf{RigDA}(S) \tag{3.1.7}$$

where the first equivalence is Proposition 3.1.6 and this functor admits a right adjoint

$$\chi\colon \mathbf{RigDA}(S) \to \mathbf{DA}(\mathcal{S}_\sigma).$$

Since ξ is symmetric monoidal [AGV22, Proposition 3.1.13], we get a commutative ring object $\chi\mathbb{1}_X \in \mathbf{CAlg}(\mathbf{DA}(\mathcal{S}_\sigma))$ by [AGV22, Corollary 3.4.2]. From now on, we fix such S and its formal model $\mathfrak{S}$ and assume they are locally of finite Krull dimension (so that categories of motives are compactly generated).

Proposition 3.1.13. *With Notations as above, the Monsky–Washnitzer functor ξ factors through the free module functor* $\mathrm{Free}\colon \mathbf{DA}(\mathcal{S}_\sigma) \to \mathbf{Mod}_{\chi\mathbb{1}}(\mathbf{DA}(\mathcal{S}_\sigma))$*; in other words, there is a functor*

$$\tilde{\xi}\colon \mathbf{Mod}_{\chi\mathbb{1}}(\mathbf{DA}(\mathcal{S}_\sigma)) \to \mathbf{RigDA}(X)$$

with a monoidal equivalence $\tilde{\xi} \circ \mathrm{Free} \simeq \xi$ in $\mathbf{CAlg}(\mathcal{P}\mathbf{r}^{\mathrm{L}}_\omega)$. Moreover, $\tilde{\xi}$ is full faithful.

Proof. The first assertion is clear. For the fully faithfulness, see [AGV22, Theorem 3.3.3]. ∎

Definition 3.1.14. We define the ∞-category of ***rigid analytic motives with good model*** over S as the full subcategory of $\mathbf{RigDA}(S)$ generated by $M(\mathfrak{X}^{\mathrm{rig}})$, where $\mathfrak{X}$ runs through smooth formal $\mathfrak{S}$-schemes, under small colimits. We will denote this category by $\mathbf{RigDA}_{\mathrm{gr}}(S)$.

Remark 3.1.15. It follows from Proposition 3.1.13, we have an equivalence $\mathbf{RigDA}_{\mathrm{gr}}(S) \simeq \mathbf{Mod}_{\chi\mathbb{1}}(\mathbf{DA}(\mathfrak{S}_\sigma))$ of ∞-categories. Roughly speaking, every rigid analytic motive with a good model arises from an algebraic motive equipped with a $\chi\mathbb{1}$-action. This provides a new perspective—distinct from analytification—for relating rigid analytic motives with good models to algebraic motives. More generally, every rigid analytic motive becomes one with a good model up to an étale extension of the base rigid analytic spaces; see [AGV22, Theorem 3.3.3(2), §3.7].

In what follows, we make this identification in Remark 3.1.15 implicitly.

Remark 3.1.16. Even when the base is a field, i.e., $S = \mathrm{Spa}(K, \mathcal{O}_K)$, the objects of $\mathbf{RigDA}_{\mathrm{gr}}(K)$ are not only relating to rigid analytic spaces with good reductions. This category includes more than just such simple motives. For example, motives associated to rigid analytic space with semi-stable reductions lie in $\mathbf{RigDA}_{\mathrm{gr}}(K)$; see also [BKV25, Proposition 3.29] for further general examples. For this reason, we prefer to call objects in $\mathbf{RigDA}_{\mathrm{gr}}(K)$ rigid analytic motives with good model.

Remark 3.1.17 (Compact Generators)**.** The ∞-category $\mathbf{RigDA}_{\mathrm{gr}}(K)$ is compactly generated, here K is a complete non-archimedean field with residue field k. More precisely, a set of compact generators of it is given by $\xi(M(X))$, up to shifts and Tate twists, where X runs through proper smooth k-varieties. Indeed, these motives associated to proper smooth k-varieties form a set of compact generators[1] for $\mathbf{DA}(k)$, and the Monsky–Washnitzer functor generates $\mathbf{RigDA}_{\mathrm{gr}}(K)$ under small colimits, by [Ayo20, Proposition 2.31]; or equivalently, one can deduce this from Proposition 3.1.13 and [HA, Lemma 5.3.2.12 together with its proof].

To understand $\mathbf{RigDA}_{\mathrm{gr}}(S)$ via its identification with $\mathbf{Mod}_{\chi\mathbb{1}}(\mathbf{DA}(\mathfrak{S}_\sigma))$, it is useful to have a good grasp of this commutative algebra $\chi\mathbb{1}$. Let's recall a useful and important

[1] This remains true in greater generality; for instance, see [CD19, Proposition 15.2.3].

computation for $\chi\mathbb{1}$ in the case of good reduction. Let's restrict to the most useful case $S = \mathrm{Spa}(K)$, where K is a complete non-archimedean field with residue field k.

Proposition 3.1.18 ([AGV22, Theorem 3.8.1, Corollary 3.8.32 (1)]). *If we choose a pseudo-uniformizer $\varpi \in \mathcal{O}_K$, then we have an identification $\chi\mathbb{1} \simeq \mathbb{1} \oplus \mathbb{1}(-1)[-1]$ in $\mathbf{DA}(k)$.*

§ 3.1.4 Rigid Analytic Motives as Algebraic Motives with Monodromy Operators

There is an another relationship between algebraic motives and rigid analytic motives. We recall it from [BGV25].

We start from a general construction: let $\mathcal{C} \in \mathcal{P}\mathbf{r}^{\mathrm{st}}_{\omega}$ and $t \in \mathcal{C}_{\omega}$. Then there is an ∞-category $\mathcal{C}^{-\otimes t}_{\mathrm{nil}} \in \mathcal{P}\mathbf{r}^{\mathrm{st}}_{\omega}$ equipped with a functor $\pi\colon \mathcal{C}^{-\otimes t}_{\mathrm{nil}} \to \mathcal{C}$. More precisely, an object in $\mathcal{C}^{-\otimes t}_{\mathrm{nil}}$ can be regarded as a pair $(X, N\colon X \to X \otimes t)$, where $X \in \mathcal{C}$ and N is a morphism, referred to as a ***monodromy operator***, in $\mathcal{C}$, such that N is ***Ind-nilpotent***, i.e. $N^k \to 0$ as $k \to \infty$. With this perspective, the functor π sends (X, N) to X, i.e. it is the forgetful functor. There is an obvious section of π which sends $X \mapsto (X, N = 0)$, denoted by $(N = 0)$. It admits a right adjoint:

Lemma 3.1.19 ([BGV25, Lemma 2.16]). *The functor $(N = 0)\colon \mathcal{C} \to \mathcal{C}^{-\otimes t}_{\mathrm{nil}}$ admits a right adjoint* $\mathrm{Fib}(N)\colon \mathcal{C}^{-\otimes t}_{\mathrm{nil}} \to \mathcal{C}$ *which is given on objects by*

$$(X, f) \mapsto \mathbf{Fib}(f).$$

Proof. In fact, there is a forgetful functor $\mathcal{C}^{-\otimes t}_{\mathrm{nil}} \to \mathrm{Fun}(\Delta^1, \mathcal{C})$ and then taking fiber of morphisms. We will apply the dual version of the local criterion [Kerodon, Proposition 02FV] for the adjunction. This means that we need to prove: $\forall (X, f) \in \mathcal{C}^{-\otimes t}_{\mathrm{nil}}$ with the canonical map $\mathrm{Fib}(f) \xrightarrow{u} X$, we have, $\forall Y \in \mathcal{C}$, the composition map

$$\mathbf{Map}_{\mathcal{C}}(Y, \mathbf{Fib}(f)) \xrightarrow{(N=0)} \mathbf{Map}_{\mathcal{C}^{-\otimes t}_{\mathrm{nil}}}\left((Y,0), (\mathbf{Fib}(f), 0)\right) \xrightarrow{\tilde{u}} \mathbf{Map}_{\mathcal{C}^{-\otimes t}_{\mathrm{nil}}}\left((Y,0), (X, f)\right),$$

is a homotopy equivalence. Here $\tilde{u}$ is induced by u, and let's explain it. Let's consider the following diagram

$$\begin{tikzcd}
\mathbf{Map}_{\mathcal{C}^{-\otimes t}_{\mathrm{nil}}}\left((Y,0), (\mathbf{Fib}(f), 0)\right) \arrow[r, shift left, "(N=0)", leftarrow] \arrow[r, shift right, "\pi"'] \arrow[d, "\tilde{u}"] & \mathbf{Map}_{\mathcal{C}}(Y, \mathbf{Fib}(f)) \arrow[r, "0_*"] \arrow[d, "u_*"] \arrow[dl, dashed] & \mathbf{Map}_{\mathcal{C}}(Y, T\,\mathbf{Fib}(f)) \arrow[d, "(Tu)_*"] \\
\mathbf{Map}_{\mathcal{C}^{-\otimes t}_{\mathrm{nil}}}\left((Y,0), (X, f)\right) \arrow[r] & \mathbf{Map}_{\mathcal{C}}(Y, X) \arrow[r, "f_*"] & \mathbf{Map}_{\mathcal{C}}(Y, TX)
\end{tikzcd} \tag{3.1.8}$$

where each row is a fiber sequence ([BGV25, Remark 2.7]); hence we have an induced map $\bar{u}$. Note that the fiber of f_* at 0 is exactly given by $\mathbf{Map}_{\mathcal{C}}(Y, \mathbf{Fib}((f)))$, hence we have a homotopy equivalence between $\mathrm{Map}_{\mathcal{C}}(Y, \mathrm{Fib}(f))$ and $\mathrm{Map}_{\mathcal{C}_{\mathrm{nil}}^{-\otimes t}}\big((Y,0),(X,f)\big)$, which is given by the dotted arrow in the diagram above. Since $(N = 0)$ is a section of π, we have

$$u_* \simeq u_* \circ \pi \circ (N = 0) \simeq \pi \circ \bar{u} \circ (N = 0),$$

which means that $(N = 0) \circ \bar{u}$ is homotopic to the dotted arrow which is a homotopy equivalence; therefore, $\bar{u} \circ (N = 0)$ is a homotopy equivalence, as desired. ■

Remark 3.1.20 (Nilpotent Operators)**.** The ∞-category $\mathcal{C}_{\mathrm{nil}}^{-\otimes t}$ is compactly generated by the essential image of $\mathcal{C}_\omega$ under $(N = 0)$. Thus, compact objects of $\mathcal{C}_{\mathrm{nil}}^{-\otimes t}$ are nilpotent, i.e, $N^k \simeq 0$ for some $k \gg 0$.

If $\mathcal{C}$ carries a symmetric monoidal structure, then so does $\mathcal{C}_{\mathrm{nil}}^{-\otimes t}$, and all of the above discussions admit monoidal refinements.

We now apply these constructions to motives. For this, we let K be a complete non-archimedean field with residue field k. We take $\mathcal{C} = \mathbf{DA}(k)$ and $t = \mathbb{1}(-1)$, and we denote $\mathbf{DA}(k)_{\mathrm{nil}}^{-\otimes \mathbb{1}(-1)}$ by $\mathbf{DA}_N(k)$.

Proposition 3.1.21. *With notations as above, the fiber functor* $\mathrm{Fib}(N)\colon \mathbf{DA}_N(k)\mathbf{DA}(k)$ *is monadic and factors through a monoidal equivalence*

$$\begin{tikzcd} \mathbf{DA}_N(k) \arrow[rr, "\simeq"] \arrow[dr, "\mathrm{Fib}(N)"'] & & \mathbf{Mod}_{\mathbb{1}\oplus\mathbb{1}(-1)[-1]}(\mathbf{DA}(k)) \arrow[dl, "\textit{forgetful}"] \\ & \mathbf{DA}(k) & \end{tikzcd}$$

In particular, a choice of a pseudo-uniformizer $\varpi \in \mathcal{O}_K$ *determines an equivalence of symmetric monoidal categories* $\mathbf{RigDA}_{\mathrm{gr}}(K) \simeq \mathbf{DA}_N(k)$.

Proof. Note that $\mathrm{Alt}^2(\mathbb{1}(-1))$ is homotopic to 0 ([Ayo14a, Lemme 11.5] and also [Ayo07b, Hypothése 3.6.1]), and the adjunction $(N = 0) \vdash \mathrm{Fib}(N)$ satisfies the projection formula ([Ayo14b, Lemme 2.8] and [Rio05]). So this is a special case of [BGV25, Proposition 2.32]. The last assertion follows from Proposition 3.1.13 and Proposition 3.1.18. ■

Remark 3.1.22. After choosing a pseudo-uniformizer of $\mathcal{O}_K$, the previous proposition shows that rigid analytic motive over K with good models can be viewed as motives over residue field with a $\chi\mathbb{1}$-action, but also such motives endowed with monodromy operators. Under the equivalence in Proposition 3.1.21, we have the following correspondences of functors:

$$\chi \leftrightsquigarrow \text{forgetful} \leftrightsquigarrow \mathrm{Fib}(N)$$
$$\xi \leftrightsquigarrow \text{free} \leftrightsquigarrow (N = 0).$$

Besides these, we have the forgetful functor $\pi\colon \mathbf{DA}_N(k) \to \mathbf{DA}(k)$, which defines the ***motivic nearby cycle functor***

$$\Psi\colon \mathbf{RigDA}_{\mathrm{gr}}(K) \to \mathbf{DA}(k). \tag{3.1.9}$$

§ 3.2 Galois-Equivariant Rigid Analytic Motives

From now on, we restrict ourselves to rigid analytic motives over complete non-archimedean fields and study the Galois action on these ∞-categories, in the sense of Definition 2.1.16.

§ 3.2.1 Galois Actions

Construction 3.2.1. Let L/K be a finite Galois extension of complete non-archimedean fields with the Galois group $\mathrm{Gal}(L/K)$. It is clear that $\mathbf{RigDA}(L)$ admits a canonical action by $\mathrm{Gal}(L/K)$ in the sense of Definition 2.1.16. More precisely, we have a functor obtained by taking the homotopy coherent nerve[1]

$$\begin{aligned} A_{L/K} \colon B_\bullet \mathrm{Gal}(L/K) &\to \mathbf{CAlg}(\mathcal{P}\mathbf{r}^{\mathrm{L}}) \\ * &\mapsto \mathbf{RigDA}(L) \\ (\sigma \in \mathrm{Gal}(L/K)) &\mapsto \big(\sigma^* \colon \mathbf{RigDA}(L) \to \mathbf{RigDA}(L)\big) \end{aligned} \tag{3.2.1}$$

Taking homotopy fixed points, we get a new category $\mathbf{RigDA}(L)^{\mathrm{h}\,\mathrm{Gal}(L/K)}$ in $\mathbf{CAlg}(\mathcal{P}\mathbf{r}^{\mathrm{L}})$. We will refer to objects in this new category as ***$\mathbf{Gal}(L/K)$-equivariant rigid analytic motives*** over L.

In the rest of this section, we fix a finite Galois extension L/K of complete non-archimedean fields with the Galois group $\mathrm{Gal}(L/K)$. Let k_L/k be the corresponding extension of residue fields, and assume they're perfect. We will denote by $e\colon \mathrm{Spa}(L) \to \mathrm{Spa}(K)$ and $\bar{e}\colon \mathrm{Spec}(k_L) \to \mathrm{Spec}(k)$ the corresponding morphisms in the geometric world.

The next lemma shows that the Galois action restricts to an action on the subcategory of those with good model.

Lemma 3.2.2 ([AGV22, Proposition 3.1.13]). *Let $e\colon \mathrm{Spa}(L) \to \mathrm{Spa}(K)$ and $\bar{e}\colon \mathrm{Spec}\, k_L \to \mathrm{Spec}\, k$ be the canonical maps. Consider the two diagrams below:*

$$\begin{tikzcd} \mathbf{DA}(k_L) \arrow[r, "\xi_L"] & \mathbf{RigDA}(L) \\ \mathbf{DA}(k) \arrow[u, "\bar{e}^*"] \arrow[r, "\xi_K"] & \mathbf{RigDA}(K) \arrow[u, "e^*"'] \end{tikzcd} \quad \text{in } \mathbf{CAlg}(\mathcal{P}\mathbf{r}^{\mathrm{L}}_{\omega}),$$

$$\begin{tikzcd} \mathbf{DA}(k_L) \arrow[r, "\xi_L"] \arrow[d, "\bar{e}_\sharp"'] & \mathbf{RigDA}(L) \arrow[d, "e_\sharp"] \\ \mathbf{DA}(k) \arrow[r, "\xi_K"] & \mathbf{RigDA}(K) \end{tikzcd} \quad \text{in } \mathcal{C}\mathbf{AT}_\infty.$$

[1] Indeed, taking the homotopy coherent nerve gives $B_\bullet \mathrm{Gal}(L/K) \to \mathcal{C}\mathbf{AT}_\infty$. Then using [HA, Corollary 2.4.1.9] and the fact that $\mathbf{RigDA}(K) \in \mathbf{CAlg}(\mathcal{P}\mathbf{r}^{\mathrm{L}})$, we get this functor.

The first diagram is commutative (up to homotopy), and the second diagram is commutative if L/K is unramified. In particular, we have restrictions

$$\begin{aligned} e^*\colon \mathbf{RigDA}_{\mathrm{gr}}(K) &\to \mathbf{RigDA}_{\mathrm{gr}}(L)\\ e_\sharp\colon \mathbf{RigDA}_{\mathrm{gr}}(L) &\to \mathbf{RigDA}_{\mathrm{gr}}(K), \end{aligned}$$

where $e_\sharp$ is well-defined if L/K is unramified.

The Galois group $\mathrm{Gal}(L/K)$ acts on $\mathbf{RigDA}_{\mathrm{gr}}(L)$ naturally thanks to Lemma 3.2.2. In other words, we have a functor

$$\begin{aligned} A_{L/K,\mathrm{gr}}\colon B_\bullet \mathrm{Gal}(L/K) &\to \mathbf{CAlg}(\mathcal{P}\mathbf{r}^{\mathrm{L}})\\ * &\mapsto \mathbf{RigDA}_{\mathrm{gr}}(L)\\ (\sigma \in \mathrm{Gal}(L/K)) &\mapsto (\sigma^*\colon \mathbf{RigDA}_{\mathrm{gr}}(L) \to \mathbf{RigDA}_{\mathrm{gr}}(L)). \end{aligned}$$

The natural inclusion $\mathbf{RigDA}_{\mathrm{gr}}(L) \hookrightarrow \mathbf{RigDA}(L)$ defines a natural transformation $\iota\colon A_{L/K,\mathrm{gr}} \to A_{L/K}$, where $A_{L/K}$ is defined in (3.2.1). Moreover, this makes $\mathbf{RigDA}_{\mathrm{gr}}(L)$ be a G-subcategory of $\mathbf{RigDA}(L)$ in the sense of Definition 2.1.20. Thus, we deduce from Corollary 2.1.22 that:

Corollary 3.2.3. *The inclusion functor induces a fully faithful functor*

$$\mathbf{RigDA}_{\mathrm{gr}}(L)^{\mathrm{h\,Gal}(L/K)} \hookrightarrow \mathbf{RigDA}(L)^{\mathrm{h\,Gal}(L/K)}.$$

§ 3.2.2 The Galois Descent

Using the Galois descent of rigid analytic motives, we can relate motives over K to motives over the Galois extension field L.

Proposition 3.2.4. *The canonical functor*

$$\tilde{e}^*\colon \mathbf{RigDA}(K) \xrightarrow{\simeq} \mathbf{RigDA}(L)^{\mathrm{h\,Gal}(L/K)}, \tag{3.2.2}$$

induced by $e^*\colon \mathbf{RigDA}(K) \to \mathbf{RigDA}(L)$, is an equivalence in $\mathbf{CAlg}(\mathcal{P}\mathbf{r}^{\mathrm{L}})$.

Proof. Since the forgetful functor $\mathbf{CAlg}(\mathcal{P}\mathbf{r}^{\mathrm{L}}) \to \mathcal{P}\mathbf{r}^{\mathrm{L}}$ is limit-preserving and conservative (Proposition 2.1.14), and limits in $\mathcal{P}\mathbf{r}^{\mathrm{L}}$ are computed as ∞-categories (Proposition 2.1.1), it suffices to prove (3.2.2) is an equivalence in $\mathcal{C}\mathbf{AT}_\infty$. In fact, this is a direct consequence of the étale descent for $\mathbf{RigDA}(-)$. We can write $B_\bullet \mathrm{Gal}(L/K)$ as a colimit of a geometric realization:

$$B_\bullet \mathrm{Gal}(L/K) \simeq \operatorname*{colim}_{\mathrm{N}_\bullet(\Delta)^{\mathrm{op}}} S_\bullet,$$

where $S_n = \mathrm{Gal}(L/K)^n$, see Lemma 2.1.23. It follows that $\mathbf{RigDA}(L)^{\mathrm{h\,Gal}(L/K)}$ can be identified with the limit of a cosimplicial object $X^\bullet$ of $\mathcal{P}\mathbf{r}^{\mathrm{L}}$, where X^n is a limit of the induced diagram

$$S_n \to B_\bullet \mathrm{Gal}(L/K) \to \mathcal{P}\mathbf{r}^{\mathrm{L}}.$$

Since S_n is discrete, we have $X^n \simeq \prod_{\sigma_1,\dots,\sigma_n \in \mathrm{Gal}(L/K)} \mathbf{RigDA}(L)$. In other words, we have

$$\mathbf{RigDA}(L)^{\mathrm{h\,Gal}(L/K)} \simeq \lim \left(\mathbf{RigDA}(L) \rightrightarrows \prod_{\mathrm{Gal}(L/K)} \mathbf{RigDA}(L) \Rrightarrow \cdots \right)$$

On the other hand, this is also the Čech nerve of the canonical map

$$e^* \colon \mathbf{RigDA}(L) \to \mathbf{RigDA}(K).$$

We therefore deduce from the étale descent (see [AGV22, Theorem 2.3.4 & Remark 2.3.5]). ■

We next study the restriction of the equivalence (3.2.2) to $\mathbf{RigDA}_{\mathrm{gr}}(L)^{\mathrm{h\,Gal}(L/K)}$.

Notation 3.2.5. We define $\mathbf{RigDA}_{L\text{-gr}}(K)$ by the pullback diagram

$$\begin{tikzcd} \mathbf{RigDA}_{L\text{-gr}}(K) \arrow[r] \arrow[d] & \mathbf{RigDA}_{\mathrm{gr}}(L) \arrow[d, hook] \\ \mathbf{RigDA}(K) \arrow[r] & \mathbf{RigDA}(L) \end{tikzcd} \tag{3.2.3}$$

in $\mathbf{CAlg}(\mathcal{P}\mathbf{r}^{\mathrm{L}})$.

Remark 3.2.6. The underlying functor $\mathbf{RigDA}_{L\text{-gr}}(K) \to \mathbf{RigDA}(K)$ is fully faithful and we can identify (up to equivalence) $\mathbf{RigDA}_{L\text{-gr}}(K)$ with the full subcategory of $\mathbf{RigDA}(K)$ spanned by those M satisfying $e^*M \in \mathbf{RigDA}_{\mathrm{gr}}(L)$, where $e\colon \mathrm{Spa}(L) \to \mathrm{Spa}(K)$ is the structure map. In particular, $\mathbf{RigDA}_{\mathrm{gr}}(K)$ is a full subcategory of $\mathbf{RigDA}_{L\text{-gr}}(K)$ thanks to Lemma 3.2.2.

We will use the identification in Remark 3.2.6 implicitly from now on.

Proposition 3.2.7. *The canonical functor $\mathbf{RigDA}_{L\text{-gr}}(K) \to \mathbf{RigDA}_{\mathrm{gr}}(L)$ factors through the canonical functor $\mathbf{RigDA}_{\mathrm{gr}}(L)^{\mathrm{h\,Gal}(L/K)} \to \mathbf{RigDA}_{\mathrm{gr}}(L)$, and induces an equivalence*

$$\begin{tikzcd} & \mathbf{RigDA}_{\mathrm{gr}}(L)^{\mathrm{h\,Gal}(L/K)} \arrow[d] \\ \mathbf{RigDA}_{L\text{-gr}}(K) \arrow[r] \arrow[ur, dashed, "\simeq"] & \mathbf{RigDA}_{\mathrm{gr}}(L) \end{tikzcd}$$

in $\mathbf{CAlg}(\mathcal{P}\mathbf{r}^{\mathrm{L}})$. Moreover, we have a commutative diagram (up to homotopy)

$$\begin{tikzcd} \mathbf{RigDA}_{L\text{-gr}}(K) \arrow[r, dashed, "\simeq"] \arrow[d, hook] & \mathbf{RigDA}_{\mathrm{gr}}(L)^{\mathrm{h\,Gal}(L/K)} \arrow[r] \arrow[d, hook] & \mathbf{RigDA}_{\mathrm{gr}}(L) \arrow[d, hook] \\ \mathbf{RigDA}(K) \arrow[r, "\simeq", "\tilde{e}^*"'] & \mathbf{RigDA}(L)^{\mathrm{h\,Gal}(L/K)} \arrow[r] & \mathbf{RigDA}(L) \end{tikzcd}$$

in $\mathbf{CAlg}(\mathcal{P}\mathbf{r}^{\mathrm{L}})$*, where the middle fully faithful functor is the one in Corollary 3.2.3.*

Proof. Using the identification in Remark 3.2.6, the pullback functor $e^* \colon \mathbf{RigDA}(K) \to \mathbf{RigDA}(L)$ restricts to a functor

$$e^*_{\mathrm{gr}} \colon \mathbf{RigDA}_{L\text{-gr}}(K) \to \mathbf{RigDA}_{\mathrm{gr}}(L),$$

which is also $\mathrm{Gal}(L/K)$-invariant. Therefore, we get a functor

$$\mathbf{RigDA}_{L\text{-gr}}(K) \to \mathbf{RigDA}_{\mathrm{gr}}(L)^{\mathrm{h\,Gal}(L/K)}$$

in $\mathbf{CAlg}(\mathcal{P}\mathbf{r}^{\mathrm{L}})$ by the definition of $\mathbf{RigDA}_{\mathrm{gr}}(L)^{\mathrm{h\,Gal}(L/K)}$. In fact, it is just the restriction of the equivalence (3.2.2). This gives the commutative diagram in the last assertion except for the dotted equivalence.

Since the forgetful functor $\mathbf{CAlg}(\mathcal{P}\mathbf{r}^{\mathrm{L}}) \to \mathcal{P}\mathbf{r}^{\mathrm{L}}$ is conservative, it suffices to show this restriction is an equivalence in $\mathcal{P}\mathbf{r}^{\mathrm{L}}$. As a restriction of an equivalence, it is clearly fully faithful. It remains to prove the essential surjectivity. Let M be an object of $\mathbf{RigDA}_{\mathrm{gr}}(L)^{\mathrm{h\,Gal}(L/K)}$. Using the equivalence (3.2.2) in Proposition 3.2.4, we can find a (unique) $N \in \mathbf{RigDA}(K)$ such that $\tilde{e}^*(N) \simeq M$. The commutativity of the diagram shows $e^*N \in \mathbf{RigDA}_{\mathrm{gr}}(L)$, i.e., $N \in \mathbf{RigDA}_{L\text{-gr}}(K)$. Therefore, this proves $M \simeq e^*N \in \mathbf{RigDA}_{\mathrm{gr}}(L)^{\mathrm{h\,Gal}(L/K)}$. ■

§ 3.2.3 The Compact Generation

Now we study the compact generation of $\mathbf{RigDA}_{\mathrm{gr}}(L)^{\mathrm{h\,Gal}(L/K)}$ and give an explicit description for its compact generators.

Lemma 3.2.8. *Let* $e\colon \mathrm{Spa}(L) \to \mathrm{Spa}(K)$ *be the structure morphism.*

(1) *The restriction of* $e_* \colon \mathbf{RigDA}(L) \to \mathbf{RigDA}(K)$ *to* $\mathbf{RigDA}_{\mathrm{gr}}(L)$ *gives a left adjoint*

$$e_* \colon \mathbf{RigDA}_{\mathrm{gr}}(L) \to \mathbf{RigDA}_{L-\mathrm{gr}}(K)$$

of the pullback functor $e^*_{\mathrm{gr}} \colon \mathbf{RigDA}_{L\text{-gr}}(K) \to \mathbf{RigDA}_{\mathrm{gr}}(L)$.

(2) *Under the equivalence* $\mathbf{RigDA}_{L\text{-gr}}(K) \simeq \mathbf{RigDA}_{\mathrm{gr}}(L)^{\mathrm{h\,Gal}(L/K)}$ *in Proposition 3.2.7, the forgetful functor*

$$\iota_L \colon \mathbf{RigDA}_{\mathrm{gr}}(L)^{\mathrm{h\,Gal}(L/K)} \to \mathbf{RigDA}_{\mathrm{gr}}(L)$$

is both left and right adjoint to the restricted functor in (1).

Proof. Let L_0 be the finite unramified extension of K inside L with the residue field k_L. Then $e\colon \mathrm{Spa}(L) \to \mathrm{Spa}(K)$ factors as $\mathrm{Spa}(L) \xrightarrow{t} \mathrm{Spa}(L_0) \xrightarrow{e_0} \mathrm{Spa}(K)$. Since t^* is an equivalence by [BKV25, Proposition 3.23], we have a commutative diagram

$$\begin{tikzcd} & \mathbf{RigDA}(L) \arrow[d, "t_*"] \\ \mathbf{DA}(k_L) \arrow[ur, "\xi_L"] \arrow[r, "\xi_{L_0}"'] \arrow[d, "\bar{e}_\sharp"'] & \mathbf{RigDA}(L_0) \arrow[d, "e_{0,\sharp}\simeq e_{0,!}\simeq e_{0,*}"] \\ \mathbf{DA}(k) \arrow[r, "\xi_K"] & \mathbf{RigDA}(K) \end{tikzcd} \tag{3.2.4}$$

where the square is commutative thanks to Lemma 3.2.2. Then we have

$$e^* e_* \xi_L \simeq e^* \xi_K \bar{e}_\sharp \simeq \xi_L \bar{e}^* \bar{e}_\sharp. \tag{3.2.5}$$

This proves (1). For part (2), the forgetful functor ι_L corresponds to the pullback functor e^* under the identification. Thus, we can deduce from (1). ■

Notation 3.2.9. From now on, we let $\mathrm{Nm}_L \colon \mathbf{RigDA}_{\mathrm{gr}}(L) \to \mathbf{RigDA}_{\mathrm{gr}}(L)^{\mathrm{h}\,\mathrm{Gal}(L/K)}$ denote the left adjoint of the forgetful functor ι_L. We call it the ***norm functor***[1].

Proposition 3.2.10. *The ∞-category $\mathbf{RigDA}_{\mathrm{gr}}(L)^{\mathrm{h}\,\mathrm{Gal}(L/K)}$ is compactly generated. A set of compact generators is given, up to negative shifts and Tate twists, by $\mathrm{Nm}_L \xi_L M(X)$, where X runs through proper smooth algebraic varieties over k_L.*

Proof. Recall that the underlying ∞-category $\mathbf{RigDA}_{\mathrm{gr}}(L)^{\mathrm{h}\,\mathrm{Gal}(L/K)}$ is the limit of

$$G_{L/K} \colon B_\bullet \mathrm{Gal}(L/K) \to \mathbf{CAlg}(\mathcal{P}\mathbf{r}^{\mathrm{L}}) \to \mathcal{P}\mathbf{r}^{\mathrm{L}}.$$

Since each σ^* is an equivalence, it is also in $\mathcal{P}\mathbf{r}^{\mathrm{R}}$. In other words, we have a commutative diagram

$$\begin{tikzcd} B_\bullet \mathrm{Gal}(L/K) \arrow[r, "G_{L/K}"] \arrow[d, "G_{L/K}"'] & \mathcal{P}\mathbf{r}^{\mathrm{L}} \arrow[d] \\ \mathcal{P}\mathbf{r}^{\mathrm{R}} \arrow[r] & \mathcal{C}\mathbf{AT}_\infty \end{tikzcd}$$

So the limit is essentially computed in $\mathcal{C}\mathbf{AT}_\infty$ (see Proposition 2.1.1), and it can be computed in $\mathcal{P}\mathbf{r}^{\mathrm{R}}$ as well.

Now we can conclude from Proposition 2.1.3 since $\mathbf{RigDA}_{\mathrm{gr}}(L)$ is compactly generated with a set of compact generators consisting of $\xi_L M(X)$, where X runs through proper smooth varieties over k_L, see Remark 3.1.17. ■

We give an explicit formula for the $\mathrm{Gal}(L/K)$-equivariant rigid analytic motives with good model.

[1]This is suggested due to Proposition 3.2.11 below. See also [HA, §6.1.6] and [HL13] for additional information.

Proposition 3.2.11. *For every proper smooth algebraic variety X over the residue field k_L of L, we have an isomorphism*

$$\iota_L \mathbf{Nm}_L(\xi_L M(X)) \simeq \bigoplus_{e_{L/K}} \xi_L M(X),$$

where $e_{L/K}$ is the ramification index of L/K. In particular, the forgetful functor

$$\iota_L : \mathbf{RigDA}_{\mathrm{gr}}(L)^{\mathrm{h\,Gal}(L/k)} \to \mathbf{RigDA}_{\mathrm{gr}}(L)$$

is in $\mathbf{CAlg}(\mathcal{P}\mathbf{r}^{\mathrm{L}}_{\omega})$.

Proof. We use notations in the proof of Lemma 3.2.8. By (3.2.5), we have

$$\iota_L \mathbf{Nm}_L(\xi_L M(X)) \simeq e^* e_*(\xi_L M(X)) \simeq \xi_L \bar{e}^* \bar{e}_\sharp M(X).$$

In fact, we can prove $\bar{e}^* \bar{e}_\sharp M(X) \simeq \oplus_{e_{L/K}} M(X)$. For this, we consider the following Cartesian diagrams of schemes:

$$\begin{tikzcd}
\tilde{X} \arrow[r, "p"] \arrow[d, "\tilde{f}"'] & X \arrow[d, "f"] \\
\mathrm{Spec}(l \otimes_k l) \arrow[r, "\tilde{e}"] \arrow[d, "\tilde{e}"'] & \mathrm{Spec}\, l \arrow[d, "\bar{e}"] \\
\mathrm{Spec}\, l \arrow[r, "\bar{e}"'] & \mathrm{Spec}\, k
\end{tikzcd}$$

where $\tilde{X}$ is the fiber product $X \otimes_k l$. Then $\bar{e}^* \bar{e}_\sharp M(X) \simeq M(\tilde{X})$, where, as a k_L-scheme, the structure map of $\tilde{X}$ is $\tilde{e} \circ \tilde{f}$. Now let's prove $M(\tilde{X})$ is the $e_{L/K}$-copies of $M(X)$:

$$M(\tilde{X}) \simeq M(X) \otimes M(\mathrm{Spec}(k_L \otimes_k k_L)).$$

Under the identification

$$\mathbf{DA}(\mathrm{Spec}(k_L \otimes_k k_L)) \simeq \mathbf{DA}(\coprod_{\mathrm{Gal}(k_L/k)} \mathrm{Spec}\, k_L) \simeq \prod_{\mathrm{Gal}(k_L/k)} \mathbf{DA}(k_L),$$

we have

$$\begin{aligned} M(\tilde{X}) &\simeq M(X) \otimes \left(\bigoplus_{\tau \in \mathrm{Gal}(k_L/k)} \tau_* \mathbb{1} \right) \simeq \bigoplus_{\tau \in \mathrm{Gal}(k_L/k)} M(X) \otimes \tau_* \mathbb{1} \\ &\simeq \bigoplus_{\tau \in \mathrm{Gal}(k_L/k)} \tau_* \tau^* M(X) \simeq \bigoplus_{\tau \in \mathrm{Gal}(k_L/k)} M(X). \end{aligned}$$

∎

Corollary 3.2.12. *Let $L/F/K$ be finite Galois extensions of complete non-archimedean fields with residue fields $k_L/k_F/k$ respectively. There is a fully faithful functor*

$$\mathbf{RigDA}_{\mathrm{gr}}(F)^{\mathrm{h\,Gal}(F/K)} \hookrightarrow \mathbf{RigDA}_{\mathrm{gr}}(L)^{\mathrm{h\,Gal}(L/K)}$$

and an equivalence

$$\mathbf{RigDA}(F)^{\mathrm{h\,Gal}(F/K)} \xrightarrow{\simeq} \mathbf{RigDA}(L)^{\mathrm{h\,Gal}(L/K)}$$

in $\mathbf{CAlg}(\mathcal{P}\mathbf{r}^{\mathrm{L}})$ such that the following diagram

$$\begin{tikzcd}
\mathbf{RigDA}_{F\text{-gr}}(K) \arrow[r, "\simeq"] \arrow[d, hook] & \mathbf{RigDA}_{\mathrm{gr}}(F)^{\mathrm{h\,Gal}(F/K)} \arrow[r, hook] \arrow[d, hook, "\exists"] & \mathbf{RigDA}(F)^{\mathrm{h\,Gal}(F/K)} \arrow[r] \arrow[d, "\simeq"] & \mathbf{RigDA}(F) \arrow[d] \\
\mathbf{RigDA}_{L\text{-gr}}(K) \arrow[r, "\simeq"] & \mathbf{RigDA}_{\mathrm{gr}}(L)^{\mathrm{h\,Gal}(L/K)} \arrow[r, hook] & \mathbf{RigDA}(L)^{\mathrm{h\,Gal}(L/K)} \arrow[r] & \mathbf{RigDA}(L)
\end{tikzcd}$$

is commutative (up to homotopy), where the right-most vertical functor is the pullback functor. Moreover, the faithful functor above is in $\mathbf{CAlg}(\mathcal{P}\mathbf{r}^{\mathrm{L}}_{\omega})$.

Proof. For finite Galois extensions L/K and F/K, both of them admit an equivalence as Proposition 3.2.4. Thus, this gives a natural equivalence between $\mathbf{RigDA}(F)^{\mathrm{h\,Gal}(F/K)}$ and $\mathbf{RigDA}(L)^{\mathrm{h\,Gal}(L/K)}$, making the right-most square commutative (up to homotopy). Similarly, using equivalences in Proposition 3.2.7, that is

$$\mathbf{RigDA}_{F\text{-gr}}(K) \simeq \mathbf{RigDA}_{\mathrm{gr}}(F)^{\mathrm{h\,Gal}(F/K)}$$
$$\mathbf{RigDA}_{L\text{-gr}}(K) \simeq \mathbf{RigDA}_{\mathrm{gr}}(L)^{\mathrm{h\,Gal}(L/K)},$$

we can get the desired fully faithful functor. Now, as the leftmost, rightmost and outer squares are commutative, the middle one is also commutative by our constructions.

Finally, let's show the faithful functor

$$\mathbf{RigDA}_{\mathrm{gr}}(F)^{\mathrm{h\,Gal}(F/K)} \hookrightarrow \mathbf{RigDA}_{\mathrm{gr}}(L)^{\mathrm{h\,Gal}(L/K)}$$

preserves compact objects. We let $\alpha\colon \mathrm{Spa}(L) \to \mathrm{Spa}(F)$ and $\beta\colon \mathrm{Spa}(F) \to \mathrm{Spa}(K)$ be obvious structure morphisms. Let X be a proper smooth algebraic variety over k_F. We need to show $e^*\beta_*\xi_F M(X) \in \mathbf{RigDA}_{\mathrm{gr}}(L)$. We can deduce from Proposition 3.2.11:

$$\begin{aligned}
e^*\beta_*\xi_F M(X) &\simeq \alpha^*\beta^*\beta_*\xi_F M(X) \\
&\simeq \alpha^*\left(\bigoplus_{e_{F/K}} \xi_F M(X)\right) \\
&\simeq \bigoplus_{e_{F/K}} \xi_L M(X \times_{k_F} k_L) \in \mathbf{RigDA}_{\mathrm{gr}}(L).
\end{aligned} \tag{3.2.6}$$

■

§ 3.2.4 The Sheafification

We now present the categorical formulation of the fact that every rigid analytic motive over K has potentially good reduction. This gives the precise form of [AGV22, Theorem 3.7.1] in the case where the base is a field. Under the equivalence

$$\mathbf{RigDA}_{\mathrm{gr}}(L)^{\mathrm{h}\,\mathrm{Gal}(L/K)} \simeq \mathbf{RigDA}_{L\text{-gr}}(K),$$

we have a fully faithful functor in $\mathbf{CAlg}(\mathcal{P}\mathbf{r}^{\mathrm{L}}_{\omega})$

$$\mathbf{RigDA}_{\mathrm{gr}}(L)^{\mathrm{h}\,\mathrm{Gal}(L/K)} \hookrightarrow \mathbf{RigDA}(K)$$

for every finite Galois extension L/K. Using Corollary 3.2.12 and taking colimits, we have a functor

$$\operatorname*{colim}_{\substack{L/K \\ \text{finite Galois}}} \mathbf{RigDA}_{\mathrm{gr}}(L)^{\mathrm{h}\,\mathrm{Gal}(L/K)} \to \mathbf{RigDA}(K) \tag{3.2.7}$$

in $\mathbf{CAlg}(\mathcal{P}\mathbf{r}^{\mathrm{L}}_{\omega})$. Now we show this is indeed an equivalence.

Proposition 3.2.13. *The functor (3.2.7) is an equivalence in* $\mathbf{CAlg}(\mathcal{P}\mathbf{r}^{\mathrm{L}}_{\omega})$. *In particular, we have an equivalence*

$$\mathbf{RigDA}(K)_{\omega} \simeq \operatorname{colim} \mathbf{RigDA}_{\mathrm{gr}}(L)_{\omega}^{\mathrm{h}\,\mathrm{Gal}(L/K)}.$$

Proof. Since the forgetful functor $\mathbf{CAlg}(\mathcal{P}\mathbf{r}^{\mathrm{L}}_{\omega}) \to \mathcal{P}\mathbf{r}^{\mathrm{L}}_{\omega}$ is conservative and preserves this filtered colimit (see Proposition 2.1.14), we prove this is an equivalence in $\mathcal{P}\mathbf{r}^{\mathrm{L}}_{\omega}$. It suffices to prove the restriction

$$\operatorname{colim} \mathbf{RigDA}_{\mathrm{gr}}(L)_{\omega}^{\mathrm{h}\,\mathrm{Gal}(L/K)} \to \mathbf{RigDA}(K)_{\omega}$$

is an equivalence of ∞-categories, where the colimit on the left-hand side is taken in $\mathcal{C}\mathbf{at}_{\infty}$ (see Proposition 2.1.6). Firstly, since this functor is obtained by taking the colimit of fully faithful functors

$$\mathbf{RigDA}_{\mathrm{gr}}(L)_{\omega}^{\mathrm{h}\,\mathrm{Gal}(L/K)} \simeq \mathbf{RigDA}_{L\text{-gr}}(K)_{\omega} \hookrightarrow \mathbf{RigDA}(K)_{\omega},$$

it is also fully faithful ([HRS25, Proposition 2.1]). It remains to show it is essentially surjective. By [Ayo15, Theorem 2.5.34] or [AGV22, Proposition 3.7.17], for every object M in $\mathbf{RigDA}(K)_{\omega}$, we can find a finite separable extension $\tilde{L}/K$ such that $M_{\tilde{L}}$ lies in $\mathbf{RigDA}_{\tilde{L}\text{-gr}}(K)$. So we can take L as the Galois closure of $\tilde{L}/K$, and then M lies in $\mathbf{RigDA}_{L\text{-gr}}(K)$. In particular, M is given by

$$\Delta^0 \to \mathbf{RigDA}_{\mathrm{gr}}(L)_{\omega}^{\mathrm{h}\,\mathrm{Gal}(L/K)} \to \operatorname{colim} \mathbf{RigDA}_{\mathrm{gr}}(F)_{\omega}^{\mathrm{h}\,\mathrm{Gal}(F/K)} \to \mathbf{RigDA}(K)_{\omega}.$$

This proves the essential surjectivity and completes the proof. ■

§ 3.3 Chow Weight Structures on Motives

We study (bounded) weight structures on the ∞-category of motives. For a scheme X, the ∞-category $\mathbf{DA}(X)$, and for a rigid analytic space S, the ∞-category $\mathbf{RigDA}(S)$, are both stable ∞-categories, since they are defined via étale sheaves valued in $\mathcal{D}(\mathbb{Q})$.

We begin by recalling the Chow weight structure on algebraic motives, as defined by Bondarko in [Bon10; Bon14] and independently by Hébert in [Héb11], as well as the weight structure on $\mathbf{RigDA}_{\mathrm{gr}}(K)$ induced by the Chow weight structure, as constructed in [BGV25]. Building on these, and using the Galois descent developed in the previous section, we can extend this to a good weight structure on $\mathbf{RigDA}(K)$, which restricts to a bounded weight structure on its subcategory of compact objects (see Theorem 3.3.10).

§ 3.3.1 The Chow Weight Structure on Algebraic Motives

We recall the Chow weight structure on algebraic motives over a general base (not simply a point).

Definition 3.3.1 ([Bon14, Definition 1.1.1]). We say a scheme X is ***reasonable*** if it is a scheme separated and of finite type over a scheme S where S is an excellent noetherian scheme of dimension ≤ 2.

This technical condition is just used to guarantee the resolution of singularities (c.f. [dJ97] and [CD19, §4]). Then, when constructing the bounded weight structure, one can check the generating condition in (2) of Proposition 2.2.17.

Example 3.3.2. Separated schemes of finite type over $\operatorname{Spec} R$, for R a field, a complete DVR, Dedekind domain with fraction field of characteristic 0, are reasonable schemes ([SP, Proposition 07QW]).

Proposition 3.3.3. *Let S be a reasonable scheme. Define*

$$\mathcal{N}_S := \left\{ f_! \mathbb{1}_X(i)[2i] \in \mathbf{DA}(S)_\omega \,\middle|\, i \in \mathbb{Z}, f\colon X \to S \text{ is proper with } X \text{ regular} \right\}.$$

Then it satisfies the condition in (2) of Proposition 2.2.17. In particular, this gives the unique bounded weight structure on $\mathbf{DA}(S)_\omega$ whose heart contains $\mathcal{N}_S$. Moreover, it can be extended to a weight structure on $\mathbf{DA}(S)$.

Proof. The first assertion follows from [Héb11, Théorème 3.3], and the second one then follows from Proposition 2.2.24. ■

Definition 3.3.4. Let S be a reasonable scheme. The weight structure in Proposition 3.3.3 is called the ***Chow weight structure*** on $\mathbf{DA}(S)$. We will use $\mathbf{CHOW}(S)$ to denote its heart and use $\mathbf{Chow}(S)$ to denote its restriction to $\mathbf{DA}(S)_\omega$.

In general, the Chow weight structure is not compatible with symmetric monoidal structure unless the base is a point. The obstruction is the regularity of schemes is an absolute concept. But if the base is a perfect field, then being regular is equivalent to saying the structure map is smooth, hence becoming a relative concept in the case.

Corollary 3.3.5 ([Aok20]). *Let k be a field. Then the Chow weight structure is compatible with monoidal structures on $\mathbf{DA}(k)$ and $\mathbf{DA}(k)_\omega$.*

Proof. We first assume k is a perfect field. In the case, we can rewrite $\mathcal{N}_k$ by

$$\mathcal{N}_k = \left\{ f_!\mathbb{1}_X(i)[2i] \in \mathbf{DA}(k)_\omega \,\middle|\, i \in \mathbb{Z}, f\colon X \to k \text{ proper smooth} \right\}.$$

Then we can show that it is compatible with monoidal structure if restricting to the compact part (using Corollary 2.2.18): let $f\colon X \to k$, $g\colon Y \to k$ be proper smooth k-schemes. To consider the pullback square

$$\begin{tikzcd} Z \arrow[r, "f'"] \arrow[d, "g'"'] & Y \arrow[d, "g"] \\ X \arrow[r, "f"] & k \end{tikzcd}$$

then we have

$$f_!\mathbb{1}_X \otimes g_!\mathbb{1}_Y \simeq f_!f^*g_!\mathbb{1}_Y \simeq f_!g'_!\mathbb{1}_Z,$$

where the first equivalence is due to the projection formula and the second one is proper base change. In the case, $f \circ g'$ is also proper smooth; hence, the tensor above lies in the heart. Therefore, we conclude this case from the last assertion of Corollary 2.2.25.

Now we pass to the non-perfect case using the semi-separatedness of motives: that is, we have an equivalence ([Ayo14a, Théorème 1.2])

$$e^*\colon \mathbf{DA}(k) \to \mathbf{DA}(k^{\mathrm{perf}})$$

in $\mathbf{CAlg}(\mathcal{P}\mathbf{r}^{\mathrm{L}}_\omega)$, where k^{perf} is the perfection of k. ■

Using the Chow weight structure and Proposition 2.2.27, we can construct a weight structure on $\mathbf{RigDA}_{\mathrm{gr}}(K)$.

Corollary 3.3.6 ([BGV25, Proposition 4.25]). *Let K be a complete non-archimedean field with residue field k. Then $\mathbf{RigDA}_{\mathrm{gr}}(K)$ has a weight structure w satisfying*

(1) *the Monsky–Washnitzer functor $\xi\colon \mathbf{DA}(k) \to \mathbf{RigDA}_{\mathrm{gr}}(K)$ is weight-exact;*

(2) *the weight structure w is compatible with the monoidal structure on $\mathbf{RigDA}_{\mathrm{gr}}(K)$.*

Proof. From Proposition 3.1.13, it suffices to define the weight structure on $\mathbf{Mod}_{\chi\mathbb{1}}(\mathbf{DA}(k))$. So we use Proposition 2.2.27. What we need to prove is tensoring $\chi\mathbb{1}$ preserves non-negative weights. Recall that we have an identification (if we choose a pseudo-uniformizer of $\mathcal{O}_k$, see Proposition 3.1.18)

$$\chi\mathbb{1} \simeq \mathbb{1} \oplus \mathbb{1}(-1)[-2][1],$$

and the construction of the Chow weight structure shows $\chi\mathbb{1} \in \mathbf{DA}(k)_{w\geq 0}$. Therefore, $\chi\mathbb{1} \oplus -\colon \mathbf{DA}(k) \to \mathbf{DA}(k)$ is right weight-exact thanks to Proposition 3.3.5. So $\chi\mathbb{1} \otimes -\colon \mathbf{DA}(k) \to \mathbf{DA}(k)$ is right weight-exact. ■

Remark 3.3.7. Furthermore, assume the field K in Corollary 3.3.6 is algebraically closed, then we have an equivalence ([AGV22, Theorem 3.7.21])

$$\mathbf{RigDA}(K) \simeq \mathbf{RigDA}_{\mathrm{gr}}(K),$$

So we get a weight structure on the whole category $\mathbf{RigDA}(K)$ in the case. In the next, we will show that we can remove the algebraically closed conditions; see Theorem 3.3.10 and Corollary 3.3.13.

Remark 3.3.8. In fact, the bounded weight structure on $\mathbf{RigDA}_{\mathrm{gr}}(K)_\omega$ constructed in Corollary 3.3.6 is uniquely characterized by the property that the Monsky–Washnitzer functor $\xi\colon \mathbf{DA}(k)_\omega \to \mathbf{RigDA}_{\mathrm{gr}}(K)_\omega$ is weight-exact. More generally, if S is a rigid analytic space with good reduction, a similar argument shows such a bounded weight structure on $\mathbf{RigDA}_{\mathrm{gr}}(S)_\omega$ exists as well.

On the other hand, if one only assume S has semistable reduction, then the argument breaks down. The reason is that, in the case, the construction of such a bounded weight structure relies essentially on the six-functor formalism of motives, whereas the Chow weight structure is only partially compatible with the six-functor formalism of algebraic motives; see [Héb11] for further discussion.

§ 3.3.2 A Weight Structure on Rigid Analytic Motives

From now on, we fix a complete non-archimedean field K with perfect residue field k.

Lemma 3.3.9. *There is a weight structure $w_{L/K}$ compatible with monoidal structure on the stable ∞-category $\mathbf{RigDA}_{\mathrm{gr}}(L)^{\mathrm{h\,Gal}(L/K)}$ such that the forgetful functor*

$$\mathbf{RigDA}_{\mathrm{gr}}(L)^{\mathrm{h\,Gal}(L/K)} \to \mathbf{RigDA}_{\mathrm{gr}}(L)$$

is weight-exact, where the weight structure on the target is given by Corollary 3.3.6. Moreover, equipped with such weight structures, the fully faithful functor in Corollary 3.2.12 is weight-exact.

Proof. We use Corollary 2.2.25 to prove this lemma. By Proposition 3.2.10, we only need to check that, for any X, Y proper smooth varieties over k_L, we have

$$\mathrm{Hom}\left(\mathrm{Nm}_L\xi_L M(X), \mathrm{Nm}_L\xi_L M(Y)[i]\right) = 0$$

for $i > 0$. By the adjunctions and Proposition 3.2.11, it follows from the negativity of the heart of the weight structure on $\mathbf{RigDA}_{\mathrm{gr}}(L)$.

For the compatibility with monoidal structure, it suffices to notice that the adjunction $\mathrm{Nm}_L \dashv \iota_L$ satisfies the projection formula from Lemma 3.2.8. Therefore, we have

$$\begin{aligned}\mathrm{Nm}_L\xi_L M(X) \otimes \mathrm{Nm}_L\xi_L M(Y) &\simeq \mathrm{Nm}_L\left(\xi_L M(X) \otimes \iota_L \mathrm{Nm}_L \xi_L M(Y)\right) \\ &\simeq \bigoplus_{e_{L/K}} \mathrm{Nm}_L\xi_L M(X \times_{k_L} Y),\end{aligned} \tag{3.3.1}$$

where the last isomorphism is also due to Proposition 3.2.11, and it is clearly in the heart by our construction. The weight-exactness is an immediate result of Proposition 3.2.11 and Corollary 2.2.26. Finally, the last assertion follows from (3.2.6). ■

Using the identification in Proposition 3.2.13, we extend the weight structure in Corollary 3.3.6 to the entire category $\mathbf{RigDA}(K)$.

Theorem 3.3.10. *There is a weight structure w on $\mathbf{RigDA}(K)$ satisfying the following conditions:*

(1) *it is compatible with monoidal structure;*

(2) *it restricts to a bounded weight structure on $\mathbf{RigDA}(K)_\omega$ which is compatible with monoidal structure;*

(3) *$\mathbf{RigDA}(K)_{w\geq 0}$ is closed under small colimits;*

(4) *the natural inclusion $\mathbf{RigDA}_{\mathrm{gr}}(K) \subseteq \mathbf{RigDA}(K)$ is weight-exact, where $\mathbf{RigDA}_{\mathrm{gr}}(K)$ is equipped with the weight structure in Corollary 3.3.6. More generally, for any finite Galois extension L/K, the fully faithful functor*

$$\mathbf{RigDA}_{\mathrm{gr}}(L)^{\mathrm{h\,Gal}(L/K)} \to \mathbf{RigDA}(K)$$

is weight-exact, where the source is equipped with weight structure $w_{L/K}$ in Lemma 3.3.9.

Proof. For each finite Galois extension L/K, we have proved that $\mathbf{RigDA}_{L\text{-}\mathrm{gr}}(K)^{\mathrm{h\,Gal}(L/K)}$ is compactly generated by $\mathrm{Nm}_L\xi_L(M(X))$, where X is proper smooth over k_L (Proposition 3.2.10). Under the identification $\mathcal{P}\mathbf{r}^{\mathrm{L}}_\omega \simeq (\mathcal{P}\mathbf{r}^{\mathrm{R}}_\omega)^{\mathrm{op}}$, we deduce that

$$\operatorname{colim} \mathbf{RigDA}_{\mathrm{gr}}(L)^{\mathrm{h\,Gal}(L/K)}$$

is compactly generated by images of $\mathrm{Nm}_L\xi_L(M(X))$ in this colimit. We let $\mathcal{N}$ be the collection of these objects. In order to use Corollary 2.2.25, it suffices to prove $\mathcal{N}$ is negative. This has been proved in Lemma 3.3.9. In particular, this also shows the fully faithful functors in (4) are weight-exact. Finally, this weight structure is compatible with the monoidal structure by (3.3.1). ∎

Corollary 3.3.11. *Let w be the weight structure in Theorem 3.3.10. Let $M \in \mathbf{RigDA}(K)$.*

(1) *$M \in \mathbf{RigDA}(K)_{w\geq 0}$ if and only if, for any finite Galois extension L/K and any X/k_L proper smooth, we have $\mathrm{Hom}(e_*\xi_L(M(X)), M[i]) \simeq 0$ for all $i > 0$, where e is the structure map.*

(2) *$M \in \mathbf{RigDA}(K)_{\omega,w\leq 0}$ if and only if, for any finite Galois extension L/K and any X/k_L proper smooth, we have $\mathrm{Hom}(M[i], e_*\xi_L(M(X))) \simeq 0$ for all $i < 0$, where e is the structure map.*

(3) *The heart of w on $\mathbf{RigDA}(K)$ (resp. $\mathbf{RigDA}(K)_\omega$) is the idempotent completion of full subcategory generated by all possible $e_*\xi_L(M(X))$ as above under direct sums (resp. finite direct sums).*

Proof. This is an immediate result of Corollary 2.2.25. ∎

Corollary 3.3.12. *Let L/K be an extension of complete non-archimedean fields with perfect residue fields k_L/k. Then the pullback functor*

$$e^*\colon \mathbf{RigDA}(K) \to \mathbf{RigDA}(L)$$

is weight-exact, where $e\colon \mathrm{Spa}(L) \to \mathrm{Spa}(K)$ is the structure morphism. Furthermore, if L/K is a finite extension, then

$$e_*\colon \mathbf{RigDA}(L) \to \mathbf{RigDA}(K)$$

is also weight-exact.

Proof. We firstly prove e^* is weight-exact. We prove this using Corollary 2.2.26. We let F/K be a finite Galois extension with residue field k_F and X be a proper smooth variety over k_F. Let $\alpha\colon \mathrm{Spa}(F) \to \mathrm{Spa}(K)$ be the structure map. We need to prove $e^*\alpha_*\xi_F M(X)$ lies in the heart of the weight structure of $\mathbf{RigDA}(L)$:

$$e^*\alpha_*\xi_F(M(X)) \simeq \xi_L\bar{e}^*\bar{\alpha}_\sharp M(X) \simeq \xi_L M(X \times_k k_L) \in \mathbf{RigDA}(L)^{\heartsuit_w}.$$

For the weight-exactness of e_* when L/K is finite, we firstly assume L/K is separable. As in Lemma 3.2.8, we can know that e_* is both the left and right adjoint of e^*. Therefore, we can conclude the weight-exactness of e_* from the weight-exactness of e^* thanks to [Bon14, Proposition 1.2.3 (9)]. In the general case, we can choose the separable closure L_0 of K in L; hence L/L_0 is purely inseparable. Thus, we conclude the general case from [AGV22, Corollary 2.9.11] and the separable case. ∎

We have seen that, if K is algebraically closed, there exists a weight structure (see Remark 3.3.7). If we replace L by the completion of an algebraic closure of K in Corollary 3.3.12, then we see that the weight structure in Theorem 3.3.10 coincides with that.

Corollary 3.3.13. *Let K be a complete non-archimedean field with perfect residue field. Assume C is its completed algebraic closure. Then the pullback functor*

$$f^*\colon \mathbf{RigDA}(K) \to \mathbf{RigDA}(C)$$

is weight-exact.

Chapter IV

Motivic Realization Functors

The main goal of this chapter is to compare motivic realizations on rigid analytic motives. We begin in §4.1 by recalling two derived ∞-categories that serve as coefficients for the motivic realizations defined later in §4.2. We then prove, in §4.4, that these realization functors are monoidally equivalent. On the other hand, we show that these motivic realization functors factor through the weight complex functor associated to the weight structure given in Theorem 3.3.10. As a consequence, we obtain weight spectral sequences converging to corresponding cohomology.

Convention. Throughout this chapter, we fix a discretely valued field K over $\mathbb{Q}_p$ with a perfect residue field k. Let $\bar{K}$ be an algebraic closure of K and denote by C its completion. In particular, the residue field $\bar{k}$ of C is an algebraic closure of k. By taking rings of Witt vectors, we denote $K_0 := \operatorname{Frac} W(k)$ and $\breve{K} := \operatorname{Frac} W(\bar{k})$ by the completion of maximal unramified extensions of $\mathbb{Q}_p$ in K and C, respectively.

Notation 4.0.1. Let $\mathcal{C}$ be an ∞-category in $\mathcal{P}\mathbf{r}^{\mathrm{L}}_{\omega}$ (or $\mathbf{CAlg}(\mathcal{P}\mathbf{r}^{\mathrm{L}}_{\omega})$). If $\mathcal{C}$ is equipped with an endofunctor $F\colon \mathcal{C} \to \mathcal{C}$ in $\mathcal{P}\mathbf{r}^{\mathrm{L}}_{\omega}$ (or $\mathbf{CAlg}(\mathcal{P}\mathbf{r}^{\mathrm{L}}_{\omega})$), then we will let $\mathcal{C}^F$ (resp. $\mathcal{C}^{F_\omega}$) denote the equalizer of F and Id_C in $\mathcal{P}\mathbf{r}^{\mathrm{L}}$ (resp. $\mathcal{P}\mathbf{r}^{\mathrm{L}}_{\omega}$).

§ 4.1 Derived ∞-Categories of Coefficients

We review two kinds of derived ∞-categories that serve as coefficient categories for motivic realization functors.

§ 4.1.1 The Derived ∞-Category of (φ, N, G_K)-Modules

We recall the derived ∞-category of (φ, N, G_K)-modules (see [Bei13, §1.15], [DN18, §2.1], [Fon94, §4.2], [FO22, §8.2]).

Definition 4.1.1. Let L/K be a finite Galois extension inside $\bar{K}$ with the Galois group $G_{L/K}$ and let k_L be the residue field of L. Let $L_0 = \operatorname{Frac} W(k_L)$ and $\sigma_L\colon L_0 \to L_0$ be the arithmetic Frobenius of L_0.

(1) The category $\mathbf{Mod}_{L_0}(\varphi, N, G_{L/K})$ of $(\boldsymbol{\varphi}, \boldsymbol{N}, \boldsymbol{G_{L/K}})$***-modules*** over L_0 is the following category:

- objects are finite dimensional L_0-vector spaces D equipped with
  - a semi-linear $G_{L/K}$-action on D;
  - a σ_L-semi-linear and $G_{L/K}$-equivariant bijection $\varphi_D \colon D \to D$;
  - a L_0-linear and $G_{L/K}$-equivariant map $N_D \colon D \to D$ satisfying $N_D\varphi_D = p\varphi_D N_D$;
- morphisms are L_0-linear maps commuting with φ, N and $G_{L/K}$-actions.

(2) If $L = K$, then the category of $(\varphi, N, G_{L/K})$-modules is called the category of $(\boldsymbol{\varphi}, \boldsymbol{N})$***-modules***, simply denoted by $\mathbf{Mod}_{K_0}(\varphi, N)$. Moreover, in the case, the full subcategory spanned by those with $N = 0$ is called the category of $\boldsymbol{\varphi}$***-modules*** over K_0, and we will denote it by $\mathbf{Mod}_{K_0}(\varphi)$.

Remark 4.1.2. The category $\mathbf{Mod}_{L_0}(\varphi, N, G_{L/K})$ is a $\mathbb{Q}_p$-linear abelian Tannakian category. The tensor products are defined by

$$(D, \varphi_D N_D, \rho_D) \otimes (E, \varphi_E, N_E, \rho_E) := (D \otimes_{K_0} E, \varphi_D \otimes \varphi_E, N_D \otimes \mathrm{Id} + \mathrm{Id} \otimes N_E, \rho_D \otimes \rho_E)$$

with the tensor unit K_0 with $\varphi_{K_0} = \sigma_K$, $N_{K_0} = 0$, and the canonical G_K-action.

Remark 4.1.3. Given a (φ, N)-module (D, φ_D, N_D), the map N_D (called ***monodromy operator***) is nilpotent since D is a finite-dimensional vector space. In particular, if $\dim_{K_0} D = 1$, then $N_D = 0$.

Example 4.1.4. (1) For every $n \in \mathbb{Z}$, we define the ***Tate twist*** $K_0(n) \in \mathbf{Mod}_{K_0}(\varphi)$ by the one-dimensional K_0-vector space K_0 with the twisted Frobenius $\varphi_{K_0(n)} = p^{-n}\sigma_K$.

(2) Let $(D, \varphi_D) \in \mathbf{Mod}_{K_0}(\varphi)$. Then, for every $n \in \mathbb{Z}$, we let $D(n)$ denote the tensor product $(D, \varphi_D) \otimes K_0(n)$ in $\mathbf{Mod}_{K_0}(\varphi)$.

(3) With the notation in (2), we can identify (φ, N)-modules over K_0 with φ-modules over K_0 with a morphism $N \colon D \to D(-1)$ in $\mathbf{Mod}_{K_0}(\varphi)$.

Notation 4.1.5. Keeping notations as in Definition 4.1.1, we will denote the bounded derived ∞-category of $(\varphi, N, G_{L/K})$-modules over L_0 by $\mathcal{D}^{\mathrm{b}}_{(\varphi,N,G_{L/K})}(L_0)$ and its Ind-completion will be denoted by $\mathcal{D}_{(\varphi,N,G_{L/K})}(L_0)$. In particular, we have some special cases:

(1) the derived ∞-category $\mathcal{D}_{(\varphi,N)}(K_0)$ of (φ, N)-modules over K_0 when $L = K$;

(2) and similarly, we have the derived ∞-category $\mathcal{D}_\varphi(K_0)$ of φ-modules over K_0.

If L runs through all possible finite Galois extension of K, then we get

$$\mathcal{D}^{\mathrm{b}}_{(\varphi,N,G_K)}(\breve{K}) := \operatorname{colim} \mathcal{D}^{\mathrm{b}}_{(\varphi,N,G_{L/K})}(L_0) \qquad \text{in } \mathbf{CAlg}(\mathbf{Cat}^{\mathrm{perf}}_\infty)$$

where transition functors are induced by base change, and the Ind-completion of it is denoted by $\mathcal{D}_{(\varphi,N,G_K)}(\breve{K})$. We will call it the derived ∞-category of ***(φ, N, G_K)-modules*** over K. But this ∞-category is indeed the derived ∞-category of ***discrete (φ, N, G_K)-modules*** in the sense of Fontaine [Fon94, §4.2.2]; see Remark 4.1.6.

These ∞-categories are presentable stable symmetric monoidal ∞-categories (see [Dre15, Proposition 3.2], [HA, Corollary 4.8.1.14]).

Remark 4.1.6. In Definition 4.1.1, we did not use the assumption L/K is finite. Thus, there is a notion of (φ, N, G_K)-modules defined as in Definition 4.1.1. Let us call them ***naive (φ, N, G_K)-modules***. However, the ∞-category $\mathcal{D}_{(\varphi,N,G_K)}(\breve{K})$ introduced above is not the derived ∞-category of all such naive (φ, N, G_K)-modules. Rather, as mentioned, objects of $\mathcal{D}^{\mathrm{b}}_{(\varphi,N,G_K)}(\breve{K})$ defined above are discrete in the sense of Fontaine; that is, they form a special subclass of naive (φ, N, G_K)-modules; see [Fon94, n° 4.2.2]. More precisely, a (φ, N, G_K)-module here is a naive (φ, N, G_K)-module for which the inertia group I_K acts with open stabilizers. This restriction does not affect the construction of the realization functors, since these functors always factor through the derived category $\mathcal{D}_{(\varphi,N,G_K)}(\breve{K})$.

Remark 4.1.7. The derived ∞-category of (φ, N)-modules is an example of objects with monodromy operators introduced in §3.1.4: in the case $\mathcal{C} = \mathcal{D}_\varphi(K_0)$, if we take $t = K_0(-1)$, then we also have a monoidal equivalence ([BGV25, Proposition 2.50])

$$\mathcal{D}_{(\varphi,N)}(K_0) \simeq \mathcal{D}_\varphi(K_0)^{-\otimes K_0(-1)}_{\mathrm{nil}} \tag{4.1.1}$$

where the right-hand side is discussed in §3.1.4.

The Derived ∞-Category of (φ, G_K)-Modules

Next we study some variants of Notation 4.1.5, which will be useful later.

Notation 4.1.8. Let L/K be a finite Galois extension. We define

$$\mathcal{D}_{(\varphi,G_{L/K})}(L_0) := \mathcal{D}_\varphi(L_0)^{\mathrm{h}\,G_{L/K}}$$

where $\mathcal{D}_\varphi(L_0) \in \mathbf{CAlg}(\mathcal{P}\mathbf{r}^{\mathrm{st}}_\omega)$ is acted on by the Galois group $G_{L/K}$ via the surjection $G_{L/K} \twoheadrightarrow G_{L_0/K_0}$. As above, let L run through all possible finite Galois extensions; we can define

$$\mathcal{D}_{(\varphi,G_K)}(\breve{K}) := \operatorname{colim} \mathcal{D}_{(\varphi,G_{L/K})}(L_0)$$

in $\mathbf{CAlg}(\mathcal{P}\mathbf{r}^{\mathrm{st}}_\omega)$.

For a finite Galois extension L/K, let $(D, \varphi_D, N_D, \rho)$ be a $(\varphi, N, G_{L/K})$-module over L_0. Then, for each $\sigma_g \in G_{L/K}$ with image $\bar{\sigma}_g \in G_{L_0/K_0}$, the map ρ_g induced an isomorphism $\rho_g\colon D \to \sigma_g^* D \simeq \bar{\sigma}_g^* D$ of (φ, N)-modules over L_0. Note that the forgetting $G_{L/K}$-action is exact, and then we have the forgetful functor

$$F\colon \mathcal{D}_{(\varphi,N,G_{L/K})}(L_0) \to \mathcal{D}_{(\varphi,N)}(L_0).$$

For each $\sigma_g \in G_{L/K}$, we also denote σ_g^* by the autofunctor of $\mathcal{D}_{(\varphi,N)}(L_0)$. Therefore, one obtains natural transformations $F \xrightarrow{\simeq} \sigma_g^* \circ F$ for all $\sigma_g \in G_{L/K}$, and they are compatible by the multiplication of $G_{L/K}$ (since ρ does). In particular, we get a functor

$$\mathcal{D}_{(\varphi,N,G_{L/K})}(L_0) \to \mathcal{D}_{(\varphi,N)}(L_0)^{\mathrm{h}\,G_{L/K}}.$$

Proposition 4.1.9. *Let L/K be a finite Galois extension. The natural functor*

$$\mathcal{D}_{(\varphi,N,G_{L/K})}(L_0) \xrightarrow{\simeq} \mathcal{D}_{(\varphi,N)}(L_0)^{\mathrm{h}\,G_{L/K}}$$

defined above is an equivalence in $\mathbf{CAlg}(\mathcal{P}\mathbf{r}^{\mathrm{st}}_{\omega})$. In particular, we get the functor forgetting monodromy operators

$$\mathcal{D}_{(\varphi,N,G_{L/K})}(L_0) \xrightarrow{\simeq} \mathcal{D}_{(\varphi,N)}(L_0)^{\mathrm{h}\,G_{L/K}} \to \mathcal{D}_{\varphi}(L_0)^{\mathrm{h}\,G_{L/K}} = \mathcal{D}_{(\varphi,G_{L/K})}(L_0).$$

Proof. It suffices to prove by restricting to the bounded parts. We first show that it is fully faithful. Let D, E be bounded complexes of $(\varphi, N, G_{L/K})$-modules. Then, by [DN18, (2.3)], the mapping space $\mathrm{Map}_{\mathcal{D}^{\mathrm{b}}_{(\varphi,N,G_{L/K})}(L_0)}(D, E)$ is the homotopy limit of the diagram

$$\begin{tikzcd}
\mathbf{Map}_{\mathcal{D}^{\mathrm{b}}(L_0)}(D,E)^{\mathrm{h}\,G_{L/K}} \arrow[r, "\varphi_{E,*}-\varphi_D^*"] \arrow[d, "N_{E,*}-N_D^*"'] & \mathbf{Map}_{\mathcal{D}^{\mathrm{b}}(L_0)}(D,\sigma_{L,*}E)^{\mathrm{h}\,G_{L/K}} \arrow[d, "N_{E,*}-pN_D^*"] \\
\mathbf{Map}_{\mathcal{D}^{\mathrm{b}}(L_0)}(D,E)^{\mathrm{h}\,G_{L/K}} \arrow[r, "p\varphi_{E,*}-\varphi_D^*"'] & \mathbf{Map}_{\mathcal{D}^{\mathrm{b}}(L_0)}(D,\sigma_{L,*}E)^{\mathrm{h}\,G_{L/K}}
\end{tikzcd}$$

Since limits commute with limits, we know that

$$\mathbf{Map}_{\mathcal{D}^{\mathrm{b}}_{(\varphi,N,G_{L/K})}(L_0)}(D,E) \simeq \mathbf{Map}_{\mathcal{D}^{\mathrm{b}}_{(\varphi,N)}(L_0)}(D,E)^{\mathrm{h}\,\mathrm{Gal}(L/K)}$$

by [DN18, Lemma 2.5] or [Bei13, §1.15]. The full faithfulness follows from Corollary 2.1.19. The essential surjectivity is deduced from the construction of the functor and the definition of $\mathcal{D}_{(\varphi,N)}(L_0)^{\mathrm{h}\,G_{L/K}}$, whose objects are equivalently given by an object in $\mathcal{D}_{(\varphi,N)}(L_0)$ equipped with an action of $G_{L/K}$ satisfying the higher coherence.

Finally, forgetting monodromy operators is compatible with the action of $G_{L/K}$, hence forgetful functors are well-defined. ■

Slopes and Weights of φ-Modules

In Example 4.1.4, we have defined n-th Tate twists of φ-modules for $n \in \mathbb{Z}$. There is a generalization:

Example 4.1.10 (Simple φ-Modules)**.** Let $\lambda \in \mathbb{Q}$ with the unique form $\lambda = m/n$ with $m \in \mathbb{Z}$, $n \in \mathbb{Z}_{>0}$ and $\gcd(m, n) = 1$. We can define a φ-module $K_0(\lambda)$ as follows: the underlying vector space is K_0^n and the bijection $\varphi_\lambda \colon K_0(\lambda) \to K_0(\lambda)$ is given by

$$\varphi_\lambda(e_1, \ldots, e_n) = (e_1, \ldots, e_n)A_\lambda, \quad \text{with } A_\lambda = \begin{pmatrix} 0 & I_{n-1} \\ p^m & 0 \end{pmatrix}$$

where $(e_1, \ldots, e_n)$ is the standard basis of K_0^n. It is not hard to see the characteristic polynomial of φ_λ is $f_\lambda(T) = (-1)^n(T^n - p^m)$. We will refer to the rational number λ as the ***slope*** of $K_0(\lambda)$.

Definition 4.1.11. (1) If k is a finite field with cardinality $q = p^f$, we say a φ-module (D, φ_D) over K_0 is ***pure of weight*** i if each generalized eigenvalue λ of the D is a q-Weil number of weight i, i.e it is an algebraic number such that $|\sigma(\lambda)| = q^{i/2}$ for every embedding $\sigma \colon \bar{\mathbb{Q}} \hookrightarrow \mathbb{C}$.

(2) If $k = \bar{\mathbb{F}}_p$, we say a φ-module (D, φ_D) over K_0 is ***pure of weight*** i if it has a model of pure weight i over $\mathbb{F}_q$ for some $q = p^f$.

We next study the relationship between slopes and weights of φ-modules. They will be used to construct the new filtration on the de Rham–Fargues–Fontaine cohomology; see Corollary 4.4.5. For this, we fix an algebraic closure $\bar{l}$ of l. Let $\breve{L}$ (resp. L_0) be the fraction field of $W(\bar{l})$ (resp. $W(l)$).

Lemma 4.1.12. *Assume $k = \mathbb{F}_q$ with $q = p^f$. Let (D, φ) be a pure φ-module over K_0 of weight i. If the φ-module $\hat{D} = D \otimes_{K_0} \breve{K}$ is semi-stable of slope λ, i.e., direct sum of $\breve{K}(\lambda)$, then $\lambda = i/2$.*

Proof. Let $\hat{\varphi}$ be the Frobenius automorphism of $\hat{D}$, i.e., $\hat{\varphi} = \varphi \otimes \sigma$. By the assumption, we can find $\alpha \in D$ and $\mu \in K_0$ such that $\varphi^f\alpha = \mu\alpha$ and $|\mu| = q^{i/2}$ for every embedding $\overline{\mathbb{Q}}_p \hookrightarrow \mathbb{C}$. Choose a basis $(e_1, \ldots, e_d)$ of D over K_0 such that the matrix of $\hat{\varphi}$ under the induced basis $(\hat{e}_1, \ldots, \hat{e}_d)$ is

$$\operatorname{diag}(A_\lambda, A_\lambda, \cdots, A_\lambda)$$

where A_λ is the one in Example 4.1.4. We write $\alpha = x_1 e_1 + \cdots + x_d e_d$ with $x_i \in K_0$. Note that we have

$$\begin{aligned}\hat{\varphi}^f(\alpha\otimes 1) &= (\hat{e}_1,\dots,\hat{e}_d)\,\mathrm{diag}(A_\lambda^f,\dots,A_\lambda^f)\begin{pmatrix}\sigma^f(x_1)\\ \vdots\\ \sigma^f(x_d)\end{pmatrix}\\ &= (\hat{e}_1,\dots,\hat{e}_d)\,\mathrm{diag}(A_\lambda^f,\dots,A_\lambda^f)\begin{pmatrix}x_1\\ \vdots\\ x_d\end{pmatrix}\end{aligned}$$

since $x_i \in K_0$, hence $\sigma^f(x_i) = x_i$. On the other hand, we have

$$\hat{\varphi}^f(\alpha\otimes 1) = \mu(\alpha\otimes 1) = (\hat{e}_1,\dots,\hat{e}_d)\mu\begin{pmatrix}x_1\\ \vdots\\ x_d\end{pmatrix}.$$

Therefore, μ is also the eigenvalue of A_λ, and then we have

$$q^{\frac{i}{2}} = |\mu| = p^{f\lambda} = q^\lambda,$$

hence $\lambda = i/2$ as desired. ■

Then we can deduce the relation in the algebraically closed case:

Proposition 4.1.13. *If (D, φ_D) is a pure φ-module over $\breve{K}$ of weight i, then it is semi-stable of slope $i/2$.*

Proof. By the assumption, we can find a model (V, φ_V) of (D, φ_D) over $\mathbb{F}_q$ for some $q = p^f$. Assume

$$V \cong \bigoplus_{j=1}^r V(\alpha_j)$$

is the slope decomposition as in [BC09, Lemma 8.1.11], i.e., the base change of the decomposition into $\breve{K}$ is the Dieudonné–Manin decomposition of (D, φ_D). Since V is pure of weight i, so is $V(\alpha_j)$ for every j. By Lemma 4.1.12, we can know that $\alpha_j = i/2$ for every j, hence D is semi-stable of slope $i/2$. ■

§ 4.1.2 Solid Quasi-Coherent Sheaves on Analytic Adic Spaces

It is well known that, due to the example of Gabber ([Con06, Example 2.1.6]), the naive definition of a quasi-coherent sheaf is ill-behaved in analytic settings. The obstruction to such a good theory is the topological issues in analytic geometry. To fix this, Andreychev used an approach developed by Clausen and Scholze in [CS19b; CS19a] to provide a theory of quasi-coherent sheaves in rigid analytic spaces, which is well-behaved in the categorical sense. We will give a brief review for that and assume familiarity with the basic theory of condensed mathematics. For more systematic discussions, we recommend [CS19b; CS19a].

Before the discussion, we fix our convention on condensed mathematics. As is well known, certain set-theoretic subtleties arise in the theory of condensed mathematics. For convenience, we fix an implicit cut-off cardinal $\mathcal{U}$ that is a Grothendieck universe. Accordingly, by a condensed objects, we mean $\mathcal{U}$-condensed object, i.e., a sheaf on $\mathcal{U}$-small extremally disconnected (compact Hausdorff) spaces. All rings considered in this context will be assumed to be commutative (not merely associative) and unital. Consequently, any condensed ring is also assumed to take values of commutative rings.

Analytic Rings

We first recall the notion of analytic (animated) rings, which constitutes a new ingredient in the foundations of condensed analytic geometry.

Notation 4.1.14. Let $\mathbf{Cond}(\mathcal{A}\mathbf{niRing})$ denote the ∞-category of condensed animated rings (defined in [CS19a, §11]) and

$$(-)^{\circ}\colon \mathbf{Cond}(\mathcal{A}\mathbf{niRing}) \to \mathbf{CAlg}(\mathcal{D}_{\geq 0}(\mathbf{Cond}(\mathcal{A}\mathbf{b})))$$

denote the natural forgetful functor ([Man22, Definition 2.2.16 (b)] and [SAG, Construction 25.1.2.1, Proposition 25.1.2.2]), where the target category is the ∞-category of commutative ring objects in condensed animated abelian groups. Then for every condensed animated ring A, we have a stable symmetric monoidal ∞-category

$$\mathcal{D}(A) := \mathbf{Mod}_{A^{\circ}}(\mathcal{D}(\mathbf{Cond}(\mathcal{A}\mathbf{b})))$$

and its full subcategory $\mathcal{D}_{\geq 0}(A)$ spanned by connective objects, i.e., the pre-stable ∞-category of animated condensed A-modules.

Definition 4.1.15. An ***uncompleted analytic animated ring*** is a pair $A = (A^{\triangleright}, \mathcal{D}(A))$ where $A^{\triangleright}$ is a condensed animated ring and $\mathcal{D}(A)$ is a full subcategory of $\mathcal{D}(A^{\triangleright})$ satisfying the following properties:

(1) $\mathcal{D}(A)$ is stable under small limits and colimits in $\mathcal{D}(A^{\triangleright})$ and $\mathcal{D}(A) \subseteq \mathcal{D}(A^{\triangleright})$ is a reflexive subcategory and we will denote the reflector (the left adjoint of the inclusion functor) by

$$- \otimes_{A^{\triangleright}} A\colon \mathcal{D}(A^{\triangleright}) \to \mathcal{D}(A);$$

there is a unique symmetric monoidal structure on $\mathcal{D}(A^{\triangleright})$ such that this reflector is symmetric monoidal;

(2) the reflector in (1) sends connective objects to connective objects;

(3) the internal hom $\underline{\mathrm{Hom}}_{A^\triangleright}(M,N) \in \mathcal{D}(A)$ for any $M \in \mathcal{D}(A^\triangleright)$ and $N \in \mathcal{D}(A)$.

Moreover, an ***analytic animated ring***[1] is an uncompleted analytic animated ring $A = (A^\triangleright, \mathcal{D}(A))$ such that $A^\triangleright \in \mathcal{D}(A)$.

Remark 4.1.16. This definition of analytic animated ring is equivalent to [Man22, Definition 2.3.1] (see also [CS19a, Proposition 12.20]). In fact, given an uncompleted analytic animated ring $A = (A^\triangleright, \mathcal{D}(A))$, we can define $A[S] := A^\triangleright[S] \otimes_{A^\triangleright} A \in \mathcal{D}_{\geq 0}(A)$ for an extremally disconnected space S, where $A^\triangleright[S] \in \mathcal{D}_{\geq 0}(A^\triangleright)$ is the free $A^\triangleright$-modules on S. The only thing we need to show is the following equivalence:

$$M \in \mathcal{D}(A) \Leftrightarrow \underline{\mathrm{Hom}}^{\underline{\mathbb{Z}}}_{A^\triangleright}(A[S], M) \xrightarrow{\simeq} \underline{\mathrm{Hom}}^{\underline{\mathbb{Z}}}_{A^\triangleright}(A^\triangleright[S], M) \text{ in } \mathcal{D}(\underline{\mathbb{Z}})$$

here $\underline{\mathrm{Hom}}^{\underline{\mathbb{Z}}}_{A^\triangleright}$ is the $\mathcal{D}(\underline{\mathbb{Z}})$-enriched hom (because $\mathcal{D}(A^\triangleright)$ is tensored over $\mathcal{D}(\underline{\mathbb{Z}})$). Assume $M \in \mathcal{D}(A)$. Using (3) in Definition 4.1.15, we know that the internal hom of $\mathcal{D}(A)$ agrees with those in $\mathcal{D}(A^\triangleright)$; thus we have

$$\underline{\mathrm{Hom}}_{A^\triangleright}(A[S], M) \simeq \underline{\mathrm{Hom}}_{A}(A[S], M) \simeq \underline{\mathrm{Hom}}_{A^\triangleright}(A^\triangleright[S], M).$$

In particular, we get the isomorphism in $\mathcal{D}(\mathbb{Z})$. Conversely, any M making the right-hand side hold is a colimit of $A[T]$ for T extremally disconnected ([Man22, Proposition 2.3.2]). As $A[T] \in \mathcal{D}(A)$ and $\mathcal{D}(A)$ is stable under colimits in $\mathcal{D}(A^\triangleright)$, we know that $M \in \mathcal{D}(A)$.

If we start from an uncompleted analytic animated ring in [Man22, Definition 2.3.1], then desired properties in Definition 4.1.15 follow from [Man22, Proposition 2.3.2, Proposition 2.3.8].

Definition 4.1.17. A ***morphism of analytic animated ring*** $f\colon A \to B$ is given by a morphism $f\colon A^\triangleright \to B^\triangleright$ of condensed animated ring such that the functor $f_*\colon \mathcal{D}(B^\triangleright) \to \mathcal{D}(A^\triangleright)$ restricts to a functor $\mathcal{D}(B) \to \mathcal{D}(A)$.

We denote by $\mathcal{A}\mathbf{nAniRing}$ the category of analytic animated rings, and its full subcategory spanned by those having static underlying condensed animated rings will be denoted by $\mathcal{A}\mathbf{nRing}$.

Proposition 4.1.18. *Let $A = (A^\triangleright, \mathcal{D}(A))$ be an analytic animated ring. Then $\mathcal{D}(A)$ is a compactly generated ∞-category with a set of compact generators $A[S]$, where S is extremally disconnected. Moreover, let $f\colon A \to B$ be a morphism of analytic animated rings. Then $f^*\colon \mathcal{D}(A) \to \mathcal{D}(B)$ is in $\mathbf{CAlg}(\mathcal{P}\mathbf{r}^{\mathrm{L}}_{\omega})$.*

Proof. These are [Man22, Proposition 2.3.2, Proposition 2.3.4]. ∎

[1] This is called a normalized analytic animated ring in [CS19a]. We here follow the terminology in [Man22].

Solid Quasi-Coherent Sheaves

We now turn to the geometric setting. In [And21], Andreychev assigns each complete Huber pair (A, A^+) to an analytic (non-animated) ring $(A, A^+)_\square$ which is compatible with the discrete case considered by Clausen and Scholze in [CS19b, Definition 9.1].

Proposition 4.1.19. *There is a fully faithful functor*

$$\begin{aligned} \{\textit{Complete Huber pairs}\} &\hookrightarrow \mathbf{AnRing} \\ (A, A^+) &\mapsto (A, A^+)_\square. \end{aligned}$$

Proof. The full faithfulness is [And21, Proposition 3.34], and [And21, Proposition 2.16, Lemma 3.19] shows every $(A, A^+)_\square$ is static. ∎

As in the previous chapter, Huber pairs are always assumed to be complete throughout the left part of this thesis.

Definition 4.1.20. For a (complete) Huber pair (A, A^+), we denote

$$\mathcal{D}_\square(A, A^+) := \mathcal{D}((A, A^+)_\square).$$

The objects of $\mathcal{D}_\square(A, A^+)$ are called the ***solid*** (A, A^+)***-modules***.

The following descent property allows us to define the solid quasi-coherent sheaves on analytic adic space:

Proposition 4.1.21 ([And21, Theorem 4.1, Theorem 5.41])**.** *Let X be an analytic adic space. Let* $\mathbf{AffOpen}(X)$ *denote the ∞-category of affinoid open subspaces of X. Then*

$$\begin{aligned} \mathbf{AffOpen}(X)^{\mathrm{op}} &\to \mathbf{CAlg}(\mathcal{P}\mathbf{r}^{\mathrm{st}}_\omega) \\ U &\mapsto \mathcal{D}_\square(\mathscr{O}_X(U), \mathscr{O}_X^+(U)) \\ \iota &\mapsto \iota^* \end{aligned}$$

is a sheaf on X valued in $\mathbf{CAlg}(\mathcal{P}\mathbf{r}^{\mathrm{st}}_\omega)$.

Definition 4.1.22. In virtue of Proposition 4.1.21, for every analytic adic space X, we can define

$$\mathbf{QCoh}(X) := \lim_{\substack{U \subseteq X \\ \text{rational open}}} \mathcal{D}_\square(\mathcal{O}_X(U), \mathcal{O}_X^+(U)),$$

where the limit is taken in $\mathbf{CAlg}(\mathcal{P}\mathbf{r}^{\mathrm{st}}_\omega)$.

Remark 4.1.23 (Analytic Stacks)**.** As mentioned earlier, analytic geometry can be modeled using condensed mathematics. More precisely, Clausen and Scholze construct analytic stacks by gluing affine pieces given by analytic (animated) rings with respect to the !-topology, which is finer than the usual analytic topology on (analytic) adic spaces. In particular, by Proposition 4.1.19, the category of analytic stacks contains analytic adic spaces as special cases.

Moreover, every analytic stack admits an ∞-category of solid quasi-coherent sheaves which has six-functor formalism. In the case of adic spaces, this specializes to the category of solid quasi-coherent sheaves defined in Definition 4.1.22. However, in this thesis, we will not work with general analytic stacks; instead, we restrict ourselves to analytic adic spaces. Accordingly, we adopt the direct definition from [And21].

In the next, we fix an affinoid analytic adic space $X = \mathrm{Spa}(A, A^+)$ and recall the analytic descent of the full subcategory of perfect complexes. We start from a fully faithful functor in $\mathcal{C}\mathbf{AT}_\infty^{\mathrm{ex}}$ ([And21, Lemma 5.7])

$$\mathrm{dCond}_A \colon \mathcal{D}(A) \to \mathcal{D}(\underline{A_{\mathrm{disc}}})$$

induced by the underline functor where A is equipped with the discrete topology on the right-hand side. The continuous identity map $A_{\mathrm{disc}} \to A$ induces a morphism $\underline{A_{\mathrm{disc}}} \to \underline{A}$ of condensed rings. Thus, we get the ***condensification functor***

$$\begin{aligned} \mathrm{Cond}_A \colon \mathcal{D}(A) &\to \mathcal{D}(\underline{A}) \\ M &\mapsto \mathrm{dCond}_A(M) \otimes_{\underline{A_{\mathrm{disc}}}} \underline{A}. \end{aligned}$$

Proposition 4.1.24. *Let (A, A^+) be a sheafy analytic (complete) Huber pair.*

(1) *The condensification functor $\mathrm{Cond}_A \colon \mathcal{D}(A) \to \mathcal{D}(\underline{A})$ is fully faithful, exact and commutes with filtered colimits. Moreover, it factors through the full subcategory $\mathcal{D}_\square(A, A^+)$ of solid (A, A^+)-modules.*

(2) *The condensification functor induces an equivalence between the full subcategory $\mathbf{Perf}_A$ of $\mathcal{D}(A)$ spanned by perfect complexes and the full subcategory of dualizable objects in $\mathcal{D}_\square(A, A^+)$. Moreover, the ∞-category of perfect complexes satisfies the analytic descent.*

Proof. The part (1) is [And21, Theorem 5.9], and the equivalence in (2) follows from [And21, Corollary 5.51.1]. The descent property is a result of the equivalence and [And21, Theorem 5.43]. ∎

Therefore, the ∞-category $\mathbf{Perf}(X) := \mathbf{Perf}(X, \mathcal{O}_X)$ of perfect complexes is a full subcategory of $\mathbf{QCoh}(X)$. Moreover, it lies in the full subcategory $\mathbf{QCoh}(X)_\omega$ of compact objects ([And21, Proposition 5.37]).

§ 4.1.3 Solid Quasi-Coherent Sheaves on the Fargues–Fontaine Curve

Next, we study Definition 4.1.22 in the case where X is the Fargues–Fontaine curve over $\mathbb{Q}_p$.

Let us recall the basic facts for the Fargues–Fontaine curve. We write $A_{\text{inf}} = W(\mathcal{O}_{C^\flat})$ and then get a pre-adic space $\mathrm{Spa}(A_{\text{inf}}, A_{\text{inf}})$ whose analytic part is

$$\mathcal{Y}_{[0,\infty]} := \big(\mathrm{Spa}(A_{\text{inf}}, A_{\text{inf}})\big)_{\text{an}} = \mathrm{Spa}(A_{\text{inf}}, A_{\text{inf}}) \setminus \{x_{\bar{k}}\}.$$

where $x_{\bar{k}}$ is induced by $A_{\text{inf}} \to \mathcal{O}_{C^\flat} \to \bar{k}$. This is an analytic adic space ([SW20, Proposition 13.1.1]) and there is a surjective continuous map ([SW20, §12.2], [FS24, Proposition II.1.16]) $\kappa\colon \left|\mathcal{Y}_{[0,\infty]}\right| \to [0,\infty]$ with the following special points

- $\kappa(x_{C^\flat}) = 0$, where $x_{C^\flat}$ corresponds to $A_{\text{inf}} \to \mathcal{O}_{C^\flat} \to C^\flat$;
- $\kappa(x_{\breve{K}}) = \infty$, where $x_{\breve{K}}$ corresponds to $A_{\text{inf}} \to W(\bar{k}) \to \breve{K}$.

We will also consider several kinds of open subspaces:

(1) $\mathcal{Y}_{(0,\infty]} := (\mathrm{Spa}\, A_{\text{inf}}) \setminus V(p)$;

(2) $\mathcal{Y}_{[0,\infty)} := (\mathrm{Spa}\, A_{\text{inf}}) \setminus V([p^\flat])$;

(3) $\mathcal{Y}_{(0,\infty)} := (\mathrm{Spa}\, A_{\text{inf}}) \setminus V(p[p^\flat])$.

(4) In general, for any sub-intervals I with rational endpoints of $[0,\infty]$, we will let $\mathcal{Y}_I$ denote the interior of the inverse image of I under the radius function κ.

The Frobenius map of $\mathrm{Spa}(A_{\text{inf}}, A_{\text{inf}})$ restricts to an automorphism φ of the open subspace $\mathcal{Y}_{(0,\infty)}$ satisfying $\kappa \circ \varphi = p\varphi$, and the map φ induces a properly discontinuous action on it; hence, one can define the quotient adic space

$$\mathbf{FF} := \mathcal{Y}_{(0,\infty)}/\varphi^{\mathbb{Z}}.$$

This quotient space is called the (absolute adic) ***Fargues–Fontaine curve*** over $\mathbb{Q}_p$.

Now, taking $X = \mathbf{FF}$ in Definition 4.1.22, we get the ∞-category $\mathbf{QCoh}(\mathbf{FF})$ of solid quasi-coherent sheaves on the Fargues–Fontaine curve. For us, its full subcategory of perfect complexes will play a crucial role in this thesis. In fact, these perfect complexes are indeed well understood by [ALB24, Proposition 2.6]: each $\mathcal{E} \in \mathbf{Perf}(\mathbf{FF})$ is quasi-isomorphic to a bounded complex of vector bundles. We next recall the computations on vector bundles on $\mathbf{FF}$ and how to relate coefficients in §4.1.1.

The Relation to φ-Modules

Recall that we can cover $\mathcal{Y}_{(0,\infty)}$ by $\mathcal{Y}_I$, where I runs through compact sub-intervals in $(0,\infty)$ with rational endpoints. Here $\Gamma(\mathcal{Y}_I, \mathcal{O}) = B_I$ is defined in [FF18, §1.6, especially Example 1.6.3]. So we have

$$\Gamma(\mathcal{Y}_{(0,\infty)}, \mathcal{O}) = \lim_{I \subseteq (0,\infty)} B_I \simeq B.$$

As B is obtained by inverting p in the completion of $A_{\mathrm{inf}}[1/p, 1/[p^\flat]]$ with respect to a family of Gauss norms (c.f. [FF18, §1.6]), there is a natural map $\breve{K} \to B$ induced by the inclusion $W(\bar{k}) \to \mathcal{O}_C$ (or equivalently, $\bar{k} \to \mathcal{O}_{C^\flat}$). Thus, we get a morphism of adic spaces:

$$e \colon \mathcal{Y}_{(0,\infty)} \to \mathrm{Spa}(\breve{K}) \tag{4.1.2}$$

which is also compatible with Frobenius morphisms. In particular, the pullback defines a functor (using similar notation in Notation 4.0.1)

$$\mathcal{E} \colon \mathcal{D}^{\mathrm{b}}_{\varphi}(\breve{K}) \simeq \mathbf{Perf}(\mathrm{Spa}(\breve{K}, \mathcal{O}_{\breve{K}}))^{\varphi} \to \mathbf{Perf}(\mathcal{Y}_{(0,\infty)})^{\varphi} \simeq \mathbf{Perf}(\mathbf{FF}),$$

where the first equivalence follows from [BGV25, Proposition 2.50] and the last equivalence follows from the analytic descent of perfect complexes, by Proposition 4.1.24. As we mentioned below Proposition 4.1.24 that each perfect complex is a compact object in $\mathbf{QCoh}(\mathbf{FF})$, we take the Ind-completion of $\mathcal{E}$ to extend it into the following functor:

$$\mathcal{E} \colon \mathcal{D}_{\varphi}(\breve{K}) \to \mathbf{QCoh}(\mathbf{FF}) \tag{4.1.3}$$

in $\mathbf{CAlg}(\mathcal{P}\mathbf{r}^{\mathrm{st}}_{\omega})$. By the construction, this is exactly the analytic version (using GAGA below) of the algebraic $\mathcal{E}$-functor defined in [FF18, §8.2.3]. By our conventions on Tate twists of φ-modules (see Example 4.1.4), we define

$$\mathcal{O}_{\mathbf{FF}}(1) := \mathcal{E}(\breve{K}(1)), \tag{4.1.4}$$

which agrees with the usual notation on the Fargues–Fontaine curve. Accordingly, as is standard, we obtain all the twists $\mathcal{O}_{\mathbf{FF}}(\lambda)$ for $\lambda \in \mathbb{Q}$.

Remark 4.1.25. We use the natural map $\breve{K} \to B$ to define a φ-equivariant morphism $e \colon \mathcal{Y}_{(0,\infty)} \to \mathrm{Spa}(\breve{K})$ in (4.1.2). Using the similar argument, we know that $\Gamma(\mathcal{Y}_{(0,\infty]}, \mathcal{O}) = B^+$ (we adopt the notation in [FF18, §1.10]). This gives a morphism $\bar{e} \colon \mathcal{Y}_{(0,\infty]} \to \mathrm{Spa}(\breve{K})$. In fact, the morphism e can be factorized into the composition of $\bar{e}$ with the pullback $\iota \colon \mathcal{Y}_{(0,\infty)} \to \mathcal{Y}_{(0,\infty]}$. In fact, the natural closed immersion $c \colon \mathrm{Spa}(\breve{K}) \to \mathcal{Y}_{(0,\infty]}$ is a section of $\bar{e}$.

Remark 4.1.26 (GAGA)**.** The original construction of the Fargues–Fontaine curve is algebraic (e.g. [FF18]) and we have the GAGA theorem for it, i.e. their category of coherent sheaves are equivalent ([KL15], [Far18], [FS24]) via analytification.

Proposition 4.1.27. *Let $\lambda, \mu \in \mathbb{Q}$. Then we have the following computations about mapping spectra:*

$$\pi_0 \mathrm{map}(\mathcal{O}_{\mathbf{FF}}(\lambda), \mathcal{O}_{\mathbf{FF}}(\mu)) = \begin{cases} 0 & \mu < \lambda \\ B^{\varphi^h = p^d} & \mu - \lambda = d/h, d, h \in \mathbb{Z}_{\geq 0}, \gcd(d,h) = 1 \end{cases}$$

$$\pi_{-1} \mathrm{map}(\mathcal{O}_{\mathbf{FF}}(\lambda), \mathcal{O}_{\mathbf{FF}}(\mu)) = 0, \quad \lambda \leq \mu$$

Proof. This follows from computations in the algebraic case ([FF18, Proposition 5.6.23, Proposition 8.2.3]) and the GAGA theorem (cf. [FS24, Proposition II.2.7]). ∎

§ 4.2 De Rham Type Realization Functors

We review several motivic realization functors defined on rigid analytic motives.

§ 4.2.1 The Relative Overconvergent de Rham Cohomology

We begin with the motivic overconvergent de Rham realization defined in [Vez18; LBV23], which serves as the source for the other motivic realizations considered later.

We fix an (admissible) analytic adic space[1] S over $\mathbb{Q}_p$. In [LBV23], authors define, for each smooth S-dagger space X ([LBV23, Definition 3.3]), a complex $\mathbf{R\Gamma}^{\dagger}_{\mathrm{dR}}(X/S)_{\square}$ in the ∞-category of solid quasi-coherent sheaves $\mathbf{QCoh}(S)$, introduced in §4.1.2. This assignment $\mathbf{R\Gamma}^{\dagger}_{\mathrm{dR}}(-/S)$ satisfies étale descent, $\mathbb{B}^{1\dagger}$-invariance, and sends $\mathbb{T}^{1\dagger}_{S}$ into an invertible object ([LBV23, Corollary 4.37]); thus, it factors through the ∞-category of dagger motives over S (whose definition is similar to étale motives over S):

$$\mathbf{R\Gamma}^{\dagger}_{\mathrm{dR},S} \colon \mathbf{RigDA}^{\dagger}(S) \to \mathbf{QCoh}(S)^{\mathrm{op}}.$$

Under the equivalence $\mathbf{RigDA}^{\dagger}(S) \xrightarrow{\simeq} \mathbf{RigDA}(S)$ ([LBV23, Theorem 3.9], [Vez18, Theorem 4.23], or [EV25, Proposition 2.16]), we get a motivic realization:

$$\mathbf{R\Gamma}^{\dagger}_{\mathrm{dR},S} \colon \mathbf{RigDA}(S) \to \mathbf{QCoh}(S)^{\mathrm{op}},$$

called the ***relative overconvergent de Rham realization***. As suggested by the name, it computes the overconvergent de Rham cohomology of rigid analytic varieties ([Vez18, Proposition 5.12]). Since this cohomology realization is the key ingredient to define further realization functors, we recollect its properties, which will be used.

Proposition 4.2.1 ([LBV23, Corollary 4.39, Theorem 4.46])**.** *Let S be a quasi-compact and quasi-separated admissible adic space over $\mathbb{Q}_p$.*

(1) *The restriction*

$$\mathbf{R\Gamma}^{\dagger}_{\mathrm{dR},S} \colon \mathbf{RigDA}(S)_{\omega} \to \mathbf{QCoh}(S)^{\mathrm{op}}$$

is symmetric monoidal and compatible with pullback functors f^* for any morphism $f \colon S' \to S$ in $\mathbf{Adic}_{\mathbb{Q}_p}$.

(2) *The realization functor $\mathbf{R\Gamma}^{\dagger}_{\mathrm{dR},S} \colon \mathbf{RigDA}(S) \to \mathbf{QCoh}(S)^{\mathrm{op}}$ sends dualizable motives to split perfect complexes (see also 4.1.24), and the cohomology groups of $\mathbf{R\Gamma}^{\dagger}_{\mathrm{dR},S}(M)$ are vector bundles on S for any dualizable motive M.*

[1] It is a subcollection of analytic adic spaces defined in [LBV23, Definition 2.1], which is used to define adic étale motives, as explained in Remark 3.1.12.

There is a special case when S is given by a complete non-archimedean field K; then each compact motive over S is dualizable (by [Ayo20, Proposition 2.31] and [Rio05]); therefore, we can get a covariant realization functor by applying $\mathrm{R}\Gamma^{\dagger}_{\mathrm{dR}}$ to the duals and then taking the Ind-completion, which, by abuse of notation, will be denoted by

$$\mathrm{R}\Gamma^{\dagger}_{\mathrm{dR},K} \colon \mathbf{RigDA}(K) \to \mathcal{D}_{\square}(K, \mathcal{O}_K).$$

In light of Proposition 4.2.1, its restriction to the compact part, the value is taken in $\mathcal{D}^{\mathrm{b}}(K)$; in particular, the realization $\mathrm{R}\Gamma^{\dagger}_{\mathrm{dR},K}$ takes value in $\mathcal{D}(K)$, we have the ***covariant overconvergent de Rham realization***:

$$\mathrm{R}\Gamma^{\dagger}_{\mathrm{dR},K} \colon \mathbf{RigDA}(K) \to \mathcal{D}(K) \tag{4.2.1}$$

We will only use this covariant version of overconvergent de Rham realization when the base is a field.

§ 4.2.2 The Rigid Realization and the Hyodo–Kato Realization

Rigid Cohomology

As in [Vez18], one can define rigid cohomology directly using overconvergent de Rham cohomology. Let k be a perfect field of characteristic $p > 0$, and let $K_0 = \operatorname{Frac} W(k)$. Composing the (covariant) overconvergent de Rham realization (4.2.1) with the Monsky–Washnitzer functor (3.1.7), we obtain a realization functor:

$$\mathrm{R}\Gamma_{\mathrm{rig}} \colon \mathbf{DA}(k) \xrightarrow{\xi} \mathbf{RigDA}(K_0) \xrightarrow{\mathrm{R}\Gamma^{\dagger}_{\mathrm{dR},K_0}} \mathcal{D}(K_0),$$

which computes the rigid cohomology by [Vez18, Proposition 5.12]. Furthermore, we can enhance this realization by taking value of φ-modules as follows: as $\mathrm{R}\Gamma_{\mathrm{rig}}$ is compatible with the Frobenius pullbacks, we have

$$\mathrm{R}\Gamma^{\varphi}_{\mathrm{rig}} \colon \mathbf{DA}(k)^{\varphi_\omega} \to \mathcal{D}_{\varphi}(K_0).$$

It is left to find the Frobenius enrichment of motives. To this end, the relative Frobenius ([SP, Section 0CC6]) allows us to obtain the following Frobenius enrichment:

$$\varphi_{\mathrm{rel}} \colon \mathbf{DA}(\bar{k}) \to \mathbf{DA}(\bar{k})^{\varphi_\omega} \tag{4.2.2}$$

see also [BGV25, Remark 4.28], [CD16, Proposition 6.3.16] and [AGV22, Theorem 2.9.7]. Here, we adopt Notation 4.0.1. This construction holds for replacing $\bar{k}$ by any scheme of characteristic $p > 0$.

Composing these two functors, we get an algebraic realization taking value of φ-modules, denoted by

$$\mathrm{R}\Gamma_{\mathrm{rig}} \colon \mathbf{DA}(k) \to \mathcal{D}_{\varphi}(K_0), \tag{4.2.3}$$

referred as the ***rigid realization***.

Overconvergent Hyodo–Kato Cohomology

Now we use the rigid realization (4.2.3) together with the perspective in §3.1.4 to give a quick definition for the Motivic Hyodo–Kato realization defined in [BGV25].

Note that the image of $\mathbb{1}(-1)$ under the rigid realization (4.2.3) is $\breve{K}(-1)$ by [BGV25, Remark 4.41, Lemma 4.30]. Then adding the monodromy operators on both sides gives the ***(overconvergent) Hyodo–Kato realization*** over K

$$\mathrm{R}\boldsymbol{\Gamma}_{\mathrm{HK},K}^{\varpi}\colon \mathbf{RigDA}_{\mathrm{gr}}(K) \to \mathcal{D}_{(\varphi,N)}(K_0), \tag{4.2.4}$$

here we use identifications $\mathbf{RigDA}_{\mathrm{gr}}(K) \simeq \mathbf{DA}_N(k)$, by Proposition 3.1.21 with a choice of the pseudo-uniformizer $\varpi \in \mathcal{O}_K$, and $\mathcal{D}_{(\varphi,N)}(K_0) \simeq \mathcal{D}_{\varphi}(K_0)_{\mathrm{nil}}^{-\otimes K_0(-1)}$ by (4.1.1).

Remark 4.2.2. The Hyodo–Kato realization (4.2.4) coincides with the one in [BGV25, Definition 4.42]. This follows from [BGV25, Proposition 4.46, Theorem 4.53] and definitions.

Note that we define rigid realization and Hyodo–Kato realization by the covariant overconvergent de Rham realization, which computes the dual of cohomology. Thus, the next proposition shows the motivic realization $\mathrm{R}\boldsymbol{\Gamma}_{\mathrm{HK}}$ does compute the duals of Hyodo–Kato cohomology.

Proposition 4.2.3. *Let ϖ be a pseudo-uniformizer of $\mathcal{O}_K$.*

(1) *We have a monoidal equivalence $\pi \circ \mathrm{R}\boldsymbol{\Gamma}_{\mathrm{HK},K} \circ \xi \simeq \mathrm{R}\boldsymbol{\Gamma}_{\mathrm{rig}}$, where $\pi\colon \mathcal{D}_{(\varphi,N)}(K_0) \to \mathcal{D}_{\varphi}(K_0)$ is forgetting the monodromy. In particular, if k is finite, for any proper smooth k-variety X, the i-cohomology (or $(-i)$-homology) of $\pi \circ \mathrm{R}\boldsymbol{\Gamma}_{\mathrm{HK}}^{\varpi} \circ \xi(M(X)^{\vee})$ is pure of weight i in the sense of Definition 4.1.11.*

(2) *Assume k is a finite field. Then the functor $\mathrm{R}\boldsymbol{\Gamma}_{\mathrm{HK},K}^{\varpi}$ computes the arithmetic overconvergent Hyodo–Kato cohomology in the sense of [CN20, §5.2.2] if* $\operatorname{char} K = p > 0$ *or* $\operatorname{char} K = 0$ *with $\varpi = p$.*

Proof. The construction of (4.2.4) shows that we have a monoidal equivalence $\pi\circ\mathrm{R}\boldsymbol{\Gamma}_{\mathrm{HK},K} \simeq \mathrm{R}\boldsymbol{\Gamma}_{\mathrm{rig}} \circ \Psi$, where $\Psi\colon \mathbf{RigDA}_{\mathrm{gr}}(K) \to \mathbf{DA}(k)$ is the motivic nearby cycle functor (3.1.9). Note that the Monsky–Washnitzer functor ξ is a section of Ψ. Therefore, the monoidal equivalence is clear. Then the part (1) follows from the Weil conjecture (see [KM74]); and the part (2) follows from [BGV25, Corollary 4.58] and [BKV25, Proposition 3.30]. ■

Notation 4.2.4. In light of Proposition 4.2.3 (2), we will adopt the notation

$$\mathrm{R}\boldsymbol{\Gamma}_{\mathrm{HK},K}\colon \mathbf{RigDA}_{\mathrm{gr}}(K) \to \mathcal{D}_{(\varphi,N)}(K_0)$$

for $\mathrm{R}\boldsymbol{\Gamma}_{\mathrm{HK},K}^{\varpi}$ when $\operatorname{char} K = p > 0$ or $\operatorname{char} K = 0$ with $\varpi = p$. In other words, when we use this abbreviation, we have already chosen a specific pseudo-uniformizer.

Remark 4.2.5. When defining the Hyodo–Kato realization functor, we have to fix a pseudo-uniformizer $\varpi \in \mathcal{O}_K$. Different choices of ϖ only affect the behavior of the monodromy operators in the Hyodo–Kato realization. In particular, if K is a local field, this dependence is solely on the valuation of ϖ in K. For further details, we refer to [BGV25, Remark 4.18].

§ 4.2.3 Motives on the Fargues–Fontaine Curve

Before giving the de Rham–Fargues–Fontaine realization, we recall the spreading-out result of motives on the Fargues–Fontaine curve from [LBV23, §5] and then get a "pullback" functor $\mathbf{RigDA}(C^\flat) \to \mathbf{RigDA}(\mathbf{FF})$. The composition of this "pullback" with the overconvergent de Rham realization on rigid analytic motives on **FF** will give the de Rham–Fargues–Fontaine realization; see the next subsection.

We have seen that there are two special points, $x_{C^\flat}$ and $x_{\breve{K}}$, on the Fargue-Fontaine curve. In the next, we will denote them by x_0 and x_∞, respectively. For $r \in \mathbb{Q}_{>0}$, we have open neighborhoods $U_{0,r} = \mathcal{Y}_{[0,r]}$ and $U_{\infty,r} = \mathcal{Y}_{[r,\infty]}$ for x_0 and x_∞ respectively. Then, for each $r \in \mathbb{Q}_{>0}$, we have morphisms:

$$j_{0,r}\colon U_{0,r} \hookrightarrow U_{0,pr}, \quad j_{\infty,r}\colon U_{\infty,r} \to U_{\infty,r/p}$$
$$\varphi_{0,r}\colon U_{0,r} \xrightarrow{\simeq} U_{0,pr}, \quad \varphi_{\infty,r}\colon U_{\infty,r} \xrightarrow{\simeq} U_{\infty,r/p}.$$

Then we can define endofunctors for $\mathbf{RigDA}(U_{\star,r})$ with $\star \in \{0,\infty\}$:

$$\iota_{\star r,\sharp} = \varphi^*_{\star r} \circ j_{\star r,\sharp}, \qquad \iota_{\star r,*} = \varphi^*_{\star r} \circ j_{\star r,*}$$
$$\iota^*_{0,r} = (\varphi^{-1}_{0,r/p})^* \circ j^*_{0,r/p}, \qquad \iota^*_{\infty,r} = (\varphi^{-1}_{\infty,pr})^* \circ j^*_{\infty,pr},$$

and let $\mathcal{Y}_0$, $\mathcal{Y}_\infty$ denote $\mathcal{Y}_{[0,\infty)}$ and $\mathcal{Y}_{(0,\infty]}$ respectively. We next use the convention in Notation 4.0.1

With notations above, we have the following spreading-out results from either x_0 or x_∞:

Proposition 4.2.6. *Let $\star \in \{0,\infty\}$ and $r \in \mathbb{Q}_{>0}$.*

(1) *The pullback along $x_\star \hookrightarrow U_{\star,r}$ induces an equivalence in $\mathbf{CAlg}(\mathcal{P}\mathbf{r}^{\mathrm{L}}_\omega)$:*

$$\operatorname{colim}\left(\mathbf{RigDA}(U_{\star,r}) \xrightarrow{\iota^*_{\star r}} \mathbf{RigDA}(U_{\star,r}) \xrightarrow{\iota^*_{\star r}} \cdots\right) \xrightarrow{\simeq} \mathbf{RigDA}(x_\star).$$

(2) *The pullbacks along $U_{\star,r} \hookrightarrow \mathcal{Y}_\star$ induces an equivalence in $\mathbf{CAlg}(\mathcal{P}\mathbf{r}^{\mathrm{L}})$:*

$$\mathbf{RigDA}(\mathcal{Y}_\star) \xrightarrow{\simeq} \lim\left(\cdots \xrightarrow{\iota^*_{\star r}} \mathbf{RigDA}(U_{\star,r}) \xrightarrow{\iota^*_{\star r}} \mathbf{RigDA}(U_{\star,r})\right)$$

(3) *We have equivalences in $\mathbf{CAlg}(\mathcal{P}\mathbf{r}^{\mathrm{L}})$:*

$$\mathbf{RigDA}(x_\star)^{\varphi_\omega} \simeq (\operatorname*{colim}_{\iota^*_{\star r}} \mathbf{RigDA}(U_{\star,r}))^{\iota^*_{\star r\omega}} \simeq \mathbf{RigDA}(U_{\star,r})^{\iota^*_{\star r\omega}}$$
$$\mathbf{RigDA}(\mathcal{Y}_\star)^{\varphi} \simeq (\lim_{\iota^*_{\star r}} \mathbf{RigDA}(U_{\star,r}))^{\iota^*_{\star r}} \simeq \mathbf{RigDA}(U_{\star,r})^{\iota^*_{\star r}}.$$

(4) *The pullbacks along the closed embeddings* $\mathcal{X}_\star \hookrightarrow \mathcal{Y}_\star$ *induce equivalences in* $\mathbf{CAlg}(\mathcal{P}\mathbf{r}^{\mathrm{L}}_\omega)$*:*

$$\mathbf{RigDA}(\mathcal{Y}_\star)^{\varphi_\omega} \xrightarrow{\simeq} \mathbf{RigDA}(\mathcal{X}_\star)^{\varphi_\omega}.$$

(5) *The pullback defines an equivalence in* $\mathbf{CAlg}(\mathcal{P}\mathbf{r}^{\mathrm{L}})$

$$\mathbf{RigDA}(\mathbf{FF}) \simeq \mathbf{RigDA}(\mathcal{Y}_{(0,\infty)})^{\varphi} \simeq \mathbf{RigDA}(\mathcal{Y}_{(0,\infty)})^{\varphi_\omega}.$$

Proof. The proof of [LBV23, Proposition 5.3] works in both cases. ∎

Corollary 4.2.7. *Let* $e\colon \mathcal{Y}_{(0,\infty)} \to \mathrm{Spa}(\breve{K})$ *defined in* (4.1.2)*. Then we have a commutative diagram*

$$\begin{tikzcd}
\mathbf{RigDA}(\breve{K})^{\varphi_\omega} \arrow[r, "e^*"] & \mathbf{RigDA}(\mathcal{Y}_{(0,\infty)})^{\varphi_\omega} \\
\mathbf{RigDA}(\mathcal{Y}_{(0,\infty]})^{\varphi_\omega} \arrow[u, "\simeq"] \arrow[ur, "j^*"'] &
\end{tikzcd}$$

in $\mathbf{CAlg}(\mathcal{P}\mathbf{r}^{\mathrm{L}}_\omega)$*, where vertical equivalence is 4.2.6 (4) and the functor* j^* *is the pullback along the inclusion.*

Proof. Recall from Remark 4.1.25, the morphism e has a factorization

$$e\colon \mathcal{Y}_{(0,\infty)} \xrightarrow{\iota} \mathcal{Y}_{(0,\infty]} \xrightarrow{\bar{e}} \mathrm{Spa}(\breve{K}, \mathcal{O}_{\breve{K}}),$$

and the closed immersion $c\colon \mathrm{Spa}(\breve{K}, \mathcal{O}_{\breve{K}}) \to \mathcal{Y}_{(0,\infty]}$ is a section of $\bar{e}$. Thus, the commutativity follows from

$$e^* \circ c^* \simeq \iota^* \circ \bar{e}^* \circ c^* \simeq \iota^* \circ (\bar{e} \circ c)^* \simeq \iota^*.$$

∎

In order to define the spreading-out functor, we need an analytic analogue of the Frobenius enrichment, similar to (4.2.2). Let S be a perfectoid space in $\mathbf{Adic}_{\bar{k}}$; there is a functor

$$\mathbf{RigDA}(S) \to \mathbf{RigDA}(S)^{\varphi_\omega}$$

where the right-hand side is the equalizer of Id and φ_S^* in $\mathbf{CAlg}(\mathcal{P}\mathbf{r}^{\mathrm{L}}_\omega)$, defined by the relative Frobenius (see [LBV23, Corollary 2.26], [AGV22, Corollary 2.9.11]). Now using Proposition 4.2.6 and the pullback $\mathbf{RigDA}(\mathcal{Y}_{[0,\infty)}) \to \mathbf{RigDA}(\mathcal{Y}_{(0,\infty)})$, we get a functor

$$\mathcal{D}_0\colon \mathbf{RigDA}(C^\flat) \to \mathbf{RigDA}(\mathbf{FF})$$

defined by

$$\begin{tikzcd}
\mathbf{RigDA}(C^\flat) \arrow[r, "\mathcal{D}_0"] \arrow[d] & \mathbf{RigDA}(\mathbf{FF}) \arrow[d, "\simeq"] \\
\mathbf{RigDA}(C^\flat)^{\varphi_\omega} \arrow[d, "\simeq"'] & \mathbf{RigDA}(\mathcal{Y}_{(0,\infty)})^{\varphi} \\
\mathbf{RigDA}(\mathcal{Y}_{[0,\infty)})^{\varphi_\omega} \arrow[r, hook] & \mathbf{RigDA}(\mathcal{Y}_{[0,\infty)})^{\varphi} \arrow[u, "j^*"']
\end{tikzcd}$$

in $\mathbf{CAlg}(\mathcal{P}\mathbf{r}^{\mathrm{L}})$. We will refer to it as the ***spreading-out functor*** from x_0.

We end this subsection by comparing the spreading out from x_0 and from x_∞. However, one of the crucial points to define the spreading-out functor from x_0 is the Frobenius enrichment $\mathbf{RigDA}(C^\flat) \to \mathbf{RigDA}(C^\flat)^{\varphi_\omega}$, which is natural in characteristic $p > 0$. In characteristic 0, we will use an alternative way to enrich into the φ-equivariant motives:

Composing the Monsky-Washinitzer functor and (4.2.2), we get

$$\mathbf{DA}(\bar{k}) \xrightarrow{\varphi_{\mathrm{rel}}} \mathbf{DA}(\bar{k})^{\varphi_\omega} \xrightarrow{\xi_{\breve{K}}} \mathbf{RigDA}(\breve{K})^{\varphi_\omega}.$$

By again applying Proposition 4.2.6 and imitating the construction of $\mathcal{D}_0$, we obtain a functor

$$\bar{\mathcal{D}}_\infty \colon \mathbf{DA}(\bar{k}) \to \mathbf{RigDA}(\breve{K})^{\varphi_\omega} \to \mathbf{RigDA}(\mathbf{FF})$$

in $\mathbf{CAlg}(\mathcal{P}\mathbf{r}^{\mathrm{L}})$. These allow us to compare spreading out from x_0 and x_∞:

Proposition 4.2.8. *We have a monoidal equivalence $\mathcal{D}_0 \circ \xi_{C^\flat} \simeq \bar{\mathcal{D}}_\infty$ in $\mathbf{CAlg}(\mathcal{P}\mathbf{r}^{\mathrm{L}})$.*

Proof. We use the equivalence $\mathbf{DA}(\bar{k}) \simeq \mathbf{FDA}(W(\mathcal{O}_C^\flat))$, where $W(\mathcal{O}_C^\flat)$ is equipped with the $(p, [p^\flat])$-adic topology, here $p^\flat \in \mathcal{O}_C^\flat$ is a pseudo-uniformizer. Then we have a commutative diagram (up to homotopy)

$$\begin{tikzcd}
\mathbf{DA}(\bar{k}) \arrow[r, "\varphi_{\mathrm{rel}}"] & \mathbf{DA}(\bar{k})^{\varphi_\omega} \arrow[r] & \mathbf{RigDA}(\breve{K})^{\varphi_\omega} \arrow[r] & \mathbf{RigDA}(\mathbf{FF}) \\
\mathbf{FDA}(W(\mathcal{O}_C^\flat)) \arrow[u, "\simeq"] \arrow[r] & \mathbf{FDA}(W(\mathcal{O}_C^\flat))^{\varphi_\omega} \arrow[u, "\simeq"] \arrow[r] & \mathbf{RigDA}(\mathcal{Y}_{[0,\infty]})^{\varphi_\omega} \arrow[u] \arrow[ur] &
\end{tikzcd}$$

where the top row from left to right is the functor $\bar{\mathcal{D}}_\infty$. Therefore, it suffices to show the bottom functor

$$\mathbf{FDA}(W(\mathcal{O}_C^\flat)) \to \mathbf{RigDA}(\mathcal{Y}_{[0,\infty]})^{\varphi_\omega} \to \mathbf{RigDA}(\mathbf{FF})$$

is equivalent to $\mathcal{D}_0 \circ \xi_{C^\flat}$. This is just [LBV23, Proposition 5.11]. ■

§ 4.2.4 The de Rham–Fargues–Fontaine Realization

We are now in a position to define the de Rham–Fargues–Fontaine realization. We will define it as a covariant functor, and so we use the same trick when defining the covariant overconvergent de Rham realization. Recall that the spreading-out functor $\mathcal{D}_0 \colon \mathbf{RigDA}(C^\flat) \to \mathbf{RigDA}(\mathbf{FF})$ (defined in §4.2.3) is in $\mathbf{CAlg}(\mathcal{P}\mathbf{r}^{\mathrm{L}})$ and it sends compact objects to dualizable objects; on the other hand, the overconvergent de Rham realization sends the dualizable motives to perfect complexes, which are compact as solid quasi-coherent sheaves. Therefore, by taking duals, we get the covariant functor

$$\mathbf{RigDA}(C)_\omega \simeq \mathbf{RigDA}(C^\flat)_\omega \xrightarrow{\mathcal{D}_0} \mathbf{RigDA}(\mathbf{FF})_{\mathrm{dual}} \xrightarrow{\mathrm{R}\Gamma^\dagger_{\mathrm{dR},\mathbf{FF}}} \mathbf{QCoh}(\mathbf{FF})_\omega,$$

where the first equivalence is the motivic tilting equivalence. Its Ind-completion will be denoted by

$$\mathrm{R}\Gamma_{\mathrm{FF}}\colon \mathbf{RigDA}(C) \to \mathbf{QCoh}(\mathbf{FF}), \tag{4.2.5}$$

called the ***de Rham–Fargues–Fontaine realization***.

In other words, the de Rham–Fargues–Fontaine cohomology is the overconvergent de Rham cohomology of the spreading-out motive from x_0. In fact, it is computed by the rigid cohomology:

Proposition 4.2.9. *We have an equivalence* $\mathrm{R}\Gamma_{\mathrm{FF}} \circ \xi_C \simeq \mathcal{E} \circ \mathrm{R}\Gamma_{\mathrm{rig}}$ *in* $\mathbf{CAlg}(\mathcal{P}\mathbf{r}^{\mathrm{L}}_{\omega})$*, where*

$$\mathcal{E}\colon \mathcal{D}_{\varphi}(\breve{K}) \to \mathbf{QCoh}(\mathbf{FF})$$

is defined in (4.1.3).

Proof. As tilting equivalence of rigid analytic motives is compatible with Monsky–Washnitzer functors ([Vez19b, Corollary 3.9]), we can prove for replacing C by $C^{\flat}$. By Proposition 4.2.8, it suffices to show we have an equivalence

$$\mathbf{R\Gamma}^{\dagger}_{\mathrm{dR},\mathbf{FF}} \circ \tilde{\mathcal{D}}_{\infty} \simeq \mathcal{E} \circ \mathbf{R\Gamma}_{\mathrm{rig}}.$$

in $\mathbf{CAlg}(\mathcal{P}\mathbf{r}^{\mathrm{L}}_{\omega})$. To this end, we use the identifications $\mathbf{QCoh}(\mathbf{FF}) \simeq \mathbf{QCoh}(\mathcal{Y}_{(0,\infty)})^{\varphi_\omega}$ and $\mathbf{RigDA}(\mathbf{FF}) \simeq \mathbf{RigDA}(\mathcal{Y}_{(0,\infty)})^{\varphi_\omega}$, and prove the equivalence above by restricting to the compact part. Now it suffices to show $\mathcal{E} \circ \mathbf{R\Gamma}^{\dagger,\varphi_\omega}_{\mathrm{dR},\breve{K}} \simeq \mathbf{R\Gamma}^{\dagger,\varphi_\omega}_{\mathrm{dR},\mathcal{Y}_{(0,\infty)}} \circ e^*$ on $\mathbf{RigDA}(\breve{K})^{\varphi_\omega}_{\omega}$, where e^* is the pullback of motives along the morphism defined in (4.1.2). This can be deduced from the commutativity of the following diagram:

$$\begin{tikzcd}
\mathbf{RigDA}(\breve{K})^{\varphi_\omega}_{\omega} \arrow[r] \arrow[dd, bend right=60, "e^*"'] & \mathcal{D}^{\mathrm{b}}_{\varphi}(\breve{K}) \arrow[dd, bend left=60, "\mathcal{E}"] \\
\mathbf{RigDA}(\mathcal{Y}_{(0,\infty]})^{\varphi_\omega}_{\omega} \arrow[u, "\simeq"'] \arrow[r] \arrow[d] & \mathbf{QCoh}(\mathcal{Y}_{(0,\infty]})^{\varphi_\omega}_{\omega} \arrow[u] \arrow[d] \\
\mathbf{RigDA}(\mathcal{Y}_{(0,\infty)})^{\varphi_\omega}_{\omega} \arrow[r] & \mathbf{QCoh}(\mathcal{Y}_{(0,\infty)})^{\varphi_\omega}_{\omega}
\end{tikzcd}$$

The middle squares are commutative due to Proposition 4.2.1 (1), and the left-most part is commutative due to Corollary 4.2.7, whose proof also works to show the commutativity of the right-most part. ■

§ 4.3 The Weight Spectral Sequence

In the section, we apply Theorem 3.3.10 to give weight filtrations of Hyodo–Kato cohomology $H^i_{\mathrm{HK}}(X)$ of smooth quasi-compact rigid analytic spaces X over K. More precisely, we will use the weight complex functor, see Construction 2.2.21, to construct a spectral sequence and deduce the weight filtration on the Hyodo–Kato realizations of compact rigid analytic motives over K. This can be regarded as the p-adic analytic analogue[1] of the weight spectral sequence studied in [RZ82] for rigid analytic spaces. Notably, we do not assume the existence of any formal models.

Following Theorem 3.3.10, we use w to denote the weight structure on $\mathbf{RigDA}(K)$ whose heart is simply denoted by $\mathcal{H}_K$. In particular, the weight complex functor gives a symmetric monoidal functor

$$W_\bullet\colon \mathbf{RigDA}(K)_\omega \to \mathcal{K}^{\mathrm{b}}(\mathrm{h}\,\mathcal{H}_K),$$

We begin by showing how to enhance the Galois action on the Hyodo–Kato realization from [BKV25, Corollary 3.35].

Construction 4.3.1. We have a realization, denoted by $\mathrm{R}\boldsymbol{\Gamma}_{\mathrm{HK}}^{\mathrm{ari}}$

$$\begin{aligned}\mathbf{RigDA}(K) \simeq \operatorname{colim} \mathbf{RigDA}_{\mathrm{gr}}(L)^{\mathrm{h}\,G_{L/K}} \to \operatorname{colim} \mathcal{D}_{(\varphi,N)}(L_0)^{\mathrm{h}\,G_{L/K}} &\simeq \operatorname{colim} \mathcal{D}_{(\varphi,N,G_{L/K})}(L_0)\\ &\simeq \mathcal{D}_{(\varphi,N,G_K)}(\breve{K}),\end{aligned}$$

where the first equivalence is Proposition 3.2.13 and the second equivalence is Proposition 4.1.9. Furthermore, we can forget monodromy operators:

$$\mathrm{R}\boldsymbol{\Gamma}_{\mathrm{HK}}^{(\varphi,G_K)}\colon \mathbf{RigDA}(K) \to \operatorname{colim} \mathcal{D}_{(\varphi,N)}(L_0)^{\mathrm{h}\,G_{L/K}} \to \operatorname{colim} \mathcal{D}_{(\varphi,G_{L/K})}(L_0) \simeq \mathcal{D}_{(\varphi,G_K)}(\breve{K}). \tag{4.3.1}$$

From now until the end of this section, we assume k is a finite field.

Lemma 4.3.2. *The realization functor (4.3.1)*

$$\mathrm{R}\boldsymbol{\Gamma}_{\mathrm{HK}}^{(\varphi,G_K)}\colon \mathbf{RigDA}(K)_\omega \to \mathcal{D}^{\mathrm{b}}_{(\varphi,G_K)}(\breve{K})$$

factors through the weight complex functor $W_\bullet\colon \mathbf{RigDA}(K)_\omega \to \mathcal{K}^{\mathrm{b}}(\mathrm{h}\,\mathcal{H}_K)$.

Proof. In the proof, we will simply write $\mathrm{R}\boldsymbol{\Gamma}_{\mathrm{HK}}$ for $\mathrm{R}\boldsymbol{\Gamma}_{\mathrm{HK}}^{(\varphi,G_K)}$ and use $\mathrm{Map}_{(\varphi,G_K)}$ or Map_φ to denote the mapping spaces between (φ,G_K)-modules or φ-modules, respectively. By [BGV25, Corollary 3.30] and Corollary 3.3.11 (3), it suffices to check that, for any finite Galois extensions L/K and L'/K and any proper smooth varieties X over k_L and Y over $k_{L'}$, we have

$$\pi_i \mathrm{Map}_{(\varphi,G_K)}(\mathrm{R}\boldsymbol{\Gamma}_{\mathrm{HK}}(e_*\xi_L M(X)), e'_*\xi_{L'}M(Y)) \simeq 0 \tag{4.3.2}$$

for $i>0$. The construction (4.3.1) shows that $\mathrm{R}\boldsymbol{\Gamma}_{\mathrm{HK}}(e_*\xi_L M(X))$ is the image of $\mathrm{R}\boldsymbol{\Gamma}_{\mathrm{HK},L}(e^*e_*\xi_L M(X))$ under the canonical functor

$$\mathcal{D}_{(\varphi,G_{L/K})}(L_0) \to \mathcal{D}_{(\varphi,G_K)}(\breve{K}).$$

[1] The semi-stable reduction case is already treated in [BGV25].

We first compute in $\mathcal{D}^{\mathrm{b}}_{(\varphi,G_{L/K})}(L_0)$:

$$\mathbf{R\Gamma}_{\mathrm{HK},L}(e^*e_*\xi_L M(X)) \simeq \bigoplus_{e_{L/K}} \mathbf{R\Gamma}_{\mathrm{HK},L}(\xi_L M(X)) \tag{4.3.3}$$

where the equivalence follows from Proposition 3.2.11. Now let F/K be a finite Galois extension containing both L and L'. Note that the canonical functor $\mathcal{D}_{(\varphi,G_{L/K})}(L_0) \to \mathcal{D}_{(\varphi,G_K)}(\breve{K})$ factors through $\mathcal{D}_{(\varphi,G_{F/K})}(F_0)$. This, together with (4.3.3), yields that $\mathbf{R\Gamma}_{\mathrm{HK}}(e_*\xi_L M(X))$ is the image of

$$\bigoplus_{e_{L/K}} \mathbf{R\Gamma}_{\mathrm{HK},F}(\xi_F M(X_{k_F}))$$

in $\mathcal{D}_{(\varphi,G_K)}(\breve{K})$; and the similar fact holds for $M(Y)$. Therefore, we have in $\mathcal{D}_{(\varphi,G_{F/K})}(F_0)$

$$\begin{aligned}&\pi_i \mathbf{Map}_{(\varphi,G_{F/K})}(\mathbf{R\Gamma}_{\mathrm{HK},F}(\xi_F M(X_{k_F})), \mathbf{R\Gamma}_{\mathrm{HK},F}(\xi_F M(Y_{k_F})))\\ &\pi_i \mathbf{Map}_{\varphi}(\mathbf{R\Gamma}_{\mathrm{HK},F}(\xi_F M(X_{k_F})), \mathbf{R\Gamma}_{\mathrm{HK},F}(\xi_F M(Y_{k_F})))^{\mathrm{h}G_{F/K}} \simeq 0\end{aligned}$$

for $i > 0$, where the vanishing follows from [BGV25, Example 3.31] together with Proposition 4.2.3. Since this holds for arbitrary such finite Galois extension F over K, this proves (4.3.2) holds. ∎

Proposition 4.3.3. *Let M be a compact motive in $\mathbf{RigDA}(K)$. There is a convergent spectral sequence starting at the first page and degenerating at the second page:*

$$E^1_{pq}(M) = H_q \mathbf{R\Gamma}^{(\varphi,G_K)}_{\mathrm{HK}}(W_p M) \Rightarrow H_{p+q}\mathbf{R\Gamma}^{(\varphi,G_K)}_{\mathrm{HK}}(M),$$

where the differential maps are induced by differential maps in the weight complex. In particular, we have a (finite) increasing filtration on $H_n \mathbf{R\Gamma}^{(\varphi,G_K)}_{\mathrm{HK}}(M)$ whose i-th graded piece is pure of weight $i-n$. Moreover, the filtration is stable under G_K-action, and the monodromy induces a map

$$\mathbf{gr}^W_i H_n \mathbf{R\Gamma}^{(\varphi,G_K)}_{\mathrm{HK}}(M) \to \mathbf{gr}^W_{i-2} H_n \mathbf{R\Gamma}^{(\varphi,G_K)}_{\mathrm{HK}}(M)(-1).$$

Proof. The naive truncation gives a filtration

$$\cdots \to \tau_{\leq i-1} W_\bullet M \to \tau_{\leq i} W_\bullet M \to \tau_{\leq i+1} W_\bullet M$$

for $W_\bullet M$. Applying the functor F, we get a filtration on $\mathbf{R\Gamma}^{(\varphi,G_K)}_{\mathrm{HK}}(M)$. Then, using the spectral sequence [HA, Proposition 1.2.2.14], we get the desired spectral sequence. For the degeneracy, it suffices to look at the underlying morphisms between φ-modules (using Corollary 2.1.19). As φ-modules, each term E^1_{pq} is pure of weight $-q$ by Proposition 4.2.3 and (4.3.3). Thus, on the second page, each differential map $E^2_{pq} \to E^2_{p-2,q+1}$ is a morphism of φ-modules with different weights, and hence it vanishes. As a consequence, the ∞-term E^∞_{pq} is pure of weight $-q$, which is the p-th graded of the induced filtration on $H_{p+q}\mathbf{R\Gamma}^{(\varphi,G_K)}_{\mathrm{HK}}(M)$.

For last assertion, the G_K-action is clear. We need to prove for the monodromy operator. As M is compact, Proposition 3.2.13 shows that the motive M lies in a full subcategory $\mathbf{RigDA}_{\mathrm{gr}}(L)^{\mathrm{h}\,G_{L/K}}$ for some finite Galois extension L of K. As we proved, the fully faithful functor

$$\mathbf{RigDA}_{\mathrm{gr}}(L)^{\mathrm{h}\,G_{L/K}}_{\omega} \hookrightarrow \mathbf{RigDA}(K)_{\omega}$$

is weight-exact (Theorem 3.3.10 (4)). Therefore, the weight complex $W_\bullet M$ of M comes from the weight complex of M in $\mathbf{RigDA}_{\mathrm{gr}}(L)^{\mathrm{h}\,G_{L/K}}$. The proof of Lemma 4.3.2 also proves that the realization

$$\mathbf{R\Gamma}^{(\varphi,G_{L/K})}_{\mathrm{HK},L} \colon \mathbf{RigDA}_{\mathrm{gr}}(L)^{\mathrm{h}\,G_{L/K}}_{\omega} \to \mathcal{D}^{\mathrm{b}}_{(\varphi,G_{L/K})}(L_0)$$

factors through the weight complex of $w_{L/K}$ (Lemma 3.3.9). Therefore, the spectral sequence associated to M is the image of the spectral sequence associated to M in $\mathbf{RigDA}_{\mathrm{gr}}(L)^{\mathrm{h}\,G_{L/K}}$. It suffices to restrict to this full subcategory and prove the monodromy induces the desired map between graded pieces. The realization

$$\mathbf{R\Gamma}_{\mathrm{HK},L} \colon \mathbf{RigDA}_{\mathrm{gr}}(L)^{\mathrm{h}\,G_{L/K}} \to \mathcal{D}_{(\varphi,N,G_{L/K})}(L_0)$$

is obtained by applying the $G_{L/K}$-homotopy fixed points functor to the Hyodo–Kato realization on $\mathbf{RigDA}_{\mathrm{gr}}(L)$. Thus, we have a commutative diagram

$$\begin{tikzcd}
\mathbf{RigDA}_{\mathrm{gr}}(L)^{\mathrm{h}\,G_{L/K}}_{\omega} \arrow[r, "W^L_\bullet"] \arrow[d, "(-)_L"'] & \mathcal{K}^{\mathrm{b}}(\mathrm{h}\,\mathcal{H}_{L/K}) \arrow[r] \arrow[d] & \mathcal{D}^{\mathrm{b}}_{(\varphi,G_{L/K})}(L_0) \arrow[d] & \mathcal{D}^{\mathrm{b}}_{(\varphi,N,G_{L/K})}(L_0) \arrow[l] \arrow[d] \\
\mathbf{RigDA}_{\mathrm{gr}}(L)_{\omega} \arrow[r, "w^L_\bullet"] & \mathcal{K}^{\mathrm{b}}(\mathrm{h}\,\mathcal{H}^{\mathrm{chow}}_L) \arrow[r] & \mathcal{D}^{\mathrm{b}}_{\varphi}(L_0) & \mathcal{D}^{\mathrm{b}}_{(\varphi,N)}(L_0) \arrow[l]
\end{tikzcd}$$

where $W^L_\bullet$ is the weight complex functor with respect to $w_{L/K}$ and $w^L_\bullet$ is the weight complex functor with respect to the weight structure induced by the Chow weight structure (Corollary 3.3.6). Note that the weight spectral sequence of M on $\mathbf{RigDA}_{\mathrm{gr}}(L)^{\mathrm{h}\,G_{L/K}}$ is given by

$$E^1_{pq} = H_q\mathbf{R\Gamma}^{(\varphi,G_{L/K})}_{\mathrm{HK},L}(W^L_p M) \Rightarrow H_{p+q}\mathbf{R\Gamma}^{(\varphi,G_{L/K})}_{\mathrm{HK},L}(M).$$

We need to show the monodromy operator on $H_n\mathbf{R\Gamma}_{\mathrm{HK},L}(M)$ induces a map

$$\mathbf{gr}_i H_n\mathbf{R\Gamma}^{(\varphi,G_{L/K})}_{\mathrm{HK},L}(M) \to \mathbf{gr}_{i+2} H_n\mathbf{R\Gamma}^{(\varphi,G_{L/K})}_{\mathrm{HK},L}(M)(-1).$$

For this, it suffices to look at their underlying φ-modules. In other words, the spectral sequence becomes

$$E^1_{pq} = H_q\mathbf{R\Gamma}_{\mathrm{HK},L}(w^L_p(M_L)) \Rightarrow H_{p+q}\mathbf{R\Gamma}_{\mathrm{HK},L}(M_L)$$

which is exactly the weight spectral sequence associated to M_L studied in [BGV25, Example 4.35, 4.36], where it is already shown that the monodromy operator induces a map from the i-th graded piece to the $(i+2)$-th graded piece. ■

Remark 4.3.4. The spectral sequence in Proposition 4.3.3 is formulated homologically, but it can be translated into a cohomological version. Throughout this thesis we adopt the homological index convention, as it is more natural and convenient in the context of stable ∞-categories, where the pervasive use of homotopy-theoretic structures makes the homological perspective more suitable.

If we take M to be a motive associated to a smooth quasi-compact rigid analytic space over K, then we get the weight filtration on the Hyodo–Kato cohomology $H^i_{\mathrm{HK}}(X)$ by the fact that $H^i_{\mathrm{HK}}(X) = H_{-i}\mathbf{R}\Gamma^{(\varphi,G_K)}_{\mathrm{HK}}(M(X)^\vee)$ because we take duals to get the covariant cohomology realizations.

Corollary 4.3.5 (Weight Filtrations for Rigid Analytic Spaces)**.** *Let X be a smooth quasi-compact K-rigid analytic space. Then, for each i, the (overconvergent) Hyodo–Kato cohomology $H^i_{\mathrm{HK}}(X)$ admits a finite increasing filtration $\mathrm{Fil}^W_k H^i_{\mathrm{HK}}(X)$ stable under G_K-action whose k-th graded piece is pure of weight $i+k$, and the monodromy induces a map $\mathrm{gr}^W_k H^i_{\mathrm{HK}}(X) \to \mathrm{gr}^W_{k-2} H^i_{\mathrm{HK}}(X)$.*

Remark 4.3.6. The filtration in Corollary 4.3.5 is referred as the ***weight filtration*** of the Hyodo–Kato cohomology $H^i_{\mathrm{HK}}(X)$. In [BGV25], the authors constructed it for a special case where X admits semi-stable reduction, which can be regarded as the p-adic analytic analogue of the weight filtration defined by Rapoport-Zink in [RZ82] for ℓ-adic cohomology of algebraic varieties. In contrast, we do not assume the existence of formal models of X, and we prove that our weight filtration is stable under the Galois action.

This filtration is closely related to Deligne's weight-monodromy conjecture. Indeed, it satisfies the conditions for Deligne's weight filtration as formulated in [Del70; Del74] and partially satisfies those for the ***monodromy filtration*** in [Del74]. More precisely, the only missing piece is whether the monodromy operator induces an isomorphism

$$N^k\colon \mathrm{gr}^W_k H^i_{\mathrm{HK}}(X) \to \mathrm{gr}^W_{-k} H^i_{\mathrm{HK}}(X). \tag{4.3.4}$$

From this perspective, if the morphism in (4.3.3) is an isomorphism, then the p-adic weight-monodromy conjecture holds.

§ 4.4 Comparisons of Realization Functors

In the section, we compare the Hyodo–Kato realization with the de Rham–Fargues–Fontaine realization. Furthermore, we provide Galois refinements for both realizations and show that, in the refined setting, the comparison isomorphism is unique.

§ 4.4.1 Monoidal Comparisons of Realization Functors

We now consider motivic realization functors defined on $\mathbf{RigDA}(C)$. The first is the de Rham–Fargues–Fontaine realization (4.2.5), and the second is the Hyodo–Kato realization obtained by replacing K with C in (4.2.4). More precisely, we have the Hyodo–Kato realization functor:

$$\mathrm{R\Gamma}_{\mathrm{HK},C} \colon \mathbf{RigDA}(C) \simeq \mathbf{RigDA}_{\mathrm{gr}}(C) \to \mathcal{D}_{(\varphi,N)}(\breve{K}),$$

where the first equivalence on the left is given by [AGV22, Theorem 3.7.21.]. To compare these two realization functors, we enhance the Hyodo–Kato realization functor by the functor $\mathcal{E} \colon \mathcal{D}_{\varphi}(\breve{K}) \to \mathbf{QCoh}(\mathbf{FF})$ (see (4.1.3)), so that both functors land in the same target category. More precisely, this yields a functor in $\mathbf{CAlg}(\mathcal{P}\mathbf{r}^{\mathrm{st}}_{\omega})$

$$\mathcal{E}_N \colon \mathcal{D}_{(\varphi,N)}(\breve{K}) \to \mathcal{D}_{\varphi}(\breve{K}) \xrightarrow{\mathcal{E}} \mathbf{QCoh}(\mathbf{FF})$$

where the first functor is forgetting[1] the monodromy operator.

Theorem 4.4.1. *There is a unique (up to homotopy) functor $F \colon \mathbf{RigDA}(C) \to \mathbf{QCoh}(\mathbf{FF})$ in $\mathbf{CAlg}(\mathcal{P}\mathbf{r}^{\mathrm{st}}_{\omega})$ satisfying $F \circ \xi_C \simeq \mathcal{E} \circ \mathrm{R\Gamma}_{\mathrm{rig}}$. In particular, we have an equivalence $\mathrm{R\Gamma}_{\mathrm{FF}} \simeq \mathcal{E}_N \circ \mathrm{R\Gamma}_{\mathrm{HK}}$ in $\mathbf{CAlg}(\mathcal{P}\mathbf{r}^{\mathrm{st}}_{\omega})$.*

Proof. The Monsky–Washnitzer functor $\xi_C \colon \mathbf{DA}(\bar{k}) \to \mathbf{RigDA}(C)$ and the functor $\mathcal{E} \circ \mathrm{R\Gamma}_{\mathrm{rig}}$ define two objects $\mathbf{RigDA}(C)$ and $\mathbf{QCoh}(\mathbf{FF})$ in $\mathbf{CAlg}(\mathcal{P}\mathbf{r}^{\mathrm{st}}_{\omega})_{\mathbf{DA}(\bar{k})/-}$. To show the uniqueness of F, it suffices to compute

$$\pi_0 \mathrm{Map}_{\mathbf{CAlg}(\mathcal{P}\mathbf{r}^{\mathrm{st}}_{\omega})_{\mathbf{DA}(\bar{k})/-}} \left(\mathbf{RigDA}(C), \mathbf{QCoh}(\mathbf{FF})\right) \tag{4.4.1}$$

We now identify $\mathbf{RigDA}(C)$ with $\mathbf{Mod}_{\chi\mathbb{1}}(\mathbf{DA}(\bar{k}))$ and use the identifications

$$\chi\mathbb{1} \simeq \mathbb{1} \oplus \mathbb{1}(-1)[-1] \simeq S(\mathbb{1}(-1)[-1])$$

in Proposition 3.1.18 and Example 2.3.1. Applying Proposition 2.3.3, we know that (4.4.1) is computed by

$$\begin{aligned}
\pi_0 \mathrm{Map}_{\mathbf{QCoh}(\mathbf{FF})}(F(\mathbb{1}(-1))[-1], \mathcal{O}_{\mathbf{FF}}) &\simeq \pi_0 \mathrm{Map}_{\mathbf{QCoh}(\mathbf{FF})}(\mathcal{E} \circ \mathrm{R\Gamma}_{\mathrm{rig}}(\mathbb{1}(-1))[-1], \mathcal{O}_{\mathbf{FF}}) \\
&\simeq \pi_0 \mathrm{Map}_{\mathbf{QCoh}(\mathbf{FF})}(\mathcal{O}_{\mathbf{FF}}(-1)[-1], \mathcal{O}_{\mathbf{FF}}) \\
&\simeq \mathrm{Ext}^1(\mathcal{O}_{\mathbf{FF}}(-1), \mathcal{O}_{\mathbf{FF}}) \simeq 0,
\end{aligned}$$

where the last equivalence is Proposition 4.1.27.

For the last assertion, it follows from the compatibilities of these two functors with the rigid realization; see Proposition 4.2.3 and 4.2.9. ∎

[1] In fact, the vector bundles associated to (φ, N)-modules only depend on the underlying φ-modules, see [FF18, Proposition 10.3.3].

Remark 4.4.2. In defining the Hyodo–Kato realization $\mathrm{R}\Gamma_{\mathrm{HK},C}$, we have chosen the canonical pseudo-uniformizer p of $\mathcal{O}_C$. In fact, different choices of pseudo-uniformizer only change the monodromy operators. However, after applying the $\mathcal{E}$ functor, they yield the same solid quasi-coherent sheaves on **FF**. Therefore, the comparison result in Theorem 4.4.1 is independent of the choice of pseudo-uniformizer.

Remark 4.4.3 (Space of Comparison Isomorphisms)**.** The previous theorem shows that we can find a monoidal natural equivalence between the de Rham–Fargues–Fontaine realization and the Hyodo–Kato realization. But this natural equivalence is not unique. Indeed, there is a big space of choices for the monoidal natural transformations:

$$\pi_1\mathrm{Map}_{\mathbf{CAlg}(\mathcal{P}\mathbf{r}^{\mathrm{st}}_{\omega})_{\mathbf{DA}(\bar{k})/-}}(\mathbf{RigDA}(C),\mathbf{QCoh}(\mathbf{FF}))\simeq\pi_0\mathrm{Map}(\mathcal{O}_{\mathbf{FF}}(-1),\mathcal{O}_{\mathbf{FF}})\simeq B^{\varphi=p}.$$

A New Filtration for Vector Bundles over FF

As an application of this comparison result, we can get a new filtration on the vector bundles arose as the de Rham–Fargues–Fontaine cohomology.

As in the previous section, we use the bounded weight structure w on $\mathbf{RigDA}(C)_\omega\simeq\mathbf{RigDA}_{\mathrm{gr}}(C)_\omega$ in Theorem 3.3.10, whose heart will be denoted by $\mathcal{H}_C$. Then we have the weight complex functor

$$W\colon\mathbf{RigDA}(C)_\omega\to\mathcal{K}^{\mathrm{b}}(\mathrm{h}\,\mathcal{H}).$$

As a consequence of Theorem 4.4.1, we know that the de Rham–Fargues–Fontaine realization factors through the weight complex functor on $\mathbf{RigDA}(C)$:

Corollary 4.4.4. *The de Rham–Fargues–Fontaine realization on the compact motives over C factors through the weight complex functor. In other words, we have a commutative diagram*

$$\begin{tikzcd}
\mathbf{RigDA}(C)_\omega \arrow[r, "\mathrm{R}\Gamma_{\mathrm{FF}}"] \arrow[d, "W"'] & \mathbf{Perf}(\mathbf{FF}) \\
\mathcal{K}^{\mathrm{b}}(\mathrm{h}\,\mathcal{H}) \arrow[ur, dashed, "\widetilde{\mathrm{R}\Gamma}_{\mathrm{FF}}"'] &
\end{tikzcd}$$

in $\mathbf{CAlg}(\mathcal{C}\mathbf{at}^{\mathrm{ex}}_\infty)$.

Proof. By [BGV25, Theorem 4.53, Corollary 4.54], we know that the functor

$$\mathbf{RigDA}(C)_\omega\xrightarrow{\mathrm{R}\Gamma_{\mathrm{HK}}}\mathcal{D}^{\mathrm{b}}_{(\varphi,N)}(\breve{K})\to\mathcal{D}^{\mathrm{b}}_{\varphi}(\breve{K})$$

has already factored through the weight complex functor. So we conclude from the comparison in . ∎

Corollary 4.4.5. *For every compact rigid analytic motive M over C, there is a convergent spectral sequence starting from the first page and degenerating at the second page:*

$$E^1_{pq} = H_q\mathbf{R\Gamma}_{\mathrm{FF}}(W_pM) \Rightarrow H_{p+q}\mathbf{R\Gamma}_{\mathrm{FF}}(M)$$

where $W_\bullet M$ is the weight complex of M. In particular, $H_n\mathbf{R\Gamma}_{\mathrm{FF}}(M)$ has a finite increasing filtration $\mathrm{Fil}_\bullet$ *whose i-th graded piece is of slope $(i-n)/2$.*

Proof. This spectral sequence is a special case of [HA, Proposition 1.2.2.14] applied to the naive filtration of the weight complex $W_\bullet M$ by truncations. We are left to show it degenerates at the second page and the slopes of E^2_{pq}.

From Corollary 3.3.6, each W_pM has a form of $\xi_C Y_p$ where Y_p is a Chow motive over $\bar{k}$. Using Proposition 4.2.9, we know that

$$H_q\mathbf{R\Gamma}_{\mathrm{FF}}(W_pM) \simeq \mathcal{E}(H_q\mathbf{R\Gamma}_{\mathrm{rig}}(Y_p))$$

where $H_q\mathbf{R\Gamma}_{\mathrm{rig}}(Y_p)$ is of weight $-q$ by Proposition 4.2.3. Thus, $H_q\mathbf{R\Gamma}_{\mathrm{FF}}(W_pM)$ has the slope of $-q/2$ by Proposition 4.1.13 since $\mathcal{E}$ preserves the slopes by our definition of Tate twists of φ-modules. Therefore, E^2_{pq} is also of slope $-q/2$; hence the differential map $E^2_{pq} \to E^2_{p-2,q+1}$ vanishes by Proposition 4.1.27. This proves that the spectral sequence degenerates at the second page. In particular, the i-th graded piece of the induced filtration on $H_n\mathbf{R\Gamma}_{\mathrm{FF}}(M)$ is $E^\infty_{i,n-i} \simeq E^2_{i,n-i}$, whose slope is $(i-n)/2$. ∎

Given a motive M over C, we put $\mathcal{H}^i_{\mathrm{FF}}(M) := H_{-i}\mathbf{R\Gamma}_{\mathrm{FF}}(M^\vee)$. In the special case where M is the associated motive of a smooth quasi-compact rigid analytic space over C, we write it simply by $\mathcal{H}^i_{\mathrm{FF}}(X)$.

Corollary 4.4.6. *Let X be a smooth quasi-compact rigid analytic variety over C. Then, for each $n \geq 0$, the vector bundle $\mathcal{H}^n_{\mathrm{FF}}(X)$ has a finite increasing filtration* $\mathrm{Fil}_\bullet$ *whose i-th graded piece is of slope $(i+n)/2$.*

Remark 4.4.7. The filtration in Corollary 4.4.6 is a new filtration on the vector bundles on the Fargues–Fontaine curve, different from the Harder-Narasimhan filtration as slopes are increasing. Even more, the set of slopes with respect to this new filtration does not agree with the set of Harder-Narasimhan slopes.

§ 4.4.2 The Uniqueness of Comparison Equivalences

As mentioned in Remark 4.4.3, the comparison natural isomorphism

$$\mathbf{R\Gamma}_{\mathrm{FF}} \simeq \mathcal{E}_N \circ \mathbf{R\Gamma}_{\mathrm{HK}}$$

is not unique, and the possible choices form a big space. To obtain a unique good comparison isomorphism, we keep track of the Galois actions. More precisely, we consider the ∞-category of $G_{\breve{K}}$-equivariant solid quasi-coherent sheaves on the Fargues–Fontaine curve; here $G_{\breve{K}} = \mathrm{Gal}(C/\breve{K})$ is the absolute Galois group of $\breve{K}$.

We firstly define this ∞-category as follows: the action of $G_{\breve{K}}$ on $A_{\mathrm{inf}} = W(\mathcal{O}_{C^\flat})$ induces an action on the Fargues–Fontaine **FF** by its construction. Thus, pullbacks along this Galois automorphisms give a $G_{\breve{K}}$-action on **FF**, which induces a $G_{\breve{K}}$-action on the ∞-category $\mathbf{QCoh}(\mathbf{FF})$. In other words, we have a functor

$$\begin{aligned} B_\bullet G_{\breve{K}} &\to \mathbf{CAlg}(\mathcal{P}\mathbf{r}^{\mathrm{st}}_\omega) \\ * &\mapsto \mathbf{QCoh}(\mathbf{FF}) \\ (g \in G_{\breve{K}}) &\mapsto (\sigma_g^*\colon \mathbf{QCoh}(\mathbf{FF}) \to \mathbf{QCoh}(\mathbf{FF})). \end{aligned} \tag{4.4.2}$$

We then define the ∞-category of ***$G_{\breve{K}}$-equivariant solid quasi-coherent sheaves*** on **FF** as the limit of the diagram above.

Next, we explain how to enhance the target of $\mathrm{R}\Gamma_{\mathrm{FF}}$ and $\mathcal{E}_n \circ \mathrm{R}\Gamma_{\mathrm{HK}}$.

From φ-modules to $\mathbf{QCoh}(\mathbf{FF})^{\mathrm{h}G_{\breve{K}}}$

We can enrich the target of $\mathcal{E}$ defined in (4.1.3): the definition of the morphism e defined in (4.1.2) shows e is stable under $G_{\breve{K}}$, and the Frobenius map is compatible with $G_{\breve{K}}$-action; that is, for every $g \in G_{\breve{K}}$, we have $e \circ \sigma_g = e$ and $\varphi \circ \sigma_g = \sigma_g \circ \varphi$. Therefore, this implies that, for each $g \in G_{\breve{K}}$, there is a functor $\sigma_g^*\colon \mathbf{Perf}(\mathcal{Y}_{(0,\infty)})^{\varphi} \to \mathbf{Perf}(\mathcal{Y}_{(0,\infty)})^{\varphi}$ and $\mathcal{E}$ is stable under these functors, i.e., $\sigma_g^* \circ \mathcal{E} \simeq \sigma_g^*$. Thus, we get a refinement of $\mathcal{E}$ in $\mathbf{CAlg}(\mathcal{P}\mathbf{r}^{\mathrm{st}}_\omega)$

$$\mathcal{E}^{\mathrm{ari}}\colon \mathcal{D}_\varphi(\breve{K}) \to \mathbf{QCoh}(\mathbf{FF})^{\mathrm{h}G_{\breve{K}}} \tag{4.4.3}$$

Clearly, the underlying solid quasi-coherent sheaves of $\mathcal{E}^{\mathrm{ari}}$ are given by the functor $\mathcal{E}$. As before, we define $\mathcal{E}^{\mathrm{ari}}_N := \mathcal{E}^{\mathrm{ari}} \circ \pi$.

The Refinement of the de Rham–Fargues–Fontaine Realization

To get a Galois refinement of the de Rham–Fargues–Fontaine realization, we have to enhance each rigid analytic motive over C into a $G_{\breve{K}}$-equivariant rigid analytic motive over C. To this end, we start from the natural functor

$$\mathbf{RigDA}_{\mathrm{gr}}(\breve{K}) \to \mathbf{RigDA}(C)^{\mathrm{h}G_{\breve{K}}} \tag{4.4.4}$$

induced by the base change functor $\mathbf{RigDA}_{\mathrm{gr}}(K) \to \mathbf{RigDA}(C)$, here $\mathbf{RigDA}(C)^{\mathrm{h}G_{\breve{K}}}$ is the limit of the $G_{\breve{K}}$-action on $\mathbf{RigDA}(C)$. More precisely, the action of $G_{\breve{K}}$ on C yields a $G_{\breve{K}}$-action on $\mathbf{RigDA}(C)$ that is a functor

$$B_\bullet G_{\breve{K}} \to \mathbf{CAlg}(\mathcal{P}\mathbf{r}^{\mathrm{st}}_\omega)$$

sending the point to $\mathbf{RigDA}(C)$, similarly to (4.4.2), and $\mathbf{RigDA}(C)^{\mathrm{h}\,G_{\breve{K}}}$ is the limit of this diagram. Since the base change $\mathbf{RigDA}_{\mathrm{gr}}(\breve{K}) \to \mathbf{RigDA}(C)$ is stable under this $G_{\breve{K}}$-action, we obtain (4.4.4).

On the other hand, since $C/\breve{K}$ is totally ramified, the base change $\mathbf{RigDA}_{\mathrm{gr}}(\breve{K}) \xrightarrow{\simeq} \mathbf{RigDA}_{\mathrm{gr}}(C)$ is an equivalence by [BKV25, Proposition 3.23], and the latter category is equivalent to $\mathbf{RigDA}(C)$ via the natural embedding, as shown in [AGV22, Theorem 3.7.21]. Thus, the composite functor $\mathbf{RigDA}_{\mathrm{gr}}(K) \xrightarrow{\simeq} \mathbf{RigDA}(C)$ is indeed an equivalence of ∞-categories. Replacing $\mathbf{RigDA}_{\mathrm{gr}}(\breve{K})$ by $\mathbf{RigDA}(C)$ via this equivalence, we get a Galois-enhanced functor

$$\alpha\colon \mathbf{RigDA}(C) \to \mathbf{RigDA}(C)^{\mathrm{h}\,G_{\breve{K}}}. \tag{4.4.5}$$

Remark 4.4.8. The intuition of (4.4.5) is that, for every rigid analytic motive M over C, we can find a unique (up to homotopy) model $\tilde{M}$ in $\mathbf{RigDA}_{\mathrm{gr}}(K)$; that is, we have an isomorphism $M \simeq \tilde{M}_C$ which gives the natural $G_{\breve{K}}$-action on M.

More generally, let F/L be a Galois extension of complete non-archimedean fields with residue fields k_F and k_L respectively. Let $I_{F/L}$ be the inertial group of this extension. Using the same argument, we can obtain an inertial enrichment:

$$\mathbf{RigDA}_{\mathrm{gr}}(F) \to \mathbf{RigDA}_{\mathrm{gr}}(F)^{\mathrm{h}\,I_{F/L}}.$$

Here the $G_{F/L}$-action (hence $I_{F/L}$-action) on $\mathbf{RigDA}(F)$ can be restricted to $\mathbf{RigDA}_{\mathrm{gr}}(F)$ due to [AGV22, Proposition 3.1.13].

Lemma 4.4.9. *The composite functor* $\mathbf{RigDA}(C) \xrightarrow{\alpha} \mathbf{RigDA}(C)^{\mathrm{h}\,G_{\breve{K}}} \to \mathbf{RigDA}(C)$*, where* α *is defined in (4.4.5) and the second map is the canonical projection, is the identity functor (up to homotopy).*

Proof. This follows from the commutativity of the following diagram

$$\begin{tikzcd}
\mathbf{RigDA}_{\mathrm{gr}}(\breve{K}) \arrow[r] \arrow[d, "\simeq"'] \arrow[dr] & \mathbf{RigDA}(C)^{\mathrm{h}\,G_{\breve{K}}} \arrow[d] \\
\mathbf{RigDA}(C) \arrow[r, equal] & \mathbf{RigDA}(C)
\end{tikzcd}.$$

∎

Note that the de Rham–Fargues–Fontaine realization

$$\mathrm{R\Gamma}_{\mathrm{FF}}\colon \mathbf{RigDA}(C) \xrightarrow{\mathcal{D}_0} \mathbf{RigDA}(\mathcal{Y}_{(0,\infty)})^{\varphi_\omega} \to \mathbf{QCoh}(\mathbf{FF})$$

is compatible with $G_{\breve{K}}$-actions. Then taking $G_{\breve{K}}$-homotopy fixed points, we get an enhanced realization functor:

$$\mathrm{R\Gamma}^{\mathrm{ari}}_{\mathrm{FF}}\colon \mathbf{RigDA}(C) \xrightarrow{\alpha} \mathbf{RigDA}(C)^{\mathrm{h}\,G_{\breve{K}}} \xrightarrow{\mathrm{R\Gamma}_{\mathrm{FF}}^{\mathrm{h}\,G_{\breve{K}}}} \mathbf{QCoh}(\mathbf{FF})^{\mathrm{h}\,G_{\breve{K}}} \tag{4.4.6}$$

In light of Lemma 4.4.9, the new realization functor $\mathbf{R\Gamma}^{\mathrm{ari}}_{\mathbf{FF}}$ computes the de Rham–Fargues–Fontaine realization of motives over C, and it also can be compared with the Galois-enriched rigid realization:

Theorem 4.4.10. (1) *There is a unique (up to a unique natural isomorphism) functor*

$$F\colon \mathbf{RigDA}(C) \to \mathbf{QCoh}(\mathbf{FF})$$

in $\mathbf{CAlg}(\mathcal{P}\mathbf{r}^{\mathrm{st}}_{\omega})$ *such that* $F \circ \xi_C \simeq \mathcal{E}^{\mathrm{ari}} \circ \mathbf{R\Gamma}_{\mathrm{rig}}$*, where* $\mathcal{E}^{\mathrm{ari}}$ *is defined in (4.4.3).*

(2) *There are canonical monoidal equivalences*

$$\mathcal{E}^{\mathrm{ari}}_{N} \circ \mathbf{R\Gamma}_{\mathrm{HK}} \circ \xi_C \simeq \mathcal{E}^{\mathrm{ari}} \circ \mathbf{R\Gamma}_{\mathrm{rig}}$$
$$\mathbf{R\Gamma}^{\mathrm{ari}}_{\mathbf{FF}} \simeq \mathcal{E}^{\mathrm{ari}} \circ \mathbf{R\Gamma}_{\mathrm{rig}}.$$

In particular, $\mathbf{R\Gamma}^{\mathrm{ari}}_{\mathbf{FF}}$ *and* $\mathcal{E}^{\mathrm{ari}}_{N} \circ \mathbf{R\Gamma}_{\mathrm{HK}}$ *are two objects in* $\mathbf{CAlg}(\mathcal{P}\mathbf{r}^{\mathrm{st}}_{\omega})_{\mathbf{DA}(\bar{k})/-}$ *via these monoidal equivalences.*

(3) *There is a unique monoidal natural isomorphism* $\mathbf{R\Gamma}^{\mathrm{ari}}_{\mathbf{FF}} \simeq \mathcal{E}^{\mathrm{ari}}_{N} \circ \mathbf{R\Gamma}_{\mathrm{HK}}$ *in* $\mathbf{CAlg}(\mathcal{P}\mathbf{r}^{\mathrm{st}}_{\omega})_{\mathbf{DA}(\bar{k})/-}$.

Proof. (1) The proof is similar to Theorem 4.4.1: we have a homotopy equivalence

$$\mathrm{Map}_{\mathbf{CAlg}(\mathcal{P}\mathbf{r}^{\mathrm{st}}_{\omega})_{\mathbf{DA}(\bar{k})/-}}(\mathbf{RigDA}(C), \mathbf{QCoh}(\mathbf{FF})^{\mathrm{h}G_{\breve{K}}}) \simeq \mathrm{Map}_{\mathbf{QCoh}(\mathbf{FF})^{\mathrm{h}G_{\breve{K}}}}(\mathcal{O}_{\mathbf{FF}}(-1)[-1], \mathcal{O}_{\mathbf{FF}}).$$

Since $\mathbf{QCoh}(\mathbf{FF})$ is $\mathbb{Q}$-linear, the space is connected, and the fundamental group of this space is $(B^{G_{\breve{K}}})^{\varphi=p}$ (see also Remark 4.4.3), which is 0 by [FF18, Corollaire 10.2.8].

(2) After identifying $\mathbf{RigDA}(C)$ with $\mathbf{RigDA}_{\mathrm{gr}}(\breve{K})$, Theorem 4.4.1 shows we have a commutative diagram

$$\begin{tikzcd}
\mathbf{RigDA}_{\mathrm{gr}}(\breve{K}) \arrow[r, "\simeq"] \arrow[d, "\pi\circ\mathbf{R\Gamma}_{\mathrm{HK},\breve{K}}"'] & \mathbf{RigDA}(C) \arrow[d, "\mathbf{R\Gamma}_{\mathbf{FF}}"] \\
\mathcal{D}_{\varphi}(\breve{K}) \arrow[r, "\mathcal{E}"'] & \mathbf{QCoh}(\mathbf{FF})
\end{tikzcd}$$

The right vertical functor is compatible with the $G_{\breve{K}}$-action, and the action fixes the left vertical functor; thus, this gives an equivalence $\mathbf{R\Gamma}^{\mathrm{ari}}_{\mathbf{FF}} \simeq \mathcal{E}^{\mathrm{ari}} \circ \pi \circ \mathbf{R\Gamma}_{\mathrm{HK}}$ in $\mathbf{CAlg}(\mathcal{P}\mathbf{r}^{\mathrm{st}}_{\omega})$. By the definition, we clearly have

$$\mathcal{E}^{\mathrm{ari}}_{N} \circ \mathbf{R\Gamma}_{\mathrm{HK}} \circ \xi_C \simeq \mathcal{E}^{\mathrm{ari}} \circ \mathbf{R\Gamma}_{\mathrm{rig}}.$$

(3) This is a direct consequence of (1) and (2).

■

Remark 4.4.11. As we seen, the monodromy operator of Hyodo–Kato realization played no role throughout: we assign a solid quasi-coherent sheaves on the Fargues–Fontaine curve via the underlying φ-modules. In [FF18, §10.3], Fargues and Fontaine gave a direct construction

$$\tilde{\mathcal{E}}_N : \mathcal{D}_{(\varphi,N)}(\breve{K}) \to \mathbf{QCoh}(\mathbf{FF})^{\mathrm{h}G_{\breve{K}}} \tag{4.4.7}$$

with a natural isomorphism $\mathrm{pr}\circ\mathcal{E}^{\mathrm{ari}}\circ\pi \xrightarrow{\simeq} \mathrm{pr}\circ\mathcal{E}^{\mathrm{ari}}_N$, where $\mathrm{pr}\colon \mathbf{QCoh}(\mathbf{FF})^{\mathrm{h}G_{\breve{K}}} \to \mathbf{QCoh}(\mathbf{FF})$ is the canonical projection; see [FF18, §10.3.2, (4)]. However, this natural isomorphism is not compatible with the base change of $\mathbf{QCoh}(\mathbf{FF})$ along the $G_{\breve{K}}$-action we defined here.

Remark 4.4.12. Using the Galois-enriched comparison in Theorem 4.4.10, we can also know it factors through the weight complex functor factors. So we have a refinement of Corollary 4.4.5; in other words, the finite filtration is also compatible with the $G_{\breve{K}}$-action.

§ 4.4.3 The Fargues–Fontaine Cohomology via the Décalage Functor

There is another Fargues–Fontaine cohomology, defined via the Décalage functor, was originally defined by Le Bras in [LB18] and later developed by Bosco using condensed mathematics in [Bos23b].

In the final part of this section, we show that this cohomology is motivic—that is, it extends to a functor on $\mathbf{RigDA}(C)$. Le Bras previously showed that it is defined on the ∞-category of effective motives $\mathbf{RigDA}^{\mathrm{eff}}(C)$ in [LB18].

We firstly give a quick review of the definition of Fargues–Fontaine cohomology defined in [LB18; Bos23b].

Let I be a compact sub-interval of $(0,\infty)$ with rational endpoints. Then we have a $\mathbb{Q}_p$-Banach space B_I, the global section of the open subspace $\mathcal{Y}_I$ of $\mathbf{FF}$. For every smooth rigid analytic variety (resp. dagger variety) X over C, Bosco defined in [Bos23b, Definition 2.38, Proposition 2.42] a solid $(B_I,\mathbb{Z})$-module $\mathrm{R}\Gamma_{B_I}(X)$, in the sense of Definition 4.1.20, via the décalage functor; see also [Bos23b, Definition 6.12, Remark 6.13]. This is called the ***B_I-cohomology*** of X.

Proposition 4.4.13. *For every compact sub-interval $I \subseteq (0,\infty)$, the B_I-cohomology of dagger varieties over C extends to a functor*

$$\mathrm{R}\Gamma_{B_I}(-)\colon \mathbf{RigDA}(C) \simeq \mathbf{RigDA}^{\dagger}(C) \to \mathcal{D}_{\square}(B_I,\mathbb{Z})^{\mathrm{op}},$$

where $\mathbf{RigDA}^{\dagger}(C)$ is the ∞-category of dagger motives over C, as defined in [Vez18; LBV23], and the first equivalence is given by [Vez18, Theorem 4.23].

Proof. Let $\mathbf{FSch}^{\dagger,\mathrm{ss}}_{\mathcal{O}_C}$ denote the category of semi-stable weak formal schemes over $\mathcal{O}_C$, as in [CN20, §2.3.1]. Then for every $\mathfrak{X} \in \mathbf{FSch}^{\dagger,\mathrm{ss}}_{\mathcal{O}_C}$, by [Bos23b, (4.50) together with Remark 6.13], there is a natural isomorphism

$$\mathrm{R}\Gamma_{B_I}(\mathfrak{X}^{\mathrm{rig}\dagger}) \simeq \left(\mathrm{R}\Gamma_{\mathrm{HK}}(\mathfrak{X}^{\mathrm{rig}\dagger}) \otimes^{\square}_{\breve{K}} B_{\log,I}\right)^{N=0}, \tag{4.4.8}$$

where $B_{\log,I}$ is the condensed period ring defined in [Bos23b, Definition 2.27] by taking $X = \mathrm{Spa}(C)$ and $\mathfrak{X}^{\mathrm{rig}\dagger}$ is the dagger generic fiber of $\mathfrak{X}$; see [LM13]. We will deduce the B_I-cohomology

$$\mathfrak{X} \mapsto \mathbf{R\Gamma}_{B_I}(\mathfrak{X}^{\mathrm{rig}\dagger})$$

has the rig-étale descent and $\mathbb{A}^1$-invariance from the comparison (4.4.8). Indeed, as shown in [BKV25, Proposition 3.8], the overconvergent Hyodo–Kato cohomology

$$\mathfrak{X} \mapsto \mathbf{R\Gamma}_{\mathrm{HK}}(\mathfrak{X}^{\mathrm{rig}\dagger})$$

has the rig-étale descent and $\mathbb{A}^1$-invariance. Thus, the $\mathbb{A}^1$-invariance is immediate. For the rig-étale descent, we need to show $(-\otimes^{\square}_{\breve{K}} B_{\log,I})^{N=0}$ commutes with descent limits. This can be deduced from [Bos23a, Corollary A.67 (ii)] together with [Bos23b, (3.16)], where the assumptions hold in our case due to the finiteness of overconvergent Hyodo–Kato cohomology and [Bos23a, Corollary A.50].

It remains to prove the T-stability: it suffices to compute $H^1_{B_I}(\mathbb{G}^{\dagger}_m)$. Using (4.4.8) and [Bos23b, Lemma 7.6]—where finiteness of overconvergent Hyodo–Kato cohomology plays a role—we know that

$$H^1_{B_I}(\mathbb{G}^{\dagger}_m) \simeq H^1_{\mathrm{HK}}(\mathbb{G}_{m,\bar{k}^0}) \otimes_{\breve{K}} B_I \simeq B_I,$$

is invertible in $\mathcal{D}_{\square}(B_I, \mathbb{Z})$.

Finally, as in [BKV25, Proposition 3.8], we can apply the dagger analogue[1] of [BKV25, Proposition 2.32] to conclude the result. ■

By the comparison (4.4.8) and the boundedness, see [Bos23b, Theorem 3.14 (ii)], of the Hyodo–Kato cohomology, we know that the motivic realization in Proposition 4.4.13 restricts to

$$\mathbf{R\Gamma}_{B_I} : \mathbf{RigDA}(C)_{\omega} \to \mathcal{D}^{\mathrm{b}}_{\square}(B_I, \mathbb{Z})^{\mathrm{op}}.$$

Therefore, as before, we get the covariant version of this motivic realization

$$\mathbf{R\Gamma}_{B_I} : \mathbf{RigDA}(C) \to \mathcal{D}_{\square}(B_I, \mathbb{Z})$$

in $\mathbf{CAlg}(\mathcal{P}\mathbf{r}^{\mathrm{L}}_{\omega})$ by taking duals.

To get the new Fargues–Fontaine cohomology from the B_I-cohomology, we need a new $\mathcal{E}$-functor rather than (4.1.3); see also constructions in [FF18, Chapter 10] and [FF14, §7.6].

We recall this construction from [LB18, §5] and [Bos23b, §6.2]. Using the functor in Proposition 4.1.19, for every compact sub-interval $I \subseteq (0, \infty)$ with rational endpoints, we have a morphism of analytic rings

$$(B_I, \mathbb{Z})_{\square} \to (B_I, B_I^{+})_{\square}.$$

[1]The proof there also holds thanks to [CN20, Proposition 2.13].

This yields a functor in $\mathbf{CAlg}(\mathcal{P}\mathbf{r}^{\mathrm{L}}_{\omega})$

$$\mathbf{CoAd}(\mathbf{Solid}_B) := \lim_{I\subseteq(0,\infty)} \mathcal{D}((B_I,\mathbb{Z})_{\square}) \to \lim_{I\subseteq(0,\infty)} \mathbf{QCoh}(\mathcal{Y}_I) \simeq \mathbf{QCoh}(\mathcal{Y}_{(0,\infty)})$$

where the source category is called the ∞-category of ***coadmissible solid modules*** over B and the last equivalence is the analytic descent of solid quasi-coherent sheaves. On the other hand, there is a Frobenius autofunctor on $\mathbf{CoAd}(\mathbf{Solid}_B)$ induced by morphisms

$$\varphi_I\colon (B_I,\mathbb{Z})_{\square} \to (B_{pI},\mathbb{Z})_{\square},$$

which are compatible with the Frobenius map on $\mathbf{QCoh}(\mathcal{Y}_{(0,\infty)})$. Thus, taking the Frobenius invariant, we obtain a functor taking values in solid quasi-coherent sheaves on the Fargues–Fontaine:

$$\mathcal{E}_B\colon \mathbf{CoAd}(\mathbf{Mod}^{\square}_B)^{\varphi_\omega} \to \mathbf{QCoh}(\mathcal{Y}_{(0,\infty)})^{\varphi_\omega} \simeq \mathbf{QCoh}(\mathbf{FF}) \tag{4.4.9}$$

in $\mathbf{CAlg}(\mathcal{P}\mathbf{r}^{\mathrm{st}}_{\omega})$, which preserves nuclear and perfect complexes ([Bos23b, Proposition 6.8]).

As shown in [Bos23b, Lemma 6.14], the family of motivic realization functors (restricted to compact parts) $\mathrm{R}\boldsymbol{\Gamma}_{B_I}$ are compatible with inclusions and Frobenius maps between B_I's. This leads to a realization functor

$$\mathbf{RigDA}(C) \to \mathbf{CoAd}(\mathbf{Solid}_B)^{\varphi_\omega}.$$

F_* is induced by composing with F, and F^* sends $\mathcal{C}^{\otimes} \to \mathcal{E}^{\otimes}$ to $\mathcal{D}^{\otimes} \otimes_{\mathcal{C}^{\otimes}} \mathcal{E}^{\otimes}$. The composition of it and $\mathcal{E}_B$ in (4.4.9) gives a new ***motivic Fargues–Fontaine cohomology***::

$$\widetilde{\mathrm{R}\Gamma}_{\mathbf{FF}}\colon \mathbf{RigDA}(C) \to \mathbf{QCoh}(\mathbf{FF}) \tag{4.4.10}$$

which is in $\mathbf{CAlg}(\mathcal{P}\mathbf{r}^{\mathrm{L}}_{\omega})$ since it sends compact motives into perfect complexes on the Fargues–Fontaine curve by [Bos23b, Theorem 6.17 (i)].

This motivic Fargues–Fontaine cohomology (4.4.10) agrees (non-canonically) with the de Rham–Fargues–Fontaine cohomology by their comparison with the Hyodo–Kato cohomology:

Proposition 4.4.14 (Comparison of Fargues–Fontaine Cohomology Theories)**.** *There is a monoidal equivalence* $\mathrm{R}\boldsymbol{\Gamma}_{\mathbf{FF}} \simeq \widetilde{\mathrm{R}\Gamma}_{\mathbf{FF}}$ *in* $\mathbf{CAlg}(\mathcal{P}\mathbf{r}^{\mathrm{st}}_{\omega})$.

Proof. By [Bos23b, Theorem 6.17], there is a natural isomorphism $\widetilde{\mathrm{R}\Gamma}_{\mathbf{FF}} \simeq \mathcal{E}_N \circ \mathrm{R}\boldsymbol{\Gamma}_{\mathrm{HK}}$. Therefore, we conclude from the comparison in Theorem 4.4.1. ∎

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