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Nonlinear Eigenvalue Problems and Bifurcation for Quasi-Linear Elliptic Operators

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Abstract. In this paper, we analyze an eigenvalue problem for quasi-linear elliptic operators involving homogeneous Dirichlet boundary conditions in a open smooth bounded domain. We show that the eigenfunctions corresponding to the eigenvalues belong to L^∞, which implies $C^{1,\alpha}$ smoothness, and the first eigenvalue is simple. Moreover, we investigate the bifurcation results from trivial solutions using the Krasnoselski bifurcation theorem and from infinity using the Leray–Schauder degree. We also show the existence of multiple critical points using variational methods and the Krasnoselski genus.

Mathematics Subject Classification. 35J20, 35J92, 35J25.

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Contents

Birkhäuser

1. Introduction

We consider $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ an open bounded domain with smooth boundary $\partial\Omega$. A classical result in the theory of eigenvalue problems guarantees that the problem

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

possesses a nondecreasing sequence of eigenvalues and a sequence of corresponding eigenfunctions which define a Hilbert basis in $L^2(\Omega)$ (see, [16]). Moreover, it is known that the first eigenvalue of problem (1.1) is characterized in the variational point of view by

$$\lambda_1^D := \inf_{u \in W_0^{1,2}(\Omega)\setminus\{0\}} \left\{ \frac{\int_\Omega |\nabla u|^2 \, \mathrm{d}x}{\int_\Omega u^2 \, \mathrm{d}x} \right\}.$$

Suppose that $p > 1$ is a given real number and consider the nonlinear eigenvalue problem with Neumann boundary condition

$$\begin{cases} -\Delta_p u = \lambda u & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ stands for the p-Laplace operator and $\lambda \in \mathbb{R}$. This problem was considered in [15], and using a direct method in calculus of variations (if $p > 2$) or a mountain-pass argument (if $p \in (\frac{2N}{N+2}, 2)$) it was shown that the set of eigenvalues of problem (1.2) is exactly the interval $[0, \infty)$. Indeed, it is sufficient to find one positive eigenvalue, say $-\Delta_p u = \lambda u$. Then a continuous family of eigenvalues can be found by the reparametrization $u = \alpha v$, satisfying $-\Delta_p v = \mu(\alpha) v$, with $\mu(\alpha) = \frac{\lambda}{\alpha^{p-2}}$.

In this paper, we consider the so-called $(p, 2)$-Laplace operator (see, [18]) with Dirichlet boundary conditions. More precisely, we analyze the following nonlinear eigenvalue problem:

$$\begin{cases} -\Delta_p u - \Delta u = \lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where $p \in (1, \infty)\setminus\{2\}$ is a real number. We recall that if $1 < p < q$, then $L^q(\Omega) \subset L^p(\Omega)$ and as a consequence, one has $W_0^{1,q}(\Omega) \subset W_0^{1,p}(\Omega)$. We will say that $\lambda \in \mathbb{R}$ is an eigenvalue of problem (1.3) if there exists $u \in W_0^{1,p}(\Omega)\setminus\{0\}$ (if $p > 2$), $u \in W_0^{1,2}(\Omega)\setminus\{0\}$ (if $1 < p < 2$) such that

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla v \, \mathrm{d}x + \int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}x = \lambda \int_\Omega u \, v \, \mathrm{d}x, \tag{1.4}$$

for all $v \in W_0^{1,p}(\Omega)$ (if $p > 2$), $v \in W_0^{1,2}(\Omega)$ (if $1 < p < 2$). In this case, such a pair (u, λ) is called an Eigenpair, and $\lambda \in \mathbb{R}$ is called an eigenvalue and $u \in W_0^{1,p}(\Omega)\setminus\{0\}$ is an eigenfunction associated with λ. We say that λ is a "first eigenvalue", if the corresponding eigenfunction u is positive or negative.

The operator $-\Delta_p - \Delta$ appears in quantum field theory (see, [5]), where it arises in the mathematical description of propagation phenomena of solitary waves. We recall that a solitary wave is a wave which propagates without any temporal evolution in shape.

The operator $-\Delta_p - \Delta$ is a special case of the so called (p, q)-Laplace operator given by $-\Delta_p - \Delta_q$ which has been widely studied; for some results related to our studies, see, e.g., [6,7,10,21,25] .

The main purpose of this work was to study the nonlinear eigenvalue problem (1.3) when $p > 2$, and $1 < p < 2$, respectively . In particular, we show in section 2 that the set of the first eigenvalues is given by the interval (λ_1^D, ∞), where λ_1^D is the first Dirichlet eigenvalue of the Laplacian. We show that the first eigenvalue of (1.3) can be obtained variationally, using a Nehari set for $1 < p < 2$, and a minimization for $p > 2$. Also in the same section, we recall some results of [15,22,23].

In Sect. 3, we prove that the eigenfunctions associated with λ belong to $L^\infty(\Omega)$: the first eigenvalue λ_1^D of problem (1.3) is simple and the corresponding eigenfunctions are positive or negative. In addition, in Sect. 3.3 we show a homeomorphism property related to $-\Delta_p - \Delta$.

In Sect. 4, we prove that λ_1^D is a bifurcation point for a branch of first eigenvalues from zero if $p > 2$, and λ_1^D is a bifurcation point from infinity if $p < 2$. Also the higher Dirichlet eigenvalues λ_k^D are bifurcation points (from 0 if $p > 2$, respectively, from infinity if $1 < p < 2$), if the multiplicity of λ_k^D is odd. Finally in Sect. 5, we prove by variational methods that if $\lambda \in (\lambda_k^D, \lambda_{k+1}^D)$, then there exist at least k nonlinear eigenvalues using Krasnoselski's genus. In what follows, we denote by $\|.\|_{1,p}$ and $\|.\|_2$ the norms on $W_0^{1,p}(\Omega)$ and $L^2(\Omega)$ defined, respectively, by

$$\|u\|_{1,p} = \left(\int_\Omega |\nabla u|^p \, \mathrm{d}x\right)^{\frac{1}{p}} \text{ and } \|u\|_2 = \left(\int_\Omega |u|^2 \, \mathrm{d}x\right)^{\frac{1}{2}}, \text{ for all } u \in W_0^{1,p}(\Omega),\ u \in L^2(\Omega).$$

We recall the Poincaré inequality, i.e., there exists a positive constant $C_p(\Omega)$ such that

$$\int_\Omega |u|^p \, \mathrm{d}x \leq C_p(\Omega) \int_\Omega |\nabla u|^p \, \mathrm{d}x \text{ for all} u \in W_0^{1,p}(\Omega),\ 1 < p < \infty. \tag{1.5}$$

2. The Spectrum of the Nonlinear Problem

We now begin with the discussion of the properties of the spectrum of the nonlinear eigenvalues problem (1.3).

Remark 2.1. Any $\lambda \leq 0$ is not an eigenvalue of problem (1.3).

Indeed, suppose by contradiction that $\lambda = 0$ is an eigenvalue of equation (1.3), then relation (1.4) with $v = u_0$ gives

$$\int_\Omega |\nabla u_0|^p \, \mathrm{d}x + \int_\Omega |\nabla u_0|^2 \, \mathrm{d}x = 0.$$

Consequently, $|\nabla u_0| = 0$; therefore, u_0 is constant on Ω and $u_0 = 0$ on Ω. And this contradicts the fact that u_0 is a nontrivial eigenfunction. Hence $\lambda = 0$ is not an eigenvalue of problem (1.3). Now it remains to show that any $\lambda < 0$ is not an eigenvalue of (1.3). Suppose by contradiction that $\lambda < 0$ is an eigenvalue of (1.3), with $u_\lambda \in W_0^{1,p}(\Omega)\backslash\{0\}$ the corresponding eigenfunction. The relation (1.4) with $v = u_\lambda$ implies

$$0 \leq \int_\Omega |\nabla u_\lambda|^p \, \mathrm{d}x + \int_\Omega |\nabla u_\lambda|^2 \, \mathrm{d}x = \lambda \int_\Omega u_\lambda^2 \, \mathrm{d}x < 0,$$

Which yields a contradiction and thus $\lambda < 0$ cannot be an eigenvalue of problem (1.3).

Lemma 2.2. *Any $\lambda \in (0, \lambda_1^D]$ is not an eigenvalue of* (1.3).

For the proof see also [15].

Proof. Let $\lambda \in (0, \lambda_1^D)$, i.e., $\lambda_1^D > \lambda$. Let us assume by contradiction that there exists a $\lambda \in (0, \lambda_1^D)$ which is an eigenvalue of (1.3) with $u_\lambda \in W_0^{1,2}(\Omega)$ $\backslash\{0\}$, the corresponding eigenfunction. Letting $v = u_\lambda$ in relation (1.4), we have on the one hand,

$$\int_\Omega |\nabla u_\lambda|^p \, \mathrm{d}x + \int_\Omega |\nabla u_\lambda|^2 \, \mathrm{d}x = \lambda \int_\Omega u_\lambda^2 \, \mathrm{d}x,$$

and on the other hand,

$$\lambda_1^D \int_\Omega u_\lambda^2 \mathrm{d}x \leq \int_\Omega |\nabla u_\lambda|^2 \, \mathrm{d}x. \tag{2.1}$$

By subtracting both sides of (2.1) by $\lambda \int_\Omega u_\lambda^2 \, \mathrm{d}x$, we obtain

$$\left(\lambda_1^D - \lambda\right) \int_\Omega u_\lambda^2 \, \mathrm{d}x \leq \int_\Omega |\nabla u_\lambda|^2 \, \mathrm{d}x - \lambda \int_\Omega u_\lambda^2 \, \mathrm{d}x,$$

$$\left(\lambda_1^D - \lambda\right) \int_\Omega u_\lambda^2 \, \mathrm{d}x \leq \int_\Omega |\nabla u_\lambda|^2 \, \mathrm{d}x - \lambda \int_\Omega u_\lambda^2 \, \mathrm{d}x + \int_\Omega |\nabla u_\lambda|^p \, \mathrm{d}x = 0.$$

Therefore, $(\lambda_1^D - \lambda)\int_\Omega u_\lambda^2 \, \mathrm{d}x \leq 0$, which is a contradiction. Hence, we conclude that $\lambda \in (0, \lambda_1^D)$ is not an eigenvalue of problem (1.3). In order to complete the proof of the Lemma 2.2 we shall show that $\lambda = \lambda_1^D$ is not an eigenvalue of (1.3).

By contradiction we assume that $\lambda = \lambda_1^D$ is an eigenvalue of (1.3). So there exists $u_{\lambda_1^D} \in W_0^{1,2}(\Omega)\backslash\{0\}$ such that relation (1.4) holds true. Letting $v = u_{\lambda_1^D}$ in relation (1.4), it follows that

$$\int_\Omega \left|\nabla u_{\lambda_1^D}\right|^p \, \mathrm{d}x + \int_\Omega \left|\nabla u_{\lambda_1^D}\right|^2 \, \mathrm{d}x = \lambda_1^D \int_\Omega u_{\lambda_1^D}^2 \, \mathrm{d}x.$$

But $\lambda_1^D \int_\Omega u_{\lambda_1^D}^2 \, \mathrm{d}x \leq \int_\Omega |\nabla u_{\lambda_1^D}|^2 \, \mathrm{d}x$; therefore

$$\int_\Omega \left|\nabla u_{\lambda_1^D}\right|^p \, \mathrm{d}x + \int_\Omega \left|\nabla u_{\lambda_1^D}\right|^2 \, \mathrm{d}x \leq \int_\Omega \left|\nabla u_{\lambda_1^D}\right|^2 \, \mathrm{d}x \Rightarrow \int_\Omega \left|\nabla u_{\lambda_1^D}\right|^p \, \mathrm{d}x \leq 0.$$

Using relation (1.5), we have $u_{\lambda_1^D} = 0$, which is a contradiction since $u_{\lambda_1^D} \in W_0^{1,2}(\Omega)\backslash\{0\}$. So $\lambda = \lambda_1^D$ is not an eigenvalue of (1.3). □

Theorem 2.3. *Assume* $p \in (1, 2)$*. Then the set of first eigenvalues of problem* (1.3) *is given by*

$$(\lambda_1^D, \infty),\ \text{where} \lambda_1^{\text{D}}\ \text{denotes the first eigenvalue of} -\Delta \text{on} \Omega.$$

Proof. Let $\lambda \in (\lambda_1^D, \infty)$, and define the energy functional

$$J_\lambda : W_0^{1,2}(\Omega) \to \mathbb{R} \text{ by } J_\lambda(u) = \int_\Omega |\nabla u|^2 \ \mathrm{d}x + \frac{2}{p}\int_\Omega |\nabla u|^p \ \mathrm{d}x - \lambda \int_\Omega u^2 \ \mathrm{d}x.$$

One shows that $J_\lambda \in C^1(W_0^{1,2}(\Omega), \mathbb{R})$ (see, [18]) with its derivatives given by

$$\langle J_\lambda'(u), v\rangle = 2\int_\Omega \nabla u \cdot \nabla v \ \mathrm{d}x + 2\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla v \ \mathrm{d}x - 2\lambda \int_\Omega u\ v \ \mathrm{d}x\ , \quad \forall\ v \in W_0^{1,2}(\Omega).$$

Thus we note that λ is an eigenvalue of problem (1.3) if and only if J_λ possesses a nontrivial critical point. Considering $J_\lambda(\rho e_1)$, where e_1 is the L^2-normalized first eigenfunction of the Laplacian, we see that

$$J_\lambda(\rho e_1) \le \lambda_1^D \rho^2 + C\rho^p - \lambda\rho^2 \to -\infty, \text{ as } \rho \to +\infty.$$

Hence, we cannot establish the coercivity of J_λ on $W_0^{1,2}(\Omega)$ for $p \in (1, 2)$, and consequently we cannot use a direct method in calculus of variations in order to determine a critical point of J_λ. To overcome this difficulty, the idea will be to analyze the functional J_λ on the so-called Nehari manifold defined by

$$\mathcal{N}_\lambda := \left\{u \in W_0^{1,2}(\Omega)\backslash\{0\}\ : \int_\Omega |\nabla u|^2 \ \mathrm{d}x + \int_\Omega |\nabla u|^p \ \mathrm{d}x = \lambda \int_\Omega u^2 \ \mathrm{d}x\right\}.$$

Note that all non-trivial solutions of (1.3) lie on $\mathcal{N}_\lambda$. On $\mathcal{N}_\lambda$ the functional J_λ takes the following form

$$\begin{aligned} J_\lambda(u) &= \int_\Omega |\nabla u|^2 \ \mathrm{d}x + \frac{2}{p}\int_\Omega |\nabla u|^p \ \mathrm{d}x - \lambda \int_\Omega u^2 \ \mathrm{d}x \\ &= \left(\frac{2}{p} - 1\right)\int_\Omega |\nabla u|^p \ \mathrm{d}x > 0. \end{aligned}$$

We have seen in Lemma 2.2 that any $\lambda \in (0, \lambda_1^D]$ is not an eigenvalue of problem (1.3); see also [15]. It remains to prove the following:

Claim: Every $\lambda \in (\lambda_1^D, \infty)$ is a first eigenvalue of problem (1.3). Indeed, we will split the proof of the claim into four steps follows:

Step 1. Here we will show that $\mathcal{N}_\lambda \neq \emptyset$ and every minimizing sequence for J_λ on $\mathcal{N}_\lambda$ is bounded. Since $\lambda > \lambda_1^D$ there exists $v_\lambda \in W_0^{1,2}(\Omega)$ such that

$$\int_\Omega |\nabla v_\lambda|^2 < \lambda \int_\Omega v_\lambda^2 \ \mathrm{d}x.$$

Then there exists $t > 0$ such that $tv_\lambda \in \mathcal{N}_\lambda \Rightarrow$

$$\int_\Omega |\nabla (tv_\lambda)|^2 \, \mathrm{d}x + \int_\Omega |\nabla (tv_\lambda)|^p \, \mathrm{d}x = \lambda \int_\Omega (tv_\lambda)^2 \, \mathrm{d}x \Rightarrow$$
$$t^2 \int_\Omega |\nabla v_\lambda|^2 \, \mathrm{d}x + t^p \int_\Omega |\nabla v_\lambda|^p \, \mathrm{d}x = t^2 \lambda \int_\Omega v_\lambda^2 \, \mathrm{d}x \Rightarrow$$
$$t = \left(\frac{\int_\Omega |\nabla v_\lambda|^p \, \mathrm{d}x}{\lambda \int_\Omega v_\lambda^2 \, \mathrm{d}x - \int_\Omega |\nabla v_\lambda|^2 \, \mathrm{d}x} \right)^{\frac{1}{2-p}} > 0.$$

With such t we have $tv_\lambda \in \mathcal{N}_\lambda$ and $\mathcal{N}_\lambda \neq \emptyset$.

Note that for $u \in B_r(v_\lambda)$, $r > 0$ small, the inequality $\lambda \int_\Omega |u|^2 \mathrm{d}x > \int_\Omega |\nabla u|^2 \mathrm{d}x$ remains valid, and then $t(u)u \in \mathcal{N}_\lambda$ for $u \in B_r(v_\lambda)$. Since $t(u) \in C^1$ we conclude that $\mathcal{N}_\lambda$ is a C^1-manifold.
Let $\{u_k\} \subset \mathcal{N}_\lambda$ be a minimizing sequence of $J_\lambda|_{\mathcal{N}_\lambda}$, i.e., $J_\lambda(u_k) \to m = \inf\limits_{w \in \mathcal{N}_\lambda} J_\lambda(w)$. Then

$$\begin{aligned} \lambda \int_\Omega u_k^2 \, \mathrm{d}x - \int_\Omega |\nabla u_k|^2 \, \mathrm{d}x \\ = \int_\Omega |\nabla u_k|^p \, \mathrm{d}x \to \left(\frac{2}{p} - 1\right)^{-1} m \text{ as } \mathrm{k} \to \infty. \end{aligned} \tag{2.2}$$

Assume by contradiction that $\{u_k\}$ is not bounded, i.e., $\displaystyle\int_\Omega |\nabla u_k|^2 \, \mathrm{d}x \to \infty$ as $k \to \infty$. It follows that $\displaystyle\int_\Omega u_k^2 \, \mathrm{d}x \to \infty$ as $k \to \infty$, thanks to relation (2.2). We set $v_k = \frac{u_k}{\|u_k\|_2}$. Since $\displaystyle\int_\Omega |\nabla u_k|^2 \, \mathrm{d}x < \lambda \int_\Omega u_k^2 \, \mathrm{d}x$, we deduce that $\displaystyle\int_\Omega |\nabla v_k|^2 \, \mathrm{d}x < \lambda$, for each k and $\|v_k\|_{1,2} < \sqrt{\lambda}$. Hence $\{v_k\} \subset W_0^{1,2}(\Omega)$ is bounded in $W_0^{1,2}(\Omega)$. Therefore, there exists $v_0 \in W_0^{1,2}(\Omega)$ such that $v_k \rightharpoonup v_0$ in $W_0^{1,2}(\Omega) \subset W_0^{1,p}(\Omega)$ and $v_k \to v_0$ in $L^2(\Omega)$. Dividing relation (2.2) by $\|u_k\|_2^p$, we get

$$\int_\Omega |\nabla v_k|^p \, \mathrm{d}x = \frac{\lambda \displaystyle\int_\Omega u_k^2 \, \mathrm{d}x - \int_\Omega |\nabla u_k|^2 \, \mathrm{d}x}{\|u_k\|_2^p} \to 0 \text{ as } \mathrm{k} \to \infty,$$

since $\lambda \displaystyle\int_\Omega u_k^2 \, \mathrm{d}x - \int_\Omega |\nabla u_k|^2 \, \mathrm{d}x \to \left(\frac{2}{p} - 1\right)^{-1} m < \infty$ and $\|u_k\|_2^p \to \infty$ as $k \to \infty$. On the other hand, since $v_k \rightharpoonup v_0$ in $W_0^{1,p}(\Omega)$, we have $\displaystyle\int_\Omega |\nabla v_0|^p \, \mathrm{d}x \leq \lim_{k \to \infty} \inf \int_\Omega |\nabla v_k|^p \, \mathrm{d}x = 0$ and consequently $v_0 = 0$. It follows that $v_k \to 0$ in $L^2(\Omega)$, which is a contradiction since $\|v_k\|_2 = 1$. Hence, $\{u_k\}$ is bounded in $W_0^{1,2}(\Omega)$.

Step 2. $m = \inf_{w \in \mathcal{N}_\lambda} J_\lambda(w) > 0$. Indeed, assume by contradiction that $m = 0$. Then, for $\{u_k\}$ as in step 1, we have

$$\begin{aligned} 0 &< \lambda \int_\Omega u_k^2 \,\mathrm{d}x - \int_\Omega |\nabla u_k|^2 \,\mathrm{d}x \\ &= \int_\Omega |\nabla u_k|^p \,\mathrm{d}x \to 0, \mathrm{as} k \to \infty. \end{aligned} \tag{2.3}$$

By Step 1, we deduce that $\{u_k\}$ is bounded in $W_0^{1,2}(\Omega)$. Therefore there exists $u_0 \in W_0^{1,2}(\Omega)$ such that $u_k \rightharpoonup u_0$ in $W_0^{1,2}(\Omega)$ and $W_0^{1,p}(\Omega)$ and $u_k \to u_0$ in $L^2(\Omega)$.

Thus $\int_\Omega |\nabla u_0|^p \le \lim_{k\to\infty} \inf \int_\Omega |\nabla u_k|^p \,\mathrm{d}x = 0$. And consequently $u_0 = 0$, $u_k \rightharpoonup 0$ in $W_0^{1,2}(\Omega)$ and $W_0^{1,p}(\Omega)$ and $u_k \to 0$ in $L^2(\Omega)$. Writing again $v_k = \frac{u_k}{\|u_k\|_2}$ we have

$$0 < \frac{\lambda \int_\Omega u_k^2 \,\mathrm{d}x - \int_\Omega |\nabla u_k|^2 \,\mathrm{d}x}{\|u_k\|_2^2} = \|u_k\|_2^{p-2} \int_\Omega |\nabla v_k|^p \,\mathrm{d}x,$$

therefore,

$$\begin{aligned} \int_\Omega |\nabla v_k|^p \,\mathrm{d}x &= \|u_k\|_2^{2-p} \left(\frac{\lambda \|u_k\|_2^2}{\|u_k\|_2^2} - \frac{\int_\Omega |\nabla u_k|^2 \,\mathrm{d}x}{\|u_k\|_2^2} \right) \\ &= \|u_k\|_2^{2-p} \left(\lambda - \int_\Omega |\nabla v_k|^2 \,\mathrm{d}x \right) \to 0 \text{ as k} \to \infty, \end{aligned}$$

since $\|u_k\|_2 \to 0$ and $p \in (1,2)$, and $\{v_k\}$ is bounded in $W_0^{1,2}(\Omega)$. Next since $v_k \rightharpoonup v_0$, we deduce that $\int_\Omega |\nabla v_0|^p \,\mathrm{d}x \le \lim_{k\to\infty} \inf \int_\Omega |\nabla v_k|^p \,\mathrm{d}x = 0$ and we have $v_0 = 0$. And it follows that $v_k \to 0$ in $L^2(\Omega)$ which is a contradiction since $\|v_k\|_2 = 1$ for each k. Hence, m is positive.

Step 3. There exists $u_0 \in \mathcal{N}_\lambda$ such that $J_\lambda(u_0) = m$.

Let $\{u_k\} \subset \mathcal{N}_\lambda$ be a minimizing sequence, i.e., $J_\lambda(u_k) \to m$ as $k \to \infty$. Thanks to Step 1, we have that $\{u_k\}$ is bounded in $W_0^{1,2}(\Omega)$. It follows that there exists $u_0 \in W_0^{1,2}(\Omega)$ such that $u_k \rightharpoonup u_0$ in $W_0^{1,2}(\Omega)$ and $W_0^{1,p}(\Omega)$ and strongly in $L^2(\Omega)$. The results in the two steps above guarantee that $J_\lambda(u_0) \le \lim_{k\to\infty} \inf J_\lambda(u_k) = m$. Since for each k we have $u_k \in \mathcal{N}_\lambda$, we have

$$\int_\Omega |\nabla u_k|^2 \,\mathrm{d}x + \int_\Omega |\nabla u_k|^p \,\mathrm{d}x = \lambda \int_\Omega u_k^2 \,\mathrm{d}x \text{ for all} \, k. \tag{2.4}$$

Assuming $u_0 \equiv 0$ on Ω implies that $\int_\Omega u_k^2 \,\mathrm{d}x \to 0$ as $k \to \infty$, and by relation (2.4) we obtain that $\int_\Omega |\nabla u_k|^2 \,\mathrm{d}x \to 0$ as $k \to \infty$. Combining this with the fact that u_k converges weakly to 0 in $W_0^{1,2}(\Omega)$, we deduce that u_k converges strongly to 0 in $W_0^{1,2}(\Omega)$ and consequently in $W_0^{1,p}(\Omega)$. Hence we infer that

$$\lambda \int_\Omega u_k^2 \,\mathrm{d}x - \int_\Omega |\nabla u_k|^2 \,\mathrm{d}x = \int_\Omega |\nabla u_k|^p \,\mathrm{d}x \to 0, \text{as} k \to \infty.$$

Next, using similar argument as the one used in the proof of Step 2, we will reach to a contradiction, which shows that $u_0 \not\equiv 0$. Letting $k \to \infty$ in relation (2.4), we deduce that

$$\int_\Omega |\nabla u_0|^2 \,\mathrm{d}x + \int_\Omega |\nabla u_0|^p \,\mathrm{d}x \leq \lambda \int_\Omega u_0^2 \,\mathrm{d}x.$$

If there is equality in the above relation, then $u_0 \in \mathcal{N}_\lambda$ and $m \leq J_\lambda(u_0)$. Assume by contradiction that

$$\int_\Omega |\nabla u|^2 \,\mathrm{d}x + \int_\Omega |\nabla u|^p \,\mathrm{d}x < \lambda \int_\Omega u^2 \,\mathrm{d}x. \tag{2.5}$$

Let $t > 0$ be such that $tu_0 \in \mathcal{N}_\lambda$, i.e.,

$$t = \left(\frac{\lambda \int_\Omega u_0^2 \,\mathrm{d}x - \int_\Omega |\nabla u_0|^2 \,\mathrm{d}x}{\int_\Omega |\nabla u_0|^p \,\mathrm{d}x} \right)^{\frac{1}{p-2}}.$$

We note that $t \in (0,1)$ since $1 < t^{p-2}$ (thanks to (2.5)). Finally, since $tu_0 \in \mathcal{N}_\lambda$ with $t \in (0,1)$ we have

$$\begin{aligned} 0 < m &\leq J_\lambda(tu_0) \\ &= \left(\frac{2}{p} - 1\right) \int_\Omega |\nabla(tu_0)|^p \,\mathrm{d}x = t^p \left(\frac{2}{p} - 1\right) \int_\Omega |\nabla u_0|^p \,\mathrm{d}x \\ &= t^p J_\lambda(u_0) \\ &\leq t^p \lim_{k\to\infty} \inf J_\lambda(u_k) = t^p m < m \text{ for } t \in (0,1), \end{aligned}$$

and this is a contradiction which assures that relation (2.5) cannot hold and consequently we have $u_0 \in \mathcal{N}_\lambda$. Hence $m \leq J_\lambda(u_0)$ and $m = J_\lambda(u_0)$.

Step 4. We conclude the proof of the claim. Let $u \in \mathcal{N}_\lambda$ be such that $J_\lambda(u) = m$ (thanks to Step 3). Since $u \in \mathcal{N}_\lambda$, we have

$$\int_\Omega |\nabla u|^2 \,\mathrm{d}x + \int_\Omega |\nabla u|^p \,\mathrm{d}x = \lambda \int_\Omega u^2 \,\mathrm{d}x,$$

and

$$\int_\Omega |\nabla u|^2 \,\mathrm{d}x < \lambda \int_\Omega u^2 \,\mathrm{d}x.$$

Let $v \in \partial B_1(0) \subset W_0^{1,2}(\Omega)$ and $\varepsilon > 0$ be very small such that $u + \delta v \neq 0$ in Ω for all $\delta \in (-\varepsilon, \varepsilon)$ and

$$\int_\Omega |\nabla(u + \delta v)|^2 \,\mathrm{d}x < \lambda \int_\Omega (u + \delta v)^2 \,\mathrm{d}x;$$

this is equivalent to

$$\lambda \int_\Omega u^2 \, dx - \int_\Omega |\nabla u|^2 \, dx > \delta \left(2 \int_\Omega \nabla u \cdot \nabla v \, dx - 2\lambda \int_\Omega uv \, dx \right) + \delta^2 \left(\int_\Omega |\nabla v|^2 \, dx - \lambda \int_\Omega v^2 \, dx \right),$$

which holds true for δ small enough since the left-hand side is positive while the function

$$h(\delta) := |\delta| \left| 2 \int_\Omega \nabla u \cdot \nabla v \, dx - 2\lambda \int_\Omega uv \, dx \right| + \delta^2 \left| \int_\Omega |\nabla v|^2 \, dx - \lambda \int_\Omega v^2 \, dx \right|$$

dominates the term from the right-hand side and $h(\delta)$ is a continuous function (polynomial in δ) which vanishes in $\delta = 0$. For each $\delta \in (-\varepsilon, \varepsilon)$, let $t(\delta) > 0$ be given by

$$t(\delta) = \left(\frac{\lambda \int_\Omega (u + \delta v)^2 \, dx - \int_\Omega |\nabla (u + \delta v)|^2 \, dx}{\int_\Omega |\nabla (u + \delta v)|^p \, dx} \right)^{\frac{1}{p-2}},$$

so that $t(\delta) \cdot (u + \delta v) \in \mathcal{N}_\lambda$. We have that $t(\delta)$ is of class $C^1(-\varepsilon, \varepsilon)$ since $t(\delta)$ is the composition of some functions of class C^1. On the other hand, since $u \in \mathcal{N}_\lambda$ we have $t(0) = 1$.

Define $\iota : (-\varepsilon, \varepsilon) \to \mathbb{R}$ by $\iota(\delta) = J_\lambda(t(\delta)(u + \delta v))$ which is of class $C^1(-\varepsilon, \varepsilon)$ and has a minimum at $\delta = 0$. We have

$$\iota'(\delta) = [t'(\delta)(u + \delta v) + vt(\delta)] \, J'_\lambda \left(t(\delta)(u + \delta v) \right) \Rightarrow$$
$$0 = \iota'(0) = J'_\lambda \left(t(0)(u) \right) [t'(0)u + vt(0)] = \langle J'_\lambda(u), v \rangle$$

since $t(0) = 1$ and $t'(0) = 0$.

This shows that every $\lambda \in (\lambda_1^D, \infty)$ is an eigenvalue of problem (1.3). □

In the next theorem we consider the case $p > 2$. For similar results for the Neumann case, (see, [22]).

Theorem 2.4. *For $p > 2$, the set of first eigenvalues of problem (1.3) is given by (λ_1^D, ∞).*

The proof of Theorem 2.4 will follow as a direct consequence of the lemmas proved below:

Lemma 2.5. *Let*

$$\lambda_1(p) := \inf_{u \in W_0^{1,p} \setminus \{0\}} \left\{ \frac{\frac{1}{p} \int_\Omega |\nabla u|^p \, dx + \frac{1}{2} \int_\Omega |\nabla u|^2 \, dx}{\frac{1}{2} \int_\Omega u^2 \, dx} \right\}.$$

Then $\lambda_1(p) = \lambda_1^D$, for all $p > 2$.

Proof. We clearly have $\lambda_1(p) \geq \lambda_1^D$ since a positive term is added. On the other hand, consider $u_n = \frac{1}{n}e_1$ (where e_1 is the first eigenfunction of $-\Delta$), we get

$$\lambda_1(p) \leq \frac{\frac{1}{2n^2}\int_\Omega |\nabla e_1|^2 \, \mathrm{d}x + \frac{1}{pn^p}\int_\Omega |\nabla e_1|^p \, \mathrm{d}x}{\frac{1}{2n^2}\int_\Omega |e_1|^2 \mathrm{d}x} \to \lambda_1^D \text{ as } n \to \infty.$$

$\square$

Lemma 2.6. *For each $\lambda > 0$, we have*

$$\lim_{\|u\|_{1,p}\to\infty} \left(\frac{1}{2}\int_\Omega |\nabla u|^2 \, \mathrm{d}x + \frac{1}{p}\int_\Omega |\nabla u|^p \, \mathrm{d}x - \frac{\lambda}{2}\int_\Omega u^2 \, \mathrm{d}x \right) = \infty.$$

Proof. Clearly,

$$\frac{1}{p}\int_\Omega |\nabla u|^p \, \mathrm{d}x + \frac{1}{2}\int_\Omega |\nabla u|^2 \, \mathrm{d}x \geq \frac{1}{p}\int_\Omega |\nabla u|^p \, \mathrm{d}x.$$

On the one hand, using Poincaré's inequality with $p = 2$, we have $\int_\Omega u^2 \, \mathrm{d}x \leq C_2(\Omega)\int_\Omega |\nabla u|^2 \, \mathrm{d}x, \forall u \in W_0^{1,p}(\Omega) \subset W_0^{1,2}(\Omega)$ and then applying the Hölder inequality to the right-hand side term of the previous estimate, we obtain

$$\int_\Omega |\nabla u|^2 \, \mathrm{d}x \leq |\Omega|^{\frac{p-2}{p}} \|u\|_{1,p}^2,$$

so $\int_\Omega u^2 \, \mathrm{d}x \leq D\|u\|_{1,p}^2$, where $D = C_2(\Omega)|\Omega|^{\frac{p-2}{p}}$. Therefore, for $\lambda > 0$,

$$\frac{1}{2}\int_\Omega |\nabla u|^2 \, \mathrm{d}x + \frac{1}{p}\int_\Omega |\nabla u|^p \, \mathrm{d}x - \frac{\lambda}{2}\int_\Omega u^2 \, \mathrm{d}x \geq C\|u\|_{1,p}^p - \frac{\lambda}{2}D\|u\|_{1,p}^2, \quad (2.6)$$

and the the right-hand side of (2.6) tends to ∞, as $\|u\|_{1,p} \to \infty$, since $p > 2$. $\square$

Lemma 2.7. *Every $\lambda \in (\lambda_1^D, \infty)$ is a first eigenvalue of problem (1.3).*

Proof. For each $\lambda > \lambda_1^D$ define $F_\lambda : W_0^{1,p}(\Omega) \to \mathbb{R}$ by

$$F_\lambda(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 \, \mathrm{d}x + \frac{1}{p}\int_\Omega |\nabla u|^p \, \mathrm{d}x - \frac{\lambda}{2}\int_\Omega u^2 \, \mathrm{d}x \ , \forall u \in W_0^{1,p}(\Omega).$$

Standard arguments show that $F_\lambda \in C^1(W_0^{1,p}(\Omega), \mathbb{R})$ (see, [18]) with its derivative given by

$$\langle F_\lambda'(u), \phi \rangle = \int_\Omega \left(|\nabla u|^{p-2} + 1 \right) \nabla u \cdot \nabla \phi \, \mathrm{d}x - \lambda \int_\Omega u\phi \, \mathrm{d}x,$$

for all $u, \phi \in W_0^{1,p}(\Omega)$. Estimate (2.6) shows that F_λ is coercive in $W_0^{1,p}(\Omega)$. On the other hand, F_λ is also weakly lower semi-continuous on $W_0^{1,p}(\Omega)$ since F_λ is a continuous convex functional (see [4], Proposition 1.5.10 and Theorem 1.5.3) . Then we can apply a calculus of variations result, in order to obtain the existence of a global minimum point of F_λ, denoted by θ_λ, i.e.,

$F_\lambda(\theta_\lambda) = \min_{W_0^{1,p}(\Omega)} F_\lambda$. Note that for any $\lambda > \lambda_1^D$ there exists $u_\lambda \in W_0^{1,p}(\Omega)$ such that $F_\lambda(u_\lambda) < 0$. Indeed, taking $u_\lambda = re_1$, we have

$$F_\lambda(re_1) = \frac{r^2}{2}\left(\lambda_1^D - \lambda\right) + \frac{r^p}{p}\int_\Omega |\nabla e_1|^p \, \mathrm{d}x < 0 \text{ for } r > 0 \text{ small.}$$

But then $F_\lambda(\theta_\lambda) \leq F_\lambda(u_\lambda) < 0$, which means that $\theta_\lambda \in W_0^{1,p}(\Omega)\backslash\{0\}$. On the other hand, we have $\langle F'_\lambda(\theta_\lambda), \phi\rangle = 0, \forall \phi \in W_0^{1,p}(\Omega)$ (θ_λ is a critical point of F_λ) with $\theta_\lambda \in W_0^{1,p}(\Omega)\backslash\{0\} \subset W_0^{1,2}(\Omega)\backslash\{0\}$. Consequently each $\lambda > \lambda_1^D$ is an eigenvalue of problem (1.3). □

A similar result of Theorem 3.1 was proved in [17] in the case of the p-Laplacian.

3. Properties of Eigenfunctions and the Operator $-\Delta_p - \Delta$

3.1. Boundedness of the Eigenfunctions

We shall prove boundedness of eigenfunctions and use this fact to obtain $C^{1,\alpha}$ smoothness of all eigenfunctions of the quasi-linear problem (1.3). The latter result is due to [17, Theorem 4.4], which originates from [13, 26].

Theorem 3.1. *Let $(u, \lambda) \in W_0^{1,p}(\Omega) \times \mathbb{R}_+^\star$ be an eigensolution of the weak formulation (1.4). Then $u \in L^\infty(\Omega)$.*

Proof. By Morrey's embedding theorem it suffices to consider the case $p \leq N$. Let us assume first that $u > 0$. For $M \geq 0$ define $w_M(x) = \min\{u(x), M\}$. Letting

$$g(x) = \begin{cases} x \text{ if } x \leq M \\ M \text{ if } x > M. \end{cases} \tag{3.1}$$

we have $g \in C(\mathbb{R})$ piecewise smooth function with $g(0) = 0$. Since $u \in W_0^{1,p}(\Omega)$ and $g' \in L^\infty(\Omega)$, then $g \circ u \in W_0^{1,p}(\Omega)$ and $w_M \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ (see, Theorem B.3 in [17]). For $k > 0$, define $\varphi = w_M^{kp+1}$, then $\nabla\varphi = (kp + 1)\nabla w_M w_M^{kp}$ and $\varphi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$.

Using φ as a test function in (1.4), one obtains

$$\begin{aligned}(kp+1)&\left[\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla w_M w_M^{kp} \, \mathrm{d}x + \int_\Omega \nabla u \cdot \nabla w_M w_M^{kp} \, \mathrm{d}x\right] \\ &= \lambda \int_\Omega u \; w_M^{kp+1} \, \mathrm{d}x.\end{aligned}$$

On the other hand, using the fact that $w_M^{kp+1} \leq u^{kp+1}$, it follows that

$$\begin{aligned}(kp+1)&\left[\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla w_M w_M^{kp} \, \mathrm{d}x + \int_\Omega \nabla u \cdot \nabla w_M w_M^{kp} \, \mathrm{d}x\right] \\ &\leq \lambda \int_\Omega |u|^{(k+1)p} \, \mathrm{d}x.\end{aligned}$$

We have $\nabla(w_M^{k+1}) = (k+1)\nabla w_M w_M^k \Rightarrow |\nabla w_M^{k+1}|^p = (k+1)^p w_M^{kp} |\nabla w_M|^p$. Since the integrals on the left are zero on $\{x : u(x) > M\}$ we can take $u = w_M$ in the previous inequality, and it follows that

$$(kp+1)\left[\int_\Omega |\nabla w_M|^p\, w_M^{kp}\ \mathrm{d}x + \int_\Omega |\nabla w_M|^2\, w_M^{kp}\ \mathrm{d}x\right] \leq \lambda \int_\Omega |u|^{(k+1)p}\ \mathrm{d}x.$$

Replacing $|\nabla w_M|^p w_M^{kp}$ by $\frac{1}{(k+1)^p}|\nabla w_M^{k+1}|^p$, we have

$$\frac{kp+1}{(k+1)^p}\int_\Omega |\nabla w_M^{k+1}|^p\ \mathrm{d}x + (kp+1)\int_\Omega |\nabla w_M|^2 w_M^{kp}\ \mathrm{d}x \leq \lambda \int_\Omega |u|^{(k+1)p}\ \mathrm{d}x,$$

which implies that

$$\frac{kp+1}{(k+1)^p}\int_\Omega |\nabla w_M^{k+1}|^p\ \mathrm{d}x \leq \lambda \int_\Omega |u|^{(k+1)p}\ \mathrm{d}x$$

and then

$$\int_\Omega \left|\nabla w_M^{k+1}\right|^p\ \mathrm{d}x \leq \left(\lambda\frac{(k+1)^p}{kp+1}\right)\int_\Omega |u|^{(k+1)p}\ \mathrm{d}x. \tag{3.2}$$

By Sobolev's embedding theorem, there is a constant $c_1 > 0$ such that

$$\left\|w_M^{k+1}\right\|_{p^\star} \leq c_1 \left\|w_M^{k+1}\right\|_{1,p}, \tag{3.3}$$

where $p^\star$ is the Sobolev critical exponent. Consequently, we have

$$\|w_M\|_{(k+1)p^\star} \leq \left\|w_M^{k+1}\right\|_{p^\star}^{\frac{1}{k+1}}, \tag{3.4}$$

and, therefore,

$$\|w_M\|_{(k+1)p^\star} \leq \left(c_1\left\|w_M^{k+1}\right\|_{1,p}\right)^{\frac{1}{k+1}} = c_1^{\frac{1}{k+1}}\left\|w_M^{k+1}\right\|_{1,p}^{\frac{1}{k+1}}. \tag{3.5}$$

But by (3.2),

$$\left\|w_M^{k+1}\right\|_{1,p} \leq \left(\lambda\frac{(k+1)^p}{kp+1}\right)^{\frac{1}{p}} \|u\|_{(k+1)p}^{k+1}, \tag{3.6}$$

and we note that we can find a constant $c_2 > 0$ such that $\left(\lambda\frac{(k+1)^p}{kp+1}\right)^{\frac{1}{p\sqrt{k+1}}} \leq c_2$, independently of k and consequently,

$$\|w_M\|_{(k+1)p^\star} \leq c_1^{\frac{1}{k+1}} c_2^{\frac{1}{\sqrt{k+1}}} \|u\|_{(k+1)p}. \tag{3.7}$$

Letting $M \to \infty$, Fatou's lemma implies

$$\|u\|_{(k+1)p^\star} \leq c_1^{\frac{1}{k+1}} c_2^{\frac{1}{\sqrt{k+1}}} \|u\|_{(k+1)p}. \tag{3.8}$$

Choosing k_1, such that $(k_1+1)p = p^\star$, then $\|u\|_{(k_1+1)p^\star} \leq c_1^{\frac{1}{k_1+1}} c_2^{\frac{1}{\sqrt{k_1+1}}} \|u\|_{p^\star}$. Next we choose k_2 such that $(k_2+1)p = (k_1+1)p^\star$; then taking $k_2 = k$ in inequality (3.8), it follows that

$$\|u\|_{(k_2+1)p^\star} \leq c_1^{\frac{1}{k_2+1}} c_2^{\frac{1}{\sqrt{k_2+1}}} \|u\|_{(k_1+1)p^\star}. \tag{3.9}$$

By induction we obtain

$$\|u\|_{(k_n+1)p^\star} \leq c_1^{\frac{1}{k_n+1}} c_2^{\frac{1}{\sqrt{k_n+1}}} \|u\|_{(k_{n-1}+1)p^\star}, \tag{3.10}$$

where the sequence $\{k_n\}$ is chosen such that $(k_n+1)p = (k_{n-1}+1)p^\star, k_0 = 0$. One gets $k_n + 1 = (\frac{p^\star}{p})^n$. As $\frac{p}{p^\star} < 1$, there is $C > 0$ (which depends on c_1 and c_2) such that for any $n = 1, 2, \ldots$

$$\|u\|_{r_n} \leq C\|u\|_{p^\star} \tag{3.11}$$

with $r_n = (k_n+1)p^\star \to \infty$ as $n \to \infty$. We note that (3.11) follows by iterating the previous inequality (3.10). We will indirectly show that $u \in L^\infty(\Omega)$. Suppose $u \notin L^\infty(\Omega)$, then there exists $\varepsilon > 0$ and a set A of positive measure in Ω such that $|u(x)| > C\|u\|_{p^\star} + \varepsilon = K$, for all $x \in A$. We then have,

$$\lim_{n\to\infty} \inf \|u\|_{r_n} \geq \lim_{n\to\infty} \inf \left(\int_A K^{r_n} \right)^{1/r_n} = \lim_{n\to\infty} \inf K|A|^{1/r_n} = K > C\|u\|_{p^\star}, \tag{3.12}$$

which contradicts (3.11). If u changes sign, we consider $u = u^+ - u^-$ where

$$u^+ = \max\{u, 0\} \text{ and } u^- = \max\{-u, 0\}. \tag{3.13}$$

We have $u^+, u^- \in W_0^{1,p}(\Omega)$. For each $M>0$ define $w_M = \min\{u^+(x), M\}$ and take again $\varphi = w_M^{kp+1}$ as a test function in (1.4). Proceeding the same way as above we conclude that $u^+ \in L^\infty(\Omega)$. Similarly, we have $u^- \in L^\infty(\Omega)$. Therefore, $u = u^+ - u^-$ is in $L^\infty(\Omega)$. □

3.2. Simplicity of the Eigenvalues

We prove an auxiliary result which will imply uniqueness of the first eigenfunction.

Let

$$I(u,v) = \left\langle -\Delta_p u, \frac{u^p - v^p}{u^{p-1}} \right\rangle + \left\langle -\Delta u, \frac{u^2 - v^2}{u} \right\rangle + \left\langle -\Delta_p v, \frac{v^p - u^p}{v^{p-1}} \right\rangle + \left\langle -\Delta v, \frac{v^2 - u^2}{v} \right\rangle,$$

for all $(u, v) \in D_I$, where

$$D_I = \left\{ (u_1, u_2) \in W_0^{1,p}(\Omega) \times W_0^{1,p}(\Omega) : u_i > 0 \text{ in } \Omega \text{ and } u_i \in L^\infty(\Omega) \text{ for } i = 1, 2 \right\} \text{ if } p > 2,$$

and

$$D_I = \left\{ (u_1, u_2) \in W_0^{1,2}(\Omega) \times W_0^{1,2}(\Omega) : \ u_i > 0 \text{in } \Omega \text{ and } u_i \in L^\infty(\Omega) \text{ for } i = 1, 2 \right\} \text{ if } 1<p<2.$$

Proposition 3.2. *For all $(u, v) \in D_I$, we have $I(u, v) \geq 0$. Furthermore, $I(u, v) = 0$ if and only if there exists $\alpha \in \mathbb{R}_+^\star$ such that $u = \alpha v$.*

Proof. We first show that $I(u, v) \geq 0$. We recall that (if $2 < p < \infty$)

$$\langle -\Delta_p u, w \rangle = \int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla w \, dx \text{ for all } w \in W_0^{1,p}(\Omega)$$
$$\langle -\Delta u, w \rangle = \int_\Omega \nabla u \cdot \nabla w \, dx \text{ for all } w \in W_0^{1,p}(\Omega),$$

and (if $1 < p < 2$)

$$\langle -\Delta_p u, w\rangle = \int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla w \, \mathrm{d}x \text{ for all } w \in W_0^{1,2}(\Omega$$

$$\langle -\Delta u, w\rangle = \int_\Omega \nabla u \cdot \nabla w \, \mathrm{d}x \text{ for all } w \in W_0^{1,2}(\Omega).$$

Let us consider $\beta = \frac{u^p - v^p}{u^{p-1}}$, $\eta = \frac{v^p - u^p}{v^{p-1}}$, $\xi = \frac{u^2 - v^2}{u}$ and $\zeta = \frac{v^2 - u^2}{v}$ as test functions in (1.4) for any $p > 1$. Straightforward computations give

$$\nabla\left(\frac{u^p - v^p}{u^{p-1}}\right) = \left\{1 + (p-1)\left(\frac{v}{u}\right)^p\right\}\nabla u - p\left(\frac{v}{u}\right)^{p-1}\nabla v$$

$$\nabla\left(\frac{v^p - u^p}{v^{p-1}}\right) = \left\{1 + (p-1)\left(\frac{u}{v}\right)^p\right\}\nabla v - p\left(\frac{u}{v}\right)^{p-1}\nabla u$$

$$\nabla\left(\frac{u^2 - v^2}{u}\right) = \left\{1 + \left(\frac{v}{u}\right)^2\right\}\nabla u - 2\left(\frac{v}{u}\right)\nabla v$$

$$\nabla\left(\frac{v^2 - u^2}{v}\right) = \left\{1 + \left(\frac{u}{v}\right)^2\right\}\nabla v - 2\left(\frac{u}{v}\right)\nabla u.$$

Therefore,

$$\begin{aligned}
&\left\langle -\Delta_p u, \frac{u^p - v^p}{u^{p-1}}\right\rangle \\
&\quad = \int_\Omega \left\{-p\left(\frac{v}{u}\right)^{p-1}|\nabla u|^{p-2}\nabla u \cdot \nabla v + \left(1 + (p-1)\left(\frac{v}{u}\right)^p\right)|\nabla u|^p\right\}\mathrm{d}x \\
&\quad = \int_\Omega \left\{p\left(\frac{v}{u}\right)^{p-1}|\nabla u|^{p-2}\left(|\nabla u||\nabla v| - \nabla u \cdot \nabla v\right) + \left(1 + (p-1)\left(\frac{v}{u}\right)^p\right)|\nabla u|^p\right\}\mathrm{d}x \\
&\qquad - \int_\Omega p\left(\frac{v}{u}\right)^{p-1}|\nabla u|^{p-1}|\nabla v| \, \mathrm{d}x
\end{aligned}$$

and

$$\begin{aligned}
&\left\langle -\Delta u, \frac{u^2 - v^2}{u}\right\rangle \\
&\quad = \int_\Omega \left\{2\left(\frac{v}{u}\right)\left(|\nabla u||\nabla v| - \nabla u \cdot \nabla v\right) + \left(1 + \left(\frac{v}{u}\right)^2\right)|\nabla u|^2 - 2\left(\frac{v}{u}\right)|\nabla u||\nabla v|\right\}\mathrm{d}x.
\end{aligned}$$

By symmetry we have

$$\begin{aligned}
&\left\langle -\Delta_p v, \frac{v^p - u^p}{v^{p-1}}\right\rangle \\
&\quad = \int_\Omega \left\{-p\left(\frac{u}{v}\right)^{p-1}|\nabla v|^{p-2}\nabla v \cdot \nabla u + \left(1 + (p-1)\left(\frac{u}{v}\right)^p\right)|\nabla v|^p\right\}\mathrm{d}x \\
&\quad = \int_\Omega \left\{p\left(\frac{u}{v}\right)^{p-1}|\nabla v|^{p-2}\left(|\nabla v||\nabla u| - \nabla v \cdot \nabla u\right) + \left(1 + (p-1)\left(\frac{u}{v}\right)^p\right)|\nabla v|^p\right\}\mathrm{d}x \\
&\qquad - \int_\Omega p\left(\frac{u}{v}\right)^{p-1}|\nabla v|^{p-1}|\nabla u| \, \mathrm{d}x
\end{aligned}$$

and

$$\begin{aligned}
\left\langle -\Delta v, \frac{v^2 - u^2}{v}\right\rangle &= \int_\Omega \left\{2\left(\frac{u}{v}\right)\left(|\nabla v||\nabla u| - \nabla v \cdot \nabla u\right)\right. \\
&\quad \left. + \left(1 + \left(\frac{u}{v}\right)^2\right)|\nabla v|^2 - 2\left(\frac{u}{v}\right)|\nabla v||\nabla u|\right\}\mathrm{d}x.
\end{aligned}$$

Thus

$$
\begin{aligned}
I(u,v) = &\int_\Omega \left\{p\left(\frac{v}{u}\right)^{p-1}|\nabla u|^{p-2}\left(|\nabla u||\nabla v| - \nabla u\cdot\nabla v\right) + \left(1+(p-1)\left(\frac{v}{u}\right)^p\right)|\nabla u|^p\right\}\,\mathrm{d}x\\
&-p\left(\frac{v}{u}\right)^{p-1}|\nabla u|^{p-1}|\nabla v|\,\mathrm{d}x\\
&+\int_\Omega \left\{p\left(\frac{u}{v}\right)^{p-1}|\nabla v|^{p-2}\left(|\nabla v||\nabla u| - \nabla v\cdot\nabla u\right) + \left(1+(p-1)\left(\frac{u}{v}\right)^p\right)|\nabla v|^p\right\}\,\mathrm{d}x\\
&-p\left(\frac{u}{v}\right)^{p-1}|\nabla v|^{p-1}|\nabla u|\,\mathrm{d}x\\
&+\int_\Omega \left\{2\left(\frac{v}{u}\right)\left(|\nabla u||\nabla v| - \nabla u\cdot\nabla v\right) + \left(1+\left(\frac{v}{u}\right)^2\right)|\nabla u|^2 - 2\left(\frac{v}{u}\right)|\nabla u||\nabla v|\right\}\,\mathrm{d}x\\
&+\int_\Omega \left\{2\left(\frac{u}{v}\right)\left(|\nabla v||\nabla u| - \nabla v\cdot\nabla u\right) + \left(1+\left(\frac{u}{v}\right)^2\right)|\nabla v|^2 - 2\left(\frac{u}{v}\right)|\nabla v||\nabla u|\right\}\,\mathrm{d}x.
\end{aligned}
$$

So

$$
I(u,v) = \int_\Omega F\left(\frac{v}{u},\nabla v,\nabla u\right)\,\mathrm{d}x + \int_\Omega G\left(\frac{v}{u},|\nabla v|,|\nabla u|\right)\,\mathrm{d}x,
$$

where

$$
\begin{aligned}
F(t,S,R) = &\, p\left\{t^{p-1}|R|^{p-2}\left(|R||S| - R\cdot S\right) + t^{1-p}|S|^{p-2}\left(|R||S| - R\cdot S\right)\right\}\\
&+2\left\{t\left(|R||S| - R\cdot S\right)\right\} + 2\left\{t^{-1}\left(|R||S| - R\cdot S\right)\right\}
\end{aligned}
$$

and

$$
\begin{aligned}
G(t,s,r) = &\,(1+(p-1)t^p)\,r^p + \left(1+(p-1)t^{-p}\right)s^p + (1+t^2)r^2\\
&+(1+t^{-2})s^2 - pt^{p-1}r^{p-1}s - pt^{1-p}s^{p-1}r - 2trs - 2t^{-1}rs,
\end{aligned}
$$

for all $t = \frac{v}{u} > 0, R = \nabla u, S = \nabla v \in \mathbb{R}^N$ and $r = |\nabla u|, s = |\nabla v| \in \mathbb{R}^+$. We clearly have that F is non-negative. Now let us show that G is non-negative. Indeed, we observe that

$$
G(t,s,0) = \left(1+(p-1)t^{-p}\right)s^p + \left(1+t^{-2}\right)s^2 \geq 0
$$

and $G(t,s,0) = 0 \Rightarrow s = 0$. If $r \neq 0$, by setting $z = \frac{s}{tr}$ we obtain

$$
\begin{aligned}
G(t,s,r) = &\, t^p r^p\left(z^p - pz + (p-1)\right) + r^p\left((p-1)z^p - pz^{p-1} + 1\right)\\
&+t^2r^2\left(z^2 - 2z + 1\right) + r^2\left(z^2 - 2z + 1\right),
\end{aligned}
$$

and G can be written as

$$
G(t,s,r) = r^p\left(t^p f(z) + g(z)\right) + r^2\left(t^2 h(z) + k(z)\right),
$$

with $f(z) = z^p - pz + (p-1)$, $g(z) = (p-1)z^p - pz^{p-1} + 1$, $h(z) = k(z) = z^2 - 2z + 1$ $\forall p > 1$. We can see that f, g, h and k are non-negative. Hence G is non-negative and thus $I(u,v) \geq 0$ for all $(u,v) \in D_I$. In addition since f, g, h and k vanish if and only if $z = 1$, then $G(t,s,r) = 0$ if and only if $s = tr$. Consequently, if $I(u,v) = 0$ then we have

$$
\nabla u\cdot\nabla v = |\nabla u||\nabla v| \text{ and } u|\nabla v| = v|\nabla u|
$$

almost everywhere in Ω. This is equivalent to $(u\nabla v - v\nabla u)^2 = 0$, which implies that $u = \alpha v$ with $\alpha \in \mathbb{R}_+^\star$. □

Theorem 3.3. *The first eigenvalues λ of Eq. (1.3) are simple, i.e., if u and v are two positive first eigenfunctions associated to λ, then $u = v$.*

Proof. By Proposition 3.2, we have $u = \alpha v$. Inserting this into the equation (1.3) implies that $\alpha = 1$. □

3.3. Invertibility of the Operator $-\Delta_p - \Delta$

To simplify some notations, here we set $X = W_0^{1,p}(\Omega)$ and its dual $X^\star = W^{-1,p'}(\Omega)$, where $\frac{1}{p} + \frac{1}{p'} = 1$.

For the proof of the following lemma, we refer to [19]:

Lemma 3.4. *Let $p > 2$. Then there exist two positive constants c_1, c_2 such that, for all $x_1, x_2 \in \mathbb{R}^n$, we have the following:*

(i) $(x_2 - x_1) \cdot (|x_2|^{p-2}x_2 - |x_1|^{p-2}x_1) \geq c_1|x_2 - x_1|^p$
(ii) $\big||x_2|^{p-2}x_2 - |x_1|^{p-2}x_1\big| \leq c_2(|x_2| + |x_1|)^{p-2}|x_2 - x_1|$

Proposition 3.5. *For $p > 2$, the operator $-\Delta_p - \Delta$ is a global homeomorphism.*

The proof is based on the previous Lemma 3.4.

Proof. Define the nonlinear operator $A : X \to X^\star$ by

$$\langle Au, v\rangle = \int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x + \int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla v \,\mathrm{d}x \text{ for all } u, v \in X.$$

To show that $-\Delta_p - \Delta$ is a homeomorphism, it is enough to show that A is a continuous strongly monotone operator, (see [9, Corollary 2.5.10]). For $p > 2$, for all $u, v \in X$, by (i), we get

$$\begin{aligned}
&\langle Au - Av, u - v\rangle \\
&\quad = \int_\Omega |\nabla(u - v)|^2 \mathrm{d}x + \int_\Omega \left(|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v\right) \cdot \nabla(u - v) \,\mathrm{d}x \\
&\quad \geq \int_\Omega |\nabla(u - v)|^2 \mathrm{d}x + c_1 \int_\Omega |\nabla(u - v)|^p \mathrm{d}x \\
&\quad \geq c_1\|u - v\|_{1,p}^p.
\end{aligned}$$

Thus A is a strongly monotone operator.

We claim that A is a continuous operator from X to $X^\star$. Indeed, assume that $u_n \to u$ in X. We have to show that $\|Au_n - Au\|_{X^\star} \to 0$ as $n \to \infty$. Indeed, using (ii) and Hölder's inequality and the Sobolev embedding theorem, one has

$$\begin{aligned}
&|\langle Au_n - Au, w\rangle| \\
&\quad \leq \int_\Omega \big||\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u\big|\,|\nabla w| \,\mathrm{d}x + \int_\Omega |\nabla(u_n - u)||\nabla w| \,\mathrm{d}x \\
&\quad \leq c_2 \int_\Omega (|\nabla u_n| + |\nabla u|)^{p-2} |\nabla(u_n - u)||\nabla w| \,\mathrm{d}x + \int_\Omega |\nabla(u_n - u)||\nabla w| \,\mathrm{d}x \\
&\quad \leq c_2 \left(\int_\Omega (|\nabla u_n| + |\nabla u|)^p \,\mathrm{d}x\right)^{p-2/p} \left(\int_\Omega |\nabla(u_n - u)|^p \mathrm{d}x\right)^{1/p} \\
&\qquad \times \left(\int_\Omega |\nabla w|^p \mathrm{d}x\right)^{1/p} + c_3\|u_n - u\|_{1,2}\|w\|_{1,2} \\
&\quad \leq c_4 \left(\|u_n\|_{1,p} + \|u\|_{1,p}\right)^{p-2} \|u_n - u\|_{1,p}\|w\|_{1,p} + c_5\|u_n - u\|_{1,p}\|w\|_{1,p}.
\end{aligned}$$

Thus $\|Au_n - Au\|_{X^\star} \to 0$, as $n \to +\infty$, and hence A is a homeomorphism. $\square$

4. Bifurcation of Eigenvalues

In the next subsection we show that for Eq. (1.3) there is a branch of first eigenvalues bifurcating from $(\lambda_1^D, 0) \in \mathbb{R}^+ \times W_0^{1,p}(\Omega)$.

4.1. Bifurcation from Zero: The Case $p > 2$

By Proposition 3.5, Eq. (1.3) is equivalent to

$$u = \lambda(-\Delta_p - \Delta)^{-1}u \text{ for } u \in W^{-1,p'}(\Omega). \tag{4.1}$$

We set

$$S_\lambda(u) = u - \lambda(-\Delta_p - \Delta)^{-1}u, \tag{4.2}$$

$u \in L^2(\Omega) \subset W^{-1,p'}(\Omega)$ and $\lambda > 0$. By $\Sigma = \{(\lambda, u) \in \mathbb{R}^+ \times W_0^{1,p}(\Omega) / \ u \neq 0\ , S_\lambda(u) = 0\}$, we denote the set of nontrivial solutions of (4.1).

A bifurcation point for (4.1) is a number $\lambda^\star \in \mathbb{R}^+$ such that $(\lambda^\star, 0)$ belongs to the closure of Σ. This is equivalent to say that, in any neighborhood of $(\lambda^\star, 0)$ in $\mathbb{R}^+ \times W_0^{1,p}(\Omega)$, there exists a nontrivial solution of $S_\lambda(u) = 0$.

Our goal is to apply the Krasnoselski bifurcation theorem [see, [1]].

Theorem 4.1. (Krasnoselski, 1964)
Let X be a Banach space and let $T \in C^1(X, X)$ be a compact operator such that $T(0) = 0$ and $T'(0) = 0$. Moreover, let $A \in \mathcal{L}(X)$ also be compact. Then every characteristic value λ^* of A with odd (algebraic) multiplicity is a bifurcation point for $u = \lambda Au + T(u)$.

We state our bifurcation result.

Theorem 4.2. *Let $p > 2$. Then every eigenvalue λ_k^D with odd multiplicity is a bifurcation point in $\mathbb{R}^+ \times W_0^{1,p}(\Omega)$ of $S_\lambda(u) = 0$, in the sense that in any neighbourhood of $(\lambda_k^D, 0)$ in $\mathbb{R}^+ \times W_0^{1,p}(\Omega)$ there exists a nontrivial solution of $S_\lambda(u) = 0$.*

Proof. We write the equation $S_\lambda(u) = 0$ as

$$u = \lambda Au + T_\lambda(u),$$

where $Au = (-\Delta)^{-1}u$ and $T_\lambda(u) = [(-\Delta_p - \Delta)^{-1} - (-\Delta)^{-1}](\lambda u)$, where we consider

$$(-\Delta_p - \Delta)^{-1} : L^2(\Omega) \subset W^{-1,p'}(\Omega) \to W_0^{1,p}(\Omega) \subset\subset L^2(\Omega)$$

and $(-\Delta)^{-1} : L^2(\Omega) \subset W^{-1,2}(\Omega) \to W_0^{1,2}(\Omega) \subset\subset L^2(\Omega)$.

For $p > 2$, the mapping

$$(-\Delta_p - \Delta)^{-1} - (-\Delta)^{-1} : L^2(\Omega) \subset W^{-1,p'}(\Omega) \to W_0^{1,p}(\Omega) \subset\subset L^2(\Omega)$$

is compact thanks to Rellich–Kondrachov theorem. We clearly have $A \in \mathcal{L}(L^2(\Omega))$ and $T_\lambda(0) = 0$. Now we have to show that

(1) $T_\lambda \in C^1$.
(2) $T'_\lambda(0) = 0$.

In order to show (1) and (2), it suffices to show that

(a) $-\Delta_p - \Delta : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ is continuously differentiable in a neighborhood $u \in W_0^{1,p}(\Omega)$.

(b) $(-\Delta_p - \Delta)^{-1}$ is a continuous inverse operator.

According to Proposition 3.5, $-\Delta_p - \Delta$ is a homeomorphism; hence $(-\Delta_p - \Delta)^{-1}$ is continuous and this shows (b). We also recall that in section 3.2, we have shown that λ_1^D is simple.

Let us show (a). We claim that $-\Delta_p : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ is Gâteaux differentiable. Indeed, for $\varphi \in W_0^{1,p}(\Omega)$ we have

$$
\begin{aligned}
&\langle -\Delta_p(u+\delta v), \varphi\rangle - \langle -\Delta_p u, \varphi\rangle \\
&\quad = \left\langle |\nabla(u+\delta v)|^{p-2}\nabla(u+\delta v), \nabla\varphi\right\rangle - \left\langle |\nabla u|^{p-2}\nabla u, \nabla\varphi\right\rangle \\
&\quad = \left\langle \left(|\nabla(u+\delta v)|^2\right)^{\frac{p-2}{2}} \nabla(u+\delta v), \nabla\varphi\rangle - \langle |\nabla u|^{p-2}\nabla u, \nabla\varphi \right\rangle \\
&\quad = \left\langle \left(|\nabla u|^2 + 2\delta\left\langle \nabla u, \nabla v\right\rangle + \delta^2|\nabla v|^2\right)^{\frac{p-2}{2}} \nabla(u+\delta v), \nabla\varphi \right\rangle \\
&\qquad - \langle |\nabla u|^{p-2}\nabla u, \nabla\varphi\rangle \\
&\quad = \langle [|\nabla u|^{p-2} + (p-2)|\nabla u|^{2(\frac{p-2}{2}-1)}\delta\langle\nabla u, \nabla v\rangle \\
&\qquad + O(\delta^2)]\nabla(u+\delta v), \nabla\varphi\rangle - \langle|\nabla u|^{p-2}\nabla u, \nabla\varphi\rangle \\
&\quad = \langle [|\nabla u|^{p-2} + (p-2)|\nabla u|^{p-4}\delta\langle\nabla u, \nabla v\rangle \\
&\qquad + O(\delta^2)]\nabla(u+\delta v), \nabla\varphi\rangle \\
&\qquad - \langle|\nabla u|^{p-2}\nabla u, \nabla\varphi\rangle \\
&\quad = (p-2)\delta|\nabla u|^{p-4}\langle\nabla u, \nabla v\rangle\langle\nabla u, \nabla\varphi\rangle \\
&\qquad + \delta\langle|\nabla u|^{p-2}\nabla v, \nabla\varphi\rangle + O(\delta^2) \\
&\quad = \delta[(p-2)|\nabla u|^{p-4}\langle\nabla u, \nabla v\rangle\langle\nabla u, \nabla\varphi\rangle \\
&\qquad + \langle|\nabla u|^{p-2}\nabla v, \nabla\varphi\rangle + O(\delta)].
\end{aligned}
$$

Define

$$
\langle B(u)v, \varphi\rangle = (p-2)|\nabla u|^{p-4}\langle\nabla u, \nabla v\rangle\langle\nabla u, \nabla\varphi\rangle + \langle|\nabla u|^{p-2}\nabla v, \nabla\varphi\rangle
$$

and let $(u_n)_{n\geq 0} \subset W_0^{1,p}(\Omega)$. Assume that $u_n \to u$, as $n \to \infty$ in $W_0^{1,p}(\Omega)$. We have

$$
\begin{aligned}
&\langle B(u_n)v - B(u)v, \varphi\rangle \\
&\quad = (p-2)\left[|\nabla u_n|^{p-4}\langle\nabla u_n, \nabla v\rangle\langle\nabla u_n, \nabla\varphi\rangle - |\nabla u|^{p-4}\langle\nabla u, \nabla v\rangle\langle\nabla u, \nabla\varphi\rangle\right] \\
&\qquad + \langle|\nabla u_n|^{p-2}\nabla v, \nabla\varphi\rangle - \langle|\nabla u|^{p-2}\nabla v, \nabla\varphi\rangle.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&|\langle B(u_n)v - B(u)v, \varphi\rangle| \\
&\quad \leq (p-2)\left||\nabla u_n|^{p-4}\langle\nabla u_n, \nabla v\rangle\langle\nabla u_n, \nabla\varphi\rangle - |\nabla u|^{p-4}\langle\nabla u, \nabla v\rangle\langle\nabla u, \nabla\varphi\rangle\right| \\
&\qquad + \left||\nabla u_n|^{p-2} - |\nabla u|^{p-2}\right| |\langle\nabla v, \nabla\varphi\rangle|.
\end{aligned}
$$

By assumption, we can assume that, up to subsequences,

$(*)$ $\nabla u_n \to \nabla u$ in $(L^p(\Omega))^N$ as $n \to \infty$ and
$(**)$ $\nabla u_n(x) \to \nabla u(x)$ almost everywhere as $n \to \infty$.

Then $|\nabla u_n|^{p-4}\langle\nabla u_n, \nabla v\rangle\langle\nabla u_n, \nabla\varphi\rangle \to |\nabla u|^{p-4}\langle\nabla u, \nabla v\rangle\langle\nabla u, \nabla\varphi\rangle$ as $n \to \infty$ and consequently $\langle B(u_n)v, \varphi\rangle \to \langle B(u)v, \varphi\rangle$as $n \to \infty$. Thus, we find that $-\Delta_p - \Delta \in C^1$ and thanks to the Inverse function theorem $(-\Delta_p - \Delta)^{-1}$ is differentiable in a neighborhood of $u \in W_0^{1,p}(\Omega)$. Therefore, according to the Krasnoselski bifurcation Theorem, we obtain that λ_k^D is a bifurcation point at zero. □

4.2. Bifurcation from Infinity: The Case $1 < p < 2$

We recall the nonlinear eigenvalue problem we are investigating

$$\begin{cases} -\Delta_p u - \Delta u = \lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.3}$$

Under a solution of (4.3) (for $1 < p < 2$), we understand a pair $(\lambda, u) \in \mathbb{R}_\star^+ \times W_0^{1,2}(\Omega)$ satisfying the integral equality

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla\varphi \ \mathrm{d}x + \int_\Omega \nabla u \cdot \nabla\varphi \ \mathrm{d}x = \lambda \int_\Omega u\varphi \ \mathrm{d}x \text{ for every } \varphi \in W_0^{1,2}(\Omega). \tag{4.4}$$

Definition 4.3. Let $\lambda \in \mathbb{R}$. We say that the pair (λ, ∞) is a bifurcation point from infinity for problem (4.3) if there exists a sequence of pairs $\{(\lambda_n, u_n)\}_{n=1}^\infty \subset \mathbb{R} \times W_0^{1,p}(\Omega)$ such that Eq. (4.4) holds and $(\lambda_n, \|u_n\|_{1,2}) \to (\lambda, \infty)$.

We now state the main theorem concerning the bifurcation from infinity.

Theorem 4.4. *The pair (λ_1^D, ∞) is a bifurcation point from infinity for the problem (4.3).*

For $u \in W_0^{1,2}(\Omega)$, $u \neq 0$, we set $v = u/\|u\|_{1,2}^{2-\frac{1}{2}p}$. We have $\|v\|_{1,2} = \frac{1}{\|u\|_{1,2}^{1-\frac{1}{2}p}}$ and

$$|\nabla v|^{p-2}\nabla v = \frac{1}{\|u\|_{1,2}^{(2-\frac{1}{2}p)(p-1)}}|\nabla u|^{p-2}\nabla u.$$

Introducing this change of variable in (4.4), we find that

$$\begin{aligned} \|u\|_{1,2}^{\left(2-\frac{1}{2}p\right)(p-2)} \int_\Omega |\nabla v|^{p-2}\nabla v \cdot \nabla\varphi \ \mathrm{d}x + \int_\Omega \nabla v \cdot \nabla\varphi \ \mathrm{d}x \\ = \lambda \int_\Omega v\varphi \ \mathrm{d}x \text{ for every } \varphi \in W_0^{1,2}(\Omega). \end{aligned} \tag{4.5}$$

But, on the other hand, we have

$$\|v\|_{1,2}^{p-4} = \frac{1}{\|u\|_{1,2}^{\left(1-\frac{1}{2}p\right)(p-4)}} = \frac{1}{\|u\|_{1,2}^{\left(2-\frac{1}{2}p\right)(p-2)}}.$$

Consequently, it follows that Eq. (4.5) is equivalent to

$$\|v\|_{1,2}^{4-p}\int_\Omega |\nabla v|^{p-2}\nabla v\cdot\nabla\varphi\ \mathrm{d}x+\int_\Omega \nabla v\cdot\nabla\varphi\ \mathrm{d}x=\lambda\int_\Omega v\varphi\ \mathrm{d}x \text{ for every } \varphi\in W_0^{1,2}(\Omega). \tag{4.6}$$

This leads to the following nonlinear eigenvalue problem (for $1<p<2$):

$$\begin{cases} -\|v\|_{1,2}^{4-p}\Delta_p v-\Delta v=\lambda v & \text{in } \Omega,\\ v=0 & \text{on } \partial\Omega. \end{cases} \tag{4.7}$$

The proof of Theorem 4.4 follows immediately from the following remark, and the proof that $(\lambda_1^D,0)$ is a bifurcation of (4.7).

Remark 4.5. With this transformation, we have that the pair (λ_1^D,∞) is a bifurcation point for the problem (4.3) if and only if the pair $(\lambda_1^D,0)$ is a bifurcation point for the problem (4.7).

Let us consider a small ball $B_r(0):=\{\ w\ \in W_0^{1,2}(\Omega)/\ \|w\|_{1,2}<r\ \}$, and consider the operator

$$T:=-\|\cdot\|_{1,2}^{4-p}\Delta_p-\Delta: W_0^{1,2}(\Omega)\to W^{-1,2}(\Omega).$$

Proposition 4.6. *Let $1<p<2$. There exists $r>0$ such that the mapping*
$T:B_r(0)\subset W_0^{1,2}(\Omega)\to W^{-1,2}(\Omega)$ is invertible, with a continuous inverse.

Proof. In order to prove that the operator T is invertible with a continuous inverse, we again rely on [9, Corollary 2.5.10]. We show that there exists $\delta>0$ such that

$$\langle T(u)-T(v),u-v\rangle\geq\delta\|u-v\|_{1,2}^2,\ \text{ for } u,v\in B_r(0)\subset W_0^{1,2}(\Omega)$$

with $r>0$ sufficiently small.

Indeed, using that $-\Delta_p$ is strongly monotone on $W_0^{1,p}(\Omega)$ on the one hand and the Hölder inequality on the other hand, we have

$$\begin{aligned} &\langle T(u)-T(v),u-v\rangle\\ &\quad=\|\nabla u-\nabla v\|_2^2+\Big(\|u\|_{1,2}^{4-p}(-\Delta_p u)-\|v\|_{1,2}^{4-p}(-\Delta_p v),u-v\Big)\\ &\quad=\|u-v\|_{1,2}^2+\|u\|_{1,2}^{4-p}\left((-\Delta_p u)-(-\Delta_p v),u-v\right)\\ &\qquad+\Big(\|u\|_{1,2}^{4-p}-\|v\|_{1,2}^{4-p}\Big)\left(-\Delta_p v,u-v\right)\\ &\quad\geq\|u-v\|_{1,2}^2-\Big|\|u\|_{1,2}^{4-p}-\|v\|_{1,2}^{4-p}\Big|\,\|\nabla v\|_p^{p-1}\|\nabla(u-v)\|_p\\ &\quad\geq\|u-v\|_{1,2}^2-\Big|\|u\|_{1,2}^{4-p}-\|v\|_{1,2}^{4-p}\Big|\,C\|v\|_{1,2}^{p-1}\|u-v\|_{1,2}. \end{aligned} \tag{4.8}$$

Now, we obtain by the Mean Value Theorem that there exists $\theta \in [0,1]$ such that

$$\begin{aligned}\left|\|u\|_{1,2}^{4-p} - \|v\|_{1,2}^{4-p}\right| &= \left|\frac{\mathrm{d}}{\mathrm{d}t}\left(\|u+t(v-u)\|_{1,2}^2\right)^{2-\frac{1}{2}p}|_{t=\theta}(v-u)\right| \\ &= \left|(2-\frac{1}{2}p)\left(\|u+\theta(v-u)\|_{1,2}^2\right)^{1-\frac{1}{2}p} 2\left(u+\theta(v-u), v-u\right)_{1,2}\right| \\ &\le (4-p)\|u+\theta(v-u)\|_{1,2}^{2-p}\|u+\theta(v-u)\|_{1,2}\|u-v\|_{1,2} \\ &= (4-p)\|u+\theta(v-u)\|_{1,2}^{3-p}\|u-v\|_{1,2} \\ &\le (4-p)\left((1-\theta)\|u\|_{1,2}+\theta\|v\|_{1,2}\right)^{3-p}\|u-v\|_{1,2} \\ &\le (4-p)r^{3-p}\|u-v\|_{1,2}.\end{aligned}$$

Hence, continuing with the estimate of Eq. (4.8), we get

$$\begin{aligned}&\langle T(u)-T(v), u-v\rangle \ge \|u-v\|_{1,2}^2\left(1-(4-p)r^{3-p}Cr^{p-1}\right) \\ &\quad = \|u-v\|_{1,2}^2\left(1-C'r^2\right),\end{aligned}$$

and thus the claim, for $r > 0$ small enough.

Hence, the operator T is strongly monotone on $B_r(0)$ and it is continuous, and hence the claim follows. □

Clearly the mappings

$$T_\tau = -\Delta - \tau\|\cdot\|_{1,2}^\gamma \Delta_p : B_r(0) \subset W_0^{1,2}(\Omega) \to W^{-1,2}(\Omega),\ 0 \le \tau \le 1$$

are also local homeomorphisms for $1 < p < 2$ with $\gamma = 4 - p > 0$. Consider now the homotopy maps

$$H(\tau, y) := (-\tau\|\cdot\|_{1,2}^\gamma \Delta_p - \Delta)^{-1}(y),\ y \in T_\tau(B_r(0)) \subset W^{-1,2}(\Omega).$$

Then we can find a $\rho > 0$ such that the ball

$$B_\rho(0) \subset \bigcap_{0\le\tau\le 1} T_\tau(B_r(0))$$

and

$$H(\tau,\cdot) : B_\rho(0) \cap L^2(\Omega) \mapsto W_0^{1,2}(\Omega) \subset\subset L^2(\Omega)$$

are compact mappings. Set now

$$\tilde{S}_\lambda(u) = u - \lambda\left(-\|u\|_{1,2}^\gamma \Delta_p - \Delta\right)^{-1} u.$$

Notice that $\tilde{S}_\lambda$ is a compact perturbation of the identity in $L^2(\Omega)$. We have $0 \notin H([0,1]\times\partial B_r(0))$. So it makes sense to consider the Leray–Schauder topological degree of $H(\tau,\cdot)$ on $B_r(0)$. And by the property of the invariance by homotopy, one has

$$\deg(H(0,\cdot), B_r(0), 0) = \deg\left(H(1,\cdot), B_r(0), 0\right). \tag{4.9}$$

Theorem 4.7. *The pair $(\lambda_1^D, 0)$ is a bifurcation point in $\mathbb{R}^+ \times L^2(\Omega)$ of $\tilde{S}_\lambda(u) = 0$, for $1 < p < 2$.*

Proof. Suppose by contradiction that $(\lambda_1^D, 0)$ is not a bifurcation for $\tilde{S}_\lambda$. Then, there exist $\delta_0 > 0$ such that for all $r \in (0, \delta_0)$ and $\varepsilon \in (0, \delta_0)$,

$$\tilde{S}_\lambda(u) \neq 0 \ \forall \ |\lambda_1^D - \lambda| \leq \varepsilon, \ \forall \ u \in L^2(\Omega), \ \|u\|_2 = r. \tag{4.10}$$

Taking into account that (4.10) holds, it follows that it make sense to consider the Leray–Schauder topological degree $\deg(\tilde{S}_\lambda, B_r(0), 0)$ of $\tilde{S}_\lambda$ on $B_r(0)$.

We observe that

$$\left(I - \left(\lambda_1^D - \varepsilon\right) H(\tau, \cdot)\right)|_{\partial B_r(0)} \neq 0 \text{ for } \tau \in [0, 1]. \tag{4.11}$$

Proving (4.11) guarantee the well posedness of $\deg(I - (\lambda_1^D \pm \varepsilon) H(\tau, \cdot), B_r(0), 0)$ for any $\tau \in [0, 1]$.

Indeed, by contradiction suppose that there exists $v \in \partial B_r(0) \subset L^2(\Omega)$ such that

$v - \left(\lambda_1^D - \varepsilon\right) H(\tau, v) = 0$, for some $\tau \in [0, 1]$.

One concludes that then $v \in W_0^{1,2}(\Omega)$, and then that

$$-\Delta v - \tau \|v\|_{1,2}^\gamma \Delta_p v = \left(\lambda_1^D - \varepsilon\right) v.$$

However, we get the contradiction,

$$\left(\lambda_1^D - \varepsilon\right) \|v\|_2^2 = \|\nabla v\|_2^2 + \tau \|v\|_{1,2}^\gamma \|\nabla v\|_p^p \geq \|\nabla v\|_2^2 \geq \lambda_1^D \|v\|_2^2.$$

By the contradiction assumption, we have

$$\deg\left(I - (\lambda_1^D + \varepsilon) H(1, \cdot), B_r(0), 0\right) = \deg\left(I - (\lambda_1^D - \varepsilon) H(1, \cdot), B_r(0), 0\right). \tag{4.12}$$

By homotopy using (4.9), we have

$$\begin{aligned}\deg\left(I - (\lambda_1^D - \varepsilon) H(1, \cdot), B_r(0), 0\right) &= \deg\left(I - (\lambda_1^D - \varepsilon) H(0, \cdot), B_r(0), 0\right) \\ &= \deg\left(I - (\lambda_1^D - \varepsilon)(-\Delta)^{-1}, B_r(0), 0\right) = 1\end{aligned} \tag{4.13}$$

Now, using (4.13) and (4.12), we find that

$$\deg(I - (\lambda_1^D + \varepsilon) H(1, \cdot), B_r(0), 0) = \deg(I - (\lambda_1^D - \varepsilon) H(0, \cdot), B_r(0), 0) = 1 \tag{4.14}$$

Furthermore, since λ_1^D is a simple eigenvalue of $-\Delta$, it is well known [see [1]] that

$$\deg\left(I - (\lambda_1^D + \varepsilon)(-\Delta)^{-1}, B_r(0)\right), 0) = \deg\left(I - (\lambda_1^D + \varepsilon) H(0, \cdot), B_r(0), 0\right) = -1 \tag{4.15}$$

In order to get contradiction (to relation (4.14)), it is enough to show that

$$\deg\left(I - (\lambda_1^D + \varepsilon) H(1, \cdot), B_r(0)\right), 0) = \deg\left(I - (\lambda_1^D + \varepsilon) H(0, \cdot), B_r(0), 0\right), \tag{4.16}$$

$r > 0$ sufficiently small. We have to show that

$$\left(I - (\lambda_1^D + \varepsilon) H(\tau, \cdot)\right)|_{\partial B_r(0)} \neq 0 \text{ for } \tau \in [0, 1].$$

Suppose by contradiction that there is $r_n \to 0$, $\tau_n \in [0,1]$ and $u_n \in \partial B_{r_n}(0)$ such that

$$u_n - \left(\lambda_1^D + \varepsilon\right) H\left(\tau_n, u_n\right) = 0$$

or equivalently

$$-\tau_n \|u_n\|_{1,2}^{\gamma} \Delta_p u_n - \Delta u_n = \left(\lambda_1^D + \varepsilon\right) u_n. \tag{4.17}$$

Dividing the Eq. (4.17) by $\|u_n\|_{1,2}$, we obtain

$$-\tau_n \|u_n\|_{1,2}^{\gamma+p-1} \Delta_p \left(\frac{u_n}{\|u_n\|_{1,2}}\right) - \Delta \left(\frac{u_n}{\|u_n\|_{1,2}}\right) = (\lambda_1^D + \varepsilon) \frac{u_n}{\|u_n\|_{1,2}},$$

and by setting $v_n = \frac{u_n}{\|u_n\|_{1,2}}$, it follows that

$$-\tau_n \|u_n\|_{1,2}^{\gamma+p-1} \Delta_p v_n - \Delta v_n = (\lambda_1^D + \varepsilon) v_n. \tag{4.18}$$

But since $\|v_n\|_{1,2} = 1$, we have $v_n \rightharpoonup v$ in $W_0^{1,2}(\Omega)$ and $v_n \to v$ in $L^2(\Omega)$. Furthermore, the first term in the left-hand side of Eq. (4.18) tends to zero in $W^{-1,p'}(\Omega)$ as $r_n \to 0$ and hence in $W^{-1,2}(\Omega)$. Equation (4.17) then implies that $v_n \to v$ strongly in $W_0^{1,2}(\Omega)$ since $-\Delta : W_0^{1,2}(\Omega) \to W^{-1,2}(\Omega)$ is a homeomorphism and thus v with $\|v\|_{1,2} = 1$ solves $-\Delta v = (\lambda_1^D + \varepsilon)v$, which is impossible because $\lambda_1^D + \varepsilon$ is not the first eigenvalue of $-\Delta$ on $W_0^{1,2}(\Omega)$ for $\varepsilon > 0$.

Therefore, by homotopy it follows that

$$\deg\left(I - (\lambda_1^D + \varepsilon) H(1, \cdot), B_r(0), 0\right) = \deg\left(I - (\lambda_1^D + \varepsilon) H(0, \cdot), B_r(0), 0\right).$$

Now, thanks to (4.15), we find that

$$\deg\left(I - (\lambda_1^D + \varepsilon) H(1, \cdot), B_r(0), 0\right) = -1,$$

which contradicts Eq. (4.14). □

Theorem 4.8. *The pair $(\lambda_k^D, 0)$ ($k > 1$) is a bifurcation point of $\tilde{S}_\lambda(u) = 0$, for $1 < p < 2$ if λ_k^D is of odd multiplicity.*

Proof. Suppose by contradiction that $(\lambda_k^D, 0)$ is not a bifurcation for $\tilde{S}_\lambda$. Then, there exist $\delta_0 > 0$ such that for all $r \in (0, \delta_0)$ and $\varepsilon \in (0, \delta_0)$,

$$\tilde{S}_\lambda(u) \neq 0 \ \forall\ |\lambda_k^D - \lambda| \leq \varepsilon,\ \forall\ u \in L^2(\Omega),\ \|u\|_2 = r. \tag{4.19}$$

Taking into account that (4.19) holds, it follows that it make sense to consider the Leray–Schauder topological degree $\deg(\tilde{S}_\lambda, B_r(0), 0)$ of $\tilde{S}_\lambda$ on $B_r(0)$.

We show that

$$\left(I - \left(\lambda_k^D - \varepsilon\right) H(\tau, \cdot)\right)|_{\partial B_r(0)} \neq 0 \text{ for } \tau \in [0,1]. \tag{4.20}$$

Proving (4.20) garantees the well posedness of $\deg(I - (\lambda_k^D \pm \varepsilon)H(\tau, \cdot), B_r(0), 0)$ for any $\tau \in [0,1]$. Indeed, consider the projections P^- and P^+ onto the spaces $\operatorname{span}\{e_1, \ldots, e_{k-1}\}$ and $\operatorname{span}\{e_k, e_{k+1}, \ldots\}$, respectively, where $e_1 \ldots, e_k, e_{k+1}, \ldots$ denote the eigenfunctions associated with the Dirichlet problem (1.1).

Suppose by contradiction that relation (4.20) does not hold. Then there exists $v \in \partial B_r(0) \subset L^2(\Omega)$ such that $v - (\lambda_k^D - \varepsilon) H(\tau, v) = 0$, for some $\tau \in [0,1]$. This is equivalent of having

$$- \Delta v - \left(\lambda_k^D - \varepsilon\right) v = \tau \|v\|_{1,2}^{\gamma} \Delta_p v. \tag{4.21}$$

Replacing v by $P^+v + P^-v$, and multiplying equation (4.21) by $P^+v - P^-v$ in the both sides, we obtain

$$\begin{aligned}
&\left\langle [-\Delta - (\lambda_k^D - \varepsilon)](P^+v + P^-v), P^+v - P^-v \right\rangle \\
&\quad = \tau \|P^+v + P^-v\|_{1,2}^{\gamma} \left\langle \Delta_p [P^+v + P^-v], P^+v - P^-v \right\rangle . \\
&\qquad\qquad\qquad\qquad \Updownarrow \\
&\quad - \left[\|\nabla P^-v\|_2^2 - (\lambda_k^D - \varepsilon) \|P^-v\|_2^2 \right] \\
&\quad + \|\nabla P^+v\|_2^2 - (\lambda_k^D - \varepsilon) \|P^+v\|_2^2 = \tau \|P^+v + P^-v\|_{1,2}^{\gamma} \\
&\quad \times \langle \Delta_p [P^+v + P^-v], P^+v - P^-v \rangle .
\end{aligned}$$

But

$$\begin{aligned}
&\left\langle \Delta_p [P^+v + P^-v], P^+v - P^-v \right\rangle \\
&\quad = - \int_\Omega \left| \nabla (P^+v + P^-v) \right|^{p-2} \nabla (P^+v + P^-v) \cdot \nabla \left(P^+v - P^-v \right) \, \mathrm{d}x,
\end{aligned}$$

and using the Hölder inequality, the embedding $W_0^{1,2}(\Omega) \subset W_0^{1,p}(\Omega)$ and the fact that P^+v and P^-v do not vanish simultaneously, there is some positive constant $C' > 0$ such that $\|P^+v - P^-v\|_{1,2} \leq C'(\|P^+v\|_{1,2}^2 + \|P^-v\|_{1,2}^2) = C'\|P^+v - P^-v\|_{1,2}^2$, since $(P^+v, P^-v)_{1,2} = 0$, we have

$$\begin{aligned}
&\left| \langle \Delta_p [P^+v + P^-v], P^+v - P^-v \rangle \right| \\
&\quad \leq \|P^+v + P^-v\|_{1,p}^{p-1} \|P^+v - P^-v\|_{1,p} \\
&\quad \leq C' \|P^+v + P^-v\|_{1,2}^{p-1} \|P^+v - P^-v\|_{1,2}^2 \\
&\quad \leq C' \|P^+v + P^-v\|_{1,2}^{p+1}, \text{ since } \|P^+v - P^-v\|_{1,2}^2 \\
&\quad = \|P^+v + P^-v\|_{1,2}^2 .
\end{aligned}$$

On the other hand, thanks to the Poincaré inequality as well as the variational characterization of eigenvalues we find

$$- \left[\|\nabla P^-v\|_2^2 - (\lambda_k^D - \varepsilon) \|P^-v\|_2^2 \right] \geq 0$$

and

$$\|\nabla P^+v\|_2^2 - (\lambda_k^D - \varepsilon) \|P^+v\|_2^2 \geq 0,$$

we can bound from below these two inequalities together by $\|\nabla P^+v\|_2^2 + \|\nabla P^-v\|_2^2$.

Finally, we have

$$\|v\|_{1,2}^2 = \|\nabla P^+v\|_2^2 + \|\nabla P^-v\|_2^2 \leq \tau C' \|P^+v + P^-v\|_{1,2}^{\gamma + p + 1}, \text{ with } \gamma = 4 - p,$$

$$\Updownarrow$$

$$\|v\|_{1,2}^2 \leq C'' \|v\|_{1,2}^{\gamma + p + 1} \Leftrightarrow 1 \leq C'' r^3 \to 0,$$

for r taken small enough. This shows that (4.20) holds.

By the contradiction assumption, we have

$$\deg\left(I-(\lambda_k^D+\varepsilon)H(1,\cdot),B_r(0),0\right)=\deg\left(I-(\lambda_k^D-\varepsilon)H(1,\cdot),B_r(0),0\right). \tag{4.22}$$

By homotopy using (4.20), we have

$$\begin{aligned}&\deg\left(I-\left(\lambda_k^D-\varepsilon\right)H(1,\cdot),B_r(0),0\right)\\&\quad=\deg\left(I-\left(\lambda_k^D-\varepsilon\right)H(0,\cdot),B_r(0),0\right)\\&\quad=\deg\left(I-(\lambda_k^D-\varepsilon)(-\Delta)^{-1},B_r(0),0\right)=(-1)^{\beta},\end{aligned} \tag{4.23}$$

where β is the sum of algebraic multiplicities of the eigenvalues $\lambda_k^D-\varepsilon<\lambda$. Similarly, if β' denotes the sum of the algebraic multiplicities of the characteristic values of $(-\Delta)^{-1}$ such that $\lambda>\lambda_k^D+\varepsilon$, then

$$\deg\left(I-\left(\lambda_k^D+\varepsilon\right)H(1,\cdot),B_r(0),0\right)=(-1)^{\beta'} \tag{4.24}$$

But since $[\lambda_k^D-\varepsilon,\lambda_k^D+\varepsilon]$ contains only the eigenvalue λ_k^D, it follows that $\beta'=\beta+\alpha$, where α denotes the algebraic multiplicity of λ_k^D. Consequently, we have

$$\begin{aligned}&\deg\left(I-\left(\lambda_k^D+\varepsilon\right)H(1,\cdot),B_r(0),0\right)\\&\quad=(-1)^{\beta+\alpha}\\&\quad=(-1)^{\alpha}\deg\left(I-\left(\lambda_k^D+\varepsilon\right)H(1,\cdot),B_r(0),0\right)\\&\quad=-\deg\left(I-\left(\lambda_k^D+\varepsilon\right)H(1,\cdot),B_r(0),0\right),\end{aligned}$$

since λ_k^D is with odd multiplicity. This contradicts (4.22). □

5. Multiple Solutions

In this section we prove multiplicity results by distinguishing again the two cases $1<p<2$ and $p>2$. We recall the following definition which will be used in this section. Let X be a Banach space and $\Omega\subset X$ an open bounded domain which is symmetric with respect to the origin of X, that is, $u\in\Omega\Rightarrow -u\in\Omega$. Let Γ be the class of all the symmetric subsets $A\subseteq X\backslash\{0\}$ which are closed in $X\backslash\{0\}$.

Definition 5.1. (Krasnoselski genus) Let $A\in\Gamma$. The genus of A is the least integer $p\in\mathbb{N}^*$ such that there exists $\Phi:A\to\mathbb{R}^p$ continuous, odd and such that $\Phi(x)\neq 0$ for all $x\in A$. The genus of A is usually denoted by $\gamma(A)$.

Theorem 5.2. *Let $1<p<2$ or $2<p<\infty$, and suppose that $\lambda\in(\lambda_k^D,\lambda_{k+1}^D)$ for any $k\in\mathbb{N}^*$. Then Eq. (1.3) has at least k pairs of nontrivial solutions.*

Proof. Case 1: $1<p<2$. In this case we will avail of [[1], Proposition 10.8]. We consider the energy functional $I_\lambda:W_0^{1,2}(\Omega)\backslash\{0\}\to\mathbb{R}$ associated with the problem (1.3) defined by

$$I_\lambda(u)=\frac{2}{p}\int_\Omega|\nabla u|^p\,\mathrm{d}x+\int_\Omega|\nabla u|^2\,\mathrm{d}x-\lambda\int_\Omega u^2\,\mathrm{d}x.$$

The functional I_λ is not bounded from below on $W_0^{1,2}(\Omega)$, so we consider again the natural constraint set, the Nehari manifold on which we minimize the functional I_λ. The Nehari manifold is given by

$$\mathcal{N}_\lambda := \Big\{u \in W_0^{1,2}(\Omega)\backslash\{0\}\Big\} : \ \langle I'_\lambda(u), u\rangle = 0\}.$$

On $\mathcal{N}_\lambda$, we have $I_\lambda(u) = (\frac{2}{p} - 1)\displaystyle\int_\Omega |\nabla u|^p \ \mathrm{d}x > 0$. We clearly have that I_λ is even and bounded from below on $\mathcal{N}_\lambda$.

Now, let us show that every (PS) sequence for I_λ has a converging subsequence on $\mathcal{N}_\lambda$. Let $(u_n)_n$ be a (PS) sequence, i.e., $|I_\lambda(u_n)| \le C$, for all n, for some $C > 0$ and $I'_\lambda(u_n) \to 0$ in $W^{-1,2}(\Omega)$ as $n \to +\infty$. We first show that the sequence $(u_n)_n$ is bounded on $\mathcal{N}_\lambda$. Suppose by contradiction that this is not true, then $\displaystyle\int_\Omega |\nabla u_n|^2 \ \mathrm{d}x \to +\infty$ as $n \to +\infty$. Since $I_\lambda(u_n) = (\frac{2}{p} - 1)\displaystyle\int_\Omega |\nabla u_n|^p \ \mathrm{d}x$ we have $\displaystyle\int_\Omega |\nabla u_n|^p \ \mathrm{d}x \le c$. On $\mathcal{N}_\lambda$, we have

$$0 < \int_\Omega |\nabla u_n|^p \ \mathrm{d}x = \lambda \int_\Omega u_n^2 \ \mathrm{d}x - \int_\Omega |\nabla u_n|^2 \ \mathrm{d}x, \tag{5.1}$$

and hence $\displaystyle\int_\Omega u_n^2 \ \mathrm{d}x \to +\infty$. Let $v_n = \frac{u_n}{\|u_n\|_2}$; then $\displaystyle\int_\Omega |\nabla v_n|^2 \ \mathrm{d}x \le \lambda$ and hence v_n is bounded in $W_0^{1,2}(\Omega)$. Therefore, there exists $v_0 \in W_0^{1,2}(\Omega)$ such that $v_n \rightharpoonup v_0$ in $W_0^{1,2}(\Omega)$ and $v_n \to v_0$ in $L^2(\Omega)$. Dividing (5.1) by $\|u_n\|_2^p$, we have

$$\frac{\lambda\displaystyle\int_\Omega u_n^2 \ \mathrm{d}x - \int_\Omega |\nabla u_n|^2 \ \mathrm{d}x}{\|u_n\|_2^p} = \int_\Omega |\nabla v_n|^p \ \mathrm{d}x \to 0,$$

since $\lambda\displaystyle\int_\Omega u_n^2 \ \mathrm{d}x - \int_\Omega |\nabla u_n|^2 \ \mathrm{d}x = (\frac{2}{p} - 1)^{-1} I_\lambda(u_n)$, $|I_\lambda(u_n)| \le C$ and $\|u_n\|_2^p \to +\infty$. Now, since $v_n \rightharpoonup v_0$ in $W_0^{1,2}(\Omega) \subset W_0^{1,p}(\Omega)$, we infer that

$$\int_\Omega |\nabla v_0|^p \ \mathrm{d}x \le \liminf_{n\to+\infty} \int_\Omega |\nabla v_n|^p \ \mathrm{d}x = 0,$$

and consequently $v_0 = 0$. So $v_n \to 0$ in $L^2(\Omega)$ and this is a contradiction since $\|v_n\|_2 = 1$. So $(u_n)_n$ is bounded on $\mathcal{N}_\lambda$.

Next, we show that u_n converges strongly to u in $W_0^{1,2}(\Omega)$.

To do this, we will use the following vector inequality for $1 < p < 2$

$$\left(|x_2|^{p-2}x_2 - |x_1|^{p-2}x_1\right) \cdot (x_2 - x_1) \ge C' \left(|x_2| + |x_1|\right)^{p-2} |x_2 - x_1|^2,$$

for all $x_1, x_2 \in \mathbb{R}^N$ and for some $C' > 0$, (see [19]).

We have $\int_\Omega u_n^2 \, \mathrm{d}x \to \int_\Omega u^2 \, \mathrm{d}x$ and since $I'_\lambda(u_n) \to 0$ in $W^{-1,2}(\Omega)$, $u_n \rightharpoonup u$ in $W_0^{1,2}(\Omega)$, we also have $I'_\lambda(u_n)(u_n - u) \to 0$ and $I'_\lambda(u)(u_n - u) \to 0$ as $n \to +\infty$. On the other hand, one has

$$\begin{aligned}
\langle I'_\lambda(u_n) - I'_\lambda(u), u_n - u\rangle &= 2\left[\int_\Omega \left(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u\right)\cdot\nabla(u_n - u)\, \mathrm{d}x\right] \\
&\quad +2\int_\Omega |\nabla(u_n-u)|^2 \, \mathrm{d}x - 2\lambda\int_\Omega |u_n - u|^2 \, \mathrm{d}x \\
&\geq C'\int_\Omega (|\nabla u_n| + |\nabla u|)^{p-2}\, |\nabla(u_n-u)|^2 \, \mathrm{d}x \\
&\quad +2\int_\Omega |\nabla(u_n-u)|^2 \, \mathrm{d}x - 2\lambda\int_\Omega |u_n - u|^2 \, \mathrm{d}x \\
&\geq 2\int_\Omega |\nabla(u_n-u)|^2 \, \mathrm{d}x - 2\lambda\int_\Omega |u_n - u|^2 \, \mathrm{d}x \\
&\geq \|u_n - u\|_{1,2}^2 - \lambda\int_\Omega |u_n-u|^2 \, \mathrm{d}x.
\end{aligned}$$

Therefore, $\|u_n - u\|_{1,2} \to 0$ as $n \to +\infty$ and u_n converges strongly to u in $W_0^{1,2}(\Omega)$.

Let $\Sigma' = \{A \subset \mathcal{N}_\lambda : \ A \text{ closed and } -A = A\}$ and $\Gamma_j = \{A \in \Sigma' : \ \gamma(A) \geq j\}$, where $\gamma(A)$ denotes the Krasnoselski's genus. We show that $\Gamma_j \neq \emptyset$.

Set $E_j = \operatorname{span}\{e_i, \ i = 1, \ldots, j\}$, where e_i are the eigenfunctions associated with the problem (1.1). Let $\lambda \in (\lambda_j^D, \lambda_{j+1}^D)$, and consider $v \in S_j := \{v \in E_j : \ \int_\Omega |v|^2 \, \mathrm{d}x = 1\}$. Then set

$$\rho(v) = \left[\frac{\int_\Omega |\nabla v|^p \, \mathrm{d}x}{\lambda\int_\Omega v^2 \, \mathrm{d}x - \int_\Omega |\nabla v|^2 \, \mathrm{d}x}\right]^{\frac{1}{2-p}}.$$

Then $\lambda\int_\Omega v^2 \, \mathrm{d}x - \int_\Omega |\nabla v|^2 \, \mathrm{d}x \geq \lambda\int_\Omega v^2 \, \mathrm{d}x - \sum_{i=1}^{j}\int_\Omega \lambda_i |e_i|^2 \, \mathrm{d}x \geq (\lambda - \lambda_j)\int_\Omega |v|^2 \, \mathrm{d}x > 0$. Hence, $\rho(v)v \in \mathcal{N}_\lambda$, and then $\rho(S_j) \in \Sigma'$, and $\gamma(\rho(S_j)) = \gamma(S_j) = j$ for $1 \leq j \leq k$, for any $k \in \mathbb{N}^*$.

It is then standard (see [1], Proposition 10.8) to conclude that

$$\sigma_{\lambda,j} = \inf_{\gamma(A)\geq j}\sup_{u\in A} I_\lambda(u), \ 1 \leq j \leq k, \text{ for any } \ k \in \mathbb{N}^*$$

yields k pairs of nontrivial critical points for I_λ, which gives rise to k nontrivial solutions of problem (1.3).

Case 2: $p > 2$.

In this case, we will rely on the following theorem:

Theorem (Clark, [11]) .
Let X be a Banach space and $G \in C^1(X, \mathbb{R})$ satisfying the Palais–Smale condition with $G(0) = 0$. Let $\Gamma_k = \{ \ A \in \Sigma \ : \ \gamma(A) \geq k \ \}$ with $\Sigma = \{ \ A \subset X \ ; \ A = -A \text{ and } A \text{ closed } \}$. If $c_k = \inf_{A\in\Gamma_k}\sup_{u\in A} G(u) \in (-\infty, 0)$, then c_k is a critical value.

Let us consider the C^1 energy functional $I_\lambda : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined as

$$I_\lambda(u) = \frac{2}{p}\int_\Omega |\nabla u|^p \,\mathrm{d}x + \int_\Omega |\nabla u|^2 \,\mathrm{d}x - \lambda \int_\Omega |u|^2 \,\mathrm{d}x.$$

We want to show that

$$-\infty < \sigma_j = \inf_{\{A\in\Sigma', \gamma(A)\geq j\}} \sup_{u\in A} I_\lambda(u) \tag{5.2}$$

is a critical point for I_λ, where $\Sigma' = \{A \subseteq S_j\}$, where $S_j = \{v \in E_j \ : \ \int_\Omega |v|^2 \,\mathrm{d}x = 1\}$.

We clearly have that $I_\lambda(u)$ is an even functional for all $u \in W_0^{1,p}(\Omega)$, and also $I_\lambda(u)$ is bounded from below on $W_0^{1,p}(\Omega)$ since $I_\lambda(u) \geq C\|u\|_{1,p}^p - C'\|u\|_{1,p}^2$.

We show that $I_\lambda(u)$ satisfies the (PS) condition. Let $\{u_n\}$ be a Palais–Smale sequence, i.e., $|I_\lambda(u_n)| \leq M$ for all n, $M > 0$ and $I'_\lambda(u_n) \to 0$ in $W^{-1,p'}(\Omega)$ as $n \to \infty$. We first show that $\{u_n\}$ is bounded in $W_0^{1,p}(\Omega)$. We have

$$M \geq |C\|u_n\|_{1,p}^p - C'\|u_n\|_{1,p}^2| \geq \left(C\|u_n\|_{1,p}^{p-2} - C'\right)\|u_n\|_{1,p}^2,$$

and so $\{u_n\}$ is bounded in $W_0^{1,p}(\Omega)$.

Therefore, $u \in W_0^{1,p}(\Omega)$ exists such that, up to subsequences that we will denote by $(u_n)_n$ we have $u_n \rightharpoonup u$ in $W_0^{1,p}(\Omega)$ and $u_n \to u$ in $L^2(\Omega)$.

We will use the following inequality for $v_1, v_2 \in \mathbb{R}^N$: there exists $R > 0$ such that

$$|v_1 - v_2|^p \leq R\left(|v_1|^{p-2}v_1 - |v_2|^{p-2}v_2\right)(v_1 - v_2),$$

for $p > 2$ (see [19]). Then we obtain

$$\begin{aligned}
\langle I'_\lambda(u_n) - I'_\lambda(u), u_n - u\rangle &= 2\int_\Omega \left(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u\right)\cdot\nabla(u_n - u)\,\mathrm{d}x \\
&\quad + 2\int_\Omega |\nabla u_n - \nabla u|^2 \,\mathrm{d}x \\
&\quad - 2\lambda\int_\Omega |u_n - u|^2 \,\mathrm{d}x \\
&\geq \frac{2}{R}\int_\Omega |\nabla u_n - \nabla u|^p \,\mathrm{d}x \\
&\quad + 2\int_\Omega |\nabla u_n - \nabla u|^2 \,\mathrm{d}x - 2\lambda\int_\Omega |u_n - u|^2 \,\mathrm{d}x \\
&\geq \frac{2}{R}\|u_n - u\|_{1,p}^p - 2\lambda\int_\Omega |u_n - u|^2 \,\mathrm{d}x.
\end{aligned}$$

Therefore, $\|u_n - u\|_{1,p} \to 0$ as $n \to +\infty$, and so u_n converges to u in $W_0^{1,p}(\Omega)$.

Next, we show that there exists sets A_j of genus $j = 1, \ldots, k$ such that $\sup_{u\in A_j} I_\lambda(u) < 0$.

Consider $E_j = \text{span}\{e_i,\ i = 1, \ldots, j\}$ and $S_j = \{v \in E_j :\ \int_\Omega |v|^2\ \mathrm{d}x = 1\}$. For any $s \in (0, 1)$, we define the set $A_j(s) := s(S_j \cap E_j)$ and so $\gamma(A_j(s)) = j$ for $j = 1, \ldots, k$. We have, for any $s \in (0, 1)$

$$\begin{aligned}\sup_{u\in A_j} I_\lambda(u) &= \sup_{v\in S_j\cap E_j} I_\lambda(sv)\\ &\le \sup_{v\in S_j\cap E_j} \left\{\frac{s^p}{p}\int_\Omega |\nabla v|^p \mathrm{d}x + \frac{s^2}{2}\int_\Omega |\nabla v|^2 \mathrm{d}x - \frac{\lambda s^2}{2}\int_\Omega |v|^2 \mathrm{d}x\right\}\\ &\le \sup_{v\in S_j\cap E_j} \left\{\frac{s^p}{p}\int_\Omega |\nabla v|^p \mathrm{d}x + \frac{s^2}{2}(\lambda_j - \lambda)\right\} < 0\end{aligned}$$

for $s > 0$ sufficiently small, since $\displaystyle\int_\Omega |\nabla v|^p\ \mathrm{d}x \le c_j$, where c_j denotes some positive constant.

Finally, we conclude that $\sigma_{\lambda,j}$ $(j = 1, \ldots, k)$ are critical values thanks to Clark's Theorem. □

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