

A SIMPLE PROOF OF GLOBAL EXISTENCE FOR THE 1D PRESSURELESS GAS DYNAMICS EQUATIONS

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ABSTRACT. Sticky particle solutions to the one-dimensional pressureless gas dynamics equations can be constructed by a suitable metric projection onto the cone of monotone maps, as was shown in recent work by Natile and Savaré. Their proof uses a discrete particle approximation and stability properties for first order differential inclusions. Here we give a more direct proof that relies on a result by Haraux on the differentiability of metric projections. We apply the same method also to the one-dimensional Euler-Poisson system, obtaining a new proof for the global existence of weak solutions.

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1. INTRODUCTION

The one-dimensional pressureless gas dynamics equations

$$\left. \begin{aligned} \partial_t \varrho + \partial_x(\varrho v) &= 0 \\ \partial_t(\varrho v) + \partial_x(\varrho v^2) &= 0 \end{aligned} \right\} \text{ in } [0, \infty) \times \mathbb{R} \quad (1.1)$$

describe the dynamics of a mass (or electric charge) distribution that moves freely in the absence of any external or internal forces. The quantity ϱ is a positive Borel measure depending on time/space (t, x) , while v is the Eulerian velocity field. We will assume in the following that $\varrho(t, \cdot)$ is a probability measure for all times. This assumption is consistent with first equation in (1.1), called the continuity equation, which describes the local conservation of mass. The second equation in (1.1) models the local conservation of momentum. We are interested in the Cauchy problem, so we assume that an initial density $\bar{\varrho}$ (which is a Borel probability measure) and an initial Eulerian velocity $\bar{v} \in \mathcal{L}^2(\mathbb{R}, \bar{\varrho})$ (with finite kinetic energy) are given.

System (1.1) is a building block for semiconductor models. Its multi-dimensional version has been proposed as a simple model describing the formation of galaxies in the early stage of the universe. Since there is no pressure, the fluid elements do not

Date: September 27, 2018.

2000 Mathematics Subject Classification. 35L65, 49J40, 82C40.

Key words and phrases. Pressureless Gas Dynamics, Optimal Transport.

interact with each other. For the applications, however, one typically augments the system (1.1) with the assumption of adhesion (also called sticky particle) dynamics; see [20]: Whenever any fluid elements meet at the same location, they stick together to form larger compounds. The density ϱ can therefore have singular parts (Dirac measures). While both mass and momentum are conserved in the collisions, kinetic energy may be lost. The assumption of adhesion dynamics can be understood as an entropy condition for the hyperbolic conservation law (1.1).

There is now a huge amount of literature studying the pressureless gas dynamics equations (1.1) and establishing global existence of solutions. Frequently, a sequence of approximate solutions is constructed by considering discrete particles: The initial mass distribution is approximated by a finite sum of Dirac measures. The dynamics of these particles are described by a finite dimensional system of ordinary differential equations between collision times. Whenever particles collide, then the new velocity of the bigger particle is determined from the conservation of mass/momentum and the assumption of the correct impact law. The general existence result is obtained by sending the number of discrete particles to infinity. To pass to the limit, several approaches are feasible. We only mention two: Brenier and Grenier [6] consider the cumulative distribution function M associated to the density ϱ :

$$M(t, x) := \int_{-\infty}^x \varrho(t, dz) \quad \text{for all } (t, x) \in [0, \infty) \times \mathbb{R},$$

and show that M is the unique entropy solution of a scalar conservation law

$$\partial_t M + \partial_x A(M) = 0 \quad \text{in } [0, \infty) \times \mathbb{R},$$

where the flux function $A: [0, 1] \rightarrow \mathbb{R}$ depends on the initial density and Eulerian velocity. In particular, the function M satisfies Oleinik's entropy condition.

A second approach, introduced by Natile and Savaré [15], uses the theory of first-order differential inclusions on the space of monotone transport maps from some reference measure space $([0, 1], \mathcal{L}^1|_{[0,1]}) =: (\Omega, \mathfrak{m})$ (where \mathcal{L}^1 is the one-dimensional Lebesgue measure) to \mathbb{R} . To every density/velocity (ϱ, v) that solves (1.1) one can associate a unique map $X \in \mathcal{L}^2(\Omega, \mathfrak{m})$ with X *monotone* such that

$$\varrho(t, \cdot) = X(t, \cdot) \# \mathfrak{m} \quad \text{for all } t \in [0, \infty). \quad (1.2)$$

Here $\#$ indicates the push-forward of measures. Then X satisfies

$$\dot{X} + \partial I_{\mathcal{C}}(X) \ni \bar{V} \quad \text{for all } t \in [0, \infty), \quad (1.3)$$

where \mathcal{C} denotes the closed convex cone of all transport maps $X \in \mathcal{L}^2(\Omega, \mathfrak{m})$ that are monotone, and where $\partial I_{\mathcal{C}}$ is the subdifferential of the indicator function of \mathcal{C} . If X satisfies (1.3) and is related to ϱ through (1.2), then the Eulerian velocity v can be recovered from the Lagrangian velocity $V := \dot{X}$ through

$$V(t, \cdot) = v(t, X(t, \cdot)) \quad \text{for all } t \in [0, \infty). \quad (1.4)$$

Assuming finite kinetic energy, it is natural to require that

$$V(t, \cdot) \in \mathcal{L}^2(\Omega, \mathfrak{m}), \quad v(t, \cdot) \in \mathcal{L}^2(\mathbb{R}, \varrho(t, \cdot)).$$

The relation (1.4) in particular determines the initial Lagrangian velocity \bar{V} in (1.3) in terms of the initial data $(\varrho, v)(0, \cdot) =: (\bar{\varrho}, \bar{v})$ of the system (1.1).

Integrating (1.3) in time and using the fact that the assumption of sticky particles implies that the subdifferential $\partial I_{\mathcal{C}}(X(t, \cdot))$ is nondecreasing in time (as a subset of

$\mathcal{L}^2(\Omega, \mathfrak{m})$), one can show that the solution of (1.3) can be written as

$$X(t, \cdot) = P_{\mathcal{C}}(\bar{X} + t\bar{V}) \quad \text{for all } t \in [0, \infty), \quad (1.5)$$

with $\bar{X} := X(0, \cdot)$ specified by (1.2); see [15]. Here $P_{\mathcal{C}}$ denotes the metric projection onto the cone \mathcal{C} . The connection between (1.1) and (1.3) makes it possible to apply classical results from the theory of first-order differential inclusions in Hilbert spaces to study the pressureless gas dynamics equations; see [7, 15] for more.

In particular, one can use stability results for first-order differential inclusions to prove that the discrete particle approximation outlined above generates (a certain class of) solutions for (1.1), as the number of particles converges to infinity. Similar results can be obtained by applying the scalar conservation law approach by Brenier and Grenier. We also refer the reader to [3–5, 8, 10, 11, 13, 14, 16–18].

In this paper, we will show directly (that is, without passing through a discrete particle approximation first) that the representation (1.5) generates solutions of the pressureless gas dynamics equations. To this end, we need a good understanding of the derivative of the map $t \mapsto X(t, \cdot)$, thus of the derivative of the metric projection away from the boundary of \mathcal{C} . A classical result by Haraux [12] gives a variational characterization of \dot{X} , which in our case implies that V is exactly of the form (1.4), where v is a suitable Eulerian velocity. Our approach is therefore a bit simpler than the ones mentioned above, which rely either on the theory of entropy solutions of scalar conservation laws, or alternatively on the theory of first-order differential inclusions. We will show how our method can be generalized to the case when (1.1) contains force terms. In particular, we consider the Euler-Poisson system.

2. DIFFERENTIABILITY OF METRIC PROJECTIONS

Let \mathcal{C} be a closed convex subset in some Hilbert space \mathcal{H} , equipped with scalar product $\langle \cdot, \cdot \rangle$ and induced norm $\|\cdot\|$. For any $Y \in \mathcal{H}$ we will denote by $P_{\mathcal{C}}(Y)$ the metric projection of Y onto the cone \mathcal{C} , so that $P_{\mathcal{C}}(Y)$ satisfies

$$\|Y - P_{\mathcal{C}}(Y)\| = \inf \left\{ \|Y - Z\| : Z \in \mathcal{C} \right\}.$$

It is well-known that the metric projection $P_{\mathcal{C}}(Y)$ exists and is uniquely determined for all $Y \in \mathcal{H}$. Moreover, it is characterized by the following property:

$$Y^* = P_{\mathcal{C}}(Y) \iff \begin{cases} Y^* \in \mathcal{C}, \\ \langle Y - Y^*, Y^* - Z \rangle \geq 0 \quad \text{for all } Z \in \mathcal{C}. \end{cases} \quad (2.1)$$

If \mathcal{C} is a cone, then one can choose $Z = 0$ and $Z = 2Y^*$ in (2.1) to obtain

$$\langle Y - Y^*, Y^* \rangle = 0, \quad \langle Y - Y^*, Z \rangle \leq 0 \quad \text{for all } Z \in \mathcal{C}. \quad (2.2)$$

It is also well-known that the metric projection is a contraction:

$$\|P_{\mathcal{C}}(Y_1) - P_{\mathcal{C}}(Y_2)\| \leq \|Y_1 - Y_2\| \quad \text{for all } Y_1, Y_2 \in \mathcal{H}. \quad (2.3)$$

We refer the reader to [19] for further information on metric projections.

For given $\bar{X}, \bar{V} \in \mathcal{H}$ we now consider the map $t \mapsto X_t := P_{\mathcal{C}}(\bar{X} + t\bar{V})$ for $t \in \mathbb{R}$, which is well-defined and Lipschitz continuous because of (2.3). We have

$$\|X_{t+h} - X_t\| \leq |h| \|\bar{V}\| \quad \text{for all } t, h \in \mathbb{R}. \quad (2.4)$$

The velocity $V_t := \lim_{h \rightarrow 0} (X_{t+h} - X_t)/h$ exists strongly for a.e. $t \in \mathbb{R}$ and satisfies the inequality $\|V_t\| \leq \|\bar{V}\|$. For any $Z \in \mathcal{C}$ we define the tangent cone

$$\mathbb{T}_Z \mathcal{C} := \overline{T_Z \mathcal{C}}, \quad T_Z \mathcal{C} := \bigcup_{h>0} h(\mathcal{C} - Z). \quad (2.5)$$

Note that $T_Z \mathcal{C}$ is a convex set: if $h_1, h_2 > 0$ and $Y_1, Y_2 \in \mathcal{C}$ are given, then

$$(1 - \lambda)h_1(Y_1 - Z) + \lambda h_2(Y_2 - Z) = h \left(((1 - \mu)Y_1 + \mu Y_2) - Z \right) \in T_Z \mathcal{C}$$

for all $\lambda \in [0, 1]$, where $h := (1 - \lambda)h_1 + \lambda h_2 > 0$ and $\mu := \lambda h_2/h \in [0, 1]$. For any $Y \in \mathcal{H}$ we denote by $[Y]^\perp$ the orthogonal complement of $\mathbb{R}Y$.

We will rely on the following result:

Theorem 2.1. *For fixed $t \in \mathbb{R}$, let V_t be any weak limit point of*

$$V_t(h_n) := (X_{t+h_n} - X_t)/h_n$$

as $h_n \rightarrow 0+$. Then $V_t \in \Sigma_{X_t} \mathcal{C}$ and $\langle \bar{V} - V_t, V_t \rangle \geq 0$, where

$$\Sigma_{X_t} \mathcal{C} := \mathbb{T}_{X_t} \mathcal{C} \cap [(\bar{X} + t\bar{V}) - X_t]^\perp. \quad (2.6)$$

Moreover, we have $\langle \bar{V} - V_t, W \rangle \leq 0$ for all $W \in T_{X_t} \mathcal{C} \cap [(\bar{X} + t\bar{V}) - X_t]^\perp$.

Remark 2.2. Theorem 2.1 follows from Proposition 1 in [12]. We include the short proof below for the reader's convenience. If the cone \mathcal{C} is *polyhedral*, so that

$$\overline{T_{X_t} \mathcal{C} \cap [(\bar{X} + t\bar{V}) - X_t]^\perp} = \Sigma_{X_t} \mathcal{C}$$

for all $t \in \mathbb{R}$, then the map $t \mapsto X_t$ is strongly right-differentiable and, denoting the right derivative again by V_t , we have $V_t = \mathbb{P}_{\Sigma_{X_t} \mathcal{C}}(\bar{V})$ for all $t \in \mathbb{R}$. Recall that the metric projection onto a closed convex set is uniquely determined, so the weak limit points of $V_t(h_n)$ for all sequences $h_n \rightarrow 0+$ have the same limit. In order to prove the claim it is sufficient to observe that, by continuity, we have $\langle \bar{V} - V_t, W \rangle \leq 0$ for all $W \in \Sigma_{X_t} \mathcal{C}$. In particular, this implies that $\langle \bar{V} - V_t, V_t \rangle = 0$.

Proof of Theorem 2.1. The difference quotients $V_t(h)$ are uniformly bounded in \mathcal{H} for all $t, h \in \mathbb{R}$ (because of (2.4)), and therefore weakly precompact. Let $h_n \rightarrow 0+$ be any sequence such that the weak limit $V_t := \lim_{n \rightarrow \infty} V_t(h_n)$ exists. Then

$$\begin{aligned} 0 &\geq \left\langle (\bar{X} + (t + h_n)\bar{V}) - X_{t+h_n}, X_t - X_{t+h_n} \right\rangle \\ &= \left\langle ((\bar{X} + t\bar{V}) + h_n\bar{V}) - (h_n V_t(h_n) + X_t), X_t - (h_n V_t(h_n) + X_t) \right\rangle \\ &= -h_n \left\langle (\bar{X} + t\bar{V}) - X_t, V_t(h_n) \right\rangle - h_n^2 \left\langle \bar{V} - V_t(h_n), V_t(h_n) \right\rangle; \end{aligned}$$

see (2.1). We used that $X_{t+h_n} = X_t + h_n V_t(h_n)$ and $X_t \in \mathcal{C}$. This implies that

$$\begin{aligned} -h_n^2 \left\langle \bar{V} - V_t(h_n), V_t(h_n) \right\rangle &\leq h_n \left\langle (\bar{X} + t\bar{V}) - X_t, V_t(h_n) \right\rangle \\ &= \left\langle (\bar{X} + t\bar{V}) - X_t, X_{t+h_n} - X_t \right\rangle \leq 0. \end{aligned} \quad (2.7)$$

We now divide by $h_n^2 > 0$ and let $n \rightarrow \infty$ to obtain $-\langle \bar{V} - V_t, V_t \rangle \leq 0$, using

$$\langle \bar{V}, V_t(h_n) \rangle \rightarrow \langle \bar{V}, V_t \rangle \quad \text{and} \quad \|V_t\|^2 \leq \liminf_{n \rightarrow \infty} \|V_t(h_n)\|^2.$$

Dividing (2.7) by $h_n > 0$ instead and letting $n \rightarrow \infty$, we obtain

$$\langle (\bar{X} + t\bar{V}) - X_t, V_t \rangle = 0 \quad (2.8)$$

because the $V_t(h_n)$ are uniformly bounded. On the other hand, since $T_{X_t}\mathcal{C}$ (recall (2.5)) is convex, its weak and strong closures coincide. Then the weak limit

$$V_t = \lim_{n \rightarrow \infty} (X_{t+h_n} - X_t)/h_n$$

satisfies $V_t \in \mathbb{T}_{X_t}\mathcal{C}$. It follows that $V_t \in \Sigma_{X_t}\mathcal{C}$.

Consider now any $Z \in \mathcal{C}$ with the property that

$$\langle (\bar{X} + t\bar{V}) - X_t, Z - X_t \rangle = 0. \quad (2.9)$$

Let $\delta_n := V_t(h_n) - V_t$ so that $\delta_n \rightarrow 0$ as $n \rightarrow \infty$. Then

$$\begin{aligned} 0 &\geq \left\langle (\bar{X} + (t+h_n)\bar{V}) - X_{t+h_n}, Z - X_{t+h_n} \right\rangle \\ &= \left\langle ((\bar{X} + t\bar{V}) - X_t) + h_n(\bar{V} - V_t) - h_n\delta_n, (Z - X_t) - h_nV_t - h_n\delta_n \right\rangle. \end{aligned}$$

Rearranging terms and dividing by $h_n > 0$, we obtain

$$\begin{aligned} \langle \bar{V} - V_t, Z - X_t \rangle &\leq -\langle (\bar{X} + t\bar{V}) - X_t, Z - X_t \rangle / h_n \\ &\quad + \left(\langle (\bar{X} + t\bar{V}) - X_t, V_t \rangle + \langle (\bar{X} + t\bar{V}) - X_t, \delta_n \rangle + \langle \delta_n, Z - X_t \rangle \right) \\ &\quad + h_n \left(\langle \bar{V} - V_t, V_t \rangle + \langle \bar{V} - V_t, \delta_n \rangle - \langle \delta_n, V_t \rangle - \|\delta_n\|^2 \right). \end{aligned}$$

Passing to the limit on the right-hand side and using (2.9)/(2.8), we obtain

$$\langle \bar{V} - V_t, Z - X_t \rangle \leq 0 \quad \text{for all } Z \in \mathcal{C}.$$

We observe now that for any $W \in T_{X_t}\mathcal{C} \cap [(\bar{X} + t\bar{V}) - X_t]^\perp$ there exist $h > 0$ and $Z \in \mathcal{C}$ with $W = h(Z - X_t)$ and (2.9). Then $\langle \bar{V} - V_t, W \rangle \leq 0$ follows. \square

3. STICKY PARTICLE SOLUTIONS

We now apply the result of the previous section to the following situation: the Hilbert space $\mathcal{H} = \mathcal{L}^2(\mathbb{R}, \mu)$ for some reference measure $\mu \in \mathcal{P}_2(\mathbb{R})$, where $\mathcal{P}_2(\mathbb{R})$ denotes the space of all Borel probability measures with finite second moment (thus $\int_{\mathbb{R}} |x|^2 \mu(dx) < \infty$). The inner product $\langle \cdot, \cdot \rangle$ and the norm $\|\cdot\|$ are the $\mathcal{L}^2(\mathbb{R}, \mu)$ -inner product and -norm, respectively. The cone is defined as

$$\mathcal{C} := \{X \in \mathcal{L}^2(\mathbb{R}, \mu) : X \text{ is monotone}\}. \quad (3.1)$$

We say that X is monotone if the support of the induced transport plan

$$\gamma_X := (\text{id}, X) \# \mu \quad (3.2)$$

is a monotone set in $\mathbb{R} \times \mathbb{R}$, where $\#$ denotes the push-forward of measures. This definition is motivated by the theory of optimal transport, where a transport plan (a probability measure of the product space) is optimal if its support is contained in the subdifferential of a convex function (see [2]), which is monotone. Recall that a subset $\Gamma \subset \mathbb{R} \times \mathbb{R}$ is monotone if for any $(m_i, x_i) \in \Gamma$, $i = 1..2$, we have

$$(m_1 - m_2)(x_1 - x_2) \geq 0.$$

For a Borel measure ν defined on some topological space Ω , we say $z \in \text{spt } \nu$ if

$$\nu(N) > 0 \quad \text{for every open neighborhood } N \text{ of } z.$$

Consequently, a point $(m, x) \in \mathbb{R}^d \times \mathbb{R}^d$ belongs to the support $\text{spt } \gamma_X$ if

$$\gamma_X(B_\delta(m) \times B_\delta(x)) > 0 \quad \text{for all } \delta > 0. \quad (3.3)$$

Lemma 3.1. *For given reference measure $\mu \in \mathcal{P}_2(\mathbb{R})$ and $X \in \mathcal{C}$ we define γ_X as in (3.2). Then there exists a Borel set $N_X \subset \mathbb{R}$ with $\mu(N_X) = 0$ such that*

$$(m, X(m)) \in \text{spt } \gamma_X \quad \text{for all } m \in \mathbb{R} \setminus N_X. \quad (3.4)$$

Proof. It is known that every finite Borel measure ν on a locally compact Hausdorff space Ω with countable basis is inner regular; see e.g. [9] for more details. It follows that $\nu(U) = 0$, where $U := \Omega \setminus \text{spt } \nu$. Indeed we have

$$\nu(U) = \sup \left\{ \nu(K) : K \subset U \text{ compact} \right\}, \quad (3.5)$$

by inner regularity of ν . For any $x \in U$ there exists an open neighborhood N_x of x with $\nu(N_x) = 0$, as follows from the definition of $\text{spt } \nu$. The family $\{N_x\}_{x \in U}$ is an open covering of the compact set K , and so a finite subcovering exists: Let $V \subset U$ be a finite set such that $K \subset \bigcup_{x \in V} N_x$. Then we can estimate

$$\nu(K) \leq \sum_{x \in V} \nu(N_x) = 0. \quad (3.6)$$

Combining this inequality with (3.5), we obtain that $\nu(U) = 0$.

Applying this observation to transport plans, we obtain (3.4). Just note that

$$\begin{aligned} \mu \left(\left\{ m \in \mathbb{R} : (m, X(m)) \notin \text{spt } \gamma_X \right\} \right) &= \mu \left((\text{id} \times X)^{-1} \left((\mathbb{R} \times \mathbb{R}) \setminus \text{spt } \gamma_X \right) \right) \\ &= \gamma_X \left((\mathbb{R} \times \mathbb{R}) \setminus \text{spt } \gamma_X \right) = 0, \end{aligned}$$

by the definition of push-forward of measures. \square

Lemma 3.2. *The set \mathcal{C} defined in (3.1) is a closed convex cone in $\mathcal{L}^2(\mathbb{R}, \mu)$. For any $X \in \mathcal{C}$ and any smooth, strictly increasing function $\zeta : \mathbb{R} \rightarrow \mathbb{R}$ that coincides with the identity map outside a compact set, we have $\zeta \circ X \in \mathcal{C}$.*

Proof. We proceed in two steps.

Step 1. Consider a sequence of $X^k \in \mathcal{C}$ converging strongly to $X \in \mathcal{L}^2(\mathbb{R}, \mu)$. The sequence of transport plans γ_{X^k} , as defined in (3.2), converges narrowly to γ_X ; see Lemma 5.4.1 in [2]. Narrow convergence implies convergence in the Kuratowski sense (see Proposition 5.1.8 in [2]): for any pair of points $(\bar{m}_i, \bar{x}_i) \in \text{spt } \gamma_X$, $i = 1..2$, there exist sequences of $(m_i^k, x_i^k) \in \text{spt } \gamma_{X^k}$ such that

$$(m_i^k, x_i^k) \rightarrow (\bar{m}_i, \bar{x}_i) \quad \text{as } k \rightarrow \infty,$$

for $i = 1..2$. Since spt_{X^k} is monotone, by definition of \mathcal{C} , we have

$$(\bar{m}_1 - \bar{m}_2)(\bar{x}_1 - \bar{x}_2) = \lim_{k \rightarrow \infty} (m_1^k - m_2^k)(x_1^k - x_2^k) \geq 0.$$

This implies that also $\text{spt } \gamma_X$ is monotone and $X \in \mathcal{C}$. Therefore \mathcal{C} is closed.

Let now $X^1, X^2 \in \mathcal{C}$ and $s \in [0, 1]$ be given and define $X_s := (1-s)X^1 + sX^2$, which is an element in $\mathcal{L}^2(\mathbb{R}, \mu)$. Let γ_{X_s} be corresponding transport plan defined by (3.2), and let N^k be the μ -null sets in Lemma 3.1 corresponding to X^k , $k = 1..2$. Pick any pair of points $(\bar{m}_i, \bar{x}_i) \in \text{spt } \gamma_{X_s}$, $i = 1..2$. For any $\delta > 0$ there exist

$$m_i \in B_\delta(\bar{m}_i) \setminus (N^1 \cup N^2)$$

such that $X_s(m_i) \in B_\delta(\bar{x}_i)$. Indeed assume that not. Then

$$\begin{aligned} 0 &= \mu\left(\{m \in B_\delta(\bar{m}_i) \setminus (N^1 \cup N^2): X_s(m) \in B_\delta(\bar{x}_i)\}\right) \\ &= \mu\left(\{m \in B_\delta(\bar{m}_i): X_s(m) \in B_\delta(\bar{x}_i)\}\right) \\ &= \gamma_{X_s}(B_\delta(\bar{m}_i) \times B_\delta(\bar{x}_i)), \end{aligned}$$

which would be a contradiction to our choice of $(\bar{m}_i, \bar{x}_i) \in \text{spt } \gamma_{X_s}$. We get that

$$\begin{aligned} (\bar{m}_1 - \bar{m}_2)(\bar{x}_1 - \bar{x}_2) &\geq (m_1 - m_2)(X_s(m_1) - X_s(m_2)) \\ &\quad - 2\delta(|\bar{m}_1| + |\bar{m}_2| + |\bar{x}_1| + |\bar{x}_2|) - 4\delta^2. \end{aligned} \quad (3.7)$$

But since $(m_i, X^k(m_i)) \in \text{spt } \gamma_{X^i}$ and $X^k \in \mathcal{C}$, and by definition of X_s , it follows that the first term on the right-hand side of (3.7) is nonnegative. Recall that $\delta > 0$ was arbitrary. Therefore the left-hand side of (3.7) is nonnegative as well. Since this construction works for any $(\bar{m}_i, \bar{x}_i) \in \text{spt } \gamma_{X_s}$, $i = 1, 2$, we have that $\text{spt } \gamma_{X_s}$ is a monotone set, and thus $X_s \in \mathcal{C}$ for any $s \in [0, 1]$. Hence \mathcal{C} is convex.

The proof that \mathcal{C} is a cone is similar and omitted.

Step 2. Consider now a smooth function ζ as above and $X \in \mathcal{C}$. Choose $M > 0$ such that $\zeta(x) = x$ for all $|x| \geq M$ and let $Y := \zeta \circ X$. Then

$$\begin{aligned} \int_{\mathbb{R}} |Y|^2 \mu &= \int_{\{|X| < M\}} |Y|^2 \mu + \int_{\{|X| \geq M\}} |Y|^2 \mu \\ &\leq \|\zeta\|_{\mathcal{L}^\infty([-M, M])}^2 + \int_{\mathbb{R}} |X|^2 \mu, \end{aligned}$$

which is finite, and so $Y \in \mathcal{L}^2(\mathbb{R}, \mu)$. With γ_X and γ_Y defined as in (3.2), we must show that $\text{spt } \gamma_Y$ is a monotone subset of $\mathbb{R} \times \mathbb{R}$. We first claim that

$$\text{spt } \gamma_Y = F(\text{spt } \gamma_X), \quad \text{where } F(m, x) := (m, \zeta(x))$$

for all $(m, x) \in \mathbb{R} \times \mathbb{R}$. Notice that ζ is invertible. Consider any $(\bar{m}, \bar{y}) \in F(\text{spt } \gamma_X)$ and let $\bar{x} := \zeta^{-1}(\bar{y})$, so that $(\bar{m}, \bar{x}) \in \text{spt } \gamma_X$. For $\delta > 0$ arbitrary we have

$$\gamma_Y(B_\delta(\bar{m}) \times B_\delta(\bar{y})) = \mu\left(\{m \in B_\delta(\bar{m}): Y(m) \in B_\delta(\bar{y})\}\right),$$

by definition of γ_Y . Since ζ is smooth, there exists $\varepsilon > 0$ with $B_\varepsilon(\bar{x}) \subset \zeta^{-1}(B_\delta(\bar{y}))$. Without loss of generality, we may assume that $\varepsilon \leq \delta$. Then

$$\begin{aligned} \gamma_Y(B_\delta(\bar{m}) \times B_\delta(\bar{y})) &\geq \mu\left(\{m \in B_\varepsilon(\bar{m}): X(m) \in B_\varepsilon(\bar{x})\}\right) \\ &= \gamma_X(B_\varepsilon(\bar{m}) \times B_\varepsilon(\bar{x})), \end{aligned}$$

which is positive since $(\bar{m}, \bar{x}) \in \text{spt } \gamma_X$; see (3.3). It follows that $F(\text{spt } \gamma_X) \subset \text{spt } \gamma_Y$. For the converse direction, we argue analogously, noting that F is invertible with $F^{-1}(m, y) := (m, \zeta^{-1}(y))$ for all $(m, y) \in \mathbb{R} \times \mathbb{R}$. Consider now $(m_i, y_i) \in \text{spt } \gamma_Y$, $i = 1, 2$. Then $(m_i, x_i) \in \text{spt } \gamma_X$ with $x_i := \zeta^{-1}(y_i)$. Assume $x_1 \neq x_2$. Then

$$(y_1 - y_2)(m_1 - m_2) = \frac{\zeta(x_1) - \zeta(x_2)}{x_1 - x_2} \cdot (x_1 - x_2)(m_1 - m_2) \geq 0.$$

Indeed the first factor is positive since ζ is strictly increasing, and the second factor is nonnegative since $\text{spt } \gamma_X$ is a monotone set. We conclude that $Y \in \mathcal{C}$. \square

We are going to prove the following result; see [15].

Theorem 3.3 (Global Existence). *Let initial data $\bar{\varrho} \in \mathcal{P}_2(\mathbb{R})$ and $\bar{v} \in \mathcal{L}^2(\mathbb{R}, \bar{\varrho})$ of the pressureless gas dynamics equations (1.1) be given. For some reference measure $\mu \in \mathcal{P}_2(\mathbb{R})$, let $\bar{X} \in \mathcal{C}$ be the unique monotone transport with $\bar{X}\#\mu = \bar{\varrho}$. Let*

$$\bar{V} := \bar{v} \circ \bar{X}, \quad X_t := \text{P}_C(\bar{X} + t\bar{V}) \quad \text{for all } t \in \mathbb{R}. \quad (3.8)$$

Then X_t is differentiable for a.e. $t \in \mathbb{R}$ and $V_t := \dot{X}_t$ can be written in the following form: there exists a velocity $v_t \in \mathcal{L}^2(\mathbb{R}, \varrho_t)$ with $\varrho_t := X_t\#\mu$, such that $V_t = v_t \circ X_t$. The pair (ϱ_t, v_t) is a weak solution of the conservation law (1.1).

Remark 3.4. A comment is in order on the role played by the reference measure μ . In Theorem 3.3 we assume the existence of a monotone transport map $\bar{X} \in \mathcal{C}$ such that $\bar{X}\#\mu = \bar{\varrho}$. Such a map may not exist (e.g. if the initial density $\bar{\varrho}$ is a diffuse measure and the reference measure μ is concentrated). In the following discussion, no particular properties of μ are used. The theorem should hence be understood in the sense that for any given $\bar{\varrho}$ there exist some reference measure μ and a monotone transport map $\bar{X} \in \mathcal{C}$ with the desired property $\bar{X}\#\mu = \bar{\varrho}$.

Let us start with three lemmas.

Lemma 3.5. *For any X_t as in Theorem 3.3 we define*

$$\mathcal{H}_{X_t} := \mathcal{L}^2(\mathbb{R}, \mu)\text{-closure of } \{\varphi \circ X_t : \varphi \in \mathcal{D}(\mathbb{R})\} \quad (3.9)$$

and $\varrho_t := X_t\#\mu$. Then the following statement is true: the function $W \in \mathcal{L}^2(\mathbb{R}, \mu)$ is in \mathcal{H}_{X_t} if and only if there exists $w \in \mathcal{L}^2(\mathbb{R}, \varrho_t)$ such that $W = w \circ X_t$.

Proof. Let $W \in \mathcal{H}_{X_t}$ be given and consider a sequence of $\varphi^k \in \mathcal{D}(\mathbb{R})$ with

$$\varphi^k \circ X_t \longrightarrow W \quad \text{in } \mathcal{L}^2(\mathbb{R}, \mu), \quad (3.10)$$

so that the $\varphi^k \circ X_t$ form a Cauchy sequence in $\mathcal{L}^2(\mathbb{R}, \mu)$. Since

$$\|\varphi^k \circ X_t\|_{\mathcal{L}^2(\mathbb{R}, \mu)} = \|\varphi^k\|_{\mathcal{L}^2(\mathbb{R}, \varrho_t)},$$

the φ^k then form a Cauchy sequence in $\mathcal{L}^2(\mathbb{R}, \varrho_t)$. Hence there exists $w \in \mathcal{L}^2(\mathbb{R}, \varrho_t)$ with the property that $\varphi^k \longrightarrow w$ in $\mathcal{L}^2(\mathbb{R}, \varrho_t)$, by completeness. This implies

$$\varphi^k \circ X_t \longrightarrow w \circ X_t \quad \text{in } \mathcal{L}^2(\mathbb{R}, \mu). \quad (3.11)$$

Combining (3.10) and (3.11), we obtain that $W = w \circ X_t$.

For the converse direction, we recall that any finite Borel measure ν on a locally compact Hausdorff space Ω with continuous base is inner regular and so the space of all continuous functions with compact support is dense in $\mathcal{L}^2(\Omega, \nu)$. We refer the reader to [9] for more information. If Ω is also a vector space, the same statement is true for smooth functions with compact support. Let $W = w \circ X_t$ be given and choose a sequence of $\varphi^k \in \mathcal{D}(\mathbb{R})$ such that $\varphi^k \longrightarrow w$ in $\mathcal{L}^2(\mathbb{R}, \varrho_t)$. Since

$$\|W - \varphi^k \circ X_t\|_{\mathcal{L}^2(\mathbb{R}, \mu)} = \|w - \varphi^k\|_{\mathcal{L}^2(\mathbb{R}, \varrho_t)} \longrightarrow 0$$

for $k \rightarrow \infty$, we conclude that $W \in \mathcal{H}_{X_t}$. This finishes the proof. \square

Lemma 3.6. *Let \mathcal{H}_{X_t} be defined as in Lemma 3.5. Then $\mathcal{H}_{X_t} \subset \mathbb{S}_{X_t}\mathcal{C}$, with*

$$\mathbb{S}_{X_t}\mathcal{C} := \mathcal{L}^2(\mathbb{R}, \mu)\text{-closure of } T_{X_t}\mathcal{C} \cap [(\bar{X} + t\bar{V}) - X_t]^\perp.$$

Proof. By density, it is enough to show that $\varphi \circ X_t \in T_{X_t}\mathcal{C} \cap [(\bar{X} + t\bar{V}) - X_t]^\perp$ for all $\varphi \in \mathcal{D}(\mathbb{R})$. We choose a constant $h > \|\varphi'\|_{\mathcal{L}^\infty(\mathbb{R})}$ and define

$$Z_h^\pm := \left(\text{id} \pm \frac{1}{h}\varphi \right) \circ X_t,$$

which is a composition of a (smooth) monotone map with X_t , and thus an element of the cone \mathcal{C} ; see Lemma 3.2. Rearranging terms, we obtain the inclusion

$$\varphi \circ X_t = h(Z_h^+ - X_t) \in T_{X_t}\mathcal{C}.$$

On the other hand, using the characterization (2.1), we find that

$$\pm \langle (\bar{X} + t\bar{V}) - X_t, \varphi \circ X_t \rangle = h \langle (\bar{X} + t\bar{V}) - X_t, Z_h^\pm - X_t \rangle \leq 0,$$

from which the orthogonality $\varphi \circ X_t \in [(\bar{X} + t\bar{V}) - X_t]^\perp$ follows. \square

Lemma 3.7. *With the notation above, we have $V_t \in \mathcal{H}_{X_t}$ for a.e. $t \in \mathbb{R}$.*

Proof. Since the map $t \mapsto X_t$ is Lipschitz continuous, it is (strongly) differentiable for a.e. $t \in \mathbb{R}$. Fix any such $t \in \mathbb{R}$. Then the velocity $V_t := \dot{X}_t$ satisfies

$$V_t = \lim_{h \rightarrow 0^+} \frac{X_{t+h} - X_t}{h} = - \lim_{h \rightarrow 0^+} \frac{X_{t-h} - X_t}{h},$$

which implies that $V_t \in \mathbb{T}_{X_t}\mathcal{C} \cap (-\mathbb{T}_{X_t}\mathcal{C})$. We now proceed in three steps.

Step 1. Since $X_t \in \mathcal{C}$, the support of $\gamma_{X_t} := (\text{id}, X_t)\#\mu$ is a monotone subset of $\mathbb{R} \times \mathbb{R}$, which can be extended to a maximal monotone set Γ ; see [1]. We denote by u the corresponding maximal monotone set-valued map:

$$u(m) := \{x \in \mathbb{R} : (m, x) \in \Gamma\} \quad \text{for } m \in \mathbb{R}.$$

Then the level sets of u are all closed intervals; see [1]. Consequently, there can be at most countably many level sets of u that contain more than one point.

Let now N_{X_t} be the μ -null set from Lemma 3.1 with the property that

$$(m, X_t(m)) \in \text{spt } \gamma_{X_t} \quad \text{for all } m \in \mathbb{R} \setminus N_{X_t}.$$

For any $x \in \mathbb{R}$ we define the level set

$$L_x := \left\{ m \in \mathbb{R} \setminus N_{X_t} : X_t(m) = x \right\}.$$

Note that the L_x are pairwise disjoint because X_t is a function (thus single-valued). Since every L_x is contained in the corresponding x -level set of u , the set

$$\mathcal{O} := \left\{ x \in \mathbb{R} : L_x \text{ has more than one element} \right\}$$

is at most countable. Then the map X_t is invertible on $\mathbb{R} \setminus \bigcup_{x \in \mathcal{O}} L_x$.

Step 2. Consider any $W \in \mathbb{T}_{X_t}\mathcal{C}$. Then there exist $Y^k \in \mathcal{C}$, $\lambda^k > 0$ with

$$W^k := Y^k - \lambda^k X_t \longrightarrow W \quad \text{in } \mathcal{L}^2(\mathbb{R}, \mu), \quad (3.12)$$

where we used that \mathcal{C} is a cone. Extracting a subsequence if necessary, we may assume that $W^k \longrightarrow W$ μ -a.e. For any $k \in \mathbb{N}$ we denote by N^k the μ -null set from Lemma 3.1, corresponding to $Y^k \in \mathcal{C}$. Then there exists a $B \subset \mathbb{R}$ with $\mu(B) = 0$

such that $W^k(m) \rightarrow W(m)$ for every $m \in \mathbb{R} \setminus B$, and B contains all N^k . For any $x \in \mathcal{O}$ consider now $m_1, m_2 \in L_x \setminus B$ with $m_1 \leq m_2$. For all $k \in \mathbb{N}$ we have

$$\begin{aligned} W^k(m_2) - W^k(m_1) &= \left((Y^k(m_2) - \lambda^k X_t(m_2)) - (Y^k(m_1) - \lambda^k X_t(m_1)) \right) \\ &= Y^k(m_2) - Y^k(m_1) \end{aligned} \quad (3.13)$$

because $X_t(m_1) = X_t(m_2) = x$. Since $Y^k \in \mathcal{C}$ and $m_1, m_2 \notin N^k$, we get

$$(m_2 - m_1)(Y^k(m_2) - Y^k(m_1)) \geq 0,$$

which implies a similar inequality for W^k . Passing to the limit $k \rightarrow \infty$, we obtain the following statement: for all $W \in \mathbb{T}_{X_t} \mathcal{C}$ and for all $x \in \mathcal{O}$, we have

$$(m_2 - m_1)(W(m_2) - W(m_1)) \geq 0$$

for μ -a.e. $m_1, m_2 \in L_x$. For $W \in -\mathbb{T}_{X_t} \mathcal{C}$ we have the opposite inequality.

Step 3. As shown above, the velocity $V_t \in \mathbb{T}_{X_t} \mathcal{C} \cap (-\mathbb{T}_{X_t} \mathcal{C})$. Then there exists a function $v_t: \mathbb{R} \rightarrow \mathbb{R}$ with $V_t = v_t \circ X_t$ μ -a.e. Indeed for all $x \in \mathbb{R} \setminus \mathcal{O}$ there exists at most one $m \in L_x$ such that $X_t(m) = x$, so we define

$$v_t(x) := \begin{cases} V_t(m) & \text{if such an } m \text{ exists, and} \\ 0 & \text{otherwise.} \end{cases}$$

For all $x \in \mathcal{O}$ we can pick a generic $m \in L_x$ and define $v_t(x) := V_t(m)$ because then V_t is constant μ -a.e. in L_x , as shown in Step 2. By construction, we have

$$V_t(m) = v_t(X_t(m)) \quad \text{for } \mu\text{-a.e. } m \in \mathbb{R} \setminus N_{X_t}.$$

Using that $\mu(N_{X_t}) = 0$, we get the claim. Finally, since $\|V_t\|_{\mathcal{L}^2(\mathbb{R}, \mu)} = \|v_t\|_{\mathcal{L}^2(\mathbb{R}, \varrho_t)}$ with $\varrho_t := X_t \# \mu$, we have that $v_t \in \mathcal{L}^2(\mathbb{R}, \varrho_t)$. Then we use Lemma 3.5. \square

Proof of Theorem 3.3. Notice first that \mathcal{H}_{X_t} is a closed subspace of $\mathcal{L}^2(\mathbb{R}, \mu)$, and that the velocity V_t is an element of that subspace for a.e. $t \in \mathbb{R}$; see Lemma 3.7. Combining Lemma 3.6 with Theorem 2.1, we obtain that $\langle \bar{V} - V_t, W \rangle \leq 0$ for all $W \in \mathcal{H}_{X_t}$, which implies that V_t is the orthogonal projection of \bar{V} onto \mathcal{H}_{X_t} .

There exists $v_t \in \mathcal{L}^2(\mathbb{R}, \varrho_t)$ with $\varrho_t := X_t \# \mu$, such that $V_t = v_t \circ X_t$ μ -a.e.; see the proof of Lemma 3.7. Notice that since $\bar{X} \in \mathcal{C}$, by assumption, we have $X_0 = \bar{X}$ and therefore $\varrho_0 = X_0 \# \mu = \bar{X} \# \mu = \bar{\varrho}$. For any $\varphi \in \mathcal{D}([0, \infty) \times \mathbb{R})$, we get

$$\begin{aligned} - \int_{\mathbb{R}} \varphi(0, x) \bar{\varrho}(dx) &= - \int_{\mathbb{R}} \varphi(0, \bar{X}(m)) \mu(dm) \\ &= \int_0^\infty \frac{d}{dt} \left(\int_{\mathbb{R}} \varphi(t, X_t(m)) \mu(dm) \right) dt. \end{aligned}$$

Recall that the map $t \mapsto X_t$ is differentiable for a.e. $t \in \mathbb{R}$, with velocity $V_t = v_t \circ X_t$ where $v_t \in \mathcal{L}^2(\mathbb{R}, \varrho_t)$. For a.e. $t \in \mathbb{R}$ we can therefore write

$$\begin{aligned} & \frac{d}{dt} \left(\int_{\mathbb{R}} \varphi(t, X_t(m)) \mu(dm) \right) \\ &= \int_{\mathbb{R}} \left(\partial_t \varphi(t, X_t(m)) + V_t(m) \partial_x \varphi(t, X_t(m)) \right) \mu(dm) \\ &= \int_{\mathbb{R}} \left(\partial_t \varphi(t, X_t(m)) + v_t(X_t(m)) \partial_x \varphi(t, X_t(m)) \right) \mu(dm) \\ &= \int_{\mathbb{R}} \left(\partial_x \varphi(t, x) + v_t(x) \partial_x \varphi(t, x) \right) \varrho_t(dx). \end{aligned}$$

It follows that for all $\varphi \in \mathcal{D}([0, \infty) \times \mathbb{R})$, we have

$$- \int_{\mathbb{R}} \varphi(0, x) \bar{\varrho}(dx) = \int_0^\infty \int_{\mathbb{R}} \left(\partial_x \varphi(t, x) + v_t(x) \partial_x \varphi(t, x) \right) \varrho_t(dx) dt.$$

This proves that (ϱ_t, v_t) satisfies the continuity equation in the distributional sense.

Similarly, for any test function $\zeta \in \mathcal{D}([0, \infty) \times \mathbb{R})$, we can write

$$\begin{aligned} - \int_{\mathbb{R}} \zeta(0, x) \bar{v}(x) \bar{\varrho}(dx) &= - \int_{\mathbb{R}} \zeta(0, \bar{X}(m)) \bar{v}(\bar{X}(m)) \mu(dm) \\ &= - \int_{\mathbb{R}} \zeta(0, \bar{X}(m)) \bar{V}(m) \mu(dm) \\ &= \int_0^\infty \frac{d}{dt} \left(\int_{\mathbb{R}} \zeta(t, X_t(m)) \bar{V}(m) \mu(dm) \right) dt. \end{aligned}$$

We used the fact that, by assumption, the initial velocity is of the form $\bar{V} = \bar{v} \circ \bar{X}$ (thus $\bar{V} \in \mathcal{H}_{\bar{X}}$; see Lemma 3.5). For a.e. $t \in \mathbb{R}$, we can write

$$\begin{aligned} & \frac{d}{dt} \left(\int_{\mathbb{R}} \zeta(t, X_t(m)) \bar{V}(m) \mu(dm) \right) \\ &= \int_{\mathbb{R}} \left(\partial_t \zeta(t, X_t(m)) + V_t(m) \partial_x \zeta(t, X_t(m)) \right) \bar{V}(m) \mu(dm) \\ &= \int_{\mathbb{R}} \left(\partial_t \zeta(t, X_t(m)) + v_t(X_t(m)) \partial_x \zeta(t, X_t(m)) \right) \bar{V}(m) \mu(dm). \end{aligned}$$

Since V_t is the orthogonal projection of \bar{V} onto \mathcal{H}_{X_t} , we get that

$$\begin{aligned} & \int_{\mathbb{R}} \left(\partial_t \zeta(t, X_t(m)) + v_t(X_t(m)) \partial_x \zeta(t, X_t(m)) \right) \bar{V}(m) \mu(dm) \\ &= \int_{\mathbb{R}} \left(\partial_t \zeta(t, X_t(m)) + v_t(X_t(m)) \partial_x \zeta(t, X_t(m)) \right) V_t(m) \mu(dm) \\ &= \int_{\mathbb{R}} \left(\partial_t \zeta(t, X_t(m)) + v_t(X_t(m)) \partial_x \zeta(t, X_t(m)) \right) v_t(X_t(m)) \mu(dm) \\ &= \int_{\mathbb{R}} \left(\partial_t \zeta(t, x) + v_t(x) \partial_x \zeta(t, x) \right) v_t(x) \varrho_t(dx). \end{aligned}$$

using the fact that $\partial_t \zeta(t, X_t(m)) + v_t(X_t(m)) \partial_x \zeta(t, X_t(m))$ is an element of \mathcal{H}_{X_t} .

It follows that for all $\zeta \in \mathcal{D}([0, \infty) \times \mathbb{R})$, we have

$$- \int_{\mathbb{R}} \zeta(0, x) \bar{v}(x) \bar{\varrho}(dx) = \int_0^\infty \int_{\mathbb{R}} \left(\partial_t \zeta(t, x) + v_t(x) \partial_x \zeta(t, x) \right) v_t(x) \varrho_t(dx) dt.$$

This proves that (ϱ_t, v_t) satisfies the momentum equation distributionally. \square

Remark 3.8. Theorem 3.3 covers the special case $\mu = \bar{\varrho}$, for which $(\bar{X}, \bar{V}) = (\text{id}, \bar{v})$. More generally, if there are two sets of initial data $(\bar{\varrho}_i, \bar{v}_i)$, $i = 1..2$, and if (\bar{X}_i, \bar{V}_i) are the monotone transport maps and initial velocities corresponding to the reference measure $\mu \in \mathcal{P}_2(\mathbb{R})$, then the transport maps $X_{i,t}$ defined as in (3.8) satisfy

$$\|X_{1,t} - X_{2,t}\|_{\mathcal{L}^2(\mathbb{R}, \mu)} \leq \|\bar{X}_1 - \bar{X}_2\|_{\mathcal{L}^2(\mathbb{R}, \mu)} + |t| \|\bar{V}_1 - \bar{V}_2\|_{\mathcal{L}^2(\mathbb{R}, \mu)}$$

for all $t \in \mathbb{R}$ since the metric projection onto closed convex subsets of Hilbert spaces is a contraction. In particular, this implies the uniqueness of the transport map X_t , from which the uniqueness of the induced density $\varrho_t := X_t \# \mu$ follows. The Eulerian velocities v_t are determined by the orthogonal projection of \bar{V} onto the space \mathcal{H}_{X_t} , which is also unique. We conclude that within the framework of solutions obtained from (3.8), we obtain both existence of solutions to (1.1), and uniqueness.

4. FORCE FIELD

The analysis above can also be applied to the case when the momentum equation in (1.1) is augmented with a force generated by the fluid itself: We consider

$$\left. \begin{aligned} \partial_t \varrho + \partial_x(\varrho v) &= 0 \\ \partial_t(\varrho v) + \partial_x(\varrho v^2) &= f[\varrho] \end{aligned} \right\} \quad \text{in } [0, \infty) \times \mathbb{R}, \quad (4.1)$$

where $f: \mathcal{P}_2(\mathbb{R}) \rightarrow \mathcal{M}(\mathbb{R})$ is the force field, with $\mathcal{M}(\mathbb{R})$ the space of finite, signed Borel measures. This equation has been studied in [5] for a suitable class of forces. Instead of stating our result in the general terms of [5] (which is possible), we focus here on the special case of the Euler-Poisson system, for simplicity. Thus

$$f[\varrho] = -\varrho \partial_x q_\varrho, \quad \text{with } q_\varrho \text{ solution of } -\partial_{xx}^2 q_\varrho = \varrho. \quad (4.2)$$

As a first step, one needs to identify a *Lagrangian* force field representing f . That means, we are looking for a functional $F: \mathcal{C} \rightarrow \mathcal{L}^2(\mathbb{R}, \mu)$ such that

$$\int_{\mathbb{R}} \varphi(x) f[\varrho](dx) = \int_{\mathbb{R}} \varphi(X(m)) F[X](m) \mu(dm) \quad \text{for all } \varphi \in \mathcal{C}_b(\mathbb{R}),$$

where $\varrho \in \mathcal{P}_2(\mathbb{R})$ and $X \in \mathcal{C}$ are related by the identity $X \# \mu = \varrho$. Whenever $f[\varrho]$ is absolute continuous with respect to ϱ , and so can be expressed in terms of the Radon-Nikodym density $f_\varrho \in \mathcal{L}^2(\mathbb{R}, \varrho)$ as $f[\varrho] = f_\varrho \varrho$, one can choose

$$F[X] := f_\varrho \circ X \quad \text{for all } X \in \mathcal{C} \text{ with } X \# \mu = \varrho.$$

To every $X \in \mathcal{C}([0, \infty), \mathcal{C})$ (with topology induced by the $\mathcal{L}^2(\mathbb{R}, \mu)$ -norm) we can associate a new variable playing the role of a modified velocity:

$$Y_t := \bar{V} + \int_0^t F[X_s] ds \quad \text{for all } t \in [0, \infty). \quad (4.3)$$

The evolution X_t for the system (4.1) is then characterized by the identity

$$X_t = P_{\mathcal{C}} \left(\bar{X} + \int_0^t Y_s ds \right) \quad \text{for all } t \in [0, \infty). \quad (4.4)$$

The existence of a curve $X \in \text{Lip}([0, \infty), \mathcal{C})$ satisfying (4.3)/(4.4) is especially easy to establish in the Euler-Poisson case (4.2) (in the general case one can use a fixed

point argument as in [5]): We assume for simplicity that $\mu = \mathcal{L}^1|_{[0,1]}$. Then

$$f_\varrho(x) = -\frac{1}{2} \left(m_\varrho^-(x) + m_\varrho^+(x) - 1 \right) \quad \text{for all } x \in \mathbb{R}$$

(see Example 6.10 in [5]), where m_ϱ^\pm are cumulative distribution functions:

$$m_\varrho^-(x) := \varrho((-\infty, x)), \quad m_\varrho^+(x) := \varrho((-\infty, x])$$

for $x \in \mathbb{R}$. By construction, it then follows that

$$F[X](m) = -\frac{1}{2}(2m - 1) \quad \text{for all } m \in [0, 1]. \quad (4.5)$$

In particular, the functional $F[X]$ does even not depend on $X \in \mathcal{C}$ anymore, and so (4.3) can be computed explicitly. Global existence of X satisfying (4.4) follows, and Lipschitz continuity in time is a consequence of the fact that the metric projection $\mathbb{P}_{\mathcal{C}}$ is a contraction in $\mathcal{L}^2(\mathbb{R}, \mu)$. To prove that (4.4) generates a solution of (4.1), we need the following result, which remains true for general $Y \in \mathcal{C}([0, \infty), \mathcal{L}^2(\mathbb{R}, \mu))$ defining X through (4.4), even those not necessarily given in terms of X .

Proposition 4.1. *Consider $\bar{X} \in \mathcal{C}$ and $Y \in \mathcal{C}([0, \infty), \mathcal{L}^2(\mathbb{R}, \mu))$ and define*

$$X_t := \mathbb{P}_{\mathcal{C}} \left(\bar{X} + \int_0^t Y_s ds \right) \quad \text{for all } t \in [0, \infty).$$

Then the derivative of X_t , denoted by V_t , exists and satisfies

$$V_t = \mathbb{P}_{\mathcal{H}_{X_t}}(Y_t) \quad \text{for a.e. } t \in [0, \infty). \quad (4.6)$$

Proof. Each step of the proof follows the same ideas of the corresponding result for the pressureless gas dynamics equations. We proceed in two steps.

Step 1. Repeating word by word the proof of Theorem 2.1, replacing $(t + h_n)\bar{V}$ by $\int_0^{t+h_n} Y(s)ds$ for all $t, h_n \in [0, \infty)$, one obtains the analogous results:

$$V_t \in \mathbb{T}_{X_t} \mathcal{C} \cap \left[\left(\bar{X} + \int_0^t Y_s ds \right) - X_t \right]^\perp,$$

$$\langle Y_t - V_t, V_t \rangle \geq 0,$$

$$\langle Y_t - V_t, W \rangle \leq 0 \quad \text{for all } W \in T_{X_t} \mathcal{C} \cap \left[\left(\bar{X} + \int_0^t Y_s ds \right) - X_t \right]^\perp.$$

Step 2. For any $t \in [0, \infty)$ for which X_t is strongly differentiable (these are a.e. t) we then deduce from Lemma 3.7 that the velocity satisfies

$$V_t \in \mathbb{T}_{X_t} \mathcal{C} \cap (-\mathbb{T}_{X_t} \mathcal{C}) \subset \mathcal{H}_{X_t}.$$

Repeating the proof of Lemma 3.6, replacing $t\bar{V}$ by $\int_0^t Y_s ds$, one gets $\mathcal{H}_{X_t} \subset \mathbb{S}_{X_t} \mathcal{C}$, where the definition of $\mathbb{S}_{X_t} \mathcal{C}$ changes slightly from the previous one:

$$\mathbb{S}_{X_t} \mathcal{C} := \mathcal{L}^2(\mathbb{R}, \mu)\text{-closure of } T_{X_t} \mathcal{C} \cap \left[\left(\bar{X} + \int_0^t Y_s ds \right) - X_t \right]^\perp.$$

We now deduce from Step 1. that

$$\langle Y_t - V_t, V_t \rangle = 0, \quad \langle Y_t - V_t, W \rangle \leq 0 \quad \text{for all } W \in \mathcal{H}_{X_t}.$$

Since $V_t \in \mathcal{H}_{X_t}$, we obtain that V_t is the orthogonal projection of Y_t onto \mathcal{H}_{X_t} . \square

Applying Proposition 4.1, we can prove the following existence result.

Theorem 4.2 (Global Existence Euler-Poisson). *Let initial data $\bar{\varrho} \in \mathcal{P}_2(\mathbb{R})$ and $\bar{v} \in \mathcal{L}^2(\mathbb{R}, \bar{\varrho})$ be given. For suitable reference measure $\mu \in \mathcal{P}_2(\mathbb{R})$, let $\bar{X} \in \mathcal{C}$ be the unique monotone transport map with $\bar{X} \# \mu = \bar{\varrho}$. Define $\bar{V} := \bar{v} \circ \bar{X}$, and let*

$$X_t := \mathbb{P}_{\mathcal{C}} \left(\bar{X} + \int_0^t Y_s ds \right), \quad Y_t = \bar{V} + \int_0^t F[X_s] ds \quad (4.7)$$

for any $t \in \mathbb{R}$, where F is given by (4.5). Then X_t is strongly differentiable for a.e. $t \in [0, \infty)$ and $V_t := \dot{X}_t$ can be written in the following form: there exists a velocity $v_t \in \mathcal{L}^2(\mathbb{R}, \varrho_t)$ with $\varrho_t := X_t \# \mu$, such that $V_t = v_t \circ X_t$. The pair (ϱ_t, v_t) is a weak solution of the conservation law (4.1) with force (4.2).

The proof is omitted; see the proof of Theorem 3.5 in [5] for details.

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