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CORRECTION

Correction to: The Néron–Severi group of a proper seminormal complex variety

Luca Barbieri-Viale[1] · Andreas Rosenschon[2] · V. Srinivas[3]

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In the paper [1] referred to in the title, the proof of Lemma 11 is not correct. The geometric observation on face maps, given in the last paragraph of the proof there, does not suffice to give the claimed splitting of complexes in (20). We are unsure if Lemma 11, as stated, is correct.

We note that Lemma 11 is used only in the proof of the subsequent Lemma 12, by invoking first an exact sequence of sheaves

$$R^1\pi_{\bullet *}\mathbb{C}^*_{X_\bullet} \to R^1\pi_{\bullet *}\mathcal{O}^*_{X_\bullet} \to R^1\pi_{\bullet *}\mathcal{F}^{1,1}_\bullet,$$

and then using the assertion of Lemma 11 to prove that the first map

$$R^1\pi_{\bullet *}\mathbb{C}^*_{X_\bullet} \to R^1\pi_{\bullet *}\mathcal{O}^*_{X_\bullet}$$

is the zero map (in [1], the notation $\mathbb{C}^*_\bullet$ is used for what we have here written as $\mathbb{C}^*_{X_\bullet}$, and the derived functors R^i are written there as $\mathbb{R}^i$; we hope this should cause no confusion).

Instead, we may directly prove the vanishing of this first map:

Lemma 1 *Let $\pi_\bullet : X_\bullet \to X$ be a smooth proper hypercovering of a complex variety X. Let $\mathbb{C}^*_X$ be the constant sheaf associated to $\mathbb{C}^*$ for the Zariski topology on X, and let $\mathbb{C}^*_{X_\bullet} = \pi^*_\bullet\mathbb{C}^*_X$ be the corresponding simplicial sheaf. Then the inclusion of simplicial sheaves $\mathbb{C}^*_{X_\bullet} \to \mathcal{O}^*_{X_\bullet}$ induces the zero map*

$$R^1\pi_{\bullet *}\mathbb{C}^*_{X_\bullet} \to R^1\pi_{\bullet *}\mathcal{O}^*_{X_\bullet}.$$

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✉ Andreas Rosenschon
Andreas.Rosenschon@mathematik.uni-muenchen.de

Luca Barbieri-Viale
barbieri@math.unimi.it

V. Srinivas
srinivas@math.tifr.res.in

1 Dipartimento di Matematica, Università di Milano, Milano, Italy

2 Mathematisches Institut, Ludwigs-Maximilians-Universität, Munich, Germany

3 Tata Institute of Fundamental Research, Mumbai, India

Proof Let X_j be the j-th component scheme of $X_\bullet$, and $\pi_j : X_j \to X$ the structure morphism induced by $\pi_\bullet$. There is a spectral sequence

$$E_1^{r,s} = R^s\pi_{r*}\mathbb{C}^*_{X_r} \Rightarrow R^{r+s}\pi_{\bullet*}\mathbb{C}^*_{X_\bullet} \tag{1}$$

of Zariski sheaves on X. Note that $\mathbb{C}^*_{X_j}$ is flasque, and hence acyclic on any nonempty open subscheme of X_j (since $\mathbb{C}^*_{X_j}$ is a constant sheaf for the Zariski topology on a locally irreducible scheme). Hence we have $E_1^{r,s} = 0$ for $s > 0$ in (1) and $E_2^{1,0} = R^1\pi_{\bullet*}\mathbb{C}^*_{X_\bullet}$ coincides with the (co)homology sheaf of $\pi_{0*}\mathbb{C}^*_{X_0} \to \pi_{1*}\mathbb{C}^*_{X_1} \to \pi_{2*}\mathbb{C}^*_{X_2}$.

If $x \in X$ is a point, we write $X_x = \operatorname{Spec}\mathcal{O}_{X,x}$ and consider the induced smooth proper hypercover

$$\pi^x_\bullet : X_{\bullet,x} = X_\bullet \times_X X_x \to X_x$$

of the local scheme X_x. Let $X_{j,x} = X_j \times_X X_x$ be the components of the simplicial scheme $X_{\bullet,x}$, and let $\pi^x_j : X_{j,x} \to X_x$ be the induced proper morphisms; note that the $X_{j,x}$ are smooth $\mathbb{C}$-schemes.

To prove the lemma, it suffices to show that for any *closed* point $x \in X(\mathbb{C})$, the induced map on the corresponding stalks

$$(R^1\pi_{\bullet*}\mathbb{C}^*_{X_\bullet})_x \to (R^1\pi_{\bullet*}\mathcal{O}^*_{X_\bullet})_x \tag{2}$$

is the zero map. We note that $(R^1\pi_{\bullet*}\mathbb{C}^*_{X_\bullet})_x = H^1(X_{\bullet,x}, \mathbb{C}^*_{X_{\bullet,x}}) = (E_2^{1,0})_x$ is the homology of the complex

$$H^0(X_{0,x}, \mathbb{C}^*_{X_{0,x}}) \to H^0(X_{1,x}, \mathbb{C}^*_{X_{1,x}}) \to H^0(X_{2,x}, \mathbb{C}^*_{X_{2,x}})$$

while

$$(R^1\pi_{\bullet*}\mathcal{O}^*_{X_\bullet})_x \cong \operatorname{Pic}(X_{\bullet,x}),$$

i.e. we need to show that the following map is the zero map

$$H^1(X_{\bullet,x}, \mathbb{C}^*_{X_{\bullet,x}}) = (E_2^{1,0})_x \to \operatorname{Pic}(X_{\bullet,x}).$$

Let $\alpha : X^h_x = \operatorname{Spec}\mathcal{O}^h_{X,x} \to X_x$ be a Henselization of the local scheme X_x and let $X^h_{\bullet x} = X_{\bullet x} \times_{X_x} X^h_x$ be the corresponding simplicial scheme over X^h_x; we write $\alpha_\bullet : X^h_{\bullet,x} \to X_{\bullet,x}$ for the induced morphism and $X^h_{j,x} = (X^h_{\bullet,x})_j = X_{j,x} \times_{X_x} X^h_x$ for the component schemes.

We have a commutative diagram

$$\begin{array}{ccc} (R^1\pi_{\bullet*}\mathbb{C}^*_{X^h_\bullet})_x & \longrightarrow & (R^1\pi_{\bullet*}\mathcal{O}^*_{X^h_\bullet})_x \\ \uparrow & & \uparrow \alpha^*_\bullet \\ (R^1\pi_{\bullet*}\mathbb{C}^*_{X_\bullet})_x & \longrightarrow & (R^1\pi_{\bullet*}\mathcal{O}^*_{X_\bullet})_x \end{array}$$

where $(R^1\pi_{\bullet*}\mathcal{O}^*_{X_\bullet})_x \cong \operatorname{Pic}(X_{\bullet,x})$ and $(R^1\pi_{\bullet*}\mathcal{O}^*_{X^h_\bullet})_x \cong \operatorname{Pic}(X^h_{\bullet,x})$.

Since the morphism $\alpha : X^h_x \to X_x$ is faithfully flat, we see that

$$\alpha^*_\bullet : \operatorname{Pic}(X_{\bullet,x}) \to \operatorname{Pic}(X^h_{\bullet,x})$$

is *injective*: for any simplicial line bundle $\mathcal{L}_{\bullet,x}$ on $X_{\bullet,x}$, we have that

$$H^0(X^h_{\bullet,x}, \alpha^*_\bullet\mathcal{L}_{\bullet,x}) \cong H^0(X_{\bullet,x}, \mathcal{L}_{\bullet,x}) \otimes_{\mathcal{O}_{X,x}} \mathcal{O}^h_{X,x},$$

and we can recognize the trivial simplicial line bundle via an appropriate pair of sections of the bundle and its dual.

Hence it follows from the above diagram that it suffices to show

$$H^1(X^h_{\bullet,x}, \mathbb{C}^*_{X^h_{\bullet,x}}) = (E_2^{1,0})^h_x \to \mathrm{Pic}(X^h_{\bullet,x})$$

is the zero map, where $(E_2^{1,0})^h_x$ is the $E_2^{1,0}$ term of the component spectral sequence for the cohomology of the sheaf $\mathbb{C}^*_{X^h_{\bullet,x}}$ on $X^h_{\bullet,x}$. We will show that in fact

$$(E_2^{1,0})^h_x = 0.$$

Let $\pi(X^h_{j,x})$ be the set of connected components of the scheme $X^h_{j,x}$. Since $\mathbb{C}^*_{X^h_{j,x}}$ is a constant sheaf on $X^h_{j,x}$, the global sections

$$H^0(X_{j,x}, \mathbb{C}^*_{X_{j,x}}) = (\mathbb{C}^*)^{\pi(X^h_{j,x})},$$

are a finite direct product of copies of $\mathbb{C}^*$.

Let $X_j(x)$ be the fibre over x of $\pi_j : X_j \to X$. Note that $X_j(x)$ coincides with the fibre over x of $X_{j,x} \to X_x$ and of $X^h_{j,x} \to X^h_x$, i.e.

$$X_j(x) = X_j \times_X \{x\} \cong X^h_j \times_{X^h_x} \{x\}$$

Then we have

- $X_\bullet(x) \to \{x\}$ is a proper hypercover of $\{x\} \cong \mathrm{Spec}\,\mathbb{C}$,
- for each j, the inclusion of the closed subscheme $X_j(x) \subset X^h_{j,x}$ yields a bijection[1] $\pi(X_j(x)) \cong \pi(X^h_{j,x})$ on connected components,
- $X_\bullet(x) \to X^h_{\bullet x}$ is a closed simplicial subscheme, augmented over $\{x\} \hookrightarrow X^h_x$.

In particular, this allows to identify $(E_2^{1,0})^h_x$, i.e. the homology of

$$H^0(X^h_{0,x}, \mathbb{C}^*_{X^h_{0,x}}) \to H^0(X^h_{1,x}, \mathbb{C}^*_{X^h_{1,x}}) \to H^0(X^h_{2,x}, \mathbb{C}^*_{X^h_{2,x}})$$

with the homology of

$$H^0(X_0(x), \mathbb{C}^*_{X_0(x)}) \to H^0(X_1(x), \mathbb{C}^*_{X_1(x)}) \to H^0(X_2(x), \mathbb{C}^*_{X_2(x)})$$

which is the $E_2^{1,0}(x)$-term in the component spectral sequence

$$E_1^{r,s}(x) = H^s(X_r(x), \mathbb{C}^*_{X_r(x)})) \Rightarrow H^{r+s}(X_\bullet(x), \mathbb{C}^*_{X_\bullet(x)}).$$

Thus it suffices to show $E_2^{1,0}(x) = 0$.

Consider now the simplicial proper analytic space $X^{an}_\bullet(x) \to \{x\}$ and the corresponding component spectral sequence

$$E_1^{r,s}(x)^{an} = H^s(X_r(x)^{an}, \mathbb{C}^*_{X_r(x)^{an}}) \Rightarrow H^{r+s}(X_\bullet(x)^{an}, \mathbb{C}^*_{X_\bullet(x)^{an}}).$$

There is a natural transformation $E_*^{r,s}(x) \to E_*^{r,s}(x)^{an}$ between these spectral sequences. Moreover, the maps $E_2^{r,0}(x) \to E_2^{r,0}(x)^{an}$ are isomorphisms for all $r \geq 0$, since the global sections of a constant sheaf in the Zariski topology coincide with the global sections of the

[1] This may be viewed as a case of proper base change for $H^0_{et}(-, \mathbb{Z}/p\mathbb{Z})$, which applies because X^h_x is strict Hensel; compare [2] II, Remark 3.8.

corresponding constant sheaf in the analytic topology. Hence it suffices to show $E_2^{1,0}(x) \cong E_2^{1,0}(x)^{an} = 0$. This follows from cohomological descent: Since $X_\bullet(x)^{an} \to \{x\}$ is a proper hypercover of analytic spaces, cohomological descent implies that the maps

$$H^n(\{x\}, \mathbb{C}^*_{\{x\}}) \to H^n(X_\bullet(x)^{an}, \mathbb{C}^*_{X_\bullet(x)^{an}}),$$

induced by the augmentation are isomorphisms for all n, so that

$$H^n(X_\bullet(x)^{an}, \mathbb{C}^*_{X_\bullet(x)^{an}}) = 0 \quad \forall\, n > 0.$$

In particular, $H^1(X_\bullet(x)^{an}, \mathbb{C}^*_{X_\bullet(x)^{an}}) = 0$. Since $E_2^{1,0}(x)^{an}$ is a subobject of this group, it follows that $E_2^{1,0}(x)^{an} = 0$, as desired.

□

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References

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