LINE GEOMETRY OF PAIRS OF SECOND-ORDER HAMILTONIAN OPERATORS AND QUASILINEAR SYSTEMS

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ABSTRACT. We show that a pair formed by a second-order homogeneous Hamiltonian structures in N components and the associated system of conservation laws is in bijective correspondence with an alternating three-form on a $N+2$ vector space. We use this result to characterise these pairs up to $N=4$. We also show that the three-form provides $N+2$ linear equations in the Plücker coordinates which define the associated line congruence.

CONTENTS

1. INTRODUCTION

This paper is devoted to show a novel correspondence in theory of Hamiltonian structures of Partial Differential Equations (PDEs) using techniques coming from differential geometry, projective algebraic geometry, and action of Lie groups. In particular, we give a geometric description of pairs formed by a second order Hamiltonian operator and its admitted system of quasilinear PDEs. Up to our knowledge this is the first time such a description appears in the literature.

The Hamiltonian formalism for PDEs is a fundamental tool when investigating nonlinear phenomema, since Hamiltonian structures imply that the solutions of a nonlinear system possess some form of regularity and they are connected to symmetries and conserved quantities [18, 21]. A modern approach to Hamiltonian formalism was presented in the second half of the last century [17]. The formalism introduced by Hamilton for Ordinary Differential

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Equations (ODEs) is extended to Partial Differential Equations (PDEs) by substituting the Poisson tensor on a symplectic manifold with integro-differential operators on a space of loops, see e.g. [19]. As in the finite dimensional case, Hamiltonian structures endow the *space* with a deep geometric structure, in both differential and algebraic sense. As a starting point, a systematic approach to Hamiltonian operators and geometry was firstly presented by B. A. Dubrovin and S. P. Novikov in [6], where the authors introduced the concept of differential-geometric Poisson structure for quasilinear systems.

We recall briefly the formalism of infinite dimensional Hamiltonian structures. Assume we are given two independent variables t, the "evolution" variable, and x, the "spatial" variable". Then, denote by $u=(u^1,\dots u^N)$ the field variables. So, an evolutionary system of l equations has the form:

$$u_t^i = f^i(x,u,u_x,\dots u_{kx}), \qquad i=1,2,\dots,l. \tag{1.1}$$

In particular, in what follows, we focus on evolutionary quasilinear systems of conservation laws:

$$u_t^i = (V^i(u))_x = V^i_{,j}(u)u_x^j, \quad V^i_{,j} := \frac{\partial V^i}{\partial u^j}, \quad i=1,\dots,N. \tag{1.2}$$

Systems of this form are also called *hydrodynamic-type systems*, and are very well studied, see e.g. [23–25].

The system (1.1) is Hamiltonian if it can be written as:

$$u_t^i = \mathscr{A}^{ij}\frac{\delta H}{\delta u^j}, \qquad i=1,2,\dots N, \tag{1.3}$$

where Einstein's summation convention on repeated indices is used, $\delta/\delta u^j$ is the variational derivate with respect to the field variable u^j, $\mathscr{A}=(\mathscr{A}^{ij})$ is a Hamiltonian differential operator, i.e. a it is skew-adjoint $\mathscr{A}^*=-\mathscr{A}$, and the Schouten bracket with itself vanishes [13]:

$$[\mathscr{A},\mathscr{A}]=0, \tag{1.4}$$

and finally H is a functional:

$$H=\int h(x,u,u_x,\dots u_{mx})\,\mathrm{d}x. \tag{1.5}$$

Alternatively, the Hamiltonian property of an operator $\mathscr{A}$ is expressed in terms of the Poisson bracket conditions [17]. That is, given a skew-adjoint operator $\mathscr{A}$, it defines a bracket between two functionals $F=\int f\,\mathrm{d}x$ and $G=\int g\,\mathrm{d}x$ as:

$$\{F,G\}_{\mathscr{A}} = \int \frac{\delta F}{\delta u^i}\mathscr{A}^{ij}\frac{\delta G}{\delta u^j}\,\mathrm{d}x. \tag{1.6}$$

Then, $\mathscr{A}$ is Hamiltonian if the associated bracket $\{.,.\}_{\mathscr{A}}$ is skew-symmetric and satisfies the Jacobi identity, i.e. the bracket (1.6) is a Poisson bracket, see also [21, Sect. 7.1].

Moreover, we will consider *local differential operators* of the form:

$$\mathscr{A}^{ij} = a^{ij\sigma}\partial_\sigma, \tag{1.7}$$

where $a^{ij\sigma} = a^{ij\sigma}(u, u_x, \dots, u_{Kx})$. On such operators a grading can be defined using the rules:

$$\deg \partial_\sigma = \sigma, \quad \deg u_{kx} = k. \tag{1.8}$$

So, the order of the operator is the maximum of the degrees of all the terms $a^{ij\sigma}\partial_\sigma$ in (1.7). An operator of this form is called *homogeneous* if all the terms have the same degree. That is, an mth-order homogeneous operator has the following form:

$$\begin{aligned}\mathscr{P}^{ij} &= g^{ij}\partial_x^m + b^{ij}_k u^k_x \partial_x^{m-1} + \left(c^{ij}_k u^k_{xx} + c^{ij}_{kl} u^k_x u^l_x\right)\partial_x^{m-2} + \dots \\ &\quad + \left(d^{ij}_k u^k_{nx} + \dots + d^{ij}_{k_1\dots k_m} u^{k_1}_x \cdots u^{k_m}_x\right).\end{aligned} \tag{1.9}$$

where $b^{ij}_k, c^{ij}_k, c^{ij}_{kl}, \dots$ depend on the field variables $u^1, \dots, u^N$. Homogeneous differential operators were studied first in [6], where the first-order case was considered, while the general expression for the mth-order was presented in [7]. The interested reader can see the review paper [19] for more details on the topic.

Second- and third-order operators were investigated independently in details by Doyle in [5] and Potëmin in [22]. They found explicitly the necessary and sufficient conditions for such operators to be Hamiltonian and proved that there exists a change of dependent variables such that $\mathscr{P}$ can be re-written into the following form

$$\mathscr{P}^{ij} = \partial_x \circ \mathscr{Q}^{ij} \circ \partial_x, \tag{1.10}$$

where $\mathscr{Q}^{ij}$ is a homogeneous operator of order 0 and 1 respectively. The present canonical form is consequently known as *Doyle-Potëmin form* of the operator. Recent developments in this direction show how this canonical form is typical of a large number of Hamiltonian operators [16] where the homogeneous operator $\mathscr{Q}$ is of arbitrary order $d \geq 0$.

In recent years, homogeneous Hamiltonian operators have been studied with a geometric approaches, coming both from differential and algebraic geometry, see [9–11, 16, 27]. For instance, in the non-degenerate case ($\det g \neq 0$) it was shown that the leading coefficient g^{ij} is invariant under projective transformations of the field variables and the whole operator is invariant under projective-reciprocal transformations of the independent variables t and x, by Ferapontov, Pavlov and Vitolo for the third-order case [9], and by Vitolo and one of the present authors for the second-order case [27]. This projective-reciprocal invariance has revealed a *deeper geometric interpretation* of the Hamiltonian operators and corresponding systems of first order PDEs in terms of Monge metrics, and alternating three-forms for the third- and second-order cases respectively.

1.1. **Content and structure of the paper.** In this paper we focus on quasilinear systems of first-order PDEs, also known as *hydrodynamic-type systems* [23], admitting a Hamiltonian structure with a second-order homogeneous operator. We will present a projective geometric interpretation of the systems and classify them in terms of projective-reciprocal transformations. We show that the pair *operator-system* defines an alternating three-form and this three-form uniquely defines the pair. We finally show a direct connection between the components of the underlying three-form and the coeffiecients of the linear system satisfyied by the Plücker coordinates of an associated line congruence, thus extending the results of [27].

The paper is structured as follows. In Section 2, we review the general geometry of systems of conservation laws admitting Hamiltonian structure with a second-order operator. Section 3 is the core of this paper, where we present a bijective correspondence between the pairs formed by a second-order structure together with the associated system of conservation laws and alternating three-forms in the $N+2$ projective space, and prove that a system of conservation laws admitting such a structure possesses projective-reciprocal invariance. We will use these two results to give a classification of the pairs operator-systems in Section 4 up to $N=4$. In Section 5, we compare our interpretation of systems as projective alternating forms to the theory of line congruences for systems of conservation laws as presented by Agafonov and Ferapontov [1,2], discussing the similarities with third-order Hamiltonian structures. Finally, in Section 6, we present some conclusions and outlook on this topic.

2. Homogeneous second-order Hamiltonian structures

We recall some facts on the geometry of systems of conservation laws admitting Hamiltonian structure with a second-order operator, see [27]. The most general operator of this form is:

$$\mathscr{P}^{ij} = g^{ij}\partial_x^2 + b^{ij}_k u^k_x \partial_x + c^{ij}_k u^k_{xx} + c^{ij}_{kh} u^k_x u^h_x, \tag{2.1}$$

where b^{ij}_k, c^{ij}_k and c^{ij}_{kl} transform as a connection and the skew-adjointness of the operator implies $g^{ij} = -g^{ji}$ and g^{ij} is non-degenerate, i.e. $\det g \neq 0$. The Doyle-Potëmin canonical form (1.10) of (2.1) is such that $\mathscr{Q}^{ij} = g^{ij}$. Due to the skew-symmetry of g^{ij}, we can have a non-degenerate g^{ij} only if $N = 2h$. Finally, defining $g_{ij} = (g^{ij})^{-1}$ we can show that [5,22]:

$$g_{ij} = T_{ijk} u^k + g^0_{ij}, \tag{2.2}$$

where T, g^0 are totally skew-symmetric tensors whose components are constants. So, we can define the two-forms:

$$g = g_{ij}\, \mathrm{d}u^i \wedge \mathrm{d}u^j, \quad g^0 = g^0_{ij}\, \mathrm{d}u^i \wedge \mathrm{d}u^j, \quad i<j, \tag{2.3}$$

and the alternating three-form:

$$T = T_{ijk}\, \mathrm{d}u^i \wedge \mathrm{d}u^j \wedge \mathrm{d}u^k, \quad i<j<k, \tag{2.4}$$

in a real or complex vector space of dimension N. Therefore, the number of independent components of the two-forms g and g^0 is $N(N-1)/2$, and the number of independent components of the alternating three-form T is $N(N-1)(N-2)/6$. In summary, every alternating two-form of type (2.2) defines a unique homogeneous Hamiltonian operator of second order.

As mentioned in the previous section, projective algebraic geometry plays a key role in investigating homogeneous Hamiltonian operators of both order 2 and 3. Let us consider the N-dimensional projective space $\mathbb{P}^N = \mathbb{P}(\mathbb{K}^{N+1})$, $\mathbb{K} = \mathbb{R}, \mathbb{C}$, and let $u^1, \dots, u^N, u^{N+1}$ be the coordinates on $\mathbb{K}^{N+1}$. Following [27], we can define a homogeneous version G of g in these coordinates, such that equation (2.2) becomes

$$G_{ij} = T_{ijk} u^k + g^0_{ij} u^{N+1}. \tag{2.5}$$

Note that G is *not* an alternating two-form. However, we can associate to G_{ij} an alternating three-form $\tilde{T}$ whose components are defined as follows:

$$\tilde{T}_{ijk} = \begin{cases} T_{ijk} & i,j,k \neq N+1, \\ +g^0_{ij} & k = N+1, \\ -g^0_{ik} & j = N+1, \\ +g^0_{jk} & i = N+1. \end{cases} \tag{2.6}$$

This construction implies the following result:

Theorem 2.1 ([27]). *There is a bijective correspondence between the leading coefficients of second order homogeneous Hamiltonian operators in Doyle-Potëmin form and the three-forms $\tilde{T}$. Moreover, the bijective correspondence is preserved by projective reciprocal transformations up to a conformal factor.*

Introducing potential coordinates $b^i_x = u^i$, the operator (2.1) takes the following simple form:

$$\mathscr{P}^{ij} = -g^{ij}. \tag{2.7}$$

Then, for a system of the form (1.2), the compatibility conditions to be Hamiltonian with a second order Hamiltonian operator (2.7) are expressed by the following theorem:

Theorem 2.2 ([26,27]). *The necessary conditions for a second-order homogeneous Hamiltonian operator $\mathscr{P}$ (2.7) to be a Hamiltonian operator for a quasilinear system of first-order conservation laws (1.2) are*

$$g_{qj} V^j_{,p} + g_{pj} V^j_{,q} = 0, \tag{2.8a}$$

$$g_{qk} V^k_{,pl} + g_{pq,k} V^k_{,l} + g_{qk,l} V^k_{,p} = 0. \tag{2.8b}$$

Note that conditions (2.8) are algebraic in g_{ij} and they can be explicitly solved for unknown V^i. Indeed, the fluxes V^i satisfying (2.8) have the form

$$V^i = g^{ij} W_j, \qquad \text{where} \quad W_j = A_{jl} u^l + B_j. \tag{2.9}$$

Here $A_{ij} = -A_{ji}$, B_i are arbitrary constants. The interested reader can see [27, Theorem 11]. Solving system (2.8) reveals also an inner mutual relation between the operator and the Hamiltonian system. In particular, we can indicate with $(\mathscr{P}, V)$ the pair *operator-system*. We denote the space of the pairs operator-system in N components by $\mathscr{Y}_N$.

Remark 2.3. We remark that the fluxes V^i are rational functions whose numerator is a polynomial of degree $N/2 = h$ in u, and the denominator is $\operatorname{Pf}(g)$, the Pfaffian of g, see [20]. Indeed, the inverse matrix of g_{ij} has rational functions entries where the numerator has degree $(N-2)/2$ in u and the denominator is $\operatorname{Pf}(g)$, whose degree in u is at most $N/2$.

3. Alternating three-forms and projective-reciprocal invariance of the pair operator-system

Based on the results we recalled in the previous section, in this section, we prove that the pair *operator-system* $(\mathscr{P}, V)$ in N components is in bijective correspondence with alternating three-forms in $N+2$ dimensions, and its the projective-reciprocal invariance.

3.1. Equivalence of the pairs $(\mathscr{P}, V)$ with alternating three-forms.

The results of [27] can be heuristically explained by saying that a second-order homogeneous Hamiltonian operator $\mathscr{P}$, see (2.2), in N components is in bijective correspondence with an alternating three-form $\tilde{T}$ on $\mathbb{K}^{N+1}$, where $\mathbb{K} = \mathbb{R}, \mathbb{C}$ as will be assumed throughout the rest of the paper.

Now, we prove that the associated system of conservation laws (1.2) can be incorporated together with its operator in an alternating three-form on $\mathbb{K}^{N+2}$. This is the content of the following theorem:

Theorem 3.1. *There exists a correspondence between the pair $(\mathscr{P}, V)$ of the second-order operator and the associated systems in N components and three-forms in $N+2$ components. Explicitly, there exists a bijective map $\Phi\colon \Lambda^3\mathbb{K}^{N+2} \to \mathscr{Y}_N$ defined as:*

$$\Lambda^3\mathbb{K}^{N+2} \ni (\omega_{ijk}) \mapsto \Big(\omega_{ijk}u^k + \omega_{ij\,N+1}, g^{is}(\omega_{ij\,N+2}u^j + \omega_{i\,N+1\,N+2})\Big) \in \mathscr{Y}_N, \tag{3.1}$$

with inverse $\Phi^{-1}\colon \mathscr{Y}_N \to \Lambda^3\mathbb{K}^{N+2}$ defined as:

$$\mathscr{Y}_N \ni (\mathscr{P}, V) \mapsto \Omega = \tilde{T}_{ijk} + A \wedge \mathrm{d}\,u^{N+2} + B \wedge \mathrm{d}\,u^{N+1} \wedge \mathrm{d}\,u^{N+2} \in \Lambda^3\mathbb{K}^{N+2}, \tag{3.2}$$

where $\tilde{T} \in \Lambda^3\mathbb{K}^{N+1}$ is defined in equation (2.6), and $A \in \Lambda^2\mathbb{K}^N$, $B \in \Lambda^1\mathbb{K}^N \simeq \mathbb{K}^N$ are the constants appearing in equation (2.9).

Proof. Let us consider an alternating three-form $\Omega \in \Lambda^3\mathbb{K}^{N+2}$ with components ω_{ijk}. Following [27, Theorem] let us define

$$g_{ij} = \omega_{ijk}u^k + \omega_{ijN+1}, \qquad i, j = 1, \dots N. \tag{3.3}$$

Note that this equation uniquely defines a second-order homogeneous Hamiltonian operator $\mathscr{P}_\omega$ in flat coordinates (2.2). Moreover, let us define $g^{ij} = (g_{ij})^{-1}$ and then

$$V^i_\omega = g^{is}\Big(\omega_{sjN+2}u^j + \omega_{sN+1N+2}\Big), \qquad i = 1, \dots N. \tag{3.4}$$

By (2.9), this covector satisfies conditions (2.8). Therefore, the couple $(\mathscr{P}_\omega, V_\omega)$ is a compatible pair operator-system.

Viceversa, let us consider a pair $(\mathscr{P}, V) \in \mathscr{Y}_N$. The bijection between operators and three-forms $\tilde{T} \in \Lambda^2\mathbb{K}^{N+1}$ has been already proved (see Theorem 2.1). Moreover, by solving conditions (2.8) we obtain that there exist an alternating two-form $A \in \Lambda^2\mathbb{K}^N$ and a 1-form $B \in \Lambda^1\mathbb{K}^N$ such that V is as in (2.9). Let us now define by direct construction of the three-form $\omega_{(\mathscr{P},V)}$:

$$\omega_{ijk} = \tilde{T}_{ijk}, \qquad i,j,k = 1,\dots,N+1, \tag{3.5a}$$

$$\omega_{ijN+2} = A_{ij}, \qquad i,j = 1,\dots N,\ k = N+2, \tag{3.5b}$$

$$\omega_{iN+1N+2} = B_i, \qquad i = 1,\dots N,\ j = N+1, k = N+2, \tag{3.5c}$$

which clearly defines a unique alternating three-form $\omega_{(\mathscr{P},V)} \in \Lambda^3\mathbb{K}^{N+2}$. □

The previous theorem can be interpreted as a decomposition of the exterior algebra $\Lambda^3\mathbb{K}$ as follows:

$$\Lambda^3\mathbb{K}^{N+2} = \Lambda^3\mathbb{K}^{N+1} \oplus \Lambda^2\mathbb{K}^N \oplus \Lambda^1\mathbb{K}^N, \tag{3.6}$$

where ω corresponds to $(\tilde{T}, A, B)$. Analogously [27, Theorem] states that:

$$\Lambda^3\mathbb{K}^{N+1} = \Lambda^3\mathbb{K}^N \oplus \Lambda^2\mathbb{K}^N. \tag{3.7}$$

For an identification with the various k-forms we refer to Table 1. The decompositions in equations (3.6) and (3.7) lead to a dimension count: the last entries of Table 1 sum to

$$\binom{N+2}{3} = \dim\Lambda^3\mathbb{K}^{N+2}, \qquad \binom{N+1}{3} = \dim\Lambda^3\mathbb{K}^{N+1}. \tag{3.8}$$

Remark 3.2. We remark that to see (3.7) we can define an alternating two-form in $\mathbb{K}^{N+1}$ as follows

$$\tilde{A} = A_{ij}\,\mathrm{d}u^i \wedge \mathrm{d}u^j + B_s\,\mathrm{d}u^s \wedge \mathrm{d}u^{N+1}, \tag{3.9}$$

where:

$$\tilde{A}_{ij} = \begin{cases} A_{ij} & 1 \le i,j \le N, \\ B_j & i = N+1, 1 \le j \le N, \\ -B_i & 1 \le i \le N, j = N+1. \end{cases} \tag{3.10}$$

Note that $\tilde{A}_{ij}$ transforms as a $(0,2)$-tensor.

3.2. **Projective-reciprocal invariance.**

In [27] the authors proved that the leading coefficient g^{ij} of a second-order homogeneous Hamiltonian operator is invariant up to a conformal factor under projective transformations of the field variables:

$$\rho : \mathbb{K}^N \longrightarrow \mathbb{K}^N, \quad \rho^i(u) = \tilde{u}^i := \frac{a^i_l u^l + a^i_{N+1}}{a^{N+1}_l u^l + a^{N+1}_{N+1}}, \tag{3.11}$$

where $a = (a^i_j) \in \mathrm{PGL}(N+1,\mathbb{K}) = \mathrm{GL}(N+1,\mathbb{K})/\{cI \mid c \in \mathbb{K}\setminus\{0\}\}$. In particular, defining:

$$A(u) := a^{N+1}_k u^k + a^{N+1}_{N+1}, \tag{3.12}$$

Space	Corresponding form	# of independent components
$\Lambda^3\mathbb{K}^N$	T	$\dfrac{N(N-1)(N-2)}{6}$
$\Lambda^2\mathbb{K}^N$	g^0	$\dfrac{N(N-1)}{2}$
$\Lambda^3\mathbb{K}^{N+1}$	$\tilde{T}$	$\dfrac{(N+1)N(N-1)}{6}$
$\Lambda^2\mathbb{K}^N$	A	$\dfrac{N(N-1)}{2}$
$\mathbb{K}^N$	B	N
$\Lambda^2\mathbb{K}^{N+1}$	$\tilde{A}$	$\dfrac{(N+1)N}{2}$
$\Lambda^3\mathbb{K}^{N+2}$	Ω	$\dfrac{(N+2)(N+1)N}{6}$

TABLE 1. Relation among the forms appearing in formulas (2.6), (3.5), and (3.9).

the two-form g transforms into $\bar{g}$ under the pullback $\rho^*(g)$ as follows:

$$\bar{g}_{ij}\,\mathrm{d}\,\tilde{u}^i \wedge d\tilde{u}^j = \frac{1}{A(u)^3}\, g_{kl}\,\mathrm{d}\,u^k \wedge \mathrm{d}\,u^l, \tag{3.13}$$

where g and $\bar{g}$ have the same structure as in equation (2.2), see [27, Corollary 5]. Thus the second-order homogeneous Hamiltonian operators are form invariant with respect to the action of the group of projective transformations $\mathrm{PGL}(N+1,\mathbb{K})$.

Moreover, in Theorem 6 of the same paper, it was proved that the whole operator in Doyle-Potëmin form (1.10) is preserved under the action of projective-reciprocal transformations. We briefly recall that a reciprocal transformation is a non-local change of independent variables of the following form:

$$\mathrm{d}\,\tilde{x} = \left(\alpha_i^0 u^i + \alpha_0^0\right)\mathrm{d}\,x + \left(\alpha_i^0 V^i + \beta\right)\mathrm{d}\,t, \tag{3.14a}$$

$$\mathrm{d}\,\tilde{t} = \left(\beta_i u^i + c\right)\mathrm{d}\,x + \left(\beta_i V^i + d\right)\mathrm{d}\,t, \tag{3.14b}$$

where α_i^0, α_0^0, β, β_i, c, and d are arbitrary constants. For a whole explanation of the theory of reciprocal transformation in the context of systems of conservation laws we refer to [8]. The combination of (3.11) and (3.14) is what is called a projective-reciprocal transformation.

Theorem 3.3. *The pair $(\mathscr{P}, V)$ is invariant in form by projective-reciprocal transformation.*

To prove this theorem we need to check the two following facts:

(1) the operator $\mathscr{P}$ is form invariant under projective-reciprocal transformations;

(2) the hydrodynamic-type system V is form invariant under projective-reciprocal transformations.

Note that (1) is the content of [27, Theorem 6], so there is nothing to be proved. We split the proof of (2) into the following lemmas:

Lemma 3.4. *The covector W_i is invariant in form under projective transformations* (3.11) *up to a conformal factor.*

Proof. Let us firstly observe that in the transformed frame

$$\bar{W} = \bar{W}_i \,\mathrm{d}\,\bar{u}^i = \left(\bar{A}_{is}\bar{u}^s + \bar{B}_i\right)\mathrm{d}\,\bar{u}^i. \tag{3.15}$$

Then by applying the transformation rule (3.11), we have:

$$\mathrm{d}\,\bar{u}^i = \frac{A(u)\,a^i_s\,\mathrm{d}\,u^s - \left(a^i_s u^s + a^i_{N+1}\right)a^{N+1}_l\,\mathrm{d}\,u^l}{A(u)^2}, \tag{3.16}$$

where $A(u)$ is defined in equation (3.12). So, using the skew-symmetry of $\bar{A}_{ij}$ we obtain the following relations:

$$\begin{gathered} a^i_l u^l \bar{A}_{ij} a^j_s u^s = 0 \qquad a^i_{N+1}\bar{A}_{ij} a^j_{N+1} = 0 \\ -\,a^i_{N+1}\bar{A}_{ij} a^j_s a^{N+1}_l = a^j_{N+1}\bar{A}_{ij} a^j_s a^{N+1}_l \end{gathered} \tag{3.17}$$

which give:

$$\begin{aligned} A_{ls} &= \frac{1}{A(u)^2}\left[a^i_s \bar{A}_{ij} a^j_l - \bar{B}_i a^i_s a^{N+1}_l + \bar{B}_i a^i_l a^{N+1}_s\right] \\ B_l &= \frac{1}{A(u)^2}\left[a^i_l \bar{A}_{ij} a^j_{N+1} - \bar{B} a^i_{N+1} a^{N+1}_l + \bar{B}_i a^i_l a^{N+1}_{N+1}\right]. \end{aligned} \tag{3.18}$$

So, we have:

$$W = W_l \,\mathrm{d}\,u^l = \left(A_{ls}u^s + B_l\right)\mathrm{d}\,u^l, \tag{3.19}$$

i.e. W has the same form of $\bar{W}$. Note that the skew-symmetry is also preserved, i.e. $A_{ij} = -A_{ji}$. □

Lemma 3.5. *A hydrodynamic-type system of the form* (2.9) *is invariant under the inversion of independent variables:*

$$\mathrm{d}\,\bar{x} = \mathrm{d}\,t, \quad \mathrm{d}\,\bar{t} = \mathrm{d}\,x. \tag{3.20}$$

Proof. The proof of this lemma is carried out with the same technique as [11, Theorem 4]. We start noting that the exchange of independent variables implies the following transformation on the dependent ones:

$$\bar{u}^i = V^i \qquad \bar{V}^i = u^i. \tag{3.21}$$

Then, we claim that the transformed system is still Hamiltonian with a second-order Hamiltonian structure given by the following two-form:

$$\bar{g}_{ij} = g_{is}V^s_j. \tag{3.22}$$

That is, we have to prove that in the new variables $\bar{g}$ has the same canonical form as g, see equation (2.2).

This follows from:

$$V^i_j = \left(V^i\right)_{,j} = g^{is}\left(A_{sl} + T_{slk}V^k\right) \tag{3.23}$$

which implies

$$\bar{g}_{ij} = g_{is}V^s_j = g_{is}g^{sl}\left(A_{lj} + T_{ljk}V^k\right) = T_{ijk}V^k + A_{ij} = T_{ijk}\bar{u}^k + A_{ij}. \tag{3.24}$$

That is, $\bar{g}$ has the same form as in equation (2.2), because it is skew-symmetric, linear in $\bar{u}$ with the identification $\bar{T}_{ijk} = T_{ijk}$ and $\bar{g}^0_{ij} = A_{ij}$, where $\bar{T}, \bar{g}^0$ are constant and skew-symmetric with respect to any exchange of indices.

Finally, we prove that the structure of the system is preserved, i.e. it has the following form:

$$\bar{V}^i = \bar{g}^{ij}\bar{W}_j = \bar{g}^{ij}(\bar{A}_{js}\bar{u}^s + \bar{B}_j). \tag{3.25}$$

By (2.9), we have that

$$T_{isk}u^kV^s + g^0_{is}V^s = A_{is}u^s + B_i, \tag{3.26}$$

and using (3.21) the transformed relation is:

$$T_{isk}\bar{V}^k\bar{u}^s + g^0_{is}\bar{u}^s = g^0_{is}\bar{V}^s + B_i. \tag{3.27}$$

So, equation (3.25) follows setting $\bar{T}_{ijk} = T_{ijk}$, $\bar{g}^0_{ij} = A_{ij}$, $\bar{A}_{ij} = g^0_{ij}$ and $\bar{B}_i = -B_i$. This ends the proof of the Lemma. □

Proof of Theorem 3.3. As noted above, proving the invariance of the system is equivalent to prove its projective invariance and its invariance under reciprocal transformations.

The first statement follows from Lemma 3.4, the explicit form of V given in equation (2.2), and the fact that g is invariant in form under projective transformations, see [27, Corollary 5].

To prove the reciprocal invariance, we recall that a general reciprocal transformation (3.14) can be viewed as

$$(x\text{-transformation}) \circ (x \leftrightarrow t) \circ (x\text{-transformation}), \tag{3.28}$$

where by x-transformation we mean a reciprocal transformation changing the variable x only, and by $x \leftrightarrow t$ we mean the independent variables exchange, see for instance [11].

In [27, Theorem 6], the invariance of the systems under x-translations has already been proved. The invariance under independent variable exchange was shown in Lemma 3.5. This concludes the proof of the Theorem. □

4. CLASSIFICATION OF THE PAIRS $(\mathscr{P}, V)$

The main consequence of projective-reciprocal invariance of the pair $(\mathscr{P}, V)$ as expressed in Theorem 3.3 is the possibility to classify them with respect to the special transformation group in the projective $N+1$ space. In this section we give such a classification for $N = 2, 4$.

Our starting point is the decomposition of $\Lambda^3\mathbb{K}^{N+2}$ given in equation (3.6): given $\Omega \in \Lambda^3\mathbb{K}^{N+2}$ we write it as follows:

$$\begin{aligned} \Omega &= \tilde{T} + A \wedge \mathrm{d}\,u^{N+2} + B \wedge \mathrm{d}\,u^{N+1} \wedge \mathrm{d}\,u^{N+2} \\ &= \tilde{T} + \tilde{A} \wedge \mathrm{d}\,u^{N+2}\,, \end{aligned} \tag{4.1}$$

where $\mathrm{d}\,u^i$, $i = 1,\ldots,N+2$ is the standard basis of $\mathbb{K}^{N+2}$ while $\tilde{T} \in \Lambda^3\mathbb{K}^{N+1}$, $A \in \Lambda^2\mathbb{K}^N$, $B \in \Lambda^1\mathbb{K}^N \cong \mathbb{K}^N$ and $\tilde{A} \in \Lambda^2\mathbb{K}^{N+1}$.

So, to classify the pairs $(\mathscr{P}, V)$ we need to classify the three-forms on $\mathbb{K}^{N+2}$ decomposed as in equation (4.1) up to the action of the projective linear group $\mathrm{PGL}(N+1,\mathbb{K})$ in $N+1$ dimensions. Moreover, we need to add the two following consistency conditions:

(1) the alternating three-form $\tilde{T} \in \Lambda^3\mathbb{K}^{N+1}$ must be non-degenerate, i.e. it must define a non-degenerate second-order homogeneous Hamiltonian operator;
(2) the alternating two-form $\tilde{A} \in \Lambda^2\mathbb{K}^{N+1}$ must be non-null, i.e. it must define a non-trivial system.

Point (1) implies that we can use the classification of second-order operators obtained in [27]. That is, we can start with a fixed alternating three-form in $\tilde{T} \in \Lambda^3\mathbb{K}^{N+1}$ in a standard form. So, we can classify $\tilde{A}$ up to the action of subgroup of $\mathrm{SL}(N+1,\mathbb{K})$ stabilising $\tilde{T}$, which we will denote by $\mathrm{stab}_{\mathrm{SL}(N+1,\mathbb{K})}(\tilde{T})$. Before starting the procedure, we observe that, since we limit ourselves to the cases $N = 2, 4$, our results will be valid for $\mathbb{K}$ being either the real or the complex numbers.

4.1. Case $N = 2$.

Let us start with the simplest case, i.e. $N = 2$. The only operator of this type is

$$\mathscr{P}_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \partial_x^2, \tag{4.2}$$

see [27]. The associated three-form is $\tilde{T}_2 = \mathrm{d}\,u^1 \wedge \mathrm{d}\,u^2 \wedge \mathrm{d}\,u^3$. Note that $\mathrm{stab}_{\mathrm{SL}(3,\mathbb{K})}(\tilde{T}_2) = \mathrm{SL}(3,\mathbb{K})$. Indeed, $\tilde{T}_2$ is a volume form of $\mathbb{K}^3$. In this case the form Ω, as in (4.1), is explicitly given by:

$$\Omega_2 = \tilde{T}_2 + A_{12}\,\mathrm{d}\,u^1 \wedge \mathrm{d}\,u^2 \wedge \mathrm{d}\,u^4 + B_1\,\mathrm{d}\,u^1 \wedge \mathrm{d}\,u^3 \wedge \mathrm{d}\,u^4 + B_2\,\mathrm{d}\,u^2 \wedge \mathrm{d}\,u^3 \wedge \mathrm{d}\,u^4. \tag{4.3}$$

Note that $A_{12} \neq 0$, otherwise we obtain only the null-system. Therefore, using the action of $\mathrm{SL}(3,\mathbb{K})$, we can rescale A_{12} to 1, and map the vector $(B_1, B_2)^T$ to the vector $(1, 0)^T$. This exhausts our possibilities.

So, the only pair $(\mathscr{P}_2, V)$ is given by (4.2) and the completely decoupled (linear) system:

$$\begin{cases} u^1_t = u^1_x, \\ u^2_t = u^2_x. \end{cases} \tag{4.4}$$

4.2. **Case** $N = 4$**.** There are two orbits of $\mathrm{SL}(5,\mathbb{K})$ on three-forms, but one of them gives g_{ij} with determinant zero. We thus restrict ourselves to the alternating three-form in $\mathbb{K}^5$:

$$\tilde{T}_4 = \mathrm{d}\,u^1 \wedge \mathrm{d}\,u^2 \wedge \mathrm{d}\,u^5 + \mathrm{d}\,u^3 \wedge \mathrm{d}\,u^4 \wedge \mathrm{d}\,u^5, \tag{4.5}$$

which determines the constant operator, see [27]:

$$\mathscr{P}_4 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \partial_x^2 \, . \tag{4.6}$$

Note that $\tilde{T} = \eta \wedge \mathrm{d}\,u^5$, where

$$\eta = \mathrm{d}\,u^1 \wedge \mathrm{d}\,u^2 + \mathrm{d}\,u^3 \wedge \mathrm{d}\,u^4 \tag{4.7}$$

is a symplectic form. Therefore, as a set:

$$\mathrm{stab}_{\mathrm{SL}(5,\mathbb{K})}(\tilde{T}_4) = \mathrm{SL}(4,\mathbb{K}) \rtimes \mathbb{K}^4. \tag{4.8}$$

A matrix representation of this group is given by the matrices $M \in \mathrm{SL}(5,\mathbb{K})$ of the following form:

$$M = \begin{pmatrix} C & 0 \\ x^T & 1 \end{pmatrix}, \quad C \in \mathrm{Sp}(4),\ x \in \mathbb{K}^4, \tag{4.9}$$

where we denoted by 0 in the null vector in $\mathbb{K}^4$.

We first consider the action of $\mathrm{Sp}(4)$ on the $A \in \Lambda^2\mathbb{K}^4$. The symplectic form η on $\mathbb{K}^4$ introduces the following splitting which is preserved by the action of $\mathrm{Sp}(4)$ on $\Lambda^2\mathbb{K}^4$:

$$\Lambda^2\mathbb{K}^4 = \mathbb{K}\eta \oplus \Theta_\eta, \tag{4.10}$$

where:

$$\Theta_\eta = \left\{\theta \in \Lambda^2\mathbb{K}^4 \,\middle|\, \eta \wedge \theta = 0\right\}. \tag{4.11}$$

To be explicit, an element $\theta \in \Theta_\eta$ can be uniquely written as:

$$\begin{aligned} \theta = \theta_0 \left(\mathrm{d}\,u^1 \wedge \mathrm{d}\,u^2 - \mathrm{d}\,u^3 \wedge \mathrm{d}\,u^4\right) + \theta_{13}\,\mathrm{d}\,u^1 \wedge \mathrm{d}\,u^3 \\ + \theta_{14}\,\mathrm{d}\,u^1 \wedge \mathrm{d}\,u^4 + \theta_{23}\,\mathrm{d}\,u^2 \wedge \mathrm{d}\,u^3 + \theta_{24}\,\mathrm{d}\,u^2 \wedge \mathrm{d}\,u^4, \end{aligned} \tag{4.12}$$

and this gives a parametrisation of the second factor in (4.10). Then, any element in the orbit of $\theta_\eta\eta + \theta$ is of the form $\theta_\eta\eta + \theta'$ for some $\theta' \in \Theta_\eta$.

Remark 4.1. We remark that

$$\dim \operatorname{stab}_{\mathrm{SL}(5,\mathbb{K})}(\tilde{T}) = \dim \mathrm{Sp}(4) + \dim \mathbb{K}^4 = 14, \tag{4.13}$$

and also:

$$\dim \Lambda^2 \mathbb{K}^5 = \binom{5}{2} = 14, \tag{4.14}$$

which at first inspection suggests that the stabilizer might have a unique orbit on the $\tilde{A}$ in $\Lambda^2\mathbb{K}^5$. However, since Sp(4) fixes the element η given in (4.7) its action cannot be transitive on $\Lambda^2\mathbb{K}^4$. This implies that the final shape of the two-form A will depend on some arbitrary parameters.

Moreover, there is a quadratic form Q on the 5-dimensional subspace Θ_η, which is invariant under the action of Sp(4), defined by

$$\theta \wedge \theta = Q(\theta)\, \mathrm{d}u^1 \wedge \mathrm{d}u^2 \wedge \mathrm{d}u^3 \wedge \mathrm{d}u^4, \tag{4.15}$$

so, with θ as in equation (4.12) we have:

$$Q(\theta) = -2\theta_0^2 - 2\theta_{13}\theta_{24} + 2\theta_{14}\theta_{23}\ . \tag{4.16}$$

Remark 4.2. We remark that if we consider θ as an alternating 4×4 matrix, then $Q(\theta)$ is twice the Pfaffian of that matrix.

It is wel-known that the image of Sp(4) in the orthogonal group of Q is the connected component of the identity element and Sp(4) is the Spin double cover of this component. Since the orthogonal group acts transitively on the two-forms θ with a given value of $Q(\theta)$, the Sp(4)-orbit of $\theta_\eta\eta+\theta$ consists of all the two-forms $\theta_\eta\eta+\theta'$ with $Q(\theta)=Q(\theta')$. Thus we may assume that:

$$A = \theta_\eta \eta + \theta_{13}\, \mathrm{d}u^1 \wedge \mathrm{d}u^3 + \mathrm{d}u^2 \wedge \mathrm{d}u^4 \tag{4.17}$$

with $Q(A) = -2\theta_{13}$. That is, we fixed the two-form A, which now depends on two arbitrary coefficients in $\mathbb{K}$, θ_η and θ_{13}.

Now, we should use the remaining freedom to fix the shape of the vector B. However, we note that this is superfluous, since being the cometric g^{ij} in equation (4.6) constant following the definition of the vectors V^i in equation (2.9) it will disappear upon differentiation. So, using the definition of quasilinear system of conservation laws (1.2) we obtained that the only pair $(\mathscr{P}_4, V)$ is given by (4.6) and the system:

$$\begin{cases} u^1_t = \theta_\eta u^1_x - u^4_x, \\ u^2_t = \theta_\eta u^2_x + \theta_{1,3} u^3_x, \\ u^3_t = u^2 + \theta_\eta u^3, \\ u^4_t = -\theta_{1,3} u^1_x + \theta_\eta u^4_x. \end{cases} \tag{4.18}$$

Note that the system is linear as in the $N=2$ case, but no longer decoupled.

5. Linear line congruences and hydrodynamic type systems

In [1,2], the authors presented an interpretation of hydrodynamic type systems of conservation laws (1.2), involving the classical theory of congruence of lines in the projective space. Using this theory, some basic concepts of homogeneous quasilinear systems such as shocks, rarefaction curves and linear degeneracy are treated by means of geometric properties of the associated algebraic variety.

In particular, it has been shown that to every conservative hydrodynamic-type system one can associate a congruence of lines

$$y^i = u^i y^{N+1} + V^i y^{N+2}, \qquad i = 1,2,\dots N, \tag{5.1}$$

in an auxiliary projective space $\mathbb{P}^{N+1}$ with homogeneous coordinates $[y^1 : \dots : y^{N+2}]$. So, for each field variable $u = (u^1, \dots, u^N)$ one considers the line $L_u \subset \mathbb{P}^{N+1}$ spanned by the two points:

$$P_u = [u^1 : \dots : u^N : 1 : 0], \quad Q_u = [V^1 : \dots : V^N : 0 : 1]. \tag{5.2}$$

The Plücker coordinates $p^{ij} = p^{ij}(u)$ of L_u are defined as the determinants of all 2×2 submatrices of

$$\begin{pmatrix} u^1 & u^2 & \dots & u^N & 1 & 0 \\ V^1 & V^2 & \dots & V^N & 0 & 1 \end{pmatrix} \tag{5.3}$$

or explicitly:

$$\begin{aligned} p^{kl} &= u^k V^l - u^l V^k, \qquad p^{r,N+1} = -V^r, \\ p^{r,N+2} &= u^r, \qquad p^{N+1,N+2} = 1\,. \end{aligned} \tag{5.4}$$

The Plücker coordinates define an embedding of the Grassmannian $\mathrm{Gr}(2, N+2)$ in a projective space $\mathbb{P}^M$ with $M = \binom{N+2}{2} - 1$.

Recall that a congruence (5.1) is said to be *linear* if its closure in $\mathrm{Gr}(2, N+2)$ is defined by linear relations between the Plücker coordinates (5.4) of its lines.

Using (2.9), one can invert g^{ij}, apply (2.2) and obtain N linear equations between the Plücker coordinates (5.4) of the lines (5.1):

$$\frac{1}{2} T_{jkl} \left(u^l V^k - u^k V^l \right) + g^0_{jk} V^k - A_{jl} u^l - B_j = 0, \quad j = 1,2,\dots N, \tag{5.5}$$

see [27, Theorem 11].

We now study in detail the line congruence associated to a simple example coming from the classification presented in the previous section.

Example 1. We consider the case $N = 2$. Then, by applying the transformation $u^3 \mapsto u^3 / g^0_{12}$ the homogeneised operator reduces to

$$\mathscr{P} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \partial_x^2, \tag{5.6}$$

as expressed previously in equation (4.2). Here, all the other terms in (2.1) vanish due to the constant entries of the leading coefficient g^{ij}. Now, from

$V^i = g^{is}(A_{sl}u^s + B_s u^{N+1})$ we obtain

$$V^1 = A_{12}u^1 - B_2 u^3, \qquad V^2 = A_{12}u^2 + B_1 u^3 \tag{5.7}$$

so that the line $L_u \subset \mathbb{P}^3$, with $u = (u^1 : u^2 : u^3) \in \mathbb{P}^2_u$, is spanned by:

$$P_u = \left[u^1 : u^2 : u^3 : 0\right] \tag{5.8a}$$

$$Q_u = \left[A_{12}u^1 - B_2 u^3 : A_{12}u^2 + B_1 u^3 : 0 : u^3\right] \ . \tag{5.8b}$$

The Plücker coordinates of L_u are obtained from the matrix

$$r\colon \begin{pmatrix} u^1 & u^2 & u^3 & 0 \\ A_{12}u^1 - B_2 u^3 & A_{12}u^2 + B_1 u^3 & 0 & u^3 \end{pmatrix} \tag{5.9}$$

and they are:

$$\begin{aligned} &p^{12} = B_1 u^1 u^3 + B_2 u^2 u^3, \quad p^{13} = -A_{12}u^1 u^3 + B_2 (u^3)^2, \quad p^{14} = u^1 u^3, \\ &p^{23} = -A_{12}u^2 u^3 - B_1 (u^3)^2, \quad p^{24} = u^2 u^3, \quad p^{34} = (u^3)^2 \ . \end{aligned} \tag{5.10}$$

The line L_u is represented by the point $(p^{12} : \ldots : p^{34}) \in \mathbb{P}^5$. Notice that all six p^{ij} are multiples of u^3 and thus the line L_u is also represented by

$$\begin{aligned} &p^{12} = B_1 u^1 + B_2 u^2, \quad p^{13} = -A_{12}u^1 + B_2 u^3, \quad p^{14} = u^1, \\ &p^{23} = -A_{12}u^2 - B_1 u^3, \quad p^{24} = u^2, \quad p^{34} = u^3 \ . \end{aligned} \tag{5.11}$$

Since all p^{ij} are linear in the u^k, the image of $\mathbb{P}^2_u$ in $\mathrm{Gr}(2,4) \subset \mathbb{P}^5$ is again a linearly embedded $\mathbb{P}^2$. It is well-known that there are two types of such planes in the Grassmannian, one type parametrizes all the lines in $\mathbb{P}^3$ through a given point and the other type parametrizes all lines in $\mathbb{P}^3$ contained in a plane. All the lines L_u in fact contain a point R:

$$L_u = \langle P_u, Q_u \rangle = \langle P_u, R \rangle, \qquad R := [-B_2 : B_1 : -A_{12} : 1] \ . \tag{5.12}$$

Thus for $N = 2$ the lines associated to a hydrodynamic type system are exactly all the lines in $\mathbb{P}^3$ through a point.

Now we consider the linear equations satisfied by the Plücker coordinates of the lines L_u. These coordinates are not all linear independent, indeed (5.5) provides two linear relations, which are the first two in (5.13), and it is not hard to find two more:

$$p^{13} + A_{12}p^{14} - B_2 p^{34} = 0, \tag{5.13a}$$

$$p^{23} + A_{12}p^{24} + B_1 p^{34} = 0 \tag{5.13b}$$

$$p^{12} - B_1 p^{14} - B_2 p^{24} = 0, \tag{5.13c}$$

$$A_{12}p^{12} + B_1 p^{13} + B_2 p^{23} = 0 \tag{5.13d}$$

The four relations (5.13) are in fact linearly dependent since multiplying the first three by B_1, B_2, and A_{12} respectively and summing gives the last equation. The first three equations are independent and they define exactly the conguence of lines L_u. In fact, the image of the four dimensional $\mathrm{Gr}(2,4)$ in $\mathbb{P}^5$ is defined by the so-called Plücker quadric:

$$p^{12}p^{34} - p^{13}p^{24} + p^{14}p^{23} = 0 \ . \tag{5.14}$$

From the first two equations, which are (5.5), we find:

$$p^{13} = -A_{12}p^{14} + B_2 p^{34}, \qquad p^{23} = -A_{12}p^{24} - B_1 p^{34}, \tag{5.15}$$

and substituting in the Plücker quadric (5.14) we obtain

$$p^{34}(p^{12} - B_1 p^{14} - B_2 p^{24}) = 0 \,. \tag{5.16}$$

Thus the first two linear equations define a $\mathbb{P}^3 \subset \mathbb{P}^5$ that cuts the quadric in the union of two planes, one is defined by the two equations and $p^{34} = 0$ wheras the other is defined by first three equations. Thus the conguence L_u is defined by the first three equations. The first two equations by themselves do not suffice to define the conguence defined a hydrodynamic type system.

In [3, Example 4.9] a three-form on $\mathbb{K}^4$ is used to define a system of three linear equations $p_{12} = p_{13} = p_{23} = 0$. One easily verifies that these define the lines in $\mathbb{P}^3$ that pass through the point $(0:0:0:1)$, similar to the L_u that pass through R.

Also our equations (5.13) can be re-written using the alternating three form $\Omega = (\omega_{ijk})$ as in (3.5), so $\omega_{ijk} = -\omega_{jik}$, $\omega_{ijk} = -\omega_{ikj}$ with $\omega_{123} = 1$, so that $\tilde{T} = \mathrm{d}u^1 \wedge \mathrm{d}u^2 \wedge \mathrm{d}u^3$, $\omega_{124} = A_{12}$, $\omega_{134} = B_1$ and $\omega_{234} = B_2$:

$$0 = \omega_{123}p^{23} + \omega_{124}p^{24} + \omega_{134}p^{34} = p^{23} + A_{12}p^{24} + B_1 p^{34} \tag{5.17a}$$

$$0 = \omega_{213}p^{13} + \omega_{214}p^{14} + \omega_{234}p^{34} = -p^{13} - A_{12}p^{14} + B_2 p^{34} \tag{5.17b}$$

$$0 = \omega_{312}p^{12} + \omega_{314}p^{14} + \omega_{324}p^{24} = p^{12} - B_1 p^{14} - B_2 p^{24} \tag{5.17c}$$

$$0 = \omega_{412}p^{12} + \omega_{413}p^{13} + \omega_{423}p^{23} = A_{12}p^{12} + B_1 p^{13} + B_2 p^{23} \tag{5.17d}$$

This can be compared to formula (5.5), where one has to recall that the 1/2 factor comes from the symmetrisation. □

Example 2. We consider the case $N = 4$. The operator in Doyle-Potëmin canonical form is determined by the following alternating 4×4 matrix $g = (g_{ij})$, which does not depend on u^4 and which we homogenize with a variable u^5:

$$g = g(u) = \begin{pmatrix} 0 & u^3 + g^0_{12}u^5 & -u^2 + g^0_{13}u^5 & g^0_{14}u^5 \\ -u^3 - g^0_{12}u^5 & 0 & u^1 + g^0_{23}u^5 & g^0_{24}u^5 \\ u^2 - g^0_{13}u^5 & -u^1 - g^0_{23}u^5 & 0 & g^0_{34}u^5 \\ -g^0_{14}u^5 & -g^0_{24}u^5 & -g^0_{34}u^5 & 0 \end{pmatrix}. \tag{5.18}$$

The fluxes V^i are determined by an alternating complex 4×4 matrix A and a $B \in \mathbb{K}^4$ as $V = g^{-1}(AU + Bu^5)$ where $U := (u^1, \ldots, u^4)$ as in (2.9). To find $g^{-1} = (g^{ij})$ we observe that the determinant of an alternating matrix is the square of its Pfaffian, in this case one has

$$\mathrm{Pf}(g) = u^5\left[g_{14}u^1 + g_{24}u^2 + g_{34}u^3 + (g_{12}g_{34} - g_{13}g_{24} + g_{14}g_{23})u^5\right]. \tag{5.19}$$

Moreover, the inverse of g can be obtained as

$$g^{-1} = \frac{1}{\mathrm{Pf}(g)}g^\sharp, \qquad g^\sharp = \begin{pmatrix} 0 & -g_{34} & g_{24} & g_{23} \\ g_{34} & 0 & -g_{14} & g_{13} \\ -g_{24} & g_{14} & 0 & -g_{12} \\ g_{23} & -g_{13} & g_{12} & 0 \end{pmatrix}. \tag{5.20}$$

The lines associated to these fluxes are the $L_u \subset \mathbb{P}^5$, with $u = (u^1 : u^2 : u^3 : u^4 : u^5) \in \mathbb{P}^4_u$, where L_u is spanned by

$$P_u = \left[u^1 : u^2 : u^3 : u^4 : u^5 : 0\right] \tag{5.21a}$$

$$Q_u = \left[V^1 : V^2 : V^3 : V^4 : 0 : u^5\right] \ . \tag{5.21b}$$

Since $Q_u \in \mathbb{P}^5$ we may multiply all coordinates by $\mathrm{Pf}(g)$ so that

$$Q_u = \left[V^1_\sharp : V^2_\sharp : V^3_\sharp : V^4_\sharp : 0 : Pf(g)u^5\right], \qquad V_\sharp := g^\sharp(AU + Bu^5) \ . \tag{5.22}$$

All coefficients of P_u, Q_u are homogeneous of degree 1 and 2 respectively in the u^i. The fifteen Plücker coordinates $p^{12}, \ldots, p^{56}$ of L_u are then homogeneous of degree three in the u^i. A computations shows that they are all divisible by u^5, so the point $(p^{12} : \ldots : p^{56}) \in \mathbb{P}^{14}$ defined by L_u has coordinates that are homogeneous of degree two in the u^i. For example:

$$p^{12} = (-g_{13}A_{14} + g_{14}A_{13})(u^1)^2 + \ldots + (g_{23}B_4 - g_{24}B_3 + g_{34}B_2)u^2u^5 \ . \tag{5.23}$$

With computer algebra one can study this Plücker map from $\mathbb{P}^4_u \to \mathrm{Gr}(2,6) \subset \mathbb{P}^{14}$, but instead we will now consider the equations defining the image.

The Plücker coordinates (5.4) of the lines L_u satisfy six linear equations of the form

$$\omega_{ijk}p^{jk} = 0, \qquad i = 1, 2, \ldots N+2, \tag{5.24}$$

where again $\omega = (\omega_{ijk})$ is an alternating three-form. In the following table, the column under p^{jk} lists the coefficients ω_{ijk}, $i = 1, \ldots, 6$ of p^{jk} in the six equations.

(5.25a)

	p^{12}	p^{13}	p^{14}	p^{15}	p^{16}	p^{23}	p^{24}	p^{25}
$\mathrm{d}u^1$	0	0	0	0	0	1	0	g^0_{12}
$\mathrm{d}u^2$	0	-1	0	$-g^0_{12}$	$-A_{12}$	0	0	0
$\mathrm{d}u^3$	1	0	0	$-g^0_{13}$	$-A_{13}$	0	0	$-g^0_{14}$
$\mathrm{d}u^4$	0	0	0	$-g_{23}$	$-A_{14}$	0	0	$-g^0_{24}$
$\mathrm{d}u^5$	g^0_{12}	g^0_{13}	g^0_{23}	0	$-B_1$	g^0_{14}	g^0_{24}	0
$\mathrm{d}u^6$	A_{12}	A_{13}	A_{14}	B_1	0	A_{23}	A_{24}	B_2

(5.25b)

	p^{26}	p^{34}	p^{35}	p^{36}	p^{45}	p^{46}	p^{56}
$\mathrm{d}u^1$	A_{12}	0	g^0_{13}	A_{13}	g^0_{23}	A_{14}	B_1
$\mathrm{d}u^2$	0	0	g^0_{14}	A_{23}	g^0_{24}	A_{24}	B_2
$\mathrm{d}u^3$	$-A_{23}$	0	0	0	g^0_{34}	A_{34}	B_3
$\mathrm{d}u^4$	$-A_{24}$	0	$-g^0_{34}$	$-A_{34}$	0	0	B_4
$\mathrm{d}u^5$	$-B_2$	g^0_{34}	0	$-B_3$	0	$-B_4$	0
$\mathrm{d}u^6$	0	A_{34}	B_3	0	B_4	0	0

The first four equations are those from (5.5).

The equations (5.25) were analyzed in [3, Example 4.11], they define a subspace $\mathbb{P}^8_\omega \subset \mathbb{P}^{14}$. The intersection $X_\omega := \mathbb{P}^8_\omega \cap \mathrm{Gr}(2,4)$ is a four dimensional subvariety X_ω of $\mathrm{Gr}(2,6)$ which is isomorphic to (the Segre image of) $\mathbb{P}^2 \times \mathbb{P}^2$. In that paper one also finds that the first four equations define a union of two (irreducible) subvarieties, X_ω and $Y = Y_{\omega, \mathrm{d}\,u^5\wedge \mathrm{d}\,u^6}$. The last two equations thus are needed to exclude the points in Y that are not in X_ω.

Since the Plücker map $\mathbb{P}^4_u \to \mathrm{Gr}(2,6)$ is easily seen to have degree one onto its image, we conclude that its image is X_ω. The Plücker map thus induces a birational isomorphism $\mathbb{P}^4_u \to \mathbb{P}^2 \times \mathbb{P}^2$ (the base locus consists of two skew lines) which is similar to the birational isomorphism between $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$ which blows up two points and contracts the line in $\mathbb{P}^2$ spanned by the base points.

Consider now the general three-form

$$\omega = (\omega_{ijk}) = \mathrm{d}\,u^1 \wedge \mathrm{d}\,u^2 \wedge \mathrm{d}\,u^3 + \mathrm{d}\,u^4 \wedge \mathrm{d}\,u^5 \wedge \mathrm{d}\,u^6\,, \tag{5.26}$$

so that the ω_{ijk} are:

$$\omega_{123} = \omega_{456} = 1, \quad \omega_{ijk} = 0 \quad \text{for} \quad (i,j,k) \neq (1,2,3), (4,5,6). \tag{5.27}$$

The $N+2=6$ equations in the Plücker equations that define $X_\omega \subset \mathrm{Gr}(2,6)$ are:

$$\pm\omega_{123}p^{12} = 0, \quad \pm\omega_{123}p^{13} = 0, \quad \pm\omega_{123}p^{23} = 0, \tag{5.28}$$

and similarly there are three equations involving ω_{456}. The lines with $p_{12} = p_{13} = p_{23} = 0$ are those which meet the plane $x_1 = x_2 = x_3 = 0$. In fact, such a line is spanned by $a = (a_1, \dots, a_6)$, $b = (b_1, \dots, b_6) \in \mathbb{K}^6$, and the vectors (a_1, a_2, a_3), $(b_1, b_2, b_3) \in \mathbb{K}^3$ are linearly dependent. So after taking a suitable linear combination of a and b we may assume that $a_1 = a_2 = a_3 = 0$ and then a lies in $x_1 = x_2 = x_3 = 0$. Similarly the lines with $p_{45} = p_{46} = p_{56} = 0$ are those which meet the plane $x_4 = x_5 = x_6 = 0$. Thus any line in X_ω is spanned by a point $P = (x_1 : x_2 : x_3 : 0 : 0 : 0)$ and a point $Q = (0,0,0, y_1 : y_2 : y_3)$. The Plücker coordinates of this line are the $x_i y_j$ so $X_\omega \cong \mathbb{P}^2 \times \mathbb{P}^2$ and X_ω is embedded via the Segre map in the $\mathbb{P}^8 \subset \mathbb{P}^{14}$ defined by $p_{12} = p_{13} = p_{23} = p_{45} = p_{46} = p_{56} = 0$. □

As observed in the Examples 1 and 2 for $N = 2,4$, besides the N linear equations for the Plücker coordinates given in (5.5), these satisfy two more linear equations. The $N+2$ equations one finds can be written as

$$\omega_{ijk}p^{jk} = 0, \qquad i = 1, 2, \dots N+2, \tag{5.29}$$

where the coefficients ω_{ijk} are constant, skew-symmetric with respect to any pair of indices.

The results of [3] imply that these linear equations, for a general alternating three-form ω, define a smooth subvariety X_ω of dimension N of $\mathrm{Gr}(2, N+2)$.

We will show in Theorem 5.2 that for any even N the Plücker coordinates of the lines L_u satisfy the equations (5.29), where ω is the three-form defined in (3.5). Thus X_ω is birationally isomorphic to $\mathbb{P}^N_u$, the projective space which parametrizes the lines L_u. In particular, we found an explicit parametrization of X_ω, which is thus a rational variety.

Remark 5.1. Notice that $\dim \mathrm{Gr}(2, N+2) = 2N$ and $\dim \mathbb{P}^N_u = \dim X_\omega = N$. However, there are $N+2$ linear forms vanishing on X_ω which are linearly independent for $N > 2$. In fact, in [3] it is shown that taking N of these linear forms defines a dimension N subset with two irreducible components one is X_ω, the other is denoted by Y. One needs two more equations to find exactly X_ω. The two components are very well visible in Example 1 but in general we do not have a good description of the 'extra' component Y defined by the first N equations in terms of Hamiltonian structures. This also explains the "dimensional gap" between the description of second-order Hamiltonian operators of [27] in terms of alternating three-forms on $\mathbb{K}^{N+1}$, and the projective interpretation of hydrodynamic-type systems (2.9) of Agafonov and Ferapontov [1, 2] on the projective space $\mathbb{P}^{N+1}$.

The equations (5.29) can also be stated more intrinsically. Let us consider $\Omega = (\omega_{ijk}) \in \Lambda^3\mathbb{K}^{N+2}$, and a line $L \subset \mathbb{P}^{N+1}$, i.e. a two-dimensional subspace in $\mathbb{K}^{N+2}$. Recall that the pullback $\Omega^* L \in \Lambda^1\mathbb{K}^{N+2}$ of Ω to L is the contraction w.r.t. two indices, so in coordinates:

$$(\Omega^* L)_i = \omega_{ijk} p^{jk}, \tag{5.30}$$

where p^{jk} are the Plücker coordinates of L in $\mathrm{Gr}(2, N+2)$. Then *annihilation set of lines* for Ω, denoted by X_Ω, are those lines $L \subset \mathbb{P}^{N+1}$ whose pullback with respect to Ω vanishes:

$$X_\Omega = \{[L] \in \mathrm{Gr}(2, N+2) : \Omega^* L = 0\}\ . \tag{5.31}$$

In coordinates, $\Omega^* L = 0$ is the set of $N+2$ linear equations in the Plücker coordinates of L $\omega_{ijk} p^{jk} = 0$. The following result shows that the lines L_u we considered satisfy the equations obtained from Ω:

Theorem 5.2. *The congruence of lines associated to a hydrodynamic-type system with second-order homogeneous Hamiltonian structure is the annihilation set of lines of the three-form Ω associated to the operator-system pair $(\mathscr{P}, V)$ in the sense of Theorem 3.1.*

Proof. Let us consider equation (5.5). Using the definition of the Plücker coordinates of the congruence of lines L_u (5.4) we can rewrite it as:

$$\sum_{k,l=1,\,k<l}^{N} T_{jkl} p^{kl} + \sum_{k=1}^{N} g^0_{jk} p^{k\,N+1} + \sum_{k=1}^{N} A_{jk} p^{k\,N+2} + B_j p^{N+1\,N+2} = 0 \tag{5.32}$$

then using the definition of Ω (3.5) we obtain:

$$\begin{aligned}\sum_{k,l=1,\,k<l}^{N} \omega_{jkl} p^{kl} + \sum_{k=1}^{N} \omega_{jk\,N+1} p^{k\,N+1} +& \\ + \sum_{k=1}^{N} \omega_{jk\,N+2} p^{k\,N+2} + \omega_{j\,N+1\,N+2} p^{N+1\,N+2} &= 0\end{aligned} \tag{5.33}$$

that is

$$\sum_{k,l=1\,k<l}^{N+2} \omega_{jkl}p^{kl} = 0 \tag{5.34}$$

thus proving the first N equations obtained from Ω are satisfied.

Let us now derive the last two equations. At first, let us consider $g_{jk}V^k = A_{jl}u^l + B_j$, then by multiplying by u^j we use the skew-symmetry to obtain:

$$(T_{jkl}u^l + g^0_{jk})V^k u^j = A_{jl}u^l u^j + B_j u^j, \tag{5.35}$$

so that after using (5.4), one obtains

$$-\frac{1}{2}g^0_{jk}(u^j V^k - u^k V^j) = B_j u^j \quad \Longrightarrow \quad \frac{1}{2}g^0_{jk}p^{jk} - B_j p^{j\,N+2} = 0. \tag{5.36}$$

Using (3.5), the previous expression is

$$\sum_{k,l=1\,k<l}^{N} \omega_{N+1\,kl}p^{kl} + \sum_{k=1}^{N} \omega_{N+1\,j\,N+2}p^{j\,N+2} = 0. \tag{5.37}$$

This is in fact the $N+1$-st equation:

$$\sum_{k,l=1\,k<l}^{N+2} \omega_{N+1\,kl}p^{kl} = 0. \tag{5.38}$$

The last relation is obtained similarly, by multiplying $g_{jk}V^k = A_{jl}u^l + B_j$ by V^j and by using the skew-symmetry:

$$(T_{jkl}u^l + g^0_{jk})V^k V^j = A_{jl}u^l V^j + B_j V^j \tag{5.39}$$

which implies $A_{jl}u^l V^j + B_j V^j = 0$, and finally

$$\frac{1}{2}A_{jl}p^{jl} - B_k p^{kN+1} = 0. \tag{5.40}$$

This is

$$\sum_{k,l=1\,k<l} \omega_{N+1\,kl}p^{kl} + \sum_{k=1}^{N} \omega_{N+2\,k\,N+2}p^{k\,N+1} = 0 \tag{5.41}$$

or equivalently,

$$\sum_{k,l=1\,k<l}^{N+2} \omega_{N+2\,kl}p^{kl} = 0. \tag{5.42}$$

This is the $N+2$-st equation. Note that by construction the coefficients are totally skew-symmetric and the theorem is proved. □

Summarising, the meaning of Theorems 3.1 and 5.2 is that the natural geometric structures of hydrodynamic-type systems admitting Hamiltonian formalism with a second-order homogeneous operators are alternating three-forms in the projective space $\mathbb{P}^{N+1}$ together with their annihilation sets of lines.

6. Conclusions

In this paper we established some novel relationships among second-order homogeneous Hamiltonian operators, their associated quasilinear systems of conservation laws, and alternating three-forms together with their intrinsic geometric structures. In particular, we proved the following results:

- a set-theoretical bijection between $\mathscr{Y}_N$, the space of pairs $(\mathscr{P}, V)$ with $\mathscr{P}$ second-order Hamiltonian operator and V the associated system of conservation laws in N components, and $\Lambda^3\mathbb{K}^{N+2}$, the space of alternating three-form in dimension $N+2$;
- a proof of the projective invariance of quasilinear system of conservation laws V admitting second-order Hamiltonian structure, justifying the above bijection, and leading to a deeper geometric interpretation of the pairs $(\mathscr{P}, V)$;
- a classification of the elements of $\mathscr{Y}_N$ for $N = 2$ and $N = 4$ through the action of the subgroup of $\mathrm{SL}(N+1, \mathbb{K})$ stabilizing $\Lambda^3\mathbb{K}^{N+1} \subset \Lambda^3\mathbb{K}^{N+2}$ corresponding to the operator $\mathscr{P}$;
- a novel interpretation of the line congruence associated to a Hamiltonian quasilinear system V as the annihilation set of lines of the alternating three-form corresponding to its pair.

A number of interesting open questions and possible extension remain to be investigated. Some of these are:

- characterise the (classes of) elements of $\mathscr{Y}_N$ for $N > 4$;
- extend the correspondence we found to the bi-Hamiltonian case, see for instance [12], and the recent results of [14, 15];
- extend the correspondence of the operator-system pairs to higher orders.

We comment that a characterisation of $\mathscr{Y}_6$ has the possibility to give many interesting results of geometric nature. Indeed, following [3, Example 4.12] the only open orbit of three-forms in $\Lambda^3\mathbb{K}^7$ has the simple Lie group G_2 as stabiliser, and the associated annihilation set of lines is the homogeneous variety of G_2 in $\mathbb{P}^{13}$. Work is in progress in this direction.

Finally, regarding the last suggested open problem, we observed that a similar construction of linear congruences has been presented in [11] for third-order Hamiltonian operators. However, a direct interpretation as annihilation set of lines was not possible, because the obtained linear equations are not governed by a totally skew-symmetric tensor. The results of this paper suggest that there might be a canonical $(0,2)$-tensor which maps such linear equations into the annihilation set of lines of a properly chosen alternating three-form.

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