THE VIETORIS FUNCTOR AND MODAL OPERATORS ON RINGS OF CONTINUOUS FUNCTIONS

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ABSTRACT. We introduce an endofunctor $\mathbb{H}$ on the category $\boldsymbol{ba\ell}$ of bounded archimedean ℓ-algebras and show that there is a dual adjunction between the category $\mathsf{Alg}(\mathbb{H})$ of algebras for $\mathbb{H}$ and the category $\mathsf{Coalg}(\mathbb{V})$ of coalgebras for the Vietoris endofunctor $\mathbb{V}$ on the category of compact Hausdorff spaces. We prove that Gelfand duality lifts to a dual equivalence between $\mathsf{Coalg}(\mathbb{V})$ and the full reflective subcategory $\mathsf{Alg}^u(\mathbb{H})$ of $\mathsf{Alg}(\mathbb{H})$. We introduce an endofunctor $\mathbb{H}^u$ on the full reflective subcategory of $\boldsymbol{ba\ell}$ consisting of uniformly complete objects of $\boldsymbol{ba\ell}$ and show that $\mathsf{Alg}(\mathbb{H}^u)$ is isomorphic to $\mathsf{Alg}^u(\mathbb{H})$, thus providing an alternate view of $\mathsf{Alg}^u(\mathbb{H})$. On the one hand, these results generalize those of [1, 20] for the category of coalgebras of the Vietoris endofunctor on the category of Stone spaces. On the other hand, they provide an alternate, more categorical proof of a recent result of [6].

1. Introduction

It is a well-known result in modal logic that the category MA of modal algebras is dually equivalent to the category DF of descriptive frames. This result has its origins in the work of Jónsson and Tarski [19], which is why it is often referred to as Jónsson-Tarski duality. In its present form it was established by Esakia [13] and Goldblatt [14] (but see also Halmos [15]).

A descriptive frame is a Stone space (compact Hausdorff zero-dimensional space) X equipped with a binary relation R that is continuous, meaning that the corresponding map $\rho_R : X \to \mathcal{V}(X)$ into the Vietoris space of X, given by

$$\rho_R(x) = R[x] = \{y \mid xRy\},$$

is a well-defined continuous map. In fact, DF is isomorphic to the category $\mathsf{Coalg}(\mathbb{V})$ of coalgebras for the Vietoris endofunctor $\mathbb{V}$ on the category Stone of Stone spaces. Abramsky [1] and Kupke, Kurz, and Venema [20] defined the dual endofunctor $\mathbb{H}$ on the category BA of boolean algebras. They showed that the category $\mathsf{Alg}(\mathbb{H})$ of algebras for $\mathbb{H}$ is isomorphic to MA, and proved that Stone duality between BA and Stone lifts to a dual equivalence between $\mathsf{Alg}(\mathbb{H})$ and $\mathsf{Coalg}(\mathbb{V})$. This yields an elegant new proof of Jónsson-Tarski duality.

Let KHaus be the category of compact Hausdorff spaces and continuous maps. Then Stone is a full subcategory of KHaus. There are several generalizations of Stone duality to KHaus. To outline one such generalization, we point out that in Stone duality we work with the boolean algebra of clopens, which correspond to continuous characteristic functions. Since

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an arbitrary compact Hausdorff space X does not have enough clopens, it is natural to work instead with the ring $C(X)$ of all continuous real-valued functions. This gives rise to the celebrated Gelfand duality between KHaus and the category of bounded archimedean ℓ-algebras that in addition are uniformly complete (see Section 2.1 for details). Up to isomorphism, these are exactly the rings $C(X)$ for $X \in \mathsf{KHaus}$.

The Vietoris endofunctor $\mathbb{V} : \mathsf{Stone} \to \mathsf{Stone}$ is the restriction of the Vietoris endofunctor $\mathbb{V} : \mathsf{KHaus} \to \mathsf{KHaus}$. We call a relation R on $X \in \mathsf{KHaus}$ continuous if $\rho_R : X \to \mathbb{V}(X)$ is a well-defined continuous map. The pairs (X, R), where $X \in \mathsf{KHaus}$ and R is a continuous relation on R, generalize descriptive frames. Following [6], we call such pairs *compact Hausdorff frames*, and denote the resulting category by KHF. Then DF is a full subcategory of KHF and the isomorphism between DF and the category of coalgebras for $\mathbb{V} : \mathsf{Stone} \to \mathsf{Stone}$ extends to an isomorphism between KHF and the category of coalgebras for $\mathbb{V} : \mathsf{KHaus} \to \mathsf{KHaus}$.

Our aim is to generalize the endofunctor $\mathbb{H} : \mathsf{BA} \to \mathsf{BA}$ that is the algebraic counterpart of $\mathbb{V} : \mathsf{Stone} \to \mathsf{Stone}$ to an endofunctor on the category $\boldsymbol{ba\ell}$ of bounded archimedean ℓ-algebras so that it is the algebraic counterpart of $\mathbb{V} : \mathsf{KHaus} \to \mathsf{KHaus}$. For this we need to overcome several obstacles. Firstly, the construction of $\mathbb{H} : \mathsf{BA} \to \mathsf{BA}$ utilizes the existence of free boolean algebras. However, as was shown in [7], free algebras on sets do not exist in $\boldsymbol{ba\ell}$. Instead we need to work with free algebras on weighted sets (see Section 2.2). Secondly, since KHaus is dually equivalent to the reflective subcategory $\boldsymbol{uba\ell}$ of $\boldsymbol{ba\ell}$ consisting of uniformly complete objects in $\boldsymbol{ba\ell}$, additional care is needed when transitioning from $\boldsymbol{ba\ell}$ to its subcategory $\boldsymbol{uba\ell}$.

We first construct the endofunctor $\mathbb{H} : \boldsymbol{ba\ell} \to \boldsymbol{ba\ell}$. This we do by viewing each $A \in \boldsymbol{ba\ell}$ as a weighted set, taking the free object in $\boldsymbol{ba\ell}$ on this weighted set, and then modding it out by the relations that are motivated by the definition of a modal operator on $A \in \boldsymbol{ba\ell}$ given in [6]. One of our main results is Theorem 5.8, which establishes that the dual compact Hausdorff space of $\mathbb{H}(A)$ is homeomorphic to the Vietoris space of the dual compact Hausdorff space of A. This paves the way to prove a dual adjunction between $\mathsf{Alg}(\mathbb{H})$ and $\mathsf{Coalg}(\mathbb{V})$. We show that this dual adjunction restricts to a dual equivalence between $\mathsf{Coalg}(\mathbb{V})$ and the full reflective subcategory $\mathsf{Alg}^u(\mathbb{H})$ of $\mathsf{Alg}(\mathbb{H})$, thus lifting Gelfand duality. We also introduce an endofunctor $\mathbb{H}^u : \boldsymbol{uba\ell} \to \boldsymbol{uba\ell}$ and show that $\mathsf{Alg}^u(\mathbb{H})$ is isomorphic to $\mathsf{Alg}(\mathbb{H}^u)$, thus providing an alternate view of the category $\mathsf{Alg}^u(\mathbb{H})$. The obtained results show that the endofunctor $\mathbb{H} : \boldsymbol{ba\ell} \to \boldsymbol{ba\ell}$ is the algebraic countrepart of the Vietoris endofunctor $\mathbb{V} : \mathsf{KHaus} \to \mathsf{KHaus}$, yielding a generalization of the results of Kupke, Kurz, and Venema [20] from Stone to KHaus.

The definition in [6] of a modal operator on $A \in \boldsymbol{ba\ell}$ has resulted in the category $\boldsymbol{mba\ell}$ of modal bounded archimedean ℓ-algebras and its reflective subcategory $\boldsymbol{muba\ell}$ consisting of uniformly complete objects. The main result of [6] establishes that there is a dual adjunction between $\boldsymbol{mba\ell}$ and KHF, which restricts to a dual equivalence between $\boldsymbol{muba\ell}$ and KHF. This is a common generalization of both Gelfand duality and Jónsson-Tarski duality. In this paper we show that $\mathsf{Alg}(\mathbb{H})$ is isomorphic to $\boldsymbol{mba\ell}$ and that $\mathsf{Alg}^u(\mathbb{H})$ is isomorphic to $\boldsymbol{muba\ell}$. From this the main result of [6] follows. In fact, our approach provides an alternate, more categorical proof of the result of [6]. Consequently, we arrive at the following diagram, where

$\simeq^{\mathrm{op}}$ represents a dual equivalence, $\cong$ an isomorphism, and $\hookleftarrow$ an embedding of categories.

$$\begin{tikzcd}
\mathsf{Alg}(\mathbb{H}) \arrow[d, leftrightarrow, "\cong"'] & \mathsf{Alg}^u(\mathbb{H}) \arrow[l, hook'] \arrow[r, leftrightarrow, "\cong"] \arrow[d, leftrightarrow, "\cong"] & \mathsf{Alg}(\mathbb{H}^u) \arrow[r, leftrightarrow, "\simeq^{\mathrm{op}}"] & \mathsf{Coalg}(\mathbb{V}) \arrow[d, leftrightarrow, "\cong"] \\
\boldsymbol{mba\ell} & \boldsymbol{muba\ell} \arrow[l, hook'] \arrow[rr, leftrightarrow, "\simeq^{\mathrm{op}}"'] & & \mathsf{KHF}
\end{tikzcd}$$

The paper is organized as follows. Section 2 provides the necessary background for the paper, including Gelfand duality, the construction of free objects in $\boldsymbol{ba\ell}$, and the definition of modal operators on objects of $\boldsymbol{ba\ell}$. In Section 3 we introduce the endofunctor $\mathbb{H}$ on $\boldsymbol{ba\ell}$, and in Section 4 we show that the category of algebras for $\mathbb{H}$ is isomorphic to $\boldsymbol{mba\ell}$. We relate $\mathbb{H}$ to the Vietoris functor $\mathbb{V}$ in Section 5 by showing that for $A \in \boldsymbol{ba\ell}$, the Yosida space of $\mathbb{H}(A)$ is homeomorphic to the Vietoris space of the Yosida space of A. We prove our main result in Section 6, establishing a dual adjunction between $\mathsf{Alg}(\mathbb{H})$ and $\mathsf{Coalg}(\mathbb{V})$. We then introduce a reflective subcategory $\mathsf{Alg}^u(\mathbb{H})$ of $\mathsf{Alg}(\mathbb{H})$ and show that this dual adjunction restricts to a dual equivalence between $\mathsf{Alg}^u(\mathbb{H})$ and $\mathsf{Coalg}(\mathbb{V})$. We also provide an alternate view of the category $\mathsf{Alg}^u(\mathbb{H})$ as $\mathsf{Alg}(\mathbb{H}^u)$. In Section 7 we derive the main result of [6], showing that there is a dual adjunction between $\boldsymbol{mba\ell}$ and KHF, which restricts to a dual equivalence between $\boldsymbol{muba\ell}$ and KHF. In Section 8 we show how the exclusion of the empty set from the construction of the Vietoris space results in the modification of the $\mathbb{H}$ functor to the functor $\mathbb{H}^*$ such that $\mathsf{Alg}(\mathbb{H}^*)$ is isomorphic to the full subcategory $\boldsymbol{mba\ell}^{\mathsf{D}}$ of those $(A, \Box)$ where $\Box$ corresponds to a serial relation. Finally, in Section 9 we relate our results to those of [20].

2. Preliminaries

In this section we provide the necessary background for the rest of the paper. In § 2.1 we recall Gelfand duality, in § 2.2 free objects in $\boldsymbol{ba\ell}$ over weighted sets, and finally in § 2.3 modal operators on algebras in $\boldsymbol{ba\ell}$ and a generalization of Gelfand duality to this setting.

2.1. Gelfand duality.

For basic facts about lattice-ordered rings and algebras we use Birkhoff's book [10, Ch. XIII and onwards] as our main reference. All rings we consider are assumed to be commutative and unital.

Definition 2.1.

(1) A ring A with a partial order $\leq$ is a *lattice-ordered ring*, or an *ℓ-ring* for short, provided $(A, \leq)$ is a lattice, $a \leq b$ implies $a + c \leq b + c$ for each c, and $0 \leq a, b$ implies $0 \leq ab$.

(2) An ℓ-ring A is an *ℓ-algebra* if it is an $\mathbb{R}$-algebra and for each $0 \leq a \in A$ and $0 \leq r \in \mathbb{R}$ we have $0 \leq r \cdot a$.

(3) An ℓ-ring A is *bounded* if for each $a \in A$ there is $n \in \mathbb{N}$ such that $a \leq n \cdot 1$ (that is, 1 is a *strong order unit*).

(4) An ℓ-ring A is *archimedean* if for each $a, b \in A$, whenever $n \cdot a \leq b$ for each $n \in \mathbb{N}$, then $a \leq 0$.

(5) An *ℓ-algebra morphism* $\alpha : A \to B$ is both an $\mathbb{R}$-algebra and lattice homomorphism. It is *unital* if $\alpha(1) = 1$.
(6) Let $\boldsymbol{ba\ell}$ be the category of bounded archimedean ℓ-algebras and unital ℓ-algebra morphisms.

Let $A \in \boldsymbol{ba\ell}$. For $a \in A$, define the *absolute value* of a by

$$|a| = a \vee (-a).$$

If we set the positive and negative parts of a to be $a^+ = a \vee 0$ and $a^- = (-a)^+$, then $a = a^+ - a^-$ and $|a| = a^+ + a^-$.

The *norm* of a is defined by

$$||a|| = \inf\{r \in \mathbb{R} \mid |a| \le r \cdot 1\}.$$

If $X \in \mathsf{KHaus}$, then $C(X) \in \boldsymbol{ba\ell}$ and the definition of the norm of $f \in C(X)$ coincides with the usual definition

$$\|f\| = \sup\{|f(x)| \mid x \in X\}.$$

If $\alpha : A \to B$ is a $\boldsymbol{ba\ell}$-morphism, it is easy to see that $\|\alpha(a)\| \le \|a\|$, $\alpha(a^+) = \alpha(a)^+$, and $\alpha(r) = r$ for all $r \in \mathbb{R}$, where we identify r with $r \cdot 1$.

Definition 2.2. We call $A \in \boldsymbol{ba\ell}$ *uniformly complete* if its norm is complete. Let $\boldsymbol{uba\ell}$ be the full subcategory of $\boldsymbol{ba\ell}$ consisting of uniformly complete objects of $\boldsymbol{ba\ell}$.

Theorem 2.3 (Gelfand duality)**.** *There is a dual adjunction between $\boldsymbol{ba\ell}$ and KHaus which restricts to a dual equivalence between $\boldsymbol{uba\ell}$ and KHaus.*

We briefly describe the functors $\mathbb{C} : \mathsf{KHaus} \to \boldsymbol{ba\ell}$ and $\mathbb{Y} : \boldsymbol{ba\ell} \to \mathsf{KHaus}$ establishing the dual adjunction of Theorem 2.3; for details see [8, Sec. 3] and the references therein. For a compact Hausdorff space X let $\mathbb{C}(X) := C(X)$ be the ring of (necessarily bounded) continuous real-valued functions on X. For a continuous map $\varphi : X \to Y$ let $\mathbb{C}(\varphi) : C(Y) \to C(X)$ be defined by $\mathbb{C}(\varphi)(f) = f \circ \varphi$ for each $f \in C(Y)$. Then $\mathbb{C} : \mathsf{KHaus} \to \boldsymbol{ba\ell}$ is a well-defined contravariant functor.

For $A \in \boldsymbol{ba\ell}$, we recall that an ideal I of A is an *ℓ-ideal* if $|a| \le |b|$ and $b \in I$ imply $a \in I$. If $A \in \boldsymbol{ba\ell}$, then we can associate to A a compact Hausdorff space as follows. Let Y_A be the space of maximal ℓ-ideals of A, whose closed sets are exactly sets of the form

$$Z_\ell(I) = \{M \in Y_A \mid I \subseteq M\},$$

where I is an ℓ-ideal of A. It follows from the work of Yosida [24] that $Y_A \in \mathsf{KHaus}$. As is customary, we refer to Y_A as the *Yosida space* of A and set $\mathbb{Y}(A) = Y_A$. For a morphism α in $\boldsymbol{ba\ell}$ we let $\mathbb{Y}(\alpha) = \alpha^{-1}$. Then $\mathbb{Y} : \boldsymbol{ba\ell} \to \mathsf{KHaus}$ is a well-defined contravariant functor, and the functors $\mathbb{C}$ and $\mathbb{Y}$ yield a dual adjunction between $\boldsymbol{ba\ell}$ and KHaus.

For $X \in \mathsf{KHaus}$ we have that $\varepsilon_X : X \to Y_{C(X)}$ is a homeomorphism where

$$\varepsilon_X(x) = \{f \in C(X) \mid f(x) = 0\}.$$

For $A \in \boldsymbol{ba\ell}$ define $\zeta_A : A \to C(Y_A)$ by $\zeta_A(a)(M) = r$ where r is the unique real number satisfying $a + M = r + M$. Then ζ_A is a monomorphism in $\boldsymbol{ba\ell}$ separating points of Y_A.

Therefore, by the Stone-Weierstrass theorem, $\zeta_A : A \to C(Y_A)$ is the uniform completion of A. Thus, if A is uniformly complete, then ζ_A is an isomorphism. Consequently, the contravariant adjunction restricts to a dual equivalence between $\boldsymbol{ubal}$ and KHaus, yielding Gelfand duality. Another consequence of these considerations is the following well-known result.

Proposition 2.4. *$\boldsymbol{ubal}$ is a full reflective subcategory of $\boldsymbol{bal}$, and the reflector assigns to each $A \in \boldsymbol{bal}$ its uniform completion $\mathbb{C}\mathbb{Y}(A) = C(Y_A) \in \boldsymbol{ubal}$.*

Remark 2.5. Since $\mathbb{C}$ and $\mathbb{Y}$ form a dual adjunction between $\boldsymbol{bal}$ and KHaus, the natural transformations ζ and ε satisfy $\mathbb{Y}(\zeta_A) \circ \varepsilon_{Y_A} = 1_{Y_A}$ and $\mathbb{C}(\varepsilon_X) \circ \zeta_{C(X)} = 1_{C(X)}$ for each $A \in \boldsymbol{bal}$ and $X \in \mathsf{KHaus}$ by [21, Thm. IV.1.1]. Moreover, since ε is a natural isomorphism, $\mathbb{Y}(\zeta_A) = \varepsilon_{Y_A}^{-1}$ and $\zeta_{C(X)} = \mathbb{C}(\varepsilon_X)^{-1}$.

If A is an ℓ-subalgebra of $B \in \boldsymbol{bal}$, we say A is *uniformly dense* in B if A is dense in B with respect to the topology induced by the norm on B. In the following lemma we collect several facts that will be used subsequently.

Lemma 2.6. [8, Lem. 2.9] *Let $\alpha : A \to B$ be a $\boldsymbol{bal}$-morphism.*

(1) $\mathbb{Y}(\alpha)$ *is onto iff* α *is* 1-1 *iff* α *is a monomorphism.*
(2) $\mathbb{Y}(\alpha)$ *is* 1-1 *iff* $\alpha[A]$ *is uniformly dense in* B *iff* α *is an epimorphism.*
(3) $\mathbb{Y}(\alpha)$ *is a homeomorphism iff* α *is a bimorphism.*

2.2. **Free objects in $\boldsymbol{bal}$.** By [7, Thm. 3.2], free objects on nonempty sets do not exist in $\boldsymbol{bal}$. To see this, observe that each $\boldsymbol{bal}$-morphism $\alpha : A \to B$ satisfies $\|\alpha(a)\| \le \|a\|$ for each $a \in A$. Now suppose that a free object $F \in \boldsymbol{bal}$ exists on $X \neq \varnothing$. Let $f : X \to F$ be the corresponding map, let $x \in X$, and let $r \in \mathbb{R}$ satisfy $r > \|f(x)\|$. Define $g : X \to \mathbb{R}$ by $g(y) = r$ for each $y \in X$. Then there is a $\boldsymbol{bal}$-morphism $\alpha : F \to \mathbb{R}$ with $\alpha \circ f = g$. Therefore, $\alpha(f(x)) = g(x) = r$, violating the inequality $\|\alpha(f(x))\| \le \|f(x)\|$. Taking this into account leads to the following notion.

Definition 2.7. [7, Def. 3.3]

- A *weight function* on a set X is a function w from X into the nonnegative real numbers.
- A *weighted set* is a pair (X, w) where X is a set and w is a weight function on X.
- A *weighted set morphism* $f : (X_1, w_1) \to (X_2, w_2)$ is a function $f : X_1 \to X_2$ satisfying $w_2(f(x)) \le w_1(x)$ for each $x \in X$.

There is a forgetful functor U from $\boldsymbol{bal}$ to the category of weighted sets that associates to each $A \in \boldsymbol{bal}$ the weighted set $(A, \|\cdot\|)$. By [7, Thm. 3.9], U has a left adjoint, which is constructed by associating to each weighted set (X, w) the free unital ℓ-algebra on the set X and then modding it out by the appropriate relations using the weight function w. Thus, we arrive at the following theorem.

Theorem 2.8. *Free objects in $\boldsymbol{bal}$ exist over weighted sets.*

2.3. **Modal operators on bounded archimedean ℓ-algebras.** In [6] the notion of a modal operator on $A \in \boldsymbol{ba\ell}$ was introduced, generalizing that of a modal operator on a boolean algebra. The motivating example comes from a continuous relation R on a compact Hausdorff space X (see Definition 2.11). If R is serial (meaning $R[x] \neq \varnothing$ for each $x \in X$), then there is a natural definition of a modal operator $\Box_R$ on $C(X)$, given by $\Box_R(f)(x) = \inf fR[x]$ for each $x \in X$. It is straightforward to see that $\Box_R$ preserves meet, $0, 1$, addition by a scalar, and multiplication by a nonnegative scalar. If R is not serial, then $\Box_R$ needs to be redefined since $R[x]$ may be empty. For an arbitrary continuous relation we define $\Box_R$ by

$$(\Box_R f)(x) = \begin{cases} \inf fR[x] & \text{if } R[x] \neq \varnothing \\ 1 & \text{if } R[x] = \varnothing. \end{cases}$$

If R is not serial, then $\Box_R 0 \neq 0$, and the properties of $\Box_R$ become more complicated. Looking carefully at those properties, we arrive at the following definition.

Definition 2.9. [6, Def. 3.10]

(1) Let $A \in \boldsymbol{ba\ell}$. We say that a unary function $\Box : A \to A$ is a *modal operator* on A provided $\Box$ satisfies the following axioms for each $a, b \in A$ and $r \in \mathbb{R}$:
  (M1) $\Box(a \wedge b) = \Box a \wedge \Box b$.
  (M2) $\Box r = r + (1 - r)\Box 0$.
  (M3) $\Box(a^+) = (\Box a)^+$.
  (M4) $\Box(a + r) = \Box a + \Box r - \Box 0$.
  (M5) $\Box(ra) = (\Box r)(\Box a)$ provided $r \geq 0$.

(2) If $\Box$ is a modal operator on $A \in \boldsymbol{ba\ell}$, then we call the pair $(A, \Box)$ a *modal bounded archimedean ℓ-algebra*.

(3) Let $\boldsymbol{mba\ell}$ be the category of modal bounded archimedean ℓ-algebras and unital ℓ-algebra homomorphisms preserving $\Box$.

(4) Let $\boldsymbol{muba\ell}$ be the full subcategory of $\boldsymbol{mba\ell}$ consisting of $(A, \Box)$ with $A \in \boldsymbol{uba\ell}$.

Remark 2.10. Let $(A, \Box) \in \boldsymbol{mba\ell}$. Axiom (M1) implies that $\Box$ is order preserving. From (M2) we have $\Box 1 = 1$. Finally (M3) shows that if $0 \leq a$, then $\Box a = (\Box a)^+$, so $0 \leq \Box a$. In particular, $0 \leq \Box 0$.

Definition 2.11. [5, Sec. 2]

(1) A binary relation R on a compact Hausdorff space X is *continuous* if:
  (a) $R[x]$ is closed for each $x \in X$.
  (b) $F \subseteq X$ closed implies $R^{-1}[F]$ is closed.
  (c) $U \subseteq X$ open implies $R^{-1}[U]$ is open.

(2) If X is compact Hausdorff and R is a continuous relation on X, then we call (X, R) a *compact Hausdorff frame*.

(3) A *bounded morphism* (or *p-morphism*) between (X, R) and (Y, S) is a map $f : X \to Y$ satisfying $f[R[x]] = S[f(x)]$ for each $x \in X$ (equivalently, $f^{-1}[S^{-1}[y]] = R^{-1}[f^{-1}[y]]$ for each $y \in Y$).

(4) Let KHF be the category of compact Hausdorff frames and continuous bounded morphisms.

Theorem 2.12. [6, Thm. 5.3] *There is a dual adjunction between* $\boldsymbol{mba\ell}$ *and* KHF *which restricts to a dual equivalence between* $\boldsymbol{muba\ell}$ *and* KHF.

The functors establishing the adjunction of Theorem 2.12 extend those of Gelfand duality. If $(A, \Box) \in \boldsymbol{mba\ell}$, define $R_\Box$ on Y_A by $xR_\Box y$ if $0 \le a \in y$ implies $\Box a \in x$. Then $\mathbb{Y}(A, \Box) := (Y_A, R_\Box) \in \mathsf{KHF}$. Going the other direction, if $(X, R) \in \mathsf{KHF}$, define $\Box_R$ on $C(X)$ as above. Then $\mathbb{C}(X, R) := (C(X), \Box_R) \in \boldsymbol{mba\ell}$.

3. The endofunctor $\mathbb{H} : \boldsymbol{ba\ell} \to \boldsymbol{ba\ell}$

In this section we define the endofunctor $\mathbb{H}$ on $\boldsymbol{ba\ell}$. Let $A \in \boldsymbol{ba\ell}$. Following [7, Def. 3.7], we call an ℓ-ideal I of A *archimedean* if A/I is archimedean (and hence $A/I \in \boldsymbol{ba\ell}$). Archimedean ℓ-ideals were studied by Banaschewski in the category of archimedean f-rings (see [3, App. 2] and [4]). It is easy to see that the intersection of archimedean ℓ-ideals is archimedean, and hence for each $S \subseteq A$ there is a least archimedean ℓ-ideal containing S. As is standard, we call it the *archimedean ℓ-ideal generated by S*.

As we pointed out in Section 2.2, for each $A \in \boldsymbol{ba\ell}$, the norm on A is a weight function on A. Below we will work with a different weight function on A.

Definition 3.1. Let $A \in \boldsymbol{ba\ell}$. Define w_A on A by $w_A(a) = \max\{\|a\|, 1\}$.

It is clear that (A, w_A) is a weighted set. We use w_A in order for a modal operator to be a weighted set morphism (see Lemma 4.2). The next definition is one of the main definitions of the paper and is motivated by the axioms defining a modal operator on $A \in \boldsymbol{ba\ell}$.

Definition 3.2. Let $A \in \boldsymbol{ba\ell}$.

(1) Let $F(A)$ be the free object in $\boldsymbol{ba\ell}$ on the weighted set (A, w_A), and let $f_A : A \to F(A)$ be the associated map. We let I_A be the archimedean ℓ-ideal of $F(A)$ generated by the following elements, where $a, b \in A$ and $r \in \mathbb{R}$:
   (a) $f_A(a \wedge b) - f_A(a) \wedge f_A(b)$;
   (b) $f_A(r) - r - (1 - r)f_A(0)$;
   (c) $f_A(a^+) - f_A(a)^+$;
   (d) $f_A(a + r) - f_A(a) - f_A(r) + f_A(0)$;
   (e) $f_A(ra) - f_A(r)f_A(a)$ if $0 \le r$.

(2) Let $\mathbb{H}(A) = F(A)/I_A$ and $h_A : A \to \mathbb{H}(A)$ be the composition of f_A with the quotient map $\pi : F(A) \to \mathbb{H}(A)$.

(3) For $a \in A$ let $\Box_a = h_A(a)$.

Remark 3.3. The set $\{\Box_a \mid a \in A\}$ generates $\mathbb{H}(A)$ (in $\boldsymbol{ba\ell}$), and these generators satisfy the following relations:

(F1) $\Box_{a \wedge b} = \Box_a \wedge \Box_b$.
(F2) $\Box_r = r + (1 - r)\Box_0$.
(F3) $\Box_{a^+} = (\Box_a)^+$.
(F4) $\Box_{a+r} = \Box_a + \Box_r - \Box_0$.
(F5) $\Box_{ra} = \Box_r \Box_a$ if $0 \le r$.

Theorem 3.4. $\mathbb{H}$ *is a covariant endofunctor on* $\boldsymbol{ba\ell}$.

Proof. Let $\alpha : A \to B$ be a $\boldsymbol{ba\ell}$-morphism. Then $\alpha : (A, w_A) \to (B, w_B)$ is a weighted set morphism since

$$w_B(\alpha(a)) = \max\{\|\alpha(a)\|, 1\} \leq \max\{\|a\|, 1\} = w_A(a)$$

for each $a \in A$. Therefore, there is a unique $\boldsymbol{ba\ell}$-morphism $\tau : F(A) \to F(B)$ making the following diagram commute.

$$\begin{tikzcd} A \arrow[r, "f_A"] \arrow[d, "\alpha"'] & F(A) \arrow[d, "\tau"] \\ B \arrow[r, "f_B"'] & F(B) \end{tikzcd}$$

We show that $\tau[I_A] \subseteq I_B$. From this it will follow that there is an induced $\boldsymbol{ba\ell}$-morphism $\overline{\tau} : \mathbb{H}(A) \to \mathbb{H}(B)$ such that $\overline{\tau} \circ h_A = h_B \circ \alpha$. To see that $\tau[I_A] \subseteq I_B$, it suffices to show that the five sets of generators (a)–(e) of I_A are sent to I_B by τ. Since the arguments are similar, we only give the argument for the generators of type (a).

Let $a, b \in A$. Then

$$\begin{aligned} \tau(f_A(a \wedge b) - f_A(a) \wedge f_A(b)) &= \tau f_A(a \wedge b) - \tau f_A(a) \wedge \tau f_A(b) \\ &= f_B\alpha(a \wedge b) - f_B\alpha(a) \wedge f_B\alpha(b) \\ &= f_B(\alpha(a) \wedge \alpha(b)) - f_B\alpha(a) \wedge f_B\alpha(b) \\ &\in I_B. \end{aligned}$$

Therefore, τ induces a $\boldsymbol{ba\ell}$-morphism $\overline{\tau} : \mathbb{H}(A) \to \mathbb{H}(B)$. We set $\mathbb{H}(\alpha) = \overline{\tau}$. It follows that $\mathbb{H}(\alpha)$ is the unique $\boldsymbol{ba\ell}$-morphism that makes the following diagram commute.

$$\begin{tikzcd} A \arrow[r, "h_A"] \arrow[d, "\alpha"'] & \mathbb{H}(A) \arrow[d, "\mathbb{H}(\alpha)"] \\ B \arrow[r, "h_B"'] & \mathbb{H}(B) \end{tikzcd}$$

It is clear that $\mathbb{H}$ sends identity morphisms to identity morphisms. If $\alpha : A \to B$ and $\gamma : B \to C$ are $\boldsymbol{ba\ell}$-morphisms, then

$$\mathbb{H}(\gamma \circ \alpha) \circ h_A = h_C \circ \gamma \circ \alpha = \mathbb{H}(\gamma) \circ h_B \circ \alpha = \mathbb{H}(\gamma) \circ \mathbb{H}(\alpha) \circ h_A.$$

Since $h_A[A]$ generates $\mathbb{H}(A)$, we see that $\mathbb{H}(\gamma \circ \alpha) = \mathbb{H}(\gamma) \circ \mathbb{H}(\alpha)$. Thus, $\mathbb{H}$ is a covariant functor. □

Remark 3.5. From the commutativity $\mathbb{H}(\alpha) \circ h_A = h_B \circ \alpha$ it follows that $\mathbb{H}(\alpha)(\Box_a) = \Box_{\alpha(a)}$ for each $a \in A$. This will be used subsequently.

4. $\mathsf{Alg}(\mathbb{H})$ AND $\boldsymbol{mba\ell}$

In this section we show that the category $\mathsf{Alg}(\mathbb{H})$ of algebras for the endofunctor $\mathbb{H}$ is isomorphic to $\boldsymbol{mba\ell}$. We start by recalling the definition of algebras for an endofunctor (see, e.g., [2, Def. 5.37]).

Definition 4.1. Let C be a category and $\mathbb{T} : \mathsf{C} \to \mathsf{C}$ an endofunctor on C.

(1) An *algebra* for $\mathbb{T}$ is a pair (A, f) where A is an object of C and $f : \mathbb{T}(A) \to A$ is a C-morphism.

(2) Let (A_1, f_1) and (A_2, f_2) be two algebras for $\mathbb{T}$. A *morphism* between (A_1, f_1) and (A_2, f_2) is a C-morphism $\alpha : A_1 \to A_2$ such that the following square is commutative.

$$\begin{tikzcd} \mathbb{T}(A_1) \arrow[r, "\mathbb{T}(\alpha)"] \arrow[d, "f_1"'] & \mathbb{T}(A_2) \arrow[d, "f_2"] \\ A_1 \arrow[r, "\alpha"'] & A_2 \end{tikzcd}$$

(3) Let $\mathsf{Alg}(\mathbb{T})$ be the category whose objects are algebras for $\mathbb{T}$ and whose morphisms are morphisms of algebras.

Lemma 4.2. *If $(A, \Box) \in \boldsymbol{mba\ell}$, then $\Box : (A, w_A) \to (A, \|\cdot\|)$ is a weighted set morphism.*

Proof. Let $0 \le r \in \mathbb{R}$. We first show that $\Box r \le \max\{r, 1\}$. If $r \le 1$, then $\Box r \le \Box 1 = 1$ by Remark 2.10. If $1 \le r$, then $\Box r = r + (1 - r)\Box 0 \le r$ since $0 \le \Box 0$, again by Remark 2.10. Therefore, $\Box r \le \max\{r, 1\}$.

We next show that $-\Box r \le \Box(-r)$. We have $\Box 0 = \Box(-r + r) = \Box(-r) + \Box r - \Box 0$, so $0 \le 2\Box 0 = \Box(-r) + \Box r$. Thus, $-\Box r \le \Box(-r)$.

To finish the proof, let $r = \|a\|$. Then $-r \le a \le r$, so $\Box(-r) \le \Box a \le \Box r$. We have $\Box r \le \max\{r, 1\}$ and $-\Box r \le \Box(-r)$. Therefore,

$$\begin{aligned} -\max\{\|a\|, 1\} = -\max\{r, 1\} \le -\Box r \le \Box(-r) \le \Box a \le \Box r \le \max\{r, 1\} \\ = \max\{\|a\|, 1\}, \end{aligned}$$

which implies that $\|\Box a\| \le \max\{\|a\|, 1\} = w_A(a)$. Thus, $\Box : (A, w_A) \to (A, \|\cdot\|)$ is a weighted set morphism. $\Box$

Lemma 4.3. *There is a covariant functor $\mathbb{M} : \mathsf{Alg}(\mathbb{H}) \to \boldsymbol{mba\ell}$ sending (A, σ) to $(A, \Box_\sigma)$, where $\Box_\sigma a = \sigma(\Box_a)$ for each $a \in A$, and an $\mathsf{Alg}(\mathbb{H})$-morphism α to itself.*

Proof. Let $(A, \sigma) \in \mathsf{Alg}(\mathbb{H})$ and define $\Box_\sigma$ on A by $\Box_\sigma a = \sigma(\Box_a)$. It follows from Definition 2.9 and Remark 3.3 that $(A, \Box_\sigma) \in \boldsymbol{mba\ell}$. If $\alpha : (A, \sigma) \to (A', \sigma')$ is an $\mathsf{Alg}(\mathbb{H})$-morphism,

$$\begin{tikzcd} \mathbb{H}(A) \arrow[r, "\sigma"] \arrow[d, "\mathbb{H}(\alpha)"'] & A \arrow[d, "\alpha"] \\ \mathbb{H}(A') \arrow[r, "\sigma'"'] & A' \end{tikzcd}$$

then

$$\alpha(\Box_\sigma a) = \alpha\sigma(\Box_a) = \sigma'\mathbb{H}(\alpha)(\Box_a) = \sigma'(\Box_{\alpha(a)}) = \Box_{\sigma'}\alpha(a),$$

where the second-to-last equality follows from Remark 3.5. Therefore, α is an $\boldsymbol{mba\ell}$-morphism. It is clear that $\mathbb{M}$ preserves identity morphisms and compositions. Thus, $\mathbb{M}$ is a covariant functor. $\Box$

Lemma 4.4. *There is a covariant functor* $\mathbb{N} : \boldsymbol{mba\ell} \to \mathsf{Alg}(\mathbb{H})$ *sending* $(A, \Box)$ *to* $(A, \sigma_\Box)$*, where* $\sigma_\Box(\Box_a) = \Box a$ *for each* $a \in A$*, and an* $\boldsymbol{mba\ell}$*-morphism* α *to itself.*

Proof. Since $\Box$ is a weighted set morphism by Lemma 4.2, there is a $\boldsymbol{ba\ell}$-morphism $\tau : F(A) \to A$ satisfying $\tau f_A(a) = \Box a$ by Theorem 2.8. It is clear from Definitions 2.9(1) and 3.2(1) that $I_A \subseteq \ker(\tau)$, so there is a $\boldsymbol{ba\ell}$-morphism $\sigma_\Box : \mathbb{H}(A) \to A$ satisfying $\sigma_\Box(\Box_a) = \Box a$. We set $\mathbb{N}(A, \Box) = (A, \sigma_\Box) \in \mathsf{Alg}(\mathbb{H})$. If $\alpha : (A, \Box) \to (A', \Box')$ is an $\boldsymbol{mba\ell}$-morphism, we show that α is an $\mathsf{Alg}(\mathbb{H})$-morphism. For this we show that the following diagram commutes.

$$\begin{array}{ccc} \mathbb{H}(A) & \xrightarrow{\sigma_\Box} & A \\ {\scriptstyle \mathbb{H}(\alpha)}\big\downarrow & & \big\downarrow{\scriptstyle \alpha} \\ \mathbb{H}(A') & \xrightarrow[\sigma_{\Box'}]{} & A' \end{array}$$

By Remark 3.5, $\mathbb{H}(\alpha)(\Box_a) = \Box_{\alpha(a)}$. Therefore, because α preserves $\Box$, we have $\alpha\sigma_\Box(\Box_a) = \alpha(\Box a) = \Box\alpha(a)$ and $\sigma_{\Box'}\mathbb{H}(\alpha)(\Box_a) = \sigma_{\Box'}(\Box_{\alpha(a)}) = \Box\alpha(a)$. As $\{\Box_a \mid a \in A\}$ generates $\mathbb{H}(A)$, we see that $\alpha \circ \sigma_\Box = \sigma_{\Box'} \circ \mathbb{H}(\alpha)$, so α is an $\mathsf{Alg}(\mathbb{H})$-morphism. It is clear that $\mathbb{N}$ preserves identity morphisms and compositions. Thus, $\mathbb{N}$ is a covariant functor. $\Box$

Theorem 4.5. *The functors* $\mathbb{M}$ *and* $\mathbb{N}$ *yield an isomorphism of categories between* $\mathsf{Alg}(\mathbb{H})$ *and* $\boldsymbol{mba\ell}$*.*

Proof. Let $(A, \sigma) \in \mathsf{Alg}(\mathbb{H})$. Then $\mathbb{M}(A, \sigma) = (A, \Box_\sigma)$. Therefore, $\mathbb{NM}(A, \sigma) = (A, \sigma_{\Box_\sigma})$ where $\sigma_{\Box_\sigma}(\Box_a) = \Box_\sigma a = \sigma(\Box_a)$. Thus, $\sigma_{\Box_\sigma} = \sigma$, and so $\mathbb{NM} = 1_{\mathsf{Alg}(\mathbb{H})}$.

Next, let $(A, \Box) \in \boldsymbol{mba\ell}$. Then $\mathbb{N}(A, \Box) = (A, \sigma_\Box)$. Therefore, $\mathbb{MN}(A, \Box) = (A, \Box_{\sigma_\Box})$. But $\Box_{\sigma_\Box} a = \sigma_\Box(\Box_a) = \Box a$ by the definition of $\sigma_\Box$, so $\Box_{\sigma_\Box} = \Box$. Thus, $\mathbb{MN} = 1_{\boldsymbol{mba\ell}}$. Consequently, $\mathbb{M}$ and $\mathbb{N}$ yield an isomorphism between $\mathsf{Alg}(\mathbb{H})$ and $\boldsymbol{mba\ell}$. $\Box$

5. $\mathbb{H}$ and the Vietoris endofunctor

In this section we relate $\mathbb{H}$ to the Vietoris endofunctor $\mathbb{V} : \mathsf{KHaus} \to \mathsf{KHaus}$ by showing that the Yosida space $Y_{\mathbb{H}(A)}$ for $A \in \boldsymbol{ba\ell}$ is homeomorphic to $\mathbb{V}(Y_A)$.

Let $X \in \mathsf{KHaus}$. We recall that the Vietoris space $\mathbb{V}(X)$ is the set of closed subsets of X, topologized as follows. If U is an open subset of X, let

$$\Box_U = \{F \in \mathbb{V}(X) \mid F \subseteq U\},$$
$$\Diamond_U = \{F \in \mathbb{V}(X) \mid F \cap U \neq \varnothing\}.$$

The Vietoris topology on $\mathbb{V}(X)$ is the topology with the subbasis

$$\{\Box_U \cap \Diamond_V \mid U, V \text{ open in } X\}.$$

We extend $\mathbb{V}$ to a functor as follows. If $\varphi : X \to Y$ is a continuous function between compact Hausdorff spaces, define $\mathbb{V}(\varphi) : \mathbb{V}(X) \to \mathbb{V}(Y)$ by $\mathbb{V}(\varphi)(F) = \varphi[F]$, the image of F under φ. It is well known that $\mathbb{V}(\varphi)$ is a well-defined continuous map.

Remark 5.1. The Vietoris space of X is usually defined as the space of nonempty closed subsets of X (see, e.g., [12, p. 120]). However, we follow [18, p. 111] in including $\varnothing$ in $\mathbb{V}(X)$. This is necessary for our considerations since the continuous relation R on X may not be serial, and hence there may be $x \in X$ with $R[x] = \varnothing$. Therefore, $\rho_R(x) = \varnothing$, and we need $\varnothing \in \mathbb{V}(X)$ for ρ_R to be well defined. However, in Section 8 we will consider $\mathbb{V}^*(X) = \mathbb{V}(X) \setminus \{\varnothing\}$ and relate it to the full subcategory of $\boldsymbol{mba\ell}$ corresponding to those (X, R) where R is a serial relation. This subcategory will be characterized by the identity $\Box 0 = 0$.

Lemma 5.2. *Let $A \in \boldsymbol{ba\ell}$. Define $g_A : A \to C(\mathbb{V}Y_A)$ by*

$$g_A(a)(F) = \begin{cases} \inf \zeta_A(a)[F] & \text{if } F \neq \varnothing; \\ 1 & \text{if } F = \varnothing. \end{cases}$$

Then $g_A : (A, w_A) \to (C(\mathbb{V}Y_A), \|\cdot\|)$ is a well-defined weighted set morphism.

Proof. To simplify notation we write g for g_A. To see that g is well defined it is sufficient to show that $g(a)$ is continuous for each $a \in A$. Let $r, s \in \mathbb{R}$ with $r < s$. We show that

$$g(a)^{-1}(r, s) = \begin{cases} \Box_{\zeta_A(a)^{-1}(r,\infty)} \cap \Diamond_{\zeta_A(a)^{-1}(-\infty,s)} & \text{if } 1 \notin (r, s) \\ (\Box_{\zeta_A(a)^{-1}(r,\infty)} \cap \Diamond_{\zeta_A(a)^{-1}(-\infty,s)}) \cup \Box_\varnothing & \text{if } 1 \in (r, s). \end{cases}$$

Suppose that $1 \notin (r, s)$. Then $g(a)(F) \in (r, s)$ implies that $F \neq \varnothing$. Therefore, since F is compact and hence $\zeta_A(a)$ attains its infimum on F, we have

$$\begin{aligned} F \in g(a)^{-1}(r, s) \quad &\text{iff} \quad r < \inf \zeta_A(a)[F] < s \\ &\text{iff} \quad r < \min \zeta_A(a)[F] < s \\ &\text{iff} \quad F \in \Box_{\zeta_A(a)^{-1}(r,\infty)} \cap \Diamond_{\zeta_A(a)^{-1}(-\infty,s)}. \end{aligned}$$

On the other hand, if $1 \in (r, s)$, then $\varnothing \in g(a)^{-1}(r, s)$. Therefore, since $\Box_\varnothing = \{\varnothing\}$, the calculation above yields the second case. Thus, $g(a)$ is continuous.

It is left to show that g is a weighted set morphism. Let $a \in A$. Then $w_A(a) = \max\{\|a\|, 1\}$. Suppose that $\|a\| = r$. Then $-r \le a \le r$. If F is nonempty, then $-r \le \inf \zeta_A(a)[F] \le r$, so $|\inf \zeta_A(a)[F]| \le r$. Also, $g(a)(\varnothing) = 1$. Therefore,

$$\begin{aligned} \|g(a)\| &= \sup\{|g(a)(F)| \mid F \in \mathbb{V}(Y_A)\} = \sup\{\{|\inf \zeta_A(a)[F]| \mid F \neq \varnothing\} \cup \{1\}\} \\ &= \max\{\sup\{|\inf \zeta_A(a)[F]| \mid F \neq \varnothing\}, 1\} \le \max\{r, 1\} = w_A(a). \end{aligned}$$

Thus, $g : (A, w_A) \to (C(\mathbb{V}Y_A), \|\cdot\|)$ is a weighted set morphism. $\Box$

Remark 5.3. In the proof of Lemma 5.4 we identify X with $Y_{C(X)}$ via the homeomorphism $\varepsilon_X : X \to Y_{C(X)}$ given in Section 2.1. We also identify Y_A with $\hom_{\boldsymbol{ba\ell}}(A, \mathbb{R})$ as follows. If $M \in Y_A$, then it is well known that $A/M \cong \mathbb{R}$ (see, e.g., [16, Cor. 2.7]), so there is a $\boldsymbol{ba\ell}$-morphism $A \to \mathbb{R}$ sending $a \in A$ to $r \in \mathbb{R}$ iff $a - r \in M$. Conversely, $\rho \in \hom_{\boldsymbol{ba\ell}}(A, \mathbb{R})$ goes to $\ker(\rho) \in Y_A$.

Lemma 5.4. *There is a (unique) $\boldsymbol{ba\ell}$-morphism $\tau_A : F(A) \to C(\mathbb{V}Y_A)$ satisfying $\tau_A \circ f_A = g_A$, the image of τ_A is uniformly dense in $C(\mathbb{V}Y_A)$, and $\ker(\tau_A)$ contains I_A. Therefore, there is a (unique) $\boldsymbol{ba\ell}$-morphism $\eta_A : \mathbb{H}(A) \to C(\mathbb{V}Y_A)$ satisfying $\eta_A \circ h_A = g_A$ and whose image* *is uniformly dense in* $C(\mathbb{V}Y_A)$*.*

$$\begin{tikzcd} & F(A) \arrow[d, "\pi"'] \arrow[dd, bend left, "\tau_A"] \\ A \arrow[ur, "f_A"] \arrow[r, "h_A"] \arrow[dr, "g_A"'] & \mathbb{H}(A) \arrow[d, "\eta_A"'] \\ & C(\mathbb{V}Y_A) \end{tikzcd}$$

Proof. The existence and uniqueness of τ_A follows from Lemma 5.2 and Theorem 2.8. To show that the image of τ_A is uniformly dense, by Lemma 2.6(2) it suffices to show that $\mathbb{Y}(\tau_A) : Y_{C(\mathbb{V}Y_A)} \to Y_{F(A)}$ is 1-1. We may identify $Y_{F(A)}$ with $\hom_{\boldsymbol{ba\ell}}(F(A), \mathbb{R})$ and $Y_{C(\mathbb{V}Y_A)}$ with $\mathbb{V}(Y_A)$ by Remark 5.3. Under these identifications, if $F \in \mathbb{V}Y_A$ we let $\rho_F \in \hom_{\boldsymbol{ba\ell}}(F(A), \mathbb{R})$ be the corresponding homomorphism. For $a \in A$ and $r \in \mathbb{R}$ we have

$$\begin{aligned} \rho_F(f_A(a)) = r \quad &\text{iff} \quad f_A(a) - r \in \mathbb{Y}(\tau_A)(\varepsilon_{\mathbb{V}Y_A}(F)) \\ &\text{iff} \quad f_A(a) - r \in \tau_A^{-1}(\varepsilon_{\mathbb{V}Y_A}(F)) \\ &\text{iff} \quad \tau_A f_A(a) - r \in \varepsilon_{\mathbb{V}Y_A}(F) \\ &\text{iff} \quad \tau_A f_A(a)(F) = r \\ &\text{iff} \quad g_A(a)(F) = r. \end{aligned}$$

Therefore, ρ_F satisfies $\rho_F(f_A(a)) = \inf \zeta_A(a)[F]$ if $F \neq \varnothing$, and $\rho_\varnothing$ is the function sending each $f_A(a)$ to 1. To see that $\mathbb{Y}(\tau_A)$ is 1-1, suppose that $C \neq D$. If one of C, D is empty, say $C = \varnothing$, then $\rho_C f_A(0) = 1$ and $\rho_D f_A(0) = \inf \zeta_A(0)[D] = 0$ since D is nonempty. Therefore, $\rho_C \neq \rho_D$. If $C, D \neq \varnothing$, without loss of generality we may assume that $C \not\subseteq D$. Then there is $y \in Y_A$ with $y \in C$ and $y \notin D$. Since Y_A is compact Hausdorff, there is $b \in C(Y_A)$ with $0 \le b \le 1$, $b[D] = \{1\}$ and $b(y) = 0$. Because $\zeta_A[A]$ is uniformly dense in $C(Y_A)$, there is $a \in A$ with $\|b - \zeta_A(a)\| < 1/3$. Therefore, $\inf \zeta_A(a)[D] \ge 2/3$ and $\inf \zeta_A(a)[C] \le 1/3$. This shows that $\rho_C f_A(a) \neq \rho_D f_A(a)$, so $\rho_C \neq \rho_D$. Thus, $\mathbb{Y}(\tau_A)$ is 1-1, and hence the image of $\tau_A : F(A) \to C(\mathbb{V}Y_A)$ is uniformly dense.

To show that $I_A \subseteq \ker(\tau_A)$, it is sufficient to show that $\ker(\tau_A)$ contains all five classes of generators of I_A. Because the proof is similar to that of [6, Lem. 3.8], we only demonstrate (a).

Let $a, b \in A$. We have

$$\begin{aligned} \tau_A(f_A(a \wedge b) - f_A(a) \wedge f_A(b)) &= \tau_A f_A(a \wedge b) - \tau_A f_A(a) \wedge \tau_A f_A(b) \\ &= g_A(a \wedge b) - g_A(a) \wedge g_A(b). \end{aligned}$$

Therefore, we need to prove that $g_A(a \wedge b) = g_A(a) \wedge g_A(b)$. Both sides send $\varnothing$ to 1. Suppose that $F \in \mathbb{V}(Y_A)$ is nonempty. Then

$$\begin{aligned} g_A(a \wedge b)(F) &= \inf(\zeta_A(a) \wedge \zeta_A(b))[F] = \min(\zeta_A(a) \wedge \zeta_A(b))[F] \\ &= \min\{(\zeta_A(a) \wedge \zeta_A(b))(x) \mid x \in F\} \\ &= \min\{\min\{\zeta_A(a)(x), \zeta_A(b)(x)\} \mid x \in F\} \\ &= \min\{\min \zeta_A(a)[F], \min \zeta_A(b)[F]\} \\ &= (g_A(a) \wedge g_A(b))(F). \end{aligned}$$

Thus, $g_A(a \wedge b) = g_A(a) \wedge g_A(b)$. $\square$

We next show that η_A is 1-1. For this we require a technical result, which is an analogue of [6, Prop. 4.8].

Definition 5.5. Let $A \in \boldsymbol{ba\ell}$.

(1) If $x \in Y_{\mathbb{H}(A)}$, set $\Box^{-1}x = \{a \in A \mid \Box_a \in x\}$.
(2) If $S \subseteq A$, set $S^+ = \{s \in S \mid 0 \le s\}$.
(3) Define a binary relation $R^\Box \subseteq Y_{\mathbb{H}(A)} \times Y_A$ by setting $xR^\Box y$ if $y^+ \subseteq \Box^{-1}x$ for each $x \in Y_{\mathbb{H}(A)}$ and $y \in Y_A$.

Proposition 5.6. *Let $A \in \boldsymbol{ba\ell}$ and $x \in Y_{\mathbb{H}(A)}$. Then $(\Box^{-1}x)^+ = \bigcup\{y^+ \mid y \in Y_A, xR^\Box y\}$.*

Proof. The proof is the same as that of [6, Prop. 4.8] after replacing $\Box a$ with $\Box_a$ and $R_\Box$ with $R^\Box$. $\square$

Lemma 5.7. *Let $\rho : \mathbb{H}(A) \to \mathbb{R}$ be a $\boldsymbol{ba\ell}$-morphism.*

(1) $\rho(\Box_0) \in \{0, 1\}$.
(2) *If $\rho(\Box_0) = 1$, then $\rho(\Box_a) = 1$ for each $a \in A$.*

Proof. (1) If we set $r = 0 = a$ in (F5), we get $\Box_0\Box_0 = \Box_0$, so $\Box_0$ is an idempotent. Therefore, $\rho(\Box_0) \in \mathbb{R}$ is an idempotent, and hence $\rho(\Box_0) \in \{0, 1\}$.

(2) Suppose that $\rho(\Box_0) = 1$. By (F5), $\Box_0\Box_a = \Box_0$ for each $a \in A$. So applying ρ to both sides yields $\rho(\Box_a) = 1$. $\square$

Theorem 5.8. *For $A \in \boldsymbol{ba\ell}$, the Yosida space of $\mathbb{H}(A)$ is homeomorphic to $\mathbb{V}(Y_A)$.*

Proof. The map $\eta_A : \mathbb{H}(A) \to C(\mathbb{V}Y_A)$ induces a continuous map $\mathbb{Y}(\eta_A) : Y_{C(\mathbb{V}Y_A)} \to Y_{\mathbb{H}(A)}$. We identify $Y_{C(\mathbb{V}Y_A)}$ with $\mathbb{V}(Y_A)$ and $Y_{\mathbb{H}(A)}$ with $\hom_{\boldsymbol{ba\ell}}(\mathbb{H}(A), \mathbb{R})$ as in Remark 5.3. As we saw in the proof of Lemma 5.4, under these identifications $\mathbb{Y}(\eta_A)(F) := \rho_F$ satisfies $\rho_F(\Box_a) = \inf \zeta_A(a)[F]$ if F is nonempty, and $\rho_F(\Box_a) = 1$ if $F = \varnothing$. By Lemma 5.4, the image of η_A is uniformly dense in $C(\mathbb{V}Y_A)$. Therefore, $\mathbb{Y}(\eta_A)$ is 1-1 by Lemma 2.6(2).

To show that $\mathbb{Y}(\eta_A)$ is onto, let $\rho : \mathbb{H}(A) \to \mathbb{R}$ be a $\boldsymbol{ba\ell}$-morphism. If $\rho(\Box_0) = 1$, then $\rho(\Box_a) = 1$ for all $a \in A$ by Lemma 5.7(2). Therefore, ρ and $\rho_\varnothing$ agree on each $\Box_a$. Since these generate $\mathbb{H}(A)$, we see that $\rho = \rho_\varnothing$. By Lemma 5.7(1), we now may assume that $\rho(\Box_0) = 0$. By (F2), $\rho(\Box_r) = r$ for each $r \in \mathbb{R}$. Let

$$S = \{(a - \rho(\Box_a))^- \mid a \in A\}$$

and $F = \{M \in Y_A \mid S \subseteq M\}$, a closed subset of Y_A. We claim that $\rho = \rho_F$. Let $a \in A$ and $y \in F$. Then $(a - \rho(\Box_a))^- \in y$. This means $0 \le (\zeta_A(a) - \rho(\Box_a))(y)$ by [9, Rem. 2.11], so $\rho(\Box_a) \le \zeta_A(a)(y)$. Since this is true for all $y \in F$, we see that $\rho(\Box_a) \le \inf \zeta_A(a)[F]$. Thus, it suffices to prove that for each $a \in A$ there is $y \in F$ with $\zeta_A(a)(y) = \rho(\Box_a)$. In other words, we need to show that there is $y \in F$ with $a - \rho(\Box_a) \in y$.

Let $x = \ker(\rho) \in Y_{\mathbb{H}(A)}$. If $a \in A$, then

$$\rho(\Box_{a-\rho(\Box_a)}) = \rho(\Box_a + \Box_{-\rho(\Box_a)} - \Box_0) = \rho(\Box_a) - \rho(\Box_a) = 0$$

by (F4) and the fact that $\rho(\Box_r) = r$. From this and (F3) we see that

$$\rho(\Box_{(a-\rho(\Box_a))^+}) = \rho(\Box^+_{a-\rho(\Box_a)}) = \rho(\Box_{a-\rho(\Box_a)})^+ = \max\{\rho(\Box_{a-\rho(\Box_a)}), 0\} = \max\{0, 0\} = 0,$$

which implies that $(a-\rho(\Box_a))^+ \in \Box^{-1}x$. By Proposition 5.6, there is $y \in Y_A$ with $xR^\Box y$ and $(a - \rho(\Box_a))^+ \in y$. We show that these two facts imply that $y \in F$ and $\rho(\Box_a) = \zeta_A(a)(y)$. Let $b \in A$. Since $A/y \cong \mathbb{R}$, there is $r \in \mathbb{R}$ with $b - r \in y$. Therefore, $(b - r)^+ \in y$, so $\Box_{(b-r)^+} \in x$. Because $x = \ker(\rho)$,

$$0 = \rho(\Box_{(b-r)^+}) = \rho(\Box^+_{b-r}) = \rho(\Box_{b-r})^+ = \max\{\rho(\Box_{b-r}), 0\} = \max\{\rho(\Box_b) - r, 0\},$$

so $\rho(\Box_b) \le r$. Consequently, $b + y = r + y \ge \rho(\Box_b) + y$, and hence $b - \rho(\Box_b) + y \ge 0 + y$. This implies that $(b - \rho(\Box_b))^- \in y$. Since this is true for all $b \in A$, we get $S \subseteq y$, so $y \in F$. Moreover, for $b = a$ we have $(a-\rho(\Box_a))^+, (a-\rho(\Box_a))^- \in y$, so $a-\rho(\Box_a) \in y$. By the above, this shows that $\rho = \rho_F$, so $\mathbb{Y}(\eta_A)$ is onto. Thus, $\mathbb{Y}(\eta_A)$ is a homeomorphism. $\square$

Remark 5.9. By Theorem 5.8, $Y_{\mathbb{H}(A)}$ is homeomorphic to $\mathbb{V}(Y_A)$. Under this homeomorphism, $R^\Box \subseteq Y_{\mathbb{H}(A)} \times Y_A$ is identified with the relation $R \subseteq \mathbb{V}(Y_A) \times Y_A$ given by FRy iff $y \in F$. From this it follows that $R[F] = F$, and for $U \subseteq Y_A$ open, we have $R^{-1}[U] = \Diamond_U$ and $R^{-1}[Y_A \setminus U] = \mathbb{V}(Y_A) \setminus \Box_U$. Consequently, R is a continuous relation, and hence so is $R^\Box$.

6. $\mathsf{Alg}(\mathbb{H})$ and $\mathsf{Coalg}(\mathbb{V})$

In this section we use Theorem 5.8 and standard algebra/coalgebra tools to lift the dual adjunction between $\boldsymbol{ba\ell}$ and KHaus to a dual adjunction between $\mathsf{Alg}(\mathbb{H})$ and $\mathsf{Coalg}(\mathbb{V})$. We show that this dual adjunction restricts to a dual equivalence between the reflective subcategory $\mathsf{Alg}^u(\mathbb{H})$ of $\mathsf{Alg}(\mathbb{H})$ and $\mathsf{Coalg}(\mathbb{V})$. The category $\mathsf{Alg}^u(\mathbb{H})$ consists of those $(A, \alpha) \in \mathsf{Alg}(\mathbb{H})$ where $A \in \boldsymbol{uba\ell}$. This dual equivalence lifts Gelfand duality. We conclude the section by giving an alternate description of $\mathsf{Alg}^u(\mathbb{H})$ as $\mathsf{Alg}(\mathbb{H}^u)$ where $\mathbb{H}^u$ is the endofunctor $\mathbb{C}\mathbb{Y}\mathbb{H} : \boldsymbol{uba\ell} \to \boldsymbol{uba\ell}$.

$$\boldsymbol{uba\ell} \xrightarrow{\mathbb{H}} \boldsymbol{ba\ell} \xrightarrow{\mathbb{Y}} \mathsf{KHaus} \xrightarrow{\mathbb{C}} \boldsymbol{uba\ell}$$

We start by recalling the definition of coalgebras (see, e.g., [23, Def. 9.1]), which is dual to the definition of algebras for an endofunctor.

Definition 6.1.

(1) A *coalgebra* for an endofunctor $\mathbb{T} : \mathsf{C} \to \mathsf{C}$ is a pair (B, g) where B is an object of C and $g : B \to \mathbb{T}(B)$ is a C-morphism.

(2) A *morphism* between two coalgebras (B_1, g_1) and (B_2, g_2) for $\mathbb{T}$ is a C-morphism $\alpha : B_1 \to B_2$ such that the following square is commutative.

$$\begin{tikzcd} B_1 \arrow[r, "\alpha"] \arrow[d, "g_1"'] & B_2 \arrow[d, "g_2"] \\ \mathbb{T}(B_1) \arrow[r, "\mathbb{T}(\alpha)"'] & \mathbb{T}(B_2) \end{tikzcd}$$

(3) Let $\mathsf{Coalg}(\mathbb{T})$ be the category whose objects are coalgebras for $\mathbb{T}$ and whose morphisms are morphisms of coalgebras.

The next result follows from [17, Thm. 2.5.9] where it is shown that, under certain conditions, an adjunction lifts to an adjunction between categories of algebras. For our purposes we reformulate it in the language of dual adjunctions.

Lemma 6.2. *Let $\mathbb{J} : \mathsf{C} \to \mathsf{C}$, $\mathbb{K} : \mathsf{D} \to \mathsf{D}$ be two endofunctors and $\mathbb{P} : \mathsf{C} \to \mathsf{D}$ a contravariant functor.*

$$\begin{tikzcd} \mathsf{C} \arrow[r, "\mathbb{P}"] \arrow[d, "\mathbb{J}"'] & \mathsf{D} \arrow[d, "\mathbb{K}"] \\ \mathsf{C} \arrow[r, "\mathbb{P}"'] & \mathsf{D} \end{tikzcd}$$

A natural transformation $\alpha : \mathbb{P} \circ \mathbb{J} \to \mathbb{K} \circ \mathbb{P}$ induces a lifting of $\mathbb{P}$ to a contravariant functor $\widehat{\mathbb{P}} : \mathsf{Alg}(\mathbb{J}) \to \mathsf{Coalg}(\mathbb{K})$.

If α is an isomorphism and $\mathbb{Q} : \mathsf{D} \to \mathsf{C}$ is a contravariant functor such that $\mathbb{P}$ and $\mathbb{Q}$ form a dual adjunction, then $\mathbb{Q}$ lifts to a contravariant functor $\widehat{\mathbb{Q}} : \mathsf{Coalg}(\mathbb{K}) \to \mathsf{Alg}(\mathbb{J})$ such that the dual adjunction between $\mathbb{P}$ and $\mathbb{Q}$ induces a dual adjunction between $\widehat{\mathbb{P}}$ and $\widehat{\mathbb{Q}}$.

We will apply Lemma 6.2 to the following diagram.

$$\begin{tikzcd} \boldsymbol{ba\ell} \arrow[r, "\mathbb{Y}"] \arrow[d, "\mathbb{H}"'] & \mathsf{KHaus} \arrow[d, "\mathbb{V}"] \\ \boldsymbol{ba\ell} \arrow[r, "\mathbb{Y}"'] & \mathsf{KHaus} \end{tikzcd}$$

In order to do so, we need a natural isomorphism $\alpha : \mathbb{Y}\mathbb{H} \to \mathbb{V}\mathbb{Y}$, which is provided by Theorem 5.8.

Lemma 6.3. *Let $A \in \boldsymbol{ba\ell}$. If $\alpha_A = (\mathbb{Y}(\eta_A) \circ \varepsilon_{\mathbb{V}Y_A})^{-1} : Y_{\mathbb{H}(A)} \to \mathbb{V}Y_A$, then $\alpha : \mathbb{Y}\mathbb{H} \to \mathbb{V}\mathbb{Y}$ is a natural isomorphism.*

Proof. The proof of Theorem 5.8 shows that α_A is a homeomorphism for each A. For naturality, let $\gamma : A \to A'$ be a morphism in $\boldsymbol{ba\ell}$. We need to prove that the outside square of the following diagram is commutative.

$$\begin{tikzcd}
Y_{\mathbb{H}(A)} \arrow[r, "\mathbb{Y}(\eta_A)^{-1}"'] \arrow[rr, bend left, "\alpha_A"] & Y_{C(\mathbb{V}Y_A)} \arrow[r, "\varepsilon^{-1}_{\mathbb{V}Y_A}"'] & \mathbb{V}Y_A \\
Y_{\mathbb{H}(A')} \arrow[u, "\mathbb{V}\mathbb{Y}(\gamma)"] \arrow[r, "\mathbb{Y}(\eta_{A'})^{-1}"] \arrow[rr, bend right, "\alpha_{A'}"'] & Y_{C(\mathbb{V}Y_{A'})} \arrow[u, "\mathbb{Y}\mathbb{C}\mathbb{V}(\gamma)"] \arrow[r, "\varepsilon^{-1}_{\mathbb{V}Y_{A'}}"] & \mathbb{V}Y_{A'} \arrow[u, "\mathbb{Y}\mathbb{H}(\gamma)"']
\end{tikzcd}$$

The right square is commutative since ε is a natural isomorphism. It then suffices to show that the left square is commutative. For this, since $\mathbb{Y}$ is a functor, it suffices to show that η is natural; that is, we need to show that the following diagram is commutative.

$$\begin{tikzcd}
\mathbb{H}(A) \arrow[r, "\eta_A"'] \arrow[d, "\mathbb{H}(\gamma)"'] & C(\mathbb{V}Y_A) \arrow[d, "\mathbb{C}\mathbb{V}\mathbb{Y}(\gamma)"] \\
\mathbb{H}(A') \arrow[r, "\eta_{A'}"] & C(\mathbb{V}Y_{A'})
\end{tikzcd}$$

Because $\{\Box_a \mid a \in A\}$ generates $\mathbb{H}(A)$, to prove commutativity it is enough to show, for each $a \in A$, that $\eta_{A'}(\mathbb{H}(\gamma)(\Box_a)) = \mathbb{C}\mathbb{V}\mathbb{Y}(\gamma)(\eta_A(a))$. By Remark 3.5 and Lemma 5.4 we have

$$\eta_{A'}(\mathbb{H}(\gamma)(\Box_a)) = \eta_{A'}(\Box_{\gamma(a)}) = g_{A'}(\gamma(a)),$$

and

$$\mathbb{C}\mathbb{V}\mathbb{Y}(\gamma)(\eta_A(a)) = \mathbb{C}\mathbb{V}\mathbb{Y}(\gamma)(g_A(a)) = g_A(a) \circ \mathbb{V}\mathbb{Y}(\gamma).$$

Let $F \in \mathbb{V}Y_{A'}$. If $F = \varnothing$, then $g_{A'}(\gamma(a))(F) = 1$ and

$$(g_A(a) \circ \mathbb{V}\mathbb{Y}(\gamma))(F) = g_A(a)(\mathbb{Y}(\gamma)(\varnothing)) = g_A(a)(\varnothing) = 1.$$

On the other hand, if $F \neq \varnothing$, then $g_{A'}(\gamma(a))(F) = \inf \zeta_{A'}(\gamma(a))[F]$ and

$$(g_A(a) \circ \mathbb{V}\mathbb{Y}(\gamma))(F) = g_A(a)(\mathbb{Y}(\gamma)(F)) = \inf \zeta_A(a)[\mathbb{Y}(\gamma)(F)].$$

To see these are equal notice that

$$\zeta_{A'}(\gamma(a)) = \mathbb{C}\mathbb{Y}(\gamma)(\zeta_A(a)) = \zeta_A(a) \circ \mathbb{Y}(\gamma),$$

where the first equality follows from the naturality of ζ. This completes the proof that $\eta_{A'}(\mathbb{H}(\gamma)(\Box_a)) = \mathbb{C}\mathbb{V}\mathbb{Y}(\gamma)(\eta_A(a))$, and so η is natural. Consequently, α is natural, and since each α_A is a homeomorphism, α is a natural isomorphism. $\square$

Lemma 6.3 allows us to utilize Lemma 6.2 to lift the dual adjunction between $\boldsymbol{ba\ell}$ and KHaus to a dual adjunction between $\mathsf{Alg}(\mathbb{H})$ and $\mathsf{Coalg}(\mathbb{V})$.

Theorem 6.4. *The dual adjunction between $\mathbb{Y} : \boldsymbol{ba\ell} \to \mathsf{KHaus}$ and $\mathbb{C} : \mathsf{KHaus} \to \boldsymbol{ba\ell}$ lifts to a dual adjunction between $\widehat{\mathbb{Y}} : \mathsf{Alg}(\mathbb{H}) \to \mathsf{Coalg}(\mathbb{V})$ and $\widehat{\mathbb{C}} : \mathsf{Coalg}(\mathbb{V}) \to \mathsf{Alg}(\mathbb{H})$.*

Remark 6.5. By adapting the proof of [17, Thm. 2.5.9], the functors $\widehat{\mathbb{C}}$ and $\widehat{\mathbb{Y}}$ and the units of the dual adjunction of Theorem 6.4 can be defined as follows:

For $(A, \alpha) \in \mathsf{Alg}(\mathbb{H})$, we have $\widehat{\mathbb{Y}}(A, \alpha) = (Y_A, \widehat{\mathbb{Y}}_\alpha) \in \mathsf{Coalg}(\mathbb{V})$, where

$$\widehat{\mathbb{Y}}_\alpha = \varepsilon^{-1}_{\mathbb{V}(Y_A)} \circ \mathbb{Y}(\eta_A)^{-1} \circ \mathbb{Y}(\alpha) : Y_A \to \mathbb{V}(Y_A).$$

$$\begin{tikzcd} Y_A \arrow[r, "\mathbb{Y}(\alpha)"] \arrow[rrr, bend right, "\widehat{\mathbb{Y}}_\alpha"'] & Y_{\mathbb{H}(A)} \arrow[r, "\mathbb{Y}(\eta_A)^{-1}"] & Y_{C(\mathbb{V}Y_A)} \arrow[r, "\varepsilon^{-1}_{\mathbb{V}(Y_A)}"] & \mathbb{V}(Y_A) \end{tikzcd}$$

If $\gamma : (A, \alpha) \to (A', \alpha')$ is an $\mathsf{Alg}(\mathbb{H})$-morphism, then $\widehat{\mathbb{Y}}(\gamma) = \mathbb{Y}(\gamma)$.

For $(X, \sigma) \in \mathsf{Coalg}(\mathbb{V})$, we have $\widehat{\mathbb{C}}(X, \sigma) = (C(X), \widehat{\mathbb{C}}_\sigma)$, where

$$\widehat{\mathbb{C}}_\sigma = \mathbb{C}(\sigma) \circ \mathbb{C}\mathbb{V}(\varepsilon_X) \circ \eta_{C(X)}.$$

$$\begin{tikzcd} \mathbb{H}C(X) \arrow[r, "\eta_{C(X)}"] \arrow[rrr, bend right, "\widehat{\mathbb{C}}_\sigma"'] & C(\mathbb{V}Y_{C(X)}) \arrow[r, "\mathbb{C}\mathbb{V}(\varepsilon_X)"] & C(\mathbb{V}X) \arrow[r, "\mathbb{C}(\sigma)"] & C(X) \end{tikzcd}$$

If $\varphi : (X, \sigma) \to (X', \sigma')$ is a $\mathsf{Coalg}(\mathbb{V})$-morphism, then $\widehat{\mathbb{C}}(\varphi) = \mathbb{C}(\varphi)$.

The units $\widehat{\zeta} : 1_{\mathsf{Alg}(\mathbb{H})} \to \widehat{\mathbb{C}}\widehat{\mathbb{Y}}$ and $\widehat{\varepsilon} : 1_{\mathsf{Coalg}(\mathbb{V})} \to \widehat{\mathbb{Y}}\widehat{\mathbb{C}}$ are given by

$$\widehat{\zeta}_{(A,\alpha)} = \zeta_A \qquad \text{and} \qquad \widehat{\varepsilon}_{(X,\sigma)} = \varepsilon_X.$$

Moreover, $\widehat{\varepsilon}$ is a natural isomorphism.

We next identify a subcategory of $\mathsf{Alg}(\mathbb{H})$ that is dually equivalent to $\boldsymbol{mubal}$.

Definition 6.6. Let $\mathsf{Alg}^u(\mathbb{H})$ be the full subcategory of $\mathsf{Alg}(\mathbb{H})$ consisting of those (A, α) with $A \in \boldsymbol{ubal}$.

Corollary 6.7.
(1) *The functors* $\widehat{\mathbb{Y}}$ *and* $\widehat{\mathbb{C}}$ *restrict to a dual equivalence between* $\mathsf{Alg}^u(\mathbb{H})$ *and* $\mathsf{Coalg}(\mathbb{V})$.
(2) $\mathsf{Alg}^u(\mathbb{H})$ *is a reflective subcategory of* $\mathsf{Alg}(\mathbb{H})$.

Proof. (1) Let $(A, \alpha) \in \mathsf{Alg}(\mathbb{H})$. Then $\widehat{\zeta}_{(A,\alpha)} = \zeta_A$ is an isomorphism iff $A \in \boldsymbol{ubal}$ iff $(A, \alpha) \in \mathsf{Alg}^u(\mathbb{H})$. Consequently, $\widehat{\zeta} : 1_{\mathsf{Alg}^u(\mathbb{H})} \to \widehat{\mathbb{C}}\widehat{\mathbb{Y}}$ is a natural isomorphism by Remark 6.5. The same remark also yields that $\widehat{\varepsilon}$ is a natural isomorphism. Therefore, $\widehat{\mathbb{Y}}$ and $\widehat{\mathbb{C}}$ restrict to a dual equivalence between $\mathsf{Alg}^u(\mathbb{H})$ and $\mathsf{Coalg}(\mathbb{V})$ by [21, Thm. IV.4.1].

(2) By (1), the functors $\widehat{\mathbb{Y}}$ and $\widehat{\mathbb{C}}$ form a dual equivalence between $\mathsf{Alg}^u(\mathbb{H})$ and $\mathsf{Coalg}(\mathbb{V})$. If $(A, \alpha) \in \mathsf{Alg}(\mathbb{H})$, then the morphism $\widehat{\zeta}_{(A,\alpha)}$ is a universal arrow from (A, α) to $\widehat{\mathbb{Y}}$ by [21, Thm. IV.1.1]. Therefore, $\mathsf{Alg}^u(\mathbb{H})$ is a reflective subcatgory of $\mathsf{Alg}(\mathbb{H})$ (see [21, p. 89]). □

Proposition 6.8. *The functors* $\mathbb{M}, \mathbb{N}$ *yield an isomorphism between* $\mathsf{Alg}^u(\mathbb{H})$ *and* $\boldsymbol{mubal}$.

Proof. If $(A, \sigma) \in \mathsf{Alg}^u(\mathbb{H})$, then $A \in \boldsymbol{ubal}$, so $\mathbb{M}(A, \sigma) = (A, \Box_\sigma) \in \boldsymbol{mubal}$. If $(A, \Box) \in \boldsymbol{mubal}$, then $A \in \boldsymbol{ubal}$, so $\mathbb{N}(A, \Box) = (A, \sigma_\Box) \in \mathsf{Alg}^u(\mathbb{H})$. Therefore, the proof of Theorem 4.5 shows that $\mathbb{M}$ and $\mathbb{N}$ restrict to $\mathsf{Alg}^u(\mathbb{H})$ and $\boldsymbol{mubal}$, respectively, to yield an isomorphism. □

We finish this section by giving an alternate view of the category $\mathsf{Alg}^u(\mathbb{H})$.

Definition 6.9. We let $\mathbb{H}^u$ be the endofunctor $\mathbb{C}\mathbb{Y}\mathbb{H}$ on $\boldsymbol{ubal}$. Therefore, if $A \in \boldsymbol{ubal}$, then $\mathbb{H}^u(A) = C(Y_{\mathbb{H}(A)})$ and if $\alpha : A \to A'$ is a $\boldsymbol{ubal}$-morphism, then $\mathbb{H}^u(\alpha) = \mathbb{C}\mathbb{Y}\mathbb{H}(\alpha)$.

Recall from Section 2.1 that if $\gamma : A \to B$ is a $\boldsymbol{ba\ell}$-morphism with $B \in \boldsymbol{uba\ell}$, then there is a unique $\boldsymbol{ba\ell}$-morphism $\gamma^u : C(Y_A) \to B$ with $\gamma^u \circ \zeta_A = \gamma$, where $\gamma^u = \zeta_B^{-1} \circ \mathbb{C}\mathbb{Y}(\gamma)$.

$$\begin{tikzcd} A \arrow[r, "\zeta_A"] \arrow[d, "\gamma"'] & C(Y_A) \arrow[d, "\mathbb{C}\mathbb{Y}(\gamma)"] \arrow[dl, "\gamma^u"] \\ B & C(Y_B) \arrow[l, "\zeta_B^{-1}"] \end{tikzcd}$$

Proposition 6.10. *There is an isomorphism of categories between* $\mathsf{Alg}^u(\mathbb{H})$ *and* $\mathsf{Alg}(\mathbb{H}^u)$.

Proof. We define $\mathbb{A} : \mathsf{Alg}^u(\mathbb{H}) \to \mathsf{Alg}(\mathbb{H}^u)$ on objects by sending (A, α) to (A, α^u). On morphisms, if γ is an $\mathsf{Alg}(\mathbb{H})$-morphism, then $\mathbb{A}(\gamma) = \gamma$.

$$\begin{tikzcd} \mathbb{H}(A) \arrow[r, "\zeta_{\mathbb{H}(A)}"'] \arrow[d, "\mathbb{H}(\gamma)"'] \arrow[rr, bend left, "\alpha"] & \mathbb{H}^u(A) \arrow[r, "\alpha^u"'] \arrow[d, "\mathbb{H}^u(\gamma)"] & A \arrow[d, "\gamma"] \\ \mathbb{H}(A') \arrow[r, "\zeta_{\mathbb{H}(A')}"] \arrow[rr, bend right, "\alpha'"'] & \mathbb{H}^u(A') \arrow[r, "(\alpha')^u"] & A' \end{tikzcd}$$

To see that γ is an $\mathsf{Alg}(\mathbb{H}^u)$-morphism, the left square of the diagram commutes by the naturality of ζ. We have

$$\begin{aligned} (\gamma \circ \alpha^u) \circ \zeta_{\mathbb{H}(A)} &= \gamma \circ \alpha = \alpha' \circ \mathbb{H}(\gamma) = (\alpha')^u \circ \zeta_{\mathbb{H}(A')} \circ \mathbb{H}(\gamma) \\ &= (\alpha')^u \circ \mathbb{H}^u(\gamma) \circ \zeta_{\mathbb{H}(A)} \end{aligned}$$

so $\gamma \circ \alpha^u = (\alpha')^u \circ \mathbb{H}^u(\gamma)$ since $\zeta_{\mathbb{H}(A)}$ is epic. This shows that γ is an $\mathsf{Alg}(\mathbb{H}^u)$-morphism. It then follows that $\mathbb{A}$ is a covariant functor.

Going in the opposite direction, we define a functor $\mathbb{B} : \mathsf{Alg}(\mathbb{H}^u) \to \mathsf{Alg}^u(\mathbb{H})$ on objects by sending (A, α) to $(A, \alpha \circ \zeta_{\mathbb{H}(A)})$. On morphisms we send a $\mathsf{Alg}(\mathbb{H}^u)$-morphism $\gamma : A \to A'$ to itself. It is clear that $\mathbb{B}$ is a covariant functor.

If $(A, \alpha) \in \mathsf{Alg}^u(\mathbb{H})$, then $\mathbb{A}(A, \alpha) = (A, \alpha^u)$, and so $\mathbb{B}\mathbb{A}(A, \alpha) = (A, \alpha^u \circ \zeta_{\mathbb{H}(A)}) = (A, \alpha)$. Therefore, $\mathbb{B}\mathbb{A} = 1_{\mathsf{Alg}^u(\mathbb{H})}$. If $(A, \alpha) \in \mathsf{Alg}(\mathbb{H}^u)$, then $(A, \alpha \circ \zeta_{\mathbb{H}(A)}) \in \mathsf{Alg}^u(\mathbb{H})$, and $(\alpha \circ \zeta_{\mathbb{H}(A)})^u = \alpha$. Therefore, $\mathbb{A}\mathbb{B} = 1_{\mathsf{Alg}(\mathbb{H}^u)}$. Thus, $\mathbb{A}, \mathbb{B}$ yield an isomorphism of categories between $\mathsf{Alg}^u(\mathbb{H})$ and $\mathsf{Alg}(\mathbb{H}^u)$. ☐

7. $\boldsymbol{mba\ell}$ AND KHF

In this section we show how to derive from our results the dual adjunction between $\boldsymbol{mba\ell}$ and KHF and the dual equivalence between $\boldsymbol{muba\ell}$ and KHF obtained in [6].

We start by recalling (see, e.g., [5, Thm. 2.16]) that there is an isomorphism of categories between $\mathsf{Coalg}(\mathbb{V})$ and KHF. The isomorphism is determined by the following functors. The functor $\mathbb{S} : \mathsf{Coalg}(\mathbb{V}) \to \mathsf{KHF}$ sends (X, σ) to $(X, R_\sigma) \in \mathsf{KHF}$, where $x R_\sigma y$ if $y \in \sigma(x)$, and $\mathbb{S}$ sends a $\mathsf{Coalg}(\mathbb{V})$ morphism to itself. The functor $\mathbb{T} : \mathsf{KHF} \to \mathsf{Coalg}(\mathbb{V})$ sends $(X, R) \in \mathsf{KHF}$ to (X, σ_R), defined by $\sigma_R(x) = R[x]$, and sends a KHF-morphism to itself.

As a consequence of this and the results of the previous section, we obtain the main result of [6].

Theorem 7.1. [6, Thm. 5.3] *There is a dual adjunction between* $\boldsymbol{mbal}$ *and* KHF *which restricts to a dual equivalence between* $\boldsymbol{mubal}$ *and* KHF*.*

Proof. By Theorem 6.4 the functors $\widehat{\mathbb{Y}}$ and $\widehat{\mathbb{C}}$ form a dual adjunction between $\mathsf{Alg}(\mathbb{H})$ and $\mathsf{Coalg}(\mathbb{V})$. By Theorem 4.5, the functors $\mathbb{M},\mathbb{N}$ yield an isomorphism of categories between $\mathsf{Alg}(\mathbb{H})$ and $\boldsymbol{mbal}$. The functors $\mathbb{S},\mathbb{T}$ yield an isomorphism of categories between $\mathsf{Coalg}(\mathbb{V})$ and KHF [5, Thm. 2.16]. We thus have the following diagram.

$$\boldsymbol{mubal} \hookrightarrow \boldsymbol{mbal} \underset{\mathbb{M}}{\overset{\mathbb{N}}{\rightleftarrows}} \mathsf{Alg}(\mathbb{H}) \underset{\widehat{\mathbb{C}}}{\overset{\widehat{\mathbb{Y}}}{\rightleftarrows}} \mathsf{Coalg}(\mathbb{V}) \underset{\mathbb{T}}{\overset{\mathbb{S}}{\rightleftarrows}} \mathsf{KHF}$$

Consequently, $\mathbb{S}\widehat{\mathbb{Y}}\mathbb{N} : \boldsymbol{mbal} \to \mathsf{KHF}$ and $\mathbb{M}\widehat{\mathbb{C}}\mathbb{T} : \mathsf{KHF} \to \boldsymbol{mbal}$ yield a dual adjunction which restricts to a dual equivalence between $\boldsymbol{mubal}$ and KHF. ☐

Utilizing the descriptions of the functors $\widehat{\mathbb{Y}}$ and $\widehat{\mathbb{C}}$ given in Remark 6.5, we can show that we obtain exactly the functors yielding the dual adjuction of [6].

Proposition 7.2. $\mathbb{S}\widehat{\mathbb{Y}}\mathbb{N}$ *and* $\mathbb{M}\widehat{\mathbb{C}}\mathbb{T}$ *are precisely the functors* $\mathbb{Y}$ *and* $\mathbb{C}$ *yielding the dual adjunction of* [6, Thm. 5.3].

Proof. Let $(A,\Box) \in \boldsymbol{mbal}$. Then $\mathbb{Y}(A,\Box) = (Y_A, R_\Box)$, where we recall from Section 2.3 that $R_\Box$ is defined by $xR_\Box y$ if $y^+ \subseteq \Box^{-1}x$. We have $\mathbb{N}(A,\Box) = (A,\sigma_\Box)$, which satisfies $\sigma_\Box(\Box_a) = \Box a$ for all $a \in A$. Then $\widehat{\mathbb{Y}}(A,\sigma_\Box) = (Y_A, \widehat{\mathbb{Y}}_{\sigma_\Box})$, where we recall (see Remark 6.5) that $\widehat{\mathbb{Y}}_{\sigma_\Box} = \varepsilon^{-1}_{\mathbb{V}(Y_A)} \circ \mathbb{Y}(\eta_A)^{-1} \circ \mathbb{Y}(\sigma_\Box)$. Finally, $\mathbb{S}$ sends this to $(Y_A, R_{\widehat{\mathbb{Y}}_{\sigma_\Box}})$, where $xR_{\widehat{\mathbb{Y}}_{\sigma_\Box}}y$ if $y \in \widehat{\mathbb{Y}}_{\sigma_\Box}(x)$. Let $x \in Y_A$ and $F = \widehat{\mathbb{Y}}_{\sigma_\Box}(x) \in \mathbb{V}(Y_A)$. If $M = \varepsilon_{\mathbb{V}(Y_A)}(F) \in Y_{C(\mathbb{V}Y_A)}$, then $M = \{g \in C(\mathbb{V}Y_A) \mid g(F) = 0\}$ and

$$\mathbb{Y}(\eta_A)(M) = \eta_A^{-1}[M] = \sigma_\Box^{-1}(x) = \mathbb{Y}(\sigma_\Box)(x).$$

We show that $R_\Box = R_{\widehat{\mathbb{Y}}_{\sigma_\Box}}$. Suppose that $xR_\Box y$, so $\Box y^+ \subseteq x$. To see that $xR_{\widehat{\mathbb{Y}}_{\sigma_\Box}}y$, we need to show that $y \in F$. If not, then by Urysohn's lemma and the fact that $\zeta_A[A]$ is uniformly dense in $C(Y_A)$, there is $a \in A$ with $\zeta_A(a)(y) = 0$ and $\zeta_A(a)[F] \geq 1/2$. By replacing a by a^+ we may assume that $a \geq 0$. Since $\zeta_A(a)(y) = 0$, we have $a \in y$. Therefore, $\Box a \in x$. This means $\sigma_\Box(\Box_a) \in x$, so $\Box_a \in \sigma_\Box^{-1}(x) = \eta_A^{-1}[M]$. Thus, $\eta_A(\Box_a) \in M$, so $g_A(a) \in M$. Therefore, $g_A(a)(F) = 0$, which is false by construction of a. This shows $y \in F$.

Conversely, if $xR_{\widehat{\mathbb{Y}}_{\sigma_\Box}}y$, then $y \in F$. Let $a \in y^+$. Then $g_A(a)(F) = 0$ because $a \in y$ and $a \geq 0$. Therefore, $\eta_A(\Box_a) \in M$, so $\Box_a \in \eta_A^{-1}(M) = \sigma_\Box^{-1}(x)$. Thus, $\Box a = \sigma_\Box(\Box_a) \in x$. This shows $\Box y^+ \subseteq x$, so $xR_\Box y$. This completes the proof that $R_{\widehat{\mathbb{Y}}_{\sigma_\Box}} = R_\Box$. Therefore, $\mathbb{Y}$ and $\mathbb{S}\widehat{\mathbb{Y}}\mathbb{N}$ agree on the objects of $\boldsymbol{mbal}$. For morphisms, if $\alpha : (A,\Box) \to (A',\Box')$ is an $\boldsymbol{mbal}$-morphism, then $\mathbb{S}\widehat{\mathbb{Y}}\mathbb{N}(\alpha) = \mathbb{S}\widehat{\mathbb{Y}}(\alpha) = \mathbb{S}(\mathbb{Y}(\alpha)) = \mathbb{Y}(\alpha)$. Thus, $\mathbb{S}\widehat{\mathbb{Y}}\mathbb{N} = \mathbb{Y}$.

In the opposite direction, if $(X,R) \in \mathsf{KHF}$, we show that $\mathbb{C}(X,R) = \mathbb{M}\widehat{\mathbb{C}}\mathbb{T}(X,R)$. First, $\mathbb{C}(X,R) = (C(X),\Box_R)$, where we recall from Section 2.3 that $\Box_R f$ is given by

$$(\Box_R f)(x) = \begin{cases} \inf fR[x] & \text{if } R[x] \neq \varnothing \\ 1 & \text{if } R[x] = \varnothing. \end{cases}$$

The functor $\mathbb{T}$ sends (X, R) to (X, σ_R), where $\sigma_R(x) = R[x]$. Then $\widehat{\mathbb{C}}$ sends this to $(C(X), \widehat{\mathbb{C}}_{\sigma_R})$, where we recall (see Remark 6.5) that $\widehat{\mathbb{C}}_{\sigma_R} = \mathbb{C}(\sigma_R) \circ \mathbb{C}\mathbb{V}(\varepsilon_X) \circ \eta_{C(X)}$. Finally, $(C(X), \widehat{\mathbb{C}}_{\sigma_R})$ is sent by $\mathbb{M}$ to $(C(X), \Box_{\widehat{\mathbb{C}}_{\sigma_R}})$, where $\Box_{\widehat{\mathbb{C}}_{\sigma_R}} f = \widehat{\mathbb{C}}_{\sigma_R}(\Box_f)$. We have

$$\begin{aligned}\widehat{\mathbb{C}}_{\sigma_R}(\Box_f) &= \mathbb{C}(\sigma_R)(\mathbb{C}\mathbb{V}(\varepsilon_X)(\eta_{C(X)}(\Box_f))) \\ &= \mathbb{C}(\sigma_R)(\mathbb{C}\mathbb{V}(\varepsilon_X)(g_{C(X)}(f))) \\ &= \mathbb{C}(\sigma_R)(g_{C(X)}(f) \circ \mathbb{V}(\varepsilon_X)) \\ &= g_{C(X)}(f) \circ \mathbb{V}(\varepsilon_X) \circ \sigma_R.\end{aligned}$$

Let $x \in X$. Then $\sigma_R(x) = R[x]$ and $\mathbb{V}(\varepsilon_X)(R[x]) = \varepsilon_X[R[x]]$. Therefore, since $f = \zeta_{C(X)}(f) \circ \varepsilon_X$ by Remark 2.5, we have

$$\begin{aligned}g_{C(X)}(f)(\varepsilon_X R[x]) &= \begin{cases} \inf \zeta_{C(X)}(f)[\varepsilon_X R[x]] & \text{if } R[x] \neq \varnothing \\ 1 & \text{if } R[x] = \varnothing \end{cases} \\ &= \begin{cases} \inf fR[x] & \text{if } R[x] \neq \varnothing \\ 1 & \text{if } R[x] = \varnothing \end{cases} \\ &= (\Box_R f)(x).\end{aligned}$$

Thus, $\mathbb{C}$ and $\mathbb{M}\widehat{\mathbb{C}}\mathbb{T}$ agree on objects of KHF. If $\sigma : (X, R) \to (X', R')$ is a KHF-morphism, then $\mathbb{M}\widehat{\mathbb{C}}\mathbb{T}(\sigma) = \mathbb{M}\mathbb{C}(\sigma) = \mathbb{C}(\sigma)$. Consequently, $\mathbb{M}\widehat{\mathbb{C}}\mathbb{T} = \mathbb{C}$. $\square$

We conclude this section with the following diagram showing the relationship between the various categories we have considered, where the curved vertical arrows are reflections.

$$\begin{tikzcd}
\boldsymbol{mba\ell} \arrow[r, shift left, "\mathbb{N}"] \arrow[d, bend right] & \mathsf{Alg}(\mathbb{H}) \arrow[l, shift left, "\mathbb{M}"] \arrow[d, bend right] & \\
\boldsymbol{muba\ell} \arrow[u, hook] \arrow[r, shift left, "\mathbb{N}"] \arrow[d, shift left, "\mathbb{Y}"] & \mathsf{Alg}^u(\mathbb{H}) \arrow[u, hook] \arrow[l, shift left, "\mathbb{M}"] \arrow[r, shift left, "\mathbb{A}"] \arrow[d, shift left, "\widehat{\mathbb{Y}}"] & \mathsf{Alg}(\mathbb{H}^u) \arrow[l, shift left, "\mathbb{B}"] \arrow[dl, bend left, "\widehat{\mathbb{Y}}\mathbb{B}"] \\
\mathsf{KHF} \arrow[u, shift left, "\mathbb{C}"] \arrow[r, shift left, "\mathbb{T}"] & \mathsf{Coalg}(\mathbb{V}) \arrow[u, shift left, "\widehat{\mathbb{C}}"] \arrow[l, shift left, "\mathbb{S}"] \arrow[ur, bend right, Rightarrow, "\mathbb{A}\widehat{\mathbb{C}}"] &
\end{tikzcd}$$

8. The serial case

We recall that a binary relation R on a set X is serial provided that $R[x] \neq \varnothing$ for each $x \in X$. If $(X, R) \in \mathsf{KHF}$ with R serial, then we can replace $\mathbb{V}(X)$ with $\mathbb{V}^*(X)$ in our considerations. It was shown in [6, Prop. 7.2] that the binary relation $R_\Box$ in the dual compact Hausdorff frame of $(A, \Box) \in \boldsymbol{mba\ell}$ is serial iff $\Box 0 = 0$. We thus arrive at the following definition, the notation of which is motivated by modal logic, where the seriality axiom is denoted by D (see, e.g., [11]).

Definition 8.1. [6, Sec. 7]

(1) Let $\boldsymbol{mba\ell}^{\mathsf{D}}$ be the full subcategory of $\boldsymbol{mba\ell}$ consisting of those $(A, \Box) \in \boldsymbol{mba\ell}$ with $\Box 0 = 0$.

(2) Let $\boldsymbol{mubal}^{\mathsf{D}}$ be the full subcategory of $\boldsymbol{mbal}^{\mathsf{D}}$ consisting of those $(A, \Box) \in \boldsymbol{mbal}^{\mathsf{D}}$ with $A \in \boldsymbol{ubal}$.

(3) Let $\mathsf{KHF}^{\mathsf{D}}$ be the full subcategory of KHF consisting of those (X, R) for which R is a serial relation.

As was pointed out in [6, Rem. 3.12], when $\Box 0 = 0$, the axioms (M2), (M4), and (M5) simplify to the following axioms:

(M2*) $\Box r = r$.
(M4*) $\Box(a + r) = \Box a + r$.
(M5*) $\Box(ra) = r(\Box a)$ provided $r \geq 0$.

Moreover, (M2*) follows from (M4*) by setting $a = 0$.

In [6, Sec. 7] we showed that the functors $\mathbb{C}$ and $\mathbb{Y}$ restrict to yield a dual adjunction between $\boldsymbol{mbal}^{\mathsf{D}}$ and $\mathsf{KHF}^{\mathsf{D}}$, which further restricts to a dual equivalence between $\boldsymbol{mubal}^{\mathsf{D}}$ and $\mathsf{KHF}^{\mathsf{D}}$. In this section we briefly outline how to derive this result from our considerations by simplifying the definition of $\mathbb{H}$ to produce a functor $\mathbb{H}^*$ such that $\mathsf{Alg}(\mathbb{H}^*)$ is isomorphic to $\boldsymbol{mbal}^{\mathsf{D}}$.

Definition 8.2. Let $A \in \boldsymbol{bal}$.

(1) Let I^*_A be the archimedean ℓ-ideal of $F(A)$ generated by the following classes of elements.
   (a): $f_A(a \wedge b) - f_A(a) \wedge f_A(b)$;
   (c): $f_A(a^+) - f_A(a)^+$;
   (d*): $f_A(a + r) - f_A(a) - r$;
   (e*): $f_A(ra) - r f_A(a)$ if $0 \leq r$.

(2) Let $\mathbb{H}^*(A) = F(A)/I^*_A$ and $h^*_A : A \to \mathbb{H}^*(A)$ be the composition of f^*_A with the quotient map $\pi : F(A) \to \mathbb{H}^*(A)$.

(3) For $a \in A$ let $\Box^*_a = h^*_A(a)$.

(4) Let $\mathbb{V}^*$ be the endofunctor on KHaus that sends X to the subspace $\mathbb{V}^*(X) = \mathbb{V}(X) \setminus \{\varnothing\}$ of $\mathbb{V}(X)$.

The table below compares the relations in $\mathbb{H}^*(A)$ to those of $\mathbb{H}(A)$.

Relations for $\mathbb{H}^*(A)$	Relations for $\mathbb{H}(A)$
(1) $\Box^*_{a\wedge b} = \Box^*_a \wedge \Box^*_b$	(1) $\Box_{a\wedge b} = \Box_a \wedge \Box_b$
(2) ———	(2) $\Box_r = r + (1 - r)\Box_0$
(3) $\Box^*_{a^+} = (\Box^*_a)^+$	(3) $\Box_{a^+} = (\Box_a)^+$
(4) $\Box^*_{a+r} = \Box^*_a + r$	(4) $\Box_{a+r} = \Box_a + \Box_r - \Box_0$
(5) $\Box^*_{ra} = r\Box^*_a$ if $0 \leq r$	(5) $\Box_{ra} = \Box_r \Box_a$ if $0 \leq r$

Remark 8.3.

(1) If we set $r = 0$ in (5) we see that $\Box^*_0 = 0$. Furthermore, setting $a = 0$ in (4) yields $\Box^*_r = r$.

(2) From the relations above it follows that $\mathbb{H}^*(A)$ is the quotient of $\mathbb{H}(A)$ by the archimedian ℓ-ideal of $\mathbb{H}(A)$ generated by $\Box_0$. Consequently, if $(A, \alpha) \in \mathsf{Alg}(\mathbb{H})$ with $\alpha(\Box_0) = 0$, then there is an induced object $(A, \alpha^*) \in \mathsf{Alg}(\mathbb{H}^*)$.

Theorem 8.4. *There is an isomorphism of categories between* $\mathsf{Alg}(\mathbb{H}^*)$ *and* $\boldsymbol{mba\ell}^{\mathsf{D}}$*.*

Proof. The functor $\mathsf{Alg}(\mathbb{H}^*) \to \boldsymbol{mba\ell}^{\mathsf{D}}$ is defined essentially the same as in Lemma 4.3. To define the functor in the other direction, if $(A, \Box) \in \boldsymbol{mba\ell}^{\mathsf{D}}$, then the induced $\boldsymbol{ba\ell}$-morphism $\sigma_\Box : \mathbb{H}(A) \to A$ satisfies $\sigma_\Box(\Box_0) = 0$, so induces an object $(A, \sigma_\Box^*) \in \mathsf{Alg}(\mathbb{H}^*)$ by Remark 8.3(2), which satisfies $\sigma_\Box^*(\Box_a^*) = \Box a$ for each $a \in A$. This gives the functor $\boldsymbol{mba\ell}^{\mathsf{D}} \to \mathsf{Alg}(\mathbb{H}^*)$. The proof that these functors yield an isomorphism is essentially the same as that of Theorem 4.5. $\square$

In parallel with Definition 6.6, let $\mathsf{Alg}^u(\mathbb{H}^*)$ be the full subcategory of $\mathsf{Alg}(\mathbb{H}^*)$ consisting of those (A, α) with $A \in \boldsymbol{uba\ell}$. The proof of the following result is similar to that of Theorem 6.4 and Corollary 6.7(1), with small changes similar to those of the previous theorem. We therefore leave out the details.

Theorem 8.5. *There is a dual adjunction between* $\mathsf{Alg}(\mathbb{H}^*)$ *and* $\mathsf{Coalg}(\mathbb{V}^*)$ *which restricts to a dual equivalence between* $\mathsf{Alg}^u(\mathbb{H}^*)$ *and* $\mathsf{Coalg}(\mathbb{V}^*)$*.*

Theorem 8.6. *There is a dual adjunction between* $\boldsymbol{mba\ell}^{\mathsf{D}}$ *and* $\mathsf{KHF}^{\mathsf{D}}$ *which restricts to a dual equivalence between* $\boldsymbol{muba\ell}^{\mathsf{D}}$ *and* $\mathsf{KHF}^{\mathsf{D}}$*.*

Proof. The proof is similar to that of Theorem 7.1 but uses Theorem 8.5 instead of Theorem 6.4 and Corollary 6.7(1). It also uses the isomorphism between $\mathsf{Coalg}(\mathbb{V}^*)$ and $\mathsf{KHF}^{\mathsf{D}}$, which is essentially the same as that between $\mathsf{Coalg}(\mathbb{V})$ and KHF. To give some detail, if $(X, \sigma) \in \mathsf{Coalg}(\mathbb{V}^*)$, then (by abusing notation) $\mathbb{S}$ sends it to (X, R_σ). Since the image of σ is in $\mathbb{V}^*(X)$, the relation R_σ is serial, so $(X, R_\sigma) \in \mathsf{KHF}^{\mathsf{D}}$. Conversely, if $(X, R) \in \mathsf{KHF}^{\mathsf{D}}$, then R is serial, so $\sigma_R(x) = R[x] \neq \varnothing$ for each $x \in X$, so $\sigma_R : X \to \mathbb{V}^*(X)$ is continuous, and hence (again abusing notation) $\mathbb{T}(X, R) = (X, \sigma_R) \in \mathsf{Coalg}(\mathbb{V}^*)$. $\square$

Remark 8.7. A slight change in the argument of Proposition 6.10 shows that $\mathsf{Alg}^u(\mathbb{H}^*)$ is isomorphic to $\mathsf{Alg}(\mathbb{H}^{*u})$, where $\mathbb{H}^{*u}$ is the composition $\mathbb{CYH}^*$.

9. Connection to modal algebras and descriptive frames

In this final section we connect our results with those of Abramsky [1] and Kupke, Kurz, and Venema [20]. We start by recalling the definition of those $A \in \boldsymbol{ba\ell}$ that are clean as rings (see, e.g., [22] and the references therein).

Definition 9.1. We call $A \in \boldsymbol{ba\ell}$ *clean* if each $a \in A$ can be written as $a = e + v$ with e an idempotent and v a unit. Let $\boldsymbol{cuba\ell}$ be the full subcategory of $\boldsymbol{uba\ell}$ consisting of clean rings.

Lemma 9.2. *If* $A \in \boldsymbol{cuba\ell}$*, then* $\mathbb{H}^u(A) \in \boldsymbol{cuba\ell}$*.*

Proof. By [8, Prop. 5.20], if $A \in \boldsymbol{cuba\ell}$, then Y_A is a Stone space. Therefore, $\mathbb{V}(Y_A)$ is a Stone space, and hence $Y_{\mathbb{H}^u(A)}$ is a Stone space by Theorem 5.8. Thus, $\mathbb{H}^u(A) \in \boldsymbol{cuba\ell}$ by [8, Prop. 5.20]. $\square$

To distinguish between $\mathbb{V}$ on KHaus and Stone, we denote the Vietoris endofunctor on Stone by $\mathbb{V}^{\mathsf{S}}$. By Lemma 9.2, $\mathbb{H}^u$ restricts to an endofunctor on $\boldsymbol{cuba\ell}$, which we denote by $\mathbb{H}^c$. The following result is then an immediate consequence of Corollary 6.7(1).

Theorem 9.3. *There is a dual equivalence between* $\mathsf{Alg}^u(\mathbb{H}^c)$ *and* $\mathsf{Coalg}(\mathbb{V}^{\mathsf{S}})$.

We let $\mathbb{H}^{\mathsf{BA}}$ be the functor of [20] that sends $B \in \mathsf{BA}$ to the free boolean algebra over its underlying meet-semilattice. It was shown in [20, Prop., 3.12] that $\mathsf{Alg}(\mathbb{H}^{\mathsf{BA}})$ is isomorphic to the category MA of modal algebras. In parallel of $\mathbb{M} : \mathsf{Alg}(\mathbb{H}) \to \boldsymbol{mba\ell}$ and $\mathbb{N} : \boldsymbol{mba\ell} \to \mathsf{Alg}(\mathbb{H})$, we denote the functors giving the isomorphism by $\mathbb{M}^{\mathsf{BA}} : \mathsf{Alg}(\mathbb{H}^{\mathsf{BA}}) \to \mathsf{MA}$ and $\mathbb{N}^{\mathsf{BA}} : \mathsf{MA} \to \mathsf{Alg}(\mathbb{H}^{\mathsf{BA}})$. By [6, Thm. 6.9], the triangle in the diagram below commutes up to natural isomorphism, where $(-)^* : \mathsf{DF} \to \mathsf{MA}$ and $(-)_* : \mathsf{MA} \to \mathsf{DF}$ are the functors yielding Jónsson-Tarski duality, and the functor Id sends $(A, \square) \in \boldsymbol{mba\ell}$ to $(\mathrm{Id}(A), \square|_{\mathrm{Id}(A)})$ (see [6, Lem. 6.5]). Therefore, there is an equivalence of categories between $\mathsf{Alg}(\mathbb{H}^c)$ and $\mathsf{Alg}(\mathbb{H}^{\mathsf{BA}})$, where the functor $\mathsf{Alg}(\mathbb{H}^c) \to \mathsf{Alg}(\mathbb{H}^{\mathsf{BA}})$ is the composition $\mathbb{N}^{\mathsf{BA}} \circ \mathrm{Id} \circ \mathbb{M}$.

$$\begin{tikzcd}
\mathsf{Alg}(\mathbb{H}^c) \arrow[rr, "\mathbb{N}^{\mathsf{BA}} \circ \mathrm{Id} \circ \mathbb{M}"] \arrow[d, shift left, "\mathbb{M}"] & & \mathsf{Alg}(\mathbb{H}^{\mathsf{BA}}) \arrow[d, shift left, "\mathbb{M}^{\mathsf{BA}}"] \\
\boldsymbol{mcuba\ell} \arrow[rr, "\mathrm{Id}"] \arrow[u, shift left, "\mathbb{N}"] \arrow[dr, shift left, "\mathbb{Y}"] & & \mathsf{MA} \arrow[u, shift left, "\mathbb{N}^{\mathsf{BA}}"] \arrow[dl, shift left, "(-)_*"] \\
& \mathsf{DF} \arrow[ul, shift left, "\mathbb{C}"] \arrow[ur, shift left, "(-)^*"] &
\end{tikzcd}$$

We conclude by summarizing our main findings. We developed an endofunctor $\mathbb{H} : \boldsymbol{ba\ell} \to \boldsymbol{ba\ell}$ and connected it to the Vietoris endofunctor $\mathbb{V} : \mathsf{KHaus} \to \mathsf{KHaus}$ by showing that there is a dual adjunction between the category $\mathsf{Alg}(\mathbb{H})$ of algebras for $\mathbb{H}$ and the category $\mathsf{Coalg}(\mathbb{V})$ of coalgebras for $\mathbb{V}$ (see Theorem 6.4). We proved that this dual adjunction restricts to a dual equivalence between $\mathsf{Coalg}(\mathbb{V})$ and the reflective subcategory $\mathsf{Alg}^u(\mathbb{H})$ of $\mathsf{Alg}(\mathbb{H})$ (see Corollary 6.7). We also constructed an endofunctor $\mathbb{H}^u : \boldsymbol{uba\ell} \to \boldsymbol{uba\ell}$ and showed that $\mathsf{Alg}^u(\mathbb{H})$ is isomorphic to $\mathsf{Alg}(\mathbb{H}^u)$ (see Proposition 6.10). Since $\mathsf{Alg}(\mathbb{H})$ is isomorphic to the category $\boldsymbol{mba\ell}$ of modal bounded archimedean ℓ-algebras and $\mathsf{Coalg}(\mathbb{V})$ to the category KHF of compact Hausdorff frames, we obtained an alternate proof of the result of [6] that there is a dual adjunction between $\boldsymbol{mba\ell}$ and KHF that restricts to a dual equivalence between KHF and the reflective subcategory $\boldsymbol{muba\ell}$ of $\boldsymbol{mba\ell}$ (see Theorem 7.1).

When we exclude $\varnothing$ from the Vietoris construction, we obtain the endofunctor $\mathbb{V}^* : \mathsf{KHaus} \to \mathsf{KHaus}$. Its algebraic counterpart is the endofunctor $\mathbb{H}^* : \boldsymbol{ba\ell} \to \boldsymbol{ba\ell}$ and we arrive at a dual adjunction between $\mathsf{Alg}(\mathbb{H}^*)$ and $\mathsf{Coalg}(\mathbb{V}^*)$ which restricts to a dual equivalence between $\mathsf{Alg}^u(\mathbb{H}^*)$ and $\mathsf{Coalg}(\mathbb{V}^*)$ (see Theorem 8.5). This yields the duality of [6] for compact Hausdorff frames whose binary relation is serial (see Theorem 8.6). Finally, restricting to the clean case yields an algebraic counterpart of the Vietoris endofunctor on Stone spaces (see Theorem 9.3), which in turn yields Jónsson-Tarski duality.

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