

K-POLYSTABILITY OF FANO 4-FOLDS WITH LARGE LEFSCHETZ DEFECT

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ABSTRACT. In this paper we study K-stability on smooth complex Fano 4-folds having large Lefschetz defect, that is greater or equal then 3, with a special focus on the case of Lefschetz defect 3. In particular, we determine whether these Fano 4-folds are K-polystable or not, and show that there are 5 families (out of 19) of K-polystable smooth Fano 4-folds with Lefschetz defect 3.

1. INTRODUCTION

The notion of K-stability was first introduced in [Tia97] as a criterion to characterize the existence of a Kähler–Einstein metric on complex Fano varieties, and has been later formulated using purely algebraic geometric terms in [Don02]. Nowadays, by the celebrated works [CDS15, Tia15], it is well known that a complex smooth Fano variety admits a Kähler–Einstein metric if and only if it is K-polystable.

This correspondence links together differential and complex algebraic geometry, and it represents one of the main motivations to investigate K-polystability of Fano varieties. Moreover, the condition of K-stability has been successfully used to construct moduli spaces of Fano varieties, thus increasing its relevance within modern algebraic geometry (see [Xu21, §2] and references therein). We refer to [Xu21] for the original definitions of K-stability involving \mathbb{C}^* -degenerations of Fano varieties, and for a survey on this topic from an algebro-geometric viewpoint. More recently, in [BtHJ17] valuation methods have been introduced to reinterpret one parameter group degenerations: these new techniques gave a fundamental development to the algebraic theory of K-stability, due to equivalent and easier ways to test K-stability notions in many situations, such as the computation of the beta invariant of divisors over the target variety (see [Fuj19, Li17]). Indeed, the beta invariant may be explicitly computed for many classes of Fano varieties whose structure of divisors in their birational models is well understood.

The situation is completely known for del Pezzo surfaces (see Corollary 2.3), while we refer to [ACC⁺23] for the case of Fano 3-folds and for a general and updated literature on this topic.

In this paper, we will use valuation methods to study K-polystability of some families of Fano 4-folds which have been first studied in [CR22] and then completely classified in [CRS22, Proposition 1.5], that is Fano 4-folds X having Lefschetz defect $\delta_X = 3$; we refer to [Cas12] for an introduction on this invariant and the first implications on the geometry of Fano varieties in the case $\delta_X \geq 3$.

From the viewpoint of K-polystability, the case of Fano 4-folds X with $\delta_X \geq 4$ easily follows from known results. Indeed, by [Cas12, Theorem 3.3] these varieties are products of two del Pezzo surfaces, and applying [Zhu20, Theorem 1.1] (see also Lemma 2.4) we see

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that a product of Fano varieties is K-polystable if and only if both of its factors are (see Remark 3.2 for details).

Thus, it arises our motivation to study the subsequent case of Fano 4-folds having Lefschetz defect $\delta = 3$: among the possible 19 families of such Fano 4-folds classified in [CR22] and [CRS22], we establish which ones are K-polystable. We state our conclusions in the following result.

Theorem 1.1. *Let X be a Fano 4-fold with $\delta_X \geq 3$. Denote by F' (resp. F) the blow-up of \mathbb{P}^2 along two (resp. three non-collinear) points. Then:*

- (i) *if $\delta_X \geq 4$, then X is K-polystable if and only if $X \not\cong S \times \mathbb{F}_1$, $X \not\cong S \times F'$, with S a del Pezzo surface having $\rho_S = \delta_X + 1$.*
- (ii) *If $\delta_X = 3$, then X is K-polystable if and only if it is one of the following:*
 - $X \cong \mathbb{P}^2 \times F$;
 - $X \cong \mathbb{P}^1 \times \mathbb{P}^1 \times F$;
 - $X \cong F \times F$;
 - X , the blow-up of $\mathbb{P}^1 \times \mathbb{P}_{\mathbb{P}^1 \times \mathbb{P}^1}(\mathcal{O} \oplus \mathcal{O}(1, -1))$ along two surfaces isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$;
 - $X \cong \mathbb{P}^1 \times Y$, where Y is the blow-up of \mathbb{P}^3 along a disjoint union of a line and a conic, and along two non-trivial fibers of the exceptional divisor over the blown-up line.

Outline of the paper. After giving some preliminaries on K-polystability, on the Lefschetz defect δ and on the structure of Fano 4-folds having $\delta = 3$ in Sections 2 and 3, we dedicate Section 4 to the proof of Theorem 1.1. As we have already observed, proving (ii) will require the most effort. Note that in (ii) all but the last one are toric varieties.

To prove our result, we distinguish between the toric and the non-toric cases, proceeding in two different ways. The key point to study the toric case is a well known criterion on K-polystability for toric Fano varieties (see Proposition 4.2). The non-toric case, on the other hand, consists of 5 possible families and is the more difficult to check: we will use the Fujita-Li's valuative criterion (see Theorem 2.2). The strategy here is to show Proposition 4.4 which gives an explicit formula to compute the beta invariant on a special exceptional divisor, denoted by \tilde{D} , that all non-toric Fano 4-folds with $\delta = 3$ contain. We introduce and describe \tilde{D} , as well as the geometry of its ambient variety, in §3.1. To deduce the formula in Proposition 4.4, we first determine the Zariski decomposition of $-K_X - t\tilde{D}$ for $t \geq 0$ (Proposition 4.10) throughout some technical and preliminary lemmas, for which we deeply use our knowledge on the birational geometry of Fano 4-folds with $\delta = 3$. Finally, we deduce that four out of five families of non-toric Fano 4-folds with $\delta = 3$ are not K-polystable, as the beta invariant on \tilde{D} turns out to be negative. The remaining case (that is the fifth variety in our list (ii) from Theorem 1.1) is isomorphic to a product and it gives the only example of non-toric K-polystable Fano variety with $\delta = 3$: we will again apply [Zhu20, Theorem 1.1] to deduce its K-polystability. Finally, we summarize our conclusions in Table 1 and Table 2.

Notations. We work over the field of complex numbers. Let X be a smooth projective variety.

- \sim denotes linear equivalence for divisors. We often will not distinguish between a Cartier divisor D and its corresponding invertible sheaf $\mathcal{O}_X(D)$.

- $\mathcal{N}_1(X)$ (resp. $\mathcal{N}^1(X)$) is the \mathbb{R} -vector space of one-cycles (resp. divisors) with real coefficients, modulo numerical equivalence, and $\rho_X := \dim \mathcal{N}_1(X) = \dim \mathcal{N}^1(X)$ is the Picard number of X . Sometimes we denote it simply by ρ .
- The *pseudoeffective cone* is the closure of the cone in $\mathcal{N}^1(X)$ generated by the classes of effective divisors on X ; its interior is the big cone. An \mathbb{R} -divisor is called pseudoeffective if its numerical class belongs to the pseudoeffective cone.
- We denote by $[C]$ the numerical equivalence class in $\mathcal{N}_1(X)$ of a one-cycle C of X .
- $\text{NE}(X) \subset \mathcal{N}_1(X)$ is the convex cone generated by classes of effective curves.
- A *contraction* of X is a surjective morphism $\varphi: X \rightarrow Y$ with connected fibers, where Y is normal and projective.
- The *relative cone* $\text{NE}(\varphi)$ of φ is the convex subcone of $\text{NE}(X)$ generated by classes of curves contracted by φ .
- We denote by δ_X , or simply by δ , the Lefschetz defect of X .
- A small \mathbb{Q} -factorial modification (SQM) among two normal projective \mathbb{Q} -factorial varieties is a birational map $g: Y \dashrightarrow Z$ that is an isomorphism in codimension one.

2. PRELIMINARIES

This section includes the preliminaries on K-polystability, see §2.1, and on the Zariski decomposition in Mori dream spaces, see §2.2.

2.1. Fujita-Li's valuative criterion. In this subsection we recall the characterization of K-semistability using valuations and we collect some preliminary results that arise from this study. A key definition is the invariant $\beta(E)$ computed on a divisor E over X , that is a divisor on a normal birational model Y over X (see [Fuj19, Li17]). For our purposes, we will focus on smooth varieties, even if the treatment can be made more general, referring to \mathbb{Q} -Fano varieties.

Definition 2.1. Let X be a smooth Fano variety and E a prime divisor on a normal birational model $\mu: Y \rightarrow X$. We define:

$$\beta(E) = A(E) - \frac{1}{(-K_X)^n} \int_0^\infty \text{vol}(-\mu^*K_X - tE) dt$$

where $A(E)$ is the log-discrepancy of X along E , namely $A(E) := 1 + \text{ord}_E(K_Y - \mu^*(K_X))$.

We refer to [Laz04, §2.2.C] for the definition of $\text{vol}(-)$. For simplicity, we set

$$S(E) := \frac{1}{(-K_X)^n} \int_0^\infty \text{vol}(-\mu^*K_X - tE) dt$$

and notice that this integral takes values in a closed set $[0, \tau]$, where $\tau = \tau(E)$ is the pseudoeffective threshold of E with respect to $-K_X$, namely:

$$\tau(E) = \sup\{s \in \mathbb{Q}_{>0} \mid -\mu^*K_X - sE \text{ is big}\}.$$

Therefore, we have $\beta(E) = A(E) - S(E)$.

The importance of the β -invariant mainly arises from the following result, that is known as the valuative criterion for K-(semi)stability, and it is due to Fujita and Li, [Fuj19, Li17] to which we also refer for a more general statement.

Theorem 2.2. *Let X be a smooth Fano variety. Then X is K-semistable if and only if $\beta(E) \geq 0$ for all divisors E over X .*

For our purposes, we will use that if X is not K -semistable, then it is not K -polystable by definition. Using the valuative criterion, it is easy to deduce that among del Pezzo surfaces, \mathbb{F}_1 and the blow-up of \mathbb{P}^2 at two points are not K -polystable (see for instance [ACC⁺23, Lemma 2.3, Lemma 2.4]). More precisely, we have the following:

Corollary 2.3. [Tia90] *Let S be a del Pezzo surface. Then S is K -polystable if and only if S is neither isomorphic to \mathbb{F}_1 nor to the blow-up of \mathbb{P}^2 at two points.*

Many varieties that we are going to study are products, and so we recall the following result. We refer to [Zhu20, Theorem 1.1] for a more general statement involving the other notions of K -stability.

Lemma 2.4. [Zhu20, Theorem 1.1] *Let X_1, X_2 be Fano varieties and let $X = X_1 \times X_2$. Then X is K -polystable if and only if X_i is K -polystable for $i = 1, 2$.*

Remark 2.5. Although the computation of the beta invariant involves the volume of divisors that are not necessarily nef (we will use the Zariski decomposition to this end, see §2.2), it may be possible to compute it explicitly for divisors whose structure in the birational models of their ambient variety is well known, thanks to the powerful tools from birational geometry. This will be our approach in the proof of Theorem 1.1 for the non-toric Fano 4-folds of our classification (see Proposition 4.4 and proof of Proposition 4.3).

2.2. Zariski decomposition in Mori dream spaces. A common approach to compute the beta invariant of an effective divisor on a Fano variety, thus its volume, is to determine its Zariski decomposition. In our case, we note that smooth Fano varieties are Mori dream spaces (MDS) by [BCHM10]. In fact, the existence of such a nice decomposition characterizes Mori dream spaces, and on such varieties the Zariski decomposition is unique, as observed in [Oka16, Remark 2.12]. To make our exposition self-contained, we start with the following basic definition, see [Oka16, §2] for details.

Definition 2.6. Let X be a normal projective variety and D a pseudoeffective \mathbb{Q} -Cartier divisor on X . A Zariski decomposition of D is given by a pair of \mathbb{Q} -Cartier divisors P and N on X which satisfy the following properties:

- P is nef;
- N is effective;
- D is \mathbb{Q} -linearly equivalent to $P + N$;
- for any sufficiently divisible $m \in \mathbb{Z}_{>0}$ the multiplication map

$$H^0(X, \mathcal{O}(mP)) \rightarrow H^0(X, \mathcal{O}(mD))$$

given by the tautological section of $\mathcal{O}(mN)$ is an isomorphism.

If X is a MDS, by [HK00, Proposition 1.11(2)] we know that there exist finitely many SQMs $g_i: X \dashrightarrow X_i$ and that the pseudoeffective cone of X is given by the union of finitely many Mori chambers \mathcal{C}_i ; each chamber is of the form $\mathcal{C}_i = g_i^* \text{Nef}(X_i) + \mathbb{R}_{\geq 0}\{E_1, \dots, E_k\}$ with E_1, \dots, E_k prime divisors contracted by g_i , and where $\text{Nef}(X_i)$ denotes the nef cone of X_i .

We may now interpret such a result as an instance of Zariski decomposition, as done in [Oka16, Proposition 2.13]. Indeed, for every \mathbb{Q} -Cartier divisor D on a MDS X , there exists a rational birational contraction $g: X \dashrightarrow Y$ (factorizing through an SQM and a birational contraction $X \xrightarrow{\psi} X' \xrightarrow{g'} Y$) and \mathbb{Q} -Cartier divisors P and N on X such that

D is \mathbb{Q} -linearly equivalent to $P + N$, $P' := \psi_* P$ is nef on X' and defines $g': X' \rightarrow Y$, $N' := \psi_* N$ is g' -exceptional, and the multiplication map $H^0(X', mP') \rightarrow H^0(X', m(\psi_* D))$ is an isomorphism for $m \gg 0$; namely P' and N' give a Zariski decomposition of $\psi_* D$ as a divisor in X' . To see this, we simply set $P := g^* g_* D$ and $N := D - P$.

3. FANO MANIFOLDS WITH LEFSCHETZ DEFECT 3

In this section we recap the classification (and construction) of smooth complex Fano varieties with Lefschetz defect $\delta = 3$.

The Lefschetz defect δ_X of a smooth Fano variety X is an invariant of X that depends on the Picard number of its prime divisors, and it was first introduced in [Cas12]. We recall its definition below, see also [Cas23] for a recent survey on this new invariant and its properties.

Definition 3.1. Let X be a complex smooth Fano variety, and D be a prime divisor on X . Consider the pushforward $\iota_*: \mathcal{N}_1(D) \rightarrow \mathcal{N}_1(X)$ induced by the inclusion and set $\mathcal{N}_1(D, X) := \iota_*(\mathcal{N}_1(D))$. The Lefschetz defect of X is

$$\delta_X := \max\{\text{codim } \mathcal{N}_1(D, X) \mid D \text{ a prime divisor in } X\}.$$

Remark 3.2. Smooth Fano varieties with high Lefschetz defect have been completely described in arbitrary dimension: indeed, X has a rigid geometry when $\delta_X \geq 4$, that is X is the product of Fano varieties of lower dimension (cf. [Cas12, Theorem 3.3]).

In particular, if X is a Fano 4-fold having $\delta_X \geq 4$, then $X \cong S_1 \times S_2$ with S_i del Pezzo surfaces, and applying [Cas12, Example 3.1] we may assume that $\rho_{S_1} = \delta_X + 1$. Then, by Corollary 2.3 and Lemma 2.4 we conclude that X is K -polystable if and only if S_2 is neither isomorphic to \mathbb{F}_1 nor to the blow-up of \mathbb{P}^2 at two points.

Thus, we consider the next case, i.e. Fano 4-folds with $\delta = 3$. The strategy to prove Theorem 1.1 is to compute the β -invariant on a particular divisor that these varieties carry out. We see that this invariant turns out to be negative in many examples, so that we understand when K -polystability fails thanks to Theorem 2.2. Although not necessarily a product, Fano varieties with $\delta = 3$ still have a very explicit description, indeed by [CRS22, Theorem 1.4] they are obtained via two possible constructions that we are going to recall below (cf. [CRS22, §3, §4]).

Let X be a smooth Fano variety with $\delta_X = 3$. Then, there exist a smooth Fano variety T with $\dim T = \dim X - 2$ and a \mathbb{P}^2 -bundle $\varphi: Z \rightarrow T$, such that X is obtained by blowing-up Z along three pairwise disjoint smooth, irreducible, codimension 2 subvarieties S_1, S_2, S_3 ; we will denote by $h: X \rightarrow Z$ the blow-up map and set $\sigma := h \circ \varphi: X \rightarrow T$. The \mathbb{P}^2 -bundle $\varphi: Z \rightarrow T$ is the projectivization of a suitable decomposable vector bundle on T , and S_2 and S_3 are sections of φ . Instead, $\varphi|_{S_1}: S_1 \rightarrow T$ is finite of degree 1 or 2: this yields two distinct constructions depending on the degree of S_1 over T , whenever the degree is 1 we refer to it as Construction A, otherwise we get Construction B.

As a consequence, in [CR22, Theorem 1.1] and [CRS22, Proposition 1.5] we get the complete classification in the case of dimension 4 and $\delta = 3$, as follows. In Theorem 1.1 we are going to analyze the K -polystability for all of these families.

Theorem 3.3. *Let X be a Fano 4-fold with $\delta_X = 3$. Then $5 \leq \rho_X \leq 8$ and there are 19 families for X , among which 14 are toric.*

- If $\rho_X = 8$, then $X \cong F \times F$, where F is the blow-up of \mathbb{P}^2 at 3 non-collinear points;

- if $\rho_X = 7$, then $X \cong F' \times F$, where F' is the blow-up of \mathbb{P}^2 at 2 points;
- if $\rho_X = 6$, there are 11 families for X , among which 8 are toric;
- if $\rho_X = 5$, there are 6 families for X , among which 4 are toric.

Remark 3.4. In view of [CRS22, Remark 6.1], the toric families of Theorem 3.3 are exactly those arising via Construction A. More precisely, they correspond to the products $F \times F$ and $F' \times F$ if $\rho \geq 7$ and, following Batyrev's classification of smooth toric Fano 4-folds and its notation (see [Bat99]), to the toric varieties of type U (eight possible families) if $\rho = 6$, and to the toric varieties of type K (four possible families) if $\rho = 5$. For most of these cases we will use a characterization result on K-polystability for toric varieties (see §4.1). Thus, the most effort will be required by the Fano 4-folds obtained via Construction B, that is the non-toric families. The two non-toric families with $\rho = 5$ have been studied in [CR22, Examples 5.1 and 5.2], while the remaining three families with $\rho = 6$ are described in [CRS22, §7].

3.1. Construction B: relative cone and relative contractions. Construction B is described in [CRS22, §4], we summarize it in the following. We have

$$\varphi: Z \cong \mathbb{P}_T(\mathcal{O}(N) \oplus \mathcal{O} \oplus \mathcal{O}) \rightarrow T,$$

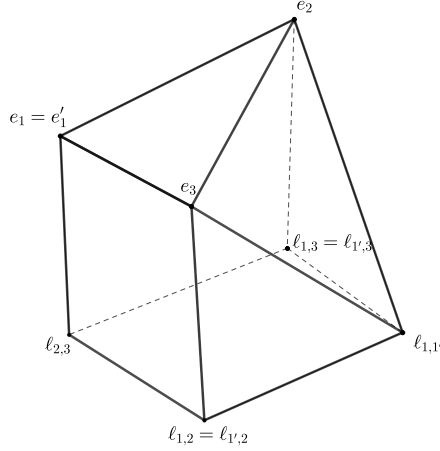
where N is a divisor on T such that $h^0(T, 2N) > 0$ and $-K_T \pm N$ is ample. We denote by H a tautological divisor of Z . Let $D := \mathbb{P}(\mathcal{O} \oplus \mathcal{O}) \hookrightarrow Z$ be the divisor given by the projection $\mathcal{O}(N) \oplus \mathcal{O} \oplus \mathcal{O} \rightarrow \mathcal{O} \oplus \mathcal{O}$, so that $D \cong \mathbb{P}^1 \times T$ and $D \sim H - \varphi^*N$. Let now $S_2, S_3 \subset D$, $S_i \cong \{pt\} \times T \subset D$, be the sections corresponding to the projections $\mathcal{O} \oplus \mathcal{O} \rightarrow \mathcal{O}$, while $\varphi|_{S_1}: S_1 \rightarrow T$ is a double cover ramified along $\Delta \in |2N|$ (see [CRS22, Remarks 4.1, 4.3]). There exists a unique smooth divisor $H_0 \in |H|$ containing S_1 such that $H_0 \cong \mathbb{P}_T(\mathcal{O}(N) \oplus \mathcal{O})$, $H|_{H_0}$ is a tautological divisor, and S_1 is linearly equivalent to $2H|_{H_0}$. Moreover, the surfaces $\{S_1, S_2, S_3\}$ are pairwise disjoint and fiber-wise in general position.

Let $h: X \rightarrow Z$ be the blow-up along $\{S_1, S_2, S_3\}$, and set $\sigma := h \circ \varphi: X \rightarrow T$. We denote by E_i the exceptional divisors over S_i , $i = 1, 2, 3$, and by \tilde{H}_0 and \tilde{D} the strict transforms of H_0 and D in X .

We now recall the description of the relative cone $\text{NE}(\sigma)$ and its elementary contractions, which are all divisorial. The corresponding exceptional divisors will be our key to study the K-polystability of the varieties obtained via Construction B. We refer to [CRS22, §6.3] for details.

Let $t \in T \setminus \Delta$, so that $X_t := \sigma^{-1}(t)$ is a smooth del Pezzo surface of degree 5 and a smooth σ -fiber. Denote by $\{p_1, p'_1, p_2, p_3\} \in Z_t := \varphi^{-1}(t)$ the points blown-up by $h|_{X_t}: X_t \rightarrow Z_t$, where $p_i = S_i \cap Z_t$ for $i = 2, 3$, and $\{p_1, p'_1\} = S_1 \cap Z_t$. The 5-dimensional cone $\text{NE}(X_t)$ is generated by the classes of the ten (-1) -curves in X_t , given by the exceptional curves and the transforms of the lines through two blown-up points. We denote by e_i (respectively e'_1) the exceptional curve over p_i (respectively p'_1), and $\ell_{i,j}$ (respectively $\ell_{1,1'}$, $\ell_{1',i}$ for $i = 2, 3$) the transform of the line $\overline{p_i p_j}$ (respectively $\overline{p_1 p'_1}$, $\overline{p'_1 p_i}$ for $i = 2, 3$). Let $\iota: X_t \hookrightarrow X$ be the inclusion; by [CRS22, Lemma 6.4] one has that every relative elementary contraction of X/T restricts to a non-trivial contraction of X_t , and $\iota_* \text{NE}(X_t) = \text{NE}(\sigma)$.

Figure 1 shows the 3-dimensional polytope obtained as a hyperplane section of the 4-dimensional cone $\text{NE}(\sigma)$, which has 7 extremal rays, and their generators. By [Wiś91, Thm. 1.2] we deduce that every relative elementary contraction of $\text{NE}(\sigma)$ is the blow-up of a smooth variety along a smooth codimension 2 subvariety. The contraction corresponding to $[e_1] = [e'_1]$ (resp. $[e_2], [e_3]$) is the blow-down of E_1 (resp. E_2, E_3), while the contractions

FIGURE 1. A section of $\text{NE}(\sigma)$

corresponding to $[\ell_{1,1'}]$ and $[\ell_{2,3}]$ have respectively exceptional divisors \tilde{H}_0 and \tilde{D} . Moreover, we denote by G_i the exceptional divisor of the contraction corresponding to $[\ell_{1,i}] = [\ell_{1',i}]$ for $i = 2, 3$; by construction, G_i has a \mathbb{P}^1 -bundle structure over S_1 whose fibers are numerically equivalent to $\ell_{1,i}$ and $\ell_{1',i}$ for $i = 2, 3$.

Lastly, we observe that $E_1 \cong G_2 \cong G_3$ and that $E_2 \cong E_3 \cong \tilde{H}_0$.

3.2. Construction B: relations among exceptional divisors. In this section we refer to [Sec23, §3.8.2].

Remark 3.5. By [CRS22, Proposition 6.6], we know that $\sigma: X \rightarrow T$ has three factorizations of the form $X \xrightarrow{h} Z \xrightarrow{\varphi} T$, where $h: X \rightarrow Z$ is the divisorial contraction of $\{E_1, E_2, E_3\}$, $\{G_2, E_3, \tilde{H}_0\}$ or $\{G_3, E_2, \tilde{H}_0\}$, and $Z \xrightarrow{\varphi} T$ is isomorphic to the \mathbb{P}^2 -bundle from §3.1. In fact, there is a \mathbb{Z}_3 -action on the set of σ -exceptional divisors

$$\{E_1, E_2, E_3, \tilde{H}_0, \tilde{D}, G_2, G_3\}$$

induced by an automorphism of a general σ -fiber X_t (see [Dol12, §8.5.4] for the description of $\text{Aut}(X_t)$), which in turn it extends to an automorphism of X over T . This action corresponds to the permutation $(1, 2, 3)$ on the triplets (E_1, G_2, G_3) and (E_2, E_3, \tilde{H}_0) , while \tilde{D} is left invariant.

The symmetry on the σ -exceptional divisors given by the three factorizations of σ allows us, for instance, to deduce computations on E_3 and \tilde{H}_0 from computations on E_2 . This will be a key tool for the computation in §4.2. Moreover, the unique behaviour of \tilde{D} among all σ -exceptional divisors led us to the computation of its β -invariant.

Recall that $H_0 - \varphi^*N \sim D$, $S_1 \subset H_0$ and $S_2, S_3 \subset D$, so that the pull-back h^* and the above Remark yield the following relations among the σ -exceptional divisors:

$$\begin{aligned} \tilde{H}_0 + E_1 - \sigma^*N &\sim \tilde{D} + E_2 + E_3, & E_2 + G_2 - \sigma^*N &\sim \tilde{D} + \tilde{H}_0 + E_3, \\ E_3 + G_3 - \sigma^*N &\sim \tilde{D} + \tilde{H}_0 + E_2. \end{aligned}$$

Moreover, E_2 , E_3 , \tilde{H}_0 and \tilde{D} are \mathbb{P}^1 -bundles over T , while E_1 , G_2 and G_3 are \mathbb{P}^1 -bundles over S_1 . We have that:

- (i) $\tilde{H}_0 \cong \mathbb{P}_T(-K_T \oplus -K_T - N)$ and $-K_X|_{\tilde{H}_0}$ is the tautological divisor; the same holds for E_2 and E_3 .
- (ii) $\tilde{D} \cong \mathbb{P}_T(-K_T - N \oplus -K_T - N)$ and $-K_X|_{\tilde{D}}$ is the tautological divisor.

Note that E_2 , E_3 and \tilde{H}_0 are pairwise disjoint, and that their intersection with \tilde{D} is a section $\{pt\} \times T$ of \tilde{D} . As a divisor in \tilde{D} , this intersections correspond to surjections $\mathcal{O}(-K_T - N) \oplus \mathcal{O}(-K_T - N) \rightarrow \mathcal{O}(-K_T - N)$, while they correspond to the projection $\mathcal{O}(-K_T) \oplus \mathcal{O}(-K_T - N) \rightarrow \mathcal{O}(-K_T - N)$, as a divisor in E_2 , E_3 and \tilde{H}_0 .

Finally, we can write $-K_X$ as

$$-K_X \sim \sigma^*(-K_T + N) + \tilde{H}_0 + 2\tilde{D} + E_2 + E_3. \quad (1)$$

4. PROOF OF THEOREM 1.1

In this section we show Theorem 1.1. We keep the notation introduced in the previous section. The case $\delta \geq 4$ has been explained in Remark 3.2, thus from now on we consider the case $\delta = 3$.

4.1. Toric case. We recall from Theorem 3.3 that there are 14 families of toric Fano 4-folds with $\delta = 3$, and from Remark 3.4 that all of them arise via Construction A. The aim of this section is to deduce which ones among them are K-polystable. Our conclusion will be the following:

Proposition 4.1. *Let X be a toric Fano 4-fold with $\delta_X = 3$. Then it is K-polystable if and only if it is one of the following varieties:*

- $X \cong \mathbb{P}^2 \times F$, where F is the blow-up of \mathbb{P}^2 along three non-collinear points;
- $X \cong \mathbb{P}^1 \times \mathbb{P}^1 \times F$;
- X is the blow-up of $\mathbb{P}^1 \times \mathbb{P}_{\mathbb{P}^1 \times \mathbb{P}^1}(\mathcal{O} \oplus \mathcal{O}(1, -1))$ along two surfaces isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$.
- $X \cong F \times F$.

In order to prove the above result, we recall that Gorenstein toric Fano varieties correspond to reflexive lattice polytopes, that is those for which the dual is also a lattice polytope. We will make use of the following characterization of K-polystability for toric Fano varieties.

Lemma 4.2. [Ber16, Corollary 1.2] *Let X_P be a toric Fano variety associated to a reflexive polytope P . Then, X_P is K-polystable if and only if 0 is the barycenter of P .*

In the following proof we follow Batyrev's notation [Bat99] for the toric Fano 4-folds of Theorem 3.3 with $\rho = 5, 6$: type K for varieties of Theorem 3.3 having $\rho = 5$, and type U for the ones with $\rho = 6$.

Proof of Proposition 4.1. Assume that X is a product of surfaces. If $\rho_X = 5$, then $X = K_4 \cong \mathbb{P}^2 \times F$ is K-polystable by Corollary 2.3 and Lemma 2.4. If $\rho_X = 6$, then either $X = U_4 \cong \mathbb{F}_1 \times F$ or $X = U_5 \cong \mathbb{P}^1 \times \mathbb{P}^1 \times F$, and applying the same results we deduce that among them only U_5 is K-polystable. For the same reason, and by Theorem 3.3, we deduce that X is not K-polystable if $\rho_X = 7$, while it is K-polystable if $\rho_X = 8$, namely if $X \cong F \times F$.

Assume now that X is not a product of surfaces. In view of Lemma 4.2 we are left to check whether 0 corresponds to the barycenter of the polytopes corresponding to the remaining varieties of our classification. To this end, we use the Graded ring database (see [BK]), giving the invariants of these varieties (computed in [Bat99]) as inputs. It turns out that among them, the only K -polystable variety is U_8 , that is the blow-up of $\mathbb{P}^1 \times \mathbb{P}_{\mathbb{P}^1 \times \mathbb{P}^1}(\mathcal{O} \oplus \mathcal{O}(1, -1))$ along two surfaces isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. ■

4.2. Non-toric case. The purpose of this section is to prove that among the five possible families of non-toric Fano 4-folds having $\delta = 3$ (see Theorem 3.3 and Remark 3.4), only one is K -polystable. More precisely, after our discussion we will deduce the following:

Proposition 4.3. *Let X be a non-toric Fano 4-fold having $\delta_X = 3$. Then X is K -polystable if and only if $X \cong \mathbb{P}^1 \times Y$, where Y is the blow-up of \mathbb{P}^3 along a disjoint union of a line and a conic, and along two non-trivial fibers of the exceptional divisor over the blown-up line.*

We recall by Remark 3.4 that all non-toric Fano 4-folds of Theorem 3.3 arise from Construction B. In particular, the variety X of Proposition 4.3 is obtained via this construction, taking $T \cong \mathbb{P}^1 \times \mathbb{P}^1$ and $N \in |\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(0, 1)|$. In order to prove Proposition 4.3, the first objective is to compute the β -invariant of \tilde{D} (see §3.1), and to this end we will show the following result.

Proposition 4.4. *Set $a = -K_X^4$; $b = N^2$; $c = (-K_T - N)^2$; $d = N \cdot (-K_T - N)$; $e = (-K_T + N)^2$; $f = N \cdot (-K_T + N)$. Then:*

$$\beta(\tilde{D}) = \frac{1}{a} \left(\frac{2}{5}b + 8c + 6d - 4e + 4f \right).$$

We start with some preliminary computations that follow from §3.1 and [Har77, Appendix A]; we will use these to prove the lemmas below.

Remark 4.5. Recall from Remark 3.5 that there is a symmetry among the exceptional divisors $\{E_2, E_3, \tilde{H}_0\}$. Denote by η a tautological divisor of $\mathbb{P}_T(N \oplus N)$ and by ξ a tautological divisor of $\mathbb{P}_T(N \oplus \mathcal{O})$. Then,

- $\tilde{D}|_{\tilde{D}} = -\eta$, so that $\tilde{D}^3 \sim \eta \cdot \sigma_{|\tilde{D}}^*(2N) - \sigma_{|\tilde{D}}^*(N)^2$ and $\tilde{D}^4 = -3N^2$;
- $\tilde{H}_0|_{\tilde{H}_0} = -\xi$, so that $(\tilde{H}_0)^3 \sim \xi \cdot \sigma_{|E_2}^*(N)$ and $(\tilde{H}_0)^4 = -N^2$; the same holds for E_2 and E_3 ;
- $-K_X|_{\tilde{D}} \sim \eta + \sigma_{|\tilde{D}}^*(-K_T - 2N)$;
- $(-K_X|_{\tilde{D}})^2 \sim -K_X|_{\tilde{D}} \cdot \sigma_{|\tilde{D}}^*(-2K_T - 2N) - \sigma_{|\tilde{D}}^*(-K_T - N)^2$;
- $-K_X|_{\tilde{H}_0} \sim \xi + \sigma_{|\tilde{H}_0}^*(-K_T - N)$; the same holds for E_2 and E_3 ;
- $(\tilde{H}_0)^2 \cdot \tilde{D} \sim 0$ and $(\tilde{H}_0)^3 \cdot \tilde{D} = 0$; the same holds for E_2 and E_3 ;
- $(\sigma^*M)^i \sim 0$ for all $M \in \text{Pic}(T)$ and $i = 3, 4$.

Notice that in all the examples of Fano 4-folds with $\delta = 3$ obtained via Construction B, the divisor N is nef (see [CRS22, §7]), therefore η and ξ are nef as well.

We will first describe the Zariski decomposition of the divisor $-K_X - t\tilde{D}$, where $t \geq 0$.

Lemma 4.6. *The restriction of $-K_X - t\tilde{D}$ to \tilde{H}_0 , E_2 and E_3 is nef for $0 \leq t \leq 1$, while $(-K_X - t\tilde{D})|_{\tilde{D}}$ is nef for $t \geq 0$.*

Proof. Recall that by construction, $-K_T \pm N$ is an ample divisor on T , so $-K_T + sN$ is ample for $-1 \leq s \leq 1$. Thus, $(-K_X - t\tilde{D})|_{\tilde{H}_0} \sim (1-t)\xi + \sigma_{|\tilde{H}_0}^*(-K_T + (t-1)N)$ is nef for $t \leq 1$. Similarly, $(-K_X - t\tilde{D})|_{\tilde{D}} \sim (1+t)(\eta - \sigma_{|\tilde{D}}^*N) + \sigma_{|\tilde{D}}^*(-K_T + (t-1)N)$; the claim follows since $\eta - \sigma_{|\tilde{D}}^*N$ is the tautological divisor of $\mathbb{P}_T(\mathcal{O} \oplus \mathcal{O})$, $-K_T - N$ is ample, and N is nef. \blacksquare

Remark 4.7. Let Γ be an irreducible curve not contained in $\tilde{H}_0 \cup E_2 \cup E_3 \cup \tilde{D}$. If Γ is contracted by σ , then by construction $\tilde{H}_0 \cdot \Gamma, E_2 \cdot \Gamma, E_3 \cdot \Gamma, \tilde{D} \cdot \Gamma \geq 0$ and at least one inequality is strict.

Lemma 4.8. *The divisor $-K_X - t\tilde{D}$ is nef for $0 \leq t \leq 1$.*

Proof. Assume $t > 1$. Let ℓ be a fiber of the restriction $\sigma_{|\tilde{H}_0} : \tilde{H}_0 \rightarrow T$. Since ℓ is a fiber of the exceptional divisor of a smoth blow-up (see §2.4), one has $(-K_X - t\tilde{D}) \cdot \ell = 1 - t < 0$. Thus, we may assume $t \leq 1$. If Γ is an irreducible curve contained in $\tilde{H}_0 \cup E_2 \cup E_3 \cup \tilde{D}$, then $(-K_X - t\tilde{D}) \cdot \Gamma \geq 0$ by Lemma 4.6. Otherwise, using equation (1) in §3.2, one has

$$(-K_X - t\tilde{D}) \cdot \Gamma = [\sigma^*(-K_T + N) + (2-t)\tilde{D} + (\tilde{H}_0 + E_2 + E_3)] \cdot \Gamma > 0,$$

and this follows from Remark 4.7 and the ampleness of $-K_T + N$. \blacksquare

Lemma 4.9. *The divisor $-K_X - \tilde{D}$ is a supporting divisor of the birational contraction $X \rightarrow W$ associated to the facet $\langle [e_2], [e_3], [l_{1,1'}] \rangle$ of $\text{NE}(\sigma)$.*

Proof. By [CRS22, Remark 6.7] we know that the contraction $X \rightarrow W$ is divisorial, and we show that $-K_X - \tilde{D}$ is a supporting divisor. From Lemma 4.8 and its proof, one has that $-K_X - \tilde{D}$ is nef and not ample, and the curves on which it vanishes are contained in $\tilde{H}_0 \cup E_2 \cup E_3 \cup \tilde{D}$. Furthermore, we see from the proof of Lemma 4.6 that $-K_X - \tilde{D}$ has zero intersection only with the fibers of σ that are contained in $\tilde{H}_0 \cup E_2 \cup E_3$. This gives the claim. \blacksquare

The following result is a consequence of the above lemmas and of the discussion done in §2.2.

Proposition 4.10. *The Zariski decomposition of the divisor $-K_X - t\tilde{D}$ is given by*

$$P(t) = \begin{cases} -K_X - t\tilde{D}, & t \in [0, 1] \\ H(t), & t \in (1, 2] \end{cases}$$

where $H(t) = [\sigma^*(-K_T + N) + (2-t)\tilde{D}] + (2-t)(\tilde{H}_0 + E_2 + E_3)$, and the pseudoeffective threshold of \tilde{D} with respect to $-K_X$ is $\tau(\tilde{D}) = 2$.

Proof. In view of Lemma 4.8, we are left to understand the decomposition of $-K_X - t\tilde{D}$ into positive and negative part for $t \geq 1$. By Lemma 4.9, being $\tilde{H}_0 \cup E_2 \cup E_3$ the exceptional locus of the divisorial contraction having $-K_X - \tilde{D}$ as a supporting divisor, we need to determine $a, b, c \geq 0$ and all values of $t \geq 1$ such that

$$P(t) = -K_X - t\tilde{D} - a\tilde{H}_0 - bE_2 - cE_3.$$

Denote by h_0, e_2, e_3 the fibers of σ contained respectively in \tilde{H}_0, E_2 , and E_3 . Requiring that $-K_X - t\tilde{D} - a\tilde{H}_0 - bE_2 - cE_3$ has zero intersection with h_0, e_2, e_3 , we get $a = b = c = t - 1$. Set $H(t) := -K_X - t\tilde{D} - (t - 1)(\tilde{H}_0 + E_2 + E_3)$. By equation (1) in §3.2, we deduce that

$$H(t) = [\sigma^*(-K_T + N) + (2 - t)\tilde{D}] + (2 - t)(\tilde{H}_0 + E_2 + E_3).$$

Let Γ be a fiber of σ contained in \tilde{D} , so that $H(t) \cdot \Gamma = 2(2 - t)$; thus, $H(t)$ is not nef for $t > 2$. Finally, we see that $H(t)$ is nef for $t \leq 2$, and this follows from Remark 4.7 and the ampleness of $-K_T + N$.

Since $H(2) \sim \sigma^*(-K_T + N)$ is a nef and not big divisor, we deduce that $\tau(\tilde{D}) = 2$, hence our claim. \blacksquare

Finally, we will use the following lemmas to compute $S(\tilde{D})$ (see §2.1 for its definition).

Lemma 4.11. *Notation as in Proposition 4.4. Then,*

$$\int_0^1 (-K_X - t\tilde{D})^4 = a - \frac{8}{5}b - 8c - 6d.$$

Proof. We compute $(-K_X - t\tilde{D})^4$. By Remark 4.5, we have:

- $-K_X^3 \cdot \tilde{D} = (-K_X|_{\tilde{D}})^3 = 3(-K_T - N)^2$;
- $-K_X^2 \cdot \tilde{D}^2 = (-K_X|_{\tilde{D}})^2 \cdot (-\eta) = -(-K_T - N)^2 - 2N \cdot (-K_T - N)$;
- $-K_X \cdot \tilde{D}^3 = (-K_X|_{\tilde{D}}) \cdot \eta^2 = 2N \cdot (-K_T - N) + N^2$.

Therefore,

$$(-K_X - t\tilde{D})^4 = a - 12ct - 6(c + 2d)t^2 - 4(b + 2d)t^3 - 3bt^4$$

and the claim follows. \blacksquare

Lemma 4.12. *Notation as in Proposition 4.4. Then,*

$$\int_1^2 H(t)^4 = \frac{6}{5}b - 4f + 4e.$$

Proof. We recall that $H(t) = [\sigma^*(-K_T + N) + (2 - t)\tilde{D}] + (2 - t)(\tilde{H}_0 + E_2 + E_3)$. In order to obtain $H(t)^4$, we compute the intersections

$$(2 - t)^i [\sigma^*(-K_T + N) + (2 - t)\tilde{D}]^{4-i} \cdot (\tilde{H}_0 + E_2 + E_3)^i,$$

for $i = 0, \dots, 4$. We use Remark 4.5 for the following computations.

Step 1. $[\sigma^*(-K_T + N) + (2 - t)\tilde{D}]^4 = -6e(2 - t)^2 + 8f(2 - t)^3 - 3b(2 - t)^4$.

Indeed:

- $\sigma^*(-K_T + N)^2 \cdot \tilde{D}^2 = \sigma_{|\tilde{D}}^*(-K_T + N)^2 \cdot (-\eta) = -(-K_T - N)^2$;
- $\sigma^*(-K_T + N) \cdot \tilde{D}^3 = \sigma_{|\tilde{D}}^*(-K_T + N) \cdot \eta^2 = 2N \cdot (-K_T + N)$.

Step 2. $[\sigma^*(-K_T + N) + (2 - t)\tilde{D}]^3 \cdot (\tilde{H}_0 + E_2 + E_3) = 9e(2 - t) - 9f(2 - t)^2 + 3b(2 - t)^3$.

Recall that $\tilde{H}_0 \cap \tilde{D}$ is a section $\{pt\} \times T$ of \tilde{D} . By restricting to \tilde{D} we obtain:

- $\sigma^*(-K_T + N)^2 \cdot \tilde{D} \cdot (\tilde{H}_0 + E_2 + E_3) = 3(-K_T + N)^2$;
- $\sigma^*(-K_T + N) \cdot \tilde{D}^2 \cdot (\tilde{H}_0 + E_2 + E_3) = -3N \cdot (-K_T + N)$;
- $\tilde{D}^3 \cdot (\tilde{H}_0 + E_2 + E_3) \cdot 3N^2$.

Step 3. $[\sigma^*(-K_T + N) + (2 - t)\tilde{D}]^2 \cdot (\tilde{H}_0 + E_2 + E_3)^2 = -3e.$

Recall that E_2 , E_3 and \tilde{H}_0 are pairwise disjoint. Thus:

- $\sigma^*(-K_T + N)^2 \cdot [(\tilde{H}_0)^2 + (E_2)^2 + (E_3)^2] = -3(-K_T + N)^2;$
- $\sigma^*(-K_T + N) \cdot \tilde{D} \cdot [(\tilde{H}_0)^2 + (E_2)^2 + (E_3)^2] = 0;$
- $\tilde{D}^2 \cdot [(\tilde{H}_0)^2 + (E_2)^2 + (E_3)^2] = 0.$

Step 4. $[\sigma^*(-K_T + N) + (2 - t)\tilde{D}] \cdot (\tilde{H}_0 + E_2 + E_3)^3 = 3f.$

Indeed:

- $\sigma^*(-K_T + N) \cdot [(\tilde{H}_0)^3 + (E_2)^3 + (E_3)^3] = 3N \cdot (-K_T + N);$
- $\tilde{D} \cdot [(\tilde{H}_0)^3 + (E_2)^3 + (E_3)^3] = 0.$

We conclude that

$$H(t)^4 = 6b(2 - t)^4 - 16f(2 - t)^3 + 12e(2 - t)^2,$$

and the claim follows. ■

Proof of Proposition 4.4. We compute $\beta(\tilde{D}) = A(\tilde{D}) - S(\tilde{D})$. Since $\tilde{D} \subset X$ is a prime divisor on X , we have $A(\tilde{D}) = 1$. Moreover, due to Proposition 4.10, we can compute

$$a \cdot S(\tilde{D}) = \int_0^2 \text{vol}(-K_X - t\tilde{D}) dt$$

by splitting it as

$$\int_0^1 (-K_X - t\tilde{D})^4 dt + \int_1^2 H(t)^4 dt.$$

Thus, the claim follows from Lemma 4.11 and Lemma 4.12. ■

We now apply Proposition 4.4 to conclude this section with the proof of Proposition 4.3. We keep the notation of such proposition.

Proof of Proposition 4.3. Assume that X is a product. Then by the classification of Fano 4-folds having $\delta = 3$ (see Theorem 3.3 and Remark 3.4) it follows that $X \cong \mathbb{P}^1 \times Y$ with Y being as in the statement. By [ACC⁺23, §5.23] we know that Y is K-polystable, then using Lemma 2.4 we conclude that X is K-polystable.

Suppose now that X is not a product. We will observe that for all the remaining four families of varieties of our classification, one has $\beta(\tilde{D}) < 0$ so that we conclude by Theorem 2.2 that they are not K-semistable, hence they are not K-polystable. Being $a > 0$, in view of Proposition 4.4 we are left to show that $\lambda := \frac{2}{5}b + 8c + 6d - 4e + 4f < 0$.

Assume first that $\rho_X = 5$. By construction B, one has $T = \mathbb{P}^2$ and by the proof of [CR22, Theorem 1.3] we know that either $N = \mathcal{O}(1)$ or $N = \mathcal{O}(2)$. Using the needed numerical invariants of the corresponding varieties computed in [CR22, Table 3.4], in the first case one can check that $\lambda = -\frac{18}{5}$, in the second case we get $\lambda = -\frac{192}{5}$.

Assume now that $\rho_X = 6$. Construction B gives either $T = \mathbb{F}_1$ or $T = \mathbb{P}^1 \times \mathbb{P}^1$. In the first case, by the proof of [CRS22, Proposition 7.1] we know that $N = \pi^*L$ where $\pi: \mathbb{F}_1 \rightarrow \mathbb{P}^2$ is the blow-up, and L general line in \mathbb{P}^2 . For this variety, using the numerical invariants of [CRS22, Table 7.1] one gets $\lambda = -\frac{38}{5}$. Otherwise, by the proof of the same proposition we have $N = \mathcal{O}(1, 1)$, and we obtain that $\lambda = -\frac{96}{5}$, hence our claim. ■

4.3. Conclusions and final table. We obtain the proof of Theorem 1.1 as a direct consequence of Remark 3.2, of the classification theorem of Fano 4-folds having $\delta = 3$ (see Theorem 3.3, Remark 3.4) and Propositions 4.1, 4.3. We summarize our results in the following tables: Table 1 gathers all Fano 4-folds with $\delta = 3$, while Fano 4-folds with $\delta \geq 4$ appear in Table 2.

The notation in the tables is as follows. In the first column we use the description of Construction B from §3.1 for the non-toric Fano 4-folds with $\delta = 3$, while we use the notation in [Bat99] for the toric case when $\delta = 3$ and $\rho = 5, 6$, explicitly showing which 4-folds are product of surfaces. The second column contains the Picard number ρ , while in the last column with the symbol \checkmark (resp. \times) we mean that the 4-fold is K-polystable (resp. not K-polystable). Table 1 contains an extra column, where we write whether $\beta(\tilde{D})$ is positive (+ve) or negative (-ve), when applicable. Finally, we recall that F' (resp. F) is the blow-up of \mathbb{P}^2 along two (resp. three non-collinear) points.

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TABLE 1. K-polystability of Fano 4-folds with $\delta = 3$

| 4-fold | ρ | $\beta(\tilde{D})$ | K-polystable |
|---|--------|--------------------|--------------|
| Non-toric | | | |
| $T = \mathbb{P}^2, N = \mathcal{O}(1)$ | 5 | -ve | \times |
| $T = \mathbb{P}^2, N = \mathcal{O}(2)$ | 5 | -ve | \times |
| $T = \mathbb{P}^1 \times \mathbb{P}^1, N = \mathcal{O}(0, 1)$ | 6 | +ve | \checkmark |
| $T = \mathbb{P}^1 \times \mathbb{P}^1, N = \mathcal{O}(1, 1)$ | 6 | -ve | \times |
| $T = \mathbb{F}_1, N = \pi^*L$ | 6 | -ve | \times |
| Toric | | | |
| K_1 | 5 | - | \times |
| K_2 | 5 | - | \times |
| K_3 | 5 | - | \times |
| $K_4 \cong \mathbb{P}^2 \times F$ | 5 | - | \checkmark |
| U_1 | 6 | - | \times |
| U_2 | 6 | - | \times |
| U_3 | 6 | - | \times |
| $U_4 \cong \mathbb{F}_1 \times F$ | 6 | - | \times |
| $U_5 \cong \mathbb{P}^1 \times \mathbb{P}^1 \times F$ | 6 | - | \checkmark |
| U_6 | 6 | - | \times |
| U_7 | 6 | - | \times |
| U_8 | 6 | - | \checkmark |
| $F' \times F$ | 7 | - | \times |
| $F \times F$ | 8 | - | \checkmark |

TABLE 2. K-polystability of Fano 4-folds with $\delta \geq 4$

| 4-fold | ρ | K-polystable |
|--|-------------------------|--------------|
| $X = S \times T$ $\rho_T \leq \delta_X + 1, T \not\cong \mathbb{F}_1, F'$ | $\delta_X + \rho_T + 1$ | \checkmark |
| $X = S \times \mathbb{F}_1$ | $\delta_X + 3$ | \times |
| $X = S \times F'$ | $\delta_X + 4$ | \times |

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