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The Two Extremal Rays of Some Hyper–Kähler Fourfolds

Federica Galluzzi[1] | Bert van Geemen[2]

[1]Dipartimento di Matematica, Università di Torino, Torino, Italy | [2]Dipartimento di Matematica, Università di Milano, Milano, Italy

Correspondence: Federica Galluzzi (federica.galluzzi@unito.it)

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ABSTRACT

We consider projective Hyper–Kähler manifolds of dimension 4 that are deformation equivalent to Hilbert squares of K3 surfaces. In case such a manifold admits a divisorial contraction, the exceptional divisor is a conic bundle over a K3 surface. A classification of lattice embeddings implies that there are five types of such conic bundles. In case the manifold has Picard rank 2 and has two (birational) divisorial contractions, we determine the types of these conic bundles. There are exactly seven cases. For the Fano varieties of lines of cubic fourfolds there are only four cases and we provide examples of these.

1 | Introduction

The divisors on a four-dimensional projective Hyper–Kähler (HK) manifold X that can be contracted provide interesting information on X itself. In this paper, we will assume that X is also of $\mathrm{K3}^{[2]}$-type, that is, it is a deformation of the Hilbert scheme of degree 2 zero-cycles on a K3 surface, and that the Picard rank of X is two. Then such a divisor E is a $\mathbf{P}^1$-fibration, so it is a conic bundle, over a projective K3 surface S of Picard rank 1. The fibration on E is induced by the contraction of this divisor. The fourfold X is then a moduli space of possibly twisted sheaves on S [1, Theorem 3.3]. This implies that the Hodge structures on $H^2(X,\mathbf{Z})$ and $H^2(S,\mathbf{Z})$ are closely related.

The conic bundles on a K3 surface of Picard rank 1 which are isomorphic to such exceptional divisors were classified in [1] and we recall this in Section 2. A main ingredient is the classification, due to Debarre and Macrì, of embeddings of the rank 2 lattice $\langle 2d\rangle \oplus \langle -2\rangle$ into the lattice defined by $H^2(X,\mathbf{Z})$ and its BBF-form. An exceptional divisor defines an isometry class of such embeddings and there are five classes which we describe now.

A well-known case is $E = \mathbf{P}\mathcal{T}_S$, the projectivization of the tangent bundle of a K3 surface S. Then $X \cong S^{[2]}$, the Hilbert square of S. This type of contraction is denoted by H in this paper. There are two other cases where $E = \mathbf{P}\mathcal{V}$ for a rank 2 vector bundle $\mathcal{V}$ on S. Then $\mathcal{V}$ is a rigid, stable, vector bundle, called a Mukai bundle. Such rank 2 bundles exist only if $h^2 = 2g-2$ with g even, where $\mathrm{Pic}(S) = \mathbf{Z}h$. We denote these cases by M_i with $i = 1,3$ according to $h^2 \equiv 2i \mod 8$. Even if the lattice embeddings of type M_1 and M_3 are quite different, the only notable difference in the exceptional divisors is the genus of the K3 surface that is the base of the conic bundle. Finally there are two cases where the Brauer class of the conic bundle is nontrivial. In that case, $E = \mathbf{P}\mathcal{U}$ for a rigid α-twisted locally free sheaf $\mathcal{U}$ of rank 2 and the Brauer class $\alpha \in Br(S)_2$ has order 2. These cases are denoted by B_i with $i = 0,1$ according to the intersection number $Bh \equiv i/2 \mod \mathbf{Z}$ where $B \in H^2(S,\mathbf{Q})$ is a B-field representative of α.

An exceptional divisor on X defines an extremal ray in $\mathrm{Pic}(X)_{\mathbf{R}} \cong \mathbf{R}^2$, which is a boundary component of the movable cone of X. The movable cone has two boundary components. The other extremal ray is either generated by an isotropic vector, in which case it defines a Lagrangian fibration, or it defines another divisorial contraction of an HK manifold birational to X. We recall the descriptions of the movable cone and the birational models of X in Section 3.

In this paper, we are interested in the types of the pairs of divisorial contractions corresponding to the two extremal rays. A priori, given the five types of exceptional divisor, there could be 15 types of such pairs. However there are only seven cases.

Theorem 1.1 (See Theorem 4.1). *Let X be a projective HK fourfold of $K3^{[2]}$ type with Picard rank 2 and assume that X has two divisorial contractions. Then the types of the two exceptional divisors are* $\mathrm{T}+\mathrm{T}$, *where* T *is any of the five types,* $\mathrm{H}+\mathrm{M}_3$, *or* $\mathrm{H}+\mathrm{B}_1$.

Notice that $\mathrm{H}+\mathrm{M}_1$ cannot occur, illustrating the difference in types M_1 and M_3. The types depend on rather subtle properties of the Picard lattice of X, like the divisibilities in $H^2(X, \mathbf{Z})$ of certain divisor classes associated to the contractions and the minimal solution of a Pell equation associated to this lattice. In the two "mixed" types $\mathrm{H}+\mathrm{M}_3$ and $\mathrm{H}+\mathrm{B}_1$, the presence of an H implies that X is birational to a Hilbert square $S^{[2]}$ for a K3 surface S.

For such a Hilbert square, if there is a second exceptional divisor E besides the projectivized tangent bundle, then E seems to play a rather important role. In case S has degree 14 or 30, the types are $\mathrm{H}+\mathrm{M}_3$ (see Sections 5.6 and 4.3) and the conic bundle E of type M_3 can be found in the literature, where also ad hoc geometrical constructions of embeddings of E in a Hilbert square are given. However, our result that such a conic bundle must be isomorphic to an exceptional divisor on a Hilbert square may simplify and clarify the constructions.

In Section 5, we consider the special, but important, case that X is birational to the Fano variety of lines of a cubic fourfold. In that case, X can have only four types of contractions.

Theorem 1.2 (See Theorem 5.4). *Let $F = F(Y)$ be the Fano variety of lines of a smooth cubic fourfold Y and assume that F has Picard rank 2.*

If F has a divisorial contraction, then its type is H, M_3, *or* B_1 *and if F has two divisorial contractions, then the types are T+T, with T one of these three, or H+*M_3.

We discuss the relation between the exceptional divisor and families of scrolls in the cubic fourfold, already explored by Hassett and Tschinkel [2] with an emphasis on the Mori cone of curves, in terms of divisor classes. We conclude with the well-known case of pfaffian cubic fourfolds. Their Fano varieties are Hilbert squares $S^{[2]}$ and the degree of S is 14. They have a contraction of type H and the second exceptional divisor is of type M_3. As mentioned above, the corresponding conic bundle (which embeds as a scroll, a family of lines parameterised by S, in $\mathbf{P}^5$) was known and it was also shown to have an embedding in $S^{[2]}$. The identification of this conic bundle with ("the second") exceptional divisor seems to be new.

2 | Exceptional Divisors on $K3^{[2]}$ Manifolds

2.1 | HK Fourfolds of $K3^{[2]}$ Type

An HK variety is a simply connected smooth complex manifold admitting a unique (up to scalar) non-degenerate holomorphic two-form. These varieties are also called irreducible holomorphic symplectic varieties. An HK fourfold X is of $K3^{[2]}$ type if it is deformation equivalent to the Hilbert scheme of length 2, zero-dimensional, subschemes of a K3 surface. The second cohomology group $H^2(X, \mathbf{Z})$ is equipped with a quadratic form q, the Beauville–Bogomolov–Fujiki (BBF) form, we write (x, y) for the associated bilinear form, so $q(x) = (x, x)$.

From now on we assume that X is of $K3^{[2]}$ type. Then the BBF form is related to the intersection product of divisors by the following relations,

$$D^4 = 3q(D)^2 = 3(D, D)^2, \qquad D^3D' = 3(D, D)(D, D') \qquad (D, D' \in \mathrm{Pic}(X)).$$

There is an isomorphism of lattices

$$H^2(X,\mathbf{Z}) \cong \Lambda_{\mathrm{K3}^{[2]}} = \Lambda_{\mathrm{K3}} \oplus \mathbf{Z}\delta = U^3 \oplus E_8(-1)^2 \oplus \langle -2\rangle\,,$$

where U is the hyperbolic plane and $E_8(-1)$ is the unique unimodular, negative definite, even lattice of rank 8 and $\Lambda_{\mathrm{K3}} \cong U^3 \oplus E_8(-1)^2$ is the K3 lattice, which is isomorphic to $H^2(S,\mathbf{Z})$ with its intersection form for any K3 surface S.

The divisibility $\mathrm{div}_{\Lambda_{\mathrm{K3}^{[2]}}}(v)$ of an element $v \in \Lambda_{\mathrm{K3}^{[2]}}$ is the positive integer such that

$$\{(v,w) : w \in \Lambda_{\mathrm{K3}^{[2]}}\} = \mathrm{div}_{\Lambda_{\mathrm{K3}^{[2]}}}(v)\mathbf{Z}\,.$$

Any primitive $v \in \Lambda_{\mathrm{K3}} \subset \Lambda_{\mathrm{K3}^{[2]}}$ has divisibility one since the K3 lattice is unimodular and hence is self-dual. The divisibility of a primitive element in $H^2(X,\mathbf{Z}) \cong \Lambda_{\mathrm{K3}^{[2]}}$ is either one or two [3, Example 3.11].

2.2 | Contractions

In the remainder of the paper, X will be a projective HK manifold of $\mathrm{K3}^{[2]}$ type with Picard group of rank 2. We recall the basic results on contractions of X which can be found in [2] and [1, section 3].

If there exists a divisorial contraction $f : X \to Y$, then the exceptional divisor E is smooth and irreducible. The class of E in $\mathrm{Pic}(X)$ is either τ or 2τ where $\tau \in \mathrm{Pic}(X)$ is a (-2)-class, that is, $q(\tau) = -2$.

The perpendicular of E in $\mathrm{Pic}(X)$ is generated by a big and nef divisor class H. A multiple of H defines the contraction, so one can identify f with the map defined by nH for some large n. The contraction f defines a positive integer d by $q(H) = 2d$. The Picard lattice then contains the sublattice

$$\mathbf{Z}H \oplus \mathbf{Z}\tau \cong \langle 2d\rangle \oplus \langle -2\rangle \qquad (\subset \mathrm{Pic}(X))\,.$$

This sublattice is either equal to $\mathrm{Pic}(X)$ or it has index 2 in it, correspondingly one has $|\det(\mathrm{Pic}(X))| = 4d$ or d [1, table 1].

The classification of the contractions relies on the classification, due to Debarre and Macrì of the embeddings of lattices

$$\mathbf{Z}H \oplus \mathbf{Z}\tau \hookrightarrow \Lambda_{\mathrm{K3}^{[2]}}\,.$$

In [4] and [5, Theorem 3.32], it is shown that there are five classes of embeddings.

2.3 | Conic Bundles

In [1, Theorem 3.3], it is shown, with case by case computations [1, Propositions 3.4–3.8], that the type of the embedding $\mathbf{Z}H \oplus \mathbf{Z}\tau \hookrightarrow \Lambda_{\mathrm{K3}^{[2]}}$ determines a moduli space of (possibly twisted) sheaves $M_v(S,B)$ on a K3 surface S which is birational to X. The Mukai vector v and the classes of H, E in $\mathrm{Pic}(M_v(S,B)) \cong \mathrm{Pic}(X)$ are explicitly determined [1, table 1]. The results of Bayer and Macrì on Bridgeland stability conditions and wall-crossings for such moduli spaces (see [6]) imply that E is isomorphic to $\mathbf{P}\mathcal{V}$, for a rank 2, locally free twisted sheaf on S. The contraction f restricts to the bundle map $p : E = \mathbf{P}\mathcal{V} \to S$ and f is an isomorphism on $X - E \cong Y - S$

$$f : X \longrightarrow Y, \qquad p := f_{|E} : E \longrightarrow S\,(\subset Y)\,.$$

In [1, Theorem 4.2, Corollary 4.3], S and the sheaf $\mathcal{V}$ are determined using [1, table 1]. The Picard rank of S is one and we write $\mathrm{Pic}(S) = \mathbf{Z}h_S$ where h_S is ample. The self-intersection number h_S^2 turns out to be either $q(H)$ or $q(H)/4$ [1, table 1].

A conic bundle $E \to S$ defines a two-torsion Brauer class [1, section 1.2], denoted by $\alpha \in Br(S)_2$, where

$$Br(S) := H^2(S,\mathcal{O}_S^\times)_{tors} \cong H^2(S,\mathbf{Q})/(\mathrm{Pic}(S)_{\mathbf{Q}} \oplus H^2(S,\mathbf{Z}))\,.$$

A Brauer class thus has a representative in $H^2(S,\mathbf{Q})$, called a B-field, and in particular a two-torsion Brauer class has a representative $B \in \frac{1}{2}H^2(S,\mathbf{Z})$. The intersection number $h_S B \in \frac{1}{2}\mathbf{Z}/\mathbf{Z}$ is an invariant of the Brauer class and, only in the

TABLE 1 | Divisorial contractions.

Type	H	M_1	M_3	B_1	B_0
$\mathrm{div}_{H^2(X,\mathbf{Z})}(H)$	1	1	2	1	1
$\mathrm{div}_{H^2(X,\mathbf{Z})}(\tau)$	2	1	1	1	1
$q(H) = 2d, \quad d \equiv$	Any	$1 \mod 4$	$3 \mod 4$	$0 \mod 4$	Any
$\lvert\det(\mathrm{Pic}(X))\rvert$	$4d$	d	$4d$	$4d$	$4d$
$T(X)$	$T(S)$	$T(S)$	$T(S)$	$T(S,\alpha)$	$T(S,\alpha)$
h_S^2	$2d$	$2d$	$2d$	$d/2$	$2d$
Invariants	0	0	0	$h_SB = \frac{1}{2},\ B^2 = \frac{1}{2}$	$h_SB = 0,\ B^2 = \frac{1}{2}$

case that $4Bh_S + h_S^2 \equiv 0 \mod 4$, also $B^2 \in \frac{1}{2}\mathbf{Z}/\mathbf{Z}$ is an invariant [1, section 2.1]). The invariant of the Brauer classes of the exceptional divisors are given in Table 1.

The transcendental lattice of S is $T(S) := \mathrm{Pic}(S)^{\perp}$. A nontrivial Brauer class $\alpha \in Br(S)_2$ corresponds to a surjective homomorphism $\alpha : T(S) \to \frac{1}{2}\mathbf{Z}/\mathbf{Z}$ given by $t \mapsto Bt$ [1, section 2.2]. It follows that the kernel of this homomorphism, denoted by $T(S,\alpha)$, is an index 2 sublattice of $T(S)$. These lattices are related to the transcendental lattice $T(X)$, see Table 1.

2.4 | The Exceptional Divisors and Rigid Rank 2 bundles

Let $E \subset X$ be the exceptional fiber of a contraction. The conic bundle $E \to S$ was determined in [1, Corollary 4.3] for each of the five types and we recall its description. The Brauer class of the conic bundle $E \to S$ is trivial for three types, denoted by H, M_1, and M_3. In these cases, $E = \mathbf{P}\mathcal{F}$ where $\mathcal{F}$ is a locally free sheaf of rank 2 on S. In the case H, the contraction is the Hilbert Chow contraction $S^{[2]} \to S^{(2)}$ and $\mathcal{F} = \mathcal{T}_S$, the tangent sheaf of S. In the other two cases, $\mathcal{F} = \mathcal{V}$ where $\mathcal{V}$ is the rank 2 rigid Mukai bundle on S. If $\mathrm{Pic}(S) = \mathbf{Z}h_S$, such a bundle exists if and only if $h_S^2 = 2g - 2$ with $g = 2s$ even, its Mukai vector is $v(\mathcal{V}) = (2, h_S, s)$ and rigidity is equivalent to $v(\mathcal{V})^2 = -2$ (see [7, section 10.3.1], [8, section 3]). The i in the type M_i is determined by $h_S^2 \equiv 2i \mod 8$.

If the Brauer class $\alpha \in Br(S)_2$ of $E \to S$ is nontrivial, there are two types denoted by B_0 and B_1. The exceptional divisor is then $E = \mathbf{P}\mathcal{U}$ for a rigid locally free rank 2 α-twisted sheaf $\mathcal{U}$ on S with Brauer class α. The i in B_i is determined by $h_SB \equiv i/2 \mod \mathbf{Z}$ where B is a B-field representative of α. In both cases one must have $B^2 \equiv 1/2$, but recall that B^2 is not an invariant unless $4Bh_S + h_S^2 \equiv 0 \mod 4$. (On a general K3 surface S, there are three types of nontrivial Brauer classes in $Br(S)_2$. In the case where B^2 is an invariant and $B^2 \equiv 0$, there does not exist a locally free rank 2 α-twisted sheaf which is rigid and such Brauer classes are not obtained from exceptional divisors.) Similar to the case $\alpha = 0$, an α-twisted sheaf $\mathcal{U}$ has a (twisted) Mukai vector $v^B(\mathcal{U})$. The rigidity of $\mathcal{U}$ is equivalent to $v^B(\mathcal{U})^2 = -2$ (cf. [1, section 2.3]).

Conversely, given a rigid, possibly twisted, locally free sheaf $\mathcal{U}$ of rank 2 on a K3 surface S with Picard rank 1 as above, there is an HK fourfold X of $\mathrm{K3}^{[2]}$-type that has an exceptional divisor E with conic bundle structure $E \cong \mathbf{P}\mathcal{U} \to S$. This X is a moduli space of twisted sheaves on S.

In Sections 2.5–2.9, we recall some results on the five types of lattice embeddings, based on Propositions 3.4–3.8 of [1, section 3]. Table 1, based on [1, table 1], collects some of these results.

2.5 | Contractions of Type H

For a contraction of type H, the fourfold X is isomorphic to a Hilbert scheme $S^{[2]}$ where S is a K3 surface with $\mathrm{Pic}(S) = \mathbf{Z}h$, let $h^2 = 2d$. The Picard lattice of $S^{[2]}$ is

$$\mathrm{Pic}(S^{[2]}) = \mathbf{Z}H \oplus \mathbf{Z}\tau, \qquad q(H) = 2d, \quad (H,\tau) = 0, \quad q(\tau) = -2,$$

here 2τ is the class of the exceptional divisor $E \cong \mathbf{P}\mathcal{T}_S$ in $S^{[2]}$ parameterised the non-reduced subschemes of length two and H is the divisor class induced by h. The embedding $\mathbf{Z}H \oplus \mathbf{Z}\tau \hookrightarrow \Lambda_{\mathrm{K3}^{[2]}}$ is such that the image of τ has divisibility 2. There is a unique class of such embeddings according to [4].

2.6 | Contractions of Type M_1

According to Table 1, the contraction of X defined by H is of type M_1 if and only if $|\det(\mathrm{Pic}(X))| = d$, where we recall that $q(H) = 2d$. Only in this case $\mathrm{Pic}(X) \neq \mathbf{Z}H \oplus \mathbf{Z}\tau$, but $\mathrm{Pic}(X)$ is generated by H and $(H+\tau)/2$ (cf. [1, Proposition 3.7]). Since $q((H+\tau)/2) = (2d-2)/4$ must be an even integer, this implies that $d \equiv 1 \mod 4$. There is an isometry $H^2(X,\mathbf{Z}) \cong \Lambda_{\mathrm{K3}} \oplus \mathbf{Z}\delta$ such that $\mathrm{Pic}(X) \subset \Lambda_{\mathrm{K3}}$, thus any primitive element in $\mathrm{Pic}(X)$ has divisibility one in $H^2(X,\mathbf{Z})$.

For example, if $d = 1$, the even, indefinite, rank 2 Picard lattice is unimodular and hence is isometric to the hyperbolic plane U. (Since there are isotropic vectors in U, X has a Lagrangian fibration, see Section 3.2, which corresponds to the other extremal ray. These fourfolds are studied in [9].)

2.7 | Contractions of Type M_3

A contraction of X defined by H is of type M_3 if and only if the divisibility γ of H in $H^2(X,\mathbf{Z})$ is two. In that case, $H = 2D + a\delta \in \Lambda_{\mathrm{K3}^{[2]}}$ for some $D \in \Lambda_{K3}$ and, since H is primitive, a is odd. Then $q(H) = 4D^2 - 2a^2 \equiv -2 \mod 8$, that is, $q(H) = 2d$ with $d \equiv 3 \mod 4$. The Picard lattice is $\mathrm{Pic}(X) = \mathbf{Z}H \oplus \mathbf{Z}\tau$. For examples, see [1, Examples 5.4, 5.5, 5.6].

2.8 | Contractions of Type B_0

If X has a contraction with associated Brauer class $\alpha \neq 0$, $Bh = 0$, then the transcendental lattice $T(X)$ has a non-cyclic discriminant group [1, Proposition 3.6]. Propositions 3.4–3.8 in [1] show that if X has another contraction, then its type must again be B_0, otherwise the discriminant group would be cyclic. For these contractions, there is no congruence condition on d.

In this case, $\mathrm{Pic}(X) = \mathbf{Z}H \oplus \mathbf{Z}\tau$ and there is an isometry $H^2(X,\mathbf{Z}) \cong \Lambda_{\mathrm{K3}} \oplus \mathbf{Z}\delta$ such that $\mathrm{Pic}(X) \subset \Lambda_{\mathrm{K3}}$, see [1, Proposition 3.6]. Therefore any nonzero primitive element in $\mathrm{Pic}(X)$ has divisibility 1 in $H^2(X,\mathbf{Z})$.

An example, due to O'Grady, is given in [1, Proposition 5.1]. In that example, $d = 1$ and the HK manifolds are double covers of EPW sextics, the covering involution permutes the two exceptional divisors.

2.9 | Contractions of Type B_1

For a contraction with a nontrivial Brauer class $\alpha \in Br(S)_2$ and $Bh \equiv 1/2 \mod \mathbf{Z}$, the transcendental lattice $T(X)$ of X is isomorphic to the index 2 sublattice $T(S,\alpha)$ of $T(S)$, see [1, Proposition 3.5]. From the description of the lattice $T(S,\alpha)$ given in [1, Proposition 3.5] one finds that its discriminant group is cyclic and $q(H) = 2d$ with $d \equiv 0 \mod 4$.

For these α, it can still happen that $T(S,\alpha) \cong T(S')$ for another K3 surface S'. In fact, $T(X) \cong T(S')$, if and only if $d \equiv 0 \mod 8$, see [1, Proposition 3.5]. In that case S' is a K3 surface of degree $d/2$.

An example, with $d = 8$, is given in [1, section 5.3], see also the case $e = 2d = 16$ in Section 4.3.

Remark 2.10. From Sections 2.6 and 2.8, one finds that if X has a contraction of type M_1 or B_0, then any other contraction of a birational model of X will be of the same type. The remaining three types, B_1, H, M_3, can be distinguished from each other by the divisibility γ of H and the one of τ:

(a) type H if the divisibility of τ is two (and the divisibility of H is one),
(b) type M_3 if the divisibility of H is two (and the divisibility of τ is one),
(c) type B_1 if the divisibility of both H and τ is one.

2.11 | The K3 Surfaces Related to Contractions

A divisorial contraction of an HK fourfold X of $\mathrm{K3}^{[2]}$-type determines a K3 surface, which is the base of the conic bundle structure $E \to S$ induced by the contraction, where E is the exceptional divisor. From the classification of these contractions, we see that $T(X) \cong T(S)$ or $T(X) \cong T(S,\alpha)$. However, these isomorphisms are in general not sufficient to determine S uniquely. In fact, two K3 surfaces with Hodge isometric transcendental lattices are called Fourier Mukai (FM) partners (see [10, 11], and [7, Definition 16.2.2]). The number of FM partners of a given K3 surface is finite. For example in [11, Proposition 1.10], one finds that the number of FM partners of a K3 surface S with $\mathrm{Pic}(S) = \mathbf{Z}h$ and $h^2 = 2d$, is $2^{p(d)-1}$ where $p(d)$ the number of distinct primes dividing d.

Assume that X has (birationally) two divisorial contractions, these define K3 surfaces S_1, S_2 with Picard rank 1 and assume that $T(X) \cong T(S_i)$ for $i = 1, 2$. From Table 1 we see that the types must be H, M_1, or M_3. From Theorem 4.1 we see that if the types are not the same they must be H+M_3. Let $\mathrm{Pic}(S_1) = \mathbf{Z}h_1$, $\mathrm{Pic}(S_2) = \mathbf{Z}h_2$, then again from Table 1 we deduce that $h_1^2 = h_2^2 = 2d$ for some $d > 0$. If d is the power of a prime number, then Oguiso's result implies that $S_1 \cong S_2$ but otherwise these K3 surfaces might not be isomorphic. It is remarkable that X then determines a specific pair of K3 surfaces whereas the condition $T(X) \cong T(S)$ determines a possibly much larger set of $2^{p(d)-1}$ distinct K3 surfaces.

The following proposition, a corollary of results of Druel, will be useful to identify exceptional divisors of contractions.

Proposition 2.12. *Let $p : Z \to S$ be a conic bundle over a K3 surface and assume that there is an embedding $Z \hookrightarrow X$ where X is an HK fourfold. Then there is a birational isomorphism $X \simeq X'$, with X' an HK fourfold, and the image of Z is the exceptional divisor of a divisorial contraction $X' \to Y'$.*

Proof. By [12, Proposition 1.4, Theorem 1.3] we only have to show that if $f \subset Z$ is a fiber of p, then $Z \cdot f < 0$. Let $i : Z \hookrightarrow X$ be the inclusion, then $Z \cdot f = Z \cdot i_* f_Z$ where f_Z is a fiber considered as a 1-cycle on Z. By the projection formula, this intersection number is $i^*Z \cdot f_Z = K_Z \cdot f_Z$ since $i^*Z = Z_{|Z} = K_Z$ by adjunction on X. As $K_Z = \omega_{Z/S}$ is also the relative dualizing sheaf (cf. [1, section 1.1]) and $\omega_{Z/S}$ restricts to ω_f on f, we get $K_Z \cdot f_Z = \deg(\omega_f) = -2 < 0$ as desired. □

3 | The Extremal Rays of the Movable Cone

3.1 | Cones and Divisorial Contractions

We recall some results on contractions of K3 surfaces, see [5, Chapter 2] or [7, Chapter 8]. Let S be a projective K3 surface with Picard number two, then the divisorial contractions of S are the contractions of a (-2)-curve $n \subset S$ and $n \cong \mathbf{P}^1$ is a smooth rational curve. The contraction $f : S \to \overline{S}$ of n is defined by a big and nef divisor $h \in \mathrm{Pic}(S)$ with intersection number $hn = 0$. In this case a primitive embedding of the rank 2 Picard lattice of S into the (unimodular) K3 lattice is unique up to isometry. In particular there are no issues with the divisibilities of the various divisor classes. The half line $\mathbf{R}_{\geq 0}h$ generated by h in $\mathrm{Pic}(S)_{\mathbf{R}}$ is a boundary ray of the nef cone of S. Since $\mathrm{Pic}(S)_{\mathbf{R}} \cong \mathbf{R}^2$ and a cone in a two-dimensional real vector space has two extremal rays, there is one other boundary ray. This ray either defines another contraction, in this case there are exactly two (-2)-curves in S, or it is $\mathbf{R}_{\geq 0}e$ for an isotropic class $e \in \mathrm{Pic}(S)$, so $e^2 = 0$, and then the ray defines a genus 1 fibration $S \to \mathbf{P}^1$. The latter case occurs if and only if the discriminant $|\det(\mathrm{Pic}(S))|$ of the Picard lattice is a square.

Let now X be a projective HK fourfold of $\mathrm{K3}^{[2]}$-type (and thus Picard rank at least one), then we have the following cones, cf. [5, section 3.7]. The positive cone $\mathrm{Pos}(X)$ of X is the connected component of the set $\{x \in \mathrm{Pic}(X)_{\mathbf{R}} : q(x) > 0\}$ that contains an ample class. The nef cone $\mathrm{Nef}(X)$, which is the closure of the ample cone, is contained in the positive cone. The movable cone $\mathrm{Mov}(X)$ is the (closed, convex) cone generated by classes of line bundles on X whose base locus has codimension at least 2. There are inclusions:

$$\mathrm{Nef}(X) \subset \mathrm{Mov}(X) \subset \mathrm{Pos}(X) \quad (\subset \mathrm{Pic}(X)_{\mathbf{R}} := \mathrm{Pic}(X) \otimes \mathbf{R}).$$

For a K3 surface, $\mathrm{Nef}(S) = \mathrm{Mov}(S)$. Whereas a birational map between K3 surfaces is an isomorphism, this is not the case for HK fourfolds. A birational map $\sigma : X \to X'$ between HK manifolds induces an isometry $\sigma^* : H^2(X', \mathbf{Z}) \to H^2(X, \mathbf{Z})$. The movable cone $\mathrm{Mov}(X)$ is the union over all such birational maps σ of the $\sigma^*(\mathrm{Nef}(X'))$. These subcones are either equal or have disjoint interiors. The manifold X has a unique smooth birational model exactly when $\mathrm{Mov}(X) = \mathrm{Nef}(X)$.

The interior of the movable cone $\mathrm{Int}(\mathrm{Mov}(X))$ is the connected component of

$$\mathrm{Pos}(X) - \bigcup_{\substack{n\in \mathrm{Pic}(X),\\ q(n)=-2}} \{x \in \mathrm{Pos}(X) :\ q(x,n) = 0\}$$

that contains an ample divisor ([5, 13, Theorem 3.16]). In particular, if there are no (-2)-classes in $\mathrm{Pic}(X)$, then $\mathrm{Int}(\mathrm{Mov}(X)) = \mathrm{Pos}(X)$.

The interior of the nef cone is the ample cone. If $\sigma : X \to X'$ is a birational map between HK fourfolds, then the interior of $\sigma^*\mathrm{Nef}(X')$ is a connected component of

$$\mathrm{Int}(\mathrm{Mov}(X)) - \bigcup_{\substack{\kappa\in \mathrm{Pic}(X),\, q(\kappa)=-10,\\ \mathrm{div}_{H^2(X,\mathbf{Z})}(\kappa)=2}} \{x \in \mathrm{Pos}(X) :\ q(x,\kappa) = 0\}$$

all connected components of this set are the ample cones of HK manifolds that are birational to X ([5, 14, Theorem 3.16]). The birational map σ is a composition of Mukai flops in (Lagrangian) planes [15].

3.2 | The Extremal Rays in the Picard Rank 2 Case

We return to the case that X is a projective HK fourfold of $\mathrm{K3}^{[2]}$-type with Picard rank 2 and we use Section 3.1 and [5, section 3.7]. In case $|\det(\mathrm{Pic}(X))|$ is not a square and there is a (-2)-class in $\mathrm{Pic}(X)$, then there are infinitely many (-2)-classes in $\mathrm{Pic}(X)$ (corresponding to solutions of a Pell-type equation, see below). The two boundary rays of the movable cone are then each defined by (-2)-classes and they are called extremal rays. An extremal ray defined by a (-2)-class is generated by a big and nef divisor H which defines a divisorial contraction on (a birational model of) X. The class of the exceptional divisor is a multiple of the (-2)-class.

In case $|\det(\mathrm{Pic}(X))|$ is a square, the boundary rays of the positive cone are spanned by isotropic vectors in $\mathrm{Pic}(X)$ (so $q(x) = 0$ for any x on this ray). If there is also a (-2)-class in $\mathrm{Pic}(X)$, then one of these boundary rays of the positive cone is a boundary ray of the movable cone, the other boundary of the movable cone is defined by a (-2)-class. The extremal ray spanned by an isotropic class defines a Lagrangian fibration on (a birational model of) X, see [16], the other one defines a divisorial contraction as above.

If there are no (-10)-classes with divisibility 2, then $\mathrm{Mov}(X) = \mathrm{Nef}(X)$, otherwise at least one, and possibly both, of the boundary rays of the nef cone are defined by such (-10)-classes, these rays are called flopping walls.

Remark 3.3. In case $\mathrm{Pic}(X) \subset \Lambda_{\mathrm{K3}} \subset H^2(X,\mathbf{Z}) \cong \Lambda_{\mathrm{K3}} \oplus \mathbf{Z}\delta$, the unimodularity of Λ_{K3} implies that there cannot exist a (primitive) class of divisibility two in $\mathrm{Pic}(X)$, so in this case the movable cone is the nef cone. This happens if X has a contraction of type B_0 or M_1 (see Sections 2.8, 2.6).

3.4 | The Pell Equation

Assume that X has a divisorial contraction defined by the primitive nef divisor H with exceptional divisor E and that $|\det(Pic(X))| = 4d$ where d is not a square. Then $\mathrm{Pic}(X) = \mathbf{Z}H \oplus \mathbf{Z}\tau$, with $q(H) = 2d$, $q(\tau) = -2$, the ray $\mathbf{R}_{\geq 0}H$ is an extremal ray and the class of E is either τ or 2τ. For a divisor class $yH + x\tau \in \mathrm{Pic}(X)$, one has

$$q(yH + x\tau) \;=\; 2dy^2 - 2x^2, \qquad \text{hence} \quad x^2 \;-\; dy^2 \;=\; 1$$

is the condition that $q(yH + x\tau) = -2$. The Pell equation $x^2 - dy^2 = 1$ has a positive solution, that is, (a, b) with integers $a, b > 0$. A positive solution with minimal a is called minimal solution, this is also the solution for which the slope b/a is minimal [5, Appendix A]. The following proposition (which generalizes [5, Example 3.18]) gives the generator H' of the second extremal ray in terms of H, τ, and the minimal positive solution of the Pell equation.

Proposition 3.5. *With the assumptions in Section 3.4, let (a, b) be the minimal positive solution of the Pell equation $x^2 - dy^2 = 1$. Then the other divisorial contraction of X is defined by the primitive nef divisor class H' and the class of its exceptional* *divisor E' is a positive multiple of τ' where:*

$$H' := aH - bd\tau, \qquad \tau' := bH - a\tau .$$

Proof. If (a,b) is any solution of the Pell equation, then $\tau' := bH - a\tau$ is a (-2)-class in $\mathrm{Pic}(X)$ and any (-2)-class is obtained from a solution. The divisor class $H' := aH - bd\tau$ is perpendicular to τ' and it is primitive since $a^2 - db^2 = 1$ implies that $GCD(a, bd) = 1$.

The description of the movable cone implies that the other extremal ray $\mathbf{R}_{>0}H'$ is the one closest to $\mathbf{R}_{>0}H$, that is with $|b/a|$ minimal, so we only need to verify that the signs of a, b in H' and in τ' are correct.

In fact, since H is big and nef, a positive multiple of H^3 is the class of a curve in X which is not contained in H' and thus $H^3H' \geq 0$. Recall that $H^3H' = 3q(H)(H,H')$. As $q(H) = 2d > 0$ one has $0 \leq (H,H') = 2ad$ hence the coefficient of H must be positive. Similarly, $E^3 = cf$ is a multiple of the class of a fiber f of the conic bundle $E \to S$ [1, Proposition 1.1]. Since by adjunction $E_{|E} = K_E$ and $K_E = \omega_{E/S}$, the relative canonical bundle, we get $Ef = K_E f = -2$. Therefore, $cEf = E^4 = 3q(E)^2 > 0$ hence $c < 0$. Since the class of E is a positive multiple of τ, f is a negative rational multiple of τ^3. Since $H' \neq E$ we have $fH' \geq 0$ and hence $0 \geq \tau^3 H' = 3q(\tau)(\tau, H') = -6 \cdot (2bd)$ so the coefficient of τ is negative.

Similarly, if E', a positive multiple of τ', is effective, then also the signs given in τ' are correct. □

4 | Pairs of Extremal Rays

Theorem 4.1. *Let X be an HK fourfold of $K3^{[2]}$ type with Picard rank 2, assume that X admits a divisorial contraction and that $|\det(Pic(X))|$ is not a square. Then the movable cone has two extremal rays defining contractions and $|\det(Pic(X))| \equiv 0, 1 \mod 4$.*

The types of the extremal rays are:

(a) In case $|\det(\mathrm{Pic}(X))| \equiv 1 \mod 4$ the types are M_1+M_1.

Now we assume that $|\det(\mathrm{Pic}(X))| \equiv 0 \mod 4$ and we write $|\det(\mathrm{Pic}(X))| = 4d$.

(b) If the discriminant group of $T(X)$ is not cyclic, the types are B_0+B_0.

Now we assume that moreover the discriminant group of $T(X)$ is cyclic. Let (H,τ), (H',τ') be the two pairs corresponding to the extremal rays. Let $(x,y) = (a,b)$ be the minimal solution of the Pell equation $x^2 - dy^2 = 1$.

(c) In case $d \equiv 0 \mod 4$ the types are:
H+H if $\mathrm{div}_{H^2(X,\mathbf{Z})}(\tau) = \mathrm{div}_{H^2(X,\mathbf{Z})}(\tau') = 2$,
B_1+B_1 *if $\mathrm{div}_{H^2(X,\mathbf{Z})}(\tau) = \mathrm{div}_{H^2(X,\mathbf{Z})}(\tau') = 1$,*
H+B_1 *if $d \equiv 0 \mod 8$ and b is odd.*

(d) In case $d \equiv 1, 2 \mod 4$ the types are H+H.

(e) In case $d \equiv 3 \mod 4$ the types are
H+H if $\mathrm{div}_{H^2(X,\mathbf{Z})}(\tau) = \mathrm{div}_{H^2(X,\mathbf{Z})}(\tau') = 2$,
M_3+M_3 *if $\mathrm{div}_{H^2(X,\mathbf{Z})}(\tau) = \mathrm{div}_{H^2(X,\mathbf{Z})}(\tau') = 1$,*
H+M_3 *if a is even.*

Proof. The rank 2 Picard lattice is indefinite and even, therefore $|\det(Pic(X))| \equiv 0, 1 \mod 4$.

(a) In Section 2.6 we observed that if $|\det(\mathrm{Pic}(X))| \equiv 1 \mod 4$, then both rays have type M_1 since this is the only case where $|\det(\mathrm{Pic}(X))| \neq 4d$, where $q(H) = 2d$.

(b) From Section 2.8 we see that if the discriminant group of $T(X)$ is not cyclic, the types are B_0+B_0. Notice that (a) and (b) also follow from Remark 2.10.
If $|\det(\mathrm{Pic}(X))| \equiv 0 \mod 4$ and the discriminant group of $T(X)$ is cyclic, there cannot be any rays of type M_1 or B_0. So we only need to consider the remaining three types: H, M_3, B_1.

(d) If there is a ray of type B_1 then $|\det(\mathrm{Pic}(X)| = 4d$ with $d \equiv 0 \mod 4$ and if the type is M_3 then $d \equiv 3 \mod 4$. So for $d \equiv 1, 2 \mod 4$ the types must be H+H.

(c) For $d \equiv 0 \mod 4$ one has types H+B_1 exactly when $\mathrm{div}_{H^2(X,\mathbf{Z})}(\tau) = 2$ and $\mathrm{div}_{H^2(X,\mathbf{Z})}(\tau') = 1$ or vice versa.

TABLE 2 | Divisorial contractions on a Hilbert square of $S^{[2]}$, S of degree e.

e	*(a,b)*	Types	*H'*	*τ'*	*e*	*(a,b)*	Types	*H'*	*τ'*
2		H			18		H		
4	(3,2)	H+H	$3H$-4τ	$2H$-3τ	20	(19,6)	H+H	$19H$-60τ	$6H$-$19\,\tau$
6	(2,1)	H+M_3	$2H$-3τ	H-2τ	22	(10,3)	H+M_3	$10H$-33τ	$3H$-10τ
8		H			24	(7,2)	H+H	$7H$-24τ	$2H$-7τ
10	(9,4)	H+H	$9H$-20τ	$4H$-9τ	26	(649,180)	H+H	$649H$-2340τ	$180H$-649τ
12	(5,2)	H+H	$5H$-12τ	$2H$-5τ	28	(15,4)	H+H	$15H$-56τ	$4H$-15τ
14	(8,3)	H+M_3	$8H$-21τ	$3H$-8τ	30	(4,1)	H+M_3	$4H$-15τ	H-4τ
16	(3,1)	H+B_1	$3H$-8τ	H-3τ	32		H		

To see that this corresponds to b odd, we use the embeddings of $\mathrm{Pic}(X)$ into $\Lambda_{K3^{[2]}} = U^3 \oplus E_8(-1)^2 \oplus \mathbf{Z}\delta$ with $q(\delta) = -2$ and we write $(r,s)_i$ for an element in the i-th copy of U:

$$\mathrm{H}: \qquad H = (1,d)_1, \quad \tau = \delta$$

[1, Proposition 3.4] and

$$\mathrm{B}_1: \qquad H = (1,d)_1, \quad \tau = (1,-d)_1 + 2(1,d/4)_2 + \delta$$

[1, Proposition 3.5]. If the first extremal ray is of type H, then, by Proposition 3.5, the second one has $\tau' = bH - a\tau = b(1,d)_1 - a\delta$. Thus, τ' has divisibility one iff b is odd. If the first ray is of type B_1, then the second one has $\tau' = bH - a\tau = (b-a,(b+a)d)_1 - 2a(1,d/4)_2 - a\delta$. The type of the second ray is H exactly when τ' has divisibility 2, that is, when $b-a$ is even. Since a,b cannot both be even because $a^2 - db^2 = 1$, it follows that both must be odd. Since $d \equiv 0 \mod 4$, a is odd, hence the type of the second ray is H if and only if b is odd.
Finally we observe that if a,b are odd, then $a^2 - db^2 \equiv 1 \mod 8$ implies that $d \equiv 0 \mod 8$.

(e) Similarly, for $d \equiv 3 \mod 4$ the types are either H+H, M_3+M_3, or H+M_3 and the latter occurs if and only if the divisibilities of τ, τ' are distinct. From [1, Proposition 3.8], the embedding of $\mathrm{Pic}(X)$ in $\Lambda_{K3^{[2]}}$ can be chosen as

$$\mathrm{M}_3: \qquad H = 2(1,(d+1)/4)_1 + \delta, \qquad \tau = (1,-1)_2\,.$$

If the first extremal ray is of type H, then the second one has $\tau' = bH - a\tau = b(1,d)_1 - a\delta$ and τ' has divisibility 1 if and only if b is odd. If the first ray is of type M_3, then the second one has $\tau' = bH - a\tau = 2b(1,(d+1)/4)_1 + b\delta - a(1,-1)_2$ which has divisibility 2 if and only if a is even. Since $a^2 - db^2 = 1$ and d is odd, by going mod 2, we see that a is even if and only if b is odd. □

4.2 | An Application to Hilbert Squares

We consider the case $X = S^{[2]}$, the Hilbert square of a K3 surface S with $\mathrm{Pic}(S) \cong \mathbf{Z}h_S$ and $h_S^2 = e$. We denote by $H \in \mathrm{Pic}(S^{[2]})$ the divisor class corresponding to h_S, which has $q(H) = e$ (see also [5, section 3.7.2] for Hilbert schemes and their extremal rays). The class H is big and nef, it contracts $E = \mathbf{P}\mathcal{T}_S \subset S^{[2]}$. The class of E is denoted by $2\tau \in \mathrm{Pic}(S^{[2]})$, one has $q(\tau) = -2$ and $q(H,\tau) = 0$. The classes H, τ are a basis of $\mathrm{Pic}(S^{[2]})$ and $|\det(\mathrm{Pic}(S^{[2]})| = 2e$, so $e = 2d$. The divisibility of H, τ in $H^2(S^{[2]}, \mathbf{Z})$ is $1, 2$, respectively. From these divisibilities and Table 1, or more directly from the definition in Section 2.5, it follows that the type of this contraction is H.

In case $|\det(Pic(S^{[2]}))|$ is not a square, there are two contractions (cf. Section 3.2). Since one of them has type H, the cases (a), (b), B_1+B_1 in (c), and M_3+M_3 in (e) of Theorem 4.1 do not occur. The other type is thus determined by $e = 2d$ and the parity of b, a. (For Hilbert squares, one can also use Remark 2.10 since the divisor class $rH + s\tau$ has divisibility 2 in $H^2(X, \mathbf{Z})$ exactly when r is even, and Proposition 3.5 which determines H' and τ' in terms of the minimal solution of $x^2 - dy^2 = 1$.) In Table 2, we list the types and other useful information.

4.3 | The Table 2

In Table 2, we consider a K3 surface S as in Section 4.2 and we determine the types of the extremal rays of $S^{[2]}$. In case $2e$ is a square, the second extremal ray defines a Lagrangian fibration. Otherwise, the second extremal ray is spanned by a big and nef divisor $H' \in \mathrm{Pic}(S^{[2]})$. The corresponding exceptional divisor is E', with class $2\tau'$ if this contraction is of type H and it is τ' in the other cases. We give the minimal solution (a, b) of the Pell equation $x^2 - dy^2$, $e = 2d$, and we write H', τ' as linear combinations of H, τ following Proposition 3.5 for the cases with $e \leq 32$.

Below we discuss some of these cases in more detail.

The case $\mathbf{e} = \mathbf{4}$, $S \subset \mathbf{P}^3$ is a quartic surface and $S^{[2]}$ has an involution (found by Beauville). It is given by sending $p + q \in S^{[2]}$ to the residual intersection of the line spanned by p, q with S. This involution does not fix the exceptional divisor E of non-reduced subschemes $2p$ in $S^{[2]}$, where $p \in S$. Thus, $S^{[2]}$ has two contractions of type H. The other exceptional divisor E' has support in the $q + r \in S^{[2]}$ such that the line in $\mathbf{P}^3$ spanned by q and r is tangent to S in a point p.

In the case $\mathbf{e} = \mathbf{6}$, the minimal solution of $x^2 - 3y^2 = 1$ is $(a, b) = (2, 1)$ so the types are $\mathrm{H}+\mathrm{M}_3$, this example is discussed in [1, section 5.4]. The equation $x^2 - 3y^2 = 5$ has no solutions modulo 3, thus there are no flopping walls and $S^{[2]}$ has a unique birational HK model.

By Proposition 3.5, the other ray is spanned by $H' := 2H - 3\tau$ and the exceptional divisor E', with class $\tau' = H - 2\tau$, is a conic bundle $E' \cong \mathbf{P}\mathcal{V}$ with trivial Brauer class where $\mathcal{V}$ is the rank 2 Mukai bundle on S.

See Example 5.6 for the case $\mathbf{e} = \mathbf{14}$.

In the case $\mathbf{e} = \mathbf{16}$, the minimal solution of the Pell equation $x^2 - 8y^2 = 1$ is $(a, b) = (3, 1)$, thus $b = 1$ is odd and the types are $\mathrm{H}+\mathrm{B}_1$. The equation $x^2 - 8y^2 = 5$ has no solutions modulo 8 and thus there are no flopping walls. The second exceptional divisor E' is a conic bundle over a K3 surface S' of degree 4. It has nontrivial Brauer class $\alpha \in Br(S')_2$ and is known as the BOSS bundle, cf. [1, Proposition 5.3] and [17]. There are Hodge isometries of transcendental lattices $T(X) \cong T(S) \cong T(S', \alpha)$.

In the case $\mathbf{e} = \mathbf{22}$, also discussed in [1, section 5.5], the minimal solution of the Pell equation $x^2 - 11y^2 = 1$ is $(a, b) = (10, 3)$ so the Hilbert scheme $X = S^{[2]}$ also admits (birationally) a contraction of type M_3 defined by the big and nef divisor H with $H' = 10H - 33\tau$. One has $q(H') = 22$ and the divisibility of H' is 2. The (singular) image of $S^{[2]}$ under map defined by H' is a Debarre–Voisin variety, see [18, section 3]. In this case X admits a flop, in fact $x^2 - 11y^2 = 5$ has the solution $(a, b) = (7, 2)$. The corresponding (-10)-class is $\kappa = 2H - 7\tau$ which has divisibility 2. The orthogonal of κ is spanned by $7H - 22\tau$ which lies in the movable cone, so the half line generated by the class is a flopping wall.

The case $\mathbf{e} = \mathbf{30}$ is related to the recent preprint [19] where it is shown that the projectivized Mukai bundle $\mathbf{P}\mathcal{U} \to S'$ on a general K3 surface S' of degree $e = 30 = 2g - 2$, so of genus $g = 16$, embeds into $S^{[2]}$, where S is the (unique) FM partner of S', that is, S, S' have isomorphic transcendental lattices, but are not isomorphic (see Section 2.11). From Proposition 2.12, it follows that the image of X is an exceptional divisor. There are no (-10)-classes in the Picard lattice, so $S^{[2]}$ has a unique HK model. It is interesting to observe that this conic bundle appears to play a crucial role in showing that these Hilbert squares are DV varieties.

5 | Pairs of Extremal Rays of Fano Fourfolds

5.1 | The Picard Lattice of a Fano Fourfold

In this section, Y will be a smooth cubic fourfold. We denote by $F = F(Y)$ its Fano variety of lines which is an HK fourfold of $\mathrm{K3}^{[2]}$ type [20, Proposition 2]. There is an isomorphism (of abelian groups only, not of lattices, and of Hodge structures), [20, section 3, Proposition 6.3], [21, section 2.5],

$$\alpha : H^4(Y, \mathbf{Z}) \xrightarrow{\cong} H^2(F(Y), \mathbf{Z}),$$

which induces an isomorphism between the group $N^2(Y)$ of codimension 2 algebraic cycles on Y and $\mathrm{Pic}(F(Y))$. Moreover, α induces a Hodge isometry between the transcendental lattices

$$\alpha : T(Y)(-1) \xrightarrow{\cong} T(F), \qquad \left(T(Y) := N^2(Y)^{\perp}, \quad T(F) := \mathrm{Pic}(F)^{\perp}\right),$$

the perpendiculars are taken with respect to the intersection form on $H^4(Y,\mathbf{Z})$ and the BBF form on $H^2(F(Y),\mathbf{Z})$, respectively, here $T(Y)(-1)$ indicates that the intersection form restricted to $T(Y)$ must be multiplied by -1.

Hassett [22, section 3] showed that if $N^2(Y)$ has rank 2, then the determinant e of the intersection form on $N^2(Y)$ determines this lattice uniquely. This lattice is denoted by K_e, one has $e > 6$ and $e \equiv 0, 2 \mod 6$, so $e = 2d$ is even. The (irreducible) Hassett divisor C_e in the moduli space of smooth cubic fourfolds parameterizes the cubic fourfolds Y such that $h_3^2 \in K_e$, where $h_3 \in H^2(Y,\mathbf{Z})$ is the hyperplane class, and K_e is primitively embedded in $N^2(Y)$. The following proposition, an easy consequence of the results in [20] and [22], describes the Picard lattice of the Fano variety of lines for $Y \in C_e$.

Proposition 5.2. *Let $Y \in C_e$ be a smooth cubic fourfold and assume that the rank of the lattice of codimension 2 algebraic cycles is 2. Let $F = F(Y)$ be the Fano fourfold of lines on Y, and let $g \in \mathrm{Pic}(F)$ the class of the Plücker polarization. Then $|\det(\mathrm{Pic}(F))| = 2e = 4d$ and there is a class $\gamma \in \mathrm{Pic}(F)$ such that the lattice $(\mathrm{Pic}(F), q)$ is*

$$\mathrm{Pic}(F) = \begin{cases} \left(\mathbf{Z}g \oplus \mathbf{Z}\gamma, \begin{pmatrix} 6 & 0 \\ 0 & -e/3 \end{pmatrix}\right), & \text{if} \quad e \equiv 0 \mod 6\,, \\ \left(\mathbf{Z}g \oplus \mathbf{Z}\gamma, \begin{pmatrix} 6 & 2 \\ 2 & (2-e)/3 \end{pmatrix}\right), & \text{if} \quad e \equiv 2 \mod 6\,. \end{cases}$$

The divisibility in $H^2(F,\mathbf{Z})$ of a primitive divisor class $rg + s\gamma \in \mathrm{Pic}(F)$ with $r, s \in \mathbf{Z}$ is:

$$\mathrm{div}_{H^2(F,\mathbf{Z})}(rg + s\gamma) = \begin{cases} 1 & \text{if} \quad s \text{ is odd}, \\ 2 & \text{if} \quad s \text{ is even}\,. \end{cases}$$

Proof. The map α induces an isomorphism $N^2(Y) \cong \mathrm{Pic}(F)$. From the classification of the rank 2 lattices $N^2(Y)$ in [22], one has

$$N^2(Y) = \left(\mathbf{Z}h_3^2 \oplus \mathbf{Z}R, K_e\right), \quad \text{with} \quad K_e = \begin{cases} \begin{pmatrix} 3 & 0 \\ 0 & e/3 \end{pmatrix} & \text{if} \quad e \equiv 0 \mod 6\,, \\ \begin{pmatrix} 3 & 1 \\ 1 & (e+1)/3 \end{pmatrix} & \text{if} \quad e \equiv 2 \mod 6\,. \end{cases}$$

Let $h_3 \in H^2(Y,\mathbf{Z})$ be the class of a hyperplane section, then α maps $h_3^2 \in H^4(Y,\mathbf{Z})$ to the Plücker class g and $q(g) = 2(h_3^2)^2 = 6$ where q is the BBF form on $H^2(F,\mathbf{Z})$. The map α induces an isomorphism between $H^4(Y,\mathbf{Z})_0$, the perpendicular of h_3^2 in $H^4(Y,\mathbf{Z})$, and $H^2(F(Y),\mathbf{Z})_0$, the perpendicular of g, which changes the sign of the intersection product. As any element in $H^4(Y,\mathbf{Z})$ can be written as $ah_3^2 + bt$ with $t \in H^4(Y,\mathbf{Z})_0$ and $3a, 3b \in \mathbf{Z}$ and $g \cdot \alpha(ah_3^2 + bt) = 6a$, it follows that $\mathrm{div}_{H^2(F)}(g) = 2$.

Next let $\gamma := \alpha(R)$. In case $e \equiv 0 \mod 6$, $R \in (h_3^2)^\perp$ and thus $q(\gamma)$ is the opposite of the self-intersection of R: $q(\gamma) = -R^2 = -d/3$.

In case $e \equiv 2 \mod 6$, $H^4(Y,\mathbf{Z})_0 \cap N^2(Y)$ is generated by $c = h_3^2 - 3R$ and $c^2 = 3 - 6 + 9 \cdot (e+1)/3 = 3e$. Therefore, $\alpha(c)^2 = -c^2 = -3e$. Since $R = (h_3^2 - c)/3$ we get

$$q(\gamma) = (1/9)(\alpha(h_3^2)^2 + \alpha(c)^2) = (1/9)(6 - 3e) = (1/3)(2 - e)\,.$$

Similarly,

$$g \cdot \gamma = (1/3)\alpha(h_3^2)(\alpha(h_3^2) - \alpha(c)) = (1/3)(6 + 0) = 2\,.$$

TABLE 3 | Divisorial contractions on the Fano variety of $Y \in \mathcal{C}_e$.

e	adm	(-2)-classes	(a, b)	Types	H	τ	$\deg(\Sigma_f)$
8	$(**)'$	Yes	$\square$	B_1	$g+\gamma$	γ	2
14	$(**)$	Yes	(8,3)	H	$3g+5\gamma$	$g+2\gamma$	5
				M_3	$3g-2\gamma$	$g-\gamma$	4
24	$(**)'$	Yes	(7,2)	B_1	$4g+3\gamma$	$g+\gamma$	6
				B_1	$4g-3\gamma$	$g-\gamma$	6
26	$(**)$	Yes	(649,180)	H	$11g-7\gamma$	$3g-2\gamma$	7
				H	$119g+137\gamma$	$33g+38\gamma$	137
38	$(**)$	Yes	(170,39)	H	$109g-61\gamma$	$25g-14\gamma$	61
				M_3	$5g+4\gamma$	$g+\gamma$	8
42	$(**)$	Yes	(55,12)	H	$14g+9\gamma$	$3g+2\gamma$	9
				H	$14g-9\gamma$	$3g-2\gamma$	9
56	$(**)'$	Yes	(127,24)	B_1	$11g-5\gamma$	$2g-\gamma$	10
				B_1	$53g+37\gamma$	$10g+7\gamma$	74
62	$(**)$	Yes	(1520,273)	H	$17g+11\gamma$	$3g+2\gamma$	11
				M_3	$451g-206\gamma$	$81g-37\gamma$	412
74	$(**)$	No	(73,12)	None	None	None	None
78	$(**)$	Yes	(25,4)	M_3	$13g+6\gamma$	$2g+\gamma$	12
				M_3	$13g-6\gamma$	$2g-\gamma$	12

Recall that the divisibility of a primitive element in $H^2(F, \mathbf{Z}) \cong \Lambda_{\mathrm{K3}[2]}$ is either one or two. The divisibility of g is two, so if s is even, the divisibility of $rg + s\gamma$ is two. The lattice $H^4(Y, \mathbf{Z})$ contains the primitive sublattice generated by h_3^2 and R, hence it has a $\mathbf{Z}$-basis $e_1 = h_3^2, e_2 = R, e_3, \dots, e_{23}$. Since $H^4(Y, \mathbf{Z})$ is unimodular (by Poincaré duality), there is a dual basis $e_1^*, \dots, e_{23}^*$ with $e_i \cdot e_j^* = \delta_{ij}$. In particular, $e_2^* \in H^4(Y, \mathbf{Z})_0$ and $R \cdot e_2^* = 1$. Since $\gamma = \alpha(e_2)$, it follows that $(rg + s\gamma) \cdot \alpha(e_2^*) = -s$ hence if s is odd, the divisibility of $rg + s\gamma$ is one. $\square$

5.3 | K3 Surfaces Associated to Cubic Fourfolds

If for $Y \in \mathcal{C}_e$ the orthogonal complement to K_e in $H^4(Y, \mathbf{Z})$ is Hodge isometric (up to sign) to the primitive cohomology of a K3 surface S (of degree e), then S is said to be a K3 surface associated to the cubic fourfold Y and e is called admissible. This is equivalent to the transcendental lattices $T(Y)(-1)$ and $T(S)$ being Hodge isometric. Hassett showed that if $N^2(Y) = K_e$, then Y has an associated K3 surface if and only if the following condition $(**)$ is satisfied ([22, Theorem 5.1.3], cf. [23, section 1.3]):

$$(**) \qquad e \text{ is even and } e/2 \text{ is not divisible by nine or by any prime } p \equiv 2 \mod 3.$$

The first admissible e are:

$$e = 14,\ 26,\ 38,\ 42,\ 62,\ 74,\ 78,\ 86,\ 98,\ 114,\ 122,\ 134,\ 146,\ 158, \dots .$$

More generally, Huybrechts [23, Theorem 1.3] showed that $T(Y) \cong T(S, \alpha)$ for a K3 surface S and a Brauer class $\alpha \in Br(S)$ if and only if the following condition (**)′ is satisfied

$$(**)' \quad e \text{ is even and in the prime factorization } e/2 = \prod p_i^{n_i} \text{ one has } n_i \equiv 0 \mod 2 \text{ for all primes } p_i \equiv 2 \mod 3.$$

If (**)′ is satisfied, then the order of $\alpha \in Br(S)$ satisfies $\mathrm{ord}(\alpha)^2 | e$ [23, Lemma 2.13].

The following theorem provides the basic results on the pairs of extremal rays of a Fano variety of lines. In combination with Theorem 4.1, it allows one to determine the types explicitly, see Table 3. The theorem also includes some well-known results on the K3 surfaces associated to these Fano varieties of lines. See [21, section 6.5] and [24, 25] for further results on the relations with (twisted) derived categories.

Theorem 5.4. *Let F be the Fano variety of lines of a smooth cubic fourfold $Y \in C_e$ with $e = 2d > 6$ and assume that* $\mathrm{Pic}(F)$ *has rank 2. Then the following hold.*

(a) If there is an exceptional divisor on F, then there is an isometry $\mathrm{Pic}(F) \cong < 2d > \oplus < -2 >$.

(b) A divisorial contraction of F has type H, M_3, or B_1. If F has a pair of extremal rays, then this pair is one of H+H, H+M_3, M_3+M_3, B_1+B_1.

(c) If F has an extremal ray of type H or M_3, then e satisfies $(**)$ and there is a K3 surface associated to Y. There is an extremal ray of type H if and only if there exist integers a, n such that $e = 2(n^2+n+1)/a^2$.

(d) If F admits a divisorial contraction of type B_1 then e satisfies $(**)'$ but not $(**)$, in fact $e/4$ is an integer and satisfies $(**)$. There is a twisted K3 surface (S, α) associated to Y with $\alpha \in Br(S)_2$, $\alpha \neq 0$.

(e) In case $e \equiv 0 \mod 6$ and F has an extremal ray, then there is another extremal ray of the same type. In case the rays both have type H, F has two divisorial contractions inducing conic bundles $E_i \to S_i$ for K3 surfaces S_1, S_2 and $F \cong S_1^{[2]} \cong S_2^{[2]}$, so F is ambiguous (see [22, Definition 6.2.1]).

Proof. (a), (b) Since the Plücker class $g \in \mathrm{Pic}(F)$ has divisibility 2 in $\mathrm{Pic}(F)$, it is not the case that $\mathrm{Pic}(F)$ embeds into Λ_{K3}. Hence extremal rays of type M_1 and B_0 are excluded. For the three remaining type of contractions, we have seen that $\mathrm{Pic}(F) \cong < 2d > \oplus < -2 >$.

(c) For the Fano variety $F = F(Y)$ one has $T(Y) \cong T(F)$. From Table 1 we see that if the type is H or M_3, then $T(F) \cong T(S)$ for some K3 surface S. Hence S is associated to Y. The Fano F has a ray of type H if and only if F is birationally isomorphic to $S^{[2]}$ for a K3 surface S, this surface is associated to Y. The existence of a, n then follows from [22, Proposition 6.1.3], see [24, Theorem 2], [21, Corollary 5.22] for the converse.

(d) In case F has a contraction of type B_1 then $d \equiv 0 \mod 4$ hence e does not satisfy $(**)$. Moreover, $T(F) \cong T(S, \alpha)$ for a K3 surface S and an $\alpha \in Br(S)$ of order 2. As $T(Y) \cong T(F)$, the twisted K3 surface (S, α) is associated to Y.

We show that if F allows a contraction, then either e or $e/4$ satisfies $(**)$. Notice that if $e/4$ satisfies $(**)$, then e satisfies $(**)'$. We verify that then the contraction is of type B_1. We consider the two cases $e \equiv 0, 2 \mod 6$ separately, even if the proofs are rather similar. Recall that for a prime number $p > 3$ the congruence $x^2 \equiv -3 \mod p$ has a solution if and only if $p \equiv 1 \mod 3$.

In case $e \equiv 2 \mod 6$, let k be such that $e = 2 + 6k = 2(1+3k)$ so that $q(xg + y\gamma) = 6x^2 + 4xy + (2-e)y^2/3 = 2(3x^2 + 2xy - ky^2)$. Since F has a contraction, there is a (-2)-class in $\mathrm{Pic}(F)$. So there are $x, y \in \mathbf{Z}$ such that $q(xg + y\gamma) = -2$, equivalently $3q(xg + y\gamma) = -6$. Let $z = 3x + y$, then this implies that

$$z^2 - (3k+1)y^2 = -3$$

has an integer solution. Let $p > 3$ be a prime divisor of e, so of $3k+1$, then $z^2 \equiv -3 \mod p$ and thus $p \equiv 1 \mod 3$. Obviously 3 does not divide e. If 2^a divides e and $a > 3$, then $e/2 = 3k+1 \equiv 0 \mod 8$, but $z^2 \equiv -3 \mod 8$ has no solutions, so $a \leq 3$. In case $a = 2$, $e/2 = 3k+1 \equiv 2 \mod 4$ and $z^2 - 2y^2 \equiv -3 \mod 4$ implies that y is even, but then we find again that $z^2 \equiv -3 \mod 8$, a contradiction. So $e = 2^a \prod p_i^{m_i}$ where $a = 1, 3$ and all $p_i \equiv 1 \mod 3$ which satisfies $(**)$ if $a = 1$ and else $a = 3$ and then e satisfies $(**)'$ while $e/4$ satisfies $(**)$. In case $a = 3$, one has the congruence $z^2 - 4y^2 \equiv -3 \mod 8$ which implies that y is odd. It follows that the divisibility of $xg + y\gamma$ in $H^2(F, \mathbf{Z})$ is equal to one, hence the type cannot be H. As $e \equiv 0 \mod 4$, the type cannot be M_3 either, so it must be B_1.

In case $e \equiv 0 \mod 6$, $q(xg + y\gamma) = 6x^2 - (e/3)y^2$. Since we can write $e = 6k$, there exists a (-2)-class iff $6x^2 - 2ky^2 = -2$, that is,

$$3x^2 - ky^2 = -1\,.$$

Notice that k obviously cannot be a multiple of 3. If a prime $p > 3$ divides k, then we get $3x^2 \equiv -1 \mod p$ and thus $p \equiv 1 \mod 3$. Thus $e/6 = k = 2^{a-1}e'$ where $e' \equiv 1 \mod 3$ and $a \geq 1$. Then we find $-2^{a-1}y^2 \equiv -1 \mod 3$ which shows that a is odd. If $a > 3$, we get $3x^2 \equiv -1 \mod 8$ which is impossible. Thus $e = 2^a \cdot 3 \cdot \prod p_i^{m_i}$ where $a = 1, 3$ and all $p_i \equiv 1 \mod 3$. If $a = 3$ we have $k = e/6 \equiv 4 \mod 8$ and $3x^2 - 4y^2 \equiv -1 \mod 8$ implies that y is odd. Hence, as for the case $e \equiv 2 \mod 6$, we get desired results.

(e) In case $e \equiv 0 \mod 6$, one has $q(xg + y\gamma) = 6x^2 - (e/3)y^2$ and g lies in the ample cone of F, hence if $H = xg + y\gamma$ generates an extremal ray one has $y \neq 0$ and $H' = xg - y\gamma$ is also an extremal ray. The corresponding (-2)-classes τ, τ' are then of the form $rg \pm s\gamma$ and thus have the same divisibility in $H^2(F, \mathbf{Z})$ hence the types are the same by Theorem 4.1. If there is a ray of type H, e satisfies $(**)$ hence $e/6 \equiv 1 \mod 3$ because any prime divisor of $e/6$ is 1 mod 3. Then there are no (-10)-classes since $3x^2 - (e/6)y^2 = -5$ has no solution modulo 3 hence F has a unique birational HK model. Therefore, a ray of type H has an exceptional divisor $E \subset F$ with $E \cong \mathbf{P}\mathcal{T}_S$ for a K3 surface S and F is isomorphic to $S^{[2]}$. See also [22, Proposition 6.2.2] and [26] for ambiguous Hilbert squares. □

5.5 | The Table 3

In Table 3, we list the contractions for the classical case $e = 8$ (the cubic fourfold has a plane), the cases $e = 14, 26, 38, 42, 62, 74, 78$ which satisfy $(**)$ as well as the cases $e = 24, 56$ which satisfy $(**)'$.

In case d is a square, where $e = 2d$, we write a □ in the column for the minimal positive solution (a, b) of the Pell equation $x^2 - dy^2 = 1$. We use the basis g, γ of $\mathrm{Pic}(F)$ as in Proposition 5.2. If a divisor $D = rg + s\gamma$ is effective, then, since g is very ample, we must have $g^3 D > 0$. Proceeding as in the proof of Proposition 3.5, this gives

$$0 \;<\; g^3(rg + s\gamma) \;=\; 3r(g,g)^2 \;+\; 3s(g,g)(g,\gamma)\,.$$

As $(g,g) = 6$ and $(g,\gamma) = 0, 2$ for $e \equiv 0, 2 \mod 6$, respectively, we see that if $rg + s\gamma$ is an effective divisor then $r > 0$ or $3r + s > 0$, respectively.

To construct the table, we found the two pairs (r, s) with $q(rg + s\gamma) = -2$, such that the lines perpendicular to the divisor classes $rg + s\gamma$ define the chamber which contains the ample class g. In case both extremal rays correspond to divisorial contractions, we checked that the results from Proposition 3.5 hold.

In the case $e = 74$, we used Magma to verify that the BBF form on $\mathrm{Pic}(F_e)$ is not equivalent to the quadratic form $< 74 > \oplus < -2 >$, hence there are no exceptional classes in $\mathrm{Pic}(F_e)$.

In case $e \equiv 0 \mod 6$ the BBF-form q on the basis g, γ of $\mathrm{Pic}(F)$ is in diagonal form, $q(xg + y\gamma) = 6x^2 - (e/3)y^2$. The ample class g lies in the movable cone and thus the two boundary, extremal, rays of the movable cone are spanned by big and nef divisor classes $ng \pm m\gamma$ with $n > 0$ and some m, similarly the (-2)-classes defining these rays are $\tau = rg \pm s\gamma$ with $r > 0$ and some s.

Each fiber f, a smooth rational curve, of the conic bundle $E \to S$ defines a scroll $\Sigma_f \subset Y$ in the cubic fourfold. This scroll is the union of the lines in Y that correspond to the points in $f \subset F(Y)$. The degree of this scroll is the intersection number $g \cdot f$, where $g \in \mathrm{Pic}(F)$ is the Plücker class, so it is the degree of the rational curve f in the Grassmannian of lines in $\mathbf{P}^5$ (cf. [2, section 7.1]). It can be computed as follows. Since H contracts E to S, we have $Hf = 0$. Recall that $Ef = -2$ (see the proof of Proposition 3.5) and that $\mathrm{Pic}(F) = \mathbf{Z}H \oplus \mathbf{Z}\tau$ (see Theorem 5.4) with $q(\tau) = -2$, in particular the Plücker class $g = rH + s\tau$ for certain integers r, s. In case $E = \tau$, a generator of $\mathrm{Pic}(F)$, we have $\tau f = -2$, otherwise the type is H and $E = 2\tau$ and $\tau f = -1$. Therefore,

$$\deg(\Sigma_f) \;=\; g \cdot f \;=\; \begin{cases} -2s & \text{if} \quad E \;=\; \tau\,, \\ -s & \text{if} \quad E \;=\; 2\tau\,, \end{cases} \qquad \text{where } g \;=\; rH \;+\; s\tau\,.$$

5.6 | Example: $e = 14$

Table 3 shows that if $e = 14$, then there is a contraction of type H. Thus, the Fano variety of lines F of a general cubic fourfold Y in $\mathcal{C}_{14}$ is birational to $S^{[2]}$ for a K3 surface S of degree 14. The transcendental lattice of such an S is isomorphic to the one of F, see Table 1. Since a K3 surface of degree 14 is uniquely determined by its transcendental lattice, S is uniquely determined by F (i.e., S has no FM-partners, cf. [11, Proposition 1.10] and Section 2.11). There are no (-10)-classes in $\mathrm{Pic}(F)$, hence F is actually isomorphic to $S^{[2]}$. See [20] and [21, section 6.2] for this result and for the geometrical relations between S, Y, and F.

Table 3 also shows that there is a second contraction of F of type M_3, so the exceptional divisor is the projectivized Mukai bundle over S. We recall some results that identify the exceptional divisor of this contraction.

The general $Y \in C_{14}$ is pfaffian, that is, the equation of Y in $\mathbf{P}^5$ is given by $pf(M_x) = 0$ where M_x is an alternating 6×6 matrix with coefficients that are linear forms in the coordinates on $\mathbf{P}^5$. Since Y is smooth, the rank of M_x in $x \in Y$ is 4 and the kernel of M_x is a two-dimensional subspace of a $\mathbf{C}^6$, so defines a line L_x in a $\mathbf{P}^5$. The union of these lines is a five-dimensional subvariety $B \subset \mathbf{P}^5 \times \mathbf{P}^5$ ([21, Chapter 6, (2.1)], [20]):

$$\begin{array}{rcl} B & := \{(v,x) : v \in L_x\} \stackrel{q}{\longrightarrow} Y & \subset \mathbf{P}^5. \\ & p \downarrow & \\ & \mathbf{P}^5 & \end{array}$$

The map p is a birational isomorphism, its inverse is the blow up of $\mathbf{P}^5$ in a threefold Z of degree 9, defined by six quartics ([21, Lemma 6.2.15], this threefold also appears in [27]).

For a point $z \in Z$, one obtains a line $q(p^{-1}(z)) \subset Y$, this gives an embedding $Z \hookrightarrow F$ [21, Remark 6.2.18]. From Proposition 2.12, one deduces that Z is the exceptional divisor of a contraction of F, this contraction is the one of type M_3. From the description of contractions of type M_3 it follows that $Z \cong \mathbf{P}\mathcal{U} \to S$ where $\mathcal{U}$ is, up to a twist by a line bundle, the Mukai bundle on S, cf. Section 2.4. This bundle has Mukai vector $v = (2, h, 4)$, so $v^2 = -2$ and the stable bundle $\mathcal{U}$ with this Mukai vector has $\det(\mathcal{U}) = h$, $c_2(\mathcal{U}_v) = 5$, and $h^0(\mathcal{U}) = 6$ (see Section 1.4.). As $H^0(\mathbf{P}\mathcal{U}_v, \mathcal{O}_{\mathbf{P}\mathcal{U}_v}(1)) \cong H^0(S, \mathcal{U}_v)$, we obtain a map from $\mathbf{P}\mathcal{U}$ to $\mathbf{P}^5$, its image should be Z. The kernel of the evaluation map $H^0(\mathcal{U}) \otimes \mathcal{O}_S \to \mathcal{U}$ will give the "Gushel" embedding $S \hookrightarrow Gr(4, H^0(\mathcal{U}_v)) \cong Gr(1,5)$, which has the property that $\mathcal{U}$ is the restriction of the universal bundle on $Gr(1,5)$ (cf. [8] and [21, Corollary 6.2.16]).

The proof of the rationality of Y given in [20] and [21, section 6.2.7] consists in showing that for a general hyperplane H in $\mathbf{P}^5$, the inverse image of H under the birational map p is mapped by q birationally onto Y. The intersection $H \cap Z$ is a K3 surface of degree 14 blown up in 5 points. This surface is defined by a global section of the Mukai bundle $\mathcal{U}_v$, which indeed has $c_2 = 5$. Fano's proof of the rationality, as presented in [28, Theorem 4.1], used that Y contains a smooth quintic del Pezzo surface V. The five quadrics that generate the ideal of V map X birationally onto $H \cong \mathbf{P}^4$. The inverse map is again the one used in [20] and [21, section 6.2.7], so both proofs use the same birational isomorphism between $\mathbf{P}^4$ and X.

The scrolls $\Sigma_f \subset Y$, where f is a fiber of $Z \to S$, are well-known, see, for example, [21, 6.2.6], they indeed have degree 4 (whereas those associated to $\mathbf{P}\mathcal{T}_S \subset S^{[2]}$ have degree 5). In [28, Theorem 4.3], one finds that Fano also used the quartic scrolls in a proof of the rationality of the general cubic fourfold in C_{14} and that these scrolls are closely related to smooth quintic del Pezzo surfaces in Y.

5.7 | Rationality of Cubic Fourfolds

The cubic fourfolds in the Hassett divisors C_e, with $e = 14, 26, 38, 42$ are known to be rational (established by Morin and Fano for $e = 14$ and by Russo and Staglianó [29, 30] in the other cases). The types of the extremal rays of the corresponding Fanos do not seem to have an obvious pattern and it is not clear that the knowledge of the types of the extremal rays is useful for establishing the rationality. However, the proofs of the rationality often use scrolls in the cubic fourfolds obtained from fibers of the conic bundle structure $p : E \to S$ of an exceptional divisor. Notice that the cubic fourfolds Y in C_e with $e = 74$ are conjectured to be rational, but the Fano variety of lines of the general Y in C_{74} does not have any extremal ray.

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Conflicts of Interest

The authors declare no conflicts of interest.

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