# What is a limit? Concept image of limits as time goes to infinity in life sciences students

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Math is scary. For life science students, maths is even scarier. Poor results in mathematics are amongst the main reasons for dropping out of STEM courses, particularly in life sciences — negative attitudes and disengagement being the main reasons for students' failure. In the absence of exogenous instrumentality — e.g., a good grade in calculus as a prerequisite to enter medical school — we should strive to engage students with active learning that could at least pretend to carry intrinsic motivations (Husman & al., 2004). Or, in other words, we want students to work on tasks that appear to be relevant (to them!) in order to build a conceptual understanding of the mathematical objects at stake.

## The theoretical framework

### Mental Pictures

A mathematical object does not exist in the real world, but only as a *concept* $C(P)$ in the mind of a person $P$, with $C(P)$ — the *mental picture* of $C$ by $P$ — consisting in a set of pictures that $P$ associates to the name of $C$.

> The word 'pictures' here is used in the broadest sense of the word and it includes any visual representation of the concept (even symbols). Thus a graph of a specific function and the symbols '$y = f(x)$' might be included (together with many other things) in someone's mental picture of the concept of function. (Vinner, 1983)

### Concept Image

A person $P$ could likely associate a set of properties (some of them correct, some of them incorrect) to each concept: e.g., $P$ might think that every odd function will be defined in 0, or she might think that any continuous function on $[0,1]$ has a maximum. Vinner (1983) calls these properties held by $P$ about $C$ together with her mental pictures of $C$ the *concept image* (of $C$ by $P$).

The definition of the concept will be just one, if any, component of the concept image held by $P$. Unless one requires definitions for definitions sake — "Students need to *know* the definition of continuous function in order to pass my course!"[1] — concept definitions are useful at best to help to generate concept images. Usually "concept definitions will remain inactive or even will be forgotten[;] in thinking, almost always the concept image will be evoked" (*ibidem*). "The formal definition [of a mathematical concept] should be only a conclusion of the various examples introduced to the students" (Vinner & Dreyfus, 1989).

[1] Emphasis on 'know:' 'understanding' is not really required.

## The context

I have been teaching *Istituzioni di matematiche* to Natural Sciences students at the University of Milan since 2015. Students are on average quite weak, with many of them not reaching the minimal level of competence in mathematics required by the degree programme. The course has been taught using the flipped classroom since 2021, with classroom time spent working on problems (Rizzo, 2022).

Almost all students graduated from an Italian high school, hence we know that they have been exposed to some calculus (at least up to derivatives) and have spent a significant amount of time learning to sketch a qualitative graph of functions.

## Limits of real functions

Following Vinner's theoretical framework, I presented the concept of $\lim_{t\to+\infty} f(t) = \ell$ in the following way: "As time goes by, the measure of $f(t)$ gets undistinguishable from $\ell$: if you get hold of better instruments, it will take some more time, but eventually it will get again undistinguishable." The formal definition is then presented as a mathematization of this image, mainly in the hope that those that learned it by heart in school will now try to make some sense out of it.

### Functions will always converge monotonically

I asked students the following questions (translated into English for the reader's sake)

Ovotransferrin — which makes up 12% of egg proteins — denatures at 62º C. We repeat an experiment with initial temperature $T(0) = A$ and $\lim_{t\to+\infty} T(t) = B$

1: If $A = 40$ and $B = 70$, what happens?

2: If $A = 90$ and $B = 25$, what happens?

3: If $A = 40$ and $B = 60$, what happens?

Students answered anonymously, using instant polling software on their phones, working in small groups or occasionally on their own. Given the classroom configuration it would have been impractical to form random groups, so they were self-selected. In Table 1 we present cumulative answers from the classes of 2021 and 2022: in total 80 students took part to the questions, divided into 41 groups. The question was open ended, and the categorisation was built ad hoc from the answers.

| Question 1 | | Question 2 | | Question 3 | |
|---|---|---|---|---|---|
| Boiled eggs | 66% | It stays boiled (but it cools down) | 61% | Proteins might have denatured | 10% |
| Denatured proteins | 7% | A cold egg | 5% | Nothing | 54% |
| Non sensical | 5% | Non sensical | 5% | Non sensical | 2% |
| No answer | 22% | No answer | 29% | No answer | 34% |

**Table 1: Categorisation of students' answers**

As it could be expected, questions 1 and 2 were quite easy, at least for those students that could make sense of the question. It could be interesting to notice that only 7% of students were not able to make the step from the denaturation of proteins to the albumen becoming solid. Some missing answer can be explained by students joining the poll individually but answering collectively; given that anonymity is part of the activity design, I have no way to distinguish such a case from students giving up to peruse social networks.

Some of the answers to question 1 point to monotonical convergence to the limit 70º C, since they are quite clearly referring to Newton's law of cooling, which I presented as the first example of limit for $t \to +\infty$ (all answers are translated by the author):

> For certain, it is not at 70º C since physics is not a matter of opinion.
>
> The egg proteins degrade and have a temperature close to but not equal to 70º C.

What is interesting is analysing answers to question 3: only three groups (and only in the class of 2022) were able to postulate that the temperature function might have exceeded 62º C, and not necessarily so in a mathematical correct sentence:

> The egg might have denatured if during the function the temperature passed 62 degrees, otherwise it stays as it was.
>
> The protein might have denatured if in the interval between 40 and 60 degrees the temperature went over 62 degrees.
>
> Ovotransferrin could have started to denature.

Notice that the last sentence could also be interpreted as pointing to a different interpretation: denaturation is actually not an on/off reaction, but a statistical one. Indeed, another answer was:

> So, some proteins denature, many others don't.

Most answers claim that nothing happens to the egg, or that it just gets warmed up. Given the answers to the first two questions, we can assume that most students mean that the temperature never passes 62º C. Some answers, though, affirm more explicitly that the $T$ increases monotonically:

> The $t$ to which it tends is not sufficient to denature proteins.
>
> The internal temperature changes but the protein will not denature.
>
> The protein will not denature. The egg warms up with no chemical reaction. We do not know what happens to other proteins.

Finally, some answers show that the concept of infinity as a mathematical object that stands for "given enough time" is clear:

> Proteins will not denature so the egg doesn't become hard, but it will rot if left for too long.

This shows that, unless we are given a data point (for example, from the computation of stationary points) that says otherwise, most student will associate the (incorrect) mental image of monotonicity to the concept of convergence at $+\infty$.

**In a neighbourhood of infinity**

In the following lecture, where actual computation techniques were shown, I asked students (again, as an open question with answers to be categorised later on):

> Suppose that the thickness of the subcutaneous layer of fat in a brooding penguin is given, in centimetres, by the function $F(t) = \exp(-0{,}05t^3 + 1{,}8)$; which meaning can we give to the limit of $F(t)$ as $t \to +\infty$?

The 2021 and 2022 classes behaved quite differently: the former formed 9 groups out of 38 students, the latter 22 groups out of 49 students. We see in Table 2 students' answers by year.

I do not know if the larger groups of the 2021 class allowed a much greater percentage of students to *get* the answer. Yet, we see that a significant number of students read the question as if it asked to *compute* the limit, showing a concept image of limits as the result of a computation.

| | **2021** | **2022** |
|---|---|---|
| The penguin died | 44% | 9% |
| The limit is 0 / the penguin gets thinner | 22% | 73% |
| It helps to understand what happens with time | 33% | 0% |
| There is no meaning | 11% | 0% |
| Non sensical | 0% | 18% |

**Table 2: Categorisation of students' answers**

## Conclusions

We can recognise three different images of the concept $\lim_{t \to +\infty}$ : a computation, the value at infinity and an approximate value given enough time. As expected, no student made explicit reference to the concept definition, although many of them most likely had learned it by heart the previous year in high school. In the ovotrasferrin case, where no computation was possible, most students were able to semiotically convert the mathematical meaning to the "in the model" meaning; in the penguin case this occurs to a much lesser extent: we could attribute this to the attraction of the "computation" image. If the learning of calculus has no exogenous instrumentality and has to be justified to life science students with intrinsic motivations, these could only be in relevant mathematical models. Hence, I believe in the importance of bringing the "approximate value given enough time" image to the forefront. Moreover, computational ability should not be overemphasised, but we should rather encourage critical thinking on what one could infer from the mathematical data.

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