

# Local limit laws for symbol statistics in bicomponent rational models \*

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## Abstract

We study the local limit distribution of the number of occurrences of a symbol in words of length  $n$  generated at random in a regular language according to a rational stochastic model. We present an analysis of the main local limits when the finite state automaton defining the stochastic model consists of two primitive components. The limit distributions depend on several parameters and conditions, such as the main constants of mean value and variance of our statistics associated with the two components, and the existence of communications from the first to the second component. The convergence rate of these results is always of order  $O(n^{-1/2})$ . We also prove an analogous  $O(n^{-1/2})$  convergence rate to a Gaussian density of the same statistic whenever the stochastic models only consists of one (primitive) component.

**Keywords:** limit distributions, local limit laws, pattern statistics, regular languages.

## 1 Introduction

In this work we present some local limit laws concerning the number of symbol occurrences in words of given length chosen at random in a regular language under suitable probabilistic hypotheses. Here the random words are assumed to be generated according to a *rational stochastic model* as introduced in [4], which is defined by a (nondeterministic) finite state automaton with real positive weights on transitions. In this setting the probability of generating a word  $w$  is proportional to the total weight of the transitions labelled by  $w$  that are accepted by the automaton; thus, the language recognized by the automaton is the family of all words having non-null probability to be generated. This model is quite general, it includes as special cases the traditional Bernoullian and Markovian sources [16, 17]. It also includes the generation of random words of length  $n$  in any regular language under uniform distribution: this occurs when the automaton is unambiguous and all transitions have the same weight.

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\*This report includes and improves the results presented in [11, 12, 13].

In order to fix ideas, consider a weighted finite state automaton  $\mathcal{A}$  over an input alphabet including a special symbol  $a$  and, for every  $n \in \mathbb{N}$ , let  $Y_n$  be the symbol statistics representing the number of occurrences of  $a$  in a word of length  $n$  generated at random according to the rational model defined by  $\mathcal{A}$ . We are interested in the asymptotic properties of the sequence of random variables  $\{Y_n\}$ . These properties turn out to be of interest for the analysis of regular patterns occurring in words generated by Markovian models [4, 16, 17] and for the asymptotic estimate of the coefficients of rational series in commutative variables [4, 5]. They are also related to the descriptive complexity of languages and computational models [6], as well as to the analysis of additive functions defined on regular languages [14]. Clearly, the behaviour of  $\{Y_n\}$  depends on automaton  $\mathcal{A}$ . It is known that if  $\mathcal{A}$  has a primitive transition matrix then  $Y_n$  has a Gaussian limit distribution [4, 16] and, under a suitable aperiodicity condition, it also satisfies a local limit theorem [4]. The limit distribution of  $Y_n$  in the global sense is known also when the transition matrix of  $\mathcal{A}$  consists of two primitive components [8].

Here we improve the results of [4, 8] presenting an analysis of the local limits of  $\{Y_n\}$  when the transition matrix of  $\mathcal{A}$  consists of two primitive components. At the cost of adding suitable aperiodicity conditions, we prove that the main convergence properties in distribution obtained in [8] also hold true in the local sense with a convergence rate of the order  $O(n^{-1/2})$ . We also obtain an analogous  $O(n^{-1/2})$  rate of convergence for the Gaussian local limit law of  $\{Y_n\}$  when the rational model consists of only one primitive component, so refining the result obtained in [4]. We recall that finding a tight convergence rate in central limit theorems is a natural goal of research [15], which measures the approximation speed of the probability values to the prescribed expression.

Our proofs are based on the analysis of the characteristic function of  $Y_n$ . In particular we apply Laplace's method to evaluate the classical integral expression of this function that yields the probability values of interest. This method is a general strategy to evaluate asymptotic expressions of integrals depending on a growing parameter (see for instance [9, Sec. B.6]). It is also used in the literature for different purposes, for instance to prove local versions of the Central Limit Theorem [10, Sec. 42] or for applications of the Saddle Point Method to combinatorial problems [9, Ch. VIII]. The main difference with respect to these classical approaches is that in our work Laplace's method is often applied to non-Gaussian integrals, yielding (local) limit distributions that are not normal.

The material we present is organized as follows. In the next section we define the problem and fix our notation. In Section 3 we consider the rational stochastic models with primitive transition matrix and show how in this case Laplace's method leads to a local limit law of Gaussian type for  $\{Y_n\}$  with a convergence rate of the order  $O(n^{-1/2})$ . In the same section we restate precisely an aperiodicity condition for the primitive stochastic model that is also necessary for the subsequent analysis. In Section 4 we study the behaviour of  $Y_n$  when the rational stochastic model consists of two primitive components and has one or more transitions from the first to the second component. In this case the following points summarize our results:

1. If the two components have different main eigenvalues (and hence there is a dominant component), a Gaussian local limit law holds true. Here the aperiodicity condition is assumed only for the dominant component.
2. If the two components have equal main eigenvalue (equipotent model) the limit distribution depends on the values of four constants:  $\beta_1, \gamma_1$  and  $\beta_2, \gamma_2$ , representing the leading terms of

mean value and variance of our statistics associated with the first and the second component, respectively. In this case we assume the aperiodicity condition for both components and the results are as follows:

- 2a.** If  $\beta_1 \neq \beta_2$  we get a local limit law for  $Y_n/n$  towards a uniform density.
- 2b.** If  $\beta_1 = \beta_2$  but  $\gamma_1 \neq \gamma_2$  we get a local limit law for  $(Y_n - \beta n)/\sqrt{n}$  towards a suitable mixture of Gaussian densities.
- 2c.** If  $\beta_1 = \beta_2$  and  $\gamma_1 = \gamma_2$  we obtain again a Gaussian local limit.

In Section 5 we study the behaviour of  $Y_n$  in the bicomponent models that have no communication between the components. Also in this case, if there is a dominant component we get a Gaussian local limit. On the contrary, in the equipotent case we get the following properties:

- 3a.** If  $\beta_1 \neq \beta_2$  and/or  $\gamma_1 \neq \gamma_2$  we get a local limit law for  $Y_n$  towards a convex linear combination of two Gaussian densities;
- 3b.** If  $\beta_1 = \beta_2$  and  $\gamma_1 = \gamma_2$  we obtain again a Gaussian local limit for  $Y_n$ .

All the local limit laws obtained above hold with a convergence rate  $O(n^{-1/2})$ .

Finally, in the last section we summarize and compare our results in a suitable table, also discussing possible goals for future investigations.

## 2 Problem setting

As usual we denote by  $\{a, b\}^*$  the set of all words over the binary alphabet  $\{a, b\}$ , including the empty word  $\varepsilon$ . Together with the operation of concatenation between words,  $\{a, b\}^*$  forms a monoid, called *free monoid* over  $\{a, b\}$ . For every word  $w \in \{a, b\}^*$  we denote by  $|w|$  the length of  $w$  and by  $|w|_a$  the number of occurrences of  $a$  in  $w$ . For each  $n \in \mathbb{N}$ , we also represent by  $\{a, b\}^n$  the set  $\{w \in \{a, b\}^* : |w| = n\}$ . A *formal series* in the non-commutative variables  $a, b$  is a function  $r : \{a, b\}^* \rightarrow \mathbb{R}_+$ , where  $\mathbb{R}_+ = \{x \in \mathbb{R} \mid x \geq 0\}$ , and for every  $w \in \{a, b\}^*$  we denote by  $(r, w)$  the value of  $r$  at  $w$ . Such a series  $r$  is called *rational* if for some integer  $m > 0$  there is a monoid morphism  $\mu : \{a, b\}^* \rightarrow \mathbb{R}_+^{m \times m}$  and two (column) arrays  $\xi, \eta \in \mathbb{R}_+^m$ , such that  $(r, w) = \xi' \mu(w) \eta$ , for every  $w \in \{a, b\}^*$  [3, 18]. In this case, as the morphism  $\mu$  is generated by matrices  $A = \mu(a)$  and  $B = \mu(b)$ , we say that the 4-tuple  $(\xi, A, B, \eta)$  is a *linear representation* of  $r$  of size  $m$ . Clearly, such a 4-tuple can be considered as a finite state automaton over the alphabet  $\{a, b\}$ , with transitions (as well as initial and final states) weighted by positive real values. Throughout this work we assume that the set  $\{w \in \{a, b\}^n : (r, w) > 0\}$  is not empty for every  $n \in \mathbb{N}_+$  (so that  $\xi \neq 0 \neq \eta$ ), and that  $A$  and  $B$  are not null matrices, i.e.  $A \neq [0] \neq B$ . Then we can consider the probability measure  $\text{Pr}$  over the set  $\{a, b\}^n$  given by

$$\text{Pr}(w) = \frac{(r, w)}{\sum_{x \in \{a, b\}^n} (r, x)} = \frac{\xi' \mu(w) \eta}{\xi' (A + B)^n \eta} \quad \forall w \in \{a, b\}^n$$

Note that, if  $r$  is the characteristic series of a language  $L \subseteq \{a, b\}^*$  then  $\text{Pr}$  is the uniform probability function over the set  $L \cap \{a, b\}^n$ . Thus we can define the random variable (r.v.)  $Y_n = |w|_a$ , where

$w$  is chosen at random in  $\{a, b\}^n$  with probability  $\Pr(w)$ . As  $A \neq [0] \neq B$ ,  $Y_n$  is not a degenerate random variable. It is clear that, for every  $k \in \{0, 1, \dots, n\}$ ,

$$p_n(k) := \Pr(Y_n = k) = \frac{\sum_{|w|=n, |w|_a=k} (r, w)}{\sum_{w \in \{a, b\}^n} (r, w)}$$

Since  $r$  is rational also the previous probability can be expressed by using its linear representation. It turns out that

$$p_n(k) = \frac{[x^k] \xi'(Ax + B)^n \eta}{\xi'(A + B)^n \eta} \quad \forall k \in \{0, 1, \dots, n\} \quad (1)$$

where, as usual, for any function  $G$  analytic in a neighbourhood of 0 with series expansion  $G(x) = \sum_{n=0}^{+\infty} g_n x^n$ , we denote by  $[x^k]G(x)$  the coefficient  $g_k$ , for every  $k \in \mathbb{N}$ .

For sake of brevity we say that  $Y_n$  is *defined* by the linear representation  $(\xi, A, B, \eta)$ . To deal with the characteristic function  $\Psi_n(t)$  of  $Y_n$  we introduce the map  $h_n(z)$  given by

$$h_n(z) = \xi'(Ae^z + B)^n \eta \quad \forall z \in \mathbb{C} \quad (2)$$

and hence we have

$$\Psi_n(t) = \sum_{k=0}^n p_n(k) e^{itk} = \frac{\xi'(Ae^{it} + B)^n \eta}{\xi'(A + B)^n \eta} = \frac{h_n(it)}{h_n(0)} \quad \forall t \in \mathbb{R} \quad (3)$$

As a consequence, mean value and variance of  $Y_n$  can be evaluated by

$$\mathbb{E}(Y_n) = \frac{h'_n(0)}{h_n(0)}, \quad \text{Var}(Y_n) = \frac{h''_n(0)}{h_n(0)} - \left( \frac{h'_n(0)}{h_n(0)} \right)^2 \quad (4)$$

Here we are mainly interested in the local limit properties of  $\{Y_n\}$ . To compare the notion of local convergence with the traditional one, we recall that a sequence of r.v.'s  $\{X_n\}$  *converges in distribution* (or in law) to a random variable  $X$  of distribution function  $F$  if  $\lim_{n \rightarrow +\infty} \Pr(X_n \leq x) = F(x)$ , for every  $x \in \mathbb{R}$  of continuity for  $F$  [10]. The central limit theorems yield classical examples of convergence in distribution to a Gaussian random variable.

On the other hand, a local limit law for a sequence of discrete r.v.'s  $\{X_n\}$  establishes, as  $n$  grows to infinity, an asymptotic expression for the probability values of  $X_n$  depending on a given density function (see for instance [2, 9, 10]). More precisely, assume that each  $X_n$  takes value in  $\{0, 1, \dots, n\}$ . We say that  $\{X_n\}$  *satisfies a local limit law* of Gaussian type if there are two real sequences  $\{a_n\}$ ,  $\{s_n\}$ , where  $\mathbb{E}(X_n) \sim a_n$ ,  $\text{Var}(X_n) \sim s_n^2$  and  $s_n > 0$  for all  $n$ , such that for some real  $\epsilon_n \rightarrow 0$ , the relation

$$\left| s_n \Pr(X_n = k) - \frac{e^{-\left(\frac{k-a_n}{s_n}\right)^2/2}}{\sqrt{2\pi}} \right| \leq \epsilon_n \quad (5)$$

holds uniformly for every  $k \in \{0, 1, \dots, n\}$  and every  $n \in \mathbb{N}$  large enough. Here,  $\epsilon_n$  yields the *convergence rate* (or the speed) of the law. A well-known example of such a property is given by the de Moivre-Laplace local limit theorem [10].

Similar definitions can be given for other (non-Gaussian) types of local limit laws. In this case the Gaussian density  $e^{-x^2/2}/\sqrt{2\pi}$  appearing in (5) is replaced by some density function  $f(x)$ ; clearly, if  $f(x)$  is not continuous at some points, the uniformity with respect to  $k$  must be adapted to the specific case.

We recall that in general convergence in distribution does not imply a local limit law; usually, some further regularity condition is necessary to guarantee a local limit behaviour [9, 10].

### 3 Primitive models

A relevant case occurs when  $M = A + B$  is primitive, i.e.  $M^k > 0$  for some  $k \in \mathbb{N}$  [19]. In this case it is known that  $Y_n$  has a Gaussian limit distribution and satisfies a local limit property [4, 16]. Here we improve this result, showing a rate of convergence  $O(n^{-1/2})$ ; we also recall some properties proved in [4, 5] that are useful in the following sections.

Since  $M$  is primitive, by Perron-Frobenius Theorem, it admits a real eigenvalue  $\lambda > 0$  greater than the modulus of any other eigenvalue. Moreover, strictly positive left and right eigenvectors  $\zeta$ ,  $\nu$  of  $M$  with respect to  $\lambda$  can be defined so that  $\zeta'\nu = 1$  [19]. Thus, we can consider the function  $u = u(z)$  implicitly defined by equation

$$\text{Det}(Iu - Ae^z - B) = 0$$

subject to condition  $u(0) = \lambda$ . It turns out that, in a neighbourhood of  $z = 0$ ,  $u(z)$  is analytic, it is a simple root of the characteristic polynomial of  $Ae^z + B$ , and  $|u(z)|$  is strictly greater than the modulus of all other eigenvalues of  $Ae^z + B$ . Moreover, a precise relationship between  $u(z)$  and function  $h(z)$ , defined in (2), is proved in [4] stating that there are two positive constants  $c$ ,  $\rho$  and a function  $r(z)$  analytic and non-null at  $z = 0$ , such that

$$h_n(z) = r(z) u(z)^n + O(\rho^n) \quad \forall z \in \mathbb{C} : |z| \leq c \quad (6)$$

where  $\rho < |u(z)|$  and in particular  $\rho < \lambda$ .

Mean value and variance of  $Y_n$  can be estimated from relations (6) and (4). It turns out [4] that the constants

$$\alpha = \xi'\nu\zeta'\eta, \quad \beta = \frac{u'(0)}{\lambda} \quad \text{and} \quad \gamma = \frac{u''(0)}{\lambda} - \left(\frac{u'(0)}{\lambda}\right)^2 \quad (7)$$

are strictly positive and satisfy the equalities

$$E(Y_n) = \beta n + O(1), \quad \beta = \frac{\nu' A \zeta}{\lambda}, \quad \text{and} \quad \text{var}(Y_n) = \gamma n + O(1)$$

Other properties concern function  $y(t) = u(it)/\lambda$ , defined for real  $t$  in a neighbourhood of 0. In particular, there exists a constant  $c > 0$  for which identity (6) holds true, satisfying the following relations [4]:

$$|y(t)| = 1 - \frac{\gamma}{2}t^2 + O(t^4), \quad \arg y(t) = \beta t + O(t^3), \quad |y(t)| \leq e^{-\frac{\gamma}{4}t^2} \quad \forall |t| \leq c \quad (8)$$

The behaviour of  $y(t)$  can be estimated precisely when  $t$  tends to 0. For any  $q$  such that  $1/3 < q < 1/2$  it can be proved [4] that

$$y(t)^n = e^{-\frac{\gamma}{2}t^2n + i\beta tn}(1 + O(t^3)n) \quad \text{for } |t| \leq n^{-q} \quad (9)$$

Now, in order to prove a local limit property for  $\{Y_n\}$  it is necessary to introduce an aperiodicity assumption for the stochastic model, studied in more detail in [5]. To state this condition properly, consider the transition graph of the finite state automaton defined by matrices  $A$  and  $B$ , i.e. the directed graph  $G$  with vertex set  $\{1, 2, \dots, m\}$  such that, for every  $i, j \in \{1, 2, \dots, m\}$ ,  $G$  has an edge from  $i$  to  $j$  labelled by a letter  $a$  ( $b$ , respectively) whenever  $A_{ij} > 0$  ( $B_{ij} > 0$ , resp.). We can denote by  $d$  the GCD of all differences in the number of occurrences of  $a$  in words representing (labels of) cycles of  $G$  having equal length. More formally, for every cycle  $\mathcal{C}$  in  $G$  let  $\ell(\mathcal{C}) \in \{a, b\}^*$  be the word obtained by concatenating the labels of all transitions in  $\mathcal{C}$  in their order; we define

$$d = \text{GCD}\{|\ell(\mathcal{C}_1)|_a - |\ell(\mathcal{C}_2)|_a : \mathcal{C}_1, \mathcal{C}_2 \text{ cycles in } G \text{ and } |\mathcal{C}_1| = |\mathcal{C}_2|\}$$

Then, we say that the pair  $(A, B)$  is *aperiodic* if  $d = 1$ . Note that such a condition is often verified, for instance  $d = 1$  whenever  $A_{ij} > 0$  and  $B_{ij} > 0$  for two (possibly equal) indices  $i, j$ . Moreover, it can be proved [5] that  $(A, B)$  is aperiodic if and only if, for every real  $t$  such that  $0 < t < 2\pi$ , we have

$$|\mu| < \lambda \quad \text{for every eigenvalue } \mu \text{ of } Ae^{it} + B \quad (10)$$

**Theorem 1** *Let  $\{Y_n\}$  be defined by a linear representation  $(\xi, A, B, \eta)$  such that matrix  $M = A + B$  is primitive,  $A \neq [0] \neq B$  and the pair  $(A, B)$  is aperiodic. Moreover, let  $\beta$  and  $\gamma$  be defined by equalities (7). Then, as  $n$  tends to  $+\infty$ , the relation*

$$\left| \sqrt{n} \text{Pr}(Y_n = k) - \frac{e^{-\frac{(k-\beta n)^2}{2\gamma n}}}{\sqrt{2\pi\gamma}} \right| = O(n^{-1/2}) \quad (11)$$

holds true uniformly for every  $k \in \{0, 1, \dots, n\}$ .

The statement is clearly meaningful when  $k$ , depending on  $n$ , varies so that  $x = \frac{k-\beta n}{\sqrt{2\gamma n}}$  lies in a finite interval. In this case, we have  $\text{Pr}(Y_n = k) = \frac{e^{-x^2}}{\sqrt{2\pi\gamma n}} + O(n^{-1})$ .

To prove the theorem, we study the characteristic function  $\Psi_n(t)$  for  $t \in [-\pi, \pi]$  by splitting this interval into three sets:

$$[-n^{-q}, n^{-q}], \quad \{t \in \mathbb{R} : n^{-q} < |t| \leq c\}, \quad \{t \in \mathbb{R} : c < |t| \leq \pi\}, \quad (12)$$

where  $c \in (0, \pi)$  is a constant satisfying relations (8) and  $q$  is an arbitrary value such that  $\frac{1}{3} < q < \frac{1}{2}$ . We get the following three propositions where we always assume the hypotheses of Theorem 1.

**Proposition 1** *For every  $c \in (0, \pi)$  there exists  $\varepsilon \in (0, 1)$  such that*

$$|\Psi_n(t)| = O(\varepsilon^n) \quad \forall t \in \mathbb{R} : c \leq |t| \leq \pi.$$

*Proof.* First note that by property (10), the aperiodicity of  $(A, B)$  implies that, for every  $c \in (0, \pi)$  there exists  $\tau \in (0, \lambda)$  such that  $|\mu| < \tau$  for every eigenvalue  $\mu$  of  $Ae^{it} + B$  and every  $t \in \mathbb{R}$  satisfying  $c \leq |t| \leq \pi$ . Also, by equality (2), the generating function of  $\{h_n(it)\}_n$  is given by

$$\sum_{n=0}^{+\infty} h_n(it)y^n = \xi' (I - (Ae^{it} + B)y)^{-1} \eta = \frac{\xi' \text{Adj}(I - (Ae^{it} + B)y) \eta}{\text{Det}(I - (Ae^{it} + B)y)}$$

and hence its singularities are the inverses of the eigenvalues of  $Ae^{it} + B$ . As a consequence,  $|h_n(it)| = O(\tau^n)$  whenever  $c \leq |t| \leq \pi$ . Moreover, from (6) we know that  $h_n(0) = \Theta(\lambda^n)$  <sup>(1)</sup> and hence, for some  $\varepsilon \in (0, 1)$ , we have

$$|\Psi_n(t)| = \left| \frac{h_n(it)}{h_n(0)} \right| = \frac{O(\tau^n)}{\Theta(\lambda^n)} = O(\varepsilon^n) \quad \forall t \in \mathbb{R} : c \leq |t| \leq \pi$$

□

**Proposition 2** *Let  $c \in (0, \pi)$  satisfy relation (8). Then, for every  $t \in \mathbb{R}$  such that  $n^{-q} \leq |t| \leq c$  we have*

$$|\Psi_n(t)| = O\left(e^{-\frac{\gamma}{4}n^{1-2q}}\right)$$

*Proof.* By relation (6), since  $y(0) = 1$ , there exists  $\rho \in (0, \lambda)$  such that

$$\Psi_n(t) = \frac{h_n(it)}{h_n(0)} = \frac{r(it)\lambda^n y(t)^n + O(\rho^n)}{r(0)\lambda^n + O(\rho^n)} \quad \forall t \in \mathbb{R} : |t| \leq c \quad (13)$$

Since  $r(z)$  is analytic in a neighbourhood of 0, we have

$$|\Psi_n(t)| = (1 + O(t))|y(t)|^n + O(\varepsilon^n), \quad \text{for } 0 < \varepsilon < 1.$$

Also, by inequality (8), we know that  $|y(t)|^n \leq e^{-\frac{\gamma}{4}t^2n}$  whenever  $|t| \leq c$ . Thus, the result follows by replacing this bound in the previous equation and recalling that  $n^{-q} \leq |t| \leq c$ .

□

**Proposition 3** *For any  $q$  such that  $1/3 < q < 1/2$ , we have*

$$\int_{|t| \leq n^{-q}} \left| \Psi_n(t) - e^{-\frac{\gamma}{2}t^2n + i\beta tn} \right| dt = O(n^{-1})$$

*Proof.* Reasoning as in Proposition 2, from relation (13) we know that  $\Psi_n(t) = (1 + O(t))y(t)^n + O(\varepsilon^n)$  for some  $\varepsilon \in (0, 1)$  and every  $t \in \mathbb{R}$  such that  $|t| \leq c$ . Thus, applying relation (9) and recalling that  $nO(t^3) = o(1)$  for  $|t| \leq n^{-q}$ , we get

$$\Psi_n(t) = (1 + O(t) + nO(t^3))e^{-\frac{\gamma}{2}t^2n + i\beta tn} + O(\varepsilon^n) \quad \forall t \in \mathbb{R} : |t| \leq n^{-q}$$

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<sup>1</sup>For any pair of sequences  $\{f_n\} \subset \mathbb{C}$  and  $\{g_n\} \subset \mathbb{R}_+$ , we write  $f_n = \Theta(g_n)$  whenever there are two positive values  $a, b$  such that  $ag_n \leq |f_n| \leq bg_n$  for every  $n$  large enough.

Thus, computing directly the primitives of simple functions, we obtain

$$\begin{aligned} & \int_{|t| \leq n^{-q}} \left| \Psi_n(t) - e^{-\frac{\gamma}{2}t^2 n + i\beta t n} \right| dt = \\ & = \int_{|t| \leq n^{-q}} |O(t) + nO(t^3)| e^{-\frac{\gamma}{2}t^2 n} dt + O(\varepsilon^n) = O(n^{-1}) \quad \square \end{aligned}$$

Now, we are able to prove the result of this section.

**Proof of Theorem 1.** It is well-known [10] that  $p_n(k) = \Pr \{Y_n = k\}$  can be computed from the inversion formula

$$p_n(k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \Psi_n(t) e^{-itk} dt \quad \forall k \in \{0, 1, \dots, n\} \quad (14)$$

To evaluate that integral, let us split  $[-\pi, \pi]$  into the three sets defined in (12). Then, by Propositions 1 and 2, for some  $\varepsilon \in (0, 1)$  we obtain

$$p_n(k) = \frac{1}{2\pi} \int_{|t| \leq n^{-q}} \Psi_n(t) e^{-itk} dt + O\left(e^{-\frac{\gamma}{4}n^{1-2q}}\right) + O(\varepsilon^n) \quad (15)$$

Moreover, setting  $v(= v_{k,n}) = \frac{k - \beta n}{\sqrt{\gamma n}}$ , by Proposition 3 we have

$$\int_{|t| \leq n^{-q}} \Psi_n(t) e^{-itk} dt = \int_{|t| \leq n^{-q}} e^{-\frac{\gamma}{2}t^2 n - itv\sqrt{\gamma n}} dt + O(n^{-1}) \quad (16)$$

By a standard computation the first term on the right-hand side becomes

$$\begin{aligned} \int_{|t| \leq n^{-q}} e^{-\frac{\gamma}{2}t^2 n - itv\sqrt{\gamma n}} dt &= \frac{1}{\sqrt{\gamma n}} \left( \int_{-\infty}^{+\infty} e^{-\frac{x^2}{2} - ivx} dx - \int_{|x| > n^{\frac{1}{2}-q} \sqrt{\gamma}} e^{-\frac{x^2}{2} - ivx} dx \right) \\ &= \frac{1}{\sqrt{\gamma n}} \left( \sqrt{2\pi} e^{-\frac{v^2}{2}} + O(e^{-\frac{\gamma}{2}n^{1-2q}}) \right) \end{aligned} \quad (17)$$

where one recognizes in the second integral the characteristic function of a Gaussian random variable. Thus, the result follows by replacing (17) in (16) and (16) in (15).  $\square$

## 4 Bicomponent models

In this section we study the behaviour of  $\{Y_n\}_{n \in \mathbb{N}}$  defined by a linear representation  $(\xi, A, B, \eta)$  of size  $m$  such that the matrix  $M = A + B$  consists of two irreducible components. Formally, there are two linear representations,  $(\xi_1, A_1, B_1, \eta_1)$  and  $(\xi_2, A_2, B_2, \eta_2)$ , of size  $m_1$  and  $m_2$  respectively, where  $m = m_1 + m_2$ , such that:

1. For some  $A_0, B_0 \in \mathbb{R}_+^{m_1 \times m_2}$  we have

$$\xi' = (\xi'_1, \xi'_2), \quad A = \begin{pmatrix} A_1 & A_0 \\ 0 & A_2 \end{pmatrix}, \quad B = \begin{pmatrix} B_1 & B_0 \\ 0 & B_2 \end{pmatrix}, \quad \eta = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}$$



2.  $M_1 = A_1 + B_1$  and  $M_2 = A_2 + B_2$  are irreducible matrices and we denote by  $\lambda_1$  and  $\lambda_2$  the corresponding Perron-Frobenius eigenvalues;
3.  $\xi_1 \neq 0 \neq \eta_2$  and matrix  $M_0 = A_0 + B_0$  is different from  $[0]$ .

Note that condition 2 is weaker than a primitivity assumption for  $M_1$  and  $M_2$ . Moreover, condition 3 avoids trivial situations and guarantees a (non-vanishing) communication from the first to the second component.

A typical example of a formal series  $r$  with a linear representation of this kind is given by the product of two rational formal series  $r_1, r_2$ , both having an irreducible linear representation, i.e.  $r = r_1 \cdot r_2$ , meaning that  $(r, w) = \sum_{w=xy} (r_1, x)(r_2, y)$  for every  $w \in \{a, b\}^*$ .

Under these hypotheses the limit distribution of  $\{Y_n\}$  first depends on whether  $\lambda_1 \neq \lambda_2$  or  $\lambda_1 = \lambda_2$ . If  $\lambda_1 \neq \lambda_2$  there is a dominant component, corresponding to the maximum between  $\lambda_1$  and  $\lambda_2$ , which determines the asymptotic behaviour of  $\{Y_n\}$ . If  $\lambda_1 = \lambda_2$  the two components are equipotent and they both contribute to the limit behaviour of  $\{Y_n\}$ . In both cases the corresponding characteristic function has some common properties we now recall briefly.

For  $j = 1, 2$ , let us define  $h_n^{(j)}(z)$ ,  $u_j(z)$ ,  $y_j(t)$ ,  $\beta_j$ , and  $\gamma_j$ , respectively, as the values  $h_n(z)$ ,  $u(z)$ ,  $y(t)$ ,  $\beta$ ,  $\gamma$  referred to component  $j$ . We also define  $H(x, y)$  as the matrix-valued function given by

$$\begin{aligned}
H(x, y) &= \sum_{n=0}^{+\infty} (Ax + B)^n y^n = \begin{bmatrix} H^{(1)}(x, y) & G(x, y) \\ 0 & H^{(2)}(x, y) \end{bmatrix}, \quad \text{where} \\
H^{(1)}(x, y) &= \frac{\text{Adj}(I - (A_1x + B_1)y)}{\text{Det}(I - (A_1x + B_1)y)}, \quad H^{(2)}(x, y) = \frac{\text{Adj}(I - (A_2x + B_2)y)}{\text{Det}(I - (A_2x + B_2)y)}, \\
\text{and} \quad G(x, y) &= H^{(1)}(x, y) (A_0x + B_0)y H^{(2)}(x, y).
\end{aligned} \tag{18}$$

Thus, the generating function of  $\{h_n(z)\}_n$  satisfies the following identities

$$\sum_{n=0}^{\infty} h_n(z) y^n = \xi' H(e^z, y) \eta = \xi_1' H^{(1)}(e^z, y) \eta_1 + \xi_1' G(e^z, y) \eta_2 + \xi_2' H^{(2)}(e^z, y) \eta_2 \tag{19}$$

Hence, setting  $g_n(z) = [y^n] \xi_1' G(e^z, y) \eta_2$ , we obtain

$$h_n(z) = h_n^{(1)}(z) + g_n(z) + h_n^{(2)}(z) \tag{20}$$

#### 4.1 Dominant case

Under the previous hypotheses let us further assume  $\lambda_1 > \lambda_2$  and  $M_1$  aperiodic (and hence primitive). In this case we say that  $\{Y_n\}$  is defined in a *dominant communicating bicomponent model* with  $\lambda_1 > \lambda_2$ . Under these assumptions it is known that if  $A_1 \neq 0 \neq B_1$  then  $0 < \beta_1 < 1$ ,  $0 < \gamma_1$  and  $\frac{Y_n - \beta_1 n}{\sqrt{\gamma_1 n}}$  converges in distribution to a normal r.v. of mean value 0 and variance 1 [8]. Here we show a Gaussian local limit law for  $\{Y_n\}$  with a convergence rate  $O(n^{-1/2})$ , under the further hypothesis that  $(A_1, B_1)$  is aperiodic. Note that, by definition, the aperiodicity of  $(A_1, B_1)$  implies  $A_1 \neq 0 \neq B_1$  (and hence  $0 < \beta_1 < 1$ ,  $0 < \gamma_1$ ). The proof is similar to that one of Theorem 1 and here we present a brief outline.

**Theorem 2** Let  $\{Y_n\}$  be defined in a dominant communicating bicomponent model with  $\lambda_1 > \lambda_2$  and assume  $(A_1, B_1)$  aperiodic. Then, as  $n$  tends to  $+\infty$ , the relation

$$\left| \sqrt{n} \Pr(Y_n = k) - \frac{e^{-\frac{(k-\beta_1 n)^2}{2\gamma_1 n}}}{\sqrt{2\pi\gamma_1}} \right| = O\left(n^{-1/2}\right)$$

holds true uniformly for every  $k \in \{0, 1, \dots, n\}$ .

Also in this case the statement is significant when  $\frac{(k-\beta_1 n)^2}{2\gamma_1 n}$  remains in a finite interval <sup>2</sup>.

As in the previous section, the proof is based on the analysis of the characteristic function  $\Psi_n(t)$  for  $t$  lying in the three sets  $|t| \leq n^{-q}$ ,  $n^{-q} < |t| \leq c$  and  $c < |t| \leq \pi$ , where  $c \in (0, \pi)$  is a suitable constant and  $q$  is an arbitrary real value such that  $\frac{1}{3} < q < \frac{1}{2}$ . First, let us consider the third set.

**Proposition 4** Under the hypotheses of Theorem 2, for every  $c \in (0, \pi)$  there exists  $\varepsilon \in (0, 1)$  such that

$$|\Psi_n(t)| = O(\varepsilon^n), \quad \forall t \in \mathbb{R} : c \leq |t| \leq \pi.$$

*Proof.* Note that, by relations (18) and (19), for every  $t \in \mathbb{R}$  the singularities of the generating function  $\xi' H(e^{it}, y) \eta = \sum_{n=0}^{\infty} h_n(it) y^n$  are the inverses of the eigenvalues of  $A_1 e^{it} + B_1$  and  $A_2 e^{it} + B_2$ . Assuming  $c \leq |t| \leq \pi$ , the first ones are in modulus smaller than  $\lambda_1$  by condition (10), while the second ones are in modulus smaller or equal to  $\lambda_2$  as a consequence of Perron-Frobenius Theorem for irreducible matrices [19, Ex. 1.9]. Thus, since  $\lambda_1 > \lambda_2$ , for some positive  $\tau < \lambda_1$  we have  $|h_n(it)| = O(\tau^n)$  for all real  $t$  such that  $c \leq |t| \leq \pi$ . For the same argument it is clear that  $h_n(0) = \Theta(\lambda_1^n)$ , and hence the result follows by reasoning as in the proof of Proposition 1.  $\square$

Concerning the other two subsets, i.e.  $|t| \leq n^{-q}$  and  $n^{-q} < |t| \leq c$ , one can study the behaviour of  $h_n(z)$  in a neighbourhood of  $z = 0$ . Reasoning as in the previous section it is easy to show that there are two positive constants  $c, \rho$  such that

$$h_n(z) = r(z) u_1(z)^n + O(\rho^n) \quad \forall z \in \mathbb{C} : |z| \leq c$$

where  $\rho < |u_1(z)|$  and  $r(z)$  is a function analytic and non-null for  $z \leq c$ . In particular  $\rho < \lambda_1$  and  $h_n(0) = r(0)\lambda_1^n + O(\rho^n)$ . These properties allow us to argue as in Section 3, replacing the values  $\lambda, \beta, \gamma, y(t)$  respectively by  $\lambda_1, \beta_1, \gamma_1$  and  $y_1(t)$ , thus proving two statements equivalent to Propositions 2 and 3, respectively. The proof of Theorem 1 can be modified in the same way and this concludes the proof of Theorem 2.

## 4.2 Equipotent case

Now let us consider the equipotent case. Formally, let  $\{Y_n\}$  be defined by a linear representation  $(\xi, A, B, \eta)$  satisfying conditions 1, 2, 3 above, assume  $\lambda_1 = \lambda_2 = \lambda$  and let both matrices  $M_1, M_2$  be aperiodic (and hence primitive). Under these hypotheses we say that  $\{Y_n\}$  is defined in an

<sup>2</sup>Even if not explicitly mentioned, similar observation also holds for the other local limit theorems presented in this work, whenever the limit function is continuous.

*equipotent communicating bicomponent model.* The limit distribution of  $\{Y_n\}$  in this case is studied in [8] and depends on the parameters  $\beta_1, \beta_2, \gamma_1, \gamma_2$ . Here we extend those results to local limit properties with a suitable convergence rate under the further assumption that both pairs  $(A_1, B_1)$  and  $(A_2, B_2)$  are aperiodic (again, such a hypothesis guarantees that  $0 < \beta_j < 1$  and  $0 < \gamma_j$ , for both  $j = 1, 2$ ). To this end we first recall some useful properties of the characteristic function of  $Y_n$  under these hypotheses [8].

First, consider equality (20) and note that both  $h_n^{(1)}(z)$  and  $h_n^{(2)}(z)$  satisfy (an analogue of) relation (6). Moreover, to evaluate  $\{g_n(z)\}$  observe that its generating function  $\xi_1' G(e^z, y)\eta_2$ , for every complex  $z$  in a neighbourhood of 0, has the singularities of smallest modulus at points  $y = u_1(z)^{-1}$  and  $y = u_2(z)^{-1}$ . Thus, for a suitable  $c > 0$  we can write

$$\xi_1' G(e^z, y)\eta_2 = \frac{s(z)y}{(1 - u_1(z)y)(1 - u_2(z)y)} + L(z, y) \quad \forall z \in \mathbb{C} : |z| \leq c$$

where again  $s(z)$  is a function analytic and non-null for  $z \leq c$ , and  $L(z, y)$  only admits singularities of modulus strictly greater than  $|u_1(z)|^{-1}$  and  $|u_2(z)|^{-1}$ . This implies

$$g_n(z) = s(z) \sum_{j=0}^{n-1} u_1(z)^j u_2(z)^{n-1-j} + O(\rho^n) \quad \forall z \in \mathbb{C} : |z| \leq c \quad (21)$$

where  $\rho < \max\{|u_1(z)|, |u_2(z)|\}$ . Replacing this expression in (20), one gets

$$h_n(z) = s(z) \sum_{j=0}^{n-1} u_1(z)^j u_2(z)^{n-1-j} + O(u_1(z)^n) + O(u_2(z)^n) \quad \forall z \in \mathbb{C} : |z| \leq c \quad (22)$$

This equality has two consequences. First, since  $u_1(0) = \lambda = u_2(0)$ , it implies

$$h_n(0) = s(0)n\lambda^{n-1}(1 + O(1/n)) \quad (s(0) \neq 0) \quad (23)$$

Second, if  $u_1(z) \neq u_2(z)$  for some  $z \in \mathbb{C}$  satisfying  $0 < |z| \leq c$ , one gets

$$h_n(z) = s(z) \frac{u_1(z)^n - u_2(z)^n}{u_1(z) - u_2(z)} + O(u_1(z)^n) + O(u_2(z)^n) \quad (24)$$

Finally, assuming the above aperiodicity condition and reasoning as in Proposition 1, the following property can be proved by using relations (18) and (23).

**Proposition 5** *Let  $\{Y_n\}$  be defined in an equipotent communicating bicomponent model and let both pairs  $(A_1, B_1)$  and  $(A_2, B_2)$  be aperiodic. Then, for every  $c \in (0, \pi)$  there exists  $\varepsilon \in (0, 1)$  such that  $|\Psi_n(t)| = O(\varepsilon^n)$  for all  $t \in \mathbb{R}$  satisfying  $c \leq |t| \leq \pi$ .*

#### 4.2.1 Local limit with different $\beta$ 's

In this subsection we assume an equipotent communicating bicomponent model with  $\beta_1 \neq \beta_2$ . In this case it is known that  $Y_n/n$  converges in distribution to a uniform r.v. over the interval of extremes  $\beta_1, \beta_2$ . Here we prove a local limit theorem toward the corresponding density function

with a convergence rate of order  $O(n^{-1/2})$ . To this end, in view of Proposition 5, we study the characteristic function  $\Psi_n(t)$  for  $|t| \leq c$ , where  $c \in (0, \pi)$  is a constant for which identity (24) holds true. Recall that in such a set both functions  $y_1(t) = u_1(it)/\lambda$  and  $y_2(t) = u_2(it)/\lambda$  satisfy relations (8), and hence for every real  $t$  such that  $|t| \leq c$ , we have

$$y_j(t) = 1 + i\beta_j t + O(t^2), \quad j = 1, 2 \quad (25)$$

$$|y_j(t)| \leq e^{-\frac{\gamma_j}{4}t^2}, \quad j = 1, 2 \quad (26)$$

As a consequence, we may assume the following relation for every  $t \in \mathbb{R}$  such that  $0 < |t| \leq c$ :

$$\Psi_n(t) = \frac{h_n(it)}{h_n(0)} = \frac{1 + O(t)}{1 + O(1/n)} \left( \frac{y_1(t)^n - y_2(t)^n}{i(\beta_1 - \beta_2)tn} \right) + \sum_{j=1,2} O\left(\frac{|y_j(t)|^n}{n}\right) \quad (27)$$

Now, for such a constant  $c$ , let us split the interval  $[-c, c]$  into sets  $S_n$  and  $V_n$  given by

$$S_n = \left\{ t \in \mathbb{R} : |t| \leq \frac{\log n}{\sqrt{n}} \right\}, \quad V_n = \left\{ t \in \mathbb{R} : \frac{\log n}{\sqrt{n}} < |t| \leq c \right\} \quad (28)$$

The behaviour of  $\Psi_n(t)$  in  $V_n$  is given by the following proposition, where we assume an equipotent communicating bicomponent model with  $\beta_1 \neq \beta_2$ .

**Proposition 6** *It turns out that  $|\Psi_n(t)| = o(n^{-3/2})$  for all  $t \in V_n$ .*

*Proof.* From equation (27), for every  $t \in V_n$ , we obtain

$$|\Psi_n(t)| \leq \frac{|y_1(t)|^n + |y_2(t)|^n}{\Theta(tn)} + \sum_{j=1,2} O\left(\frac{|y_j(t)|^n}{n}\right) = \sum_{j=1,2} o\left(\frac{|y_j(t)|^n}{\sqrt{n}}\right)$$

Taking  $a = \frac{\min\{\gamma_1, \gamma_2\}}{4}$  by relations (26) we get  $|\Psi_n(t)| = o\left(n^{-\frac{1}{2}-a(\log n)^2}\right)$ , which proves the result.  $\square$

Now, let us evaluate  $\Psi_n(t)$  for  $t \in S_n$ . To this end we need the following

**Lemma 1** *For  $k, m \in \mathbb{N}$ ,  $k < m$ , let  $g : [2k\pi, 2m\pi] \rightarrow \mathbb{R}_+$  be a monotone function, and let  $I_{k,m} = \int_{2k\pi}^{2m\pi} g(x) \sin x \, dx$ . Then:*

a) *if  $g$  is non-increasing we have  $0 \leq I_{k,m} \leq 2[g(2k\pi) - g(2m\pi)]$ ;*

b) *if  $g$  is non-decreasing we have  $2[g(2k\pi) - g(2m\pi)] \leq I_{k,m} \leq 0$ .*

*In both cases  $|I_{k,m}| \leq 2|g(2k\pi) - g(2m\pi)|$ .*

*Proof.* If  $g$  is non-increasing, for each integer  $j \in [k, m)$  we have  $0 \leq I_{j,j+1}$  and

$$I_{j,j+1} = \int_{2j\pi}^{(2j+1)\pi} g(x) \sin x \, dx - \int_{(2j+1)\pi}^{2(j+1)\pi} g(x) |\sin x| \, dx \leq 2[g(2j\pi) - g(2(j+1)\pi)]$$

Thus a) follows by summing the expressions above for  $j = k, \dots, m-1$ . Part b) is proved by applying a) to function  $h(x) = g(2m\pi) - g(x)$ .  $\square$

Going back to the analysis of  $\Psi_n(t)$  in  $S_n$ , let us define

$$K_n(t) = \frac{e^{-\frac{\gamma_1}{2}t^2n+i\beta_1tn} - e^{-\frac{\gamma_2}{2}t^2n+i\beta_2tn}}{i(\beta_1 - \beta_2)tn} \quad (29)$$

and consider relation (27). Since for  $t \in S_n$  one has  $nO(t^3) = o(1)$ , relation (9) applies to both  $y_1(t)$  and  $y_2(t)$ , yielding

$$y_j(t)^n = e^{-\frac{\gamma_j}{2}t^2n+i\beta_jtn}(1 + nO(t^3)) \quad \forall t \in S_n, \quad j = 1, 2$$

Replacing these values in (27), for some  $a > 0$  one gets

$$\Psi_n(t) = [1 + O(t) + nO(t^3) + O(1/n)] K_n(t) + O(n^{-1}e^{-at^2n}) \quad \forall t \in S_n \quad (30)$$

Such an equality allows to determine the properties of  $\Psi_n(t)$  in  $S_n$ .

**Proposition 7** *Assume an equipotent communicating bicomponent model with  $\beta_1 \neq \beta_2$  and let  $S_n$  and  $K_n(t)$  be defined as in (28) and (29), respectively. Then, we have*

$$\left| \int_{S_n} (\Psi_n(t) - K_n(t)) dt \right| = O(n^{-3/2})$$

*Proof.* Integrating both sides of (30), we obtain

$$\int_{S_n} [\Psi_n(t) - K_n(t)] dt = \int_{S_n} \left\{ [O(t) + nO(t^3) + O(n^{-1})] K_n(t) + O(n^{-1}e^{-at^2n}) \right\} dt \quad (31)$$

In order to evaluate the integral in the right hand side observe that, for any constant  $a > 0$  and  $\tau_n = n^{-1/2}(\log n)$ , we have

$$\int_{S_n} e^{-at^2n} dt \leq \frac{2}{\sqrt{n}} + 2 \int_{n^{-1/2}}^{\tau_n} \sqrt{nt} e^{-at^2n} dt = \Theta(n^{-1/2}) \quad (32)$$

which implies (for a suitable  $a > 0$ )

$$\left| \int_{S_n} tK_n(t) dt \right| \leq O\left( \int_{S_n} n^{-1} e^{-at^2n} dt \right) = O(n^{-3/2}) \quad (33)$$

Moreover, using similar bounds one gets

$$\begin{aligned} \left| \int_{S_n} nt^3 K_n(t) dt \right| &= O\left( \int_{S_n} t^2 e^{-at^2n} dt \right) = O\left( n^{-3/2} + \int_{n^{-1/2}}^{\tau_n} t^2 e^{-at^2n} dt \right) \\ &= O\left( n^{-3/2} + \frac{1}{n} \left\{ -te^{-at^2n} \Big|_{n^{-1/2}}^{\tau_n} + \int_{n^{-1/2}}^{\tau_n} e^{-at^2n} dt \right\} \right) = O(n^{-3/2}) \end{aligned} \quad (34)$$

Using (32), (33) and (34) in (31) one easily see that the result is proved once we show

$$\int_{S_n} K_n(t) dt = O(1/n) \quad (35)$$

To this end, define  $\delta = \beta_1 - \beta_2$ . Since  $\cos x$  and  $\sin x$  are respectively even and odd function, we can write

$$\left| \int_{S_n} K_n(t) dt \right| \leq \sum_{j=1,2} \left| \int_{S_n} \frac{e^{-\frac{\gamma_j}{2} t^2 n + i\beta_j t n} - 1}{i\delta t n} dt \right| = \sum_{j=1,2} \left| \int_{S_n} \frac{e^{-\frac{\gamma_j}{2} t^2 n} \sin(\beta_j t n)}{\delta t n} dt \right|$$

Setting  $u = \beta_j n t$ , each integral in the last sum becomes (for some  $a, b > 0$ )

$$\frac{2}{|\delta|n} \int_0^{bn\tau_n} \frac{e^{-a\frac{u^2}{n}} \sin u}{u} du \leq \frac{2}{|\delta|n} \left\{ \int_0^{2\pi} \frac{\sin u}{u} du + \int_{2\pi}^{bn\tau_n} \frac{e^{-a\frac{u^2}{n}} \sin u}{u} du \right\} \quad (36)$$

Thus, we can apply Lemma 1 to  $g(u) = u^{-1} e^{-a\frac{u^2}{n}}$  in the last expression, and get

$$\int_{2\pi}^{bn\tau_n} u^{-1} e^{-a\frac{u^2}{n}} \sin u du = 2(g(2\pi) - g(bn\tau_n) + o(1)) = \pi^{-1} + o(1)$$

Replacing this value in (36) we obtain equality (35) and the proof is complete.  $\square$

Now, we are able to prove the local limit in the present case. Set  $b_1 = \min\{\beta_1, \beta_2\}$ ,  $b_2 = \max\{\beta_1, \beta_2\}$  and denote by  $f_U(x)$  the density function of a uniform r.v.  $U$  in the interval  $[b_1, b_2]$ , that is

$$f_U(x) = \frac{1}{b_2 - b_1} \chi_{[b_1, b_2]}(x) \quad \forall x \in \mathbb{R}$$

where  $\chi_I$  denotes the indicator function of interval  $I \subset \mathbb{R}$ .

**Theorem 3** *Let  $\{Y_n\}_{n \in \mathbb{N}}$  be defined in an equipotent communicating bicomponent model with  $\beta_1 \neq \beta_2$  and assume aperiodic both pairs  $(A_1, B_1)$  and  $(A_2, B_2)$ . Then, for  $n$  tending to  $+\infty$ ,  $Y_n$  satisfies the relation*

$$|n \Pr(Y_n = k) - f_U(x)| = O\left(n^{-1/2}\right) \quad (37)$$

for every  $k = k(n) \in \mathbb{N}$  such that  $\lim_{n \rightarrow \infty} k/n = x$ , where  $x$  is a constant different from  $\beta_1$  and  $\beta_2$ .

*Proof.* We start again from the inversion formula (14). To evaluate the integral therein we split the interval  $[-\pi, \pi]$  into the three sets  $\{t \in \mathbb{R} : c < |t| \leq \pi\}$ ,  $V_n$ , and  $S_n$ , where  $c > 0$  is a constant for which relation (27) holds true, while  $S_n$  and  $V_n$  are defined in equations (28). Then, by Propositions 5, 6, 7, we obtain

$$p_n(k) = \frac{1}{2\pi} \int_{S_n} \left( \frac{e^{-\frac{\gamma_2}{2} t^2 n + i\beta_2 t n} - e^{-\frac{\gamma_1}{2} t^2 n + i\beta_1 t n}}{it(\beta_2 - \beta_1)n} \right) e^{-ikt} dt + O\left(n^{-3/2}\right) \quad (38)$$

Now, set  $v = k/n$  and note that for  $n \rightarrow +\infty$ ,  $v$  converges to a constant  $x$  different from  $\beta_1$  and  $\beta_2$ . Thus, defining

$$\Delta_n(v) = \int_{S_n} \frac{e^{i(\beta_2 - v)tn - \frac{\gamma_2}{2} t^2 n} - e^{i(\beta_1 - v)tn - \frac{\gamma_1}{2} t^2 n}}{i(\beta_2 - \beta_1)t} dt$$

we are done once we prove that

$$\Delta_n(v) = 2\pi f_U(x) + O(n^{-1/2}) \quad (39)$$

To this end, without loss of generality assume  $\beta_1 < \beta_2$  and set  $\delta = \beta_2 - \beta_1$ . Then,  $\Delta_n(v)$  is an integral of the difference between two functions of the form

$$A_n(t, v) = \frac{e^{i(\beta-v)tn - \frac{\gamma}{2}t^2n} - 1}{i\delta t}$$

where  $\beta$  and  $\gamma$  take the values  $\beta_2, \gamma_2$  and  $\beta_1, \gamma_1$ , respectively. Since the real and the imaginary part of  $A_n$  are (respectively) an even and an odd function in  $t$ , recalling that  $\tau_n = n^{-1/2}(\log n)$  and setting  $u = (\beta - v)tn$ , we get

$$\begin{aligned} \int_{S_n} A_n(t, v) dt &= \frac{2}{\delta} \int_0^{\tau_n} \frac{e^{-\frac{\gamma}{2}t^2n} \sin((\beta - v)tn)}{t} dt = \\ &= \frac{2}{\delta} \left\{ \int_0^{(\beta-v)\tau_n n} \frac{\sin(u)}{u} du - \int_0^{(\beta-v)\tau_n n} \left(1 - e^{-\frac{\gamma u^2}{2(\beta-v)^2n}}\right) \frac{\sin(u)}{u} du \right\} \end{aligned} \quad (40)$$

By Lemma 1 the first term of (40) can be written as

$$\begin{aligned} \frac{2}{\delta} \int_0^{(\beta-v)\tau_n n} \frac{\sin(u)}{u} du &= \frac{2 \operatorname{sgn}(\beta - v)}{\delta} \left( \int_0^{+\infty} \frac{\sin(u)}{u} du - \int_{|\beta-v|\tau_n n}^{+\infty} \frac{\sin(u)}{u} du \right) \\ &= \frac{\pi}{\delta} \operatorname{sgn}(\beta - v) - O\left(n^{-1/2}(\log n)^{-1}\right) \end{aligned} \quad (41)$$

Now we use again Lemma 1 to deal with the second term of (40), which has the form

$$\frac{2}{\delta} \int_0^{(\beta-v)\tau_n n} B_n(u) \sin(u) du \quad (42)$$

where  $B_n(u) = u^{-1} \left(1 - e^{-\frac{\gamma u^2}{2(\beta-v)^2n}}\right)$ . Note that  $B_n(u) > 0$  for all  $u > 0$ , and

$$\lim_{u \rightarrow 0} B_n(u) = 0 = \lim_{u \rightarrow +\infty} B_n(u)$$

Moreover in the set  $(0, +\infty)$  its derivative is null only at the point  $u_n = \alpha|\beta - v|\sqrt{n/\gamma}$ , for a constant  $\alpha \in (1, 2)$  independent of  $n$  and  $v$ . Thus, for  $n$  large enough,  $u_n$  belongs to the interval  $(0, |\beta - v|\tau_n n)$ ,  $B_n(u)$  is increasing in the set  $(0, u_n)$  and decreasing in  $(u_n, +\infty)$ , while its maximum value is

$$B_n(u_n) = \frac{1 - e^{-\frac{\alpha^2}{2}}}{\alpha|\beta - v|} \sqrt{\frac{\gamma}{n}} = \Theta(n^{-1/2})$$

Defining  $k_n = \lfloor \frac{u_n}{2\pi} \rfloor$  and  $K = \lfloor \frac{|\beta-v|\tau_n n}{2\pi} \rfloor$ , we can apply Lemma 1 to the intervals  $[0, 2k_n\pi]$  and  $[2k_n\pi + 2\pi, 2K\pi]$ , to get

$$\begin{aligned} \left| \int_0^{|\beta-v|\tau_n} B_n(u) \sin u \, du \right| &\leq 2B_n(2k_n\pi) + \left| \int_{2k_n\pi}^{2(k_n+1)\pi} B_n(u) \sin u \, du \right| + \\ &+ 2[B_n(2(k_n+1)\pi) - B_n(2K\pi)] + \int_{2K\pi}^{|\beta-v|\tau_n} B_n(u) \sin u \, du = \\ &\leq 6B_n(u_n) = \frac{c\sqrt{\gamma}}{|\beta-v|\sqrt{n}} \end{aligned}$$

where  $c$  is a positive constant independent of  $v$  and  $n$ .

This implies that, for any  $v$  approaching a constant different from  $\beta_1$  and  $\beta_2$ , the second term of (40) is  $O(n^{-1/2})$ . Therefore, applying (41) and recalling that  $v$  converges to a constant  $x$  different from  $\beta_1$  and  $\beta_2$ , we get

$$\begin{aligned} \Delta_n(v) &= \frac{2}{\delta} \left[ \int_0^{(\beta_2-v)n\tau_n} \frac{\sin u}{u} \, du - \int_0^{(\beta_1-v)n\tau_n} \frac{\sin u}{u} \, du \right] + O(n^{-1/2}) \\ &= \frac{\pi}{\delta} [\operatorname{sgn}(\beta_2 - v) - \operatorname{sgn}(\beta_1 - v)] + O(n^{-1/2}) = 2\pi f_U(x) + O(n^{-1/2}) \end{aligned}$$

This proves equation (39) and hence the proof is complete.  $\square$

The theorem clearly holds also when  $k/n$  definitely lies in a finite interval not including  $\beta_1$  nor  $\beta_2$  (the proof being the same).

As an example, consider the rational stochastic model defined by the weighted finite automaton of Figure 1, where each transition is labelled by a pair  $(\sigma, p)$ , for a symbol  $\sigma \in \{a, b, c\}$  and a weight  $p > 0$ , together with the arrays  $\xi = (1, 0, 0, 0)$  and  $\eta = (0, 0, 1, 1)$ . Such an automaton recognizes the set of all words  $w \in \{a, b, c\}^*$  of the form  $w = xcy$ , such that  $x, y \in \{a, b\}^*$  and the strings  $aa$  and  $bb$  do not occur in  $x$  and  $y$ , respectively. Clearly this is a bicomponent model, with both pairs  $(A_1, B_1)$  and  $(A_2, B_2)$  aperiodic. Moreover  $M_1 = M_2$ , while  $A_1 \neq A_2$ . Hence the two components are equipotent and  $\beta_1 \neq \beta_2$ . This means that  $Y_n/n$  converges in distribution to a uniform r.v. of extremes  $\beta_1, \beta_2$ , and  $Y_n$  satisfies Theorem 3. Note that simple changes may modify the limit distribution: for instance, setting to 3 the weight of transition  $2 \xrightarrow{b} 1$  makes dominant the first component, implying a Gaussian local limit law (Theorem 2).

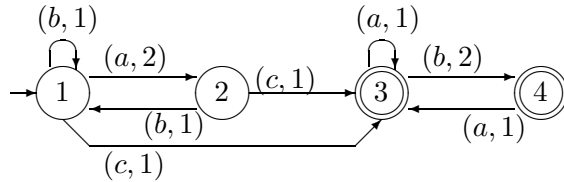


Figure 1: Weighted finite automaton defining an equipotent bicomponent model ( $\lambda_1 = \lambda_2 = 2$ ) with  $\beta_1 = 1/3$  and  $\beta_2 = 2/3$ .



#### 4.2.2 Local limit with equal $\beta$ 's and different $\gamma$ 's

In this section we present a local limit theorem for  $\{Y_n\}$  defined in an equipotent communicating bicomponent model with  $\beta_1 = \beta_2$  and  $\gamma_1 \neq \gamma_2$ . In this case, setting  $\beta = \beta_1 = \beta_2$  and  $\gamma = \frac{\gamma_1 + \gamma_2}{2}$ , it is proved that the distribution of  $\frac{Y_n - \beta n}{\sqrt{\gamma n}}$  converges to a mixture of Gaussian laws having mean 0 and variance uniformly distributed over an interval of extremes  $\frac{\gamma_1}{\gamma}$  and  $\frac{\gamma_2}{\gamma}$  [8].

Formally, we consider a r.v.  $T$  having density function

$$f_T(x) = \frac{\gamma}{\gamma_2 - \gamma_1} \int_{\frac{\gamma_1}{\gamma}}^{\frac{\gamma_2}{\gamma}} \frac{e^{-\frac{x^2}{2s}}}{\sqrt{2\pi s}} ds \quad \forall x \in \mathbb{R} \quad (43)$$

In passing, we observe that for each  $x \in \mathbb{R}$ ,  $f_T(x)$  may be regarded as the mean value of the ‘‘heat kernel’’  $K(x, t) = (4\pi t)^{-1/2} e^{-\frac{x^2}{4t}}$  at point  $x$  in the time interval of extremes  $\gamma_1/(2\gamma)$  and  $\gamma_2/(2\gamma)$  [7].

Note that  $E(T) = 0$  and  $var(T) = 1$ , while its characteristic function is

$$\Phi_T(t) = \int_{-\infty}^{+\infty} f_T(x) e^{itx} dx = 2\gamma \frac{e^{-\frac{\gamma_1}{2\gamma}t^2} - e^{-\frac{\gamma_2}{2\gamma}t^2}}{(\gamma_2 - \gamma_1)t^2} \quad (44)$$

Clearly,  $f_T(x)$  can be expressed in the form

$$f_T(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \Phi_T(t) e^{-itx} dt = \frac{1}{2\pi} \int_{-\infty}^{+\infty} 2\gamma \frac{e^{-\frac{\gamma_1}{2\gamma}t^2} - e^{-\frac{\gamma_2}{2\gamma}t^2}}{(\gamma_2 - \gamma_1)t^2} e^{-itx} dt$$

As in the previous section, we assume aperiodic both pairs  $(A_1, B_1)$  and  $(A_2, B_2)$ , which implies Proposition 5,  $c \in (0, \pi)$  is a constant for which relation (24) holds true and both functions  $y_1(t)$ ,  $y_2(t)$  satisfy relations (8), which can now be refined as

$$y_j(t) = \frac{u_j(it)}{\lambda} = 1 + i\beta t - \frac{\gamma_j + \beta^2}{2} t^2 + O(t^3), \quad \forall t \in \mathbb{R} : |t| \leq c, j = 1, 2$$

Applying these values in (24), by identity (23) for some  $c \in (0, \pi)$  and every  $t \in \mathbb{R}$  satisfying  $0 < |t| \leq c$  we obtain

$$\Psi_n(t) = \frac{h_n(it)}{h_n(0)} = 2 \frac{1 + O(t)}{1 + O(1/n)} \frac{y_1(t)^n - y_2(t)^n}{(\gamma_2 - \gamma_1)nt^2 + nO(t^3)} + \sum_{j=1,2} O\left(\frac{y_j(t)^n}{n}\right) \quad (45)$$

Now, for such a constant  $c$ , we split the interval  $[-c, c]$  into sets  $S_n$  and  $V_n$  defined in (28). The behaviour of  $\Psi_n(t)$  in these sets is studied in the two propositions below, where we always assume an equipotent communicating bicomponent model with  $\beta_1 = \beta_2 = \beta$  and  $\gamma_1 \neq \gamma_2$ .

**Proposition 8** *For some  $a > 0$  we have  $|\Psi_n(t)| = o(n^{-3/2})$  for all  $t \in V_n$ .*

*Proof.* From equation (45), taking  $a = \min\{\gamma_1, \gamma_2\}/4$  and using (26), we can write

$$|\Psi_n(t)| \leq O\left(\frac{|y_1(t)|^n + |y_2(t)|^n}{t^2 n}\right) + O\left(\sum_{j=1,2} \frac{|y_j(t)|^n}{n}\right) = O\left(\frac{e^{-a(\log n)^2}}{(\log n)^2}\right), \quad \forall t \in V_n$$

which proves the result.  $\square$

As regards the behaviour of  $\Psi_n(t)$  in  $S_n$ , we define

$$H_n(t) = 2 \frac{e^{-\frac{\gamma_1}{2}t^2n} - e^{-\frac{\gamma_2}{2}t^2n}}{(\gamma_2 - \gamma_1)t^2n} e^{i\beta tn}, \quad \forall t \in \mathbb{R} \quad (46)$$

It is easy to see that  $|H_n(t)| \leq 2 \sum_{j=1,2} \left( \frac{1 - e^{-\frac{\gamma_j}{2}t^2n}}{|\gamma_2 - \gamma_1|t^2n} \right)$  for every  $t \in \mathbb{R}$ . Both addends take their maximum value at  $t = 0$ , where they have a removable singularity, and such values are independent of  $n$ . As a consequence we can state that  $|H_n(t)| \leq \frac{\gamma_1 + \gamma_2}{|\gamma_2 - \gamma_1|}$ , for every  $n \in \mathbb{N}_+$  and every  $t \in S_n$ .

**Proposition 9** *Let  $S_n$  and  $H_n(t)$  be defined by (28) and (46), respectively. Then, we have*

$$\int_{S_n} |\Psi_n(t) - H_n(t)| dt = O(n^{-1})$$

*Proof.* Starting again from equation (45) and applying relations (9) to both  $y_1(t)$  and  $y_2(t)$ , one can prove that

$$\Psi_n(t) = [1 + O(t) + O(1/n) + nO(t^3)] H_n(t) + O(n^{-1}e^{-at^2n})$$

Applying relation (32) we get

$$\int_{S_n} (\Psi_n(t) - H_n(t)) dt = \int_{S_n} (O(t) + O(1/n) + nO(t^3)) H_n(t) dt + O(n^{-3/2}) \quad (47)$$

Now, recalling that  $H_n(t) = O(1)$  for a suitable  $a > 0$  we obtain the following relations

$$\begin{aligned} \int_{S_n} n^{-1} H_n(t) dt &= O(n^{-1}) [t]_0^{\frac{\log n}{\sqrt{n}}} = o(n^{-1}) \\ \int_{S_n} t H_n(t) dt &\leq 2 [t^2]_0^{n^{-1/2}} + O\left(\int_{n^{-1/2}}^{\frac{\log n}{\sqrt{n}}} \frac{e^{-at^2n}}{\sqrt{n}} dt\right) = \Theta(n^{-1}) \\ \int_{S_n} nt^3 H_n(t) dt &= \int_0^{\frac{\log n}{\sqrt{n}}} \Theta(te^{-at^2n}) dt = \Theta(n^{-1}) \end{aligned}$$

Thus, the result follows by applying the previous relations in (47).  $\square$

We are now able to prove the main result in the present case.

**Theorem 4** *Let  $\{Y_n\}_{n \in \mathbb{N}}$  be defined in an equipotent communicating bicomponent model with  $\beta_1 = \beta_2 = \beta$  and  $\gamma_1 \neq \gamma_2$ . Set  $\gamma = (\gamma_1 + \gamma_2)/2$  and assume aperiodic both pairs  $(A_1, B_1)$  and  $(A_2, B_2)$ . Then, for  $n$  tending to  $+\infty$ ,  $Y_n$  satisfies the relation*

$$\left| \sqrt{\gamma n} \Pr(Y_n = k) - f_T\left(\frac{k - \beta n}{\sqrt{\gamma n}}\right) \right| = O(n^{-1/2}) \quad (48)$$

uniformly for  $k \in \{0, 1, \dots, n\}$ , where  $f_T$  is defined in (43).

*Proof.* Again we start from equation (14) and split  $[-\pi, \pi]$  into the three sets  $\{t \in \mathbb{R} : c < |t| \leq \pi\}$ ,  $V_n$  and  $S_n$ , where  $S_n$  and  $V_n$  are defined in equalities (28),  $c$  being a constant for which relation (45) holds true. Then, by Propositions 5, 8 and 9, we obtain

$$p_n(k) = \frac{1}{2\pi} \int_{S_n} H_n(t) e^{-ikt} dt + O(n^{-1})$$

where  $H_n(t)$  is defined in (46). Now, setting  $v = \frac{k-\beta n}{\sqrt{\gamma n}}$  in the previous integral we get

$$p_n(k) = \frac{1}{2\pi} \int_{S_n} 2 \frac{e^{-\frac{\gamma_1}{2} t^2 n} - e^{-\frac{\gamma_2}{2} t^2 n}}{(\gamma_2 - \gamma_1) n t^2} e^{-iv\sqrt{\gamma n} t} dt + O(n^{-1}) \quad (49)$$

By setting  $x = t\sqrt{\gamma n}$  and recalling (44), we obtain

$$\begin{aligned} \int_{S_n} 2 \frac{e^{-\frac{\gamma_1}{2} t^2 n} - e^{-\frac{\gamma_2}{2} t^2 n}}{(\gamma_2 - \gamma_1) n t^2} e^{-iv\sqrt{\gamma n} t} dt &= 2\sqrt{\frac{\gamma}{n}} \int_{|x| \leq \sqrt{\gamma} \log n} \frac{e^{-\frac{\gamma_1}{2\gamma} x^2} - e^{-\frac{\gamma_2}{2\gamma} x^2}}{(\gamma_2 - \gamma_1) x^2} e^{-ixv} dx = \\ &= \frac{1}{\sqrt{\gamma n}} \left\{ \int_{-\infty}^{+\infty} \Phi_T(x) e^{-ixv} dx - \int_{|x| > \sqrt{\gamma} \log n} \Phi_T(x) e^{-ixv} dx \right\} = \frac{2\pi f_T(v)}{\sqrt{\gamma n}} + o(n^{-2}) \end{aligned}$$

The result follows by replacing this value in (49).  $\square$

### 4.2.3 Local limit with equal $\beta$ 's and equal $\gamma$ 's

In this section we study the local limit properties of  $\{Y_n\}$  assuming an equipotent communicating bicomponent model with  $\beta_1 = \beta_2 = \beta$  and  $\gamma_1 = \gamma_2 = \gamma$ . In this case, it is known [8] that  $\frac{Y_n - \beta n}{\sqrt{\gamma n}}$  converges in distribution to a Gaussian r.v. of mean 0 and variance 1 and here we present a local limit law with a convergence rate of the order  $O(n^{-1/2})$ .

Again we assume  $c \in (0, \pi)$  constant for which equality (22) holds true, so that both functions  $y_1(t)$  and  $y_2(t)$  satisfy relations (8) and (9), which can be restated as

$$\begin{aligned} |y_j(t)| &\leq e^{-\frac{\gamma}{4} t^2} \quad \forall t \in \mathbb{R} : |t| \leq c, \quad j = 1, 2 \quad (50) \\ y_j(t)^n &= e^{-\frac{\gamma}{2} t^2 n + i\beta t n + nO(t^3)} \quad \forall t \in \mathbb{R} : |t| \leq n^{-q}, \quad j = 1, 2 \quad (51) \end{aligned}$$

where  $q$  is an arbitrary value such that  $1/3 < q < 1/2$ .

In the following two propositions the characteristic function  $\Psi_n(t)$  is studied under conditions  $|t| \leq n^{-q}$  and  $n^{-q} < |t| \leq c$ , respectively, assuming an equipotent communicating bicomponent model with  $\beta_1 = \beta_2 = \beta$  and  $\gamma_1 = \gamma_2 = \gamma$ .

**Proposition 10** *For every  $q \in (1/3, 1/2)$ , we have*

$$|\Psi_n(t)| = O\left(e^{-\frac{\gamma}{4} n^{1-2q}}\right) \quad \forall t \in \mathbb{R} : n^{-q} < |t| \leq c$$

*Proof.* Applying relations (50) to equality (22), we obtain

$$\begin{aligned} |h_n(it)| &= \left| s(it)\lambda^{n-1} \sum_{j=0}^{n-1} y_1(t)^j y_2(t)^{n-1-j} + \lambda^n \sum_{j=1,2} O(y_j(t)^n) \right| \\ &\leq |s(it)| n \lambda^{n-1} e^{-\frac{\gamma}{4}t^2(n-1)} + \lambda^n O(e^{-\frac{\gamma}{4}t^2n}) \end{aligned}$$

and hence, by (23), we have

$$|\Psi_n(t)| = \left| \frac{h_n(it)}{h_n(0)} \right| \leq \frac{1 + O(t)}{1 + O(1/n)} e^{-\frac{\gamma}{4}t^2(n-1)} + O\left(e^{-\frac{\gamma}{4}t^2n/n}\right)$$

which implies the result since  $n^{-q} < |t| \leq c$ .  $\square$

**Proposition 11** *For every  $q \in (1/3, 1/2)$ , we have*

$$\int_{|t| \leq n^{-q}} \left| \Psi_n(t) - e^{-\frac{\gamma}{2}t^2n + i\beta tn} \right| dt = O(n^{-1})$$

*Proof.* From relations (22) and (23), applying (51) and recalling that  $nO(t^3) = o(1)$  for  $|t| \leq n^{-q}$ , we obtain

$$\begin{aligned} \Psi_n(t) &= \frac{h_n(it)}{h_n(0)} = \frac{1 + O(t)}{n(1 + O(1/n))} e^{-\frac{\gamma}{2}t^2n + i\beta tn} \sum_{j=0}^{n-1} e^{jO(t^3) + (n-1-j)O(t^3)} + O\left(\frac{e^{-\frac{\gamma}{2}t^2n}}{n}\right) \\ &= (1 + O(t) + O(n^{-1}) + nO(t^3)) e^{-\frac{\gamma}{2}t^2n + i\beta tn} + O\left(e^{-\frac{\gamma}{2}t^2n/n}\right) \end{aligned}$$

Therefore, a straightforward computation shows that

$$\begin{aligned} &\int_{|t| \leq n^{-q}} \left| \Psi_n(t) - e^{-\frac{\gamma}{2}t^2n + i\beta tn} \right| dt = \\ &= \int_0^{n^{-q}} (O(t) + O(n^{-1}) + nO(t^3)) e^{-\frac{\gamma}{2}t^2n} dt + O(n^{-1-q}) = O(n^{-1}) \end{aligned}$$

$\square$

Now we are able to state the local limit theorem in the present case. For the proof one can argue as in Theorem 1, replacing Propositions 1, 2 and 3 by Propositions 5, 10 and 11, respectively.

**Theorem 5** *Let  $\{Y_n\}_{n \in \mathbb{N}}$  be defined in an equipotent communicating bicomponent model with  $\beta_1 = \beta_2 = \beta$  and  $\gamma_1 = \gamma_2 = \gamma$ , and assume aperiodic both pairs  $(A_1, B_1)$  and  $(A_2, B_2)$ . Then, for  $n$  tending to  $+\infty$  the relation*

$$\left| \sqrt{n} \Pr(Y_n = k) - \frac{e^{-\frac{(k-\beta n)^2}{2\gamma n}}}{\sqrt{2\pi\gamma}} \right| = O(n^{-1/2})$$

*holds true uniformly for every  $k \in \{0, 1, \dots, n\}$ .*

## 5 Sum models

In this section we study the problem assuming a bicomponent rational model without communication. Formally, the linear representation  $(\xi, A, B, \eta)$  defining  $\{Y_n\}$  satisfies conditions 1. and 2. of Section 4, for two suitable 4-tuples  $(\xi_1, A_1, B_1, \eta_1)$ ,  $(\xi_2, A_2, B_2, \eta_2)$ , together with the further condition

**3'**.  $\xi_1 \neq 0 \neq \eta_1$ ,  $\xi_2 \neq 0 \neq \eta_2$  and  $A_0 = [0] = B_0$ .

In this case, for every  $w \in \{a, b\}^*$  we have

$$\xi' \mu(w) \eta = \xi'_1 \mu_1(w) \eta_1 + \xi'_2 \mu_2(w) \eta_2$$

where  $\mu$ ,  $\mu_1$  and  $\mu_2$  are the morphisms defined by pairs  $(A, B)$ ,  $(A_1, B_1)$  and  $(A_2, B_2)$ , respectively. This means that the formal series  $r$  with linear representation  $(\xi, A, B, \eta)$  is the sum of two rational formal series  $r_1$ ,  $r_2$  with irreducible linear representation, i.e.  $(r, w) = (r_1, w) + (r_2, w)$  for every  $w \in \{a, b\}^*$ .

Under these hypotheses, for sake of brevity, we say that  $\{Y_n\}_n$  is defined in a *sum model*. Adopting the same notation of Section 4, here we have  $G(x, y) = 0$  in relations (18) implying, for every  $z \in \mathbb{C}$  and  $t \in \mathbb{R}$ , the identities

$$h_n(z) = h_n^{(1)}(z) + h_n^{(2)}(z) \quad \Psi_n(it) = \frac{h_n^{(1)}(it) + h_n^{(2)}(it)}{h_n(0)} \quad (52)$$

Again the simplest case occurs when there exists a dominant component. Recall that in this case  $\{Y_n\}$  has a Gaussian limit distribution [8] and this result can be extended to a local limit law as stated in the following statement, whose proof is similar to that one of Theorem 2.

**Theorem 6** *Let  $\{Y_n\}$  be defined in a sum model with  $\lambda_1 > \lambda_2$  and  $M_1$  aperiodic (and hence primitive). Also assume aperiodic the pair  $(A_1, B_1)$ . Then  $0 < \beta_1 < 1$ ,  $0 < \gamma_1$  and, as  $n$  tends to  $+\infty$ , the relation*

$$\left| \sqrt{n} Pr(Y_n = k) - \frac{e^{-\frac{(k-\beta_1 n)^2}{2\gamma_1 n}}}{\sqrt{2\pi\gamma_1}} \right| = O(n^{-1/2})$$

*holds true uniformly for every  $k \in \{0, 1, \dots, n\}$ .*

### 5.1 Equipotent sum models

We now study the local limit properties of our statistics for non-communicating bicomponent models in the equipotent case. More precisely, let  $\{Y_n\}$  be defined in a sum model with  $\lambda_1 = \lambda_2 = \lambda$  and both matrices  $M_1$ ,  $M_2$  aperiodic (and hence primitive). Under these hypotheses we say that  $\{Y_n\}$  is defined in an *equipotent sum model*. The limit distribution of  $\{Y_n\}$  in this case is studied in [8] and depends on the parameters  $\alpha_1$ ,  $\alpha_2$ ,  $\beta_1$ ,  $\beta_2$ ,  $\gamma_1$ ,  $\gamma_2$  defined in (7). Here we prove local limit properties, with a convergence rate  $O(n^{-1/2})$ , under the further assumption that both pairs  $(A_1, B_1)$  and  $(A_2, B_2)$  are aperiodic. To this end we first determine some identities for function  $h_n(z)$  in the present case.

By properties of the primitive matrices [19] it is easy to see that

$$\begin{aligned} h_n(0) = \xi' M^n \eta &= \xi_1' \nu_1 \zeta_1' \eta_1 \cdot \lambda^n + \xi_2' \nu_2 \zeta_2' \eta_2 \cdot \lambda^n + O(\rho^n) \\ &= (\alpha_1 + \alpha_2) \lambda^n + O(\rho^n), \quad 0 \leq \rho < \lambda \end{aligned}$$

where  $\zeta_j$  and  $\nu_j$  are the eigenvectors defined in Section 3, for  $j = 1, 2$ . Also note that  $\alpha_j = r_j(0)$  for each  $j$ ,  $r_j(z)$  being the same as in (6). Using these identities function  $\Psi_n(t)$  can be evaluated from (52).

Clearly, also the type of local limit law we present depends on parameters  $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2$ . In general we obtain a local limit law towards a convex combination of two Gaussian distributions, which coincide when  $\beta_1 = \beta_2$  and  $\gamma_1 = \gamma_2$ .

**Theorem 7** *Let  $\{Y_n\}$  be defined in an equipotent sum model and assume that both pairs  $(A_1, B_1)$ ,  $(A_2, B_2)$  are aperiodic. Then, as  $n$  tends to  $+\infty$ , the relation*

$$\left| \sqrt{n} \Pr(Y_n = k) - \left( \frac{\alpha_1}{\alpha_1 + \alpha_2} \frac{e^{-\frac{(k-\beta_1 n)^2}{2\gamma_1 n}}}{\sqrt{2\pi\gamma_1}} + \frac{\alpha_2}{\alpha_1 + \alpha_2} \frac{e^{-\frac{(k-\beta_2 n)^2}{2\gamma_2 n}}}{\sqrt{2\pi\gamma_2}} \right) \right| = O(n^{-1/2})$$

holds true uniformly for every  $k \in \{0, 1, \dots, n\}$ .

**Proof.** Again the main idea is to study the characteristic function  $\Psi_n(t)$  for  $t \in [-\pi, \pi]$  by splitting this interval into the three sets given in (12), where  $c \in (0, \pi)$  is a constant satisfying relations (8) for both  $y_1(t)$  and  $y_2(t)$ , and  $q$  is an arbitrary value such that  $\frac{1}{3} < q < \frac{1}{2}$ . The behaviour of  $\Psi_n(t)$  in these sets is characterized by the following properties:

a. For some  $\varepsilon \in (0, 1)$  we have

$$|\Psi_n(t)| = O(\varepsilon^n) \quad \forall t \in \mathbb{R} : c < |t| \leq \pi \quad (53)$$

b. There exists  $a > 0$  such that

$$|\Psi_n(t)| = O\left(e^{-an^{1-2q}}\right) \quad \forall t \in \mathbb{R} : n^{-q} < |t| \leq c \quad (54)$$

c.

$$\int_{|t| \leq n^{-q}} \left| \Psi_n(t) - \frac{\alpha_1}{\alpha_1 + \alpha_2} e^{-\frac{\gamma_1}{2} t^2 n + i\beta_1 t n} - \frac{\alpha_2}{\alpha_1 + \alpha_2} e^{-\frac{\gamma_2}{2} t^2 n + i\beta_2 t n} \right| dt = O(n^{-1}) \quad (55)$$

*Proof of (53).* We can argue as in Proposition 4. The only difference is that now the eigenvalues of  $A_2 e^{it} + B_2$  are smaller than  $\lambda = \lambda_2$ , and this simplifies the proof.

*Proof of (54).* By relation (6), for some  $\varepsilon \in (0, 1)$  and all  $t \in \mathbb{R}$  satisfying  $|t| \leq c$ , we have

$$\Psi_n(t) = \frac{h_n(it)}{h_n(0)} = \frac{r_1(it)u_1(it)^n + r_2(it)u_2(it)^n}{(r_1(0) + r_2(0))\lambda^n} + O(\varepsilon^n) = \sum_{j=1,2} c_j y_j(t)^n + O(\varepsilon^n) \quad (56)$$

where  $c_1$  and  $c_2$  are positive constants. Also, setting  $a = \min\{\gamma_1/4, \gamma_2/4\}$ , by inequality (8) recalling  $n^{-q} \leq |t| \leq c$  we obtain  $|y_j(t)|^n \leq e^{-an^{1-2q}}$ , for each  $j = 1, 2$ , which implies the result.

*Proof of (55).* From equality (56), applying relation (9) and recalling that  $nO(t^3) = o(1)$  for  $|t| \leq n^{-q}$ , in the same interval for  $t$  we get

$$\Psi_n(t) = \sum_{j=1,2} \frac{r_j(0) + O(t)}{r_1(0) + r_2(0)} (1 + nO(t^3)) e^{-\frac{\gamma_j}{2} t^2 n + i\beta_j t n} + O(\varepsilon^n)$$

Thus, since  $r_j(0) = \alpha_j$  for each  $j$ , reasoning as in the proof of Proposition 3 we obtain

$$\begin{aligned} & \int_{|t| \leq n^{-q}} \left| \Psi_n(t) - \sum_{j=1,2} \frac{\alpha_j}{\alpha_1 + \alpha_2} e^{-\frac{\gamma_j}{2} t^2 n + i\beta_j t n} \right| dt = \\ & = \sum_{j=1,2} \int_{|t| \leq n^{-q}} |O(t) + nO(t^3)| e^{-\frac{\gamma_j}{2} t^2 n} dt + O(\varepsilon^n) = O(n^{-1}) \end{aligned} \quad (57)$$

Now consider our main goal. Defining  $p_n(k) = \Pr \{Y_n = k\}$ , from the inversion formula (14), by relations (53), (54) and (55), we obtain

$$\begin{aligned} p_n(k) &= \frac{1}{2\pi} \int_{|t| \leq n^{-q}} \Psi_n(t) e^{-itk} dt + O\left(e^{-an^{1-2q}}\right) + O(\varepsilon^n) \\ &= \frac{1}{2\pi} \sum_{j=1,2} \frac{\alpha_j}{\alpha_1 + \alpha_2} \int_{|t| \leq n^{-q}} e^{-\frac{\gamma_j}{2} t^2 n + i\beta_j t n - itk} dt + O(n^{-1}) \end{aligned} \quad (58)$$

Moreover, defining the variables  $v_j = \frac{k - \beta_j n}{\sqrt{\gamma_j n}}$ , for  $j = 1, 2$ , the last integrals can be evaluated as in (16) and (17), obtaining

$$\int_{|t| \leq n^{-q}} e^{-\frac{\gamma_j}{2} t^2 n + i\beta_j t n - itk} dt = \frac{1}{\sqrt{\gamma_j n}} \left( \sqrt{2\pi} e^{-\frac{v_j^2}{2}} + O(e^{-\frac{\gamma_j}{2} n^{1-2q}}) \right)$$

which replaced in (58) yields the result.  $\square$

We observe that if  $\beta_1 = \beta_2$  and  $\gamma_1 = \gamma_2$  then the limit density given by Theorem 7 reduces to a Gaussian law. This yields the following

**Corollary 1** *Let  $\{Y_n\}$  be defined in an equipotent sum model with  $\beta_1 = \beta_2 = \beta$ ,  $\gamma_1 = \gamma_2 = \gamma$  and assume aperiodic both pairs  $(A_1, B_1)$ ,  $(A_2, B_2)$ . Then, as  $n$  tends to  $+\infty$ , the relation*

$$\left| \sqrt{n} \Pr(Y_n = k) - \frac{e^{-\frac{(k - \beta n)^2}{2\gamma n}}}{\sqrt{2\pi\gamma}} \right| = O(n^{-1/2})$$

*holds true uniformly for every  $k \in \{0, 1, \dots, n\}$ .*

On the contrary, when  $\beta_1 \neq \beta_2$  or  $\gamma_1 \neq \gamma_2$  (or both) the previous result yields a local limit law toward a convex combination of two Gaussian distributions that differ by their mean value or by their variance. More precisely, in this case we obtain the distribution of a r.v.  $\mathcal{L}$  defined by

$$\mathcal{L} = [\beta_1 \mathcal{B}_p + \beta_2 (1 - \mathcal{B}_p)] n + [\mathcal{B}_p \mathcal{N}_{0, \gamma_1} + (1 - \mathcal{B}_p) \mathcal{N}_{0, \gamma_2}] \sqrt{n} \quad (59)$$

where  $\mathcal{B}_p$  is a Bernoullian r.v. of parameter  $p = \frac{\alpha_1}{\alpha_1 + \alpha_2}$ , and  $\mathcal{N}_{0,\gamma_j}$  is a Gaussian r.v. of mean 0 and variance  $\gamma_j$ , assuming mutually independent all variables  $\mathcal{B}_p, \mathcal{N}_{0,\gamma_1}, \mathcal{N}_{0,\gamma_2}$ . In particular from the analysis above it is clear that

$$\frac{Y_n - [\beta_1 \mathcal{B}_p + \beta_2 (1 - \mathcal{B}_p)] n}{[\mathcal{B}_p \sqrt{\gamma_1} + (1 - \mathcal{B}_p) \sqrt{\gamma_2}] \sqrt{n}} \longrightarrow \mathcal{N}_{0,1} \quad \text{in distribution}$$

which specializes into the following cases:

- if  $\beta_1 = \beta_2 = \beta$  and  $\gamma_1 \neq \gamma_2$  then

$$\frac{Y_n - \beta n}{\sqrt{n}} \longrightarrow [\mathcal{B}_p \mathcal{N}_{0,\gamma_1} + (1 - \mathcal{B}_p) \mathcal{N}_{0,\gamma_2}] \quad \text{in distribution}$$

- if  $\beta_1 \neq \beta_2$  and  $\gamma_1 = \gamma_2 = \gamma$  then

$$\frac{Y_n - [\beta_1 \mathcal{B}_p + \beta_2 (1 - \mathcal{B}_p)] n}{\sqrt{n}} \longrightarrow \mathcal{N}_{0,\gamma} \quad \text{in distribution}$$

A curious fact is that  $\mathcal{L}$  as defined in (59) also depends on the weights of initial and final states  $(\xi, \eta)$ . This does not occur in the equipotent bicomponent models with communication, studied in Section 4.2, and seems to suggest that the present model is not ergodic (in the sense that the limit distribution depends on the starting states).

As an example, consider the rational model defined by the weighted finite automaton of Figure 2, together with  $\xi = (1, 0, 1, 0)$  and  $\eta = (0, 1, 1, 1)$ . Such an automaton recognizes the set of all words  $\{w \in \{a, b\}^*\}$  such that pattern  $aa$  or pattern  $bb$  (or both) do not occur in  $w$ . Clearly this is a bicomponent model, with both pairs  $(A_1, B_1)$  and  $(A_2, B_2)$  aperiodic. Moreover  $M_1 = M_2$ , while  $A_1 \neq A_2$ . Hence the two components are equipotent and  $\beta_1 \neq \beta_2$ , however one can show that  $\gamma_1 = \gamma_2$  [4]. This implies a local limit law towards a convex combination of two Gaussian laws having different main constant of the mean value (but equal main constant of the variance). Note that simple changes may modify the limit distribution: for instance, setting to 2 the weight of transition  $2 \xrightarrow{b} 1$  makes dominant the first component, implying a Gaussian local limit law (Theorem 6).



Figure 2: Weighted finite automaton defining a non-communicating bicomponent model with  $\lambda_1 = \lambda_2 = 2$ ,  $\alpha_1 = 2/3$ ,  $\alpha_2 = 4/3$ ,  $\beta_1 = 1/3$ ,  $\beta_2 = 2/3$ ,  $\gamma_1 = \gamma_2 = 2/27$ .



## 6 Conclusions

In this work we have studied the local limit laws of symbol statistics defined in primitive rational models and in bicomponent rational models with or without communication. These laws, summarized in Table 6, yield a convergence rate of the order  $O(n^{-1/2})$  and are obtained assuming suitable aperiodicity conditions concerning the number of symbol occurrences in cycles of equal length.

1.	Primitive models	Bicomponent models with communication			
		dominant	equipotent		
			$\beta_1 \neq \beta_2$	$\beta_1 = \beta_2$ $\gamma_1 \neq \gamma_2$	$\beta_1 = \beta_2$ $\gamma_1 = \gamma_2$
Local limit distribution	$\mathcal{N}_{0,1}$	$\mathcal{N}_{0,1}$	$U_{\beta_1, \beta_2}$	$T$	$\mathcal{N}_{0,1}$
2.		Bicomponent models without communication			
		dominant	equipotent		
			$\beta_1 \neq \beta_2$ $\gamma_1 \neq \gamma_2$	$\beta_1 = \beta_2 = \beta$ $\gamma_1 \neq \gamma_2$	$\beta_1 \neq \beta_2$ $\gamma_1 = \gamma_2 = \gamma$
Local limit distribution	$\mathcal{N}_{0,1}$	$\mathcal{L}$	$\mathcal{B}_p \mathcal{N}_{0, \gamma_1} + (1 - \mathcal{B}_p) \mathcal{N}_{0, \gamma_2}$	$\mathcal{N}_{0, \gamma}$	$\mathcal{N}_{0,1}$

Table 1: Symbols  $\mathcal{N}_{0,1}$ ,  $U_{\beta_1, \beta_2}$  and  $T$  denote respectively a Gaussian, uniform and  $T$ -type local limit,  $T$  being defined in Section 4.2.2. Also, the r.v.'s  $\mathcal{L}$  and  $\mathcal{B}_p$  are defined in (59) .

Our analysis of the bicomponent models includes the main cases but is not exhaustive, since it does not contain the degenerate cases, i.e. when either  $A_i = 0$  or  $B_i = 0$  for a dominant component  $i \in \{1, 2\}$ . In these cases a large variety of different limit distributions is obtained [8, Section 8], most of them related to natural matrix extensions of the geometric distribution. This large range of possible limit behaviours is also mentioned in [1] and it seems natural to study common properties of these distributions, determining a suitable classification.

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