Estimation of the reliability parameter for a Poisson-exponential stress-strength model

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Abstract

In this paper, we consider the problem of estimating the reliability parameter of a mixed-type stress-strength model, i.e., the probability $\boldsymbol{R} = \boldsymbol{P}(\boldsymbol{X} < \boldsymbol{Y})$ where $\boldsymbol{X}$ and $\boldsymbol{Y}$ are a discrete and a continuous random variable, respectively. We focus on the specific case of Poisson stress and exponential strength, deriving the expression of $\boldsymbol{R}$ and its maximum likelihood estimator (MLE) and its uniformly minimum-variance unbiased estimator (UMVUE), based on simple random samples independently drawn from $\boldsymbol{X}$ and $\boldsymbol{Y}$. For the MLE, we are able to provide an expression for the cumulative distribution function, which allows us to compute its expected value, bias, and variance. We derive asymptotic properties of the MLE, which we exploit for constructing approximate confidence intervals based on different approaches. A simulation study empirically compares such estimators and provides advice for their correct use, which is also illustrated through an application to real data.

Keywords: exponential family; interference theory; MLE; profile likelihood; UMVUE

1 Introduction

In its simplest form, the so-called stress-strength model considers a unit or a system with an intrinsic strength Y that is subjected to an external stress X. The unit/system works if and only if $Y > X$. If X and Y are modeled as random variables (rvs), then the probability that the unit/system works is given by $R = P(X < Y)$, which is called the reliability parameter of the stress-strength model. This stochastic model can encompass many other applications with a broader meaning of the reliability parameter. In a clinical study, X may model the response of a control group and Y the response of a treatment group; then, R measures the effectiveness of the treatment. If X and Y are the remission times of two chemicals when they are administered to two kinds of mechanical systems, then R presents a comparison of the effectiveness of the two chemicals. If X and Y are future observations on the stability of an engineering design, then R is the predictive probability that X is less than Y. Similarly, if X and Y represent the lifetimes of two electronic devices, then R is the probability that the former fails before the latter.

The stress-strength problem was first considered by Birnbaum (1956), and since then, a huge number of contributions have been produced on the computation of R and its sample estimation, under many parametric assumptions about the distributions of X and Y (Kotz et al., 2003). Most of them model stress and strength as (independent) continuous rvs. This is because the stress-strength model generally has applications in the engineering field, in which variables are usually measured on a quantitative continuous scale (Gnedenko and Ushakov, 1995). Among continuous models, an exponential distribution is often used to describe the lifetime of complex electronic equipments. For example, Tong (1974) and Tong (1975) considered the point estimation of R when X and Y are independent (one-parameter) exponential rvs; under the same assumptions, Enis and Geisser (1971) derived an exact confidence interval (CI) for R. Varde (1969) and Krishnamoorthy et al. (2007) studied the case of two-parameter exponential stress and strength; Raqab et al. (2008) focused on stress and strength following the three-parameter generalized exponential distribution.Cortese and Ventura (2013); Jian and Wong (2008) suggested improvements in inferential methods for the estimation of R based on higher-order methods, with a focus on small samples and also using an exponential distribution.

However, even in the engineering field, it sometimes happens that measurements can be done on a discrete scale, e.g., when the lifetime of a device is measured in terms of the number of days of functioning, or when the number of cycles or runs it can sustain before failing is considered. Furthermore, continuous data may be discretized or grouped into classes so that they can be regarded as discrete. Therefore, stress-strength models in which at least one component is categorical/discrete are worth exploring. They can be of interest when comparisons are made of two times (lifetimes, times to failure, remission times, etc) that are measured on different scales (discrete and continuous). They are proved to be "useful for the reliability situation in which the time of failure of one component is recorded continuously while that of the other discretely" (Tong, 1977). An example is provided in Hu et al. (2021): if X is the unit of time that it takes to reach out to a patient and Y is the survival time of the patient, then R is the probability of successfully reaching out to a patient who is still alive.

To the best of our knowledge, little work has been done on this topic. Maiti (1995) studied the inference of $P(X < Y)$ when X and Y are independent geometric rvs; Barbiero (2013) considered the same problem when X and Y are independent Poisson rvs; and Obradovic et al. (2014) focused on $P(X \leq Y)$ with X a geometric and Y a Poisson rv. Jovanović (2016) considered $P(X < Y)$ when X and Y follow a geometric and an exponential distribution, respectively. The author derived the maximum likelihood estimator (MLE) of R, its asymptotic distribution, and the confidence intervals based on it, as well as the uniformly minimum-variance unbiased estimator (UMVUE) of R and of its variance; the Bayes estimator of R was investigated, and its Lindley's approximation was obtained. For the same model, Hu et al. (2021), taking into consideration parametric and nonparametric methods, developed two-stage and modified two-stage sampling procedures, respectively, to determine the necessary sample sizes for constructing a CI with the required accuracy. Singh et al. (2023) discussed a stress-strength model with geometric stress and strength following a continuous Lindley distribution.

In this paper, a stress-strength model with stress and strength distributed as a Poisson and an exponential rv, respectively, is considered. These two random distributions represent popular choices for modeling discrete and continuous lifetimes; in addition, they are strictly connected since they are the basis of the genesis of the Poisson process. The paper is structured as follows. In the next section the expression of the reliability R for the Poisson-exponential model is derived. The problem of sample estimation under the frequentist paradigm is then examined in Section 3, where the expressions of the MLE and the UMVUE are provided as well as the exact and approximate distributions of the former. In Section 4, asymptotic confidence intervals for R relying on the asymptotic normality of its MLE or on the asymptotic behavior of the profile log-likelihood ratio are proposed, the latter incorporating a numerical search procedure. Section 5 illustrates a Monte Carlo simulation plan, aimed at investigating and comparing the performance of point and interval estimators under different settings.

Section 6 applies the estimation techniques previously discussed to two real data sets taken from the literature.

Some final remarks conclude the paper.

2 Reliability parameter

Let us consider a stress-strength model in which the stress X follows a Poisson distribution with parameter $\lambda_1 > 0$, with probability mass function (pmf)

$$p_X(x) = P(X = x) = \frac{\lambda_1^x e^{-\lambda_1 x}}{x!}, \quad x = 0, 1, 2, \ldots,$$

and the strength Y follows an exponential distribution, independently of X, with rate parameter $\lambda_2 > 0$, with probability density function (pdf):

$$f_Y(y) = \begin{cases} \lambda_2 \exp(-\lambda_2 y) & y > 0 \\ 0 & y \leq 0. \end{cases}$$

The reliability parameter $R = P(X < Y)$ of this model is then given by:

$$R = \sum_{x=0}^{\infty} \int_x^{\infty} f_Y(y)\mathrm{d}y \cdot p_X(x) = \sum_{x=0}^{\infty} \int_x^{\infty} \lambda_2 e^{-\lambda_2 y}\mathrm{d}y \cdot \frac{\lambda_1^x e^{-\lambda_1}}{x!} = \sum_{x=0}^{\infty} e^{-\lambda_2 x} \cdot \frac{\lambda_1^x e^{-\lambda_1}}{x!}$$
$$= e^{-\lambda_1} \sum_{x=0}^{\infty} \frac{(\lambda_1 e^{-\lambda_2})^x}{x!} = e^{-\lambda_1} e^{\lambda_1 e^{-\lambda_2}} = e^{-\lambda_1(1-e^{-\lambda_2})}. \quad (1)$$

Note that $R > e^{-\lambda_1}$.

3 Point estimation

Suppose an i.i.d. sample of size n, $\boldsymbol{x} = (x_1, \ldots, x_n)$, is available from X and an i.i.d. sample of size m, $\boldsymbol{y} = (y_1, \ldots, y_m)$, is available from Y. We can calculate the MLE as well as the UMVUE of R.

3.1 Maximum likelihood estimator

The MLE of R is obtained by exploiting its invariance property and then by simply plugging the MLEs of λ_1 and λ_2 into (1). Since for a Poisson rv with parameter λ_1 and an exponential rv of parameter λ_2 these MLEs are $\hat{\lambda}_1 = \frac{1}{n}\sum_{i=1}^n x_i = \bar{x}$ and $\hat{\lambda}_2 = m/\sum_{j=1}^m y_j = 1/\bar{y}$, respectively, the MLE of R is given by

$$\hat{R} = e^{-\hat{\lambda}_1(1-e^{-\hat{\lambda}_2})} = e^{-\bar{x}(1-e^{-1/\bar{y}})}. \quad (2)$$

If we define $U = \sum_{i=1}^n x_i = n\bar{x}$ and $V = \sum_{j=1}^m y_j = m\bar{y}$, we know that $U \sim \mathrm{Pois}(n\lambda_1)$ and $V \sim \mathrm{Gamma}(m, \lambda_2)$. We can then consider the rv

$$L = -\ln \hat{R} = \bar{x}(1 - e^{-1/\bar{y}}) = \frac{U}{n}(1 - e^{-m/V}),$$

and compute the distribution of $T = 1 - e^{-m/V}$. Since $V = -\dfrac{m}{\log(1-T)}$, the cdf of T is given by

$$F_T(t) = 1 - F_V\left(-\frac{m}{\log(1-t)}\right), 0 < t < 1,$$

with F_V being the cdf of V. Then, the cdf of $L = -\log \hat{R}$ can be computed as

$$\begin{aligned} F_L(l) &= P(U \cdot T/n < l) = P(U = 0) + P\left(T < \frac{nl}{U}\right) \\ &= e^{-n\lambda_1} + \sum_{s=1}^{\infty} \frac{(n\lambda_1)^s e^{-n\lambda_1}}{s!} \cdot F_T\left(\frac{nl}{s}\right) \quad 0 \le l < \infty, \end{aligned}$$

and then the cdf of $\hat{R}$ is given by

$$
\begin{aligned}
F_{\hat{R}}(r) &= P(\log \hat{R} < \log r) = 1 - P(-\log \hat{R} < -\log r) \\
&= 1 - F_L(-\log r) = 1 - e^{-n\lambda_1} - \sum_{s=1}^{\infty} \frac{(n\lambda_1)^s e^{-n\lambda_1}}{s!} \cdot F_T\left(\frac{-n\log r}{s}\right) \\
&= \sum_{s=1}^{\infty} \frac{(n\lambda_1)^s e^{-n\lambda_1}}{s!} \cdot F_V(a(r)), \qquad (3)
\end{aligned}
$$

with $0 < r < 1$ and

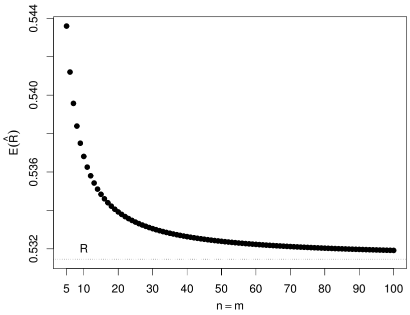
$$
a(r) = \begin{cases} -\frac{m}{\log(1+\frac{n}{s}\log r)} & 1 + \frac{n}{s}\log r > 0 \\ 0 & 1 + \frac{n}{s}\log r \leq 0. \end{cases}
$$

Since the cdf of a Gamma rv like F_V is implemented under any statistical software, such as `R` (R Core Team, 2023), the cdf in (3) can be easily evaluated numerically for any $0 < r < 1$. Note that at $r = 1$, $\hat{R}$ has a probability mass equal to $\exp(-n\lambda_1)$, which corresponds to the probability that all the x values are zero (because $\bar{y} > 0$ almost surely, the MLE of $R = P(X < Y)$ is 1). Thus, if the values of the parameters λ_1 and λ_2 and the sample sizes n and m are known, one can compute numerically the entire cdf of the MLE of R, and by inverting it, any α-quantile, $0 < \alpha < 1$. The moments of $\hat{R}$ can be calculated, recalling that for a non-negative rvs X with cdf F_X, the following expressions hold:

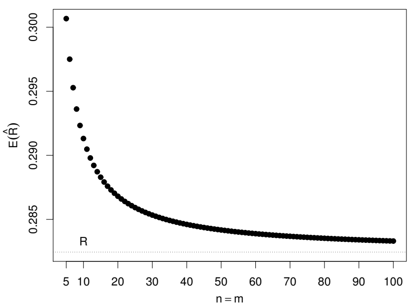
$$
\mathbb{E}(X) = \int_0^{\infty} 1 - F_X(x)\mathrm{d}x \qquad (4a)
$$

$$
\mathrm{var}(X) = \int_0^{\infty} 2x(1 - F_X(x))\mathrm{d}x. \qquad (4b)
$$

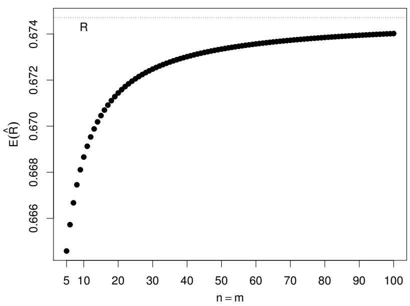
The expected value of $\hat{R}$ can therefore be numerically recovered, and this allows us to assess the bias of the MLE of $\hat{R}$ for any choice of the stress and strength distributions' parameters and sample sizes. In the panels of Figure 1, the expected value of the MLE of R is reported for several choices of (λ_1, λ_2) and for equal sample sizes $n = m = 1, \ldots, 100$. Note that once the stress and strength distributions are fixed, for the parameter combinations examined here, $\mathbb{E}(\hat{R})$ is a monotone (increasing or decreasing) function of the common sample size, and it tends asymptotically to the value R (as expected, the MLE is asymptotically unbiased). However, for other combinations of λ_1 and λ_2, the monotonicity feature can be lost: for example, when $\lambda_1 = 3$ and $\lambda_2 = 0.5$, $\mathbb{E}(\hat{R})$ is first decreasing and then increasing. As a general rule, one can observe that keeping λ_1 fixed, by decreasing λ_2 (and then increasing the reliability parameter R), $\mathbb{E}\hat{R}$ changes from decreasing to increasing, possibly passing through the non-monotone trend described previously. Similarly to the expectation, the variance and the mean squared error $\mathrm{rmse}(\hat{R}) = \mathbb{E}[(\hat{R} - R)^2]$ can be evaluated for any combination $(n, m, \lambda_1, \lambda_2)$.



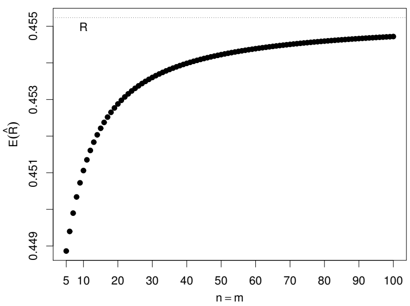
(a) $\lambda_1 = 1, \lambda_2 = 1$



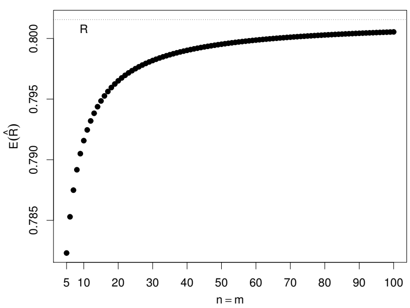
(b) $\lambda_1 = 2, \lambda_2 = 1$



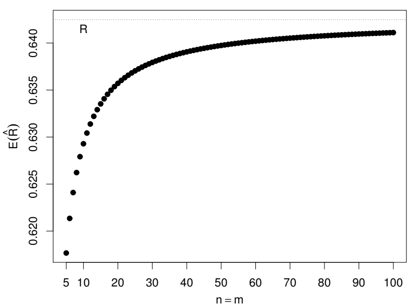
(c) $\lambda_1 = 1, \lambda_2 = 0.5$



(d) $\lambda_1 = 2, \lambda_2 = 0.5$



(e) $\lambda_1 = 1, \lambda_2 = 1/4$



(f) $\lambda_1 = 2, \lambda_2 = 1/4$

Fig. 1: Expected value of the MLE of R for several parameter configurations as a function of common sample size $n = m$ (N.B.: the scale of the ordinate is not the same across the six graphs).

3.2 Uniformly minimum-variance unbiased estimator

The exact expression of the UMVUE of R can be derived as well, and it is given by

$$\tilde{R} = \sum_{j=0}^{\lfloor \min(u,v) \rfloor} \binom{v}{j} \left(\frac{1}{n}\right)^j \left(\frac{n-1}{n}\right)^{v-j} (1 - j/u)^{m-1}, \tag{5}$$

with $u = \sum_{i=1}^n x_i$ and $v = \sum_{j=1}^m y_j$ (see Appendix A for details).

4 Interval estimation

4.1 Asymptotic distribution of $\hat{R}$

In the next two theorems, we will establish the asymptotic distribution of $(\hat{\lambda}_1, \hat{\lambda}_2)$ and $\hat{R}$, when two independent samples of size n and m are available from X and Y, respectively.

The likelihood function of λ_1 and λ_2 can be written as

$$L(\lambda_1, \lambda_2; \boldsymbol{x}, \boldsymbol{y}) = \prod_{i=1}^n p(x_i; \lambda_1) \prod_{j=1}^m f(y_j; \lambda_2) = \frac{\lambda_1^{\sum x_i} \exp(-n\lambda_1)}{\prod x_i!} \cdot \lambda_2^m e^{-\lambda_2 \sum y_j},$$

and the log-likelihood function, omitting $\boldsymbol{x}$ and $\boldsymbol{y}$ from the notation, as

$$\ell(\lambda_1, \lambda_2) = \sum x_i \log \lambda_1 - n\lambda_1 - \sum \log x_i! + m \log \lambda_2 - \lambda_2 \sum y_j, \tag{6}$$

so that

$$\frac{\partial^2 \ell(\lambda_1, \lambda_2)}{\partial \lambda_1^2} = -\frac{\sum_{i=1}^n x_i}{\lambda_1^2}$$

and

$$\frac{\partial^2 \ell(\lambda_1, \lambda_2)}{\partial \lambda_2^2} = -\frac{m}{\lambda_2^2}.$$

Theorem 1. *Let the ratio $\frac{n}{m}$ converge to a positive number s, when both n and m tend to infinity. Then,*

$$(\sqrt{n}(\hat{\lambda}_1 - \lambda_1), \sqrt{n}(\hat{\lambda}_2 - \lambda_2)) \xrightarrow[n\to\infty]{d} N_2(\mathbf{0}, J(\lambda_1, \lambda_2)),$$

where

$$J(\lambda_1, \lambda_2) = \begin{pmatrix} \lambda_1 & 0 \\ 0 & s\lambda_2^2 \end{pmatrix}.$$

Proof. Since

$$-\mathbb{E}\left(\frac{\partial^2 \ell(\lambda_1, \lambda_2)}{\partial \lambda_1^2}\right) = \frac{n}{\lambda_1}$$

and

$$-\mathbb{E}\left(\frac{\partial^2 \ell(\lambda_1, \lambda_2)}{\partial \lambda_2^2}\right) = \frac{m}{\lambda_2^2}$$

from the asymptotic normality of the MLE, it follows that

$$\sqrt{n}(\hat{\lambda}_1 - \lambda_1) \xrightarrow[n\to\infty]{d} N(0, \lambda_1)$$

and

$$\sqrt{m}(\hat{\lambda}_2 - \lambda_2) \xrightarrow[m\to\infty]{d} N(0, \lambda_2^2)$$

and then

$$\sqrt{n}(\hat{\lambda}_2 - \lambda_2) \xrightarrow[n\to\infty]{d} N(0, s\lambda_2^2).$$

From the independence of $\hat{\lambda}_1$ and $\hat{\lambda}_2$, we obtain the statement of the theorem. □

Theorem 2. *Under the same hypothesis as that of the Theorem 1,*

$$\sqrt{n}(\hat{R} - R) \xrightarrow[n\to\infty]{d} N(0, e^{-2\lambda_1(1-e^{-\lambda_2})}\lambda_1[1 + (1 + s\lambda_1\lambda_2^2)e^{-2\lambda_2} - 2e^{-\lambda_2}]).$$

Proof. In order to prove this theorem, let us consider

$$B = \left[\frac{\partial R}{\partial \lambda_1}, \frac{\partial R}{\partial \lambda_2}\right] = \left[-e^{-\lambda_1(1-e^{-\lambda_2})}(1 - e^{-\lambda_2}), e^{-\lambda_1(1-e^{-\lambda_2})}\lambda_1 e^{-\lambda_2}\right].$$

Then, we have

$$\sqrt{n}(\hat{R} - R) \xrightarrow[n\to\infty]{d} N(0, BJB').$$

Plugging in the values of B and J, we obtain the statement of the theorem. □

One can compare the exact value of the variance of $\hat{R}$, whose calculation was described in Section 3, with the approximate value provided by the second theorem for large n and m, for any possible configurations.

4.2 Interval estimators

Using the previous theorem, we can construct the asymptotic confidence intervals for R. Define

$$\widehat{\sigma}^2 = e^{-2\hat{\lambda}_1(1-e^{-\hat{\lambda}_2})}\hat{\lambda}_1[1 + (1 + \hat{\lambda}_1\hat{\lambda}_2^2)e^{-2\hat{\lambda}_2} - 2e^{-\hat{\lambda}_2}].$$

Then, the estimator of the variance of $\hat{R}$ is

$$\widehat{\text{var}}(\widehat{R}) = \frac{\widehat{\sigma}^2}{n}. \tag{7}$$

A symmetric Wald-type CI with level $1 - \alpha$ is therefore provided as

$$\left(\hat{R} - z_{1-\alpha/2}\sqrt{\frac{\widehat{\text{var}}(\hat{R})}{n}}, \hat{R} + z_{1-\alpha/2}\sqrt{\frac{\widehat{\text{var}}(\hat{R})}{n}}\right), \tag{8}$$

with z_γ being the γ quantile of the standard normal distribution. This CI, however, may have a poor performance for small sample sizes or point estimates $\hat{R}$ close to 0 or 1, and it may contain invalid values for R, if the lower bound falls below 0 or the upper bound exceeds 1.

Thus, an alternative CI based on the logit transformation is here suggested; it has been used in the estimation of the reliability parameter of stress-strength models in Mukherjee and Mahiti (1998). Letting $\theta = \log\left(\frac{R}{1-R}\right)$ and $\hat{\theta} = \log\left(\frac{\hat{R}}{1-\hat{R}}\right)$, one can first construct an approximate $(1-\alpha)100\%$ CI for θ, as

$$(\theta_L, \theta_U) = \left(\hat{\theta} - z_{1-\alpha/2}\frac{\sqrt{\widehat{\mathrm{var}}(\hat{R})}}{\hat{R}(1-\hat{R})}, \hat{\theta} + z_{1-\alpha/2}\frac{\sqrt{\widehat{\mathrm{var}}(\hat{R})}}{\hat{R}(1-\hat{R})}\right),$$

and then the corresponding CI for R by inverting the logit function:

$$(R_L, R_U) = \left(\frac{\exp(\theta_L)}{1+\exp(\theta_L)}, \frac{\exp(\theta_U)}{1+\exp(\theta_U)}\right). \tag{9}$$

Such a CI, in contrast to the Wald-type CI in (8), is no longer symmetrical around $\hat{R}$, and it is expected on average to have a coverage rate closer to the nominal level and/or a smaller width. However, Wald's statistic and then the Wald-type CI in (8) are not invariant with respect to reparametrizations. It is shown, for instance, in Mantel (1987), that for exponential families, "with use of the Wald test, according to the parameterization employed, the same data can be both consistent with all possible null values for the parameter and inconsistent with all possible values" (see also Fears et al. (1996)). In view of this, some caution must be used even when resorting to the logit transformation.

A possibly better interval estimator of R can be obtained by relying on the asymptotic distribution of the profile log-likelihood ratio, which may provide satisfactory results especially for small sample sizes (Cortese and Ventura, 2013; Díaz-Francés and Montoya, 2011; Ventura and Racugno, 2011) or, in general, when there is not enough data available to reach the asymptotic properties of the MLE. We know that the maximum value of the log-likelihood function, $\ell_{\max}(\lambda_1, \lambda_2; \boldsymbol{x}, \boldsymbol{y}) = \max_{\lambda_1,\lambda_2} \ell(\lambda_1, \lambda_2; \boldsymbol{x}, \boldsymbol{y})$, is attained at the MLEs ($\hat{\lambda}_1 = \bar{x}, \hat{\lambda}_2 = 1/\bar{y}$). The log-likelihood function in (6) can be re-parametrized in terms of R and λ_2 by solving (1) for λ_1, $\lambda_1 = -\frac{\log R}{1-e^{-\lambda_2}}$, and substituting its expression in (6), thus obtaining

$$\ell(R, \lambda_2; \boldsymbol{x}, \boldsymbol{y}) = \sum_{i=1}^{n} x_i \cdot \log\left(-\frac{\log R}{1-e^{-\lambda_2}}\right) + \frac{n \log R}{1-e^{-\lambda_2}} - \sum_{i=1}^{n} \log x_i! + m \log \lambda_2 - \lambda_2 \sum_{j=1}^{m} y_j. \tag{10}$$

The profile log-likelihood of R is by definition obtained from (10) by substituting for λ_2 its restricted MLE for a specified R, $\hat{\lambda}_2(R)$:

$$\ell_p(R; \boldsymbol{x}, \boldsymbol{y}) = \max_{\lambda_2} \ell(R, \lambda_2; \boldsymbol{x}, \boldsymbol{y}) = \ell(R, \hat{\lambda}_2(R); \boldsymbol{x}, \boldsymbol{y}).$$

In order to obtain $\hat{\lambda}_2(R)$, one can compute the first-order derivative of the profile log-likelihood in (10) with respect to λ_2, equate it to zero, and solve for its unique root:

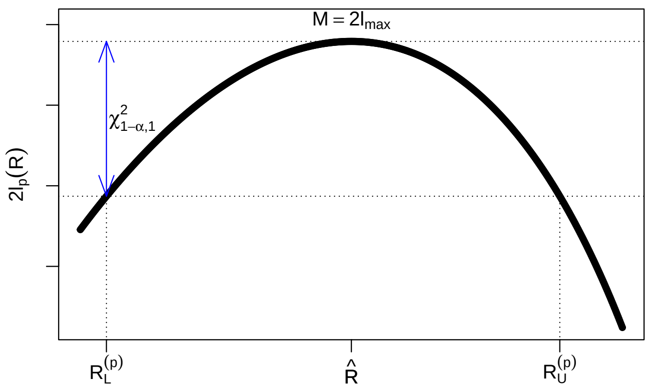
$$\frac{\partial \ell(R, \lambda_2; \boldsymbol{x}, \boldsymbol{y})}{\partial \lambda_2} = -\frac{n\bar{x}e^{-\lambda_2}}{1-e^{-\lambda_2}} - \frac{ne^{-\lambda_2}\log R}{(1-e^{-\lambda_2})^2} + \frac{m}{\lambda_2} - m\bar{y} = 0.$$

However, this non-linear equation in λ_2 can be solved only numerically, so the value $\ell_p(R, \boldsymbol{x}; \boldsymbol{y})$ can be calculated numerically as well. Now the statistic

$$2[\ell_p(\hat{R}; \boldsymbol{x}, \boldsymbol{y}) - \ell_p(R; \boldsymbol{x}, \boldsymbol{y})] = 2[\ell_{\max}(\boldsymbol{x}, \boldsymbol{y}) - \ell_p(R; \boldsymbol{x}, \boldsymbol{y})],$$

which is the profile log-likelihood ratio, is asymptotically distributed as a chi-square rv with 1 degree of freedom. Then, an (approximate) CI for R at level $1-\alpha$ is given by $(R_L^{(p)}, R_U^{(p)})$, with $R_L^{(p)}$ and $R_U^{(p)}$ being the ordered roots of the following equation: $2[\ell_p(\hat{R}; \boldsymbol{x}, \boldsymbol{y}) - \ell_p(R; \boldsymbol{x}, \boldsymbol{y})] = \chi^2_{1-\alpha,1}$ (see also Figure 2). Determining such lower and

Fig. 2: Computation of the lower and upper bounds of the confidence intervals based on the profile log-likelihood. The solid curve is the graph of twice the profile log-likelihood function, $2\ell_p(R; \boldsymbol{x}, \boldsymbol{y})$ for assigned $\boldsymbol{x}$ and $\boldsymbol{y}$; the function takes on its maximum value $M = 2\ell_{\max}(\lambda_1, \lambda_2; \boldsymbol{x}, \boldsymbol{y})$ at $\hat{R}$. The two values $R_L^{(p)}$ and $R_U^{(p)}$ are obtained by equating twice the profile log-likelihood function to $M - 3.84$, with 3.84 being the value of the 95% quantile of level of a chi-square rv with 1 degree of freedom.



upper bounds for R requires a numerical search procedure, which can easily be implemented (see also Jazi et al. (2010) for a similar procedure for finding the MLE). Note that inference based on the profile log-likelihood ratio should be theoretically preferred to that based on the Wald statistic, presented in the previous paragraph. Although they are asymptotically equivalent, the Wald-type CI can be interpreted as a CI based on a quadratic approximation of the log-likelihood, whereas the profile likelihood confidence interval is constructed by exact computation of the profile log-likelihood. Moreover, the profile log-likelihood ratio test is invariant to a change in parametrization and the log-likelihood-based CI always contains valid values of the parameter, in contrast to the Wald interval (8).

An approximate $(1-\alpha)100\%$ CI for R can also be built resorting to the results of Section 3 related to the exact distribution of R. One can consider the interval $(\hat{F}_{\hat{R}}^{-1}(\alpha/2), \hat{F}_{\hat{R}}^{-1}(1-\alpha/2))$, where $\hat{F}_{\hat{R}}^{-1}$ denotes the MLE of the generalized inverse function of $F_{\hat{R}}$, i.e., the generalized inverse (or quantile function) of $F_{\hat{R}}$, with the MLEs of λ_1 and λ_2 plugged in.

5 A Monte Carlo simulation experiment

In this section, some Monte Carlo (MC) simulations are performed to compare the two different point estimators and the four different interval estimators of R. We study different sample sizes, $(n, m) \in \{(10, 10), (10, 20), (20, 10), (20, 20), (100, 100)\}$ (making the same choice as Jovanović (2016) for the geometric-exponential stress-strength model), and different values of parameter λ_1 and λ_2, $\lambda_1 \in \{1, 2, 3\}$, $\lambda_2 \in \{1/8, 1/4, 1/2\}$. The values of the two parameters were chosen in order to produce round expected values for the stress and strength rvs and several different values for the corresponding reliability parameter R. For each combination of n, m, λ_1, and λ_2, we simulated one random sample from $X \sim \text{Pois}(\lambda_1)$ and one random sample from $Y \sim \text{Exp}(\lambda_2)$ independently and calculated the MLE of R according to (2) and the UMVUE of R using (5). We also calculated a 95% CI according to the four techniques described in Section 4 (Wald-type CI, Eq.(8); logit-transformed Wald-type, Eq.(9); profile-log-likelihood-based CI; estimated cdf-based CI). This procedure was repeated $T = 50,000$ times, and the MC averages for MLE, $\text{avg}(\hat{R})$, and UMVUE, $\text{avg}(\tilde{R})$, were calculated. Over the T estimates of $\hat{R}$ and $\tilde{R}$, we also computed an MC estimate of their rmse, as $\widehat{\text{rmse}}(\hat{R}) = \sqrt{\frac{1}{T}\sum_{t=1}^{T}(\hat{R}_t - R)^2}$ and $\widehat{\text{rmse}}(\tilde{R}) = \sqrt{\frac{1}{T}\sum_{t=1}^{T}(\tilde{R}_t - R)^2}$; these two quantities can be used to compare the precision of the two point estimators. Note that the value of $\text{rmse}(\hat{R})$ could be calculated exactly, since we know the exact distribution of $\hat{R}$: $\text{rmse}(\hat{R}) = \text{var}(\hat{R}) + (\mathbb{E}(\hat{R}) - R)^2$, with $\mathbb{E}(\hat{R})$ and $\text{var}(\hat{R})$ calculated through equations (4a) and (4b). However, since the distribution of $\tilde{R}$ cannot be recovered exactly, we preferred to compare the two estimators through their empirical rmse. Over the T MC runs, we calculated the actual coverage for each type of CI, i.e., the proportion of the T confidence intervals containing the true value of R, along with the average width.

Table 1 displays the results of the two point estimators; for the MLE, we also computed the expected value $\mathbb{E}(\hat{R})$ and the standard deviation $\sigma_{\hat{R}} = \sqrt{\text{var}(\hat{R})}$. We can notice that in almost all cases, the UMVUE has an MC average closer to R than the MLE, as one may expect: $\tilde{R}$ is unbiased by construction, and the very small biases in the simulation results are the consequence of rounding off and the finite number of iterations. The MLE, in contrast, is not theoretically unbiased, and by comparing the values of R with those of $\mathbb{E}(\hat{R})$ or $\text{avg}(\hat{R})$, we realize that its (MC) bias is always negative except for one scenario. One can also note that the MC bias (in absolute value), for a given combination (λ_1, λ_2), decreases as the two sample sizes increase, becoming almost zero when $n = m = 100$. As for the rmse, one can note that the

UMVUE tends to present a lower rmse than the MLE for the scenarios leading to larger values of R, so in these cases the UMVUE would be preferable to the MLE, being theoretically unbiased and with a smaller rmse. In contrast, the UMVUE tends to present a higher rmse for the scenarios leading to a smaller value of R. Increasing the sample sizes makes the performances of the two estimators more alike under each scenario here considered.

Figure 3 displays the scatter plot of the MC distribution of $(\hat{R}, \tilde{R})$ and the histograms of the MC marginal distributions of $\hat{R}$ and $\tilde{R}$ when $\lambda_1 = 1$, $\lambda_2 = 1/8$ and $n = m = 10$, corresponding to a value of the reliability parameter $R = 0.8891$. The scatter plot indicates that for each pair of samples drawn from the stress and strength rvs, the values of $\hat{R}$ and $\tilde{R}$ are very close to each other, although we always have $\hat{R} < \tilde{R}$ (all the points lie very close to each other but above the 45-degree bisector; the maximum absolute difference between the two estimates is about 0.0164).

Table 2 displays the results concerning the interval estimation of R. Trying to summarize them, we first remark that when both n and m are equal to the largest value examined here, 100, then the four methods are almost equivalent: they all present an actual coverage very close to the nominal one and a very similar average length. Focusing on the smaller sample sizes, in particular, $n = m = 10$, it is evident that the profile-log-likelihood-based estimator shows overall the best performance, in terms of closeness of the actual coverage to the nominal coverage and in terms of average width. The second best is the interval estimator based on the logit transformation, which exhibits results very close to the profile-log-likelihood-based CI. The CI based on the inversion of the estimated cdf usually shows, among the four estimators, the largest average width, though performing quite well in terms of actual coverage (but this performance has an unclear relationship with sample sizes). It turns out to be the best estimator and is on par with the profile-log-likelihood-based estimator only when $\lambda_1 = 3$ and $\lambda_2 = 0.5$, which corresponds to the lowest value of R considered. The classical Wald-type CI often suffers from under-coverage (the actual coverage rate falls down to $92 - 93\%$ when $n = m = 10$), though performing well in terms of average width. Since the additional computational effort required for the calculation of the profile-log-likelihood based CI is minimal, the use of this interval estimator is recommended especially for small samples.

Table 1: Monte Carlo simulation results. For different parameter values and sample sizes, the Monte Carlo average and rmse for the MLE and UMVUE of R are reported

parameters		reliability	sample sizes		MLE				UMVUE	
λ_1	λ_2	R	n	m	$\mathbb{E}(\hat{R})$	$\sigma_{\hat{R}}$	$\mathrm{avg}(\hat{R})$	$\widehat{\mathrm{rmse}}(\hat{R})$	$\mathrm{avg}(\tilde{R})$	$\widehat{\mathrm{rmse}}(\tilde{R})$
			10	10	0.8808	0.0504	0.8811	0.0507	0.8895	0.0477
			10	20	0.8854	0.0416	0.8853	0.0419	0.8891	0.0407
1	0.125	0.8891	20	10	0.8804	0.0436	0.8804	0.0446	0.8892	0.0417
			20	20	0.8850	0.0338	0.8850	0.0340	0.8891	0.0329
			100	100	0.8883	0.0145	0.8883	0.0145	0.8891	0.0144
			10	10	0.7916	0.0789	0.7921	0.0789	0.8021	0.0773
			10	20	0.7976	0.0676	0.7974	0.0678	0.8014	0.0673
1	0.25	0.8016	20	10	0.7904	0.0673	0.7905	0.0685	0.8016	0.0666
			20	20	0.7965	0.0544	0.7964	0.0545	0.8014	0.0538
			100	100	0.8005	0.0238	0.8005	0.0237	0.8016	0.0237
			10	10	0.6687	0.1060	0.6692	0.1055	0.6753	0.1077
			10	20	0.6742	0.0954	0.6740	0.0954	0.6745	0.0967
1	0.5	0.6747	20	10	0.6658	0.0881	0.6659	0.0887	0.6748	0.0902
			20	20	0.6714	0.0741	0.6712	0.0750	0.6745	0.0757
			100	100	0.6740	0.0335	0.6740	0.0335	0.6747	0.0335
			10	10	0.7770	0.0748	0.7776	0.0754	0.7911	0.0725
			10	20	0.7844	0.0577	0.7843	0.0594	0.7904	0.0584
2	0.125	0.7906	20	10	0.7763	0.0682	0.7765	0.0699	0.7907	0.0667
			20	20	0.7838	0.0509	0.7838	0.0511	0.7905	0.0499
			100	100	0.7892	0.0220	0.7892	0.0220	0.7906	0.0219

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Table 1 – continued from previous page

parameters		reliability	sample sizes		MLE				UMVUE	
λ_1	λ_2	R	n	m	$\mathbb{E}(\hat{R})$	$\sigma_{\hat{R}}$	$\text{avg}(\hat{R})$	$\widehat{\text{rmse}}(\hat{R})$	$\text{avg}(\tilde{R})$	$\widehat{\text{rmse}}(\tilde{R})$
			10	10	0.6293	0.1032	0.6291	0.1039	0.6423	0.1048
			10	20	0.6374	0.0852	0.6373	0.0854	0.6424	0.0860
2	0.25	0.6425	20	10	0.6275	0.0931	0.6278	0.0945	0.6427	0.0947
			20	20	0.6357	0.0725	0.6356	0.0725	0.6424	0.0726
			100	100	0.6411	0.0322	0.6411	0.0321	0.6425	0.0321
			10	10	0.4511	0.1147	0.4515	0.1149	0.4557	0.1204
			10	20	0.4565	0.0996	0.4566	0.0997	0.4553	0.1022
2	0.5	0.4552	20	10	0.4474	0.1008	0.4477	0.1013	0.4555	0.1059
			20	20	0.4529	0.0824	0.4527	0.0822	0.4550	0.0841
			100	100	0.4547	0.0374	0.4547	0.0374	0.4552	0.0376
			10	10	0.6865	0.0912	0.6862	0.0925	0.7027	0.0905
			10	20	0.6955	0.0707	0.6956	0.0711	0.7031	0.0705
3	0.125	0.7029	20	10	0.6856	0.0853	0.6859	0.0872	0.7032	0.0848
			20	20	0.6947	0.0629	0.6945	0.0633	0.7027	0.0624
			100	100	0.7013	0.0275	0.7012	0.0277	0.7028	0.0276
			10	10	0.5023	0.1124	0.5032	0.1129	0.5159	0.1166
			10	20	0.5103	0.0910	0.5102	0.0912	0.5149	0.0929
3	0.25	0.5150	20	10	0.5002	0.1041	0.5006	0.1053	0.5154	0.1080
			20	20	0.5084	0.0801	0.5087	0.0807	0.5153	0.0819
			100	100	0.5136	0.0360	0.5137	0.0358	0.5150	0.0359
			10	10	0.3068	0.1053	0.3064	0.1051	0.3067	0.1113
			10	20	0.3104	0.0892	0.3111	0.0895	0.3079	0.0920
3	0.5	0.3072	20	10	0.3032	0.0951	0.3031	0.0952	0.3070	0.1008
			20	20	0.3068	0.0762	0.3062	0.0761	0.3065	0.0784
			100	100	0.3071	0.0347	0.3071	0.0348	0.3072	0.0350

Table 2: Monte Carlo simulation results. For different parameter values and sample sizes, the average length and coverage of four types of confidence intervals are reported

parameters		reliability	sample sizes		Wald		logit		profile		estimated cdf	
λ_1	λ_2	R	n	m	coverage	avg.len	coverage	avg.len	coverage	avg.len	coverage	avg.len
			10	10	0.9265	0.1857	0.9538	0.1941	0.9490	0.1893	0.9490	0.2025
			10	20	0.9274	0.1583	0.9555	0.1645	0.9478	0.1608	0.9450	0.1648
1	0.125	0.8891	20	10	0.9353	0.1604	0.9464	0.1653	0.9473	0.1618	0.9444	0.1761
			20	20	0.9389	0.1288	0.9507	0.1320	0.9491	0.1302	0.9486	0.1349
			100	100	0.9483	0.0565	0.9514	0.0568	0.9512	0.0566	0.9511	0.0570
			10	10	0.9251	0.2963	0.9554	0.2958	0.9493	0.2930	0.9494	0.3076
			10	20	0.9272	0.2588	0.9567	0.2596	0.9480	0.2572	0.9442	0.2631
1	0.25	0.8016	20	10	0.9323	0.2527	0.9472	0.2521	0.9473	0.2497	0.9454	0.2641
			20	20	0.9375	0.2087	0.9513	0.2089	0.9489	0.2079	0.9492	0.2135
			100	100	0.9479	0.0928	0.9515	0.0928	0.9511	0.0927	0.9511	0.0932
			10	10	0.9221	0.4054	0.9578	0.3886	0.9501	0.3899	0.9487	0.4018
			10	20	0.9259	0.3669	0.9584	0.3548	0.9477	0.3558	0.9429	0.3640
1	0.5	0.6747	20	10	0.9283	0.3379	0.9478	0.3275	0.9479	0.3282	0.9476	0.3355
			20	20	0.9361	0.2906	0.9525	0.2841	0.9490	0.2848	0.9486	0.2896
			100	100	0.9478	0.1311	0.9512	0.1305	0.9507	0.1306	0.9508	0.1311
			10	10	0.9356	0.2780	0.9517	0.2757	0.9499	0.2747	0.9463	0.2951
			10	20	0.9373	0.2261	0.9532	0.2255	0.9499	0.2251	0.9497	0.2327
2	0.125	0.7906	20	10	0.9380	0.2526	0.9466	0.2508	0.9478	0.2489	0.9381	0.2701
			20	20	0.9437	0.1944	0.9499	0.1940	0.9494	0.1935	0.9483	0.2011
			100	100	0.9483	0.0859	0.9505	0.0859	0.9500	0.0858	0.9501	0.0865

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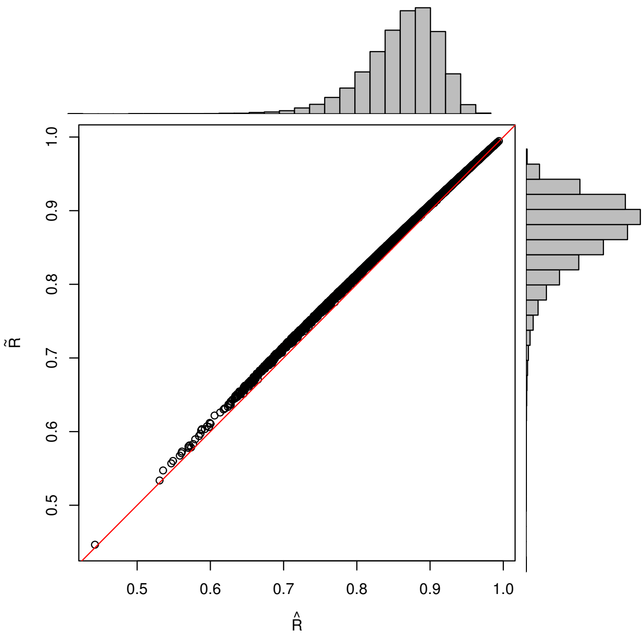
Table 2 – continued from previous page

parameters		reliability	sample sizes		Wald		logit		profile		estimated cdf	
λ_1	λ_2	R	n	m	coverage	avg.len	coverage	avg.len	coverage	avg.len	coverage	avg.len
			10	10	0.9288	0.3926	0.9534	0.3751	0.9501	0.3782	0.9476	0.3937
			10	20	0.9340	0.3286	0.9532	0.3183	0.9489	0.3206	0.9489	0.3286
2	0.25	0.6425	20	10	0.9299	0.3529	0.9477	0.3400	0.9475	0.3416	0.9399	0.3558
			20	20	0.9400	0.2798	0.9514	0.2733	0.9497	0.2746	0.9484	0.2810
			100	100	0.9481	0.1257	0.9509	0.1251	0.9505	0.1252	0.9502	0.1259
			10	10	0.9195	0.4434	0.9515	0.4181	0.9481	0.4224	0.9480	0.4264
			10	20	0.9299	0.3860	0.9541	0.3687	0.9492	0.3711	0.9492	0.3784
2	0.5	0.4552	20	10	0.9214	0.3901	0.9468	0.3723	0.9479	0.3771	0.9424	0.3745
			20	20	0.9354	0.3209	0.9519	0.3107	0.9502	0.3126	0.9494	0.3143
			100	100	0.9456	0.1462	0.9496	0.1452	0.9491	0.1454	0.9490	0.1456
			10	10	0.9346	0.3422	0.9517	0.3313	0.9498	0.3328	0.9417	0.3545
			10	20	0.9374	0.2711	0.9511	0.2659	0.9485	0.2671	0.9479	0.2758
3	0.125	0.7029	20	10	0.9358	0.3183	0.9485	0.3095	0.9484	0.3096	0.9335	0.3320
			20	20	0.9414	0.2410	0.9497	0.2374	0.9486	0.2379	0.9466	0.2464
			100	100	0.9473	0.1072	0.9496	0.1069	0.9493	0.1070	0.9490	0.1078
			10	10	0.9215	0.4305	0.9527	0.4068	0.9484	0.4112	0.9429	0.4202
			10	20	0.9346	0.3526	0.9537	0.3390	0.9495	0.3416	0.9497	0.3478
3	0.25	0.5150	20	10	0.9234	0.3980	0.9497	0.3789	0.9481	0.3826	0.9348	0.3899
			20	20	0.9356	0.3100	0.9506	0.3006	0.9481	0.3026	0.9463	0.3067
			100	100	0.9479	0.1406	0.9513	0.1397	0.9509	0.1399	0.9509	0.1403
			10	10	0.9098	0.4086	0.9517	0.3935	0.9492	0.3944	0.9455	0.3864
			10	20	0.9271	0.3467	0.9530	0.3369	0.9497	0.3354	0.9496	0.3377
3	0.5	0.3072	20	10	0.9103	0.3700	0.9465	0.3584	0.9478	0.3627	0.9376	0.3484
			20	20	0.9296	0.2969	0.9518	0.2908	0.9503	0.2912	0.9481	0.2879
			100	100	0.9452	0.1360	0.9501	0.1354	0.9495	0.1355	0.9490	0.1351

Fig. 3: Scatter plot and histograms for the bivariate distribution of $(\hat{R}, \tilde{R})$ when $\lambda_1 = 1$, $\lambda_2 = 0.125$, and $n = m = 10$. The red line superimposed on the scatter plot is the bisector of the first and third quadrants.



6 A real-data application

We consider the data set 3 and the data set 4 reported in Table VII of Gupta et al. (2023), containing the times to breakdown of an insulating fluid under different experimental conditions (different voltages between electrodes). The authors of Gupta et al. (2023) show that both data sets are fitted by the one-parameter exponential distribution more than adequately. In order to apply the estimation techniques for the Poisson-exponential stress-strength model presented and assessed in the previous sections, we regard the values in data set 3 as a sample from an exponential strength distribution, with estimated parameter $\hat{\lambda}_2 = 0.2171$, whereas we truncate the (continuous) values in data set 4 to their integer part, and then we regard them as a sample from a Poisson stress distribution. The "transformed" integer data are

$$0, 0, 0, 0, 0, 1, 1, 2,$$

and have a mean of 0.5 (corresponding to $\hat{\lambda}_1$) and a variance of 0.571. The Poisson distribution is a good fit for them, which can be checked by resorting to the distance-based tests of Poissonity relying on the Cramér-von Mises distance and on

Table 3: Point and interval estimates of the reliability parameter $R = P(X < Y)$ for the real-data application. Second row: estimates; third row (between brackets): standard error of the point estimate and length of the confidence interval

$\hat{R}$	$\tilde{R}$	Wald CI	logit CI	profile CI	est.cdf CI
0.9070	0.9107	$(0.8115, 1.000)$	$(0.7585, 0.9680)$	$(0.7754, 0.9724)$	$(0.7859, 0.9817)$
(0.0514)	-	(0.1885)	(0.2095)	(0.1970)	(0.1958)

the energy distance, proposed by Szekely and Rizzo (2004) and available in the package `energy`) (Szekely and Rizzo, 2022); for both tests, the p-value is far greater than 5% (89.94% and 77.36%, respectively). We further assume that the stress and strength random variables underlying these two samples are independent. Then, we compute the point estimates $\hat{R}$ and $\tilde{R}$ of the reliability parameter $R = P(X < Y)$, as well as the four different CIs presented in Section 4, with a confidence level of 95%; the standard error of $\hat{R}$ was calculated as well. The results are displayed in Table 3, in which the length for each CI is also reported. One can note that the two point estimates provide slightly different results (0.9070 vs. 0.9107); this discrepancy among the different interval estimates is more accentuated, a fact that could have been expected considering the small sample sizes. We underline that since the upper bound of the symmetrical Wald-type CI exceeded the natural bound 1, we truncated it to 1 (this is a typical problem with Wald-type CIs for a probability or a proportion, as we mentioned in Section 4). The Wald-type CI shows the narrowest length, but also has a lower bound that is much larger than that of the others, whereas the logit CI is the one with the largest width. The CIs based on the profile log-likelihood and on the estimated cdf are quite similar and show almost the same widths.

The statistical environment `R` (R Core Team, 2023) was used for carrying out the computations. The relevant code developed for implementing all the required routines and for running the Monte Carlo simulation study and the real-data application to real data of this section is freely available from https://tinyurl.com/IJSA-D23-00390.

7 Conclusions

In this paper, the estimation of the reliability parameter R for a stress-strength model was discussed, focusing on exponential strength and Poisson stress. The maximum likelihood estimator (MLE) and the uniformly minimum-variance unbiased estimator (UMVUE) of R, based on simple random samples drawn independently from the stress and strength distributions, were derived and discussed. They both have a closed-form expression, and for the former it is possible to derive its cumulative distribution function numerically, which enables the exact evaluation of the expected value and variance. The asymptotic distribution of the MLE is also presented, based on which one can construct an asymptotic Wald-type confidence interval (CI); to improve its performance, which is expected to be poor especially for small sample sizes, a logit transformation is considered. Moreover, a CI based on the profile likelihood and the asymptotic chi-square distribution of the profile log-likelihood ratio is also suggested. A Monte Carlo simulation study, which explores several combinations of distribution parameters and sample sizes, was carried out in order to assess and compare the statistical performance of the MLE and UMVUE and of CIs. It showed that in some settings the UMVUE is better than the MLE, the former being by construction unbiased and exhibiting a smaller root-mean-square error, but their behaviors are similar and tend to converge as the sample sizes increase. The CI based on the profile log-likelihood ratio performs overall the best in terms of coverage rate and average length, especially for small samples (closely followed by the logit-transformed interval estimates). Future research will focus on inference based on non-complete (i.e., censored) samples and on dependence between stress and strength for the Poisson-exponential model.

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Availability of data and materials

Not applicable.

Ethics approval and consent to participate

Not applicable.

Competing interests

The author declares that there is no competing interest to disclose.

Appendix A Derivation of the UMVUE

Because the sum $U = \sum_{i=1}^n X_i$ of n i.i.d. Poisson rvs X_i of parameter λ_1 is still a Poisson rv with parameter $n\lambda_1$, then it follows that

$$P\left(X_1 = x \middle| \sum_{i=1}^n X_i = u\right) = \frac{P\left(X_1 = x \cap \sum_{i=1}^n X_i = u\right)}{P\left(\sum_{i=1}^n X_i = u\right)} = \frac{P\left(X_1 = x\right) P\left(\sum_{i=2}^n X_i = u - x\right)}{P\left(\sum_{i=1}^n X_i = u\right)} \tag{A1}$$

$$= \frac{\lambda_1^x e^{-\lambda_1}}{x!} \frac{[(n-1)\lambda_1]^{u-x} e^{-(n-1)\lambda_1}}{(u-x)!} \Bigg/ \frac{(n\lambda_1)^u e^{-n\lambda_1}}{u!}$$

$$= \binom{u}{x} \left(\frac{1}{n}\right)^x \left(\frac{n-1}{n}\right)^{u-x}, \quad x = 0, 1, \ldots, u,$$

which is a binomial distribution with parameters u and $1/n$.

Analogously, we can deduce the following result regarding the conditional pdf of rv Y_j given $V = v$, where $V = \sum_{j=1}^{m} Y_j$, and Y_j, $j = 1, \ldots, m$, are m i.i.d. exponential rvs with parameter λ_2:

$$f_{Y|V=v}(y|v) = \frac{f_Y(y) f_{V|Y=y}(v|y)}{f_V(v)} = \frac{\lambda_2 \exp(-\lambda_2 y) \frac{\lambda_2^{m-1}}{\Gamma(m-1)} (v-y)^{m-2} \exp(-\lambda_2(v-y))}{\frac{\lambda_2^m}{\Gamma(m)} v^{m-1} \exp(-\lambda_2 v)} \tag{A2}$$

$$= (m-1) \frac{(v-y)^{m-2}}{v^{m-1}} \quad 0 < y < v.$$

Recalling formula (5) for the computation of the UMVUE of R in Tong (1977), which holds when the distributions of X and Y both belong to the exponential family, and adapting it for the specific context of exponential/Poisson rvs, we obtain

$$\tilde{R} = \int_{y=0}^{v} \sum_{x=0}^{u} \tau(x, y) f(y|v) P\left(X = x|u\right) \mathrm{d}y, \tag{A3}$$

with

$$\tau(x, y) = \begin{cases} 0 & x \geq y \\ 1 & x < y \end{cases}.$$

Then, rewriting (A3) by taking into account (A1) and (A2), we have:

$$\tilde{R} = \sum_{x=0}^{\lfloor \min(u,v) \rfloor} \binom{u}{x} \left(\frac{1}{n}\right)^x \left(\frac{n-1}{n}\right)^{u-x} \int_x^v (m-1) \frac{(v-y)^{m-2}}{v^{m-1}} \mathrm{d}y$$

$$= \sum_{x=0}^{\lfloor \min(u,v) \rfloor} \binom{u}{x} \left(\frac{1}{n}\right)^x \left(\frac{n-1}{n}\right)^{u-x} \left[\left(\frac{v-y}{v}\right)^{m-1} \right]_v^x,$$

where $\lfloor \cdot \rfloor$ denotes the integer part function. Thus, the UMVUE is finally given by

$$\tilde{R} = \sum_{j=0}^{\lfloor \min(u,v) \rfloor} \binom{u}{j} \left(\frac{1}{n}\right)^j \left(\frac{n-1}{n}\right)^{u-j} (1 - j/v)^{m-1}.$$

References

Barbiero, A. (2013) Inference on reliability of stress-strength models for Poisson data, *Journal of Probability and Statistics* 2013: 1-8, https://doi.org/10.1155/2013/530530.

Birnbaum, Z. W. (1956). On a use of Mann-Whitney statistics. Proceedings of the Third Berkley Symposium in Mathematics, Statistics and Probability 1:13–17. Berkley, CA: University of California Press.

Cortese, G., Ventura, L. (2013). Accurate higher-order likelihood inference on $P(Y < X)$, *Computational Statistics* 28(3): 1035–1059.

Díaz-Francés, E., Montoya, J.A. (2012). The simplicity of likelihood based inferences for $P(X < Y)$ and for the ratio of means in the exponential model. *Statistical Papers* 54(2): 499–522.

Enis, P., Geisser, S. (1971). Estimation of the probability that $X < Y$, *Journal of the American Statistical Association* 66(333): 162–168.

Fears, T.R., Benichou, J., Gail, M.H. (1996). A reminder of the fallibility of the Wald statistic, *American Statistician* 50(3): 226–227.

Gnedenko, B.V., Ushakov, I.A. (1995). *Probabilistic Reliability Engineering.* New York: John Wiley & Sons.

Gupta, T. R., Pauly, M., Kumar, S. (2023). Estimation of a New Stress Strength Index for One Parameter Exponential Family. *IEEE Transactions on Reliability*, doi: 10.1109/TR.2022.3233897.

Hu, J., Zhuang, Y., Goldiner, C. (2021). Fixed-accuracy confidence interval estimation of $P(X < Y)$ under a geometric-exponential model. *Japanese Journal of Statistics and Data Science*, 4(2), 1079–1104.

Jazi, M.A., Lai, C.-D., Alamatsaz, M.H. (2010). A discrete inverse Weibull distribution and estimation of its parameters. *Statistical Methodology* 7(2): 121–132.

Jiang, L., Wong, A.C.M. (2008). A note on inference for $P(X < Y)$ for right truncated exponentially distributed data, *Statistical Papers* 49: 637–651

Jovanović, M. (2016) Estimation of $P\{X < Y\}$ for geometric-exponential model based on complete and censored samples, *Communications in Statistics - Simulation and Computation*, 46(4), 3050–3066.

Kotz, S., Lumelskii, M., Pensky, M. (2003). *The stress-strength model and its generalizations: theory and applications.* River Edge, NJ: World Scientific.

Krishnamoorthy, K., Mukherjee, S., Guo, H. (2007). Inference on reliability in two-parameter exponential stress-strength model. *Metrika*, 65(3), 261–273.

Maiti, S.S. (1995). Estimation of $P(X \leq Y)$ in the Geometric Case, *Journal of Indian Statistical Association* 33(2), 87–91.

Mantel N. (1987). Understanding Wald's test for exponential families, *American Statistician* 41(2): 147–149.

Mukherjee, S.P., Maiti, S.S. (1998). Stress-strength reliability in the Weibull case. In: *Frontiers In Reliability*, 4. World Scientific, Singapore, pp.231–248.

Obradovic, M., Jovanovic, M., Milosevic, B., Jevremovic, V. (2014) Estimation of $P(X \leq Y)$ for Geometric-Poisson Model, *Hacettepe Journal of Mathematics and Statistics*, 44(4), 949–964.

R Core Team (2023). R: A language and environment for statistical computing. R Foundation for Statistical Computing, Vienna, Austria. URL https://www.R-project.org/.

Raqab, M. Z., Madi, T., Kundu, D. (2008). Estimation of $P(Y < X)$ for the three-parameter generalized exponential distribution. *Communications in Statistics - Theory and Methods* 37(18): 2854–2865.

Szekely, G. J., Rizzo, M. L. (2004) Mean Distance Test of Poisson Distribution. *Statistics and Probability Letters* 67(3), 241–247.

Szekely, G. J., Rizzo, M. L. (2022) energy: E-Statistics: Multivariate Inference via the Energy of Data. R package version 1.7-11. https://CRAN.R-project.org/package=energy

Singh, B., Nayal, A. S., Tyagi, A. (2023). Estimation of $P\,[Y < Z]$ under Geometric-Lindley model. *Ricerche di Matematica*, 1–32, https://doi.org/10.1007/s11587-022-00753-x.

Tong, H. (1974). A note on the estimation of $Pr\,\{Y < X\}$ in the exponential case. *Technometrics* 16: 625.

Tong, H. (1975). A note on the estimation of $Pr\,\{Y < X\}$ in the exponential case, Errata. *Technometrics* 17: 395.

Tong, H. (1977). On The Estimation of $Pr\,\{Y \leq X\}$ for Exponential Families, *IEEE Transactions on Reliability* 26(1): 54–56.

Varde, S.D. (1969). Life Testing and Reliability Estimation for the Two Parameter Exponential Distribution. *Journal of the American Statistical Association* 64(326): 621–631.

Ventura, L., Racugno, W. (2011) Recent advances on Bayesian inference for $P(X < Y)$. *Bayesian Analysis* 6(3): 411–428.