GRASP approaches to the Weighted Safe Set Problem

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1 The problem

The *Weighted Safe Set Problem* (*WSSP*) is one of several graph optimization models recently proposed to characterise the safety of networks and the interplay between the subnetworks that compose them, in particular referring to communities in social networks. It is defined as follows [2]: given a connected undirected graph $G = (V, E)$ and a positive weight function $w : V \rightarrow \mathbb{Q}_+$ defined on its vertices, find a subset of vertices $S \subseteq V$ of minimum weight, such that each maximal connected component of $G[S]$ has weight nonsmaller than each adjacent connected component of $G[V \setminus S]$. The weights of subsets and components are obtained summing the weights of their vertices.

The complexity of the problem has been investigated from the theoretical point of view for several special graph topologies, both in the weighted and the unweighted case [1–3]. Exact branch-and-cut approaches [7, 8] and Mixed Integer Linear Programming (*MILP*) formulations [6] have been proposed. Currently, the only existing heuristic algorithm is a randomised destructive heuristic ancillary to one of the exact approaches [7].

2 The GRASP metaheuristics

We develop two metaheuristics based on the *Greedy Randomized Adaptive Search Procedure* (*GRASP*). They have a similar structure, sequentially composed by a constructive and a destructive phase. The former starts from the empty set and progressively updates a current infeasible solution by adding vertices until all constraints are satisfied. The latter removes vertices as long as feasibility can be maintained.

More in detail, let S be the current subset in the constructive phase. By *candidate vertices* we denote the vertices in $V \setminus S$ that are more likely to reduce the violation of the constraints, if moved to S. They are adjacent to a component of $G[S]$ of insufficient weight or they belong to a component of $G[V \setminus S]$ whose weight exceeds that of an adjacent component of $G[S]$. The *unsafe degree* of a candidate vertex v is the number of adjacent vertices belonging to $V \setminus S$, $\delta^{V \setminus S}(v) = |\{u \in V \setminus S \mid (u, v) \in E\}|$. One of these vertices is randomly selected and inserted in S, favouring those with larger unsafe degree, that are more likely to split the components of $G[V \setminus S]$ and yield a feasible solution. The two *GRASP* metaheuristics adopt different randomisation schemes:

Restricted Candidate List (*RCL*) [5]: compute a convex combination of the minimum and the maximum unsafe degree of the candidate vertices, $(1 - \mu)\delta_{\min} + \mu\delta_{\max}$, with $\mu \in [0, 1]$, identify the vertices whose degree is not strictly smaller than this combination, and select one of them at random with uniform probability;

Heuristic-Biased Stochastic Sampling (*HBSS*) [4]: assign to each candidate vertex v a probability proportional to $(\delta^{V \setminus S}(v))^{\alpha} + 1$, with $\alpha \geq 0$, and select a random vertex with these probabilities.

The destructive phase refines the feasible solution obtained by the constructive one by iteratively trying to remove its vertices in nondecreasing weight order without violating feasibility; vertices of equal weight are considered by increasing degree.

Notice that the objective function of the problem is considered only in the destructive phase, in a purely deterministic way. In fact, the weight can be misleading in the constructive phase, as smaller values favour optimality, whereas larger ones favour feasibility. In the destructive phase, that preserves feasibility and usually performs few iterations, the weight provides a useful information.

3 The computational results

We have considered the 126 benchmark instances used by [7], that consist in random weighted and unweighted graphs from 10 to 30 vertices with various densities. Since these instances are small, we have also considered the 40 largest instances introduced in [6] (50 vertices) and generated 200 new larger instances, up to 300 vertices, with the same structure. Benchmarks and detailed results are available at https://homes.di.unimi.it/cordone/research/wssp.html.

In summary, our computational experiments show that:

1. both RCL and HBSS approaches always produce results equal or better than the destructive procedure in [7] in a time one order of magnitude smaller;
2. the quality of both methods is robust in a range of parameter values;
3. RCL produces average better results than HBSS;
4. RCL often provides near-optimal solutions in a few seconds.

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