A note on the continuity in the Hurst index of the solution of rough differential equations driven by a fractional Brownian motion

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Abstract

Within the rough path framework we prove the continuity of the solution to random differential equations driven by fractional Brownian motion with respect to the Hurst parameter H when $H \in (1/3, 1/2]$.

1 Introduction

The importance of the study of stochastic equations driven by a fractional Brownian motion with parameter $H \in (0, 1)$ naturally arises from the observation of many phenomena for which the assumption of independence of increments which is intrinsic, for example in the case of the standard Brownian motion, cannot be supposed ([7]). Indeed, in biology, meteorology, telecommunications, queueing theory and finance evidence of memory and autocorrelation effects are shown ([6,10,15]). The estimation of H is very important, since it determines the magnitude of the self-correlation of the noise in the models. As emphasized in [8], not only one has to deal with the problem of the estimation of the Hurst parameter H of the noise, as in [9,11,12], but one needs to check that the model does not exhibit a large sensitivity with respect to the values of H. Hence, the study of the continuity problem is important in the case of both time (SDE) and time-space (SPDE) stochastic differential equations driven by fractional noises, and it is a very interesting problem not only from a theoretical point of view, but also in the modeling applications ([1, 2, 4, 13, 14]).

Here we investigate a continuity problem for a stochastic differential equation (SDE) driven by a fractional Brownian motion Y^H in the rough paths theory setting ([1–3,5]). In particular, the central object is the following equation

$$dX_t = \alpha(X_t)dt + \beta(X_t) \circ dY_t, \tag{1}$$

where $Y : [0,T] \to \mathbb{R}^d$ is a driving signal, $X : [0;T] \to \mathbb{R}^n$, $\alpha : \mathbb{R}^n \to \mathbb{R}^n$, and $\beta : \mathbb{R}^n \to \mathbb{R}^{n \times d}$ are smooth functions. As usual the solution process $\{X_t\}_{t \in [0,T]}$ has to be interpreted in the integral form, given an initial datum X_0 , i.e.

$$X_t = X_0 + \int_0^t \alpha(X_t)dt + \int_0^t \beta(X_t) \circ dY_t,$$
(2)

where the integral has to be understood in the sense of canonical rough integral. We recall that integration in the rough path sense involves functions with low regularity, in particular which are Hölder continuous.

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We consider the particular case in which the noise is given by a fractional Brownian motion, that is the process Y in (1) is a fractional Brownian motion W^H with Hurst parameter $H \in (0, 1)$. The continuity of solution of stochastic differential equations driven by a fractional Brownian motion and its functional, linear and not linear, has been already investigated in [13, 14], in the case $H \downarrow \frac{1}{2}$. We recall that when H > 1/2 the integral in equation (2) is a pathwise Stieltjes integral in the sense of Young ([16]). In the applications, however, the estimation of H shows that it may take values less then 1/2 ([6]). In the present work we consider the case $H \to H_{\infty} \in (\frac{1}{3}, \frac{1}{2}]$ and we specialize to the case of fractional Brownian motion some convergence results obtained for a more general class of noise in [2, 3]. The integral is understood within a rough path approach and the weak convergence is considered with respect to the *p*-variation topology.

The result is precisely the following: for $H \in \left(\frac{1}{3}, \frac{1}{2}\right]$ let us consider the solution X^H of (1) where $Y \equiv W^H$, which defines a probability distribution on the space $C^{1/3}([0,T])$ of $\frac{1}{3}$ -Hölder continuous functions. We show that whenever $H \to H_{\infty} \in \left(\frac{1}{3}, \frac{1}{2}\right]$, it holds that $X^H \xrightarrow{d} X^{H_{\infty}}$, where \xrightarrow{d} denotes the convergence in distribution on $C^{1/3}([0,T])$.

The proof relies on the observation that the solution operator which maps the lift (W^H, \mathbb{W}^H) of the noise W^H into the solution X^H can be made continuous ([1,2]), so that it is sufficient to show that $(W^H, \mathbb{B}^H) \to (B^{H_{\infty}}, \mathbb{B}^{H_{\infty}})$ and to exploit the continuity of the solution map to deduce that $X^H \to X^{H_{\infty}}$. In the present work, we prove the continuity of the lift by following the standard scheme, that is by first establishing the tightness property and then by identifying the limit. Our main contribution is the proof of the tightness in the specific case of the fractional Brownian motion taking advantage of some fundamental results of the rough path theory ([2]).

2 Hölder spaces and lifted paths spaces

Let us recall now the main functional spaces useful within the rough path theory.

Definition 1. Let $\alpha > 0$. Given a Banach space $(E, |\cdot|_E)$, a function $Y : [0, T] \to E$ is a α -*Hölder continuous* function if the seminorm

$$\|Y\|_{\alpha} := \sup_{t \neq s} \frac{|Y_t - Y_s|_E}{|t - s|^{\alpha}}$$
(3)

is finite. Let $\mathcal{C}^{\alpha}([0,T]; E)$ be the space of all α -Hölder continuous functions from [0,T] into E. A norm on \mathcal{C}^{α} is define as follows

$$\|Y\|_{\mathcal{C}^{\alpha}} = |Y_0|_E + \|Y\|_{\alpha}.$$
(4)

We may extend the space introduced in Definition 1 to the functions defined on $[0, T]^2$. **Definition 2.** Let $\alpha > 0$. Given a Banach space $(E, |\cdot|_E)$, we define the space C_2^{α} as the set of functions $W : [0, T]^2 \to E$ such that the seminorm

$$\|W\|_{\mathcal{C}_{2}^{\alpha}} := \sup_{t \neq s} \frac{|W(s,t)|_{E}}{|t-s|^{\alpha}}$$
(5)

is finite.

Definition 3. Let $\alpha > 0$. Given a Banach space $(E, |\cdot|_E)$, the vector space $\mathcal{C}^{\alpha} \oplus \mathcal{C}_2^{2\alpha}$ is the set of the pair functions (X, W) with $X : [0, T] \to E$ and $W : [0, T]^2 \to E$, endowed by the norm

$$\|(X,W)\|_{\mathcal{C}^{\alpha}\oplus \mathcal{C}_{2}^{2\alpha}} := \|X\|_{\mathcal{C}^{\alpha}} + \|W\|_{\mathcal{C}_{2}^{2\alpha}}.$$
(6)

Such a space is a Banach space.

The rough path may be seen as a subspace of the Banach space given in Definition 3.

Definition 4. Let $\alpha \in (\frac{1}{3}, \frac{1}{2}]$. An α -Hölder rough path is a pair of functions $(X, \mathbb{X}) \in \mathcal{C}^{\alpha}([0, T], E) \oplus \mathcal{C}^{2\alpha}_{2}([0, T]^{2}, E)$ such that the so called Chen's relation is satisfied, i.e. for any $s, u, t \in [0, T]$,

$$\mathbb{X}_{s,t} - \mathbb{X}_{s,u} - \mathbb{X}_{u,t} = (X_u - X_s) \otimes (X_t - X_u).$$

$$\tag{7}$$

We denote by \mathscr{C}^{α} the subspace of $\mathcal{C}^{\alpha}([0,T], E) \oplus \mathcal{C}_{2}^{2\alpha}([0,T]^{2}, E)$ such that the Chen's relation (7) is satisfied, endowed by the distance

$$\rho_{\alpha} \left((X, \mathbb{X}), (Y, \mathbb{Y}) \right) = \|X - Y\|_{\alpha} + \|\mathbb{X} - \mathbb{Y}\|_{C_{2}^{2\alpha}}$$

$$= \sup_{t \neq s} \frac{|X_{t} - X_{s} - (Y_{t} - Y_{s})|_{E}}{|t - s|^{\alpha}} + \sup_{t \neq s} \frac{|\mathbb{X}_{s,t} - \mathbb{Y}_{s,t}|_{E}}{|t - s|^{2\alpha}}.$$
(8)

Remark 5. The space \mathscr{C}^{α} is a subset of the vector space $\mathcal{C}^{\alpha} \oplus \mathcal{C}_2^{2\alpha}$, but it is not a linear subspace, due to the non-linear scaling given by (7). In detail, for $(X, \mathbb{X}) \in \mathscr{C}^{\alpha}$ and $\lambda \in \mathbb{R}$ we have that

$$\begin{split} \lambda \mathbb{X}_{s,t} - \lambda \mathbb{X}_{u,t} &= \lambda (\mathbb{X}_{s,t} - \mathbb{X}_{s,u} - \mathbb{X}_{u,t}) \\ &= \lambda ((X_u - X_s) \otimes (X_t - X_u)) \\ &\neq \lambda^2 ((X_u - X_s) \otimes (X_t - X_u)) \\ &= (\lambda X_t - \lambda X_u) \otimes (\lambda X_u - \lambda X_s). \end{split}$$

Hence, the Chen's relation is not satisfied by $\lambda(X, \mathbb{X})$, except for $\lambda = 0, 1$. On the contrary, if $(X, \mathbb{X}) \in \mathscr{C}^{\alpha}$, then $(\lambda X, \lambda^2 \mathbb{X})$ satisfied the Chen relation (7).

The non-linear scaling property given by $(X, \mathbb{X}) \to (\lambda X, \lambda^2 \mathbb{X})$ suggests the definition of the following quantity, which is homogeneous with respect to (7).

Definition 6. We define on \mathscr{C}^{α} the α -Hölder rough path norm as the quantity given by

$$||(X, \mathbb{X})||_{\mathscr{C}^{\alpha}} := ||X||_{\alpha} + \sqrt{||\mathbb{X}||_{C_{2}^{2\alpha}}}.$$
 (9)

Remark 7. The quantity $||(X, \mathbb{X})||_{\mathscr{C}^{\alpha}}$ is not a norm in the usual sense, because $||\lambda(X, \mathbb{X})||_{\mathscr{C}^{\alpha}} \neq |\lambda| \cdot ||(X, \mathbb{X})||_{\mathscr{C}^{\alpha}}$, but it scales correctly with respect to (7) by preserving the transformation $(X, \mathbb{X}) \to (\lambda X, \lambda^2 \mathbb{X})$. Indeed, we have that

$$||(\lambda X, \lambda^2 \mathbb{X})||_{\mathscr{C}^{\alpha}} = |\lambda| \cdot ||(X, \mathbb{X})||_{\mathscr{C}^{\alpha}}.$$

Let us observe that neither (7) nor the definition of \mathscr{C}^{α} imply any type of chain rule or integration by parts formula.

Definition 8. Let $E = \mathbb{R}$. We define the space \mathscr{C}_g^{α} of geometric rough paths as the space of rough paths in \mathscr{C}^{α} which moreover satisfy the following condition

$$\mathbb{X}_{s,t} = \frac{1}{2} (X_t - X_s)^2.$$
(10)

Remark 9. We note that in the case $E = \mathbb{R}^d$, d = 1, the geometric rough path condition (10) completely determines the form of X. If we consider paths with values in \mathbb{R}^d , the function X becomes $\mathbb{R}^d \otimes \mathbb{R}^d$ -valued (matrix-valued) and condition (10) becomes $\operatorname{Sym}(X) = \frac{1}{2}(X_{s,t} \otimes X_{s,t})$. We refer to [1] for a precise description of the multidimensional case \mathbb{R}^d , d > 1.

2.1 Gaussian processes as rough paths

We consider a canonical rough path structure for a class of continuous Gaussian processes, which satisfy a specific condition upon the covariance structure. In order to define the properties upon the covariance, one needs to introduce the right definition of variation ([1,2]). The fractional Brownian motion belongs to such a class of processes.

Definition 10. Let $\Delta_T := \{0 \le s \le t \le T\}$ and consider a map $\omega : \Delta_T \times \Delta_T \to [0, \infty)$. We say that ω is a 2D control if it is super-additive in the following way: given a rectangle $R \subset [0, T]^2$ and any finite partition $\{R_j, 1 \le j \le n\}$ of R, we have

$$\omega(R) \ge \sum_{j \le n} \omega(R_j).$$

Given a function f defined on rectangles, we say that f is *controlled by* the control ω if, for any rectangle $R \subset [0, T]^2$, the following estimate holds

$$|f(R)| \le \omega(R)$$

Definition 11. Given a function $f: [0,T]^2 \to \mathbb{R}$, we denote by

$$R := \begin{pmatrix} s, t \\ u, v \end{pmatrix} := [s, t] \times [u, v]$$

a rectangle of $[0,T]^2$ and

$$f\binom{s,t}{u,v} := f(t,v) - f(t,u) - f(s,v) + f(s,u)$$

the rectangular increment of f where $0 \le s \le t \le T$ and $0 \le u \le v \le T$.

Definition 12. Let us denote by $R_{i,j} = (t_i, t_{i+1}] \times (t'_j, t'_{j+1}] \subseteq R$, such that $\{t_i\}_i \in \mathcal{D}([s,t]), \{t'_j\}_j \in \mathcal{D}([u,v])$, where $\mathcal{D}([s,t])$ is the family of partitions of the interval [s,t] and $\pi(R)$ a (generic) partition of R.

In the following we might denote a partition $\widetilde{\pi}(R) = \{t_i, t'_i\}_{i,j} \in \mathcal{D}^2(R)$, i.e.

$$\{t_i, t'_j\}_{i,j} := \{R_{i,j}\}_{i,j} = \widetilde{\pi}(R).$$

The set $\mathcal{D}^2(R)$ is the family of regular or grid-like partitions of R,

$$\mathcal{D}^{2}(R) = \left\{ \widetilde{\pi}(R) = \{R_{i,j}\}_{i,j} : \bigcup_{i,j \in \mathbb{N}} R_{i,j} = R \right\}$$

$$= \mathcal{D}([s,t]) \times \mathcal{D}([u,v])$$
(11)

The set $\mathcal{P}(R)$ denotes the family of all rectangular partitions or tessellations of R, i.e. all families π such that

$$\mathcal{P}(R) = \left\{ \pi(R) = \{R_j\}_{j \in \mathbb{N}} : R_j \neq \emptyset; \mathring{R}_i \cap \mathring{R}_i = \emptyset, i \neq j; \bigcup_{j \in \mathbb{N}} R_j = R \right\}.$$

Since not any partition is of grid-like type, one has trivially that, for any $R \subseteq [0, T]^2$,

$$\mathcal{D}^2(R) \subset \mathcal{P}(R). \tag{12}$$

Definition 13. Let $f : [0,T]^2 \to \mathbb{R}$ and $p \in [1,\infty)$. For any rectangle $R \subset [0,T]^2$ the following quantity

$$V_p(f,R) := \left(\sup_{\{t_i, t'_j\}_{i,j} \in \mathcal{D}^2(R)} \sum_{i,j} \left| f\binom{t_i, t_{i+1}}{t_j, t_{j+1}} \right|^p \right)^{\frac{1}{p}}$$
(13)

is called the *p*-variation of f over $R \subseteq [0, T]^2$. The function f has finite *p*-variation if it holds that

$$V_p(f, [0, T]^2) < \infty.$$

Definition 14. Let $f : [0,T]^2 \to \mathbb{R}$ and let $p \in [1,\infty)$. For any rectangle $R \subset [0,T]^2$ we define the *controlled p-variation* as

$$|f|_{p\text{-var},R} := \left(\sup_{\pi \in \mathcal{P}(R)} \sum_{A \in \pi} \left| f(A) \right|^p \right)^{\frac{1}{p}}.$$
(14)

Remark 15. Recall that if $x : [0,T] \to \mathbb{R}$, then for any $[u,v] \subset [0,T]$, and any p > 0 $V_p(x, [u,v]) = |x|_{p-var, [u,v]}$. Furthermore, defining ω_1 , for any $s, t \in [0,T]$, as $\omega_1(s,t) = |x|_{p-var, [u,v]}^p$, we have that ω_1 is a 1D control and it controls x, i.e.

$$\omega_1(s,u) + \omega_1(u,t) \leq \omega_1(s,t); \tag{15}$$

$$|x(t) - x(s)| \leq \omega_1(s,t)^{1/p}.$$
 (16)

Remark 16. Since for any f and any $R \subseteq [0,T]^2$ inclusion (12) holds, for any $p \ge 1$ the following inequality holds

$$V_p(f,R) \le |f|_{p\text{-var},R}.$$
(17)

Whenever p > 1,

$$V_p(f, R) < |f|_{p-\operatorname{var}, R}$$

We will see an example of this behavior in the case of the fractional Brownian motion W^H in Proposition 23.

Even if the *p*-variation and the controlled *p*-variation are different concepts, it is know they are ε -close concepts. Indeed, by means of the Young-Towghi's maximal inequality, in [3] the authors prove the following result.

Proposition 17 ([3], Theorem 1-4). Let $p \ge 1$ and $\varepsilon > 0$. There exists an explicit constant $C(p,\varepsilon) \ge 1$ such that for every $f:[0,T]^2 \to \mathbb{R}$ and for every R rectangle in $[0,T]^2$ it holds

$$\frac{1}{C(p,\varepsilon)}|f|_{p+\varepsilon\text{-}var,R} \le V_p(f,R) \le |f|_{p\text{-}var,R}.$$
(18)

Introducing $\alpha_p = p(p + \varepsilon)$, the constant C is given by

$$C(p,\varepsilon) = \left\{ \left[1 + \zeta \left(1 + \frac{\varepsilon}{2\alpha_p + \varepsilon} \right) \right]^{1 + \frac{\varepsilon}{2\alpha_p}} \times \zeta \left(1 + \frac{\varepsilon}{2\alpha_p} \right) + \left[1 + \zeta \left(1 + \frac{\varepsilon}{\alpha_p} \right) \right] \right\},$$
(19)

where ζ denotes the Riemann zeta function.

Furthermore, if $|f|_{p-var,R}$ is finite, then it is superadditive as function of R.

Remark 18. Note that, for any fixed $\varepsilon > 0$, $C(p, \varepsilon)$ is continuous as function of $p \in [1, \infty)$. Indeed, since $\zeta(x) \to \infty$ when $x \to 1^+$ it only diverges when $p \to \infty$.

Remark 19. From (17) and (18) one obtain the following inequality

$$V_{p+\varepsilon}(f,R) \le C(p,\varepsilon)V_p(f,R).$$
(20)

Now let $\{X_t, t \in [0, T]\}$ be a real-valued centered continuous Gaussian process with covariance structure given, for $s, t \in [0, T]$, by

$$K(s,t) := \mathbb{E}[X_t X_s].$$

Given a continuous and centered Gaussian process X with covariance K, it is possible to construct a canonical rough path (X, \mathbb{X}) , provided that the covariance function K has some p-variation regularity, and that the p-variation of K is controlled by some 2D control ω . We make it more precise.

Theorem 20 ([2], Theorem 15.33). Let X_t , for $t \in [0, T]$, be a centered continuous Gaussian process with values in \mathbb{R} . Suppose that there exists a $\rho \in [1, 2)$ such that the covariance K of X, given 2D control ω such that $\omega([0, T]^2) < \infty$

$$|K|_{\rho\text{-var},R} \le \omega(R), \qquad \forall R \subseteq [0,T]^2, \tag{21}$$

that is the covariance K has finite controlled ρ -variation dominated by a 2D control ω .

Then, there exists a unique process (X, \mathbb{X}) in \mathscr{C}^{α} such that (X, \mathbb{X}) lifts X, in the sense that $\pi_1((X, \mathbb{X})_t) = X_t - X_0$. Moreover, there exists a constant $C = C(\rho)$ such that for every $s \leq t$ and for every $q \geq 1$ it holds

$$\mathbf{E}\left[\left(\left|X_{s,t}\right| + \left|\mathbb{X}_{s,t}\right|^{1/2}\right)^{q}\right]^{\frac{1}{q}} \le C(\rho)\sqrt{q}\,\omega([s,t]^{2})^{\frac{1}{2\rho}}.$$
(22)

The lift (X, \mathbb{X}) is unique and natural in the sense that it is the limit in the space of rough paths \mathscr{C}_g^{α} of any sequence X_n of piecewise linear or mollified approximations to X such that $||X_n - X||_{\infty} \to 0$ almost surely.

Remark 21. Regarding the approximations to a rough path (X, \mathbb{X}) via regular functions, we refer to ([2], Chapter 15), in which there is a large discussion about piecewise linear and mollified approximations of a Gaussian process. A complete discussion about this topic would exceed the scope of this work.

3 RDEs driven by a fractional Brownian motion

The fractional Brownian motion with Hurst parameter $H \in [0, 1]$ is a zero mean Gaussian process with covariance given by

$$\mathbb{E}[W_t^H W_s^H] = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t - s|^{2H}) =: K^H(s, t).$$
(23)

The parameter H is responsible of the strength and the sign of the correlations between the increments. Indeed, for $H \in (0,1) \setminus \{\frac{1}{2}\}$, set $\tilde{H} = H - \frac{1}{2}$, for any $t_1 < t_2 < t_3 < t_4$, one may express the covariance of the increments in an integral form

$$E\left[(W_{t_{2}}^{H} - W_{t_{1}}^{H})(W_{t_{4}}^{H} - W_{t_{3}}^{H})\right] = 2\widetilde{H}H \int_{t_{1}}^{t_{2}} \int_{t_{3}}^{t_{4}} (u - v)^{2\widetilde{H} - 1} du \, dv$$

$$= \frac{1}{2} \left(|t_{4} - t_{1}|^{2H} + |t_{3} - t_{2}|^{2H} - |t_{4} - t_{1}|^{2H} + |t_{3} - t_{1}|^{2H}\right).$$
(24)

Since in the above integral form the integrand is a positive function and H > 0, it follows that the sign of the correlation depends only upon \tilde{H} , being positive when $\tilde{H} > 0$, i.e. $H \in (\frac{1}{2}, 1)$, and negative when $\tilde{H} < 0$, i.e. $H \in (0, \frac{1}{2})$.

For any $(s,t) \in [0,T]^2$, we denote by $|\pi_{s,t}| = \max_j |t_j - t_{j-1}|$ the width of any partition $\pi_{s,t} \in \mathcal{D}([s,t])$. Then, given the process $\{W_t^H\}_{t \in \mathbb{R}_+}$, let $\{\mathbb{W}_{s,t}^H\}_{s,t}$ be the iterated integral operator defined as

$$\mathbb{W}_{s,t}^{H} = \int_{s}^{t} (W_{\tau}^{H} - W_{s}^{H}) \circ dW_{\tau}^{H} := \frac{1}{2} (X_{t} - X_{s})^{2}.$$
(25)

According with Definition 4, the process $\mathbf{W}^{H} = (W^{H}, \mathbb{W}^{H})$ is a geometric rough path.

Let us now consider the RDE (1) with driven signal given by a fractional Brownian motion W^H of index $\frac{1}{3} < H < 1$.

$$dX_t^H = \mu(X_t^H)dt + \sigma(X_t^H) \circ dW_t^H.$$
(26)

The solution X_t^H is interpreted in the following integral form

$$X_t^H = X_0^H + \int_0^t \mu(X_t^H) dt + \int_0^t \sigma(X_t^H) \circ dW_t^H,$$
(27)

with initial condition $X_0^H \in L^2(\Omega)$, where the stochastic integral is an integral with respect to a rough path $\mathbf{W}^H = (W^H, \mathbb{W}^H)$ defined over the process W^H . The following relevant results concerning existence, uniqueness and a notable continuity property of the above stochastic equation holds ([2]).

Theorem 22. Given $H \in (\frac{1}{3}, \frac{1}{2}]$, let $X_0^H = x_0 \in \mathbb{R}$ be a constant and let $\mu, \sigma \in \mathcal{C}_b^3(\mathbb{R})$ (three times differentiable bounded functions). Then there exists an unique solution $X^H = (X_t^H)_{t \in [0,T]}$ to equation (26) with initial condition x_0 . Moreover, the solution X^H is a continuous function of $\mathbf{W}^H = (W_t^H, \mathbb{W}_{s,t}^H)$, in the sense that the solution map S, given by

$$S: \mathscr{C}^{\alpha} \longrightarrow \mathcal{C}^{\alpha}([0,T])$$
$$\mathbf{W}^{H} \longmapsto X^{H},$$
(28)

is continuous, for any $0 < \alpha < H$.

3.1 A weak continuity result with respect to the noise

Here we provide the main results of the paper. Given the existence and uniqueness result of a solution X^H to (26) stated by Theorem 22, a natural question that can be addressed is about the continuity of such a solution X^H with respect to the parameter H.

In order to use the rough paths techniques in a non-trivial way we restrict to the most interesting case of W^H , fractional Brownian motion of Hurst parameter $H \in (\frac{1}{3}, \frac{1}{2}]$. Indeed, when $H > \frac{1}{2}$ the regularity of the noise allows for a classical solution theory in the sense of Young integration. By Theorem 22 we have that a solution to (26) exists and it is unique and, moreover, the solution operator is continuous from \mathscr{C}^{α} to $\mathscr{C}^{\alpha}([0,T])$, for any $0 < \alpha < H$. When $H = \frac{1}{2}$, the solution $X^{\frac{1}{2}}$ to (26) becomes a Stratonovich solution of an SDE driven by a standard Brownian motion (sBm). This is a direct consequence of the well-known fact that when we lift a sBm $W^{\frac{1}{2}}$ to a geometric rough path, one obtains the Stratonovich integral.

We first consider some boundedness results upon the covariance K^H given by (23) with respect to the *p*-variation and the controlled *p*-variation. We see how their behavior may be completely different, as first point out in [3]. In the following, in order to prove the uniformity of the involved estimates, we always make explicit the dependence upon the parameter H.

Proposition 23. The covariance K^H of a a fractional Brownian motion of parameter $H \in (0, \frac{1}{2}]$, given by (23), has bounded $\frac{1}{2H}$ -variation $V_{\frac{1}{2H}}(K^H, [0, T]^2)$, which, moreover, for any s < t satisfies

$$V_{\frac{1}{2H}}(K^{H}, [s, t]^{2}) \le 3|t - s|^{2H}.$$
(29)

Moreover, one has that the controlled $\frac{1}{2H}$ -variation is infinite, that is, for any $R \subset [0,T]^2$

$$\left|K^{H}\right|_{\frac{1}{2H} \cdot var, R} = \infty.$$

$$(30)$$

Proof. Let us prove the inequality (29). For the proof of the unboundedness of the controlled $1\backslash 2H$ -variation, we refer to ([3], Example 2).

Without loss of generality, we consider T = 1. For any $u_1, u_2 \in [0, 1]$, with $u_1 \leq u_2$, we use the symbol W_{u_1,u_2}^H for the increment $W_{u_2}^H - W_{u_1}^H$. Furthermore we put $p = 1/2H \geq 1$.

Fixing the interval $[s,t] \subset [0,1]$, let us take two partitions $\pi_1 = \{t_i\}_i, \pi_2 = \{t'_j\}_j \in \mathcal{D}([s,t])$, with $n_j = card(\pi_j), j = 1, 2$. For any fixed $\{t_i, t_{i+1}\} \in \pi_1$, we consider the function $f^i := \mathbb{E}\left[W^H_{t_i,t_{i+1}}, W^H_{\cdot}\right]$. Since the 1D *p*-variation $|f^i|_{p-var,[t'_j,t'_{j+1}]}^p$ is a control, by (16) we obtain

$$f^{i}(t'_{j+1}) - f^{i}(t'_{j}) \leq \left[\left| f^{i} \right|_{p-var,[t'_{j},t'_{j+1}]}^{p} \right]^{\frac{1}{p}},$$

and, by super additivity of the control,

$$\sum_{j=1}^{n_2} \left| f^i(t'_{j+1}) - f^i(t'_j) \right|^p \leq \sum_{j=1}^{n_2} \left| f^i \right|^p_{p-var,[t'_j,t'_{j+1}]} \leq \left| f^i \right|^p_{p-var,[s,t]}.$$
(31)

By considering the subpartition of [s, t] given by $[s, t_i], [t_i, t_{i+1}], [t_{i+1}, t] \subset [s, t]$, since when $p \ge 1$ it holds that $(a + b + c)^p \le 3^{p-1}(a^p + b^p + c^p)$, we have

$$\left|f^{i}\right|_{p-var,[s,t]}^{p} \leq 3^{p-1} \left(\left|f^{i}\right|_{p-var,[s,t_{i}]}^{p} + \left|f^{i}\right|_{p-var,[t_{i},t_{i+1}]}^{p} + \left|f^{i}\right|_{p-var,[t_{i+1},t]}^{p}\right).$$
(32)

Now we need to estimate the three *p*-variations on the right hand side. First, we observe that the first and third terms are the *p*-variations of the covariance of the increments of disjoint time increments, despite the second one.

For the estimation of the latter, we notice that, whenever $[u, v] \subset [t_i, t_{i+1}]$, with $0 \le s \le u \le v \le t \le 1$, we obtain

$$\begin{split} \left| \mathbb{E} \left[W_{t_{i},t_{i+1}}^{H} W_{u,v}^{H} \right] \right| &= \left| \mathbb{E} \left[(W_{t_{i+1}}^{H} - W_{t_{i}}^{H})(W_{v}^{H} - W_{u}^{H}) \right] \right| \\ &= \left| \mathbb{E} \left[(W_{t_{i+1}}^{H} - W_{v}^{H} + W_{v}^{H} - W_{u}^{H} + W_{u}^{H} - W_{t_{i}}^{H})(W_{v}^{H} - W_{u}^{H}) \right] \right| \\ &= \left| \mathbb{E} \left[(W_{t_{i+1}}^{H} - W_{v}^{H})(W_{v}^{H} - W_{u}^{H}) \right] + \mathbb{E} \left[(W_{v}^{H} - W_{u}^{H})^{2} \right] \right. \\ &+ \mathbb{E} \left[(W_{u}^{H} - W_{t_{i}}^{H})(W_{v}^{H} - W_{u}^{H}) \right] \right| \\ &= \frac{1}{2} \left| |t_{i+1} - u|^{2H} - |t_{i+1} - v|^{2H} + |v - t_{i}|^{2H} - |u - t_{i}|^{2H} \right| \\ &\leq |u - v|^{2H} = |u - v|^{\frac{1}{p}}. \end{split}$$

The last inequality is due to the fact that, since $0 < 2H \le 1$, if $h_1 \le h_2 \le h_3$, $|h_3 - h_1|^{2H} = |h_3 - h_2 + h_2 - h_1|^{2H} \le |h_3 - h_2|^{2H} + |h_2 - h_1|^{2H}$, i.e. $|h_3 - h_1|^{2H} - |h_3 - h_2|^{2H} \le |h_2 - h_1|^{2H}$. As a consequence by definition of p-variation

$$\left|f^{i}\right|_{p-var,[t_{i},t_{i+1}]}^{p} = \sup_{\{u_{j}\}_{j} \in \mathcal{D}\left([t_{i},t_{i+1}]\right)} \sum_{j} \left|\mathbb{E}\left[W_{t_{i},t_{i+1}}^{H}W_{u_{j},u_{j+1}}^{H}\right]\right|^{p} \le |t_{i+1} - t_{i}|.$$
(33)

From (24) it is clear that in the case $H \leq \frac{1}{2}$ the disjoint increments of the fractional Brownian motion have negative correlations. This implies

$$\begin{split} \left| \mathbb{E} \left[W_{t_i,t_{i+1}}^H W_{\cdot}^H \right] \right|_{p-var;[s,t_i]}^p &\leq \sup_{\{u_j\}_j \in \mathcal{D}([s,t_i])} \left| \mathbb{E} \left[\sum_j W_{t_i,t_{i+1}}^H W_{u_j,u_{j+1}}^H \right] \right|^p \\ &= \left| \mathbb{E} \left[W_{t_i,t_{i+1}}^H W_{s,t_i}^H \right] \right|^p. \end{split}$$

Again by (24)

$$\mathbb{E}\left[W_{t_{i},t_{i+1}}^{H}W_{s,t_{i}}^{H}\right] = \frac{1}{2}\left(|t_{i+1}-s|^{2H}-|t_{i+1}-t_{i}|^{2H}-|t_{i}-s|^{2H}\right) \\
\leq \frac{1}{2}\left(|t_{i+1}-s|^{2H}-|t_{i}-s|^{2H}\right) \leq \frac{1}{2}|t_{i+1}-t_{i}|^{2H} \\
\leq |t_{i+1}-t_{i}|^{2H} = |t_{i+1}-t_{i}|^{\frac{1}{p}}.$$

As a consequence

$$\left|f^{i}\right|_{p-var,[s,t_{i}]}^{p} = \sup_{\{u_{j}\}_{j}\in\mathcal{D}([s,t_{i}])}\sum_{j}\left|\mathbb{E}\left[W_{t_{i},t_{i+1}}^{H}W_{u_{j},u_{j+1}}^{H}\right]\right|^{p} \le |t_{i+1}-t_{i}|.$$
(34)

In the same way, one con prove that

$$\left|f^{i}\right|_{p-var,[t_{i+1},t]}^{p} = \sup_{\{u_{j}\}_{j} \in \mathcal{D}\left([t_{i+1},t]\right)} \sum_{j} \left|\mathbb{E}\left[W_{t_{i},t_{i+1}}^{H}W_{u_{j},u_{j+1}}^{H}\right]\right|^{p} \le |t_{i+1} - t_{i}|.$$
(35)

Finally, by (13), (31) and (32), one obtains

$$\left(V_p(K^H, [s, t]^2) \right)^p = \sup_{\{t_i, t'_j\}_{i,j} \in \mathcal{D}^2(R)} \sum_{i,j} \left| \mathbb{E} \left[W^H_{t_i, t_{i+1}} W^H_{t_j, t_{j+1}} \right] \right|^p \\ \leq \sup_i \sum_i \left| f^i \right|_{p-var, [s, t]}^p \\ \leq 3^{p-1} \sup_{\{t_i\} \in \mathcal{D}([s, t])} \sum_i \left(3 \left| t_{i+1} - t_i \right| \right) = 3^p |t - s|,$$

i.e.

$$V_p\left(K^H, [s,t]^2\right) \leq 3|t-s|^{\frac{1}{p}} = 3|t-s|^{2H}.$$

	-	-	

The following theorem states the main result of the present work, that is the continuity of the solution of the equation (26) with respect to the Hurst parameter in the case $H \in (1/3, 1/2]$. **Theorem 24.** Let $\{H_n\}_{n \in \mathbb{N}} \in (1/3, 1/2]^{\mathbb{N}}$ be a sequence of Hurst parameters and let $H_{\infty} \in (1/3, 1/2]$. Let $\{W^{H_n}\}_{n \in H}$ be a family of fractional Brownian motions, each of them independent upon a random variable $X_0 \in L^2(\Omega)$. For any $n \in \mathbb{N}$, let us denote by $X_n^{X_0}$ the solution to the equation (26) with $H = H_n$, for $t \in [0, 1]$ and with initial condition X_0 . Suppose that $\mu, \sigma \in C_b^3(\mathbb{R})$. Then, the sequence $\{X_n^{X_0}\}_{n \in \mathbb{N}}$ converges to $X_{\infty}^{X_0}$ in distribution in the space $C^{\frac{1}{3}}([0, 1])$.

Proof. Given the continuity of the solution map stated in Theorem 22, we have only to show that

$$\mathbf{W}^{H_n} = (W_{\tau}^{H_n}, \mathbb{W}_{s,t}^{H_n}) \xrightarrow{n \to \infty} \mathbf{W}^{H_\infty} = (W_{\tau}^{H_\infty}, \mathbb{W}_{s,t}^{H_\infty}),$$
(36)

in $\mathscr{C}^{\frac{1}{3}}([0,1]).$

The first step is to prove the tightness of $\{\mathbf{W}^{H_n}\}_n$. The thesis follows by the Kolmogorov-Lamberti criterion (see [2], Corollary A.11), if we establish that there exist constants M > 0, q > r > 1 such that $\frac{1}{r} - \frac{1}{q} > \frac{1}{3}$ and, moreover, that the following estimate holds

$$\sup_{n\in\mathbb{N}} \mathbb{E}\left[d(\mathbf{W}_t^{H_n}, \mathbf{W}_s^{H_n})^q\right]^{\frac{1}{q}} \le M \left|t-1\right|^{1/r}.$$
(37)

Let us denote for any $n \in \mathbb{N}$ $p_n = \frac{1}{2H_n} \in [1, 3/2)$. Using the hypothesis $H_n \to H_\infty > \frac{1}{3}$, there exist a $\delta > 0$ and an $n_0(\delta) \in \mathbb{N}$ such that $H_n > \frac{1}{3} + \delta$ for any $n > n(\delta)$. Defining the following constants

$$\rho := \sup_{n \ge n_0(\delta)} p_n < \frac{3}{2}, \quad \varepsilon_n = \rho + \epsilon - p_n > 0, \tag{38}$$

where $0 < \varepsilon < \frac{3}{2} - \frac{3}{2(1+3\delta)}$ is some fixed real number, for any $n \in \mathbb{N}$ and any $R \in [0,T]^2$, by (18) it holds that

$$\left| K^{H_n} \right|_{\rho + \varepsilon \text{-var}, R} = \left| K^{H_n} \right|_{p_n + \varepsilon_n \text{-var}, R} \le C \left(p_n, \varepsilon_n \right) V_{p_n}(K^{H_n}, R) \le \overline{C} V_{p_n}(K^{H_n}, R) < \infty.$$
(39)

where $\bar{C} = \sup_{n > n_0(\delta)} C(p_n, \varepsilon_n)$, which is finite since $p_n < \rho$, $\epsilon_n > \epsilon$ and by Remark 18 $C(p, \tilde{\epsilon})$, is a continuous function for $p < \rho$ and $\tilde{\epsilon} > \epsilon > 0$. Furthermore the boundedness of the p_n -variation $V_{p_n}(K^{H_n}, R)$ is guaranteed by Proposition 23.

For any $n > n(\delta)$ the 2D control

$$\omega_{H_n} := |K^{H_n}|_{\rho+\varepsilon\text{-var},R}^{\rho+\varepsilon}.$$

is an Hölder dominated control, uniformly in $n > n(\delta)$. Indeed by (39)

$$\omega_{H_n}([s,t]^2) \leq \overline{C}^{\rho+\varepsilon} V_{p_n} \left(K^{H_n}, [s,t]^2 \right)^{\rho+\varepsilon} \\
\leq \left(3\overline{C} \right)^{\rho+\varepsilon} |t-s|^{\frac{\rho+\varepsilon}{p_n}} \\
\leq C|t-s|.$$
(40)

The last two inequalities are due to Proposition 23, to the fact that $(\rho + \varepsilon) \setminus p_n > 1$ together with the assumption $|t - s| \leq 1$.

In particular we obtain that

$$\omega_{H_n}([0,T]^2) \le M_1. \tag{41}$$

Moreover, since for any $n > n(\delta)$, $|K^{H_n}|_{(\rho+\epsilon)\text{-var},R} \leq \omega_{H_n}(R)$, condition (21) in Theorem 20 is satisfied and so by (22) and (40) we obtain that there exists a constant $\widetilde{C} = \widetilde{C}(\rho+\epsilon)$ such that for every $q \in [1,\infty)$ and for every $s, t \in [0,1]$

$$\sup_{n\in\mathbb{N}} \mathbb{E}\left[d(\mathbf{W}_{t}^{H_{n}}, \mathbf{W}_{s}^{H_{n}})^{q}\right]^{\frac{1}{q}} \leq \sup_{n\in\mathbb{N}}\left[\widetilde{C}\sqrt{q} \ \omega_{H_{n}}\left([s,t]^{2}\right)^{\frac{1}{(\rho+\epsilon)}}\right]$$

$$\leq \widetilde{C}\sqrt{q} \sup_{n\in\mathbb{N}}|t-s|^{\frac{1}{(\rho+\epsilon)}}.$$
(42)

Since there exists ϵ such that $\rho + \epsilon \in (1, 3/2)$, then $\frac{1}{r} = \frac{1}{(\rho + \epsilon)} \in (1/3, 1/2)$. As a consequence, by choosing q >> 1 such that $\frac{1}{r} - \frac{1}{q} > \frac{1}{3}$ and defining $M = \widetilde{C}\sqrt{q}$, from (42) we finally get the inequality (37). Kolmogorov-Lamperti tightness criterion implies that the sequence \mathbf{W}^{H_n} is tight in $\mathscr{C}^{\frac{1}{3}}$, and thus it possesses a subsequence converging to some limit \mathbf{Y} .

The last step is in the identification of the limit \mathbf{Y} as $\mathbf{W}^{H_{\infty}}$.

Once the tightness has been achieved, we only need to show that the finitely dimensional distributions of \mathbf{W}^{H_n} converge to the ones of \mathbf{W}^{H_∞} , i.e. for any $m, \ell \in \mathbb{N}$, and for any increasing times $\{t_j\}_{j=1}^m, \{s_i\}_{i=1}^\ell, \{k_i\}_{i=1}^\ell$, such that, for any $i = 1, \ldots, \ell, s_i \leq k_i$, we have that $(W_{t_1}^{H_n}, \ldots, W_{t_m}^{H_n}, \mathbb{W}_{s_1,k_1}^{H_n}, \ldots, \mathbb{W}_{s_\ell,k_\ell}^{H_n})$ converges in probability to $(W_{t_1}^{H_\infty}, \ldots, W_{t_m}^{H_\infty}, \mathbb{W}_{s_1,k_1}^{H_\infty}, \ldots, \mathbb{W}_{s_\ell,k_\ell}^{H_\infty})$.

First of all we note that since $W^n = \{W^{H_n}\}_n$ is a Gaussian process for any $n \in \mathbb{N}$ and the covariance of W^n converges to the covariance of W^{∞} , the finitely dimensional distributions $(W_{t_1}^{H_n}, \ldots, W_{t_m}^{H_n})$ converge in probability to $(W_{t_1}^{H_{\infty}}, \ldots, W_{t_m}^{H_{\infty}})$. As far as concern the convergence of $(\mathbb{W}_{s_1,k_1}^{H_n}, \ldots, \mathbb{W}_{s_\ell,k_\ell}^{H_n})$ we prove the one dimensional case, i.e. that, given $s \leq k \in [0, 1]$, the random variable $\mathbb{W}_{s,k}^{H_n}$ converges to $\mathbb{W}_{s,k}^{H_{\infty}}$. The general case is a straightforward generalization.

Let D be a partition of [0,1] having width δ_D and let $S_3(W^{H_n}, D)_{s,k}$ be the piecewise linear approximations of $\mathbb{W}_{s,k}^{H_n}$. By (41) we obtain that $\sup_{n \in \mathbb{N}} \omega_{H_n}([0,1]^2) \leq M_1 < +\infty$; hence, by Theorem 15.42 in [2], fixed an arbitrary $p \in (2(\rho + \epsilon), 4)$, for any $\eta \in (0, \frac{1}{2(\rho + \epsilon)} - \frac{1}{p})$, there exists a constant $C_1(\rho + \epsilon, p, M_1, \eta)$ such that

$$\mathbb{E}\left[\left\|S_3(W^{H_n}, D)_{s,k} - \mathbb{W}_{s,k}^{H_n}\right\|_{L^q}\right] \le C_1 C \sqrt{q} \,\delta_D^{\eta/3}.$$

Let F be a C^1 bounded function which is globally Lipschitz with Lipschitz constant L, then we have

$$\left| \mathbb{E} \left[F(\mathbb{W}_{s,k}^{H_n}) \right] - \mathbb{E} \left[F(S_3(W^{H_n}, D)_{s,k}) \right] \right| \le KC_1 C \, \delta_D^{\eta/3}. \tag{43}$$

From (43) we get

$$\begin{split} \limsup_{n \to +\infty} \left| \mathbb{E} \left[F(\mathbb{W}_{s,k}^{H_n}) \right] - \mathbb{E} \left[F(\mathbb{W}_{s,k}^{H_\infty}) \right] \right| &\leq \limsup_{n \to +\infty} \left| \mathbb{E} \left[F(\mathbb{W}_{s,k}^{H_n}) \right] - \mathbb{E} \left[F(S_3(W^{H_n}, D)_{s,k}) \right] \right| \\ &+ \limsup_{n \to +\infty} \left| \mathbb{E} \left[F(\mathbb{W}_{s,k}^{H_\infty}) \right] - \mathbb{E} \left[F(S_3(W^{H_\infty}, D)_{s,k}) \right] \right| \\ &+ \limsup_{n \to +\infty} \left| \mathbb{E} \left[F(S_3(W^{H_n}, D)_{s,k}) \right] \\ &- \mathbb{E} \left[F(S_3(W^{H_\infty}, D)_{s,k}) \right] \right| \\ &\leq 2LC_1 C \, \delta_D^{\eta/3}. \end{split}$$

$$(44)$$

The last step derived by the facts that W^{H_n} converges in probability to $W^{H_{\infty}}$ and that $S_3(W^{H_n}, D)_{s,k}$ and $S_3(W^{H_{\infty}}, D)_{s,k}$ are polynomial approximations of W^{H_n} and $W^{H_{\infty}}$, respectively. Since δ_D can be chosen in an arbitrary way the thesis is proven.

Remark 25. Note that the restriction to $t \in [0, 1]$ is only a technical simplification, since one can always reformulate an equation on [0, T] as an equation on [0, 1] via a reparametrization (see [2]).

Remark 26. When $H > \frac{1}{2}$, the result can be proven following the same steps, but without the need of rough paths theory. Indeed, the solution map $W^H \to X^H$ is continuous, since we are in the framework of Young integration theory. This means that whenever $H > \frac{1}{2}$ it is sufficient to show that, for some $\alpha > \frac{1}{2}$, when $H \to H_{\infty}$ it holds that $W^H \to W^{H_{\infty}}$ in $\mathcal{C}^{\alpha}([0,T])$. Latter fact can be shown again via Kolmogorov-Lamperti criterion (Corollary A.11, [2]).

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