A Terminating Sequent Calculus for Intuitionistic Strong Löb Logic with the Subformula Property

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Abstract. Intuitionistic Strong Löb logic iSL is an intuitionistic modal logic with a provability interpretation. We introduce $\mathsf{GbuSL}_\Box$, a terminating sequent calculus for iSL with the subformula property. $\mathsf{GbuSL}_\Box$ modifies the sequent calculus $\mathsf{G3iSL}_\Box$ for iSL based on G3i, by annotating the sequents to distinguish rule applications into an unblocked phase, where any rule can be backward applied, and a blocked phase where only right rules can be used. We prove that, if proof search for a sequent σ in $\mathsf{GbuSL}_\Box$ fails, then a Kripke countermodel for σ can be constructed.

1 Introduction

Intuitionistic Strong Löb Logic iSL is the intuitionistic modal logic obtained by adding both the Gödel-Löb axiom $\Box(\Box\varphi \to \varphi) \to \Box\varphi$ and the completeness axiom $\varphi \to \Box\varphi$ to $\mathbf{K}_\Box$, the $\Box$-fragment of Intuitionistic Modal Logic. Equivalently, iSL is the extension of $\mathbf{K}_\Box$ with the Strong Löb axiom $(\Box\varphi \to \varphi) \to \varphi$. Logic iSL has prominent relevance in the study of provability of Heyting Arithmetic HA. It is well known that the Gödel-Löb Logic, obtained by extending classical modal logic with Gödel-Löb axiom, is the provability logic of Peano Arithmetic [11]. However, it is an open problem what the provability logic of HA should be; a solution to this problem is claimed in a preprint paper [8]. In [16], it is shown that iSL is the provability logic of an extension of HA with respect to slow provability. Moreover, iSL plays an important role in the Σ_1-provability logic of HA [1]. We stress that iSL, as well as other related logics (such as the logics iGL, mHC and KM investigated in [13,14]), only treats the $\Box$-modality, connected with the provability interpretation; it is not clear what interpretation $\Diamond$ should have and which laws it should obey.

In this paper we investigate proof search for iSL. Recently, in [13,15] some sequent calculi for iSL have been introduced, obtained by enhancing the sequent calculus G3i [12] for IPL (Intuitionistic Propositional Logic) with the rule $R\Box$ to treat right $\Box$ (actually, four variants of such a rule are proposed). We start

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by presenting the sequent calculus $\mathsf{G3iSL}^+_\Box$ (see Fig. 1), a polished version of the calculus $\mathsf{G3iSL}_\Box$ [13,15] where rule $R\Box$ avoids some redundant duplications of formulas. The calculus $\mathsf{G3iSL}^+_\Box$ has the *subformula property*, namely: every formula occurring in a $\mathsf{G3iSL}^+_\Box$-tree is a subformula of a formula in the root sequent. However, $\mathsf{G3iSL}^+_\Box$ is not well-suited for proof search. This is mainly due to the rule $L\rightarrow$ for left implication, which has applications where the sequent $\alpha \rightarrow \beta, \Gamma \Rightarrow \alpha$ is both the conclusion and the left premise, and this yields loops in backward proof search. We are interested in a sequent calculus $\mathcal{C}$ where backward proof search always terminates, that is: given a sequent of $\mathcal{C}$ and repeatedly applying the rules of $\mathcal{C}$ upwards, proof search eventually halts, no matter which strategy is used. A calculus of this kind is called *(strongly) terminating* and can be characterized as follows: there exists a well-founded relation $\prec$ on sequents of $\mathcal{C}$ such that, for every application ρ of a rule of $\mathcal{C}$, if the sequent σ is the conclusion of ρ and σ' is any of the premises, then $\sigma' \prec \sigma$. Clearly, any calculus containing rule $L\rightarrow$ is not terminating; in this case, to get a terminating proof search procedure for $\mathcal{C}$ some machinery must be introduced (for instance, loop-checking). A calculus $\mathcal{C}$ is *weakly terminating* if it admits a terminating proof search strategy. The calculus $\mathsf{G3i}$ is weakly terminating. A well-known terminating calculus for IPL is $\mathsf{G4i}$ [2]; this is obtained from $\mathsf{G3i}$ by replacing the looping rule $L\rightarrow$ with more specialized rules: basically, the left rule with main formula $\alpha \rightarrow \beta$ is defined according to the structure of α. The same approach is used in [13,15], where the $\mathsf{G4}$-variants of the $\mathsf{G3}$-calculi for iSL are introduced. The obtained calculi are weakly (but not strongly) terminating and the proof search procedure yields a countermodel in case of failure. This means that, if proof search for a sequent $\sigma = \Gamma \Rightarrow \delta$ fails, one gets a Kripke model for σ (as defined in [1,7]) certifying that δ is not an iSL-consequence of Γ. These results have been definitely improved in [10], where the $\mathsf{G4}$-style (strongly) terminating calculus $\mathsf{G4iSLt}$ for iSL is presented. Notably, the proofs of termination and completeness (via cut-admissibility) have been formalized in the Coq Proof Assistant.

So far, it seems that the only way to design a (weakly or strongly) terminating calculus for iSL is to throw rule $L\rightarrow$ away and to comply with $\mathsf{G4}$-style. As a side effect, the obtained calculi lack the subformula property. Now, an intriguing question is: is it possible to get a terminating variant of $\mathsf{G3iSL}^+_\Box$ still preserving the subformula property? To address this issue, we follow the approach discussed in [4,5], where (strongly) terminating variants of the intuitionistic calculus $\mathsf{G3i}$ are introduced: the crucial expedient is to decorate the sequents with one of the labels b (blocked) and u (unblocked). In backward proof search, if a sequent has label b, the (backward) application of left rules is blocked, so that only right rules can be applied. Accordingly, bottom-up proof search alternates between an unblocked phase, where both left and right rules can be applied, and a blocked phase, where the focus is on the right formula (the application of left rules is forbidden). We call the obtained calculus $\mathsf{GbuSL}_\Box$ (see Fig. 2). The subformula property for $\mathsf{GbuSL}_\Box$ can be easily checked; to ascertain that $\mathsf{GbuSL}_\Box$ is terminating, we introduce the well-founded relation $\prec_{\mathrm{bu}}$ on labelled sequents (Definition 2). We show that a $\mathsf{GbuSL}_\Box$-derivation can be translated into a $\mathsf{G3iSL}^+_\Box$-derivation; as a corollary, the calculus $\mathsf{G3iSL}^+_\Box$ is weakly terminating. To prove the completeness of $\mathsf{GbuSL}_\Box$, we show that, if proof search for a sequent σ with label u fails, then a countermodel for σ can be built. An implementation of the proof search procedure, based on the Java framework JTabWb [6], is available at https://github.com/ferram/jtabwb_provers/tree/master/isl_gbuSL; the repository also contains the online appendix we refer to henceforth.

2 The Logic iSL

Formulas, denoted by lowercase Greek letters, are built from an enumerable set of propositional variables $\mathcal{V}$, the constant $\bot$ and the connectives $\wedge$, $\vee$, $\to$ and $\Box$; $\neg\alpha$ is an abbreviation for $\alpha \to \bot$. Let α be a formula and Γ a multiset of formulas. By $\Box\Gamma$ we denote the multiset $\{\Box\alpha \mid \alpha \in \Gamma\}$. By $\mathrm{Sf}(\alpha)$ we denote the set of the subformulas of α, including α itself; $\mathrm{Sf}(\Gamma)$ is the union of the sets $\mathrm{Sf}(\alpha)$, for every α in Γ. The size of α, denoted by $|\alpha|$, is the number of symbols in α; the size of Γ, denoted by $|\Gamma|$, is the sum of the sizes of formulas α in Γ, taking into account their multiplicity. A relation R is *well-founded* iff there is no infinite descending chain $\dots Rx_2Rx_1Rx_0$; R is *converse well-founded* if the converse relation R^{-1} is well-founded.

An iSL-*(Kripke) model* $\mathcal{K}$ is a tuple $\langle W, \leq, R, r, V\rangle$ where W is a non-empty set (worlds), $\leq$ (the intuitionistic relation) and R (the modal relation) are subsets of $W \times W$, r (the root) is the minimum element of W w.r.t. $\leq$, V (the valuation function) is a map from W to $2^{\mathcal{V}}$ such that:

- (M1) $\leq$ is reflexive and transitive;
- (M2) R is transitive and converse well-founded;
- (M3) R is a subset of $\leq$;
- (M4) if $w_0 \leq w_1$ and w_1Rw_2, then w_0Rw_2;
- (M5) V is persistent, namely: $w_0 \leq w_1$ implies $V(w_0) \subseteq V(w_1)$.

Given an iSL-model $\mathcal{K}$, the forcing relation $\Vdash$ between worlds of $\mathcal{K}$ and formulas is defined as follows:

$\mathcal{K}, w \Vdash p$ iff $p \in V(w)$, $\forall p \in \mathcal{V}$ $\qquad$ $\mathcal{K}, w \nVdash \bot$

$\mathcal{K}, w \Vdash \alpha \wedge \beta$ iff $\mathcal{K}, w \Vdash \alpha$ and $\mathcal{K}, w \Vdash \beta$ $\qquad$ $\mathcal{K}, w \Vdash \alpha \vee \beta$ iff $\mathcal{K}, w \Vdash \alpha$ or $\mathcal{K}, w \Vdash \beta$

$\mathcal{K}, w \Vdash \alpha \to \beta$ iff $\forall w' \geq w$, if $\mathcal{K}, w' \Vdash \alpha$ then $\mathcal{K}, w' \Vdash \beta$

$\mathcal{K}, w \Vdash \Box\alpha$ iff $\forall w' \in W$, if wRw' then $\mathcal{K}, w' \Vdash \alpha$.

We write $w \Vdash \varphi$ instead of $\mathcal{K}, w \Vdash \varphi$ when the model $\mathcal{K}$ at hand is clear from the context. One can easily prove that forcing is persistent, i.e.: if $w \Vdash \varphi$ and $w \leq w'$, then $w' \Vdash \varphi$. Let Γ be a (multi)set of formulas. By $w \Vdash \Gamma$ we mean that $w \Vdash \varphi$, for every φ in Γ. The iSL-consequence relation $\models_{\mathsf{iSL}}$ is defined as follows:

$$\Gamma \models_{\mathsf{iSL}} \varphi \quad \text{iff} \quad \forall \mathcal{K}\, \forall w \ (\mathcal{K}, w \Vdash \Gamma \implies \mathcal{K}, w \Vdash \varphi)\,.$$

$$\frac{}{p, \Gamma \Rightarrow p}\,\mathrm{Id} \qquad \frac{}{\bot, \Gamma \Rightarrow \delta}\,L\bot \qquad \frac{\alpha, \beta, \Gamma \Rightarrow \delta}{\alpha \wedge \beta, \Gamma \Rightarrow \delta}\,L\wedge$$

$$\frac{\Gamma \Rightarrow \alpha \qquad \Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \wedge \beta}\,R\wedge \qquad \frac{\alpha, \Gamma \Rightarrow \delta \qquad \beta, \Gamma \Rightarrow \delta}{\alpha \vee \beta, \Gamma \Rightarrow \delta}\,L\vee \qquad \frac{\Gamma \Rightarrow \alpha_k}{\Gamma \Rightarrow \alpha_0 \vee \alpha_1}\,R\vee_k$$

$$\frac{\alpha \to \beta, \Gamma \Rightarrow \alpha \qquad \beta, \Gamma \Rightarrow \delta}{\alpha \to \beta, \Gamma \Rightarrow \delta}\,L\to \qquad \frac{\alpha, \Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \to \beta}\,R\to \qquad \frac{\Box\alpha, \Gamma, \Delta \Rightarrow \alpha}{\Gamma, \Box\Delta \Rightarrow \Box\alpha}\,R\Box$$

Fig. 1. The calculus $\mathsf{G3iSL}^+_\Box$ ($p \in \mathcal{V}$, $k \in \{0, 1\}$).

The logic iSL is the set of formulas φ such that $\emptyset \models_{\mathsf{iSL}} \varphi$. Accordingly, if $\varphi \notin \mathsf{iSL}$, there exists an iSL-model $\mathcal{K}$ such that $r \nVdash \varphi$, with r the root of $\mathcal{K}$; we call $\mathcal{K}$ a *countermodel* for φ. We stress that iSL satisfies the finite model property [16]; thus, we can assume that iSL-models are finite and condition (M2) can be rephrased as "R is transitive and irreflexive".

Example 1. Figure 5 defines a formula ψ and a countermodel $\mathcal{K}$ for ψ. The worlds of $\mathcal{K}$ are w_2 (the root), w_7, w_{12}, w_{15}, w_{19}, w_{24}. The relations $\leq$ and R of $\mathcal{K}$ can be inferred by the displayed arrows, as accounted for in the figure. For instance $w_2 \leq w_{19}$, since there is a path from w_2 and w_{19} (actually, a unique path); $w_2 \leq w_{15}$ and $w_2 R w_{15}$, since the path from w_2 and w_{15} ends with the solid arrow $\to$. However, it is not the case that $w_2 R w_{19}$, since the path from w_2 to w_{19} ends with the dashed arrow $\dashrightarrow$. In each world w_k, the first line displays the value of $V(w_k)$, the remaining lines report (separated by commas) some of the formulas forced and not forced in w_k. Since $w_2 \nVdash \psi$, $\mathcal{K}$ is a countermodel for ψ.

We remark that, if we replace a dashed arrow with a solid arrow, or vice-versa, we get $w_2 \Vdash \psi$, thus $\mathcal{K}$ is no longer a countermodel for ψ. For instance, let us set $w_2 \to w_7$. Then, $w_2 R w_7$ and, since $w_7 \nVdash s$, we get $w_2 \nVdash \Box s$, hence $w_2 \nVdash \alpha$. Since $w_7 \Vdash \gamma$ and $w_{12} \Vdash \beta$, it follows that $w_2 \Vdash \psi$. Similarly, assume $w_{15} \to w_{19}$, which implies $w_{15} R w_{19}$. Then $w_{15} \nVdash \Box\neg p$ (indeed, $w_{15} R w_{19}$ and $w_{19} \nVdash \neg p$) and, by the fact that $w_2 R w_{15}$, we get $w_2 \nVdash \Box\Box\neg p$, thus $w_2 \nVdash \alpha$; as in the previous case, we conclude $w_2 \Vdash \psi$. Let us set $w_2 \to w_{12}$. Since $w_{12} \nVdash \Box\neg p$ and $w_2 R w_{12}$, we get $w_2 \nVdash \Box\Box\neg p$; this implies that $w_2 \Vdash \psi$. ◊

In the paper we introduce some sequent calculi for iSL. For the notation and the terminology about a generic calculus $\mathcal{C}$ (e.g., the notions of $\mathcal{C}$-tree, $\mathcal{C}$-derivation, branch, depth of a $\mathcal{C}$-tree), we refer to [12]. By $\vdash_{\mathcal{C}} \sigma$ we mean that the sequent σ is derivable in the calculus $\mathcal{C}$. Let $\mathcal{C}$ be a calculus and let $\prec$ be a relation on the sequents of $\mathcal{C}$. A rule $\mathcal{R}$ of $\mathcal{C}$ is *decreasing w.r.t.* $\prec$ iff, for every application ρ of $\mathcal{R}$, if σ is the conclusion of ρ and σ' is any of the premises of ρ, then $\sigma' \prec \sigma$. A calculus $\mathcal{C}$ is *terminating* iff there exists a well-founded relation $\prec$ such that every rule of $\mathcal{C}$ is decreasing w.r.t. $\prec$.

The calculus $\mathsf{G3iSL}^+_\Box$ in Fig. 1 is obtained by adding the rule $R\Box$ to the intuitionistic calculus $\mathsf{G3i}$ [12]. Sequents of $\mathsf{G3iSL}^+_\Box$ have the form $\Gamma \Rightarrow \delta$, where Γ is a finite multiset of formulas and δ is a formula. The calculus is very close to the variant $\mathsf{G3iSL}^a_\Box$ of the calculus $\mathsf{G3iSL}_\Box$ for iSL presented in [13,15]. The notable difference is in the presentation of rule $R\Box$: given the conclusion $\Gamma, \Box\Delta \Rightarrow \Box\alpha$, in $\mathsf{G3iSL}^a_\Box$ the premise is $\Box\alpha, \Gamma, \Box\Delta, \Delta \Rightarrow \alpha$, in $\mathsf{G3iSL}^+_\Box$ the redundant multiset $\Box\Delta$ is omitted. The calculus $\mathsf{G3iSL}^+_\Box$ is sound and complete for iSL:

Theorem 1. $\vdash_{\mathsf{G3iSL}^+_\Box} \Gamma \Rightarrow \delta$ *iff* $\Gamma \models_{\mathsf{iSL}} \delta$.

The soundness of $\mathsf{G3iSL}^+_\Box$ (the only-if side of Theorem 1) immediately follows from the soundness of $\mathsf{G3iSL}^a_\Box$ (for a semantic proof, see the online appendix); the completeness is discussed in Sect. 4.[1] It is easy to check that $\mathsf{G3iSL}^+_\Box$ enjoys the subformula property; however, as discussed in the Introduction, $\mathsf{G3iSL}^+_\Box$ is not terminating, due to the presence of rule $L\rightarrow$.

3 The Sequent Calculus $\mathsf{GbuSL}_\Box$

The sequent calculus $\mathsf{GbuSL}_\Box$ is obtained from $\mathsf{G3iSL}^+_\Box$ by refining the sequent definition: we decorate sequents by a label l, where l can be b (blocked) or u (unblocked). Thus, a $\mathsf{GbuSL}_\Box$-sequent σ has the form $\Gamma \overset{l}{\Rightarrow} \delta$, with $l \in \{\mathrm{b}, \mathrm{u}\}$; Γ and δ are referred to as the lhs and the rhs (left/right hand side) of σ respectively. We call l-sequent a sequent with label l; $\mathrm{Sf}(\Gamma \overset{l}{\Rightarrow} \delta)$ denotes the set $\mathrm{Sf}(\Gamma \cup \{\delta\})$. To define the calculus, we introduce the following evaluation relation.

Definition 1 (Evaluation). *Let Γ be a multiset of formulas and φ a formula. We say that Γ* evaluates *φ, written $\Gamma \rhd \varphi$, iff φ matches the following BNF:*

$$\varphi \coloneqq \gamma \mid \varphi \wedge \varphi \mid \varphi \vee \alpha \mid \alpha \vee \varphi \mid \alpha \rightarrow \varphi \mid \Box\varphi \quad \text{with } \gamma \in \Gamma \text{ and } \alpha \text{ any formula.}$$

By $\Gamma \rhd \Delta$ we mean that $\Gamma \rhd \delta$, for every $\delta \in \Delta$. We state some properties of evaluation.

Lemma 1.

(i) If $\Gamma \rhd \varphi$ and $\Gamma \subseteq \Gamma'$, then $\Gamma' \rhd \varphi$.
(ii) If $\Gamma \cup \Delta \rhd \varphi$ and $\Gamma' \rhd \Delta$, then $\Gamma \cup \Gamma' \rhd \varphi$.
(iii) If $\Gamma \rhd \varphi$, then $\Gamma \cap \mathrm{Sf}(\varphi) \rhd \varphi$.
(iv) If $\Gamma \rhd \varphi$, then $\vdash_{\mathsf{G3iSL}^+_\Box} \Gamma \Rightarrow \varphi$.
(v) If $\Gamma \rhd \varphi$ and $\mathcal{K}, w \Vdash \Gamma$, then $\mathcal{K}, w \Vdash \varphi$.

Proof. All the assertions are proved by induction on the structure of φ.
(i). Let $\Gamma \rhd \varphi$ and $\Gamma \subseteq \Gamma'$; we prove $\Gamma' \rhd \varphi$. If $\varphi \in \Gamma$, then $\varphi \in \Gamma'$, hence $\Gamma' \rhd \varphi$. Let us assume $\varphi \notin \Gamma$. If $\varphi = \alpha \wedge \beta$, then $\Gamma \rhd \alpha$ and $\Gamma \rhd \beta$. By the induction hypothesis, we get $\Gamma' \rhd \alpha$ and $\Gamma' \rhd \beta$, hence $\Gamma' \rhd \alpha \wedge \beta$. The other cases are similar.
(ii). Let $\Gamma \cup \Delta \rhd \varphi$ and $\Gamma' \rhd \Delta$; we prove $\Gamma \cup \Gamma' \rhd \varphi$. Let us assume $\varphi \in \Gamma \cup \Delta$. If $\varphi \in \Gamma$, then $\Gamma \cup \Gamma' \rhd \varphi$. Otherwise, it holds that $\varphi \in \Delta$. Since $\Gamma' \rhd \Delta$, we

[1] We stress that the completeness of $\mathsf{G3iSL}^+_\Box$ is not a consequence of the one of $\mathsf{G3iSL}^a_\Box$, since rule $R\Box$ of $\mathsf{G3iSL}^+_\Box$ is a restriction of rule $R\Box$ of $\mathsf{G3iSL}^a_\Box$.

get $\Gamma' \triangleright \varphi$; by point (i), we conclude $\Gamma \cup \Gamma' \triangleright \varphi$. Let us assume $\varphi \notin \Gamma \cup \Delta$. If $\varphi = \alpha \wedge \beta$, then $\Gamma \cup \Delta \triangleright \alpha$ and $\Gamma \cup \Delta \triangleright \beta$. By the induction hypothesis we get $\Gamma \cup \Gamma' \triangleright \alpha$ and $\Gamma \cup \Gamma' \triangleright \beta$, hence $\Gamma \cup \Gamma' \triangleright \alpha \wedge \beta$. The other cases are similar.
(iii). Let $\Gamma \triangleright \varphi$: we prove $\Gamma \cap \mathrm{Sf}(\varphi) \triangleright \varphi$. If $\varphi \in \Gamma$, then $\varphi \in \Gamma \cap \mathrm{Sf}(\varphi)$, which implies $\Gamma \cap \mathrm{Sf}(\varphi) \triangleright \varphi$. Let $\varphi \notin \Gamma$. If $\varphi = \alpha \wedge \beta$, then $\Gamma \triangleright \alpha$ and $\Gamma \triangleright \beta$. By the induction hypothesis, we get $\Gamma \cap \mathrm{Sf}(\alpha) \triangleright \alpha$ and $\Gamma \cap \mathrm{Sf}(\beta) \triangleright \beta$. Since $\mathrm{Sf}(\alpha) \subseteq \mathrm{Sf}(\alpha \wedge \beta)$ and $\mathrm{Sf}(\beta) \subseteq \mathrm{Sf}(\alpha \wedge \beta)$, by point (i) we get $\Gamma \cap \mathrm{Sf}(\alpha \wedge \beta) \triangleright \alpha$ and $\Gamma \cap \mathrm{Sf}(\alpha \wedge \beta) \triangleright \beta$; we conclude $\Gamma \cap \mathrm{Sf}(\alpha \wedge \beta) \triangleright \alpha \wedge \beta$. The other cases are similar.
(iv). We prove the assertion by outlining an effective procedure to build a $\mathsf{G3iSL}^+_\Box$-derivation of the sequent $\Gamma \Rightarrow \varphi$. We start by showing that:

$$(*) \quad \vdash_{\mathsf{G3iSL}^+_\Box} \varphi, \Gamma \Rightarrow \varphi, \text{ for every formula } \varphi \text{ and every multiset of formulas } \Gamma.$$

We prove (*) by induction on the structure of φ. If $\varphi \in \mathcal{V} \cup \{\bot\}$, a $\mathsf{G3iSL}^+_\Box$-derivation of $\varphi, \Gamma \Rightarrow \varphi$ is obtained by applying rule Id or rule $L\bot$. Otherwise, a $\mathsf{G3iSL}^+_\Box$-derivation of $\varphi, \Gamma \Rightarrow \varphi$ can be built as follows, according to the form of φ, where the omitted $\mathsf{G3iSL}^+_\Box$-derivations are given by the induction hypothesis:

$$\dfrac{\dfrac{\vdots \atop \alpha, \beta, \Gamma \Rightarrow \alpha}{\alpha \wedge \beta, \Gamma \Rightarrow \alpha}\,L\wedge \qquad \dfrac{\vdots \atop \alpha, \beta, \Gamma \Rightarrow \beta}{\alpha \wedge \beta, \Gamma \Rightarrow \beta}\,L\wedge}{\alpha \wedge \beta, \Gamma \Rightarrow \alpha \wedge \beta}\,R\wedge \qquad \dfrac{\dfrac{\vdots \atop \alpha, \Gamma \Rightarrow \alpha}{\alpha, \Gamma \Rightarrow \alpha \vee \beta}\,R\vee_0 \qquad \dfrac{\vdots \atop \beta, \Gamma \Rightarrow \beta}{\beta, \Gamma \Rightarrow \alpha \vee \beta}\,R\vee_1}{\alpha \vee \beta, \Gamma \Rightarrow \alpha \vee \beta}\,L\vee$$

$$\dfrac{\dfrac{\vdots \atop \alpha, \alpha \to \beta, \Gamma \Rightarrow \alpha \qquad \vdots \atop \alpha, \beta, \Gamma \Rightarrow \beta}{\alpha, \alpha \to \beta, \Gamma \Rightarrow \beta}\,L\to}{\alpha \to \beta, \Gamma \Rightarrow \alpha \to \beta}\,R\to \qquad \dfrac{\vdots \atop \Box\alpha, \alpha, \Gamma \Rightarrow \alpha}{\Box\alpha, \Gamma \Rightarrow \Box\alpha}\,R\Box$$

Let $\Gamma \triangleright \varphi$; we show that $\Gamma \Rightarrow \varphi$ is provable in $\mathsf{G3iSL}^+_\Box$. If $\varphi \in \Gamma$, the assertion follows by (*). Let us assume $\varphi \notin \Gamma$. According to the shape of φ, a $\mathsf{G3iSL}^+_\Box$-derivation of $\Gamma \Rightarrow \varphi$ can be built as follows:

$$\dfrac{\vdots \atop \Gamma \Rightarrow \alpha \qquad \vdots \atop \Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \wedge \beta}\,R\wedge \qquad \dfrac{\vdots \atop \Gamma \Rightarrow \alpha_k}{\Gamma \Rightarrow \alpha_0 \vee \alpha_1}\,R\vee_k \qquad \dfrac{\vdots \atop \alpha, \Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \to \beta}\,R\to \qquad \dfrac{\vdots \atop \Box\alpha, \Gamma \Rightarrow \alpha}{\Gamma \Rightarrow \Box\alpha}\,R\Box$$

The omitted $\mathsf{G3iSL}^+_\Box$-derivations exist by the induction hypothesis; for instance, if $\varphi = \alpha \wedge \beta$, then $\Gamma \triangleright \alpha$ and $\Gamma \triangleright \beta$, hence both $\Gamma \Rightarrow \alpha$ and $\Gamma \Rightarrow \beta$ are provable in $\mathsf{G3iSL}^+_\Box$. In the cases $\varphi = \alpha \to \beta$ and $\varphi = \Box\alpha$, we also have to use point (i). For instance, let $\varphi = \alpha \to \beta$; then, $\Gamma \triangleright \beta$ and, by point (i), we get $\Gamma \cup \{\alpha\} \triangleright \beta$, hence the $\mathsf{G3iSL}^+_\Box$-derivation of $\alpha, \Gamma \Rightarrow \beta$ exists by the induction hypothesis.
(v). Let $\Gamma \triangleright \varphi$ and $w \Vdash \Gamma$ (in $\mathcal{K}$); we prove that $w \Vdash \varphi$. The case $\varphi \in \Gamma$ is trivial. Let $\varphi \notin \Gamma$. If $\varphi = \alpha \wedge \beta$, then $\Gamma \triangleright \alpha$ and $\Gamma \triangleright \beta$. By the induction hypothesis, we get $w \Vdash \alpha$ and $w \Vdash \beta$, hence $w \Vdash \alpha \wedge \beta$. The other cases are similar. ■

$$\dfrac{}{\Gamma \stackrel{l}{\Rightarrow} \alpha}\,\mathrm{Ax}^{\triangleright} \quad \text{if } \Gamma \triangleright \alpha \qquad\qquad \dfrac{}{\bot, \Gamma \stackrel{\mathrm{u}}{\Rightarrow} \delta}\,L\bot$$

$$\dfrac{\alpha, \beta, \Gamma \stackrel{\mathrm{u}}{\Rightarrow} \delta}{\alpha \wedge \beta, \Gamma \stackrel{\mathrm{u}}{\Rightarrow} \delta}\,L\wedge \qquad\qquad \dfrac{\Gamma \stackrel{l}{\Rightarrow} \alpha \qquad \Gamma \stackrel{l}{\Rightarrow} \beta}{\Gamma \stackrel{l}{\Rightarrow} \alpha \wedge \beta}\,R\wedge$$

$$\dfrac{\alpha, \Gamma \stackrel{\mathrm{u}}{\Rightarrow} \delta \qquad \beta, \Gamma \stackrel{\mathrm{u}}{\Rightarrow} \delta}{\alpha \vee \beta, \Gamma \stackrel{\mathrm{u}}{\Rightarrow} \delta}\,L\vee \qquad\qquad \dfrac{\Gamma \stackrel{\mathrm{b}}{\Rightarrow} \alpha_k}{\Gamma \stackrel{l}{\Rightarrow} \alpha_0 \vee \alpha_1}\,R\vee_k$$

$$\dfrac{\alpha \rightarrow \beta, \Gamma \stackrel{\mathrm{b}}{\Rightarrow} \alpha \qquad \beta, \Gamma \stackrel{\mathrm{u}}{\Rightarrow} \delta}{\alpha \rightarrow \beta, \Gamma \stackrel{\mathrm{u}}{\Rightarrow} \delta}\,L\!\rightarrow$$

$$\dfrac{\Gamma \stackrel{l}{\Rightarrow} \beta}{\Gamma \stackrel{l}{\Rightarrow} \alpha \rightarrow \beta}\,R^{\triangleright}_{\rightarrow} \quad \text{if } \Gamma \triangleright \alpha \qquad\qquad \dfrac{\alpha, \Gamma \stackrel{\mathrm{u}}{\Rightarrow} \beta}{\Gamma \stackrel{l}{\Rightarrow} \alpha \rightarrow \beta}\,R^{\not\triangleright}_{\rightarrow} \quad \text{if } \Gamma \not\triangleright \alpha$$

$$\dfrac{\Gamma, \Delta \stackrel{\mathrm{u}}{\Rightarrow} \alpha}{\Gamma, \Box\Delta \stackrel{\mathrm{u}}{\Rightarrow} \Box\alpha}\,R^{\Box}_{\mathrm{u}} \qquad\qquad \dfrac{\Box\alpha, \Gamma, \Delta \stackrel{\mathrm{u}}{\Rightarrow} \alpha}{\Gamma, \Box\Delta \stackrel{\mathrm{b}}{\Rightarrow} \Box\alpha}\,R^{\Box}_{\mathrm{b}} \quad \text{if } \Gamma \cup \Box\Delta \not\triangleright \Box\alpha$$

Fig. 2. The calculus $\mathsf{GbuSL}_{\Box}$ ($l \in \{\mathrm{b}, \mathrm{u}\}$, $k \in \{0, 1\}$).

The calculus $\mathsf{GbuSL}_{\Box}$ (see Fig. 2) consists of the axiom rules $\mathrm{Ax}^{\triangleright}$ and $L\bot$, together with left/right rules for each logical operator. The calculus is oriented to backward proof search, where rules are applied bottom-up. If the conclusion of a rule has label b, the (bottom-up) application of left rules is blocked. There are two rules for right implication, namely $R^{\triangleright}_{\rightarrow}$ and $R^{\not\triangleright}_{\rightarrow}$; the choice between them is settled by the evaluation relation $\triangleright$. Right $\Box$-formulas are handled by rules $R^{\Box}_{\mathrm{u}}$ and $R^{\Box}_{\mathrm{b}}$; here the choice is determined by the label of the conclusion. We remark that if $\sigma = \Gamma, \Box\Delta \stackrel{\mathrm{b}}{\Rightarrow} \Box\alpha$ and $\Gamma \cup \Box\Delta \triangleright \Box\alpha$, then σ is an axiom sequent (see rule $\mathrm{Ax}^{\triangleright}$) and an application of rule $R^{\Box}_{\mathrm{b}}$ to σ is prevented by the side condition of $R^{\Box}_{\mathrm{b}}$. Rule $R^{\Box}_{\mathrm{b}}$ is similar to rule $R\Box$ of $\mathsf{G3iSL}^{+}_{\Box}$: both rules introduce in the lhs of the premise a copy of the main formula $\Box\alpha$ (also called *diagonal formula*); in rule $R^{\Box}_{\mathrm{u}}$ such a duplication is not required. In backward proof search, a b-sequent starts the construction of a branch only containing b-sequents, where only right rules are applied. This phase ends either when an axiom sequent is obtained or when no rule can be applied or when one of the rules turning a label b into u is applied (namely, rules $R^{\not\triangleright}_{\rightarrow}$ and $R^{\Box}_{\mathrm{b}}$).

Example 2. We show a $\mathsf{GbuSL}_{\Box}$-derivation of the u-sequent $\sigma_0 = \stackrel{\mathrm{u}}{\Rightarrow} \neg\neg\Box p$.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\Box p, \neg\Box p \stackrel{\mathrm{b}}{\Rightarrow} \Box p_{\,(4)}}\,\mathrm{Ax}^{\triangleright} \qquad \dfrac{}{\Box p, \bot \stackrel{\mathrm{u}}{\Rightarrow} p_{\,(5)}}\,L\bot}{\Box p, \neg\Box p \stackrel{\mathrm{u}}{\Rightarrow} p_{\,(3)}}\,L\!\rightarrow}{\neg\Box p \stackrel{\mathrm{b}}{\Rightarrow} \Box p_{\,(2)}}\,R^{\Box}_{\mathrm{b}} \qquad \dfrac{}{\bot \stackrel{\mathrm{u}}{\Rightarrow} \bot_{\,(6)}}\,L\bot}{\dfrac{\neg\Box p \stackrel{\mathrm{u}}{\Rightarrow} \bot_{\,(1)}}{\stackrel{\mathrm{u}}{\Rightarrow} \neg\neg\Box p_{\,(0)}}\,R^{\not\triangleright}_{\rightarrow}}\,L\!\rightarrow$$

In the derivations each sequent is marked with an index (n) so that we can refer to it as σ_n. The above derivation highlights some of the peculiarities of $\mathsf{GbuSL}_\Box$. In backward proof search, σ_2 is obtained by a (backward) application of rule $L\to$ to σ_1; the label b in σ_2 is crucial to block the application of rule $L\to$, which would generate an infinite branch. The sequent σ_3 is obtained by the application of rule $R^{\Box}_{\mathrm{b}}$ to σ_2. In this case, the key feature is the presence of the diagonal formula $\Box p$; without it, the sequent σ_3 would be $\neg\Box p \stackrel{\mathrm{u}}{\Rightarrow} p$ and, after the application of $L\to$ (the only applicable rule), the left premise would be $\sigma_4 = \neg\Box p \stackrel{\mathrm{b}}{\Rightarrow} \Box p$, which yields a loop $(\sigma_4 = \sigma_2)$. ◊

We state the main properties of $\mathsf{GbuSL}_\Box$.

Theorem 2.

(i) $\mathsf{GbuSL}_\Box$ *has the subformula property.*
(ii) $\mathsf{GbuSL}_\Box$ *is terminating.*
(iii) $\vdash_{\mathsf{GbuSL}_\Box} \Gamma \stackrel{l}{\Rightarrow} \delta$ *implies* $\Gamma \models_{\mathsf{iSL}} \delta$ *(Soundness).*
(iv) $\Gamma \models_{\mathsf{iSL}} \delta$ *implies* $\vdash_{\mathsf{GbuSL}_\Box} \Gamma \stackrel{\mathrm{u}}{\Rightarrow} \delta$ *(Completeness).*

We remark that in soundness l is any label; instead, in completeness the label is set to u. For instance, since $p \vee q \models_{\mathsf{iSL}} q \vee p$, completeness guarantees that the u-sequent $\sigma^{\mathrm{u}} = p \vee q \stackrel{\mathrm{u}}{\Rightarrow} q \vee p$ is provable in $\mathsf{GbuSL}_\Box$. A $\mathsf{GbuSL}_\Box$-derivation of σ^{u} is obtained by first (upwards) applying rule $L\vee$ to σ^{u} and then one of the rules $R\vee_0$ or $R\vee_1$; if we first apply a right rule, we are stuck (e.g., if we apply $R\vee_0$ to σ^{u}, we get the unprovable sequent $p \vee q \stackrel{\mathrm{u}}{\Rightarrow} q$). On the contrary, the b-sequent $p \vee q \stackrel{\mathrm{b}}{\Rightarrow} q \vee p$ is not provable in $\mathsf{GbuSL}_\Box$, since the label b inhibits the application of rule $L\vee$ and forces the application of a right rule.

The subformula property of $\mathsf{GbuSL}_\Box$ can be easily checked by inspecting the rules; termination is discussed below and completeness in the next section. Soundness can be proved in different ways. One can exploit semantics, relying on the fact that rules preserve the consequence relation $\models_{\mathsf{iSL}}$ (see the online appendix). Here we prove the soundness of $\mathsf{GbuSL}_\Box$ by showing that $\mathsf{GbuSL}_\Box$-derivations can be mapped to $\mathsf{G3iSL}^{+}_{\Box}$-derivations.

Proposition 1. *If* $\mathsf{GbuSL}_\Box \vdash \Gamma \stackrel{l}{\Rightarrow} \delta$, *then* $\mathsf{G3iSL}^{+}_{\Box} \vdash \Gamma \Rightarrow \delta$.

Proof. Let $\mathcal{T}$ be a $\mathsf{GbuSL}_\Box$-tree with root sequent $\sigma = \Gamma \stackrel{l}{\Rightarrow} \delta$; $\mathcal{T}$ can be translated into a $\mathsf{G3iSL}^{+}_{\Box}$-tree $\tilde{\mathcal{T}}$ having root sequent $\tilde{\sigma} = \Gamma \Rightarrow \delta$ by erasing the labels and weakening the lhs of sequents when rules $R\xrightarrow{\triangleright}$ and $R^{\Box}_{\mathrm{u}}$ are applied. Assume now that the $\mathsf{GbuSL}_\Box$-tree $\mathcal{T}$ is a $\mathsf{GbuSL}_\Box$-derivation of σ and let $\sigma^\star = \Delta \Rightarrow \varphi$ be a leaf of $\tilde{\mathcal{T}}$ which is not an axiom of $\mathsf{G3iSL}^{+}_{\Box}$. Note that $\Delta \triangleright \varphi$, hence by Lemma 1(iv) we can build a $\mathsf{G3iSL}^{+}_{\Box}$-derivation $\mathcal{D}^\star$ of $\sigma^\star$. By replacing in $\tilde{\mathcal{T}}$ every leaf $\sigma^\star$ with the corresponding derivation $\mathcal{D}^\star$, we eventually get a $\mathsf{G3iSL}^{+}_{\Box}$-derivation of $\tilde{\sigma}$. ■

To prove the termination of $\mathsf{GbuSL}_\Box$ we have to introduce a proper well-founded relation $\prec_{\mathrm{bu}}$ on labelled sequents. As mentioned in the Introduction, the main problem stems from rule $L\to$. Let σ and σ' be the conclusion and the left premise of an application of rule $L\rightarrow$; we stipulate that $\sigma' \prec_{\rm bu} \sigma$ since σ' has label b and σ has label u; thus, we establish that b weighs less than u. Now, we need a way out to accommodate rules $R\xrightarrow{\not\triangleright}$ and $R^{\Box}_{\rm b}$ that, read bottom-up, switch b with u. In both cases, we observe that the lhs of the premise evaluates a new formula; e.g., in the application of rule $R\xrightarrow{\not\triangleright}$ having premise $\alpha, \Gamma \overset{\rm u}{\Rightarrow} \beta$ and conclusion $\Gamma \overset{l}{\Rightarrow} \alpha \rightarrow \beta$, it holds that $\Gamma \not\triangleright \alpha$ (side condition) and $\Gamma \cup \{\alpha\} \triangleright \alpha$ (definition of $\triangleright$); this suggests that here we can exploit the evaluation relation. Let Ev be defined as follows:

$$\mathrm{Ev}(\Gamma \overset{l}{\Rightarrow} \delta) \;=\; \{\, \varphi \mid \varphi \in \mathrm{Sf}(\Gamma \cup \{\delta\}) \text{ and } \Gamma \triangleright \varphi \,\}$$

Note that $\mathrm{Ev}(\sigma) \subseteq \mathrm{Sf}(\sigma)$. We also have to take into account the size of a sequents, where $|\Gamma \overset{l}{\Rightarrow} \delta| = |\Gamma| + |\delta|$. This leads to the definition of $\prec_{\rm bu}$:

Definition 2 **($\prec_{\rm bu}$).** *$\sigma' \prec_{\rm bu} \sigma$ iff one of the following conditions holds:*

(a) $\mathrm{Sf}(\sigma') \subset \mathrm{Sf}(\sigma)$*;*
(b) $\mathrm{Sf}(\sigma') = \mathrm{Sf}(\sigma)$ *and* $\mathrm{Ev}(\sigma') \supset \mathrm{Ev}(\sigma)$*;*
(c) $\mathrm{Sf}(\sigma') = \mathrm{Sf}(\sigma)$ *and* $\mathrm{Ev}(\sigma') = \mathrm{Ev}(\sigma)$ *and* $\mathrm{label}(\sigma') = \mathrm{b}$ *and* $\mathrm{label}(\sigma) = \mathrm{u}$*;*
(d) $\mathrm{Sf}(\sigma') = \mathrm{Sf}(\sigma)$ *and* $\mathrm{Ev}(\sigma') = \mathrm{Ev}(\sigma)$ *and* $\mathrm{label}(\sigma') = \mathrm{label}(\sigma)$ *and* $|\sigma'| < |\sigma|$*.*

Proposition 2. *The relation $\prec_{\rm bu}$ is well-founded.*

Proof. Assume, by contradiction, that there is an infinite descending chain of the kind $\ldots \prec_{\rm bu} \sigma_1 \prec_{\rm bu} \sigma_0$. Since $\mathrm{Sf}(\sigma_0) \supseteq \mathrm{Sf}(\sigma_1) \supseteq \ldots$ and $\mathrm{Sf}(\sigma_0)$ is finite, the sets $\mathrm{Sf}(\sigma_j)$ eventually stabilize, namely: there is $k \geq 0$ such that $\mathrm{Sf}(\sigma_j) = \mathrm{Sf}(\sigma_k)$ for every $j \geq k$. Since $\mathrm{Ev}(\sigma_j) \subseteq \mathrm{Sf}(\sigma_j)$, we get $\mathrm{Ev}(\sigma_k) \subseteq \mathrm{Ev}(\sigma_{k+1}) \subseteq \ \ldots\ \subseteq \mathrm{Sf}(\sigma_k)$. Since $\mathrm{Sf}(\sigma_k)$ is finite, there is $m \geq k$ such that $\mathrm{Ev}(\sigma_j) = \mathrm{Ev}(\sigma_m)$ for every $j \geq m$. This implies that there exists $n \geq m$ such that all the sequents $\sigma_n, \sigma_{n+1}, \ldots$ have the same label; accordingly $|\sigma_n| > |\sigma_{n+1}| > |\sigma_{n+2}| > \ldots \geq 0$, a contradiction. We conclude that $\prec_{\rm bu}$ is well-founded. ■

To prove that the rules of $\mathsf{GbuSL}_\Box$ are decreasing w.r.t.$\prec_{\rm bu}$, we need the following property.

Lemma 2. *Let ρ be an application of a rule of $\mathsf{GbuSL}_\Box$, let σ be the conclusion of ρ and σ' any of the premises. For every formula φ, if $\mathrm{lhs}(\sigma) \triangleright \varphi$ then $\mathrm{lhs}(\sigma') \triangleright \varphi$.*

Proof. The assertion can be proved by applying Lemma 1. For instance, let $\sigma = \Gamma, \Box\Delta \overset{\rm u}{\Rightarrow} \Box\alpha$ and $\sigma' = \Gamma, \Delta \overset{\rm u}{\Rightarrow} \alpha$ be the conclusion and the premise of rule $R^{\Box}_{\rm u}$; assume that $\Gamma \cup \Box\Delta \triangleright \varphi$. Since $\Delta \triangleright \Box\Delta$, by Lemma 1(ii) get $\Gamma \cup \Delta \triangleright \varphi$. ■

Proposition 3. *Every rule of the calculus $\mathsf{GbuSL}_\Box$ is decreasing w.r.t. $\prec_{\rm bu}$.*

Proof. Let σ and σ' be the conclusion and one of the premises of an application of a rule of $\mathsf{GbuSL}_\Box$. Note that $\mathrm{Sf}(\sigma') \subseteq \mathrm{Sf}(\sigma)$; moreover, if $\mathrm{Sf}(\sigma') = \mathrm{Sf}(\sigma)$, by Lemma 2 we get $\mathrm{Ev}(\sigma') \supseteq \mathrm{Ev}(\sigma)$. We can prove $\sigma' \prec_{\rm bu} \sigma$ by a case analysis; we only detail two significant cases.

Γ^{at} is a multiset of propositional variables, $\Gamma^{\to}$ is a multiset of $\to$-formulas

$$\frac{}{\sigma}\,\mathrm{Irr}\quad \text{if } \sigma \text{ is irreducible} \qquad \frac{\alpha, \beta, \Gamma \overset{\mathrm{u}}{\not\Rightarrow} \delta}{\alpha \wedge \beta, \Gamma \overset{\mathrm{u}}{\not\Rightarrow} \delta}\, L\wedge \qquad \frac{\Gamma \overset{l}{\not\Rightarrow} \alpha_k}{\Gamma \overset{l}{\not\Rightarrow} \alpha_0 \wedge \alpha_1}\, R\wedge_k$$

$$\frac{\alpha_k, \Gamma \overset{\mathrm{u}}{\not\Rightarrow} \delta}{\alpha_0 \vee \alpha_1, \Gamma \overset{\mathrm{u}}{\not\Rightarrow} \delta}\, L\vee_k \qquad \frac{\Gamma \overset{\mathrm{b}}{\not\Rightarrow} \alpha \qquad \Gamma \overset{\mathrm{b}}{\not\Rightarrow} \beta}{\Gamma \overset{\mathrm{b}}{\not\Rightarrow} \alpha \vee \beta}\, R\vee \qquad \frac{\beta, \Gamma \overset{\mathrm{u}}{\not\Rightarrow} \delta}{\alpha \to \beta, \Gamma \overset{\mathrm{u}}{\not\Rightarrow} \delta}\, L\to$$

$$\frac{\Gamma \overset{l}{\not\Rightarrow} \beta}{\Gamma \overset{l}{\not\Rightarrow} \alpha \to \beta}\, R\overset{\rhd}{\to} \quad \Gamma \rhd \alpha \qquad \frac{\alpha, \Gamma \overset{\mathrm{u}}{\not\Rightarrow} \beta}{\Gamma \overset{l}{\not\Rightarrow} \alpha \to \beta}\, R\overset{\not\rhd}{\to} \quad \Gamma \not\rhd \alpha$$

$$\frac{\Box\alpha, \Gamma^{\mathrm{at}}, \Gamma^{\to}, \Delta \overset{\mathrm{u}}{\not\Rightarrow} \alpha}{\underbrace{\Gamma^{\mathrm{at}}, \Gamma^{\to}, \Box\Delta}_{\Gamma} \overset{\mathrm{b}}{\not\Rightarrow} \Box\alpha}\, R^{\Box}_{\mathrm{b}} \quad \Gamma \not\rhd \Box\alpha \qquad \frac{\{\, \Gamma \overset{\mathrm{b}}{\not\Rightarrow} \alpha \,\}_{\alpha\to\beta \in \Gamma^{\to}}}{\Gamma^{\mathrm{at}}, \Gamma^{\to}, \Box\Delta \overset{\mathrm{u}}{\not\Rightarrow} \delta}\, \mathrm{S}^{\mathrm{At}}_{\mathrm{u}} \quad \begin{array}{l}\Gamma^{\to} \neq \emptyset \\ \delta \in (\mathcal{V} \cup \{\bot\}) \setminus \Gamma^{\mathrm{at}}\end{array}$$

$$\frac{\{\, \Gamma \overset{\mathrm{b}}{\not\Rightarrow} \alpha \,\}_{\alpha\to\beta \in \Gamma^{\to}} \quad \Gamma \overset{\mathrm{b}}{\not\Rightarrow} \delta_0 \quad \Gamma \overset{\mathrm{b}}{\not\Rightarrow} \delta_1}{\Gamma^{\mathrm{at}}, \Gamma^{\to}, \Box\Delta \overset{\mathrm{u}}{\not\Rightarrow} \delta_0 \vee \delta_1}\, \mathrm{S}^{\vee}_{\mathrm{u}} \qquad \frac{\{\, \Gamma \overset{\mathrm{b}}{\not\Rightarrow} \alpha \,\}_{\alpha\to\beta \in \Gamma^{\to}} \quad \Gamma^{\mathrm{at}}, \Gamma^{\to}, \Delta \overset{\mathrm{u}}{\not\Rightarrow} \delta}{\Gamma^{\mathrm{at}}, \Gamma^{\to}, \Box\Delta \overset{\mathrm{u}}{\not\Rightarrow} \Box\delta}\, \mathrm{S}^{\Box}_{\mathrm{u}}$$

Fig. 3. The refutation calculus $\mathsf{RbuSL}_{\Box}$ ($l \in \{\mathrm{b}, \mathrm{u}\}$, $k \in \{0, 1\}$).

$$\frac{\sigma' \;=\; \alpha \to \beta, \Gamma \overset{\mathrm{b}}{\Rightarrow} \alpha \qquad \beta, \Gamma \overset{\mathrm{u}}{\Rightarrow} \delta}{\sigma \;=\; \alpha \to \beta, \Gamma \overset{\mathrm{u}}{\Rightarrow} \delta}\, L\to$$

If $\mathrm{Sf}(\sigma') \subset \mathrm{Sf}(\sigma)$, then $\sigma' \prec_{\mathrm{bu}} \sigma$ by point (a) of the definition. Otherwise, it holds that $\mathrm{Sf}(\sigma') = \mathrm{Sf}(\sigma)$ and $\mathrm{Ev}(\sigma') \supseteq \mathrm{Ev}(\sigma)$. If $\mathrm{Ev}(\sigma') \supset \mathrm{Ev}(\sigma)$, then $\sigma' \prec_{\mathrm{bu}} \sigma$ by point (b); otherwise, $\sigma' \prec_{\mathrm{bu}} \sigma$ follows by point (c).

$$\frac{\sigma' \;=\; \Box\alpha, \Gamma, \Delta \overset{\mathrm{u}}{\Rightarrow} \alpha}{\sigma \;=\; \Gamma, \Box\Delta \overset{l}{\Rightarrow} \Box\alpha}\, R^{\Box}_{\mathrm{b}} \qquad \Gamma \cup \Box\Delta \not\rhd \Box\alpha$$

If $\mathrm{Sf}(\sigma') \subset \mathrm{Sf}(\sigma)$, then $\sigma' \prec_{\mathrm{bu}} \sigma$ by point (a). Otherwise, $\mathrm{Sf}(\sigma') = \mathrm{Sf}(\sigma)$ and $\mathrm{Ev}(\sigma') \supseteq \mathrm{Ev}(\sigma)$. Note that $\Box\alpha \in \mathrm{Ev}(\sigma')$ and, by the side condition, $\Box\alpha \notin \mathrm{Ev}(\sigma)$. This implies that $\mathrm{Ev}(\sigma') \supset \mathrm{Ev}(\sigma)$, hence $\sigma' \prec_{\mathrm{bu}} \sigma$ by point (b). ■

By Proposition 2 and 3, we conclude that the calculus $\mathsf{GbuSL}_{\Box}$ is terminating.

4 The Refutation Calculus $\mathsf{RbuSL}_{\Box}$

A common technique to prove the completeness of a sequent calculus $\mathcal{C}$ consists in showing that, whenever a sequent σ is not provable in $\mathcal{C}$, then a counter-model for σ can be built (see, e.g., the proof of completeness of $\mathsf{G4iSL}_{\Box}$ discussed in [13,15]); we prove the completeness of $\mathsf{GbuSL}_{\Box}$ according with this plan. Following the ideas in [3–5,9], we formalize the notion of "non-provability in $\mathsf{GbuSL}_{\Box}$" by introducing the refutation calculus $\mathsf{RbuSL}_{\Box}$, a dual calculus to $\mathsf{GbuSL}_{\Box}$. Sequents of $\mathsf{RbuSL}_{\Box}$, called *antisequents*, have the form $\Gamma \overset{l}{\not\Rightarrow} \delta$. Intuitively, a derivation in $\mathsf{RbuSL}_{\Box}$ of $\Gamma \overset{l}{\not\Rightarrow} \delta$ witnesses that the sequent $\Gamma \overset{l}{\Rightarrow} \delta$ is refutable, that is, not provable, in $\mathsf{GbuSL}_\Box$. Henceforth, Γ^{at} denotes a finite multiset of propositional variables, $\Gamma^{\to}$ denotes a finite multiset of $\to$-formulas (i.e., formulas of the kind $\alpha \to \beta$). The axioms of $\mathsf{RbuSL}_\Box$ are the *irreducible antisequents*, namely the antisequents $\Gamma \overset{l}{\not\Rightarrow} \delta$ such that the corresponding dual sequents $\Gamma \overset{l}{\Rightarrow} \delta$ are not the conclusion of any of the rules of $\mathsf{GbuSL}_\Box$. Irreducible antisequents are characterized as follows:

Definition 3. *An antisequent σ is* irreducible *iff $\sigma = \Gamma^{\mathrm{at}}, \Gamma^{\to}, \Box\Delta \overset{l}{\not\Rightarrow} \delta$ and both (i) $\delta \in (\mathcal{V} \cup \{\bot\}) \setminus \Gamma^{\mathrm{at}}$ and (ii) $l = \mathrm{b}$ or $\Gamma^{\to} = \emptyset$.*

The rules of $\mathsf{RbuSL}_\Box$ are displayed in Fig. 3. In rules $\mathrm{S}_{\mathrm{u}}^{\mathrm{At}}$, $\mathrm{S}_{\mathrm{u}}^{\vee}$ and $\mathrm{S}_{\mathrm{u}}^{\Box}$ (we call *Succ rules*) the notation $\{\Gamma \overset{\mathrm{b}}{\not\Rightarrow} \alpha\}_{\alpha\to\beta\in\Gamma^{\to}}$ means that, for every $\alpha \to \beta \in \Gamma^{\to}$, the b-antisequent $\Gamma \overset{\mathrm{b}}{\not\Rightarrow} \alpha$ is a premise of the rule. Note that all of the Succ rules have at least one premise (in rule $\mathrm{S}_{\mathrm{u}}^{\mathrm{At}}$ this is imposed by the condition $\Gamma^{\to} \neq \emptyset$). The next theorem, proved below, states the soundness of $\mathsf{RbuSL}_\Box$:

Theorem 3 (Soundness of $\mathsf{RbuSL}_\Box$). *If $\vdash_{\mathsf{RbuSL}_\Box} \Gamma \overset{\mathrm{u}}{\not\Rightarrow} \delta$, then $\Gamma \not\models_{\mathsf{iSL}} \delta$.*

Example 3. Figure 4 displays the $\mathsf{RbuSL}_\Box$-derivation $\mathcal{D}$ of $\sigma_0 = \overset{\mathrm{u}}{\not\Rightarrow} \psi$. The (backward) application of rule $\mathrm{S}_{\mathrm{u}}^{\vee}$ to σ_2 has three premises, the left-most one is related to the formula $p \to q$ in Θ. The application of rule $\mathrm{S}_{\mathrm{u}}^{\mathrm{At}}$ to σ_7 has only the premise σ_8, generated by the formula $\neg s$ in Λ. To σ_{13} we must apply $R\overset{\triangleright}{\to}$, since $\Sigma \triangleright q$. The application of rule $\mathrm{S}_{\mathrm{u}}^{\mathrm{At}}$ to σ_{24} gives rise to two premises, corresponding to the formulas $\neg\neg q$ and $\neg p$ in Ω. By Theorem 3, we get $\not\models_{\mathsf{IPL}} \psi$, namely $\psi \notin \mathsf{iSL}$. $\Diamond$

Countermodel Extraction. An iSL-model $\mathcal{K}$ with root r is a *countermodel for* $\sigma = \Gamma \overset{\mathrm{u}}{\not\Rightarrow} \delta$ iff $r \Vdash \Gamma$ and $r \nVdash \delta$; thus $\mathcal{K}$ certifies that $\Gamma \not\models_{\mathsf{iSL}} \delta$. Let $\mathcal{D}$ be an $\mathsf{RbuSL}_\Box$-derivation of a u-antisequent σ_0^{u}; we show that from $\mathcal{D}$ we can extract a countermodel $\mathrm{Mod}(\mathcal{D})$ for σ_0^{u}. A u-antisequent σ of $\mathcal{D}$ is *prime* iff σ is the conclusion of rule Irr or of a Succ rule. We introduce the relations $\preceq$, $\prec$ and $\prec_{\mathrm{R}}$ between antisequents occurring in $\mathcal{D}$:

- $\sigma_1 \prec \sigma_2$ iff σ_1 and σ_2 belong to the same branch of $\mathcal{D}$ and σ_1 is below σ_2;
- $\sigma_1 \preceq \sigma_2$ iff either $\sigma_1 = \sigma_2$ or $\sigma_1 \prec \sigma_2$;
- $\sigma_1 \prec_{\mathrm{R}} \sigma_2$ iff there exists a u-antisequent σ' such that $\sigma_1 \prec \sigma' \preceq \sigma_2$ and σ' is either the premise of rule $R_{\mathrm{b}}^{\Box}$ or the rightmost premise of $\mathrm{S}_{\mathrm{u}}^{\Box}$.

We define $\mathrm{Mod}(\mathcal{D})$ as the structure $\langle W, \leq, R, \sigma_r^{\mathrm{u}}, V\rangle$ where:

- W is the set of the prime antisequents of $\mathcal{D}$;
- $\leq$ and R are the restrictions of $\preceq$ and $\prec_{\mathrm{R}}$ to W respectively;
- σ_r^{u} is the $\leq$-minimum prime antisequent of $\mathcal{D}$;
- $V(\,\Gamma \overset{\mathrm{u}}{\not\Rightarrow} \delta\,) = \Gamma \cap \mathcal{V}$.

It is easy to check that $\mathrm{Mod}(\mathcal{D})$ is an iSL-model; in particular, σ_r^{u} exists since the antisequent at the root of $\mathcal{D}$ has label u. We introduce a *canonical map* Ψ between the u-antisequents of $\mathcal{D}$ and the worlds of $\mathrm{Mod}(\mathcal{D})$:

$\psi = \alpha \to (\beta \vee (\gamma \vee q))$ $\quad \alpha = (p \to q) \wedge \Box s \wedge \Box\Box\neg p \wedge \Box\Box\Box\neg q$
$\beta = \neg(p \wedge \neg s)$ $\quad \gamma = \neg\neg q \to \Box\delta$ $\quad \delta = \neg p \vee \Box\neg\neg p$
$\Theta = p \to q,\ \Box s,\ \Box\Box\neg p,\ \Box\Box\Box\neg q$ $\quad \Lambda = p,\ q,\ \neg s,\ \Box s,\ \Box\Box\neg p,\ \Box\Box\Box\neg q$
$\Sigma = q,\ \neg\neg q,\ \Box s,\ \Box\Box\neg p,\ \Box\Box\Box\neg q$ $\quad \Upsilon = q,\ s,\ \neg\neg q,\ \Box\neg p,\ \Box\Box\neg q$
$\Omega = q,\ s,\ \neg\neg q,\ \neg p,\ \Box\neg\neg p,\ \Box\neg q$ $\quad$ antisequents marked by $\star$ are prime
In $L\to$ application (†) the main formula is $p \to q$ (thus, $p \to q$ is replaced with q)

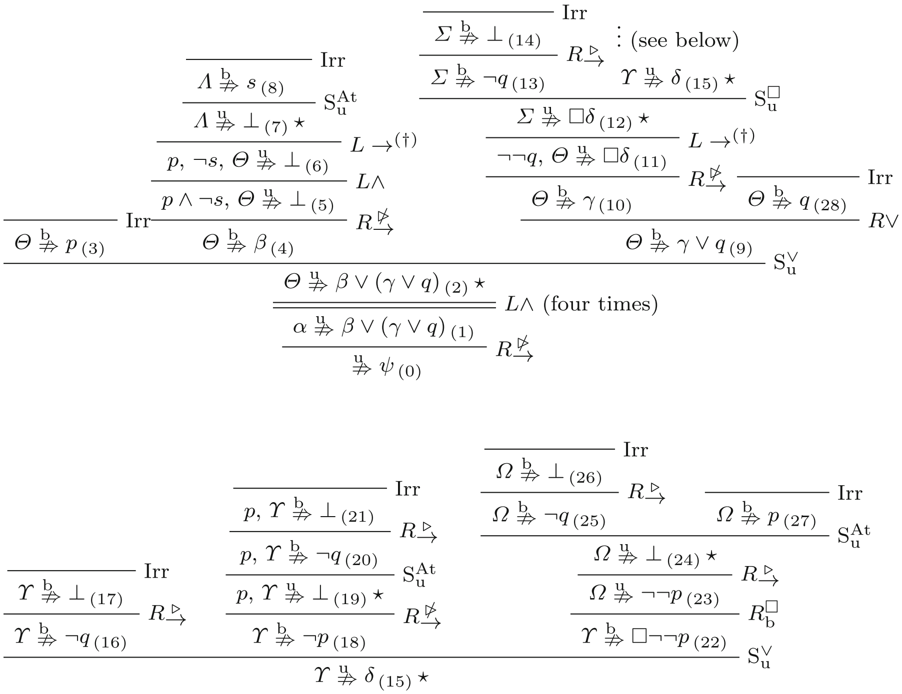


Fig. 4. The $\mathsf{RbuSL}_\Box$-derivation $\mathcal{D}$ of $\sigma_0 = \ \stackrel{\mathrm{u}}{\not\Rightarrow} \psi$ (see Example 3).

– $\Psi(\sigma^{\mathrm{u}}) = \sigma_p^{\mathrm{u}}$ iff σ_p^{u} is the $\preceq$-minimum prime antisequent σ such that $\sigma^{\mathrm{u}} \preceq \sigma$.

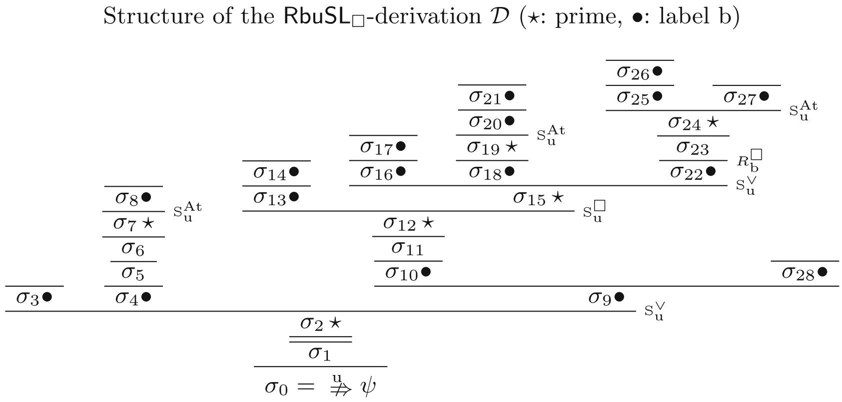
One can easily check that Ψ is well-defined and $\Psi(\sigma_p) = \sigma_p$, for every prime σ_p. We state the main properties of $\mathrm{Mod}(\mathcal{D})$.

Theorem 4. *Let $\mathcal{D}$ be an $\mathsf{RbuSL}_\Box$-derivation of a* u*-antisequent σ_0^{u}.*

(i) For every u*-antisequent $\sigma^{\mathrm{u}} = \Gamma \stackrel{\mathrm{u}}{\not\Rightarrow} \delta$ in $\mathcal{D}$, $\Psi(\sigma^{\mathrm{u}}) \Vdash \Gamma$ and $\Psi(\sigma^{\mathrm{u}}) \nVdash \delta$.*
(ii) $\mathrm{Mod}(\mathcal{D})$ is a countermodel for σ_0^{u}.

Point (ii) follows from (i) and the fact that $\Psi(\sigma_0^{\mathrm{u}})$ is the root of $\mathrm{Mod}(\mathcal{D})$. The proof of (i) is deferred below. We remark that point (ii) of Theorem 4 immediately implies the soundness of $\mathsf{RbuSL}_\Box$ (Theorem 3).

Example 4. At the top of Fig. 5 we represent the structure of the $\mathsf{RbuSL}_\Box$-derivation $\mathcal{D}$ of Fig. 4, displaying the information relevant to the definition of $\mathrm{Mod}(\mathcal{D})$. The countermodel $\mathrm{Mod}(\mathcal{D})$ for σ_0 coincides with the iSL-model in the figure and described in Example 1; the figure also reports the canonical map Ψ. $\Diamond$



$$\psi \;=\; \alpha \to (\beta \lor (\gamma \lor q)) \qquad \alpha \;=\; (p \to q) \land \Box s \land \Box\Box\neg p \land \Box\Box\Box\neg q$$
$$\beta \;=\; \neg(p \land \neg s) \qquad \gamma \;=\; \neg\neg q \to \Box\delta \qquad \delta \;=\; \neg p \lor \Box\neg\neg p$$

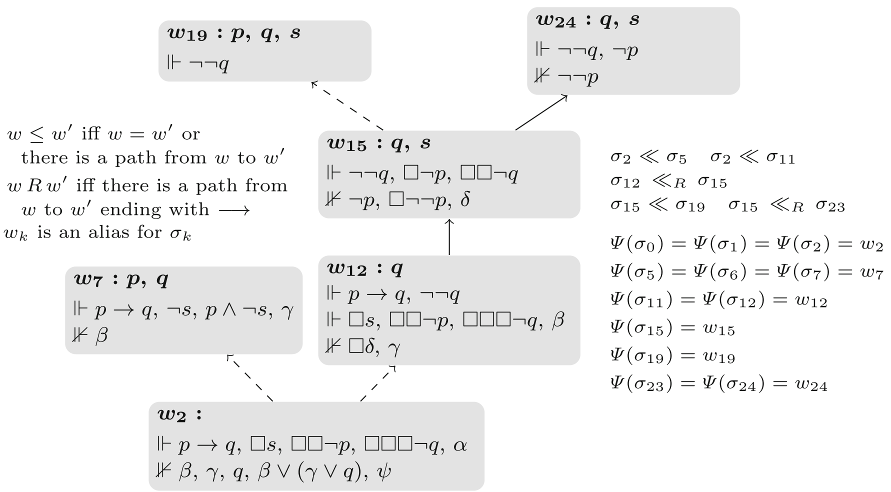


Fig. 5. The countermodel $\mathrm{Mod}(\mathcal{D})$ for ψ (see Examples 1, 4).

Proof Search. We investigate more deeply the duality between $\mathsf{GbuSL}_\Box$ and $\mathsf{RbuSL}_\Box$. A sequent $\sigma = \Gamma \overset{l}{\Rightarrow} \delta$ is *regular* iff $l = \mathrm{u}$ or $\Gamma = \Gamma^{\mathrm{at}}, \Gamma^{\to}, \Box\Delta$; by $\overline{\sigma}$ we denote the antisequent $\Gamma \overset{l}{\not\Rightarrow} \delta$. Let σ be a regular sequent; in the next proposition we show that either σ is provable in $\mathsf{GbuSL}_\Box$ or $\overline{\sigma}$ is provable in $\mathsf{RbuSL}_\Box$. The proof conveys a proof search strategy to build the proper derivation, based on backward application of the rules of $\mathsf{GbuSL}_\Box$. We give priority to the *invertible rules* of $\mathsf{GbuSL}_\Box$, namely: $L\wedge$, $R\wedge$, $L\vee$, $R\overset{\rhd}{\rightarrow}$, $R\overset{\not\rhd}{\rightarrow}$, $R^\Box_{\mathrm{b}}$; as discussed in the proof of Proposition 4, the application of such rules does not require backtracking. If the search for a $\mathsf{GbuSL}_\Box$-derivation of σ fails, we get an $\mathsf{RbuSL}_\Box$-derivation of $\overline{\sigma}$. The proof search procedure is detailed in the online appendix.

Proposition 4. *Let σ be a regular sequent. One can build either a* $\mathsf{GbuSL}_\Box$*-derivation of σ or an* $\mathsf{RbuSL}_\Box$*-derivation of $\overline{\sigma}$.*

Proof. Since $\prec_{\mathrm{bu}}$ is well-founded (Proposition 2), we can inductively assume that the assertion holds for every regular sequent σ' such that $\sigma' \prec_{\mathrm{bu}} \sigma$ (IH). If σ or $\overline{\sigma}$ is an axiom (in the respective calculus), the assertion immediately follows. If an invertible rule ρ of $\mathsf{GbuSL}_\Box$ is (backward) applicable to σ, we can build the proper derivation by applying ρ or its dual image in $\mathsf{RbuSL}_\Box$. For instance, let us assume that rule $L\vee$ of $\mathsf{GbuSL}_\Box$ is applicable with conclusion $\sigma = \alpha_0 \vee \alpha_1, \Gamma \overset{\mathrm{u}}{\Rightarrow} \delta$ and premises $\sigma_k = \alpha_k, \Gamma \overset{\mathrm{u}}{\Rightarrow} \delta$. Let $k \in \{0,1\}$; since $\sigma_k \prec_{\mathrm{bu}} \sigma$ (see Proposition 3), by (IH) there exists either a $\mathsf{GbuSL}_\Box$-derivation $\mathcal{D}_k$ of σ_k or an $\mathsf{RbuSL}_\Box$-derivation $\mathcal{E}_k$ of $\overline{\sigma_k}$. According to the case, we can build one of the following derivations:

$$\dfrac{\begin{matrix}\mathcal{D}_0 \\ \alpha_0, \Gamma \overset{\mathrm{u}}{\Rightarrow} \delta\end{matrix} \qquad \begin{matrix}\mathcal{D}_1 \\ \alpha_1, \Gamma \overset{\mathrm{u}}{\Rightarrow} \delta\end{matrix}}{\alpha_0 \vee \alpha_1, \Gamma \overset{\mathrm{u}}{\Rightarrow} \delta}\, L\vee \qquad \dfrac{\begin{matrix}\mathcal{E}_0 \\ \alpha_0, \Gamma \overset{\mathrm{u}}{\not\Rightarrow} \delta\end{matrix}}{\alpha_0 \vee \alpha_1, \Gamma \overset{\mathrm{u}}{\not\Rightarrow} \delta}\, L\vee_0 \qquad \dfrac{\begin{matrix}\mathcal{E}_1 \\ \alpha_1, \Gamma \overset{\mathrm{u}}{\not\Rightarrow} \delta\end{matrix}}{\alpha_0 \vee \alpha_1, \Gamma \overset{\mathrm{u}}{\not\Rightarrow} \delta}\, L\vee_1$$

Let us assume that no invertible rule can be applied to σ; then:

- $\sigma = \Gamma \overset{\mathrm{u}}{\Rightarrow} \delta$ with $\Gamma = \Gamma^{\mathrm{at}}, \Gamma^{\rightarrow}, \Box\Delta$ and $\delta \in \mathcal{V} \cup \{\,\bot,\, \delta_0 \vee \delta_1,\, \Box\delta_0\,\}$.

We only discuss the case $\delta = \Box\delta_0$. Let $\sigma_0 = \Gamma^{\mathrm{at}}, \Gamma^{\rightarrow}, \Delta \overset{\mathrm{u}}{\Rightarrow} \delta_0$ be the premise of the application of rule $R^\Box_{\mathrm{u}}$ of $\mathsf{GbuSL}_\Box$ to σ; for every $\alpha \rightarrow \beta \in \Gamma^{\rightarrow}$, let $\sigma_\alpha = \Gamma \overset{\mathrm{b}}{\Rightarrow} \alpha$ and $\sigma_\beta = \Gamma \setminus \{\alpha \rightarrow \beta\}, \beta \overset{\mathrm{u}}{\Rightarrow} \delta$ be the two premises of an application of rule $L\rightarrow$ of $\mathsf{GbuSL}_\Box$ to σ with main formula $\alpha \rightarrow \beta$. By the (IH):

- we can build either a $\mathsf{GbuSL}_\Box$-der. $\mathcal{D}_0$ of σ_0 or an $\mathsf{RbuSL}_\Box$-der. $\mathcal{E}_0$ of $\overline{\sigma_0}$.
- for every $\alpha \rightarrow \beta \in \Gamma^{\rightarrow}$ and for every $\omega \in \{\alpha, \beta\}$, we can build either a $\mathsf{GbuSL}_\Box$-derivation $\mathcal{D}_\omega$ of σ_ω or an $\mathsf{RbuSL}_\Box$-derivation $\mathcal{E}_\omega$ of $\overline{\sigma_\omega}$.

One of the following four cases holds:

(A) We get $\mathcal{D}_0$.
(B) There is $\alpha \rightarrow \beta \in \Gamma^{\rightarrow}$ such that we get both $\mathcal{D}_\alpha$ and $\mathcal{D}_\beta$.
(C) There is $\alpha \rightarrow \beta \in \Gamma^{\rightarrow}$ such that we get $\mathcal{E}_\beta$.
(D) We get $\mathcal{E}_0$ and, for every $\alpha \rightarrow \beta \in \Gamma^{\rightarrow}$, $\mathcal{E}_\alpha$.

According to the case, we can build one of the following derivations:

$$\text{(A)}\ \dfrac{\begin{matrix}\mathcal{D}_0 \\ \sigma_0\end{matrix}}{\sigma}\, R^\Box_{\mathrm{u}} \qquad \text{(B)}\ \dfrac{\begin{matrix}\mathcal{D}_\alpha \\ \sigma_\alpha\end{matrix} \ \begin{matrix}\mathcal{D}_\beta \\ \sigma_\beta\end{matrix}}{\sigma}\, L\rightarrow \qquad \text{(C)}\ \dfrac{\begin{matrix}\mathcal{E}_\beta \\ \overline{\sigma_\beta}\end{matrix}}{\overline{\sigma}}\, L\rightarrow \qquad \text{(D)}\ \dfrac{\ldots \begin{matrix}\mathcal{E}_\alpha \\ \overline{\sigma_\alpha}\end{matrix} \ldots \begin{matrix}\mathcal{E}_0 \\ \overline{\sigma_0}\end{matrix}}{\overline{\sigma}}\, \mathrm{S}^\Box_{\mathrm{u}}$$

In the proof search strategy, this corresponds to a backtrack point, since we cannot predict which case holds. ■

Let us assume $\Gamma \models_{\mathsf{iSL}} \delta$ and let $\sigma = \Gamma \overset{\mathrm{u}}{\Rightarrow} \delta$. By Soundness of $\mathsf{RbuSL}_\Box$ (Theorem 3) $\overline{\sigma}$ is not provable in $\mathsf{RbuSL}_\Box$, hence, by Proposition 4, σ is provable in $\mathsf{GbuSL}_\Box$; this proves the Completeness of $\mathsf{GbuSL}_\Box$ (Theorem 2(iv)). By Proposition 1 it follows that $\mathsf{G3iSL}^+_\Box$ is complete as well.

Properties of $\mathsf{RbuSL}_\Box$. It remains to prove point (i) of Theorem 4. By $\mathrm{Sf}^-(\alpha)$ we denote the set $\mathrm{Sf}(\alpha) \setminus \{\alpha\}$; $w < w'$ means that $w \leq w'$ and $w \neq w'$.

Lemma 3. *Let $\mathcal{T}^{\mathrm{b}}$ be an $\mathsf{RbuSL}_\Box$-tree only containing* b*-antisequents having root $\Gamma^{\mathrm{at}}, \Gamma^{\rightarrow}, \Box\Delta \overset{\mathrm{b}}{\not\Rightarrow} \delta$; let $\mathcal{K} = \langle W, \leq, R, r, V\rangle$ and $w \in W$ such that:*

(I1) $w \nVdash \delta'$, for every leaf $\Gamma^{\mathrm{at}}, \Gamma^{\rightarrow}, \Box\Delta \overset{\mathrm{b}}{\not\Rightarrow} \delta'$ of $\mathcal{T}^{\mathrm{b}}$;
(I2) $w \Vdash (\Gamma^{\rightarrow} \cap \mathrm{Sf}^-(\delta)) \cup \Box\Delta$;
(I3) $V(w) = \Gamma^{\mathrm{at}}$.

Then, $w \nVdash \delta$.

Proof. By induction on $\mathrm{depth}(\mathcal{T}^{\mathrm{b}})$. The case $\mathrm{depth}(\mathcal{T}^{\mathrm{b}}) = 0$ is trivial, since the root of $\mathcal{T}^{\mathrm{b}}$ is also a leaf. Let $\mathrm{depth}(\mathcal{T}^{\mathrm{b}}) > 0$; we only discuss the case where

$$\mathcal{T}^{\mathrm{b}} \;=\; \dfrac{\begin{matrix}\mathcal{T}^{\mathrm{b}}_0\\ \sigma^{\mathrm{b}}_0 \;=\; \Gamma \overset{\mathrm{b}}{\not\Rightarrow} \beta\end{matrix}}{\Gamma \overset{\mathrm{b}}{\not\Rightarrow} \alpha \rightarrow \beta}\; R\overset{\triangleright}{\rightarrow} \qquad \begin{array}{l}\Gamma \;=\; \Gamma^{\mathrm{at}}, \Gamma^{\rightarrow}, \Box\Delta\\ \Gamma \triangleright \alpha\end{array}$$

By applying the induction hypothesis to the $\mathsf{RbuSL}_\Box$-tree $\mathcal{T}^{\mathrm{b}}_0$, having root σ^{b}_0 and the same leaves as $\mathcal{T}^{\mathrm{b}}$, we get $w \nVdash \beta$. Let $\Gamma_\alpha = \Gamma \cap \mathrm{Sf}(\alpha)$; by Lemma 1(iii), $\Gamma_\alpha \triangleright \alpha$. Since $\mathrm{Sf}(\alpha) \subseteq \mathrm{Sf}^-(\alpha \rightarrow \beta)$, by hypotheses (I2)– (I3) we get $w \Vdash \Gamma_\alpha$, which implies $w \Vdash \alpha$ (Lemma 1(v)). This proves $w \nVdash \alpha \rightarrow \beta$. ■

Let $\mathcal{D}$ be an $\mathsf{RbuSL}_\Box$-derivation having a Succ rule at the root. To display $\mathcal{D}$, we introduce the schema (1) below; at the same time, we define the relations $\ll$ and $\ll_R$ between u-antisequents in $\mathcal{D}$ (for exemplifications, see Fig. 5).

$$\mathcal{D} = \quad \dfrac{\cdots \quad \begin{matrix}\mathcal{D}_\chi\\ \sigma^{\mathrm{b}}_\chi \;=\; \Gamma^{\mathrm{at}}, \Gamma^{\rightarrow}, \Box\Delta \overset{\mathrm{b}}{\not\Rightarrow} \chi\end{matrix} \quad \cdots \qquad \begin{matrix}\vdots\\ \sigma^{\mathrm{u}}_\psi \;=\; \Gamma^{\mathrm{at}}, \Gamma^{\rightarrow}, \Delta \overset{\mathrm{u}}{\not\Rightarrow} \psi\end{matrix}}{\sigma^{\mathrm{u}} \;=\; \Gamma^{\mathrm{at}}, \Gamma^{\rightarrow}, \Box\Delta \overset{\mathrm{u}}{\not\Rightarrow} \delta}\;\mathrm{Succ} \qquad (1)$$

- σ^{b}_χ is any of the premises of Succ having label b.
- σ^{u}_ψ is only defined if Succ is $\mathrm{S}^\Box_{\mathrm{u}}$ (thus $\delta = \Box\psi$); in this case we set $\sigma^{\mathrm{u}} \ll_R \sigma^{\mathrm{u}}_\psi$.

- The $\mathsf{RbuSL}_\Box$-derivation $\mathcal{D}_\chi$ of σ^{b}_χ has the form

$$\dfrac{\dfrac{\vdots}{\sigma^{\mathrm{u}}_1}}{\sigma^{\mathrm{b}}_1}\,\rho_1 \quad \cdots \quad \dfrac{\dfrac{\vdots}{\sigma^{\mathrm{u}}_m}}{\sigma^{\mathrm{b}}_m}\,\rho_n \quad \dfrac{}{\tau^{\mathrm{b}}_1}\,\mathrm{Irr} \quad \cdots \quad \dfrac{}{\tau^{\mathrm{b}}_n}\,\mathrm{Irr} \qquad \begin{array}{l} m+n \geq 0 \\ \mathcal{T}^{\mathrm{b}}_\chi \text{ only contains} \\ \text{b-antisequents} \\ \Gamma = \Gamma^{\mathrm{at}}, \Gamma^{\to}, \Box\Delta \end{array}$$
$$\mathcal{T}^{\mathrm{b}}_\chi$$
$$\sigma^{\mathrm{b}}_\chi \; = \; \Gamma \overset{\mathrm{b}}{\not\Rightarrow} \chi$$

  - The $\mathsf{RbuSL}_\Box$-tree $\mathcal{T}^{\mathrm{b}}_\chi$ has root σ^{b}_χ and leaves $\sigma^{\mathrm{b}}_1, \ldots, \sigma^{\mathrm{b}}_m$, $\tau^{\mathrm{b}}_1, \ldots, \tau^{\mathrm{b}}_n$.
  - For every $i \in \{1, \ldots, m\}$, either (A) $\rho_i = R^{\not\triangleright}_{\to}$ or (B) $\rho_i = R^{\Box}_{\mathrm{b}}$, namely:

$$\text{(A)}\ \dfrac{\sigma^{\mathrm{u}}_i \; = \; \alpha, \Gamma \overset{\mathrm{u}}{\not\Rightarrow} \beta}{\sigma^{\mathrm{b}}_i \; = \; \Gamma \overset{\mathrm{b}}{\not\Rightarrow} \alpha \to \beta}\, R^{\not\triangleright}_{\to} \qquad \text{or}$$
$$\text{(B)}\ \dfrac{\sigma^{\mathrm{u}}_i \; = \; \Box\alpha, \Gamma^{\mathrm{at}}, \Gamma^{\to}, \Delta \overset{\mathrm{u}}{\not\Rightarrow} \alpha}{\sigma^{\mathrm{b}}_i \; = \; \Gamma \overset{\mathrm{b}}{\not\Rightarrow} \Box\alpha}\, R^{\Box}_{\mathrm{b}}$$

  In case (A) we set $\sigma^{\mathrm{u}} \ll \sigma^{\mathrm{u}}_i$, in case (B) we set $\sigma^{\mathrm{u}} \ll_R \sigma^{\mathrm{u}}_i$.

Lemma 4. *Let $\mathcal{D}$ be an $\mathsf{RbuSL}_\Box$-derivation of $\sigma^{\mathrm{u}} = \Gamma \overset{\mathrm{u}}{\not\Rightarrow} \delta$ having form (1) where $\Gamma = \Gamma^{\mathrm{at}}, \Gamma^{\to}, \Box\Delta$; let $\mathcal{K} = \langle W, \leq, R, r, V\rangle$ and $w \in W$ such that:*

(J1) for every $w' \in W$ such that $w < w'$, it holds that $w' \Vdash \Gamma^{\to}$.
(J2) For every $w' \in W$ such that wRw', it holds that $w' \Vdash \Delta$.
(J3) For every $\sigma' = \alpha, \Gamma \overset{\mathrm{u}}{\not\Rightarrow} \beta$ such that $\sigma^{\mathrm{u}} \ll \sigma'$, there exists $w' \in W$ such that $w \leq w'$ and $w' \Vdash \alpha$ and $w' \nVdash \beta$.
(J4) For every $\sigma' = \Box\alpha, \Gamma^{\mathrm{at}}, \Gamma^{\to}, \Delta \overset{\mathrm{u}}{\not\Rightarrow} \alpha$ such that $\sigma^{\mathrm{u}} \ll_R \sigma'$, there exists $w' \in W$ such that wRw' and $w' \nVdash \alpha$.
(J5) $V(w) = \Gamma^{\mathrm{at}}$.

Then, $w \Vdash \Gamma$ and $w \nVdash \delta$.

Proof. We show that:

(P1) $w \nVdash \chi$, for every premise $\sigma^{\mathrm{b}}_\chi = \Gamma \overset{\mathrm{b}}{\not\Rightarrow} \chi$ of Succ;
(P2) $w \Vdash \alpha \to \beta$, for every $\alpha \to \beta \in \Gamma^{\to}$.

We introduce the following induction hypothesis:

(IH1) to prove Point (P1) for a formula χ, we inductively assume that Point (P2) holds for every formula $\alpha \to \beta$ such that $|\alpha \to \beta| < |\chi|$;
(IH2) to prove Point (P2) for a formula $\alpha \to \beta$, we inductively assume that Point (P1) holds for every formula χ such that $|\chi| < |\alpha \to \beta|$.

We prove Point (P1). Let σ^{b}_χ be the premise of Succ displayed in schema (1). We show that the $\mathsf{RbuSL}_\Box$-tree $\mathcal{T}^{\mathrm{b}}_X$ and w match the hypotheses (I1)–(I3) of Lemma 3, so that we can apply the lemma to infer $w \nVdash \chi$.

We prove (I1). Assume $m \geq 1$ and let $i \in \{1, \ldots, m\}$; then either (A) $\sigma^{\mathrm{b}}_i = \Gamma \overset{\mathrm{b}}{\not\Rightarrow} \alpha \to \beta$ or (B) $\sigma^{\mathrm{b}}_i = \Box\alpha, \Gamma^{\mathrm{at}}, \Gamma^{\to}, \Delta \overset{\mathrm{b}}{\not\Rightarrow} \Box\alpha$. In case (A) we have $\sigma^{\mathrm{u}}_i = \alpha, \Gamma \overset{\mathrm{u}}{\not\Rightarrow} \beta$ and $\sigma^{\mathrm{u}} \ll \sigma^{\mathrm{u}}_i$; by hypothesis (J3), there is $w' \in W$ such that $w \leq$

w' and $w' \Vdash \alpha$ and $w' \nVdash \beta$, hence $w \nVdash \alpha \to \beta$. In case (B), we have $\sigma_i^{\mathrm{u}} = \Box\alpha, \Gamma^{\mathrm{at}}, \Gamma^{\to}, \Delta \overset{\mathrm{u}}{\not\Rightarrow} \alpha$ and $\sigma^{\mathrm{u}} \ll_R \sigma_i^{\mathrm{u}}$; by hypothesis (J4), there is w' such that wRw' and $w' \nVdash \alpha$, hence $w \nVdash \Box\alpha$. Assume $n \geq 1$, let $j \in \{1, \ldots, n\}$ and $\tau_j^{\mathrm{b}} = \Gamma \overset{\mathrm{b}}{\not\Rightarrow} \delta_j$. Since τ_j^{b} is irreducible and $V(w) = \Gamma^{\mathrm{at}}$ (hypothesis (J5)), we get $w \nVdash \delta_j$. This proves that hypothesis (I1) holds.

We prove (I2). Let $\gamma \in \Gamma^{\to} \cap \mathrm{Sf}^{-}(\chi)$; since $|\gamma| < |\chi|$, by (IH1) we get $w \Vdash \gamma$. Moreover, $w \Vdash \Box\Delta$ by (J2), thus (I2) holds. Finally, (I3) coincides with (J5). We can apply Lemma 3 and conclude $w \nVdash \chi$, and this proves Point (P1).

We prove Point (P2). Let $\alpha \to \beta \in \Gamma^{\to}$, let $w' \in W$ be such that $w \leq w'$ and $w' \Vdash \alpha$; we show that $w' \Vdash \beta$. Note that $\sigma_\alpha^{\mathrm{b}} = \Gamma \overset{\mathrm{b}}{\not\Rightarrow} \alpha$ is a premise of Succ; since $|\alpha| < |\alpha \to \beta|$, by (IH2) we get $w \nVdash \alpha$. This implies that $w < w'$. By hypothesis (J1), $w' \Vdash \alpha \to \beta$, hence $w' \Vdash \beta$; this proves (P2).

We prove the assertion of the lemma. By (P2) and hypotheses (J2) and (J5), we get $w \Vdash \Gamma$. The proof that $w \nVdash \delta$ depends on the specific rule Succ at hand and follows from Point (P1) and hypothesis (J5). ■

Proof (Theorem 4(i)). By induction on the depth of the sequent $\sigma^{\mathrm{u}} = \Gamma \overset{\mathrm{u}}{\Rightarrow} \delta$ in $\mathcal{D}$. Let ρ be the rule of $\mathsf{RbuSL}_\Box$ having conclusion σ^{u}. We proceed by a case analysis, only detailing some significant cases.

If $\rho = \mathrm{Irr}$, then $\Gamma = \Gamma^{\mathrm{at}}, \Box\Delta$ and $\delta \in (\mathcal{V} \cup \{\bot\}) \setminus \Gamma^{\mathrm{at}}$ and $\Psi(\sigma^{\mathrm{u}}) = \sigma^{\mathrm{u}}$. Since $V(\sigma^{\mathrm{u}}) = \Gamma^{\mathrm{at}}$ and σ^{u} is R-maximal, it follows that $\Psi(\sigma^{\mathrm{u}}) \Vdash \Gamma$ and $\Psi(\sigma^{\mathrm{u}}) \nVdash \delta$.

Let us assume that $\rho = R\overset{\triangleright}{\to}$. Then, $\sigma^{\mathrm{u}} = \Gamma \overset{\mathrm{u}}{\not\Rightarrow} \alpha \to \beta$, where $\Gamma \triangleright \alpha$, and the premise of ρ is $\sigma_1^{\mathrm{u}} = \Gamma \overset{\mathrm{u}}{\not\Rightarrow} \beta$. By the induction hypothesis, $\Psi(\sigma_1^{\mathrm{u}}) \Vdash \Gamma$ and $\Psi(\sigma_1^{\mathrm{u}}) \nVdash \beta$. By Lemma 1(v) we get $\Psi(\sigma_1^{\mathrm{u}}) \Vdash \alpha$, which implies $\Psi(\sigma_1^{\mathrm{u}}) \nVdash \alpha \to \beta$. Since $\Psi(\sigma^{\mathrm{u}}) = \Psi(\sigma_1^{\mathrm{u}})$, we conclude $\Psi(\sigma^{\mathrm{u}}) \Vdash \Gamma$ and $\Psi(\sigma^{\mathrm{u}}) \nVdash \alpha \to \beta$.

Let us assume $\rho = \mathrm{S}_{\mathrm{u}}^{\Box}$. We have $\sigma^{\mathrm{u}} = \Gamma \overset{\mathrm{u}}{\not\Rightarrow} \Box\delta$, where $\Gamma = \Gamma^{\mathrm{at}}, \Gamma^{\to}, \Box\Delta$, and $\Psi(\sigma^{\mathrm{u}}) = \sigma^{\mathrm{u}}$. Let $\mathcal{D}^{\mathrm{u}}$ be the subderivation of $\mathcal{D}$ having root sequent σ^{u}; we apply Lemma 4 setting $\mathcal{D} = \mathcal{D}^{\mathrm{u}}$, $\mathcal{K} = \mathrm{Mod}(\mathcal{D})$ and $w = \sigma^{\mathrm{u}}$. We check that hypotheses (J1)–(J5) hold.

Let w' be a world of $\mathrm{Mod}(\mathcal{D})$ such that $\sigma^{\mathrm{u}} < w'$. There exists an u-sequent $\sigma' = \Gamma' \overset{\mathrm{u}}{\Rightarrow} \delta'$ such that $\sigma^{\mathrm{u}} \prec \sigma' \preceq w'$ and $\Gamma^{\to} \subseteq \Gamma'$. Since $\mathrm{depth}(\sigma') < \mathrm{depth}(\sigma^{\mathrm{u}})$, by the induction hypothesis we get $\Psi(\sigma') \Vdash \Gamma'$, hence $\Psi(\sigma') \Vdash \Gamma^{\to}$. Since $\Psi(\sigma') \leq w'$, we conclude $w' \Vdash \Gamma^{\to}$, and this proves hypothesis (J1).

Let w' be a world of $\mathrm{Mod}(\mathcal{D})$ such that $\sigma^{\mathrm{u}} R w'$. There exists an u-sequent $\sigma' = \Gamma' \overset{\mathrm{u}}{\Rightarrow} \delta'$ such that $\sigma^{\mathrm{u}} \prec \sigma' \preceq w'$ and $\Delta \subseteq \Gamma'$. Reasoning as in the previous case, we get $w' \Vdash \Delta$, and this proves hypothesis (J2).

Let $\sigma^{\mathrm{u}} \ll \sigma' = \alpha, \Gamma \overset{\mathrm{u}}{\not\Rightarrow} \beta$. By the induction hypothesis, $\Psi(\sigma') \Vdash \alpha$ and $\Psi(\sigma') \nVdash \beta$. Since $\sigma^{\mathrm{u}} = \Psi(\sigma^{\mathrm{u}}) \leq \Psi(\sigma')$, hypothesis (J3) holds. The proof for hypothesis (J4) is similar. Hypothesis (J5) holds by the definition of V. By applying Lemma 4, we conclude that $\sigma^{\mathrm{u}} \Vdash \Gamma$ and $\sigma^{\mathrm{u}} \nVdash \delta$. ■

Conclusions. In this paper we have presented a terminating sequent calculus $\mathsf{GbuSL}_\Box$ for iSL enjoying the subformula property; iSL is obtained by adding labels to $\mathsf{G3iSL}_\Box^{+}$, a variant of the calculus $\mathsf{G3iSL}_\Box$ [13,15]. If a sequent σ is not derivable in $\mathsf{GbuSL}_\Box$, then σ is derivable in the dual calculus $\mathsf{RbuSL}_\Box$, and from

	Lineage	Termination	Subf. property	Other features
$\mathsf{GbuSL}_\Box$	G3i	Strong	✓	`Count`
$\mathsf{G3iSL}_\Box^+$	G3i	Weak	✓	
G4iSLt [10]	G4i	Strong	✗	`Cut`
$\mathsf{G3iSL}_\Box$ [13,15]	G3i	Weak	✓	`Cut`
$\mathsf{G4iSL}_\Box$ [13,15]	G4i	Weak	✗	`Count`

Fig. 6. Overview of the main sequent calculi for iSL. `Cut`: syntactic proof of cut-admissibility; `Count`: proof search procedure with countermodel generation.

the $\mathsf{RbuSL}_\Box$-derivation we can extract a countermodel for σ. In Fig. 6 we compare the known sequent calculi for iSL. We leave as future work the investigation of cut-admissibility for $\mathsf{GbuSL}_\Box$; this is a rather tricky task since labels impose strict constraints on the shape of derivations. We also aim to extend our approach to other provability logics related with iSL, such as the logics iGL, mHC and KM (for an overview, see e.g. [13]).

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