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QUASILINEAR LOGARITHMIC CHOQUARD EQUATIONS WITH EXPONENTIAL GROWTH IN $\mathbb{R}^N$

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ABSTRACT. We consider the N-Laplacian Schrödinger equation strongly coupled with higher order fractional Poisson's equations. When the order of the Riesz potential α is equal to the Euclidean dimension N, and thus it is a logarithm, the system turns out to be equivalent to a nonlocal Choquard type equation. On the one hand, the natural function space setting in which the Schrödinger energy is well defined is the Sobolev limiting space $W^{1,N}(\mathbb{R}^N)$, where the maximal nonlinear growth is of exponential type. On the other hand, in order to have the nonlocal energy well defined and prove the existence of finite energy solutions, we introduce a suitable *log*-weighted variant of the Pohozaev-Trudinger inequality which provides a proper functional framework where we use variational methods.

1. INTRODUCTION AND MAIN RESULTS

Consider the following system of elliptic equations

$$\begin{cases} -\Delta_m u + V(x)|u|^{m-2}u = f(u)v, \\ \\ -\Delta^{\frac{\alpha}{2}} v = F(u) \ , \end{cases} \qquad x \in \mathbb{R}^N, \quad N \geq 2 \tag{1.1}$$

where Δ_m, $m \geq 2$, is the m-Laplace operator defined as follows

$$\Delta_m u = \mathrm{div}(|\nabla u|^{m-2}\nabla u),$$

$V : \mathbb{R}^N \to \mathbb{R}$ is the external Schrödinger potential, F is the primitive of f vanishing at zero and where $(-\Delta)^{\frac{\alpha}{2}}$, $\alpha > 0$, is the fractional Laplacian, see Section 2.1. System (1.1) is in gradient form as the nonlinearity in the right hand side of (1.1) is the gradient of the potential function $G(u, v) = F(u)v$. It is also strongly coupled as $u = 0 \iff v = 0$. However, (1.1) does not possess in general a variational structure because of the presence of the nonlocal operator in the second equation, which prevents solutions of the system to be critical points of an energy functional $E(u, v)$, which may not exist or may not be well defined.

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The parameters m, N, α play an important role from the theoretical point of view as well as from that of applications, see [24] and references therein. For $m = 2$ we have the linear Schrödinger operator in the left hand side of the first equation and the problem has been widely studied in dimension $N > 2$ and for $\alpha < N$, see [24] for a survey and [11, 13] for related critical cases. The case of dimension $N = 2$ and $\alpha < N$ has been studied in [4, 5]. More recently in [6, 10, 14] it has been considered the limiting case $\alpha = N = 2$; see also [3, 9] for related results.
A major difficulty in the limiting case is to construct a proper function space framework in which to settle the problem. As developed in [10] in dimension $N = 2$, in order to consider the maximal exponential growth, a suitable functional framework can be obtained by means of *log*-weighted versions of the Pohozaev–Trudinger inequality [26, 30].
In this paper we tackle the general limiting case

$$\boxed{\alpha = N = m > 2}\,. \tag{1.2}$$

This leads from one side to handle a quasilinear Schrödinger equation in the system [12] and on the other side demands for a more general function space setting. A key ingredient for this purpose, is to extend the fundamental functional inequality established in [10], in the special case $\alpha = N = m = 2$, to the general case (1.2).
Let $I_N \colon \mathbb{R}^N \setminus \{0\} \to \mathbb{R}$ be the logarithmic Riesz kernel

$$I_N(x) = \frac{1}{\gamma_N} \log \frac{1}{|x|}$$

with

$$\gamma_N = 2^{N-1} \pi^{\frac{N}{2}} \Gamma\left(\frac{N}{2}\right).$$

By setting $v := I_N * F(u)$, (1.1) is formally equivalent (see Section 2.1) to the following quasilinear Choquard type equation

$$-\Delta_N u + V|u|^{N-2}u = (I_N * F(u))f(u) \qquad \text{in } \mathbb{R}^N, \tag{1.3}$$

which does have a variational structure.
Indeed, (1.3) is the Euler-Lagrange equation related to the energy functional

$$E(u) = \frac{1}{N}\int_{\mathbb{R}^N} |\nabla u|^N + V|u|^N \, dx - \frac{1}{2}\int_{\mathbb{R}^N} (I_N * F(u))(x) F(u(x)) \, dx \,,$$

provided such energy is well defined in a suitable function space which we are going to construct in the sequel as one of our main results.

Before stating our main results let us introduce a few assumptions:

- (V) $V \colon \mathbb{R}^N \to \mathbb{R}$ is continuous, 1-periodic and there exists $V_0 > 0$ such that $V(x) \geq V_0$;
- (f_1) $f \colon \mathbb{R} \to \mathbb{R}$ is continuous and differentiable, such that

(i) $f(s) \geq 0$, for all $s \geq 0$ and we may also assume (as we look for positive solutions) $f(s) = 0$ for $s \leq 0$;

(ii) there exists $C > 0$ such that $f(s) \leq C s^p e^{\alpha_N s^{\frac{N}{N-1}}}$ as $s \to +\infty$, for some $p > 0$ and where α_N is given below;

(iii) $f(s) \asymp s^{q-1}$, as $s \to +\infty$, for some $q > N$;

(f_2) there exist $C > \delta > 0$ such that

$$\frac{N-2}{N} + \delta \leq \frac{F(s)f'(s)}{f^2(s)} \leq C, \quad s > 0;$$

(f_3) $\lim_{s\to+\infty} \frac{F(s)f'(s)}{f^2(s)} = 1$, or equivalently $\lim_{s\to+\infty} \frac{d}{ds}\frac{F(s)}{f(s)} = 0$;

(f_4) there exists $\beta > 0$ such that

$$\lim_{s\to+\infty} \frac{s^{\frac{2N-1}{N-1}} f(s)F(s)}{e^{2N(\omega_{N-1})^{\frac{1}{N-1}} s^{\frac{N}{N-1}}}} \geq \beta > \boldsymbol{\nu},$$

where $\boldsymbol{\nu}$ will be explicitly given in Sect. 4.

Notice that from the assumptions on f we also deduce the following:

- there exists $s_0 > 1$ such that

$$0 \leq F(s) \leq C \begin{cases} |s|^q, & s \leq s_0, \\ s^{p-\frac{1}{N-1}} e^{\alpha_N s^{\frac{N}{N-1}}}, & s > s_0; \end{cases} \tag{1.4}$$

- $f(s)$ is monotone increasing, hence $F(s) = \int_0^s f(\tau)d\tau \leq sf(s)$, while the quantity $\frac{F(s)}{f(s)}$ is well defined and vanishes only at $s = 0$. Furthermore,

$$\frac{d}{ds}\left(\frac{F(s)}{f(s)}\right) = \frac{f^2(s) - F(s)f'(s)}{f^2(s)} \leq \frac{2}{N} - \delta \tag{1.5}$$

which implies $F(s) \leq (\frac{2}{N} - \delta)sf(s)$;

- (f_3) implies a fine lower bound on the quotient $\dfrac{Ff'}{f^2}$, as $s \to +\infty$. Indeed, for any $\varepsilon > 0$ there exists $s_\varepsilon > 0$ such that

$$\frac{Ff'}{f^2}(s) \geq \begin{cases} \left(\dfrac{2}{N} + \delta\right) s, & s \leq s_\varepsilon \\ (1-\varepsilon)s, & s > s_\varepsilon; \end{cases} \tag{1.6}$$

- (f_4) is in the spirit of the de Figueiredo–Miyagaki–Ruf condition [15] and turns out to be a suitable compactness condition in this context. The role of condition (f_4) will be detailed in Section 4.1.

Examples of functions $F(s)$ satisfying our set of assumptions are given below:

$$F(s) = \begin{cases} s^q, s \leq s_0 \\ e^{\alpha s^{\frac{N}{N-1}}}, s > s_0 \end{cases}, \ \forall\, q > N;$$

$$F(s) = s^p e^{\alpha s^{\frac{N}{N-1}}},\ \forall\, p > N;$$

$$F(s) = \begin{cases} s^q, & s \le s_0 \\ \alpha s^p e^{\beta s^{\frac{N}{N-1}}}, & s > s_0 \end{cases}$$

for $q \ge N$, $p > 1$ and suitable constants $\alpha, \beta > 0$.

Let us now introduce some basic notation:

$$\|u\|_N := \|u\|_{L^N(\mathbb{R}^N)}$$

and

$$\|u\| := \|u\|_{W^{1,N}(\mathbb{R}^N)} = \left(\|\nabla u\|_N^N + \|u\|_N^N\right)^{\frac{1}{N}}.$$

Let $w(x) := \log(e+|x|)$ and define the weighted Sobolev space $W^{1,N}L^q_w(\mathbb{R}^N)$ as the completion of smooth compactly supported functions with respect to the norm

$$\begin{aligned}\|u\|_{q,w}^N &= \|\nabla u\|_N^N + \|u\|_{L^q(wdx)}^N \\ &= \int_{\mathbb{R}^N} |\nabla u|^N\, dx + \left(\int_{\mathbb{R}^N} |u|^q \log(e+|x|)\, dx\right)^{N/q}.\end{aligned}$$

When $q = N$, for simplicity we denote $W^{1,N}_w(\mathbb{R}^N) := W^{1,N}L^N_w(\mathbb{R}^N)$ and

$$\|u\|_w^N := \|\nabla u\|_N^N + \|u\|_{L^N(wdx)}^N = \int_{\mathbb{R}^N} |\nabla u|^N dx + \int_{\mathbb{R}^N} |u|^N \log(e+|x|)\, dx.$$

Let us set

$$\|u\|_V := \left(\int_{\mathbb{R}^N} |\nabla u|^N + V|u|^N\, dx\right)^{\frac{1}{N}},$$

and we use $W^{1,N}_V(\mathbb{R}^N)$ to denote the set of all functions with bounded $\|\cdot\|_V$ norm. Let us also set $w_0(x) := \log(1+|x|)$, and

$$\|u\|_{q,V,w_0}^N := \|u\|_V^N + \|u\|_{L^q(w_0dx)}^N = \|u\|_V^N + \left(\int_{\mathbb{R}^N} |u|^q \log(1+|x|) dx\right)^{\frac{N}{q}},$$

and consider $W^{1,N}_V L^q_{w_0}(\mathbb{R}^N)$ as the completion of smooth compactly supported functions with respect to the norm $\|\cdot\|_{q,V,w_0}$.

The proper function space setting in which the energy and the variational framework turns out to be well defined, will be a consequence of the following weighted version of the Pohozaev–Trudinger inequality, which we state here for simplicity in the case $q = N$ (see Section 3 for the case $q > N$):

Theorem 1.1. *The weighted Sobolev space $W^{1,N}_w(\mathbb{R}^N)$ embeds into the weighted Orlicz space $L_{\phi_N}(\mathbb{R}^N, \log(e+|x|)dx)$ where*

$$\phi_N(t) = e^t - \sum_{j=0}^{N-2} \frac{t^j}{j!}.$$

More precisely, the following holds

$$\int_{\mathbb{R}^N} \phi_N \left(\alpha |u|^{\frac{N}{N-1}}\right) \log(e+|x|)dx < \infty \ , \tag{1.7}$$

for any $u \in W^{1,N}_w(\mathbb{R}^N)$ and any $\alpha > 0$. Moreover, the following uniform bound holds

$$\sup_{\|u\|_w^N \le 1} \int_{\mathbb{R}^N} \phi_N \left(\alpha_N \left(\frac{N}{N+1}\right)^{1/(N-1)} |u|^{\frac{N}{N-1}}\right) \log(e+|x|)dx < +\infty \ , \tag{1.8}$$

where $\alpha_N = N\omega_{N-1}^{\frac{1}{N-1}}$ is the sharp Moser exponent, and ω_{N-1} is the $(N-1)-$dimensional surface of the unit sphere in $\mathbb{R}^N$.

Inequality (1.8) and its version in the case $q > N$ (Theorem 3.3), turn out to be key ingredients to obtain the following result:

Theorem 1.2. *Suppose the nonlinearity f satisfies (f_1)–(f_4) and that the potential V enjoys (V). Then, problem (1.3) possesses a nontrivial mountain pass solution which has finite energy in the weighted Sobolev space $W^{1,N}_V L^q_{w_0}(\mathbb{R}^N)$.*

Overview. In the next section we collect some preliminary material. Special attention is devoted to discuss equivalence between (1.3) and (1.1). This is a quite delicate matter and still with some shadows which prevent to obtain optimal results. We are motivated by a very recent debate on this topic, towards a better understanding of the higher order fractional context.

In Section 3, we prove some fundamental results which from one side extend classical embeddings from Functional Analysis, due independently to Pohozaev and Trudigner in late sixties, on the other side provide a new tool in the 'variational toolbox' to prove existence results by variational methods; we are confident these results will be useful in other situations. Here we extend in a non trivial fashion to any dimension, previous results obtained in [10] in dimension two and then applied to prove the existence of finite energy solutions to Schrödinger–Newton systems by variational techniques.

In Section 4, we exploit the abstract results of Section 3 to provide a suitable variational framework in which we can prove the existence of a mountain pass solution to (1.3). Due to the presence of exponential growth in the nonlinearity and of a sign-changing logarithmic kernel in the nonlocal part of the equation, here even the most standard variational steps become somehow delicate. We take care of stressing differences with the two dimensional case, in particular passing from semilinear, in dimension $N = 2$, to quasilinear nonlocal Schrödinger equations in higher dimensions $N \ge 3$, where some new ideas and efforts are needed.

2. On the equivalence between nonlocal equations and higher order fractional systems

Here we discuss the equivalence between the Choquard type equation (1.3) and the higher order fractional system (1.1). Formally, if in (1.3) we set

$$\Phi_u := I_N * F(u),$$

then the function u solves the equation

$$-\Delta_N u + V|u|^{N-2}u = \Phi_u f(u), \qquad \text{in} \quad \mathbb{R}^N$$

and moreover, Φ_u is the unique solution in $\mathbb{R}^N$ to the following fractional equation

$$(-\Delta)^{\frac{N}{2}}\phi = F(u)\ .$$

However, this argument is affected by the notion of solution we deal with, this is somehow a delicate matter and not yet completely understood. Having in mind the commitment to make precise in the sequel what we mean by solution, we have the following, and for the moment heuristic

Proposition 2.1. *Let $u \in W_V^{1,N}L^1_{w_0}(\mathbb{R}^N)$ be a solution of (1.3). Then u is a solution to*

$$-\Delta_N u + V|u|^{N-2}u = \phi f(u) \quad \text{in } \mathbb{R}^N, \tag{2.1}$$

where ϕ is the unique solution to

$$(-\Delta)^{\frac{N}{2}}\phi = F(u) \quad \text{in } \mathbb{R}^N.$$

Let us begin by recalling the definition in the distributional sense of the fractional Laplacian of any order. Set for $s > 0$,

$$L_s(\mathbb{R}^N) := \left\{ u \in L^1_{\mathrm{loc}}(\mathbb{R}^N) \;\middle|\; \int_{\mathbb{R}^N} \frac{|u(x)|}{1+|x|^{N+2s}}\,dx < \infty \right\},$$

the operator $(-\Delta)^s u$ is defined for all $u \in L_s(\mathbb{R}^N)$ via duality, as

$$\langle (-\Delta)^s u, \varphi \rangle = \int_{\mathbb{R}^N} u\,(-\Delta)^s \varphi\,dx, \qquad \forall \varphi \in \mathcal{S}(\mathbb{R}^N), \tag{2.2}$$

where

$$(-\Delta)^s \varphi = \mathcal{F}^{-1}\left(|\xi|^{2s}\mathcal{F}\varphi(\xi)\right), \qquad \forall \varphi \in \mathcal{S}(\mathbb{R}^N)$$

denoting by $\mathcal{F}$ the Fourier transform and where $\mathcal{S}(\mathbb{R}^N)$ denotes the Schwartz space of rapidly decreasing functions. We remark that the right hand side of (2.2) is well defined, thanks to the fact that for $\varphi \in \mathcal{S}(\mathbb{R}^N)$,

$$|(-\Delta)^s \varphi(x)| \le \frac{C}{|x|^{N+2s}},$$

see e.g. [18, Proposition 2.1].

Let us consider the fractional Poisson's equation

$$(-\Delta)^s u = f \quad \text{in } \mathbb{R}^N, \text{ with } 0 < s \le \frac{N}{2} \tag{2.3}$$

If $s \in (0,1)$, the setting and the representation formulas for solutions of this equation are settled in literature, among which recent studies carried out in [1,8]. The case $s > 1$ has been considered in very recent papers [2,7,29], whereas a general approach, based on the notion of distributional solution dates back to classical works [20,27,28], see also [16].

Definition 2.2. *Given $f \in \mathcal{S}'(\mathbb{R}^N)$, we say that $u \in L_{\frac{N}{2}}(\mathbb{R}^N)$ is a distributional solution of (2.3) if*

$$\int_{\mathbb{R}^N} u(-\Delta)^{\frac{N}{2}}\varphi\, dx = \langle f, \varphi\rangle$$

for all $\varphi \in \mathcal{S}(\mathbb{R}^N)$.

It is well known that, if $s < \frac{N}{2}$, the distributional solution of (2.3) is given by

$$u(x) = \mathcal{F}^{-1}\left(\frac{1}{|\xi|^{2s}}\mathcal{F}f(\xi)\right)(x)$$

which is realized also by convolution with the Newtonian potential

$$u(x) = (I_{2s} * f)(x), \qquad \text{where } \; I_{2s}(x) = \frac{1}{\gamma_{N,s}}|x|^{2s-N}$$

for some $\gamma_{N,s} > 0$, (see [27, Chapter 5]). Note that the Newtonian potential, in the case $2s = N$, is given by

$$I_N(x) = \frac{1}{\gamma_N}\log\frac{1}{|x|} = \mathcal{F}|\xi|^{-N}(x),$$

Nevertheless, when $2s = N$ it is not possible to define the solution to (2.3) by Fourier transform in $\mathcal{S}(\mathbb{R}^N)$ in general, due to the singularity of $|\xi|^{-N}$ in zero. However, various assumptions on f, which improve the regularity of its Fourier transform, allow to recover the notion of distributional solution, for instance, the assumption

$$\mathcal{F}f(0) = 0, \qquad \text{that is} \qquad \int_{\mathbb{R}^N} f(x)dx = 0.$$

The notion of Fourier transform has to be settled in a suitable framework, such as Lizorkin's spaces, defined as the subspace of Schwartz functions which are orthogonal to polynomials, namely :

$$\Phi = \left\{\varphi(x)\,|\varphi \in \mathcal{S}(\mathbb{R}^N),\ \int x^j\varphi(x) = 0,\ \forall\,|j| \in \mathbb{N}_0\right\}.$$

Again, the convolution with a log-kernel does not yield enough L^1_{loc}-regularity to provide a notion of distributional solution in the general context of $\mathcal{S}(\mathbb{R}^N)$. See [27, Chapter 5] for more details.
Notice that when N is even, $(-\Delta)^{N/2}$ is an integer order operator, so its fundamental solution in $\mathbb{R}^N$ is known, see e.g. [22, Proposition 22].
When N is odd, the fractional case, an alternative approach to circumvent the loss of regularity in the borderline case $2s = N$ is given in [18] (see

also [17, 19, 23]). We next recall some ideas from [18]. The argument is simple, the $N/2$-Laplacian can be seen as the composition of the $1/2$-Laplacian with the Laplacian of integer order $(N-1)/2$. The following proposition ensures that the logarithmic potential I_N is the fundamental solution of the $N/2$-Laplacian in this sense.

Proposition 2.3. *[18, Lemma A.2] Let $N \geq 3$ be an odd integer number and define*

$$\Phi(x) := (-\Delta)^{\frac{N-1}{2}} I_N(x) = \frac{c_n}{|x|^{N-1}}.$$

Then Φ is the fundamental solution of $(-\Delta)^{\frac{1}{2}}$ in $\mathbb{R}^N$, in the sense that for all $f \in L^1(\mathbb{R}^N)$ it holds that $\Phi * f \in L_{\frac{1}{2}}(\mathbb{R}^N)$ and that

$$\langle (-\Delta)^{\frac{1}{2}}(\Phi * f), \varphi \rangle := \int_{\mathbb{R}^N} (\Phi * f)(x)(-\Delta)^{\frac{1}{2}}\varphi(x)\, dx = \int_{\mathbb{R}^N} f\varphi\, dx,$$

for all $\varphi \in \mathcal{S}(\mathbb{R}^N)$.

However, it is not straightforward from here that I_N is the fundamental solution of $(-\Delta)^{N/2}$ in the sense of Definition 2.2. Actually, by interpreting the $N/2$-Laplacian (when N is odd) as the composition of the Laplacian of (integer) order $(N-1)/2$ and the $1/2$-Laplacian, one has that Definition 2.2 turns out to be equivalent to the following

Definition 2.4. *[18, Definition 1.1] Given $f \in \mathcal{S}'(\mathbb{R}^N)$, we say u is a solution of (2.3) if*

$$u \in W^{N-1,1}_{\mathrm{loc}}(\mathbb{R}^N), \qquad \Delta^{\frac{N-1}{2}} u \in L_{\frac{1}{2}}(\mathbb{R}^N),$$

and

$$\int_{\mathbb{R}^N} (-\Delta)^{\frac{N-1}{2}} u (-\Delta)^{\frac{1}{2}} \varphi\, dx = \langle f, \varphi \rangle$$

for all $\varphi \in \mathcal{S}(\mathbb{R}^N)$.

Indeed, we have

Proposition 2.5. *[18, Proposition 2.6] Let $f \in L^1(\mathbb{R}^N)$. Then u is a solution of (2.3) in the sense of Definition 2.4 if and only if u is a solution in the sense of Definition 2.2.*

As already pointed out, the convolution $I_N * f$ itself does not provide in general a distributional solution of (2.3). Nevertheless, a suitable modification of the logarithmic potential I_N, from one side yields enough L^1_{loc}-regularity, on the other side it is the fundamental solution of the $N/2$-Laplacian, in the distributional sense of Definition 2.2.

Lemma 2.6. *[18, Lemma 2.3] Let $f \in L^1(\mathbb{R}^N)$, and for all $x \in \mathbb{R}^N$,*

$$\tilde{v}(x) := \frac{1}{\gamma_N} \int_{\mathbb{R}^N} \log\left(\frac{1+|y|}{|x-y|}\right) f(y)\, dy.$$

Then $\tilde v \in W^{N-1,1}_{\mathrm{loc}}(\mathbb{R}^N)$ *and*

$$\int_{\mathbb{R}^N} \tilde v(x)(-\Delta)^{\frac{N}{2}}\varphi(x)\,dx = \int_{\mathbb{R}^N} (-\Delta)^{\frac{N-1}{2}}\tilde v(x)(-\Delta)^{\frac{1}{2}}\varphi(x)\,dx$$
$$= \int_{\mathbb{R}^N} f(x)\varphi(x)\,dx$$

for all $\varphi \in \mathcal{S}(\mathbb{R}^N)$, *i.e,* $\tilde v$ *is a distributional solution (in the sense of Definitions 2.4, 2.2) of* (2.3).

Lemma 2.7. *[18, Lemma 2.4] Let* u *be a solution of* (2.3) *in the sense of Definition 2.2 with* $f \in L^1(\mathbb{R}^N)$. *Then*

$$u = \tilde v + p,$$

where p *is a polynomial of degree at most* $n-1$.

As a consequence of what we have recalled here from [18], we are now in the position to proof Proposition 2.1.

Proof of Proposition 2.1. Let $u \in W^{1,N}L^q_{w_0}(\mathbb{R}^N)$ be a (weak) solution of (1.3). Then $F(u) \in L^p(\mathbb{R}^N)$ for any $p \geq 1$, as a consequence of Theorems 1.1 and 3.3. Let

$$\tilde v(x) = \int_{\mathbb{R}^N} \log\left(\frac{1+|y|}{|x-y|}\right) F(u(y))\,dy.$$

Let us rewrite equation (1.3) as follows

$$(-\Delta)_N u(x) + V(x)|u(x)|^{N-2}u(x)$$
$$= \tilde v(x) f(u(x)) + \big[(I_N * F(u))(x) - \tilde v(x)\big] f(u(x)) \,.$$

Set

$$\tilde\Phi(x) := I_N * F(u)(x) = \tilde v(x) + \big[(I_N * F(u))(x) - \tilde v(x)\big]$$

and recall that

$$\begin{aligned}
&(I_N * F(u))(x) - \tilde v(x) = \\
&= \frac{1}{\gamma_N}\int_{\mathbb{R}^N}\left(\log\frac{1}{|x-y|} - \log\left(\frac{1+|y|}{|x-y|}\right)\right) F(u(y))\,dy \\
&= -\frac{1}{\gamma_N}\int_{\mathbb{R}^N} \log(1+|y|)F(u(y))\,dy\,.
\end{aligned}$$

Since $u \in W^{1,N}L^q_{w_0}(\mathbb{R}^N)$, according to Theorem 3.3

$$\int_{\mathbb{R}^N} \log(1+|y|)F(u(y))\,dy < \infty,$$

hence $\kappa_N := (I_N * F(u))(x) - \tilde v(x)$ is a constant function. Hence

$$\tilde\Phi(x) = \tilde v(x) + \kappa_N \in W^{N-1,1}_{\mathrm{loc}}(\mathbb{R}^N)$$

and according to Lemma 2.6, for any $\varphi \in \mathcal{S}(\mathbb{R}^N)$ we have

$$\begin{aligned}\int_{\mathbb{R}^N} \tilde{\Phi}(-\Delta)^{\frac{N}{2}}\varphi\,dx &= \int_{\mathbb{R}^N}(-\Delta)^{\frac{N-1}{2}}\tilde{\Phi}(-\Delta)^{\frac{1}{2}}\varphi\,dx\\ &= \int_{\mathbb{R}^N}(-\Delta)^{\frac{N-1}{2}}(\tilde{v}+\kappa_N)(-\Delta)^{\frac{1}{2}}\varphi\,dx\\ &= \int_{\mathbb{R}^N}(-\Delta)^{\frac{N-1}{2}}\tilde{v}(-\Delta)^{\frac{1}{2}}\varphi\,dx\\ &= \int_{\mathbb{R}^N} F(u)\varphi\,dx.\end{aligned}$$

Indeed, we point out that since N is odd, $(-\Delta)^{\frac{N-1}{2}}$ is an integer order operator and hence $(-\Delta)^{\frac{N-1}{2}}\kappa_N = 0$. Therefore, $\tilde{\Phi}$ is a distributional solution of $(-\Delta)^{\frac{N}{2}}\phi = F(u)$, in the sense of Definition 2.2. □

Remark 2.8. Notice that

$$I_N * F(u) \in L^1_{\text{loc}}(\mathbb{R}^N)$$

since

$$\int_\Omega |I_N * F(u)|dx \le \int_{\Omega\times\mathbb{R}^N} |\log|x-y||F(u)dxdy \le$$
$$\le C_\mu \int_{\Omega\times\mathbb{R}^N}\left[\frac{1}{|x-y|^\mu} + \log(1+|x|)+\log(1+|y|)\right]F(u(y))dxdy < +\infty$$

where $\mu > 0$. Boundedness follows by the Hardy-Littlewood-Sobolev inequality and from Theorems 1.1 and 3.3. Similarly one also has

$$I_N * F(u) \in L_{\frac{N}{2}}(\mathbb{R}^N)$$

and that it is a distributional solution of

$$(-\Delta)^{N/2}\phi = F(u).$$

We conclude this preliminary section by recalling two classical versions of the Hardy–Littlewood–Sobolev inequality which will be used later on:

Proposition 2.9 (HLS inequality)**.** *Let $s, r > 1$ and $0 < \mu < N$ with $1/s + \mu/N + 1/r = 2$, $f \in L^s(\mathbb{R}^N)$ and $g \in L^r(\mathbb{R}^N)$. There exists a constant $C(s, N, \mu, r)$, independent of f, h, such that*

$$\int_{\mathbb{R}^N}\left[\frac{1}{|x|^\mu} * f(x)\right]g(x) \le C(s,N,\mu,r)\|f\|_s\|g\|_r.$$

Proposition 2.10 (Logarithmic HLS inequality)**.** *Let f, g be two non-negative functions belonging to $L\ln L(\mathbb{R}^N)$, such that $\int f\log(1+|x|) < \infty$, $\int g\log(1+|x|) < \infty$ and $\|f\|_1 = \|g\|_1 = 1$. There exists a constant C_N, independent of f, g, such that*

$$2N\int_{\mathbb{R}^N}\left[\log\frac{1}{|x|} * f(x)\right]g(x) \le C_N + \int_{\mathbb{R}^N} f\log f\,dx + \int_{\mathbb{R}^N} g\log g\,dx\ .$$

3. A log-weighted Pohozaev–Trudinger type inequality in $\mathbb{R}^N$

In this section, we prove a Pohozaev–Trudinger type inequality in the whole $\mathbb{R}^N$, with a logarithmic weight which appears just in the mass part of the energy. The prototype weight is

$$w = \log(e + |x|),$$

which plays a role only as $|x| \to +\infty$. The main result of this section is a quite involved extension of [10, Theorem 3.1], where the two dimensional case was considered. We begin with the case $q = N$ whence the case $q > N$ will be covered in Theorem 3.3.

Proof of Theorem 1.1. We perform a change of variables, by using hyperspherical coordinates in $\mathbb{R}^N$, to pass from $W^{1,N}_{w_0}(\mathbb{R}^N)$ to $W^{1,N}(\mathbb{R}^N)$ as follows

$$x = \begin{cases} x_1 = |x| \sin\theta_1 \sin\theta_2 \dots \sin\theta_{N-2} \sin\theta_{N-1} \\ x_2 = |x| \sin\theta_1 \sin\theta_2 \dots \sin\theta_{N-2} \cos\theta_{N-1} \\ x_3 = |x| \sin\theta_1 \sin\theta_2 \dots \cos\theta_{N-2} \\ \dots \\ x_N = |x| \cos\theta_1, \end{cases}$$

where $\theta_1, \dots \theta_{N-2} \in [0, \pi]$, whence $\theta_{N-1} \in [0, 2\pi)$, $|x|^2 = x_1^2 + \dots + x_N^2$. By acting only on the radial component of a point in $\mathbb{R}^N$, set

$$T(|x|) = |y|, \quad \frac{y}{|y|} = \frac{x}{|x|}, \quad |y| = |x| \sqrt[N]{\log(e + |x|)}.$$

We set $r = |x|$ and $s = |y|$, hence $s = T(r) = r\sqrt[N]{\log(e + r)}$. We obtain

$$T'(r) = \frac{N\log(e + r) + \frac{r}{e+r}}{N[\log(e + r)]^{\frac{N-1}{N}}} > 0, \quad T(0) = 0, \quad \lim_{r\to+\infty} T(r) = +\infty$$

and thus T is invertible on $\mathbb{R}^N$ (though the inverse map is not explicitly known). Set

$$v(y) := u(x)$$

or, equivalently

$$u(r \sin\theta_1 \dots \sin\theta_{N-1}, \dots, r\cos\theta_1)$$
$$v\left(T(r) \sin\theta_1 \dots \sin\theta_{N-1}, \dots, T(r)\cos\theta_1\right) .$$

Then, denoting $\theta = (\theta_1, \dots, \theta_{N-1})$ and

$$\begin{aligned} w(r, \theta) &:= u(r \sin\theta_1 \dots \sin\theta_{N-1}, \dots, r\cos\theta_1) \\ \widetilde{w}(s, \theta) &:= v\left(s \sin\theta_1 \dots \sin\theta_{N-1}, \dots, s\cos\theta_1\right), \\ w(r, \theta) &= \widetilde{w}\left(T(r), \theta\right), \end{aligned}$$

we compute

$$\begin{aligned} &w_r(r, \theta) = \widetilde{w}_s\left(T(r), \theta\right) T'(r), \\ &w_{\theta_i}\left(T(r), \theta\right) = \widetilde{w}_{\theta_i}\left(T(r), \theta\right) \quad i = 1, \dots, N-1. \end{aligned}$$

Therefore

$$
\begin{aligned}
&\int_{\mathbb{R}^N} |\nabla v|^N \, dy \\
= &\int_0^{2\pi} \int_0^{\pi} \sin\theta_{N-2} \cdots \int_0^{\pi} \sin^{N-2}\theta_1 \int_0^{+\infty} \Bigg[\widetilde{w}_s^2(s,\theta) + \frac{\widetilde{w}_{\theta_1}^2(s,\theta)}{s^2} + \dots \\
&+ \frac{\widetilde{w}_{\theta_{N-1}}^2(s,\theta)}{s^2 \sin^2\theta_1 \dots \sin^2\theta_{N-2}} \Bigg]^{\frac{N}{2}} s^{N-1} ds \, d\theta_1 \dots d\theta_{N-2} \, d\theta_{N-1} \\
= &\int_0^{2\pi} \int_0^{\pi} \sin\theta_{N-2} \cdots \int_0^{\pi} \sin^{N-2}\theta_1 \int_0^{+\infty} \Bigg[\widetilde{w}_s^2(T(r),\theta) + \frac{\widetilde{w}_{\theta_1}^2(T(r),\theta)}{T^2(r)} + \dots \\
&+ \frac{\widetilde{w}_{\theta_{N-1}}^2(T(r),\theta)}{T^2(r) \sin^2\theta_1 \dots \sin^2\theta_{N-2}} \Bigg]^{\frac{N}{2}} T^{N-1}(r) T'(r) dr \, d\theta_1 \dots d\theta_{N-2} \, d\theta_{N-1} \\
= &\int_0^{2\pi} \int_0^{\pi} \sin\theta_{N-2} \cdots \int_0^{\pi} \sin^{N-2}\theta_1 \int_0^{+\infty} \Bigg[\frac{w_r^2(r,\theta)}{[T'(r)]^2} + \frac{w_{\theta_1}^2(r,\theta)}{r^2} \frac{r^2}{T^2(r)} + \dots \\
&+ \frac{w_{\theta_{N-1}}^2(r,\theta)}{r^2} \frac{r^2}{T^2(r) \sin^2\theta_1 \dots \sin^2\theta_{N-2}} \Bigg]^{\frac{N}{2}} T^{N-1} T' dr d\theta_1 \dots d\theta_{N-2} d\theta_{N-1}.
\end{aligned}
$$

Now, since

$$
\frac{1}{[T'(r)]^2} = \frac{[\log(e+r)]^{\frac{2(N-1)}{N}}}{\left[\log(e+r) + \frac{r}{N(e+r)}\right]^2}, \quad \frac{r^2}{T^2(r)} = \frac{1}{[\log(e+r)]^{2/N}}
$$

we get

$$
\frac{N}{N+1} \frac{r^2}{T^2(r)} < \frac{1}{[T'(r)]^2} < \frac{r^2}{T^2(r)} \ .
$$

Thus we have

$$
\begin{aligned}
&\left(\frac{N}{N+1}\right)^{\frac{N}{2}} \int_0^{2\pi}\int_0^{\pi} \sin\theta_{N-2}\cdots\int_0^{\pi} \sin^{N-2}\theta_1 \\
&\qquad\cdot \int_0^{+\infty}\left[w_r^2(r,\theta)+\frac{w_{\theta_1}^2(r,\theta)}{r^2}+\dots\right.\\
&\qquad\left.+\frac{w_{\theta_{N-1}}^2(r,\theta)}{r^2\sin^2\theta_1\dots\sin^2\theta_{N-2}}\right]^{\frac{N}{2}}\frac{r^N T'(r)}{T(r)}dr\,d\theta_1\dots d\theta_{N-2}\,d\theta_{N-1}\\
&\qquad\le \int_{\mathbb{R}^N}|\nabla v|^N dy_1\dots dy_N\\
&\le \int_0^{2\pi}\int_0^{\pi}\sin\theta_{N-2}\cdots\int_0^{\pi}\sin^{N-2}\theta_1\int_0^{+\infty}\left[w_r^2(r,\theta)+\frac{w_{\theta_1}^2(r,\theta)}{r^2}+\dots\right.\\
&\qquad\left.+\frac{w_{\theta_{N-1}}^2(r,\theta)}{r^2\sin^2\theta_1\dots\sin^2\theta_{N-2}}\right]^{\frac{N}{2}}\frac{r^N T'(r)}{T(r)}dr\,d\theta_1\dots d\theta_{N-2}\,d\theta_{N-1}\ .
\end{aligned}
$$

On the one hand, from

$$
\frac{r^N T'(r)}{T(r)} = r^{N-1}\left[1+\frac{r}{N(e+r)\log(e+r)}\right]
$$

one has

$$
r^{N-1} < \frac{r^N T'(r)}{T(r)} < \frac{N+1}{N} r^{N-1} \tag{3.1}
$$

and then

$$
\left(\frac{N}{N+1}\right)^{\frac{N}{2}}\int_{\mathbb{R}^N}|\nabla u|^N\,dx < \int_{\mathbb{R}^N}|\nabla v|^N\,dy < \frac{N+1}{N}\int_{\mathbb{R}^N}|\nabla u|^N\,dx. \tag{3.2}
$$

On the other hand,

$$
\begin{aligned}
&\int_{\mathbb{R}^N}|v|^N\,dy\\
&= \int_0^{2\pi}\int_0^{\pi}\sin\theta_{N-2}\cdots\int_0^{\pi}\sin^{N-2}\theta_1\int_0^{+\infty}|\widetilde{w}(s,\theta)|^N s^{N-1}ds d\theta_1\dots d\theta_{N-1}\\
&= \int_0^{2\pi}\int_0^{\pi}\sin\theta_{N-2}\cdots\int_0^{\pi}\sin^{N-2}\theta_1\int_0^{+\infty}|\widetilde{w}(T(r),\theta)|^N T' T^{N-1}dr\dots d\theta_{N-1}\\
&= \int_0^{2\pi}\int_0^{\pi}\sin\theta_{N-2}\cdots\int_0^{\pi}\sin^{N-2}\theta_1\int_0^{+\infty}|w(r,\theta)|^N T' T^{N-1}dr\,d\theta_1\dots d\theta_{N-1}.
\end{aligned}
$$

Notice that

$$\begin{aligned} T'(r)T^{N-1}(r) &= r^{N-1}\left[\log(e+r) + \frac{r}{N(e+r)}\right] \\ &= r^{N-1}\log(e+r)\left[1 + \frac{r}{N(e+r)\log(e+r)}\right] \\ &= \frac{r^N T'(r)}{T(r)}\log(e+r) \end{aligned} \tag{3.3}$$

and hence

$$\int_{\mathbb{R}^N} |v|^N dy = \int_0^{2\pi}\int_0^{\pi} \sin\theta_{N-2}\dots$$
$$\dots\int_0^{\pi}\sin^{N-2}\theta_1\int_0^{+\infty}|w(r,\theta)|^N\frac{r^N T'}{T}\log(e+r)dr\dots d\theta_{N-1}\ .$$

By (3.1),

$$\int_{\mathbb{R}^N}|u|^N\log(e+|x|)dx < \int_{\mathbb{R}^N}|v|^N\,dy < \frac{N+1}{N}\int_{\mathbb{R}^N}|u|^N\log(e+|x|)dx\ .$$

Finally,

$$\left(\frac{N}{N+1}\right)^{\frac{N}{2}}\|u\|_w^N < \|v\|^N < \frac{N+1}{N}\|u\|_w^N\ . \tag{3.4}$$

We have hence proved that the map

$$\begin{array}{rcl} \mathfrak{T}: W_w^{1,N}(\mathbb{R}^N) & \to & W_0^{1,N}(\mathbb{R}^N) \\ u & \mapsto & v \end{array}$$

is invertible, continuous and with continuous inverse. Then, similarly as above,

$$\int_{\mathbb{R}^N}\phi_N\left(\alpha|u(x)|^{\frac{N}{N-1}}\right)\log(e+|x|)dx$$
$$= \int_0^{2\pi}\cdots\int_0^{\pi}\sin^{N-2}\theta_1$$
$$\int_0^{+\infty}\phi_N\left(\alpha|w(r,\theta)|^{\frac{N}{N-1}}\right)\log(e+r)r^{N-1}dr\,d\theta_1\dots d\theta_{N-1}$$
$$= \int_0^{2\pi}\cdots\int_0^{\pi}\sin^{N-2}\theta_1$$
$$\int_0^{+\infty}\phi_N\left(\alpha|\widetilde{w}(T(r),\theta)|^{\frac{N}{N-1}}\right)\frac{\log(e+r)r^{N-1}}{T'(r)T(r)^{N-1}}T'(r)T(r)^{N-1}dr\,d\theta_1\dots d\theta_{N-1}$$
$$\le \int_0^{2\pi}\cdots\int_0^{\pi}\sin^{N-2}\theta_1\int_0^{+\infty}\phi_N\left(\alpha|\widetilde{w}(\rho,\theta)|^{\frac{N}{N-1}}\right)\rho^{N-1}d\rho\,d\theta_1\dots d\theta_{N-1}$$
$$= \int_{\mathbb{R}^N}\phi_N\left(\alpha|v|^{\frac{N}{N-1}}\right)dx < +\infty$$

for any $\alpha > 0$, where we have used (3.1), (3.3), and [21, Theorem 1.1] in the last line. The uniform bound (1.8) follows directly from (3.4). Indeed, for any $u \in W^{1,N}_w(\mathbb{R}^N)$ and $\alpha \le \alpha_N \left(\frac{N}{N+1}\right)^{1/(N-1)}$ one has

$$\begin{aligned}
&\int_{\mathbb{R}^N} \phi_N\left(\alpha\left(\frac{|u|}{\|u\|_w}\right)^{\frac{N}{N-1}}\right)\log(e+|x|)dx \\
&\qquad = \int_0^{2\pi}\cdots\int_0^{\pi} \sin^{N-2}\theta_1 \\
&\int_0^{+\infty} \phi_N\left(\alpha\left(\frac{|w(r,\theta)|}{\|u\|_w}\right)^{\frac{N}{N-1}}\right)\log(e+r)r^{N-1}dr\, d\theta_1\ldots d\theta_{N-1} \\
&\qquad = \int_0^{2\pi}\cdots\int_0^{\pi} \sin^{N-2}\theta_1 \\
&\int_0^{+\infty} \phi_N\left(\alpha\left(\frac{|\widetilde{w}(T(r),\theta)|}{\|u\|_w}\right)^{\frac{N}{N-1}}\right)\frac{\log(e+r)r^{N-1}}{T'(r)T(r)^{N-1}}T'(r)T(r)^{N-1}dr\ldots d\theta_{N-1} \\
&\qquad \le \int_0^{2\pi}\cdots\int_0^{\pi} \sin^{N-2}\theta_1 \\
&\int_0^{+\infty} \phi_N\left(\alpha_N\left(\frac{|\widetilde{w}(\rho,\theta)|}{\|v\|}\right)^{\frac{N}{N-1}}\right)\rho^{N-1}d\rho\, d\theta_1\ldots d\theta_{N-1} \\
&\qquad = \int_{\mathbb{R}^N} \phi_N\left(\alpha_N\left(\frac{|v|}{\|v\|}\right)^{\frac{N}{N-1}}\right)dx < C
\end{aligned}$$

by (3.4) and using [21, Theorem 1.1] for the last inequality. □

As a byproduct of this embedding result one has the continuity of a weighted Pohozaev–Trudinger functional on $W^{1,N}_w(\mathbb{R}^N)$, namely we have the following

Corollary 3.1. *For any $\alpha > 0$, the functional*

$$u \longmapsto \int_{\mathbb{R}^N} \phi_N\left(\alpha|u|^{\frac{N}{N-1}}\right)\log(e+|x|)\,dx$$

is continuous on $W^{1,N}_w(\mathbb{R}^N)$.

Remark 3.2. The value $\alpha_N\left(\frac{N}{N+1}\right)^{1/(N-1)}$ in (1.8) is not sharp and we conjecture that the sharp value is α_N as in the Moser case [25], though it is somehow delicate and still out of reach.

Next we consider the case in which the asymptotic growth of the nonlinearity near zero is a power $q > N$. We prove the following

Theorem 3.3. *Let $f\colon \mathbb{R} \to [0,+\infty)$ satisfying (f_1) and let $q > N$. Then, the space $W^{1,N}L^q_w(\mathbb{R}^N)$ embeds into the weighted Orlicz space $L_F(\mathbb{R}^N, \log(e+$*

$|x|)dx)$. *More precisely,*

$$\int_{\mathbb{R}^N} F(\alpha|u|)\log(e+|x|)\,dx<\infty, \quad \forall u\in W^{1,N}L^q_w(\mathbb{R}^N),\ \forall\alpha>0\ .$$

Moreover, for any $\alpha<\frac{N}{q}\left(\frac{N}{N+1}\right)^{1/N}$ *the following uniform bound holds*

$$\sup_{\|u\|^N_{q,w}\le 1}\int_{\mathbb{R}^N} F\left(\alpha|u|\right)\log(e+|x|)dx<+\infty\ . \tag{3.5}$$

Proof. For $u\in W^{1,N}L^q_w(\mathbb{R}^N)$ set

$$v:=\begin{cases}|u|^{\frac{q}{N}}, & |u|<1,\\ |u|, & |u|\ge 1,\end{cases}$$

which belongs to $W^{1,N}_w(\mathbb{R}^N)$. Indeed,

$$\begin{aligned}\|v\|^N_w &= \int_{\mathbb{R}^N}|\nabla v|^N\,dx+\int_{\mathbb{R}^N}|v|^N\log(e+|x|)\,dx\\ &\le\left(\frac{q}{N}\right)^N\int_{\mathbb{R}^N}|\nabla u|^N\,dx+\int_{\mathbb{R}^N}|u|^q\log(e+|x|)\,dx\le\left(\frac{q}{N}\right)^N\|u\|^N_{q,w}.\end{aligned}$$

Now recall from (1.4) that for any $\|u\|_{q,w}\le 1$ one has:

- if $p\le\frac{1}{N-1}$,

$$\begin{aligned}&\int_{\mathbb{R}^N}F(\alpha|u|)\log(e+|x|)\,dx\\ &\le C\left(\int_{\mathbb{R}^N}\alpha^q|u|^q\log(e+|x|)\,dx\right.\\ &\left.+\alpha^{p-\frac{1}{N-1}}\int_{\{|u|>1\}}e^{\alpha_N(\alpha|u|)^{\frac{N}{N-1}}}\log(e+|x|)\,dx\right)\\ &=C\left(\int_{\mathbb{R}^N}\alpha^q|u|^q\log(e+|x|)\,dx\right.\\ &\left.+\alpha^{p-\frac{1}{N-1}}\int_{\{|v|>1\}}e^{\alpha_N(\alpha\|v\|_w)^{\frac{N}{N-1}}(|v|/\|v\|_w)^{\frac{N}{N-1}}}\log(e+|x|)\,dx\right)\\ &\le C\left(\alpha^q\|u\|^q_{L^q(wdx)}\right.\\ &\left.+\alpha^{p-\frac{1}{N-1}}\int_{\{|v|>1\}}\phi_N\left(\alpha_N\left(\frac{N}{N+1}\right)^{\frac{1}{N-1}}\left(\frac{|v|}{\|v\|_w}\right)^{\frac{N}{N-1}}\right)\log(e+|x|)\,dx\right)\le C\end{aligned}$$

(where the last bound is independent of u in the unit ball of $W^{1,N}L^q_w(\mathbb{R}^N)$);

• if $p > \frac{1}{N-1}$,

$$
\begin{aligned}
&\int_{\mathbb{R}^N} F(\alpha|u|)\log(e+|x|)\,dx \\
&\leq C\Big(\int_{\mathbb{R}^N} \alpha^q |u|^q \log(e+|x|)\,dx \\
&+\int_{\{|u|>1\}} (\alpha|u|)^{p-\frac{1}{N-1}}\, e^{\alpha_N(\alpha|u|)^{\frac{N}{N-1}}} \log(e+|x|)\,dx\Big) \\
&= C\Big(\alpha^q \|u\|^q_{L^q(wdx)} \\
&+\int_{\{|v|>1\}} (\alpha|v|)^{p-\frac{1}{N-1}}\, e^{\alpha_N(\alpha\|v\|_w)^{\frac{N}{N-1}}(|v|/\|v\|_w)^{\frac{N}{N-1}}} \log(e+|x|)\,dx\Big) \\
&\leq C\Big(\alpha^q \|u\|^q_{L^q(wdx)} \\
&+\alpha^{p-\frac{1}{N-1}} \int_{\{|u|>1\}} |u|^{r'(p-\frac{1}{N-1})} \int_{\{|v|>1\}} e^{r\alpha_N(\alpha\|v\|_w)^{\frac{N}{N-1}}(|v|/\|v\|_w)^{\frac{N}{N-1}}} \log(e+|x|)\,dx\Big)
\end{aligned}
$$

where $r = \left(\frac{N}{N+1}\right)^{\frac{1}{N-1}}\left(\frac{N}{\alpha q}\right)^{\frac{N}{N-1}} > 1$, provided $\alpha < \frac{N}{q}\left(\frac{N}{N+1}\right)^{\frac{1}{N}}$ and r' is the Young conjugate of r. Hence,

$$
\begin{aligned}
&\int_{\mathbb{R}^N} F(\alpha|u|)\log(e+|x|)\,dx \\
&\leq C\Big(\alpha^q \|u\|^q_{L^q(wdx)} + \\
&\alpha^{p-\frac{1}{N-1}} \int_{\{|v|>1\}} \phi_N\left(\left(\frac{N}{N+1}\right)^{\frac{1}{N-1}} \alpha_N \left(\frac{|v|}{\|v\|_w}\right)^{\frac{N}{N-1}}\right) \log(e+|x|)\,dx\Big) \leq C
\end{aligned}
$$

(where the constant C does not depend on u in the unit ball of $W^{1,N}L^q_w(\mathbb{R}^N)$). □

Remark 3.4. The analogous of Corollary 3.1 holds also in the case $q > N$.

4. The variational framework: proof of Theorem 1.2

The energy functional we consider is the following

$$\mathcal{I}_V(u) = \frac{1}{N}\|u\|_V^N - \mathcal{F}(u),$$

with

$$
\begin{aligned}
\mathcal{F}(u) &= \frac{1}{2}\int_{\mathbb{R}^N} (I_N * F(u))(x) F(u(x))\,dx \\
&= \frac{1}{2\gamma_N}\int_{\mathbb{R}^N} \left(\log\frac{1}{|\cdot|} * F(u)\right)(x) F(u(x))\,dx \qquad (4.1)\\
&= \frac{1}{2\gamma_N}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \log\frac{1}{|x-y|} F(u(y))F(u(x))\,dx\,dy.
\end{aligned}
$$

The regularity of $\mathcal{I}_V$ can be proved following line by line [10, Lemma 4.2], namely one has

Theorem 4.1. *The energy functional $\mathcal{I}_V$ is of class C^1 on $W^{1,N}_V L^q_{w_0}(\mathbb{R}^N)$.*

4.1. **Mountain pass geometry.** Let us focus here on the geometry of the energy functional $\mathcal{I}_V$.

Lemma 4.2. *The energy functional $\mathcal{I}_V$ satisfies the following:*

(*i*) *there exist $\rho, \delta_0 > 0$ such that $\mathcal{I}_V|_{S_\rho} \geq \delta_0$ for all $u \in S_\rho$*

$$S_\rho := \{u \in W^{1,N}_V L^q_{w_0}(\mathbb{R}^N) \mid \|u\|_{q,V,w_0} = \rho\};$$

(*ii*) *there exists $e \in W^{1,N}_V L^q_{w_0}(\mathbb{R}^N)$, $\|e\|_{q,V,w_0} > \rho$ such that $\mathcal{I}_V(e) < 0$.*

Proof. Throughout the proof, constants may change from line to line. Notice that from the logarithmic Hardy-Littlewood-Sobolev inequality, Proposition 2.10, we have

$$\mathcal{F}(u) \leq \|F(u)\|_{L^1(\mathbb{R}^N)} \bigg(C_N \|F(u)\|_{L^1(\mathbb{R}^N)} + \frac{1}{N} \int_{\mathbb{R}^N} F(u) \log F(u)\, dx \\ - \frac{1}{N} \|F(u)\|_{L^1(\mathbb{R}^N)} \log \|F(u)\|_{L^1(\mathbb{R}^N)} \bigg).$$

Since $\|u\|_V \leq \|u\|_{q,V,w_0} = \rho$, for ρ small, by (1.4) and noting that for any $s > s_0$ and $p > 0$

$$s^{p-\frac{1}{N-1}} \phi_N \left(\alpha_N s^{\frac{N}{N-1}} \right) \leq C_{N,p} s^{2N} \phi_N \left(2\alpha_N s^{\frac{N}{N-1}} \right)$$

we have, for some $r > 1$,

$$\|F(u)\|_{L^1(\mathbb{R}^N)} \leq C \|u\|_q^q + C \left[\int_{\mathbb{R}^N} |u|^{2Nr'} dx \right]^{\frac{1}{r'}} \cdot \\ \cdot \left[\int_{\mathbb{R}^N} \phi_N \left(2\alpha_N r \|\nabla u\|_N^{\frac{N}{N-1}} \left(\frac{|u|}{\|\nabla u\|_N} \right)^{\frac{N}{N-1}} \right) dx \right]^{\frac{1}{r}} \\ \leq C \left(\|u\|_V^q + \|u\|_V^{2N} \right) \leq C \left(\|u\|_{q,V,w_0}^N + \|u\|_{q,V,w_0}^{2N} \right) \leq C \|u\|_{q,V,w_0}^N$$

and in turn,

$$\left| \|F(u)\|_{L^1(\mathbb{R}^N)} \log \|F(u)\|_{L^1(\mathbb{R}^N)} \right| \leq C \|u\|_{q,V,w_0}^N \left| \log \|u\|_{q,V,w_0} \right|.$$

Moreover, combining (f_1) with elementary estimates for the function $t|\log t|$, and since $q > N$, if $p \geq \frac{1}{N-1}$ we have

$$\begin{aligned}\int_{\mathbb{R}^N} F(u)\log F(u)\,dx &\leq C\int_{\mathbb{R}^N} F(u)|\log F(u)|\,dx\\ &\leq C\left(\int_{\{|u|<1\}}|u|^N+\int_{\{|u>1|\}}F(u)\log|F(u)|\,dx\right)\\ &\leq C\left(\|u\|_V^N+\int_{\{|u>1|\}}|u|^{p+1}e^{\alpha_N|u|^{\frac{N}{N-1}}}\,dx\right)\leq C\left(\|u\|_V^N+\|u\|_V^{p+1}\right),\end{aligned}$$

for small $\|u\|_V$, where we have also applied the classical Moser inequality on the whole $\mathbb{R}^N$. Similarly, when $p < \frac{1}{N-1}$ we obtain

$$\begin{aligned}\int_{\mathbb{R}^N} F(u)\log F(u)\,dx &\leq C\int_{\mathbb{R}^N} F(u)|\log F(u)|\,dx\\ &\leq C\left(\int_{\{|u|<1\}}|u|^N+\int_{\{|u>1|\}}F(u)\log|F(u)|\,dx\right)\\ &\leq C\left(\|u\|_V^N+\int_{\{|u>1|\}}|u|^{\frac{N}{N-1}}e^{\alpha_N|u|^{\frac{N}{N-1}}}\,dx\right)\leq C\left(\|u\|_V^N+\|u\|_V^{\frac{N}{N-1}}\right).\end{aligned}$$

Combining the two previous estimates we end up with the following

$$\begin{aligned}\mathcal{F}(u) &\leq C\|u\|_{q,V,w_0}^N\left(\|u\|_{q,V,w_0}^N+\|u\|_{q,V,w_0}^N|\log\|u\|_{q,V,w_0}|+\|u\|_{q,V,w_0}^{\frac{N}{N-1}}\right)\\ &\leq C\|u\|_{q,V,w_0}^{\frac{N^2}{N-1}}.\end{aligned}$$

Hence, for ρ small enough one has

$$\mathcal{I}_V(u)\geq\frac{1}{N}\|u\|_{q,V,w_0}^N-C\|u\|_{q,V,w_0}^{\frac{N^2}{N-1}}=\delta_0>0,$$

with δ_0 depending only on ρ, which proves (i).
In order to prove (ii), let us consider a smooth function $e\in W^{1,N}_{q,V,w_0}(\mathbb{R}^N)$, supported in $B_{1/4}$. Since $F(e(x)), F(e(y))\neq 0$ only for x, y $B_{1/4}$, let us evaluate

$$\begin{aligned}\mathcal{F}(e) &= \frac{1}{2}\int_{\mathbb{R}^N}(I_N * F(e))(x)F(e(x))\,dx\\ &= \frac{1}{2\gamma_N}\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\log\frac{1}{|x-y|}F(e(y))\,dy\right)F(e(x))\,dx\\ &\geq\frac{\log 2}{2\gamma_N}\left(\int_{\{|x|\leq\frac{1}{4}\}}F(e)\,dx\right)^2,\end{aligned}$$

from which we have

$$\begin{aligned}\mathcal{I}_V(te) &= \frac{1}{N}t^N\|e\|_V^N - \mathcal{F}(te)\\ &\leq \frac{1}{N}t^N\|e\|_V^N - \frac{\log 2}{2\gamma_N}\left(\int_{\{|x|\leq\frac{1}{4}\}} F(te)\,dx\right)^2 \to -\infty,\end{aligned}$$

as $t \to +\infty$, since F has exponential growth. □

By Ekeland's Variational Principle, there exists a Palais-Smale (PS in the sequel) sequence $\{u_n\} \in W_V^{1,N}L_{w_0}^q(\mathbb{R}^N)$ such that

$$\mathcal{I}_V'(u_n) \to 0, \qquad \mathcal{I}_V(u_n) \to m_V,$$

where m_V is the mountain pass level,

$$0 < m_V := \inf_{\gamma\in\Gamma}\max_{t\in[0,1]}\mathcal{I}_V(\gamma(t)),$$

and

$$\Gamma = \left\{\gamma \in C^1\left([0,1], W_{q,V,w_0}^{1,N}(\mathbb{R}^N)\right) \mid \gamma(0) = 0, \mathcal{I}_V(\gamma(1)) < 0\right\}.$$

Next, a few efforts are needed to extend to the higher dimensional case $N \geq 3$, the mountain pass level estimates carried out in [10, Lemma 5.2].

Lemma 4.3. *The mountain pass level m_V satisifes*

$$m_V < \frac{1}{N}\ .$$

Proof. We are reduced to exhibit a function $v \in W_V^{1,N}L_{w_0}^q(\mathbb{R}^N)$ with unitary norm and such that

$$\max_{t\geq 0}\mathcal{I}_V(tv) < \frac{1}{N}.$$

For this purpose let us introduce the following Moser type functions for all $n \geq 1$, supported in B_ρ for some $\rho > 0$,

$$\overline{w}_n = \begin{cases} C_n \log n, & 0 \leq |x| \leq \dfrac{\rho}{n},\\ C_n \log \dfrac{\rho}{|x|}, & \dfrac{\rho}{n} \leq |x| \leq \rho,\\ 0, & \rho \leq |x|,\end{cases}$$

with

$$C_n = (\omega_{N-1}\log n)^{-\frac{1}{N}}. \tag{4.2}$$

We have

$$\int_{\mathbb{R}^N} |\nabla\overline{w}_n|^N\,dx = \omega_{N-1}C_n^N\int_{\frac{\rho}{n}}^{\rho} r^{-1}\,dr = \omega_{N-1}C_n^N\log n = 1,$$

as well as

$$
\begin{aligned}
&\int_{\mathbb{R}^N} V(x)|\overline{w}_n|^N\,dx \\
&\quad \le \sup_{B_\rho} V \left(C_n^N (\log n)^N \int_{B_{\rho/n}} dx + \int_{B_\rho\setminus B_{\rho/n}} C_n^N \left(\log\frac{\rho}{|x|}\right)^N dx \right) \\
&\quad = C_n^N \omega_{N-1} \sup_{B_\rho} V \left((\log n)^N \int_0^{\frac{\rho}{n}} r^{N-1}\,dr + \int_{\frac{\rho}{n}}^{\rho} \left(\log\frac{\rho}{r}\right)^N r^{N-1}\,dr \right) \\
&\qquad\qquad \le \frac{\sup_{B_\rho} V}{\log n} \left(\frac{(\log n)^N}{N n^N}\rho^N + \int_{\frac{\rho}{n}}^{\rho} \left(\log\frac{\rho}{r}\right)^N r^{N-1}\,dr \right),
\end{aligned}
$$

recalling from (4.2) that $C_n^N\omega_{N-1} = 1/\log n$. Now

$$
\begin{aligned}
&\int_{\mathbb{R}^N} |\overline{w}_n|^q \log(1+|x|)\,dx = (C_n \log n)^q \int_{B_{\rho/n}} \log(1+|x|)\,dx \\
&\qquad + C_n^q \int_{B_\rho\setminus B_{\rho/n}} \log(1+|x|) \left|\log\frac{\rho}{|x|}\right|^q dx \\
&= C_n^q\omega_{N-1} \left(\log^q n \int_0^{\frac{\rho}{n}} \log(1+r) r^{N-1}\,dr + \int_{\frac{\rho}{n}}^{\rho} \log(1+r)\log^q\left(\frac{\rho}{r}\right) r^{N-1}\,dr \right) \\
&\quad \le C_n^q\omega_{N-1}\log^q n \left(\frac{\rho}{n}\right)^{N+1} + C_n^q\omega_{N-1}\log(e+\rho) \int_{\frac{\rho}{n}}^{\rho} \left(\log\frac{\rho}{r}\right)^q r^{N-1}\,dr.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\|\overline{w}_n\|_{q,V,w_0}^N &= \int_{\mathbb{R}^N} \left(|\nabla \overline{w}_n|^N + V|\overline{w}_n|^N \right) dx + \\
&\qquad \left(\int_{\mathbb{R}^N} |\overline{w}_n|^q \log(1+|x|)\,dx \right)^{\frac{N}{q}} \\
&\le 1 + \frac{\sup_{B_\rho} V}{\log n} \int_{\frac{\rho}{n}}^{\rho} \left(\log\frac{\rho}{r}\right)^N r^{N-1}\,dr + O\left(\frac{(\log n)^{N-1}}{n^N}\right) \\
&\quad + \frac{\omega_{N-1}^{\frac{N-q}{q}}}{\log n}(\log(1+\rho))^{\frac{N}{q}} \left(\int_{\frac{\rho}{n}}^{\rho} \left(\log\frac{\rho}{r}\right)^q r^{N-1}\,dr \right)^{\frac{N}{q}} \\
&\qquad + O\left(\frac{1}{\log n} \left(\frac{\log n}{n^{\frac{N+1}{q}}} \right)^N \right).
\end{aligned}
$$

Let us estimate explicitly integrals in the above inequality, as for $k\in\mathbb{N}$ we have

$$
\int \left(\log\frac{\rho}{r}\right)^k r^{N-1}\,dr = \frac{r^N}{N} \sum_{j=0}^{k} \left(\log\frac{\rho}{r}\right)^{k-j} \frac{k(k-1)\dots(k-j+1)}{N^j},
$$

and from the estimate

$$\int \left(\log\frac{\rho}{r}\right)^{q} r^{N-1}\,dr \;\le\; \int \left(\log\frac{\rho}{r}\right)^{[q]} r^{N-1}\,dr + \int \left(\log\frac{\rho}{r}\right)^{[q+1]} r^{N-1}\,dr,$$

we get

$$1 \le \|\overline{w}_n\|^N_{q,V,w_0} \le 1+\delta_n, \qquad \text{with } \delta_n \to 0, \text{ as } n\to\infty$$

and

$$\delta_n = \frac{1}{\log n}\frac{\rho^N}{N}\left[\sup_{B_\rho} V\,\frac{N!}{N^N} + \omega_{N-1}^{\frac{N-q}{q}}(\log(e+\rho))^{\frac{N}{q}}\left(\frac{[q]!}{N^{[q]}} + \frac{[q+1]!}{N^{[q+1]}}\right)\right] + o\left(\frac{1}{\log n}\right) \ . \tag{4.3}$$

Then,

$$w_n = \frac{\overline{w}_n}{\sqrt[N]{1+\delta_n}}, \qquad \|w_n\|^N_{q,V,w} \le 1.$$

Claim:

$$\exists\, n\in\mathbb{N} \text{ such that } \max_{t\ge 0}\mathcal{I}_V(tw_n) < \frac{1}{N}. \tag{4.4}$$

By contradiction, suppose that for all n

$$\mathcal{I}_V(t_n w_n) := \max_{t\ge 0}\mathcal{I}_V(tw_n) \ge \frac{1}{N},$$

together with

$$\frac{d}{dt}\mathcal{I}_V(tw_n)|_{t=t_n} = 0 \ .$$

As a consequence we obtain

$$\frac{t_n^N}{N} \ge \frac{1}{N} + \frac{1}{2\gamma_N}\int_{\mathbb{R}^{2N}} \log\frac{1}{|x-y|} F(t_n w_n(y))F(t_n w_n(x))\,dx\,dy, \tag{4.5}$$

and

$$t_n^N \ge \frac{1}{\gamma_N}\int_{\mathbb{R}^{2N}} \log\frac{1}{|x-y|} F(t_n w_n(y)) f(t_n w_n(x)) t_n w_n(x)\,dx\,dy. \tag{4.6}$$

Assume $\rho \le 1/2$, thus if $x, y \in B_\rho$, then $\log(1/|x-y|) \ge 0$. Observe from (4.5) that, since w_n is supported in B_ρ,

$$t_n \ge 1.$$

Next we prove the following

$$\liminf_{n\to+\infty} t_n \le 1 \ . \tag{4.7}$$

Indeed, if not there exists some $\delta_n > 0$ such that for n large enough

$$t_n^N \ge 1+\delta_n \ . \tag{4.8}$$

Notice that

$$
\begin{aligned}
I &:= \int_{\mathbb{R}^{2N}} \log\frac{1}{|x-y|} F(t_n w_n(y)) f(t_n w_n(x)) t_n w_n(x)\, dx\, dy \\
&= \int_{B_{\frac{\rho}{n}} \times B_{\frac{\rho}{n}}} \log\frac{1}{|x-y|} F(t_n w_n(y)) f(t_n w_n(x)) t_n w_n(x)\, dx\, dy \\
&\quad + \int_{\mathbb{R}^{2N} \setminus \left(B_{\frac{\rho}{n}} \times B_{\frac{\rho}{n}}\right)} \log\frac{1}{|x-y|} F(t_n w_n(y)) f(t_n w_n(x)) t_n w_n(x)\, dx\, dy \\
&\geq \int_{B_{\frac{\rho}{n}} \times B_{\frac{\rho}{n}}} \log\frac{1}{|x-y|} F(t_n w_n(y)) f(t_n w_n(x)) t_n w_n(x)\, dx\, dy,
\end{aligned}
$$

again since $\log(1/|x-y|) \geq 0$ for $x, y \in B_\rho$, where w_n is supported. By (f_4), for any $\varepsilon \in (0, \beta/2)$ small enough, there exists $s_\varepsilon > 0$ such that for all $s > s_\varepsilon$,

$$
s^{\frac{2N-1}{N-1}} f(s) F(s) > \frac{\beta}{2} e^{2N\omega_{N-1}^{\frac{1}{N-1}} s^{\frac{N}{N-1}}}
$$

and in turn

$$
s f(s) F(s) > \frac{\beta}{2} s^{-\frac{N}{N-1}} e^{2N\omega_{N-1}^{\frac{1}{N-1}} s^{\frac{N}{N-1}}}.
$$

Therefore, by using the explicit value of w_n and the fact that $|x-y| \leq 2\rho/n$ for $x, y \in B_{\frac{\rho}{n}} \times B_{\frac{\rho}{n}}$, we have

$$
\begin{aligned}
I &\geq \frac{\beta}{2} \log\left(\frac{n}{2\rho}\right) \frac{\omega_{N-1}^2}{N^2} \left(\frac{\rho}{n}\right)^{2N} e^{2N\omega_{N-1}^{\frac{1}{N-1}} \left(\frac{t_n C_n \log n}{\sqrt[N]{1+\delta_n}}\right)^{\frac{N}{N-1}}} \left(\frac{t_n C_n \log n}{\sqrt[N]{1+\delta_n}}\right)^{\frac{-N}{N-1}} \\
&= \frac{\beta}{2} \frac{\omega_{N-1}^2}{N^2} \log\left(\frac{n}{2\rho}\right) \rho^{2N} e^{-2N\log n} e^{2N \log n \left(\frac{t_n}{\sqrt[N]{1+\delta_n}}\right)^{\frac{N}{N-1}}} \cdot \\
&\qquad \cdot \left(\frac{t_n}{\sqrt[N]{1+\delta_n}}\right)^{\frac{-N}{N-1}} \omega_{N-1}^{\frac{1}{N-1}} \frac{1}{\log n} \\
&\geq \frac{\beta}{2N^2} \omega_{N-1}^{\frac{2N-1}{N-1}} \rho^{2N} \frac{e^{2N\log n \left[\left(\frac{t_n}{\sqrt[N]{1+\delta_n}}\right)^{\frac{N}{N-1}} - 1\right]}}{\left(\frac{t_n}{\sqrt[N]{1+\delta_n}}\right)^{\frac{N}{N-1}}},
\end{aligned}
$$

recalling the value of C_n given in (4.2). It follows from (4.6) that

$$
t_n^N \geq \frac{\beta}{2N^2 \gamma_N} \omega_{N-1}^{\frac{2N-1}{N-1}} \rho^{2N} \frac{e^{2N\log n \left[\left(\frac{t_n}{\sqrt[N]{1+\delta_n}}\right)^{\frac{N}{N-1}} - 1\right]}}{\left(\frac{t_n}{\sqrt[N]{1+\delta_n}}\right)^{\frac{N}{N-1}}}. \tag{4.9}
$$

In both cases when $t_n \to \infty$, as $n \to \infty$ or when t_n stays bounded, (4.8) yields a contradiction. Thus (4.7) holds and hence

$$
\lim_{n\to\infty} t_n = 1\ .
$$

Now, from one side we have

$$e^{\log n\left[\left(\frac{t_n}{\sqrt[N]{1+\delta_n}}\right)^{\frac{N}{N-1}}-1\right]}\leq C$$

and thus

$$\left(\frac{t_n}{\sqrt[N]{1+\delta_n}}\right)^{\frac{N}{N-1}}\leq 1+O\left(\frac{1}{\log n}\right).$$

On the other side, from (4.9) we obtain

$$1+o(1)\geq t_n^{\frac{N^2}{N-1}}\geq\frac{\beta}{2N^2\gamma_N}\omega_{N-1}^{\frac{2N-1}{N-1}}\rho^{2N}e^{2N\log n\left[\left(\frac{t_n}{\sqrt[N]{1+\delta_n}}\right)^{\frac{N}{N-1}}-1\right]}(1+\delta_n)^{\frac{1}{N-1}}.$$

As a consequence we finally get

$$\begin{aligned}1+o(1)&\geq\frac{\beta}{2N^2\gamma_N}\omega_{N-1}^{\frac{2N-1}{N-1}}\rho^{2N}e^{2N\log n\left[\left(\frac{t_n}{\sqrt[N]{1+\delta_n}}\right)^{\frac{N}{N-1}}-1\right]}(1+\delta_n)^{\frac{1}{N-1}}\\&=\frac{\beta}{2N^2\gamma_N}\omega_{N-1}^{\frac{2N-1}{N-1}}\rho^{2N}e^{2N\log n\left[-\frac{\delta_n}{N-1}+o(\delta_n)\right]}.\end{aligned}$$

By substituting (4.3) in the previous inequality and letting $n\to\infty$, we end up with

$$1\geq\frac{\beta}{2N^2\gamma_N}\omega_{N-1}^{\frac{2N-1}{N-1}}\rho^{2N}e^{\frac{-2N}{N-1}\frac{\rho^N}{N}\left[\sup_{B_\rho}V\frac{N!}{N^N}+\omega_{N-1}^{\frac{N-q}{q}}(\log(1+\rho))^{\frac{N}{q}}\left(\frac{[q]!}{N^{[q]}}+\frac{[q+1]!}{N^{[q+1]}}\right)\right]}.$$

For a fixed $\rho\leq 1/2$, set

$$\nu:=\sup_{\rho\leq\frac{1}{2}}\frac{2N^2\gamma_N}{\rho^{2N}}\omega_{N-1}^{\frac{-2N+1}{N-1}}e^{\frac{2N}{N-1}\frac{\rho^N}{N}\left[\sup_{B_\rho}V\frac{N!}{N^N}+\omega_{N-1}^{\frac{N-q}{q}}(\log(1+\rho))^{\frac{N}{q}}\left(\frac{[q]!}{N^{[q]}}+\frac{[q+1]!}{N^{[q+1]}}\right)\right]}$$

to get a contradiction from (f_4), since $\beta>\nu$. □

4.2. **On the Ekeland Palais-Smale sequence.** In this section we study the behavior of the PS sequence provided by Ekeland's Variational Principle. In particular, it is a non trivial fact, in this context, that the weak limit turns out to be a nontrivial solution of the equation. Boundedness of PS sequences buys the line of [10, Lemma 6.1], to which we refer for the proof of the next

Lemma 4.4. *Assume* (V) *and* (f_1)*–*(f_4)*. Let* $\{u_n\}\subset W_V^{1,N}L_{w_0}^q(\mathbb{R}^N)$ *be an arbitrary PS sequence for* $\mathcal{I}_V$ *at level* c*, namely*

$$\mathcal{I}_V(u_n)\to c\quad\text{and}\quad\mathcal{I}_V'(u_n)\to 0\quad\text{in}\ \left(W_V^{1,N}L_{w_0}^q(\mathbb{R}^N)\right)',\quad\text{as } n\to+\infty,$$

the dual space of $W_V^{1,N}L_{w_0}^q(\mathbb{R}^N)$*. Then, the following hold:*

(*i*) $\|u_n\|_V\leq C$;

(ii)

$$\left| \int_{\mathbb{R}^N} \left[\log \frac{1}{|x|} * F(u_n) \right] F(u_n) dx \right| \leq C \ ;$$

(iii)

$$\left| \int_{\mathbb{R}^N} \left[\log \frac{1}{|x|} * F(u_n) \right] u_n f(u_n) dx \right| \leq C \ .$$

Remark 4.5. Note that, as a consequence of Lemma 4.4, we may assume the PS sequence at level c to be positive. Indeed, since u_n is bounded, we can test $\mathcal{I}'_V(u_n)$ against $u_n^- = \max(-u_n, 0)$ to get

$$\left| \int_{\{u_n<0\}} |\nabla u_n|^N + V|u_n|^N dx \right| \leq \tau_n C$$

Hence, the positive sequence $\{u_n^+\}$ is still a PS sequence at the same level c, since $F(s) = 0$ for $s \leq 0$.

From now on we will consider only positive PS sequences. Because of the exponential nonlinearity and the presence of a sign-changing logarithmic kernel, we cannot exploit standard arguments to obtain the existence of a solution as byproduct of boundedness of a PS sequence. Here it is fundamental to take advantage of the key estimate for the mountain pass level of Lemma 4.3. The next lemma is an extension of [10, Lemma 6.2]. However, it is not a virtual transcription, so that, for convenience of the reader, we recall the main steps of the proof.

Lemma 4.6. *Assume* (V) *and* (f_1)*–*(f_4)*. Let* $\{u_n\} \subset W_V^{1,N} L_{w_0}^q(\mathbb{R}^N)$ *be a (positive) PS sequence for* $\mathcal{I}_V$ *at level* $0 < c < 1/N$*. Then, for any* $1 \leq \alpha < 1/(Nc)$ *the following uniform bound holds*

$$\sup_{n\in\mathbb{N}} \int_{\mathbb{R}^N} F^\alpha(u_n)\, dx < \infty \ .$$

Proof. From Lemma 4.4, there exists $u \in W_V^{1,N}(\mathbb{R}^N)$ such that:

$$\begin{aligned} u_n &\rightharpoonup u \quad \text{in } W_V^{1,N}(\mathbb{R}^N); \\ u_n &\to u \quad \text{in } L^s_{\text{loc}}(\mathbb{R}^N) \quad \text{for any } 1 \leq s < \infty; \\ u_n &\to u \quad \text{a.e. in } \mathbb{R}^N, \end{aligned}$$

with

$$\lim_{n\to+\infty} \|u_n\|_V^N = A^N \geq \|u\|_V^N. \tag{4.10}$$

Let $G\colon \mathbb{R}^+ \to \mathbb{R}^+$,

$$G(t) := \int_0^t \sqrt[N]{\frac{N}{2}\frac{F(s)f'(s)}{f^2(s)} - \frac{N-2}{2}} ds,$$

and notice that $G \in C^1(\mathbb{R}^+)$ thanks to (f_2). By Hölder's inequality we have

$$G^N(t) \leq \left(\int_0^t ds\right)^{N-1} \int_0^t \left(\frac{N}{2}\frac{F(s)f'(s)}{f^2(s)} - \frac{N-2}{2}\right) ds$$
$$= t^N - \frac{N}{2}t^{N-1}\frac{F(t)}{f(t)} \ . \tag{4.11}$$

Set

$$v_n := G(u_n) > 0 \ .$$

Since u_n is bounded in $W_V^{1,N}(\mathbb{R}^N)$ and thanks to (f_2),

$$\int_{\mathbb{R}^N} |\nabla v_n|^N dx = \int_{\mathbb{R}^N} |\nabla u_n|^N \left(\frac{N}{2}\frac{F(u_n)f'(u_n)}{f^2(u_n)} - \frac{N-2}{2}\right) dx \leq C$$

and

$$\int_{\mathbb{R}^N} V v_n^N dx = \int_{\mathbb{R}^N} V G^N(u_n) dx \leq C \int_{\mathbb{R}^N} V u_n^N \, dx \leq C.$$

We claim that for n large enough

$$\|\nabla v_n\|_V^N \leq 1.$$

Combining the facts $\mathcal{I}_V(u_n) \to c$, (4.1) and (4.10), we have

$$\lim_{n\to+\infty} \frac{1}{\gamma_N} \int_{\mathbb{R}^N} \left[\log\left(\frac{1}{|x|}\right) * F(u_n)\right] F(u_n) dx = 2\left(\frac{A^N}{N} - c\right).$$

Moreover, since $\mathcal{I}'_V(u_n) \to 0$ in $\left(W_V^{1,N} L^q_{w_0}(\mathbb{R}^N)\right)'$, we have

$$\mathcal{I}'_V(u_n)\left[\frac{F(u_n)}{f(u_n)}\right] \to 0,$$

and hence

$$\begin{aligned} &\int_{\mathbb{R}^N} |\nabla u_n|^N \left(1 - \frac{F(u_n)f'(u_n)}{f^2(u_n)}\right) dx + \int_{\mathbb{R}^N} V u_n^{N-1}\frac{F(u_n)}{f(u_n)} dx \\ &- \frac{1}{\gamma_N}\int_{\mathbb{R}^N} \left[\log\left(\frac{1}{|x|}\right) * F(u_n)\right] F(u_n) dx = \mathrm{o}(1). \end{aligned} \tag{4.12}$$

Again by (4.10) we get

$$\begin{aligned} &\int_{\mathbb{R}^N} |\nabla u_n|^N \left(1 - \frac{F(u_n)f'(u_n)}{f^2(u_n)}\right) dx + \int_{\mathbb{R}^N} V u_n^{N-1}\frac{F(u_n)}{f(u_n)} dx + 2c \\ &- \frac{2}{N}\int_{\mathbb{R}^N} |\nabla u_n|^N dx - \frac{2}{N}\int_{\mathbb{R}^N} V u_n^N dx = \mathrm{o}(1). \end{aligned}$$

Therefore, thanks to (4.11)

$$
\begin{aligned}
\|v_n\|_V^N &= \int_{\mathbb{R}^N} |\nabla G(u_n)|^N dx + \int_{\mathbb{R}^N} V G^N(u_n) dx \\
&= \int_{\mathbb{R}^N} |\nabla u_n|^N \left(\frac{N}{2}\frac{F(u_n)f'(u_n)}{f^2(u_n)} - \frac{N-2}{2} \right) dx + \int_{\mathbb{R}^N} V G^N(u_n) dx \\
&= Nc + \int_{\mathbb{R}^N} V \left(\frac{N}{2} u_n^{N-1} \frac{F(u_n)}{f(u_n)} - u_n^N + G^N(u_n) \right) dx + \mathrm{o}(1) \\
&\le Nc + \mathrm{o}(1) < 1
\end{aligned}
\tag{4.13}
$$

for n large enough.
At this point, we are able to improve the exponential integrability of u_n. Thanks to (f_3), for any $\epsilon > 0$ there exists $t_\epsilon > 0$ large enough such that

$$
1 - \epsilon < \sqrt[N]{\frac{N}{2}\frac{F(t)f'(t)}{f^2(t)} - \frac{N-2}{2}} \le 1 + \epsilon, \qquad \text{for all } \ t \ge t_\epsilon \ .
$$

Next by (f_2) we also have either $u_n(x) \le t_\epsilon$ or $u_n(x) \ge t_\epsilon$ which implies

$$
\begin{aligned}
v_n &\ge \int_0^{t_\epsilon} \sqrt[N]{\delta \frac{N}{2}} dt + \int_{t_\epsilon}^{u_n} (1-\epsilon) dt \\
&\qquad \ge \sqrt[N]{\delta \frac{N}{2}} t_\epsilon + (1-\epsilon)(u_n - t_\epsilon) \ge (1-\epsilon)(u_n - t_\epsilon)
\end{aligned}
\tag{4.14}
$$

and in turn

$$
u_n \le t_\epsilon + \frac{v_n}{1-\epsilon}, \qquad \text{for any } x \in \mathbb{R}^N \ .
$$

Hence, by (1.4) and since F is an increasing function, and recalling that $q > N$,

$$
\begin{aligned}
\int_{\mathbb{R}^N} F^\alpha(u_n) dx &= \int_{0 \le u_n \le t_\epsilon} F^\alpha(u_n) dx + \int_{u_n \ge t_\epsilon} F^\alpha(u_n) dx \\
&\le C_\epsilon \int_{u_n \le t_\epsilon} u_n^{N\alpha} dx \\
&\quad + \int_{u_n \ge t_\epsilon} \left[F\left(t_\epsilon + \frac{v_n}{1-\epsilon} \right) \right]^\alpha dx \\
\le C_\epsilon \int_{u_n \le t_\epsilon} u_n^N dx &+ C \int_{u_n \ge t_\epsilon} \left(t_\epsilon + \frac{v_n}{1-\epsilon} \right)^{\alpha(p - \frac{1}{N-1})} \phi_N \left(\alpha \alpha_N (t_\varepsilon + \frac{v_n}{1-\epsilon})^{\frac{N}{N-1}} \right) dx \\
&\le C_\epsilon \|u_n\|_N^N + C_\epsilon \int_{u_n \ge t_\epsilon} \phi_N \left(\alpha \alpha_N (1+\epsilon)(t_\epsilon + \frac{v_n}{1-\epsilon})^{\frac{N}{N-1}} \right) dx \\
&\le C_\epsilon \|u_n\|_N^N + C_\epsilon \int_{\mathbb{R}^N} \phi_N \left(\alpha \alpha_N (1+\epsilon)^{\frac{N}{N-1}} \frac{v_n^{\frac{N}{N-1}}}{(1-\epsilon)^{\frac{N}{N-1}}} \right) dx,
\end{aligned}
\tag{4.15}
$$

where $C_\epsilon > 0$ may change from line to line. (The last inequality can be verified just observing that for large values of u_n, also v_n is large so that $(t_\epsilon + \frac{v_n}{1-\epsilon})^{\frac{N}{N-1}} \sim (\frac{v_n}{1-\epsilon})^{\frac{N}{N-1}})$.
Set

$$\eta := \frac{1}{Nc} - \alpha > 0$$

and let us fix $0 < \epsilon_\alpha < 1$, depending on $\alpha < \frac{1}{Nc}$ such that

$$\frac{(1+\epsilon_\alpha)^{\frac{N}{N-1}}}{(1-\epsilon_\alpha)^{\frac{N}{N-1}}} \left(1 - \eta^2 (Nc)^2\right) < 1.$$

With these choices we obtain

$$\int_{\mathbb{R}^N} F^\alpha(u_n) dx \le C_\alpha \|u_n\|_N^N +$$
$$C_\alpha \int_{\mathbb{R}^N} \phi_N \left(\alpha \alpha_N \frac{(1+\epsilon_\alpha)^{\frac{N}{N-1}}}{(1-\epsilon_\alpha)^{\frac{N}{N-1}}} \|v_n\|_V^{\frac{N}{N-1}} \frac{|v_n|^{\frac{N}{N-1}}}{\|v_n\|_V^{\frac{N}{N-1}}} \right) dx \ .$$

By (4.13), $\|v_n\|_V^N \le Nc + \mathrm{o}(1)$ as n is large enough, so that

$$\|v_n\|_V^N \le Nc + (Nc)^2 \eta, \qquad \text{as } n \to +\infty \ .$$

Thus,

$$\alpha \frac{(1+\epsilon_\alpha)^{\frac{N}{N-1}}}{(1-\epsilon_\alpha)^{\frac{N}{N-1}}} \|v_n\|_V^N \le \left(\frac{1}{Nc} - \eta \right) \frac{(1+\epsilon_\alpha)^{\frac{N}{N-1}}}{(1-\epsilon_\alpha)^{\frac{N}{N-1}}} Nc(1 + Nc\eta)$$
$$= \frac{(1+\epsilon_\alpha)^{\frac{N}{N-1}}}{(1-\epsilon_\alpha)^{\frac{N}{N-1}}} \left(1 - (Nc)^2 \eta^2\right) < 1,$$

and finally we obtain

$$\int_{\mathbb{R}^N} F^\alpha(u_n) dx \le C_\alpha \|u_n\|_N^N + C_\alpha \int_{\mathbb{R}^N} \phi_N \left(\alpha_N \frac{|v_n|^{\frac{N}{N-1}}}{\|v_n\|_V^{\frac{N}{N-1}}} \right) dx \le C_\alpha \ .$$

□

Proposition 4.7. *Assume that conditions (V) and (f_1)–(f_4) are satisfied. Let $\{u_n\} \subset W^{1,N}_{q,w}$ be a PS sequence for $\mathcal{I}_V$ at level $c < 1/N$, weakly converging to u in $W^{1,N}_V(\mathbb{R}^N)$. If $u \neq 0$, then $u \in W^{1,N}_V L^q_{w_0(\mathbb{R}^N)}$ and $u_n \rightharpoonup u$ weakly in $W^{1,N}_V L^q_{w_0(\mathbb{R}^N)}$. Furthermore, as $n \to \infty$*

$$\Big(\log|x| * F(u_n)\Big) f(u_n) \longrightarrow \Big(\log|x| * F(u)\Big) f(u) \quad \textit{in } L^1_{loc}(\mathbb{R}^N) \quad (4.16)$$

and u is a weak solution to (1.3).

For the proof we refer to [10, Proposition 6.3].

4.3. **Proof of Theorem 1.2.** The functional $\mathcal{I}_V$ satisfies the Mountain Pass geometry thanks to Lemma 4.2. This yields a (PS) sequence $\{u_n\} \subset W_V^{1,N} L_{w_0}^q(\mathbb{R}^N)$ at level m_V. Then, by Lemma 4.6 we have that $\{u_n\}$ is bounded in $u \in W_V^{1,N}(\mathbb{R}^N)$ and it weakly converges to some $u \in W_V^{1,N}(\mathbb{R}^N)$. It remains to prove that $u \neq 0$.

Either $\{u_n\}$ is vanishing, that is for any $r > 0$

$$\lim_{n\to+\infty} \sup_{y\in\mathbb{R}^N} \int_{B_r(y)} |u_n|^N dx = 0$$

or, there exist $r, \delta > 0$ and a sequence $\{y_n\} \subset \mathbb{Z}^N$ such that

$$\lim_{n\to\infty} \int_{B_r(y_n)} |u_n|^N dx \geq \delta.$$

If $\{u_n\}$ is vanishing, by Lions' concentration-compactness principle we have

$$u_n \to 0 \quad \text{in} \quad L^s(\mathbb{R}^N) \quad \forall\, s > N, \tag{4.17}$$

as $n \to \infty$. In this case it is standard to show that

$$\|F(u_n)\|_\gamma, \|u_n f(u_n)\|_\gamma \to 0$$

for some values of $\gamma > 1$ and close to 1, thanks to the improved exponential integrability given by Lemma 4.6 and the growth assumption $F(t) < \frac{N}{2} t f(t)$, see (1.5)). Hence, by applying the HLS inequality (Proposition 2.9) we obtain as $n \to \infty$, similarly to Proposition 4.7:

$$\int_{\mathbb{R}^{2N}} \log\left(1 + \frac{1}{|x-y|}\right) F(u_n(x))F(u_n(y))dxdy \quad \to 0 \tag{4.18}$$

$$\int_{\mathbb{R}^{2N}} \log\left(1 + \frac{1}{|x-y|}\right) F(u_n(x))u_n(y)f(u_n(y))dxdy \quad \to 0 \tag{4.19}$$

Combining (4.18)-(4.19) and the facts $\mathcal{I}_V(u_n) \to c$ and $\mathcal{I}_V'(u_n)[v] \to 0$ on $C_c^\infty(\mathbb{R}^N)$ test functions, we obtain

$$\frac{1}{\gamma_N} \int_{\mathbb{R}^{2N}} \log\left(1 + |x-y|\right) F(u_n(x)) \left[F(u_n(y)) - \frac{2}{N} u_n(y) f(u_n(y))\right] dxdy \\ = 2m_V + \mathrm{o}(1)$$

so that $m_v \leq 0$ thanks to (1.5), which is not possible. Therefore, the vanishing case does not occur.

Now set $v_n := u_n(\cdot - y_n)$, then

$$\int_{B_r(0)} |v_n|^2 dx \geq \delta\ . \tag{4.20}$$

Using the periodicity assumption, $\mathcal{I}_V$ and $\mathcal{I}_V'$ are both invariant by the $\mathbb{Z}^N$-action, therefore $\{v_n\}$ is still a PS sequence at level m_V. Then $v_n \rightharpoonup v$ in $W_V^{1,N}(\mathbb{R}^N)$ with $v \neq 0$ by using (4.20), since $v_n \to v$ in $L_{loc}^N(\mathbb{R}^N)$. We conclude by Proposition 4.7 that $v \in W_V^{1,N} L_{w_0}^q(\mathbb{R}^N)$ is a nontrivial critical point of $\mathcal{I}_V$ and $\mathcal{I}_V(v) = m_V$, which completes the proof of Theorem 1.2.

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