Prenormal categories

Sandra Mantovani[*] Mariano Messora[*]
`sandra.mantovani@unimi.it` `mariano.messora@unimi.it`
[*]Department of Mathematics, University of Milan, Via Cesare Saldini 50, 20133 Milan, Italy

---

Abstract. In this paper we introduce the notion of (pointed) *prenormal category*, modelled after regular categories, but with the key notions of coequaliser and kernel pair replaced by those of cokernel and kernel. This framework provides a natural setting for extending certain classical results in algebra. We study the fundamental properties of prenormal categories, including a characterisation in terms of a factorisation system involving normal epimorphisms, and a categorical version of Noether's so-called 'third isomorphism theorem'. We also present a range of examples, with the category of commutative monoids constituting a central one. In the second part of the paper we extend prenormality and its related properties to the non-pointed context, using kernels and cokernels defined relative to a distinguished class of trivial objects.

Keywords: prenormal category; normal category; factorisation system; trivial objects
2000 MSC: 18G50; 18A20; 18A32; 18E99; 20M32

Contents

Accepted for publication in *Applied Categorical Structures* December 2025

Introduction

It is a straightforward observation that in any additive category, regular epimorphisms and normal epimorphisms define the same class of maps: indeed, the coequaliser of a pair of morphisms is also the cokernel of their difference. Similarly, in this setting, a map is a monomorphism if and only if its kernel is trivial. Consequently, if an additive category is also regular, normal epimorphisms and morphisms with a trivial kernel form a stable factorisation system, which coincides with the one determined by regularity. This phenomenon – in which normal epimorphisms coincide with regular epimorphisms and are pullback-stable – is not limited to regular additive categories; it also occurs, for instance, in the category of groups, and, more in general, in any homological category ([3]), as well as in several other well-studied contexts, where it is known to give rise to a range of useful properties. These recurring features, along with the properties they entail, have motivated the definition of *normal categories* ([16]), denoting regular pointed categories in which every regular epimorphism is normal.

Although the notion of normal category indeed applies to numerous commonly encountered settings, it frames the behaviour of normal epimorphisms only in relation to regularity. In contrast, we wish to emphasize that many structural properties of normal categories can persist even outside the context of regular categories. In particular, we observe that normal epimorphisms and maps with a trivial kernel can form a stable factorisation system entirely independently of any connection to the one given by regular epimorphisms and monomorphisms. A remarkable example illustrating this independence is provided by the category of commutative monoids, where one can show that the two factorisation systems just mentioned coexist, are both stable, and yet remain distinct, leading to several interesting consequences (see Example 2.20). This example, along with others, suggests that the stability of normal epimorphisms can be meaningfully studied on its own, without relying on regularity or any identification with regular epimorphisms.

Building on the above considerations, we propose a definition of (pointed) *prenormal category*, modelled on the classical definition of regular category, but where the key notions of coequaliser and kernel pair are replaced by those of cokernel and kernel: a prenormal category is thus defined as a pointed, finitely complete category where cokernels of kernels exist and normal epimorphisms are pullback-stable (Definition 2.1). This notion is strictly weaker than that of normal category ([16]), as proven by the example of commutative monoids discussed above, and provides a natural setting for extending certain classical results in algebra. The aim of this paper is then to study prenormality and its associated categorical properties in full generality, irrespective of regularity or any specific behaviour of regular epimorphisms. In particular, we prove that many properties that hold for regular categories can be carried over to prenormal categories by replacing regular epimorphisms with normal epimorphisms, and monomorphisms with morphisms with a trivial kernel. Notably, we find a characterisation of prenormal categories as those pointed, finitely complete categories where normal epimorphisms form (with maps with a trivial kernel) a stable factorisation system (Proposition 2.4). Other interesting properties in prenormal categories include the fact that pullback squares of normal epimorphisms are pushouts (Proposition 2.5), a pullback cancellation principle involving normal epimorphisms (Proposition 2.6) and a categorical version of Noether's so-called 'third isomorphism theorem' (Proposition 2.10). We also present new examples that fit within this broad framework, further exhibiting the independence of prenormality and regularity (the two can coexist and be distinct, one can exist without the other, and so on).

The second part of this paper is devoted to a generalisation of prenormal categories to the non-pointed setting. A key motivation for this extension arises from the behaviour of *slices*: a well-known and useful property of regular categories is their stability under taking slices. It is then natural to ask how prenormal categories behave in this regard. However, the slice of a pointed category is generally no longer pointed, and as a result, the study of slices of prenormal categories naturally leads us beyond the realm of pointed categories. To pursue this line of inquiry, we develop a non-pointed generalisation of prenormality by adopting the standard approach of defining kernels and cokernels relative to some distinguished class $\mathcal{Z}$ of trivial objects, rather than with respect to a zero object. This leads to a notion of $\mathcal{Z}$-*prenormal category* (Definition 4.1), which reduces to standard prenormality when the ambient category is pointed and $\mathcal{Z}$ is the class of zero objects.

This generalised approach indeed allows us to recover a form of slice stability: the slice $\mathcal{C}/X$ of a $\mathcal{Z}$-prenormal category $\mathcal{C}$ over an object X is naturally a $(\mathcal{Z}/X)$-prenormal category (Proposition 6.11). Moreover, many of the key properties of prenormal categories are retained in this broader setting, most notably the characterisation in terms of a stable factorisation system (Proposition 5.1). These and other properties, along with the additional subtleties involved with working in a non-pointed setting, are investigated in Sections 4 to 7. This non-pointed generalisation also greatly broadens the range of possible examples, with a detailed list provided in Section 9. Moreover, this framework appears to be particularly well-suited to the study of certain characterisations of *pretorsion theories* (see [8, 19]) – a direction that will be addressed in future work.

Other interesting approaches to both pointed and non-pointed 'normality' exist in the literature, including [13, 11, 22, 4, 17]. In [11, 22], the notions considered require the ambient category to be regular, while [4] develops a notion of normality relative to a given proper stable factorisation system. In [17] the authors provide non-pointed notions of normal monomorphism and normal epimorphism that are not based on freely chosen 'trivial objects', but are instead defined naturally in terms of initial and terminal objects in any category with enough limits and colimits. Accordingly, the premises of these four approaches differ significantly from the one adopted in the present paper. In [13], the author considers an axiom of partial stability of (possibly non-pointed) normal epimorphisms, along with other additional hypotheses. Section 8 is devoted to a comparison with this last notion.

1 Preliminaries

In this section we establish some notation and we recall a few known results which will be useful later. We begin by setting up the notation for certain special classes of arrows.

Notation 1.1. For any category $\mathcal{C}$ we use the following notation:

- $\mathsf{StrEpi}(\mathcal{C})$ denotes the class of strong epimorphisms in $\mathcal{C}$.
- $\mathsf{RegEpi}(\mathcal{C})$ denotes the class of regular epimorphisms in $\mathcal{C}$. These are graphically represented using the special arrow '$\relbar\!\!\blacktriangleright$'.
- $\mathsf{Mono}(\mathcal{C})$ denotes the class of monomorphisms in $\mathcal{C}$. These are graphically represented using the special arrow '$\rightarrowtail$'.

We also represent (ordinary) epimorphisms with the special arrow '$\twoheadrightarrow$'. If moreover $\mathcal{C}$ is pointed we introduce the following additional notation:

- $\mathsf{NEpi}(\mathcal{C})$ denotes the class of normal epimorphism in $\mathcal{C}$. These are graphically represented using the special arrow '$\relbar\!\!\vartriangleright$'.

- $\mathsf{NMono}(\mathcal{C})$ denotes the class of normal monomorphisms in $\mathcal{C}$. These are graphically represented using the special arrow ‘ $\rightarrowtriangle$ ’.

- $\mathsf{TKer}(\mathcal{C})$ denotes the class of maps in $\mathcal{C}$ whose kernel is reduced to zero. These are graphically represented using the special arrow ‘ $\circ\!\!\rightarrow$ ’.

Next, we fix the terminology for factorisation systems that we will use throughout this paper. See for example [1] for details.

Definition 1.2. Let $\mathcal{C}$ be any category. By *factorisation system on* $\mathcal{C}$ we mean a pair $(\mathscr{E}, \mathscr{M})$ of classes of morphisms in $\mathcal{C}$ satisfying the following conditions:

1. both $\mathscr{E}$ and $\mathscr{M}$ are closed under composition with isomorphisms;

2. every map $f\colon X \to Y$ in $\mathcal{C}$ factors as $f = m \cdot e$, with $e \in \mathscr{E}$ and $m \in \mathscr{M}$;

3. every $f\colon X \to Y$ in $\mathscr{E}$ satisfies the *left lifting property* with respect to every $f'\colon X' \to Y'$ in $\mathscr{M}$, i.e. given any $x\colon X \to X'$ and any $y\colon Y \to Y'$ such that $y \cdot f = f' \cdot x$, there exists a unique $d\colon Y \to X'$ in $\mathcal{C}$ such that $d \cdot f = x$ and $f' \cdot d = y$ (we also say that f' satisfies the *right lifting property* with respect to f).

$$\begin{tikzcd}
X \arrow[r, "x"] \arrow[d, "f"'] & X' \arrow[d, "f'"] \\
Y \arrow[r, "y"'] \arrow[ur, dashed, "d" description] & Y'
\end{tikzcd}$$

(This is sometimes referred to as an *orthogonal factorisation system*.)

Furthermore, the factorisation system $(\mathscr{E}, \mathscr{M})$ is said to be *stable* if the pullback of a map in $\mathscr{E}$ along any map in $\mathcal{C}$ is again in $\mathscr{E}$.

Classes of maps involved in factorisation systems enjoy many useful properties. We recall a few of them that we will need in the next sections.

Proposition 1.3. Let $\mathcal{C}$ be a category and let $(\mathscr{E}, \mathscr{M})$ be a factorisation system on $\mathcal{C}$. Then the following properties hold.

1. If a map in $\mathcal{C}$ is both in $\mathscr{E}$ and $\mathscr{M}$, then it is an isomorphism.

2. $\mathscr{E}$ is the class of maps in $\mathcal{C}$ satisfying the left lifting property with respect to all the maps in $\mathscr{M}$, and, dually, $\mathscr{M}$ is the class of maps in $\mathcal{C}$ satisfying the right lifting property with respect to all the maps in $\mathscr{E}$.

3. If some composable maps f and g in $\mathcal{C}$ are such that $g \cdot f$ and f lie in $\mathscr{E}$, then $g \in \mathscr{E}$ as well. Dually, if $g \cdot f$ and g lie in $\mathscr{M}$, then $f \in \mathscr{M}$ as well.

2 The pointed case: prenormal categories

We now begin the main development of the paper by formally stating the definition of prenormality, initially within the pointed setting, followed by a presentation of its fundamental properties. While a more general, non-pointed version will be presented later, the pointed setting provides a more intuitive framework with less technicalities and includes many interesting examples, making it a natural starting point.

As previously discussed, the notion of prenormality is based on that of regular categories, with cokernels taking the place of coequalisers and kernels that of kernel pairs.

Definition 2.1. A *prenormal* category is a pointed category $\mathcal{C}$ such that:

1. $\mathcal{C}$ has finite limits;
2. $\mathcal{C}$ has cokernels of kernels;
3. $\mathcal{C}$ has pullback-stable normal epimorphisms.

Let us also introduce the weaker notion of a *semi-prenormal* category, which is closely related to axioms considered in [13] (see also Section 8).

Definition 2.2. A *semi-prenormal* category is a pointed category $\mathcal{C}$ such that:

1. $\mathcal{C}$ has pullbacks along normal monomorphisms;
2. $\mathcal{C}$ has cokernels of kernels;
3. the pullback of a normal epimorphism along a normal monomorphism is again a normal epimorphism.

We also recall the definition of a *normal* category from [16], as it will be referenced throughout the text.

Definition 2.3. A *normal* category is a pointed regular category $\mathcal{C}$ such that every regular epimorphism in $\mathcal{C}$ is a normal epimorphism.

With these definitions in place, we now proceed to establish a number of general properties of (semi-)prenormal categories. Most proofs are deferred until Section 4, where we study prenormality in greater generality. Many of these properties mirror familiar results from the theory of regular categories (for example in [2, 10]), but are new in this context.

Proposition 2.4. Let $\mathcal{C}$ be a pointed category with finite limits. The following are equivalent:

(i) $\mathcal{C}$ is prenormal;

(ii) $\mathcal{C}$ admits $(\mathsf{NEpi}(\mathcal{C}), \mathsf{TKer}(\mathcal{C}))$ as a stable factorisation system;

(iii) $\mathcal{C}$ admits a stable factorisation system of the form $\big(\mathsf{NEpi}(\mathcal{C}), \mathscr{M}\big)$, for some class $\mathscr{M}$ of morphisms in $\mathcal{C}$;

(iv) $\mathcal{C}$ admits a stable factorisation system of the form $\big(\mathscr{E}, \mathsf{TKer}(\mathcal{C})\big)$, for some class $\mathscr{E}$ of morphisms in $\mathcal{C}$.

Proposition 2.5. In a prenormal category, consider the following pullback square.

$$\begin{tikzcd}
\cdot \arrow[r, "f'"] \arrow[d, "x"'] & \cdot \arrow[d, "y"] \\
\cdot \arrow[r, "f"'] & \cdot
\end{tikzcd}$$

The above diagram is also a pushout whenever one of the following conditions hold. (a) f is a normal epimorphism. (b) x is a normal epimorphism, f is a regular epimorphism, and f' is an epimorphism.

Proposition 2.6. In a prenormal category, consider then the following commutative diagram, where f' is a normal epimorphism and both the outer rectangle and left-hand square are pullbacks.

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "a"'] & B \arrow[r, "g"] \arrow[d, "b"'] & C \arrow[d, "c"'] \\
A' \arrow[r, "f'"', -{Triangle[open]}] & B' \arrow[r, "g'"'] & C'
\end{tikzcd}$$

Then the right-hand square is also a pullback.

Proposition 2.7. Let $\mathcal{C}$ be a prenormal category, and consider the following commutative diagram where f is the kernel of g.

$$\begin{tikzcd}
A' \arrow[r, "f'"] \arrow[d, "a"'] & B' \arrow[r, "g'"] \arrow[d, "b"'] & C' \arrow[d, "c"'] \\
A \arrow[r, "f"', {Triangle[open]}->] & B \arrow[r, "g"'] & C
\end{tikzcd}$$

If the left square is a pullback and g' is the cokernel of f', then $c \in \mathsf{TKer}(\mathcal{C})$. Conversely, if $c \in \mathsf{TKer}(\mathcal{C})$ and f' is the kernel of g', then the left square is a pullback.

Proposition 2.8. In a prenormal category, consider the following commutative diagram where f is a normal epimorphism, (r_0, r_1) is the kernel pair of f and (s_0, s_1) is the kernel pair of g.

$$\begin{tikzcd}
\cdot \arrow[r, shift left, "r_0"] \arrow[r, shift right, "r_1"'] \arrow[d] & \cdot \arrow[r, "f", -{Triangle[open]}] \arrow[d] & \cdot \arrow[d] \\
\cdot \arrow[r, shift left, "s_0"] \arrow[r, shift right, "s_1"'] & \cdot \arrow[r, "g"'] & \cdot
\end{tikzcd}$$

If either of the left-hand squares is a pullback, then the right-hand square is a pullback.

Proposition 2.9. Let $\mathcal{C}$ be a prenormal category. Then the following properties hold.

1. The product of two normal epimorphisms is a normal epimorphism. Furthermore, the product of two exact sequences is again an exact sequence.

2. Let $E\colon A \to B \to C$ be an exact sequence, and let $E'\colon A' \to B' \to C'$ be the sequence obtained by pulling back E along a given map $c\colon C' \to C$. Then E' is exact if and only if $c \in \mathsf{TKer}(\mathcal{C})$.

Proposition 2.10. In a prenormal category, Noether's third isomorphism theorem holds: given normal monomorphisms $m\colon M \rightarrowtail A$ and $n\colon N \rightarrowtail A$, with n factoring through m, then the induced map on the cokernels $N/M \to A/M$ is a normal monomorphism whose cokernel $(A/M)/(N/M)$ is canonically isomorphic to the cokernel A/N of n.

Proposition 2.11. Let $\mathcal{C}$ be a prenormal category. For any category $\mathcal{B}$, the functor category $\mathcal{C}^{\mathcal{B}}$ is prenormal.

Remark 2.12. Some of the properties of prenormal categories actually extend to semi-prenormal categories (in particular Propositions 2.7, 2.10, 2.11 and a version of Proposition 2.4; see Section 8 for more details).

Remark 2.13. As mentioned in the introduction, the slice of a pointed category is not pointed, in general, and therefore the slice of a prenormal category is in general not prenormal. It is however prenormal in the non-pointed sense discussed in Section 4 (see Proposition 6.11).

Prenormality is in general independent of both regularity and of normality in the sense of [16], as illustrated in Examples 2.17, 2.20, 2.23 and 2.25. However, the three notions are interconnected, as established in Propositions 2.14 and 2.15 and Example 2.16 below.

Proposition 2.14. Let $\mathcal{C}$ be a prenormal category. Then the following are equivalent: (i) $\mathcal{C}$ is normal in the sense of [16]; (ii) $\mathsf{StrEpi}(\mathcal{C}) = \mathsf{NEpi}(\mathcal{C})$; (iii) $\mathsf{RegEpi}(\mathcal{C}) = \mathsf{NEpi}(\mathcal{C})$; (iv) $\mathsf{Mono}(\mathcal{C}) = \mathsf{TKer}(\mathcal{C})$.

Proposition 2.15. Let $\mathcal{C}$ be a pointed protomodular category (as in [3]). Then the following are equivalent: (i) $\mathcal{C}$ is prenormal; (ii) $\mathcal{C}$ is homological; (iii) $\mathcal{C}$ is normal (in the sense of [16]); (iv) $\mathcal{C}$ is regular.

Proof. In the presence of protomodularity we have

$$\mathsf{Mono}(\mathcal{C}) = \mathsf{TKer}(\mathcal{C}), \qquad \mathsf{RegEpi}(\mathcal{C}) = \mathsf{NEpi}(\mathcal{C}).$$

Therefore prenormality, normality and regularity are the same notion in the presence of protomodularity. Then $\mathcal{C}$ is homological if it is protomodular and regular, by definition. ☐

We devote the rest of the section to a collection of various examples and non-examples.

Example 2.16. We have the following chain of implications:

abelian $\Longrightarrow$ semi-abelian $\Longrightarrow$ homological $\Longrightarrow$ normal $\Longrightarrow$ prenormal $\Longrightarrow$ semi-prenormal.

Therefore any abelian, semi-abelian, homological or normal category is prenormal.

Example 2.17. The category CMon of commutative monoids is prenormal.

As is well known, CMon is pointed, complete and cocomplete. It is also a regular category and regular epimorphisms are simply surjective monoid homomorphisms. Given a kernel $k\colon K \hookrightarrow M$, the cokernel of k is obtained by taking the quotient of M by the congruence

$$x \sim y \text{ if } x + a = y + b \text{ for some } a, b \in K,$$

so that normal epimorphisms can be characterised as those surjective monoid homomorphisms $f\colon M \to N$ with the following additional property:

$$\begin{gathered}\text{for every } x, y \in M, \text{ if } f(x) = f(y) \text{ then } x + a = y + b \text{ for some} \\ a, b \in M \text{ such that } f(a) = f(b) = 0.\end{gathered} \tag{2.18}$$

It is a simple check to verify that surjective maps satisfying Property 2.18 are stable under pullback. Therefore CMon is indeed prenormal.

Notice that Property 2.18 is not automatically satisfied by regular epimorphisms. Consider for example the operation of sum of natural numbers, seen as a map of commutative monoids

$$+\colon \mathbb{N} \times \mathbb{N} \to \mathbb{N}. \tag{2.19}$$

Since this is a surjective map not satisfying Property 2.18, we conclude that regular epimorphisms in CMon do not coincide with normal epimorphisms, and therefore, CMon is a regular pointed category which is prenormal but not normal in the sense of [16]. We then have two different stable factorisation systems: the familiar regular epi-mono factorisation system, plus the factorisation system described in Proposition 2.4. The above map 2.19 is surjective and has a trivial kernel: it is thus in the left part of the first factorisation system and in the right part of the second.

Example 2.20. The category Mon of not necessarily commutative monoids is a variety of algebras which is *not* prenormal – and not even semi-prenormal.

Consider the following short exact sequence of monoids

$$\begin{tikzcd} K \arrow[r, hook] & M \arrow[r, "p"] & N, \end{tikzcd}$$

where $M = \langle x, y, z \,|\, x^2 = y^2 \rangle$ is the quotient of the free monoid on the generators x, y and z by the relation $x^2 = y^2$, and $K = \langle z \rangle$ is the free monoid generated by z, seen as a submonoid of M. Clearly we have that $N = M/K = \langle x, y \,|\, x^2 = y^2 \rangle$. Consider now the submonoid N' of N consisting of those elements in N where only an even number of xs appear (this is well-defined, because the relation $x^2 = y^2$ does not change the parity of the number of xs). It is easy to check that N' is a normal submonoid of N (it is the kernel of the map $N \to \langle x \,|\, x^2 = 0 \rangle$ given by $x \mapsto x$ and $y \mapsto 0$). Call M' the pullback of p along the inclusion $N' \hookrightarrow N$. Obviously M' consists of those elements of M where only an even number of xs appear.

We claim that the pullback projection $p'\colon M' \to N'$ is not a normal epimorphism. The kernel of p' is still K, but we show that the kernel pair of p' is not the smallest equivalence relation R' on M' that satisfies $(w, 0) \in R'$ for all $w \in K$. This smallest equivalence relation, R', can be explicitly described as the transitive closure of the following relation $\sim$:

(2.21)

for $a, a' \in M'$, we say $a \sim a'$ if we we can write

- $a = u_1 k_1 \cdots u_n k_n$,
- $a' = u'_1 k'_1 \cdots u'_n k'_n$,
- $u_1 h_1 \cdots u_n h_n = u'_1 h'_1 \cdots u'_n h'_n$,

for some $n \in \mathbb{N}$, $u_i, u'_i \in M'$ and $k_i, k'_i, h_i, h'_i \in K$, for all $i \in \{1, \ldots n\}$.

Consider $a = xzx \in M'$ and $b = yzy \in M'$. Clearly $p'(b) = y^2 = x^2 = p'(a)$, so that (a, b) is in the kernel pair of p'. However, one can show that $(a, b) \notin R'$. The idea of the proof is as follows.

For each $w \in M'$ define $\chi_x(w)$, the maximal number of xs that can appear in w (for any $w \in M'$, w can be rewritten using the relation $x^2 = y^2$ in a finite number of ways, and so there is a way to write w such that the number of xs is maximal). Similarly, define $\chi_y(w)$ for each $w \in M'$. By using the explicit description 2.21 of $\sim$ and the fact that the u_is in 2.21 cannot contain an odd number of xs, one proves that if $w_1, w_2 \in M'$ are such that $w_1 \sim w_2$, $\chi_x(w_1) = 2$ and $\chi_y(w_1) = 0$, then $\chi_x(w_2) = 2$ and $\chi_y(w_2) = 0$ as well. Applying this to a, one finds that if it was the case that $(a, b) \in R'$, then b would satisfy $\chi_y(b) = 0$. But clearly $\chi_y(b) = 2$.

Example 2.22. The example of commutative monoids 2.17 can be generalised as follows. Consider a classical algebraic theory $\Omega = (\Sigma, T)$ consisting of a set Σ of operations of finite positive arities and a set T of equational axioms. In analogy with Ω-groups, we consider the notion of a (commutative) Ω-monoid, defined as a quadruple $\big(M, +, 0, (\sigma_M)_{\sigma\in\Sigma}\big)$ such that:

- $(M, +, 0)$ is a (commutative) monoid;
- $\big(M, (\sigma_M)_{\sigma\in\Sigma}\big)$ is an Ω-algebra – i.e. for each n-ary operation $\sigma \in \Sigma$, $\sigma_M \colon M^n \to M$ is a (set-theoretic) function, and the σ_Ms satisfy the axioms specified by T;
- for each n-ary operation $\sigma \in \Sigma$, the function $\sigma_M \colon M^n \to M$ is a monoid homomorphism in each variable separately (in other words, the σ_Ms are distributive over the base monoid operation and have the base monoid unit as an absorbing element).

Examples of commutative Ω-monoids for various Ωs include commutative monoids themselves (where Ω is empty), non-unital semirings (where Ω consists of one binary operation satisfying associativity) and R-semimodules for a fixed semiring R (with Ω containing a unary operation for every element of R and some additional axioms).

For a fixed theory Ω, consider Ω-CMon, the variety of commutative Ω-monoids (here 'variety' is again understood in the sense of universal algebra). It is easily verified that in such a category, normal epimorphisms are characterised in the same way as in CMon, and therefore Ω-CMon is prenormal.

Note that, for instance, the category $\mathsf{SemiRng}$ of non-unital semirings is another example of a category that is prenormal but not normal in the sense of [16], because any commutative monoid $(M, +, 0)$ can be seen as a non-unital semiring $(M, +, 0, *)$ with $*$ given by $x * y = 0$ for every $x, y \in M$. So the same counterexample seen in Example 2.17 for CMon holds for $\mathsf{SemiRng}$.

Example 2.23. Consider the category $\mathcal{P} = \mathsf{PreOrd}(\mathsf{CMon})$ of preordered commutative monoids. Its objects are quadruples $(M, +, 0, \leq)$ (which we will usually denote by just M) where $(M, +, 0)$ is a commutative monoid, $(M, \leq)$ is a preordered set, and the following holds for every $x, y, a \in M$:

$$x \leq y \implies x + a \leq y + a.$$

Morphisms in $\mathcal{P}$ are functions that are monoid morphisms and order morphisms at the same time. We prove that this category is prenormal but not regular.

First, $\mathcal{P}$ is pointed and limits in $\mathcal{P}$ exist and are computed as in CMon. Given a kernel $k \colon K \hookrightarrow M$ in $\mathcal{P}$, the cokernel of k is obtained by taking the cokernel $q \colon M \to Q$ in CMon and equipping Q with the preorder $\leq_Q$ given by

$$\begin{gathered} u \leq_Q v \text{ if there exist } x, y \in M \text{ such that} \\ q(x) = u,\ q(y) = v \text{ and } x \leq_M y. \end{gathered}$$

The fact that $\leq_Q$ is transitive follows from the characterisation in 2.18 of normal epimorphisms of commutative monoids. In fact, suppose $u \leq_Q v$ and $v \leq_Q w$. Then there exist $x, y, y', z \in M$ such that $q(x) = u$, $q(y) = q(y') = v$, $q(z) = w$, $x \leq_M y$ and $y' \leq_M z$. Since $q(y) = q(y')$, we find that $y + a = y' + a'$ for some $a, a' \in K$. Thus we obtain

$$x + a \leq_M y + a = y' + a' \leq_M z + a'.$$

Since $q(x + a) = q(x) = u$ and $q(z + a') = q(z) = w$, we deduce that $u \leq_Q w$ by definition. The rest of the proof that $q\colon M \to Q$ is indeed the cokernel of k is now easy to carry out. We can then characterise normal epimorphisms in $\mathcal{P}$ as those maps $f\colon M \to N$ satisfying Property 2.18 as well as the following.

(2.24) If $u \leq_N v$ for some $u, v \in N$, then there exist $x, y \in M$ such that
$f(x) = u$, $f(y) = v$ and $x \leq_M y$.

Properties 2.18 and 2.24 are together pullback-stable, and so we conclude that $\mathcal{P}$ is prenormal.

To show that $\mathcal{P}$ is not regular consider the following pullback square in $\mathcal{P}$,

$$\begin{tikzcd}
D \arrow[r, "p'"] \arrow[d, hook, "i'"'] & C \arrow[d, hook, "i"] \\
A \arrow[r, -latex, "p"'] & B
\end{tikzcd}$$

where the objects and maps are defined as follows.

- A is the set $\{0, 1, 1', 2\}$. We equip A with the partial order $\leq$ defined by

  $$0 \leq 1, \quad 1' \leq 2$$

  and with the binary operation $\curlyvee$, defined as the join with respect to the the total order $\preceq$ given by $0 \preceq 1 \preceq 1' \preceq 2$. One can check that with these definitions, $(A, \curlyvee, 0, \leq)$ is a preordered commutative monoid.

- B is the set $\{0, 1, 2\}$ equipped with the total order $0 \leq 1 \leq 2$ and the binary operation $\vee$, defined as the join with respect to the same order $\leq$ on B. Clearly $(B, \vee, 0, \leq)$ is a preordered commutative monoid.

- The map $p\colon A \to B$ is defined as

  $$p(0) = 0, \quad p(1) = p(1') = 1, \quad p(2) = 2.$$

  It easy easy to check that p is a regular epimorphism in $\mathcal{P}$ (it is the coequalizer of its kernel pair).

- C is the subset $\{0, 2\}$ of B. We equip C with the restrictions of the operation $\vee$ and the preorder $\leq$ of B, so that we have a full inclusion $i\colon C \hookrightarrow B$.

- Finally, (D, p', i') is the pullback of p along i. One can easily check that we can take D to be the set $\{0, 2\}$ with the discrete preorder and the operation

  $$0 + 0 = 0, \quad 0 + 2 = 2 + 0 = 2, \quad 2 + 2 = 2;$$

  i' is the inclusion of $\{0,2\}$ into $\{0, 1, 1', 2\}$ and p' is the identity on the underlying set $\{0, 2\}$.

Since p' is a monomorphism but not an isomorphism (D and C are not isomorphic as preordered sets), it cannot be a regular epimorphism. But as p is a regular epimorphism, we conclude that $\mathcal{P}$ is not regular.

Example 2.25. The category $\mathsf{ParOrd}(\mathsf{CMon})$ of partially ordered commutative monoids, like $\mathsf{PreOrd}(\mathsf{CMon})$, is prenormal but not regular.

Limits in $\mathsf{ParOrd}(\mathsf{CMon})$ are computed in the same way as in $\mathsf{PreOrd}(\mathsf{CMon})$. As shown in [9], $\mathsf{ParOrd}(\mathsf{CMon})$ is a reflective subcategory of $\mathsf{PreOrd}(\mathsf{CMon})$ and the reflection has stable units. Given a kernel $k\colon K \hookrightarrow M$ in $\mathsf{ParOrd}(\mathsf{CMon})$, it is easy to check that the cokernel of k is obtained by first taking the cokernel $q\colon M \to Q$ of k in $\mathsf{PreOrd}(\mathsf{CMon})$, and then composing q with a reflection $\rho_Q\colon Q \to rQ$ of P in $\mathsf{ParOrd}(\mathsf{CMon})$:

$$\begin{tikzcd} K \arrow[r, hook, "k"] & M \arrow[r, "q"] & Q \arrow[r, "\rho_Q"] & rQ. \end{tikzcd}$$

One can then verify that a map $f\colon M \to N$ of partially ordered commutative monoids is a normal epimorphism if and only if it factors in $\mathsf{PreOrd}(\mathsf{CMon})$ as $f = \rho \cdot g$, with $g\colon M \to N'$ a normal epimorphism in $\mathsf{PreOrd}(\mathsf{CMon})$, and $\rho\colon N' \to N$ a reflection of N' in $\mathsf{ParOrd}(\mathsf{CMon})$. Since normal epimorphisms are stable in $\mathsf{PreOrd}(\mathsf{CMon})$ (by Example 2.23) and the reflector $\mathsf{PreOrd}(\mathsf{CMon}) \to \mathsf{ParOrd}(\mathsf{CMon})$ has stable units, we conclude that normal epimorphisms are stable in $\mathsf{ParOrd}(\mathsf{CMon})$.

The fact that $\mathsf{ParOrd}(\mathsf{CMon})$ is not regular follows from the same counterexample given in Example 2.23, as all the involved preorders are actually partial orders.

Example 2.26. The category Set_* of pointed sets is pointed, complete and cocomplete. A map of pointed sets $f\colon (X, x_0) \to (Y, y_0)$ is a normal epimorphism if and only if for each $y \in Y$, if $y \neq y_0$ then the fibre $f^{-1}(y)$ has cardinality exactly 1.

In general, the pullback of a normal epimorphism is not necessarily a normal epimorphism again. Consider for example the set $2 = \{0, 1\}$ (pointed in 0) and the terminal object $1 = \{0\}$; then, the unique map $p\colon 2 \to 1$ is a normal epimorphism, but the pullback of p along itself is a product projection $2 \times 2 \to 2$, which is clearly not a normal epimorphism according to the above characterisation.

However, it is easy to check that the pullback of a normal epimorphism along a normal monomorphism (actually along any monomorphism) is again a normal epimorphism. This makes Set_* a semi-prenormal category which is not prenormal.

Example 2.27. Consider the category $\mathsf{mpParOrd}$ of *minimally pointed posets*, whose objects are triples $(X, \leq, x_0)$, where $(X, \leq)$ is a poset and x_0 is a minimal element in $(X, \leq)$, and whose morphisms are monotone functions that preserve the distinguished minimal element. A function in $\mathsf{mpParOrd}$ is a normal epimorphism if and only if it is one in both Set_* and $\mathsf{PreOrd}(\mathsf{CMon})$. Using arguments similar to those in Examples 2.25 and 2.26, one can show that $\mathsf{mpParOrd}$ is semi-prenormal but neither prenormal nor regular.

We conclude this section by summing up in a table the main examples and non-examples we have considered, each with its properties highlighted.

	Ab	CMon	Mon	SemiRng	PreOrd(CMon)	ParOrd(CMon)	Set$_*$	mpParOrd
	2.16	2.17	2.20	2.22	2.23	2.25	2.26	2.27
Regular?	Yes	Yes	Yes	Yes	No	No	Yes	No
Semi-prenormal?	Yes	Yes	No	Yes	Yes	Yes	Yes	Yes
Prenormal?	Yes	Yes	No	Yes	Yes	Yes	No	No
Normal?	Yes	No	No	No	No	No	No	No

As was already mentioned, the notion of prenormality for pointed categories, while interesting in its own right, can be generalised to the broader setting of (non-pointed) categories equipped with a class of trivial objects. The next sections are devoted to this generalisation, beginning with the necessary preliminaries.

3 Categories with a distinguished class of trivial objects

This section contains the necessary background on kernels and cokernels defined relative to a class of trivial objects. We omit most of the proofs as they concern well-known results. A more in-depth discussion of this topic can be found for example in [13, 14, 15].

Throughout this section, we fix a category $\mathcal{C}$ and a class $\mathcal{Z}$ of objects in $\mathcal{C}$, which we identify with the full subcategory of $\mathcal{C}$ spanned by those objects.

Definition 3.1. We say that a morphism in $\mathcal{C}$ is $\mathcal{Z}$-*trivial* if it factors through an object in $\mathcal{Z}$. Given any map $f\colon A \to B$ in $\mathcal{C}$, a $\mathcal{Z}$-*kernel* of f is given by an object K and a map $k\colon K \to A$ such that $f \cdot k$ is $\mathcal{Z}$-trivial and for any other map $x\colon X \to A$ such that $f \cdot x$ is $\mathcal{Z}$-trivial, there exists a unique $x'\colon X \to K$ such that $k \cdot x' = x$. By a slight abuse of notation, we may refer to the kernel (K, k) simply as K or k, when the distinction is clear from context. Of course, $\mathcal{Z}$-*cokernels* are defined dually. A $\mathcal{Z}$-*exact sequence* is given by a pair of composable maps $A \xrightarrow{f} B \xrightarrow{g} C$ such that (A, f) is the $\mathcal{Z}$-kernel of g and (C, g) is the $\mathcal{Z}$-cokernel of f.

Remark 3.2. Given a $\mathcal{Z}$-kernel (A, f), if $f\colon A \to B$ admits a $\mathcal{Z}$-cokernel (C, g), then (A, f) is the $\mathcal{Z}$-kernel of g.

Proposition 3.3. Let f be the underlying map of a $\mathcal{Z}$-kernel. Then f is a monomorphism. Furthermore, if g is a map such that $f \cdot g$ is defined and $\mathcal{Z}$-trivial, then g itself is $\mathcal{Z}$-trivial.

Proof. The first property is well known. For the second property, suppose f is the $\mathcal{Z}$-kernel of a map u. Since $f \cdot g$ is $\mathcal{Z}$-trivial, we can write $f \cdot g = t \cdot s$ for some maps s and t such that $\operatorname{cod} s = \operatorname{dom} t$ is in $\mathcal{Z}$. Observe that $u \cdot t$ is $\mathcal{Z}$-trivial, and hence t factors uniquely as $t = f \cdot v$ for some map v. It follows that $f \cdot v \cdot s = t \cdot s = f \cdot g$. As f is a monomorphism, we obtain $g = v \cdot s$, which shows that g is $\mathcal{Z}$-trivial. □

Notation 3.4. We call $\mathcal{Z}$-*normal monomorphisms* the monomorphisms in $\mathcal{C}$ underlying $\mathcal{Z}$-kernels. We denote by $\mathsf{NMono}(\mathcal{C}, \mathcal{Z})$ the class of $\mathcal{Z}$-normal monomorphisms in $\mathcal{C}$, and we graphically represent them using the same special arrow ‘ $\triangleright\!\!\rightarrow$ ’ we used for (ordinary) normal monomorphisms. Similarly, we call $\mathcal{Z}$-*normal epimorphisms* the epimorphisms underlying $\mathcal{Z}$-cokernels. We denote by $\mathsf{NEpi}(\mathcal{C}, \mathcal{Z})$ the class of $\mathcal{Z}$-normal epimorphisms, and we use again the special arrow ‘ $\rightarrow\!\!\triangleright$ ’ to graphically represent them.

Proposition 3.5. Let $f\colon X \to Y$ be a morphism in $\mathcal{C}$ and let $k\colon K \to X$ be its $\mathcal{Z}$-kernel. Given any morphism $x\colon X' \to X$, if the pullback of k along x exists, then it is the $\mathcal{Z}$-kernel of the composite $f \cdot x$. In particular, $\mathcal{Z}$-normal monomorphisms are stable under pullbacks.

Remark 3.6. The class of $\mathcal{Z}$-trivial morphisms in $\mathcal{C}$ constitutes an *ideal* of maps ([7, 14, 15]). Replacing $\mathcal{Z}$ with its closure under retracts (i.e. replacing $\mathcal{Z}$ with the class $\mathcal{Z}'$ of objects X such that there exists a split monomorphism with domain X and codomain in $\mathcal{Z}$) does not change the corresponding ideal of trivial maps, nor the associated notions of kernel, cokernel and exactness. On the other hand, working with a class $\mathcal{Z}$ closed under retracts allows for cleaner statements of certain results. Accordingly, we may make the following assumption without loss of generality.

From now on, and throughout this section, we assume that $\mathcal{Z}$ is closed under retracts in $\mathcal{C}$.

Proposition 3.7. Given an object B in $\mathcal{C}$ and a monomorphism $\varepsilon_B\colon S_B \rightarrowtail B$, then (S_B, ε_B) is a coreflection of B in $\mathcal{Z}$ if and only if it is a $\mathcal{Z}$-kernel of id_B. In this case, given maps $K \xrightarrow{k} A \xrightarrow{f} B$, then (K, k) is the $\mathcal{Z}$-kernel of f if and only if there exists a (unique) map $h\colon K \to S_B$ such that the following square is a pullback.

$$\begin{tikzcd}
K \arrow[r, "h"] \arrow[d, "k"'] & S_B \arrow[d, "\varepsilon_B"] \\
A \arrow[r, "f"'] & B
\end{tikzcd}$$

Finally, $\mathcal{Z}$ is a mono-coreflective subcategory of $\mathcal{C}$ if and only if every identity morphism in $\mathcal{C}$ admits a $\mathcal{Z}$-kernel.

(In this paper, reflective and coreflective subcategories are always assumed to be closed under isomorphisms – and therefore under retracts.)

Proposition 3.8. Given a $\mathcal{Z}$-normal monomorphism $f\colon A \to B$ in $\mathcal{C}$, then the following are equivalent. (i) The map f is $\mathcal{Z}$-trivial; (ii) A is in $\mathcal{Z}$; (iii) (A, f) is the $\mathcal{Z}$-kernel of an isomorphism; (iv) (A, f) is a coreflection of B in $\mathcal{Z}$.

Notation 3.9. We denote by $\mathsf{TKer}(\mathcal{C}, \mathcal{Z})$ the class of maps whose $\mathcal{Z}$-kernel exists and is $\mathcal{Z}$-trivial (according to any of the equivalent statements in Proposition 3.8), and we graphically represent a map with a $\mathcal{Z}$-trivial $\mathcal{Z}$-kernel by using again the special arrow ‘ $\circ\!\!\rightarrow$ ’.

Proposition 3.10. Suppose f is a map in $\mathcal{C}$ admitting a $\mathcal{Z}$-kernel. Then the $\mathcal{Z}$-kernel of f is $\mathcal{Z}$-trivial if and only if, for every map g such that $f \cdot g$ is defined and $\mathcal{Z}$-trivial, the map g itself is $\mathcal{Z}$-trivial.

Proof. Let k be the $\mathcal{Z}$-kernel of f. If k is $\mathcal{Z}$-trivial and g is such that $f \cdot g$ is defined and $\mathcal{Z}$-trivial, then g factors through k, which is a $\mathcal{Z}$-trivial map, and so g is itself $\mathcal{Z}$-trivial. The converse is immediate. $\square$

Proposition 3.11. In $\mathcal{C}$, $\mathcal{Z}$-normal epimorphisms satisfy the left lifting property with respect to maps with $\mathcal{Z}$-trivial $\mathcal{Z}$-kernel.

Proof. Consider the following commutative square,

$$\begin{tikzcd}
\cdot \arrow[r, "v"] \arrow[d, "f"', -{Triangle[open]}] & \cdot \arrow[d, "g", o->] \\
\cdot \arrow[r, "u"'] & \cdot
\end{tikzcd}$$

where f is the $\mathcal{Z}$-cokernel of a map a and and g is a map whose $\mathcal{Z}$-kernel b is $\mathcal{Z}$-trivial. Since $g \cdot v \cdot a$ is $\mathcal{Z}$-trivial, it follows from Proposition 3.10 that $v \cdot a$ is $\mathcal{Z}$-trivial as well. Hence, there exists a unique map d such that $v = d \cdot f$, and by the epimorphicity of f, we further deduce that $g \cdot d = u$. $\square$

4 Non-pointed prenormal categories

Building on the preparatory material of the previous section, we are now ready to formalise the generalisation of prenormal categories to the non-pointed setting.

Definition 4.1. Let $\mathcal{C}$ be a category and $\mathcal{Z}$ a subcategory. We say that $\mathcal{C}$ is *prenormal with respect to* $\mathcal{Z}$, or that $\mathcal{C}$ is $\mathcal{Z}$*-prenormal*, if the following properties hold.

0. $\mathcal{Z}$ is mono-coreflective in $\mathcal{C}$.
1. $\mathcal{C}$ has finite limits.
2. $\mathcal{C}$ has $\mathcal{Z}$-cokernels of $\mathcal{Z}$-kernels.
3. $\mathcal{Z}$-normal epimorphisms are pullback-stable in $\mathcal{C}$.

In what follows, when we say that $\mathcal{C}$ is a $\mathcal{Z}$-prenormal category, it is understood that $\mathcal{Z}$ is a subcategory of $\mathcal{C}$, and that $\mathcal{C}$ and $\mathcal{Z}$ satisfy the conditions stated above.

Remark 4.2. Since (pointed) kernels are limits, in (pointed) prenormal categories the existence of kernels follows by finite completeness. In the non-pointed case the situation is less straightforward: the existence of $\mathcal{Z}$-kernels of identity maps is not automatic, but it is equivalent to the mono-coreflectivity of $\mathcal{Z}$, by Proposition 3.7. Once the existence of such kernels is ensured, the existence of every other $\mathcal{Z}$-kernel follows from finite completeness, again by Proposition 3.7.

We can now state one of the main features of prenormal categories in the non-pointed case.

Proposition 4.3. Let $\mathcal{C}$ be a $\mathcal{Z}$-prenormal category. Then $\mathcal{C}$ admits the pair $(\mathsf{NEpi}(\mathcal{C}, \mathcal{Z}), \mathsf{TKer}(\mathcal{C}, \mathcal{Z}))$ as a stable factorisation system.

Proof. Clearly, $\mathsf{NEpi}(\mathcal{C}, \mathcal{Z})$ and $\mathsf{TKer}(\mathcal{C}, \mathcal{Z})$ are closed under compositions with isomorphisms. Moreover, we have already verified in Proposition 3.11 that every $\mathcal{Z}$-normal epimorphism is orthogonal to every map with $\mathcal{Z}$-trivial $\mathcal{Z}$-kernel. Now, given any morphism f in $\mathcal{C}$, consider the following diagram,

$$\begin{tikzcd}
\cdot \arrow[r, tail, "k"] \arrow[d, "e'"'] & \cdot \arrow[r, "f"] \arrow[d, "e"] & \cdot \\
\cdot \arrow[r, tail, "h"'] & \cdot \arrow[ur, bend right, "m"'] &
\end{tikzcd}$$

where k is the $\mathcal{Z}$-kernel of f, e is the $\mathcal{Z}$-cokernel of k and m is the unique map such that $f = m \cdot e$, by the universal property of e. The morphism h is the $\mathcal{Z}$-kernel of m and e' is the unique map such that $e \cdot k = h \cdot e'$ by the universal property of h. The left square is a pullback, and therefore e' is a $\mathcal{Z}$-normal epimorphism. Since $h \cdot e' = e \cdot k$ is $\mathcal{Z}$-trivial, it follows from the dual of Proposition 3.3, that h is itself $\mathcal{Z}$-trivial, and thus f factors as a $\mathcal{Z}$-normal epimorphism followed by a map with a $\mathcal{Z}$-trivial $\mathcal{Z}$-kernel, as desired. □

Before proceeding further, we pause to make a few general remarks on some features of the definition of non-pointed prenormal category and alternative approaches to similar ideas.

Remark 4.4. In this remark we discuss the choice of working with a distinguished class of trivial objects rather than an ideal of morphisms (see again [7, 14, 15]). Of course, Definition 4.1 could easily be reformulated using ideals of morphisms rather than trivial objects. However, some crucial results would not carry over. In particular, Proposition 4.3 no longer holds in the ideal-based approach: what fails is the dual of the cancellation property in Proposition 3.3. The following example illustrates this failure. (Notice that, while Proposition 4.3 is stated for finitely complete categories, the proof only requires the existence of pullbacks, which exist in the following example.)

Consider the category $\mathcal{C}$ having just one object $*$, whose endomorphisms are all positive integers, and with composition given by multiplication. This category has pullbacks. In fact, given any positive integers m and n, let d be their greatest common divisor, and write $m = ad$ and $n = bd$. Then it is easy to check that the following square is the pullback of m along n.

$$\begin{tikzcd}
* \arrow[r, "a"] \arrow[d, "b"'] & * \arrow[d, "n"] \\
* \arrow[r, "m"'] & *
\end{tikzcd}$$

Consider now the ideal $\mathcal{N}$ of arrows in $\mathcal{C}$ consisting of positive integers divisible by 10. This ideal is obviously not *closed*, meaning that there is no class of objects $\mathcal{Z}$ such that $\mathcal{N}$ is the class of $\mathcal{Z}$-trivial maps. Given any map n, its $\mathcal{N}$-kernel and $\mathcal{N}$-cokernel are as follows:

$$\mathcal{N}\text{-ker}(n) = \mathcal{N}\text{-coker}(n) = 2^{D(n,5)} \cdot 5^{D(n,2)},$$

with $D(n,k) = 0$ if k divides n and $D(n,k) = 1$ otherwise. We notice then that the set of $\mathcal{N}$-normal epimorphisms is just $\{1, 2, 5, 10\}$, and it coincides with the set of $\mathcal{N}$-normal monomorphisms. Since $\mathcal{N}$-normal monomorphisms are always pullback-stable (as one quickly checks), we readily conclude that $\mathcal{N}$-normal epimorphisms are pullback-stable in $\mathcal{C}$.

We show now that in this category, despite it having pullbacks, $\mathcal{N}$-kernels, $\mathcal{N}$-cokernels and stable $\mathcal{N}$-normal epimorphisms, we do not have a $(\mathsf{NEpi}(\mathcal{C}, \mathcal{N}), \mathsf{TKer}(\mathcal{C}, \mathcal{N}))$-factorisation. First of all, notice that if f is a map which factors as $f = m \cdot q$, with $q \in \mathsf{NEpi}(\mathcal{C}, \mathcal{N})$ and $m \in \mathsf{TKer}(\mathcal{C}, \mathcal{N})$ and k is the $\mathcal{N}$-kernel of f, then q is the $\mathcal{N}$-cokernel of k (this is an easily proved general fact).

Consider now the map 25 and suppose it factors as $25 = m \cdot q$ with $q \in \mathsf{NEpi}(\mathcal{C}, \mathcal{N})$ and $m \in \mathsf{TKer}(\mathcal{C}, \mathcal{N})$. The $\mathcal{N}$-kernel of 25 is $k = 2$. Since q must be the $\mathcal{N}$-cokernel of k, we have $q = 5$, and so m must be 5 as well. However, the $\mathcal{N}$-kernel of 5 is 2, which is not in $\mathcal{N}$, and so $m = 5 \notin \mathsf{TKer}(\mathcal{C}, \mathcal{N})$.

Remark 4.5. Different approaches exist in the literature that introduce non-pointed notions of normal epimorphisms (or normal monomorphisms) that can participate in (possibly stable) factorisation systems. In particular, in the very recent work [17], a 'normal monomorphism' is defined – in a category with enough (co)limits – as a map $f \colon X \to Y$ whose pushout along the terminal map $X \to 1$ is also a pullback, thereby bypassing the necessity for a (reflective) subcategory $\mathcal{Z}$ of trivial objects altogether. Note however that if $\mathcal{Z}$ is taken to be the subcategory of terminal objects, then $\mathcal{Z}$-normal monomorphisms (in the sense of 3.4) are normal monomorphisms in the sense of [17], though the converse is not true in general (for instance, in Set, any function $1 \to 1 + 1$ provides a counterexample). Moreover, such $\mathcal{Z}$ is not generally coreflective. These features highlight notable differences between [17] and the present paper. Nevertheless, the normal monomorphisms from [17], and their dual notion of normal epimorphisms, can still be part of (not necessarily stable) factorisation systems, as established in [17, Theorem 4.2] and examples in the same work.

We conclude the section by highlighting a few properties of trivial objects in non-pointed prenormal categories, most of which are trivial or invisible in the pointed case.

In the definition of a $\mathcal{Z}$-prenormal category $\mathcal{C}$, we require the subcategory $\mathcal{Z}$ of trivial objects to be mono-coreflective, but not necessarily reflective, as the latter is not needed in general (see Example 9.b). However, $\mathcal{Z}$ does satisfy a weaker but closely related condition. In the remainder of the section, we make this condition precise and explore its implications.

Definition 4.6. Given a category $\mathcal{E}$ and a subcategory $\mathcal{A}$, we say that $\mathcal{A}$ is *subreflective* in $\mathcal{E}$ if the following holds: letting $\mathcal{E}^{\to \mathcal{A}}$ denote the full subcategory of $\mathcal{E}$ consisting of those objects that admit a morphism into some object of $\mathcal{A}$, then every object in $\mathcal{E}^{\to \mathcal{A}}$ admits a reflection into $\mathcal{A}$; in other words, $\mathcal{A}$ is a reflective subcategory of $\mathcal{E}^{\to \mathcal{A}}$.

Proposition 4.7. If $\mathcal{C}$ is a $\mathcal{Z}$-prenormal category, then $\mathcal{Z}$ is subreflective in $\mathcal{C}$.

Proof. It suffices to observe that for every object X in $\mathcal{C}^{\to \mathcal{Z}}$, id_X is a $\mathcal{Z}$-normal monomorphism, and therefore admits a $\mathcal{Z}$-cokernel. □

Remark 4.8. In finitely complete categories, the notions of reflectivity and subreflectivity are related as follows: a subreflective subcategory of a finitely complete category is reflective if and only if it is closed under (finite) limits, if and only if it contains the terminal object.

Moreover, if a subcategory $\mathcal{Z}$ of a category $\mathcal{C}$ is both subreflective and mono-coreflective, then it satisfies several interesting structural properties:

1. $\mathcal{Z}$ is closed under colimits and non-empty limits existing in $\mathcal{C}$.
2. If $\mathcal{C}$ admits colimits of a given shape, then maps from a colimit of that shape are $\mathcal{Z}$-trivial if and only if their components are.
3. If $\mathcal{C}$ admits limits of a given non-empty shape, then maps to limits of that shape are $\mathcal{Z}$-trivial if and only if their components are.
4. If an object X admits a reflection (R_X, η_X), then η_X is an epimorphism (and therefore (R_X, η_X) is the $\mathcal{Z}$-cokernel of id_X).

5. Every strong monomorphism in $\mathcal{C}$ has a $\mathcal{Z}$-trivial $\mathcal{Z}$-kernel.

Items 1 to 4 can be established essentially as in the reflective case, with minor adaptations, while Item 5 follows from Item 4.

5 Prenormality and factorisation systems

This section is devoted to establishing and proving the non-pointed analogue of Proposition 2.4, providing a characterisation of prenormal categories in terms of certain factorisation systems.

Proposition 5.1. Let $\mathcal{C}$ be a category with finite limits, and let $\mathcal{Z}$ be a mono-coreflective subcategory of $\mathcal{C}$. Then the following are equivalent.

(i) $\mathcal{C}$ is $\mathcal{Z}$-prenormal.

(ii) $\mathcal{C}$ admits $(\mathsf{NEpi}(\mathcal{C},\mathcal{Z}),\mathsf{TKer}(\mathcal{C},\mathcal{Z}))$ as a stable factorisation system.

If $\mathcal{Z}$ is subreflective (see Definition 4.6), then the following are also equivalent to Items (i) and (ii).

(iii) $\mathcal{C}$ admits a stable factorisation system of the form $\big(\mathsf{NEpi}(\mathcal{C},\mathcal{Z}),\mathscr{M}\big)$ for some class $\mathscr{M}$ of arrows in $\mathcal{C}$.

(iv) $\mathcal{C}$ admits a stable factorisation system of the form $\big(\mathscr{E},\mathsf{TKer}(\mathcal{C},\mathcal{Z})\big)$ for some class $\mathscr{E}$ of arrows in $\mathcal{C}$.

Proof. We have already proved that (i) $\Longrightarrow$ (ii) in Proposition 4.3, and obviously (ii) $\Longrightarrow$ (iii) and (ii) $\Longrightarrow$ (iv). We now prove the remaining implications. Let us begin by showing that (ii) $\Longrightarrow$ (i).

- Suppose (ii) holds. To show that $\mathcal{C}$ is $\mathcal{Z}$-prenormal we just need to prove that $\mathcal{C}$ admits $\mathcal{Z}$-cokernels of $\mathcal{Z}$-kernels. Let (A,f) be the $\mathcal{Z}$-kernel of a map $g\colon B\to C$. Consider the factorisation $g=m\cdot e$, with $e\in\mathsf{NEpi}(\mathcal{C},\mathcal{Z})$ and $m\in\mathsf{TKer}(\mathcal{C},\mathcal{Z})$. It follows from Proposition 3.10 that (A,f) is also the $\mathcal{Z}$-kernel of e, and therefore e is the $\mathcal{Z}$-cokernel of f by the dual of Remark 3.2.

We now proceed to prove that (iii) $\Longrightarrow$ (ii) and that (iv) $\Longrightarrow$ (ii) under the assumption that $\mathcal{Z}$ is subreflective.

- Suppose (iii) holds. We show that $\mathscr{M}=\mathsf{TKer}(\mathcal{C},\mathcal{Z})$. We know from Proposition 3.11 that $\mathcal{Z}$-normal epimorphisms satisfy the left lifting property with respect to maps with a $\mathcal{Z}$-trivial $\mathcal{Z}$-kernel. It then follows from Item 2 in Proposition 1.3 that $\mathsf{TKer}(\mathcal{C},\mathcal{Z})\subseteq\mathscr{M}$. It remains to prove the reverse inclusion. Let $f\colon A\to B$ be a map in $\mathscr{M}$, and let (K,k) be its $\mathcal{Z}$-kernel (whose existence is guaranteed by Remark 4.2). Since the composite $f\cdot k\colon K\to B$ is $\mathcal{Z}$-trivial, subreflectivity of $\mathcal{Z}$ ensures that K admits a reflection (R_K,η_K) in $\mathcal{Z}$. We thus obtain the following commutative square for some map $R_K\to B$.

$$\begin{tikzcd} K \arrow[r, "k"] \arrow[d, "\eta_K"'] & A \arrow[d, "f"] \\ R_K \arrow[r] & B \end{tikzcd}$$

  By Item 4 in Remark 4.8, η_K is a $\mathcal{Z}$-normal epimorphism, and hence it satisfies the left lifting property with respect to maps in $\mathscr{M}$. We then obtain a map $d\colon R_K\to A$ such that $k=d\cdot\eta_K$, and so k is trivial, as desired.

- Suppose now that (iv) holds. We show that $\mathscr{E} = \mathsf{NEpi}(\mathcal{C}, \mathcal{Z})$. Arguing as in the previous point, we have that $\mathsf{NEpi}(\mathcal{C}, \mathcal{Z}) \subseteq \mathscr{E}$. We need to prove the reverse inclusion. We proceed in steps.
  a) First let us show that if $f\colon A \to B$ lies in $\mathscr{E}$, and $g\colon B \to C$ is such that the composite $g \cdot f$ is $\mathcal{Z}$-trivial, then g is itself $\mathcal{Z}$-trivial. Consider a coreflection (S_C, ε_C) of C in $\mathcal{Z}$, so that we have the following factorisation for some map $A \to S_C$.

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "f"'] & S_C \arrow[d, "\varepsilon_C"] \\
B \arrow[r, "g"'] & C
\end{tikzcd}$$

  Since ε_C has a $\mathcal{Z}$-trivial $\mathcal{Z}$-kernel (as it is a $\mathcal{Z}$-normal monomorphism – see Section 3) and $f \in \mathscr{E}$, there exists a map $d\colon B \to S_C$ such that $g = \varepsilon_C \cdot d$. Hence, g is $\mathcal{Z}$-trivial.

  b) Now let $f\colon A \to B$ be a map in $\mathscr{E}$ and let (K, k) be its $\mathcal{Z}$-kernel. We claim that if $g\colon A \to C$ is an arrow such that $g \cdot k$ is $\mathcal{Z}$-trivial, then there exists a map g' such that $g = g' \cdot f$; in other words, (B, f) satisfies the universal property of a $\mathcal{Z}$-cokernel of k, but without uniqueness.

  Consider the product $B \times C$, with projections p_B and p_C, and let $h\colon A \to B \times C$ be the map with components f and g, as shown in the following diagram.

$$\begin{tikzcd}
 & B & \\
A \arrow[ur, "f"] \arrow[rr, "h" description] \arrow[dr, "g"'] & & B \times C \arrow[ul, "p_B"'] \arrow[dl, "p_C"] \\
 & C &
\end{tikzcd}$$

  We can factor h as $h = m \cdot e$, for some $e\colon A \to M$ in $\mathscr{E}$ and $m\colon M \to B \times C$ in $\mathsf{TKer}(\mathcal{C}, \mathcal{Z})$. Since f and e are both in $\mathscr{E}$, we can apply Item 3 of Proposition 1.3 to the equality $f = p_B \cdot m \cdot e$ and immediately conclude that $\varphi := p_B \cdot m \in \mathscr{E}$.

  Next, let (S_B, ε_B) be a coreflection of B in $\mathcal{Z}$. Since (K, k) is the $\mathcal{Z}$-kernel of f, there exists a (unique) map $w\colon K \to S_B$ such that the square of sides f, ε_B, w and k is a pullback (see Proposition 3.7). We can then construct the diagram below, where (L, l, y) is the pullback of ε_B and $p_B \cdot m$, and x is the unique map such that $y \cdot x = w$ and $l \cdot x = e \cdot k$.

$$\begin{tikzcd}
K \arrow[r, "x"] \arrow[d, "k"] \arrow[rr, bend left, "w"] & L \arrow[r, "y"] \arrow[d, "l"] & S_B \arrow[d, "\varepsilon_B"] \\
A \arrow[r, "e"'] \arrow[rr, bend right, "f"'] & M \arrow[r, "p_B \cdot m"'] & B
\end{tikzcd}$$

  First, since the right-hand square is a pullback, it follows from Proposition 3.7 that (L, l) is a $\mathcal{Z}$-kernel of $p_B \cdot m$. Second, since the outer rectangle is a pullback (as mentioned above), we deduce the left-hand square is also a pullback. Hence, $x \in \mathscr{E}$ as it is a pullback of $e \in \mathscr{E}$.

Consider now the morphism $m \cdot l \cdot x = m \cdot e \cdot k = h \cdot k$ into the product $B \times C$. The components of this morphism are $f \cdot k$ and $g \cdot k$, both $\mathcal{Z}$-trivial by hypothesis. By subreflectivity of $\mathcal{Z}$ it follows that $k \cdot h$, and hence $m \cdot l \cdot x$, is $\mathcal{Z}$-trivial (see Remark 4.8, Item 3). We can then apply Item a) above and Proposition 3.10 to cancel m on the left and x on the right, and conclude that l itself is $\mathcal{Z}$-trivial.

As previously observed, (L, l) is the $\mathcal{Z}$-kernel of $\varphi = p_B \cdot m$, and so we have shown that $\varphi \in \mathsf{TKer}(\mathcal{C}, \mathcal{Z})$. We also established earlier that $\varphi \in \mathscr{E}$ as well, and hence φ is an isomorphism (use Proposition 1.3). We can therefore define the map $g' = p_C \cdot m \cdot \varphi^{-1}$, which satisfies $g' \cdot f = g$.

c) Finally, we show that any map in $\mathscr{E}$ is an epimorphism. By Remark 4.8 (Item 5), every strong monomorphism has a $\mathcal{Z}$-trivial $\mathcal{Z}$-kernel. It follows that every map in $\mathscr{E}$ satisfies the left lifting property with respect to strong monomorphisms, which implies – in the presence of equalisers – that they are epimorphisms.

Combining Items b) and c) we obtain that any map in $\mathscr{E}$ is the $\mathcal{Z}$-cokernel of its $\mathcal{Z}$-kernel, as required. ☐

6 General properties of (non-pointed) prenormal categories

In this section, we extend to the general non-pointed case the properties previously established for prenormal categories in Section 2, this time providing full proofs. We begin with a result generalising some well-known properties of pullbacks holding for regular categories.

Lemma 6.1. Let $\mathcal{C}$ be a category with pullbacks.

1. Consider the following pullback diagram in $\mathcal{C}$.

(6.2)
$$\begin{tikzcd}
\cdot \arrow[r, two heads, "q"] \arrow[d, two heads, "g"'] & \cdot \arrow[d, "f"] \\
\cdot \arrow[r, -latex, "p"'] & \cdot
\end{tikzcd}$$

If p is a regular epimorphism, g is a pullback-stable epimorphism and q is an epimorphism, then 6.2 is also a pushout square.

2. Consider then the following commutative diagram in $\mathcal{C}$.

(6.3)
$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "a"] & B \arrow[r, "g"] \arrow[d, "b"] & C \arrow[d, "c"] \\
A' \arrow[r, -latex, "f'"'] & B' \arrow[r, "g'"'] & C'
\end{tikzcd}$$

If f' is a pullback-stable regular epimorphism and both the outer rectangle and the left-hand square are pullbacks, then the right-hand square is also a pullback.

3. (Generalised Barr-Kock theorem) Consider the following commutative diagram where f is a pullback-stable regular epimorphism, (r_0, r_1) is the kernel pair of f and (s_0, s_1) is the kernel pair of g.

$$\begin{tikzcd}
\cdot \arrow[r, shift left, "r_0"] \arrow[r, shift right, "r_1"'] \arrow[d] & \cdot \arrow[r, "f", -latex] \arrow[d] & \cdot \arrow[d] \\
\cdot \arrow[r, shift left, "s_0"] \arrow[r, shift right, "s_1"'] & \cdot \arrow[r, "g"'] & \cdot
\end{tikzcd} \tag{6.4}$$

If either of the left-hand squares is a pullback, then the right-hand square is a pullback.

Proof. Item 1 may be proved essentially as in the regular case. Item 3 may also be proved as in the regular case, using Item 2 (see, for example, [10, Theorem 1.16]). For Item 2, we adapt the proof given in [10] for the analogous result. Consider the following commutative diagram.

$$\begin{tikzcd}
A \arrow[rr, "f", -latex] \arrow[dr, "\alpha"] \arrow[dd, "a"'] & & B \arrow[rr, "g"] \arrow[dr, "\beta"] \arrow[dd, "b"' near start] & & C \arrow[dr, equal] \arrow[dd, "c"' near start] & \\
& P \arrow[rr, "p" near start, -latex] \arrow[dl, "p'"] & & Q \arrow[rr, "q" near start] \arrow[dl, "q'"] & & C \arrow[dl, "c"] \\
A' \arrow[rr, "f'"', -latex] & & B' \arrow[rr, "g'"'] & & C' &
\end{tikzcd}$$

Here (Q, q, q') is the pullback of g' and c, (P, p, p') is the pullback of f' and q', while α and β are the induced maps. Note that p and f are regular epimorphisms, as they arise as pullbacks of f', while α is an isomorphism, since the outer rectangle is a pullback. Therefore, the following square

$$\begin{tikzcd}
A \arrow[r, "f", -latex] \arrow[d, "\alpha"', "\cong"] & B \arrow[d, "\beta"] \\
P \arrow[r, "p"', -latex] & Q
\end{tikzcd} \tag{6.5}$$

is a pullback square where p and f are regular epimorphisms and α is an isomorphism. We can then apply Item 1 and deduce that 6.5 is also a pushout, which proves that β is an isomorphism. □

We can now establish some pullback properties for non-pointed prenormal category, most following from Lemma 6.1.

Proposition 6.6. *In a $\mathcal{Z}$-prenormal category $\mathcal{C}$, consider the following pullback diagram.*

$$\begin{tikzcd}
X' \arrow[r, "f'"] \arrow[d, "x"'] & Y' \arrow[d, "y"] \\
X \arrow[r, "f"'] & Y
\end{tikzcd} \tag{6.7}$$

Diagram 6.7 is also a pushout whenever one of the following conditions holds. (a) f is a $\mathcal{Z}$-normal epimorphism and the $\mathcal{Z}$-coreflection of y is an isomorphism. (b) f is a $\mathcal{Z}$-normal epimorphism and y is a $\mathcal{Z}$-normal monomorphism. (c) f is a regular epimorphism, x is a $\mathcal{Z}$-normal epimorphism and f' is an epimorphism.

Proof. Item (c) follows from Lemma 6.1, Item 1. Item (b) is a special case of Item (a), as the $\mathcal{Z}$-coreflection of a $\mathcal{Z}$-normal monomorphism is always an isomorphism. To prove Item (a), let $\varepsilon\colon S \Rightarrow \mathrm{id}_{\mathcal{C}}$ be a coreflection of $\mathcal{Z}$ in $\mathcal{C}$, and consider the following diagram, where the left square is the pullback of f' along $\varepsilon_{Y'}$.

$$\begin{tikzcd}
K \arrow[r, "k", tail] \arrow[d, "f''", -{Triangle[open]}] & X' \arrow[r, "x"] \arrow[d, "f'", -{Triangle[open]}] & X \arrow[d, "f", -{Triangle[open]}] \\
SY' \arrow[r, "\varepsilon_{Y'}"', tail] & Y' \arrow[r, "y"'] & Y
\end{tikzcd}$$

By Proposition 7.1, the left-hand square is a pushout. Since the $\mathcal{Z}$-coreflection of y is an isomorphism, it follows that $y \cdot \varepsilon_{Y'} = \varepsilon_Y \cdot Sy \cong \varepsilon_Y$ is a $\mathcal{Z}$-coreflection of Y. Therefore, again by Proposition 7.1, the outer rectangle in the above diagram is a pushout. We thus conclude that Diagram 6.7 is a pushout. ☐

Proposition 6.8. In a $\mathcal{Z}$-prenormal category where $\mathcal{Z}$-normal epimorphisms are also regular epimorphisms, the following properties hold.

1. In any diagram as in 6.3, if f' is a $\mathcal{Z}$-normal epimorphism and both the outer rectangle and the left-hand square are pullbacks, it follows that the right-hand square is also a pullback.

2. In any diagram as in 6.4, if f is a $\mathcal{Z}$-normal epimorphism and either of left-hand squares is a pullback, it follows that the right-hand square is also a pullback.

We now resume our list of properties.

Proposition 6.9. Let $\mathcal{C}$ be a $\mathcal{Z}$-prenormal category, and consider the following commutative diagram where f is the $\mathcal{Z}$-kernel of g.

$$\begin{tikzcd}
A' \arrow[r, "f'"] \arrow[d, "a"'] & B' \arrow[r, "g'"] \arrow[d, "b"'] & C' \arrow[d, "c"'] \\
A \arrow[r, "f"', tail] & B \arrow[r, "g"'] & C
\end{tikzcd}$$

1. If the left square is a pullback and g' is the $\mathcal{Z}$-cokernel of f', then $c \in \mathsf{TKer}(\mathcal{C})$.

2. Conversely, if $c \in \mathsf{TKer}(\mathcal{C})$ and f' is the $\mathcal{Z}$-kernel of g', then the left square is a pullback.

Proof. To prove Item 1, notice that by Proposition 3.5 we have that f' is the $\mathcal{Z}$-kernel of the map $d = g \cdot b = c \cdot g'$. Since g' is the $\mathcal{Z}$-cokernel of f', by the construction provided in the proof of Proposition 4.3, we find that $c \cdot g'$ is the $(\mathsf{NEpi}(\mathcal{C}, \mathcal{Z}), \mathsf{TKer}(\mathcal{C}, \mathcal{Z}))$-factorisation of d, and so $c \in \mathsf{TKer}(\mathcal{C}, \mathcal{Z})$.

The partial converse in Item 2 does not rely on $\mathcal{Z}$-prenormality. It readily follows from the universal property of $\mathcal{Z}$-kernels and the characterisation of maps in $\mathsf{TKer}(\mathcal{C}, \mathcal{Z})$ given in Proposition 3.10. ☐

Lemma 6.10. In a $\mathcal{Z}$-prenormal category $\mathcal{C}$, if $f\colon A \rightarrowtriangle B$ and $f'\colon A' \rightarrowtriangle B'$ are $\mathcal{Z}$-normal epimorphism, then $f \times f'\colon A \times A' \to B \times B'$ is also a $\mathcal{Z}$-normal epimorphism.

Proposition 6.11. Let $\mathcal{C}$ be a $\mathcal{Z}$-prenormal category. If C is an object in $\mathcal{C}$, then the slice category $\mathcal{C}/C$ is a a $(\mathcal{Z}/C)$-prenormal category. If $\mathcal{B}$ is any category, then the functor category $\mathcal{C}^{\mathcal{B}}$ is a $\mathcal{Z}^{\mathcal{B}}$-prenormal category.

Proof. See Examples 9.c and 9.d. ☐

In the final part of this section, we highlight some relationships between the notions introduced in this paper and the one in [16]. The main difference between prenormal categories and normal categories as defined in [16] is that, in the former, we do not ask for any relation between normal epimorphisms and regular epimorphisms. This distinction becomes particularly relevant if one wants to work in a non-pointed setting, as highlighted by the following propositions.

Proposition 6.12. Let $\mathcal{C}$ be a finitely complete category and $\mathcal{Z}$ a class of trivial objects (closed under retracts). If $\mathsf{RegEpi}(\mathcal{C}) \subseteq \mathsf{NEpi}(\mathcal{C}, \mathcal{Z})$, then every object in Z in $\mathcal{Z}$ is subterminal.

Proof. In the presence of kernel pairs, maps having the right lifting property with respect to split epimorphisms are monomorphisms. Therefore, if $\mathsf{RegEpi}(\mathcal{C}) \subseteq \mathsf{NEpi}(\mathcal{C}, \mathcal{Z})$, then by Proposition 3.11 we have that $\mathsf{Mono}(\mathcal{C}) \supseteq \mathsf{TKer}(\mathcal{C}, \mathcal{Z})$. In particular, for any object Z in $\mathcal{Z}$ the terminal map $Z \to 1$ is a monomorphism, since it has a $\mathcal{Z}$-trivial $\mathcal{Z}$-kernel – namely (Z, id_Z). ☐

Proposition 6.13. Let $\mathcal{C}$ be a pointed category, and suppose that $\mathcal{C}$ is $\mathcal{Z}$-prenormal for some class $\mathcal{Z}$ of trivial objects. The following conditions are equivalent: (i) $\mathsf{TKer}(\mathcal{C}, \mathcal{Z}) \subseteq \mathsf{Mono}(\mathcal{C})$, (ii) $\mathsf{NEpi}(\mathcal{C}, \mathcal{Z}) \supseteq \mathsf{StrEpi}(\mathcal{C})$, (iii) $\mathsf{NEpi}(\mathcal{C}, \mathcal{Z}) \supseteq \mathsf{RegEpi}(\mathcal{C})$, (iv) $\mathcal{Z}$ is the class of zero objects of $\mathcal{C}$ and $\mathcal{C}$ is normal (in the sense of [16]).

Proof. The only non-obvious implication is (iii) $\implies$ (iv). Suppose that (iii) holds. By Proposition 6.12, it follows that $\mathcal{Z}$ is the class of zero objects (subterminal objects are obviously zero objects in a pointed category), and that $\mathsf{NEpi}(\mathcal{C}, \mathcal{Z}) = \mathsf{NEpi}(\mathcal{C}) = \mathsf{RegEpi}(\mathcal{C})$. Therefore, $\mathcal{C}$ is a pointed, finitely complete category where regular epimorphisms are stable and coincide with normal epimorphisms, i.e. $\mathcal{C}$ is normal. ☐

7 Exact sequences in prenormal categories

In this section, we study the properties of exact sequences in (non-pointed) prenormal categories. We also consider functors that preserve such sequences along with finite limits, and show that they preserve much of the extra structure of prenormal categories.

Proposition 7.1. Let $\mathcal{C}$ be a $\mathcal{Z}$-prenormal category, and let $A \overset{f}{\rightarrowtriangle} B \overset{g}{\twoheadrightarrow} C$ be a $\mathcal{Z}$-exact sequence in $\mathcal{C}$. Then the composite $g \cdot f$ factors through an object Z, i.e. there exist morphisms $\eta \colon A \to Z$ and $\varepsilon \colon Z \to C$ such that the following diagram commutes.

(7.2)
$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "\eta"'] & B \arrow[d, "g"] \\
Z \arrow[r, "\varepsilon"'] & C
\end{tikzcd}$$

For any such factorisation, the following conditions are equivalent: (a) Diagram 7.2 is both a pullback and a pushout; (b) (Z, ε) is a $\mathcal{Z}$-coreflection of C; (c) (Z, η) is a $\mathcal{Z}$-reflection of A.

Proof. By Proposition 3.7, we may consider a $\mathcal{Z}$-coreflection $\varepsilon\colon Z \to C$ of C, and obtain a map $\eta\colon A \to Z$ such that Diagram 7.2 is a pullback. Since g is a $\mathcal{Z}$-normal epimorphism and Diagram 7.2 is a pullback, it follows that η is also a $\mathcal{Z}$-normal epimorphism. By the dual of Proposition 3.8, η is a $\mathcal{Z}$-epi-reflection of A, and by the dual of Proposition 3.7, Diagram 7.2 is a pushout.

Now, given any diagram of the form of 7.2 satisfying either (b) or (c), it must be isomorphic to the one constructed in the previous paragraph. This shows that Conditions (b) and (c) are equivalent, and that each of them implies (a).

Conversely, given a diagram like 7.2 satisfying Condition (a), then η is a $\mathcal{Z}$-trivial $\mathcal{Z}$-normal epimorphism, and thus (b) holds by the dual of Proposition 3.8. □

Proposition 7.3. In a $\mathcal{Z}$-prenormal category $\mathcal{C}$:

1. binary products of $\mathcal{Z}$-exact sequences are $\mathcal{Z}$-exact;
2. the pullback of a $\mathcal{Z}$-exact sequence along a morphism c is $\mathcal{Z}$-exact if and only if $c \in \mathsf{TKer}(\mathcal{C}, \mathcal{Z})$.

Proof. Given two $\mathcal{Z}$-exact sequences

$$\begin{tikzcd} A \arrow[r, "f", {Triangle[open]}->] & B \arrow[r, "g", -{Triangle[open]}] & C \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} A' \arrow[r, "f'", {Triangle[open]}->] & B' \arrow[r, "g'", -{Triangle[open]}] & C', \end{tikzcd}$$

we know that $g \times g'$ is a $\mathcal{Z}$-normal epimorphism by Lemma 6.10. The fact that $f \times f'$ is the $\mathcal{Z}$-kernel of $g \times g'$ follows from the observation that the product of mono-coreflections is a mono-coreflection of the product, together with Proposition 3.7.

Consider the following diagram,

$$\begin{tikzcd} A' \arrow[r, "f'"] \arrow[d, "a"'] & B' \arrow[r, "g'"] \arrow[d, "b"'] & C' \arrow[d, "c"'] \\ A \arrow[r, "f"', {Triangle[open]}->] & B \arrow[r, "g"', -{Triangle[open]}] & C \end{tikzcd}$$

with both squares being pullbacks and the lower row $\mathcal{Z}$-exact. In general, $g' \in \mathsf{NEpi}(\mathcal{C}, \mathcal{Z})$ by $\mathcal{Z}$-prenormality, and f' is the $\mathcal{Z}$-kernel of $g \cdot b = c \cdot g'$ by Proposition 3.5. Moreover, if $c \in \mathsf{TKer}(\mathcal{C}, \mathcal{Z})$ one readily verifies that f' is also the $\mathcal{Z}$-cokernel of g'. The converse implication follows from Proposition 6.9. □

Proposition 7.4. Let $\mathcal{C}$ and $\mathcal{D}$ be categories, and let $\mathcal{U}$ and $\mathcal{V}$ be subcategories of $\mathcal{C}$ and $\mathcal{D}$ respectively. Suppose $\mathcal{C}$ is $\mathcal{U}$-prenormal and $\mathcal{D}$ is $\mathcal{V}$-prenormal. If $F\colon \mathcal{C} \to \mathcal{D}$ is a functor preserving finite limits and sending $\mathcal{U}$-exact sequences to $\mathcal{V}$-exact sequences, then F preserves:

1. trivial objects and trivial maps ($F(\mathcal{U}) \subseteq \mathcal{V}$ and if f is $\mathcal{U}$-trivial, then Ff is $\mathcal{V}$-trivial);
2. coreflections into trivial objects (if (S_X, ε_X) is a coreflection of an object X of $\mathcal{C}$ in $\mathcal{U}$, then $(FS_X, F\varepsilon_X)$ is a coreflection of FX in $\mathcal{V}$);
3. kernels (if (K, k) is a $\mathcal{U}$-kernel of f in $\mathcal{C}$, then (FK, Fk) is a $\mathcal{V}$-kernel of Ff);
4. the normal-epi-trivial-kernel factorisations ($F(\mathsf{TKer}(\mathcal{C}, \mathcal{U})) \subseteq \mathsf{TKer}(\mathcal{D}, \mathcal{V})$ and $F(\mathsf{NEpi}(\mathcal{C}, \mathcal{U})) \subseteq \mathsf{NEpi}(\mathcal{D}, \mathcal{V})$).

Proof. Items 1 and 2 follow from the fact that F sends $\mathcal{U}$-exact sequences to $\mathcal{V}$-exact sequences, together with the following observations:

- an object X in $\mathcal{C}$ (respectively, in $\mathcal{D}$) lies in $\mathcal{U}$ (respectively, in $\mathcal{V}$) if and only if the sequence $X = X = X$ is $\mathcal{U}$-exact (respectively, $\mathcal{V}$-exact);
- for an object X in $\mathcal{C}$ (respectively, in $\mathcal{D}$), we have that (S_X, ε_X) is a $\mathcal{U}$-coreflection (respectively, a $\mathcal{V}$-coreflection) if and only if the sequence $S_X \xrightarrow{\varepsilon_X} X = X$ is $\mathcal{U}$-exact (respectively, $\mathcal{V}$-exact).

Item 3 follows immediately from the fact that F preserves finite limits, from Item 2 above and from the description of kernels in Proposition 3.7.

To prove that $F(\mathsf{TKer}(\mathcal{C}, \mathcal{U})) \subseteq \mathsf{TKer}(\mathcal{D}, \mathcal{V})$ just use Items 1 and 3. To prove that $F(\mathsf{NEpi}(\mathcal{C}, \mathcal{U})) \subseteq \mathsf{NEpi}(\mathcal{D}, \mathcal{V})$, just notice that any $\mathcal{U}$-normal epimorphism f is part of a $\mathcal{U}$-exact sequence $\cdot \xrightarrow{k} \cdot \xrightarrow{f} \cdot$, where k is the $\mathcal{U}$-kernel of f. □

8 A weaker notion: semi-prenormal categories

In this section we want to explore a relaxation of the axioms of prenormality, which we refer to as *semi-prenormality*. Condition 3 in Definition 8.1 below was first studied – even in the non-pointed context – by Grandis ([13, 14]), albeit with different hypotheses, purposes and terminology. Closely related conditions in the pointed context were also investigated in [20]. Besides the examples found in Sections 2 and 9, further instances of semi-prenormal categories appear in [13], and a more recent one in [6].

Definition 8.1. Let $\mathcal{C}$ be a category and let $\mathcal{Z}$ be a subcategory of $\mathcal{C}$. We say that $\mathcal{C}$ is $\mathcal{Z}$-*semi-prenormal* if the following properties hold.

0. $\mathcal{Z}$ is mono-coreflective in $\mathcal{C}$;
1. $\mathcal{C}$ admits pullbacks along $\mathcal{Z}$-normal monomorphisms;
2. $\mathcal{C}$ admits $\mathcal{Z}$-cokernels of $\mathcal{Z}$-kernels;
3. the pullback of a $\mathcal{Z}$-normal epimorphism along a $\mathcal{Z}$-normal monomorphism is a $\mathcal{Z}$-normal epimorphism.

In his works [13, 14], Grandis did not relate the partial stability of normal epimorphisms to the factorisation system we have considered in this paper. Nevertheless, the characterisation given in Proposition 5.1 still holds for semi-prenormal categories, though in the following weaker form.

Proposition 8.2. Let $\mathcal{C}$ be a category and let $\mathcal{Z}$ be a mono-coreflective subcategory of $\mathcal{C}$. Suppose $\mathcal{C}$ admits pullbacks along $\mathcal{Z}$-normal monomorphisms. Then the following are equivalent.

1. $\mathcal{C}$ is $\mathcal{Z}$-semi-prenormal.
2. $\mathcal{C}$ admits $(\mathsf{NEpi}(\mathcal{C}, \mathcal{Z}), \mathsf{TKer}(\mathcal{C}, \mathcal{Z}))$ as a factorisation system, and the class $\mathsf{NEpi}(\mathcal{C}, \mathcal{Z})$ is stable under pullbacks along $\mathcal{Z}$-normal monomorphisms.

If $\mathcal{C}$ admits binary products and equalisers and $\mathcal{Z}$ is subreflective, then the following are also equivalent to Items 1 and 2.

3. $\mathcal{C}$ admits a factorisation system of the form $\big(\mathsf{NEpi}(\mathcal{C},\mathcal{Z}),\mathscr{M}\big)$ for some class $\mathscr{M}$ of arrows in $\mathcal{C}$, and the class $\mathsf{NEpi}(\mathcal{C},\mathcal{Z})$ is stable under pullbacks along $\mathcal{Z}$-normal monomorphisms.
4. $\mathcal{C}$ admits a factorisation system of the form $\big(\mathscr{E},\mathsf{TKer}(\mathcal{C},\mathcal{Z})\big)$ for some class $\mathscr{E}$ of arrows in $\mathcal{C}$ stable under pullbacks along $\mathcal{Z}$-normal monomorphisms.

Proof. The proof of Proposition 5.1 essentially works without any changes for this proposition as well. ☐

Some other properties we proved for $\mathcal{Z}$-prenormal categories (in particular those in Propositions 4.7, 6.6.(b), 6.9, 6.11, 7.1) extend to $\mathcal{Z}$-semi-prenormal categories with no changes, as they neither require the full stability of normal epimorphisms nor the existence of all finite limits.

Furthermore, Grandis, in his work [13], also considered another interesting property which is closely related to semi-prenormality: a non-pointed generalisation of Noether's first isomorphism theorem. Suppose $\mathcal{C}$ is a category with a mono-coreflective subcategory $\mathcal{Z}$ satisfying Conditions 1 and 2 of Definition 8.1 of $\mathcal{Z}$-semi-prenormality. Then, for any diagram of the form shown below on the left, where m, n (and therefore j) are $\mathcal{Z}$-normal monomorphisms, we can construct the corresponding diagram on the right, where p, q and r are the $\mathcal{Z}$-cokernels of m, n and j respectively, and φ and ψ are the induced maps making the diagram commutative.

$$\begin{tikzcd} & N \arrow[ld, bend right, "j"'] \arrow[d, "n"] \\ M \arrow[r, "m"'] & A \end{tikzcd} \qquad \begin{tikzcd} N \arrow[r, equal] \arrow[d, "j"'] & N \arrow[d, "n"] & \\ M \arrow[r, "m"'] \arrow[d, "r"] & A \arrow[r, "p"] \arrow[d, "q"] & A/M \arrow[d, equal] \\ M/N \arrow[r, "\varphi"'] & A/N \arrow[r, "\psi"'] & A/M \end{tikzcd} \tag{8.3}$$

The following proposition holds, and its core argument can be found in [13], though formulated in a different setting.

Proposition 8.4. Let $\mathcal{C}$ be a category and $\mathcal{Z}$ a mono-coreflective subcategory satisfying Conditions 1 and 2 of Definition 8.1 of $\mathcal{Z}$-semi-prenormality. The following are equivalent.

1. $\mathcal{C}$ and $\mathcal{Z}$ satisfy Condition 3 of Definition 8.1 (i.e. $\mathcal{C}$ is $\mathcal{Z}$-semi-prenormal).
2. The following properties hold for $\mathcal{C}$:
    - 2a) the composition of $\mathcal{Z}$-normal epimorphisms is a $\mathcal{Z}$-normal epimorphism;
    - 2b) in a situation like the one depicted in Diagram 8.3, the diagram on the right has $\mathcal{Z}$-exact rows (i.e. M/N is a $\mathcal{Z}$-normal subobject of A/N and $\frac{A/N}{M/N} \cong A/M$).

Proof. A similar equivalence is proved by Grandis in [13] under the hypothesis of composable $\mathcal{Z}$-normal epimorphisms, which we know is satisfied for $\mathcal{Z}$-semi-prenormal categories, since $\mathcal{Z}$-normal epimorphisms are part of a factorisation system. ☐

9 Examples

9.a Pointed examples and non-examples

In a pointed category $\mathcal{C}$, the subcategory $\mathcal{Z}$ of zero objects is automatically epi-reflective and mono-coreflective, and $\mathcal{C}$ is $\mathcal{Z}$-prenormal (respectively $\mathcal{Z}$-semi-prenormal) if and only if it is prenormal (respectively semi-prenormal).

9.b Trivial examples

For every category $\mathcal{C}$, the largest coreflective subcategory of $\mathcal{C}$ is $\mathcal{C}$ itself. If $\mathcal{C}$ has finite limits then it is $\mathcal{C}$-prenormal. $\mathcal{C}$-normal epimorphisms are exactly isomorphisms, and every map has a $\mathcal{C}$-trivial $\mathcal{C}$-kernel.

If a category $\mathcal{C}$ has an initial object, then the full subcategory $\mathcal{Z}$ of initial objects is the smallest coreflective subcategory of $\mathcal{C}$. If the initial object is strict (meaning that every morphism in $\mathcal{C}$ having the initial object as codomain is an isomorphism) and $\mathcal{C}$ is finitely complete, then $\mathcal{C}$ is $\mathcal{Z}$-prenormal, and once again $\mathcal{Z}$-normal epimorphisms coincide with isomorphisms, and every map has a $\mathcal{Z}$-trivial $\mathcal{Z}$-kernel. In this case $\mathcal{Z}$ is not reflective in $\mathcal{C}$ (unless $\mathcal{Z} = \mathcal{C}$).

9.c Slices

Let $\mathcal{C}$ be a $\mathcal{Z}$-prenormal category and C an object in $\mathcal{C}$. It is well-known that when $\mathcal{C}$ is finitely complete, then so is the the slice category $\mathcal{C}/C$. In particular, non-empty connected limits (such as pullbacks) are computed as in $\mathcal{C}$. The category $\mathcal{Z}/C$ is a mono-coreflective subcategory of $\mathcal{C}/C$, where the coreflection in $\mathcal{Z}/C$ of an object $x\colon X \to C$ in $\mathcal{C}/C$ is simply $(x \cdot \varepsilon_X\colon S_X \to C, \varepsilon_X)$, with (S_X, ε_X) a coreflection of X in $\mathcal{Z}$. It is easy to check that a map in $\mathcal{C}/C$ is $(\mathcal{Z}/C)$-trivial if and only if the same map, seen as a map in $\mathcal{C}$, is $\mathcal{Z}$-trivial.

By the above description of pullbacks and coreflections, it follows that $(\mathcal{Z}/C)$-kernels can be computed as $\mathcal{Z}$-kernels in $\mathcal{C}$.

Suppose now that $(h\colon H \to C, k)$ is the $(\mathcal{Z}/C)$-kernel of a map f in $\mathcal{C}/C$ from $x\colon X \to C$ to $y\colon Y \to C$. By the previous paragraph, we have that k is the $\mathcal{Z}$-kernel of f in $\mathcal{C}$. We can thus consider its $\mathcal{Z}$-cokernel (P, q) in $\mathcal{C}$. Since $f \cdot k$ is $(\mathcal{Z}/C)$-trivial in $\mathcal{C}/C$, then it is also $\mathcal{Z}$-trivial in $\mathcal{C}$. Therefore, there exists a map f' in $\mathcal{C}$ such that $f = f' \cdot q$. Call p the composite $y \cdot f'$.

$$\begin{tikzcd}
H \arrow[r, "k", {Triangle[open]}->] \arrow[ddr, bend right, "h"'] & X \arrow[r, "q", -{Triangle[open]}] \arrow[d, "f"] \arrow[dd, bend right, "x"'] & P \arrow[dl, bend left, "f'"] \arrow[ddl, bend left, "p"] \\
 & Y \arrow[d, "y"] & \\
 & C &
\end{tikzcd}$$

One can show that $(p\colon P \to C, q)$ is a $(\mathcal{Z}/C)$-cokernel of k. Since every $(\mathcal{Z}/C)$-cokernel is also the $(\mathcal{Z}/C)$-cokernel of its $(\mathcal{Z}/C)$-kernel, we can conclude that a map is a $(\mathcal{Z}/C)$-normal epimorphism in $\mathcal{C}/C$ if and only if the same map is a $\mathcal{Z}$-normal epimorphism in $\mathcal{C}$. Stability of $(\mathcal{Z}/C)$-normal epimorphisms in $\mathcal{C}/C$ then follows from the stability of $\mathcal{Z}$-normal epimorphisms in $\mathcal{C}$.

9.d Functor categories

Let $\mathcal{C}$ be a $\mathcal{Z}$-prenormal category, and let $\mathcal{B}$ be any category. Then the functor category $\mathcal{C}^{\mathcal{B}}$ is $(\mathcal{Z}^{\mathcal{B}})$-prenormal. First, the subcategory $\mathcal{Z}^{\mathcal{B}}$ is mono-coreflective in $\mathcal{C}^{\mathcal{B}}$: if $S\colon \mathcal{Z} \to \mathcal{C}$ is a right adjoint to the inclusion $I\colon \mathcal{Z} \to \mathcal{C}$, then $S \cdot (-)\colon \mathcal{C}^{\mathcal{B}} \to \mathcal{Z}^{\mathcal{B}}$ is right adjoint to the inclusion $I \cdot (-)\colon \mathcal{Z}^{\mathcal{B}} \to \mathcal{C}^{\mathcal{B}}$. The component of the counit of this adjunction at a functor $F\colon \mathcal{B} \to \mathcal{C}$ is obtained as the whiskering $\varepsilon \cdot F$, where ε is the counit of the adjunction $I \dashv S$. Since limits existing in $\mathcal{C}$ also exist in $\mathcal{C}^{\mathcal{B}}$ and they are computed pointwise in $\mathcal{C}^{\mathcal{B}}$, the latter is finitely complete, and $(\mathcal{Z}^{\mathcal{B}})$-kernels in $\mathcal{C}^{\mathcal{B}}$ are also computed pointwise. Finally, one easily checks that $(\mathcal{Z}^{\mathcal{B}})$-cokernels of $(\mathcal{Z}^{\mathcal{B}})$-kernels exist and are computed pointwise as well.

9.e Categories of monomorphisms

For any category $\mathcal{E}$, we denote by $\mathsf{Arr}(\mathcal{E})$ the category whose objects are the arrows of $\mathcal{E}$, and where morphisms from $x\colon X \to X_0$ to $y\colon Y \to Y_0$ are pairs (f, f_0), with $f\colon X \to Y$ and $f_0\colon X_0 \to Y_0$ such that $y \cdot f = f_0 \cdot x$.

Definition 9.1. Let $\mathcal{E}$ be a category, and let $\mathscr{S}$ be a class of monomorphisms in $\mathcal{E}$ containing all identities.

a) Suppose $\mathcal{E}$ has pullbacks. We say that $\mathscr{S}$ is a *semi-stable class of monomorphisms* if the following properties hold.

   1. For all f and g, composable maps in $\mathcal{E}$, if g and $g \cdot f$ are in $\mathscr{S}$, then f is in $\mathscr{S}$;
   2. the pullback of a map in $\mathscr{S}$ along any map in $\mathcal{E}$ is again in $\mathscr{S}$.

b) Suppose $\mathcal{E}$ is finitely complete and call $\mathcal{S}$ the full subcategory of $\mathsf{Arr}(\mathcal{E})$ generated by $\mathscr{S}$. We say that $\mathscr{S}$ is a *stable class of monomorphisms* if $\mathcal{S}$ is closed under finite limits in $\mathsf{Arr}(\mathcal{E})$.

Lemma 9.2. Let $\mathcal{E}$ be a finitely complete category. Then every stable class of monomorphisms $\mathscr{S}$ on $\mathcal{E}$ is semi-stable.

Proof. First, we show that, if $f\colon A \to B$ and $g\colon B \to C$ are maps in $\mathcal{E}$, with $h = g \cdot f \in \mathscr{S}$ and g a monomorphism, then f is in $\mathscr{S}$. Notice that the following square is a pullback in $\mathsf{Arr}(\mathcal{E})$.

$$\begin{tikzcd}
f \arrow[r, "(\mathrm{id}_A, g)"] \arrow[d, "(f, \mathrm{id}_B)"'] & h \arrow[d, "(h, \mathrm{id}_C)"] \\
\mathrm{id}_B \arrow[r, "(g,g)"'] & \mathrm{id}_C
\end{tikzcd}$$

Since $\mathcal{S}$ is closed under limits in $\mathsf{Arr}(\mathcal{E})$, and h, id_B and id_C are all in $\mathscr{S}$, we conclude that $f \in \mathscr{S}$ by closure under limits.

Suppose we are now given a pullback square in $\mathcal{E}$ like the one below on the left, with $f \in \mathscr{S}$. Then the square on the right is a pullback square in $\mathsf{Arr}(\mathcal{E})$.

$$\begin{tikzcd}
A' \arrow[r, "a"] \arrow[d, "f'"'] & A \arrow[d, "f"] \\
B' \arrow[r, "b"'] & B
\end{tikzcd}
\qquad
\begin{tikzcd}
f' \arrow[r, "(a,b)"] \arrow[d, "(f', \mathrm{id}_{B'})"'] & f \arrow[d, "(f, \mathrm{id}_B)"] \\
\mathrm{id}_{B'} \arrow[r, "(b,b)"'] & \mathrm{id}_B
\end{tikzcd}$$

Since f, id_B and $\mathrm{id}_{B'}$ are all in $\mathscr{S}$, we conclude that f' is also in $\mathscr{S}$ by closure under limits. □

Example 9.3. In a category $\mathcal{E}$ with pullbacks, the following classes of monomorphisms are all semi-stable: 1. all monomorphisms; 2. strong monomorphisms; 3. regular monomorphisms; 4. normal monomorphisms, when $\mathcal{E}$ is pointed; 5. more in general, $\mathcal{Z}$-normal monomorphisms for any reflective subcategory $\mathcal{Z}$ of $\mathcal{E}$ (reflectivity ensures that identity maps are $\mathcal{Z}$-normal monomorphisms).

If $\mathcal{E}$ is finitely complete, the classes of monomorphisms and of strong monomorphisms, and any class $\mathscr{R}$ of monomorphisms in $\mathcal{E}$ which is part of a factorisation system $(\mathscr{L},\mathscr{R})$ on $\mathcal{E}$ are all stable classes of monomorphisms.

Proposition 9.4. Let $\mathcal{E}$ be any category and let $\mathscr{S}$ be a class of monomorphisms in $\mathcal{E}$ containing all identity maps. Call $\mathcal{Z}$ the full subcategory of $\mathsf{Arr}(\mathcal{E})$ whose objects are all isomorphisms in $\mathcal{E}$, and call $\mathcal{S}$ the full subcategory of $\mathsf{Arr}(\mathcal{E})$ generated by $\mathscr{S}$.

1. If $\mathcal{E}$ has pullbacks and $\mathscr{S}$ is a semi-stable class of monomorphisms, then $\mathcal{S}$ is $\mathcal{Z}$-semi-prenormal.
2. If $\mathcal{E}$ is finitely complete and $\mathscr{S}$ is a stable class of monomorphisms, then $\mathcal{S}$ is $\mathcal{Z}$-prenormal.

Proof. As $\mathscr{S}$ contains all identities, it is straightforward to verify that $\mathcal{Z}$ is both mono-coreflective and epi-reflective in $\mathcal{S}$. The computation of $\mathcal{Z}$-kernels and $\mathcal{Z}$-cokernels in similar contexts has been studied in various sources (such as [13, 15]), and it is easy to check that, when $\mathcal{E}$ has pullbacks and $\mathscr{S}$ is a semi-stable class of monomorphisms, $\mathcal{Z}$-kernels and $\mathcal{Z}$-cokernels of $\mathcal{Z}$-kernels exist in $\mathcal{S}$. Moreover, $\mathcal{Z}$-normal monomorphisms and $\mathcal{Z}$-normal epimorphisms in $\mathcal{S}$ can be characterised (up to isomorphism) as morphisms of the form (id, s) and (s, id), respectively, with $s \in \mathscr{S}$. Morphisms with $\mathcal{Z}$-trivial $\mathcal{Z}$-kernel are commutative square which are pullbacks in $\mathcal{E}$.

A direct check shows that all the relevant limits in $\mathcal{S}$ required for $\mathcal{Z}$-semi-prenormality or $\mathcal{Z}$-prenormality can be computed level-wise in $\mathcal{E}$. As a result, the stability conditions for $\mathcal{Z}$-normal epimorphisms follow immediately. □

9.f Groupoids

In the category Grpd of small groupoids and functors consider the full subcategory $\mathcal{Z}$ of discrete groupoids (i.e. $\mathcal{Z} \cong \mathsf{Set}$). Then, with this definition of $\mathcal{Z}$, Grpd is $\mathcal{Z}$-prenormal. The construction of $\mathcal{Z}$-kernels and $\mathcal{Z}$-cokernels is described in [13]; a functor $F\colon \mathcal{G} \to \mathcal{H}$ is a $\mathcal{Z}$-normal epimorphism if and only if it is strictly surjective (on arrows) and for any arrows g and g' in $\mathcal{G}$ such that $Fg = Fg'$ there exist arrows u and u' in $\mathcal{G}$ such that Fu and Fu' are identity morphisms in $\mathcal{H}$ and such that the compositions $u \cdot g$ and $g' \cdot u'$ are defined and equal (cf. the analogous condition 2.18 for commutative monoids). It is a straightforward check that this class of morphisms is pullback-stable in Grpd.

9.g Categories of relations

Let $\mathcal{R}$ denote any of the following categories of relations on sets, where morphisms are relation-preserving functions: (1) the category of reflexive relations, (2) the category of preorders, (3) the category of equivalence relations.

Let $\mathcal{Z}$ be the full subcategory of $\mathcal{R}$ consisting of sets equipped with the discrete equivalence relation. For any morphism $f\colon (X,\rho) \to (Y,\rho)$ in $\mathcal{R}$, its $\mathcal{Z}$-kernel is given by $\big((X, \mathrm{K}_f \wedge \sigma), \mathrm{id}_X\big)$, where K_f denotes the kernel pair of f (in Set) and $\wedge$ denotes the intersection of relations. On the other hand, its $\mathcal{Z}$-cokernel is given by $\big((Y_0,\sigma_0),p\big)$, where $p\colon Y\to Y_0$ is the quotient in Set of Y by the equivalence relation on Y generated by $f(\rho)$, and σ_0 is the relation of type $\mathcal{R}$ generated by $p(\sigma)$ on Y_0. By this construction, it follows that a morphism $f\colon (X,\rho)\to(Y,\sigma)$ in $\mathcal{R}$ is a $\mathcal{Z}$-normal epimorphism if and only if f is surjective, σ is the relation of type $\mathcal{R}$ generated by $f(\rho)$ on Y and the kernel pair K_f of f is the smallest equivalence relation on X containing $\mathrm{K}_f\wedge\rho$.

It can be shown that when $\mathcal{R}$ is either the category of reflexive relations or of equivalence relations (cases (1) and (3)), then it is $\mathcal{Z}$-prenormal. The key point in proving the pullback-stability of $\mathcal{Z}$-normal epimorphisms is that, under the hypotheses of Item (1) or (3) above, the image $f(\rho)$ of a relation ρ of type $\mathcal{R}$ under any $\mathcal{Z}$-normal epimorphism $f\colon (X,\rho)\to(Y,\sigma)$ in $\mathcal{R}$ is automatically a relation of type $\mathcal{R}$: namely, a reflexive relation in case (1) and an equivalence relation in case (3).

The category of preordered sets, instead, is not even $\mathcal{Z}$-semi-prenormal. To prove this, consider the morphisms of preordered sets

$$\left\{\ 1\to 2\leftarrow 2'\to 3\ \right\}\xrightarrow{f}\left\{\ 1 \rightleftarrows 2\ \right\}$$

defined by $f(1)=f(3)=1$ and $f(2)=f(2')=2$. Using the above constructions of $\mathcal{Z}$-kernels and $\mathcal{Z}$-cokernels, the morphism f can be explicitly factorised as $f=m\cdot q$, with q the $\mathcal{Z}$-cokernel of the $\mathcal{Z}$-kernel of f, and m the induced map. Computing the $\mathcal{Z}$-kernel of m, one easily finds that it is not $\mathcal{Z}$-trivial, and therefore $\mathcal{R}$ cannot be $\mathcal{Z}$-semi-prenormal (nor $\mathcal{Z}$-prenormal).

9.h Inverse commutative monoids

A commutative monoid M is called an *inverse commutative monoid* if for every $x\in M$ there exists an element $x^{-1}\in M$, called the *inverse* of x, such that

$$xx^{-1}x=x \quad\text{and}\quad x^{-1}xx^{-1}=x^{-1}. \tag{9.5}$$

(in this subsection we shall use multiplicative notation). Thanks to commutativity, such an inverse is unique (see [18]). We denote by ICMon the category of inverse commutative monoids and monoid morphisms (inverses are then automatically preserved).

Remark 9.6. Inverse commutative monoids can be seen as commutative Ω-monoids with Ω consisting of a unique unary operation $(-)^{-1}$ satisfying the axioms in 9.5 above. It immediately follows that ICMon is prenormal (see Example 2.22).

Despite ICMon being pointed, we are now interested in a different class of trivial objects. In a monoid, an element e is said to be *idempotent* if $e^2=e$. A monoid is said to be *idempotent* if all of its elements are idempotent. We denote by ECMon the category of idempotent commutative monoids with monoid morphisms. Clearly, every idempotent commutative monoid is an inverse monoid, where the inverse of an element e is e itself. We now show that ICMon is prenormal with respect to ECMon.

We have functors

$$\begin{tikzcd}\mathsf{ECMon} \arrow[r, bend right, "I"'] & \mathsf{ICMon}, \arrow[l, bend right, "E"']\end{tikzcd}$$

where I denotes the inclusion functor, while E denotes the restriction to idempotents (for any inverse monoid M, the subset $E(M)$ of idempotent elements is an idempotent submonoid, and monoid morphisms obviously preserve idempotents). The functor E is both left and right adjoint to I. The unit η of the adjunction $E \dashv I$ and the counit ε of the adjunction $I \dashv E$ are given by

$$\begin{tikzcd} M \arrow[r, "\eta_M"] & E(M) & & E(M) \arrow[r, hook, "\varepsilon_M"] & M \\ x \arrow[r, mapsto] & xx^{-1} & & e \arrow[r, mapsto] & e \end{tikzcd}$$

for all inverse commutative monoids M. Clearly, η_M is an epimorphism and ε_M a monomorphism (in fact, $\eta_M \cdot \varepsilon_M = \mathrm{id}_{E(M)}$). Taking $\mathcal{Z} = \mathsf{ECMon}$, we have the following characterisations.

- A $\mathcal{Z}$-normal monomorphism is, up to isomorphism, the inclusion of a normal inverse submonoid $A \hookrightarrow M$, that is a submonoid A closed under inverses and conjugation and containing all idempotents of M (see [18]).

- A morphism $f \colon M \to N$ is a $\mathcal{Z}$-normal epimorphism if and only if it is surjective and the restriction $E(f)$ to idempotents is an isomorphism. Indeed, every $\mathcal{Z}$-normal epimorphism is an epimorphism, and its $\mathcal{Z}$-reflection is always an isomorphism. Vice versa, suppose $f \colon M \to N$ is surjective and that $E(f)$ is an isomorphism. Consider its $\mathcal{Z}$-kernel $k \colon K \hookrightarrow M$, consisting of elements $x \in M$ such that $f(x)$ is idempotent. Let $g \colon M \to P$ be a morphism in ICMon such that $g \cdot k$ is $\mathcal{Z}$-trivial (i.e. $g(x)$ is idempotent for all $x \in K$). We must show that if $f(x) = f(y)$, then $g(x) = g(y)$, so that g factors uniquely through f. A simple computation shows that

$$xx^{-1} = xy^{-1}x^{-1}y = yy^{-1},$$

  as they are idempotents and their images via f are equal. Since $f(xy^{-1})$ is obviously idempotent, we have $xy^{-1} \in K$, so $g(xy^{-1})$ is itself idempotent. We can thus write

$$\begin{aligned} g(x) = g(xx^{-1}x) = g(xy^{-1}x^{-1}yx) = g(xy^{-1})g(x^{-1}y)g(x) \\ = g(x^{-1}y)g(x) = g(xx^{-1}y) = g(yy^{-1}y) = g(y). \end{aligned}$$

With these characterisations in place, it is straightforward to verify that the category of inverse commutative monoids is $\mathcal{Z}$-prenormal. Indeed, ICMon is a variety of universal algebra, and is therefore complete and cocomplete. We have seen that $\mathcal{Z} = \mathsf{ECMon}$ is both mono-coreflective and epi-reflective. In particular, all $\mathcal{Z}$-cokernels exist, and can be obtained as pushout along $\mathcal{Z}$-reflections (dual of Proposition 3.7). Now, let f' be the pullback of a $\mathcal{Z}$-normal epimorphism f along an arbitrary morphism. Clearly, f' is surjective (since pullbacks are computed as in Set and thus preserve surjective maps). Moreover, because E is a right adjoint, it preserves pullbacks. Hence, $E(f')$ appears as the pullback of the isomorphism $E(f)$, and is therefore itself an isomorphism. We conclude that f' is a $\mathcal{Z}$-normal epimorphism.

Remark 9.7. This example was inspired by the recent talk [21], where the speaker observed that for any regular epimorphism $f \colon X \to Y$ in the category of inverse monoids, the pullback of f along $E(Y) \hookrightarrow Y$ is also a pushout. This observation led us to wonder whether the category might be prenormal.

Remark 9.8. Note that, with the exception of Remark 9.6 which involves pointedness, everything in this subsection also applies to inverse commutative semigroups. See [18] for more details.

9.i Preordered groups

Let OrdGrp be the category whose objects are preordered groups, namely groups G equipped with a preorder relation $\leq$ satisfying $xx' \leq yy'$ for all $x \leq x'$ and $y \leq y'$ in G, and whose arrows are monotone group homomorphisms. This category is equivalent to the one having as objects the pairs (G, M), where G is a group and M is a submonoid of G closed under conjugation, and morphisms $(G, M) \to (H, N)$ given by a group homomorphisms $f\colon G \to H$ such that $f(M) \subseteq N$. Many remarkable properties of this category have been studied in [5]. In particular, it was proved that OrdGrp is normal (in the sense of [16]), and therefore prenormal as well.

Alongside the usual class of zero objects, in [12], the authors consider another class of trivial objects in OrdGrp, namely the class $\mathcal{Z}$ of pairs of the form $(G, 0)$, with G any group (these pairs correspond exactly to groups equipped with the discrete preorder). In what follows, we prove that OrdGrp is $\mathcal{Z}$-prenormal.

The subcategory $\mathcal{Z}$ is mono-coreflective, with the coreflection of a pair (G, M) given by the inclusion $\mathrm{id}_G\colon (G, 0) \to (G, M)$. Since pullbacks are computed level-wise, we can characterise $\mathcal{Z}$-normal monomorphisms as maps of the form $\mathrm{id}_G\colon (G, M) \to (G, N)$, where M and N are submonoids of the group G closed under conjugation and with $M \subseteq N$. We have that the $\mathcal{Z}$-cokernel of such a $\mathcal{Z}$-normal monomorphism is given by $q\colon (G, N) \to (Q, P)$, where $q\colon G \to Q$ is the quotient in Grp of G by the subgroup M' generated by M in G, and P is simply $q(N)$ (i.e. the regular image of $q|\colon N \to Q$ in Mon). Note that M' is explicitly given by the set of finite products $x_1 x_2 \cdots x_n$ in G, with $x_i \in M$ or $x_i^{-1} \in M$ for all $i \in \{1, \ldots, n\}$; this subgroup is automatically normal in G, since M is closed under conjugation. The universal property of $((Q, P), q)$ can be proved directly using, in particular, the left lifting property of regular epimorphisms with respect to monomorphisms in Mon. We can then characterise $\mathcal{Z}$-normal epimorphisms as maps $f\colon (G, M) \to (H, N)$ such that $f(G) = H$, $f(M) = N$ and such that the kernel of $f\colon G \to H$ is generated in G by the kernel of the monoid morphism $f|\colon M \to N$ (using the above description of $\mathcal{Z}$-cokernels, one can prove that this is the $\mathcal{Z}$-cokernel of $\mathrm{id}_G\colon (G, L) \to (G, M)$, with L being the kernel of the monoid morphism $f|\colon M \to N$). Finally, one can show that this class of maps is pullback-stable, using, in particular, the explicit description of the (normal) subgroup generated by a subset provided above.

Acknowledgments

This research was conducted while both authors were affiliated with INdAM – Istituto Nazionale di Alta Matematica 'Francesco Severi', Gruppo Nazionale per le Strutture Algebriche, Geometriche e le loro Applicazioni (GNSAGA).

References

[1] J. Adámek, H. Herrlich and G. Strecker. *Abstract and concrete categories: The joy of cats*. Repr. Theory Appl. Categ. 17. Theory Appl. Categ., 2006.

[2] F. Borceux. *Handbook of Categorical Algebra 2. Categories and Structure*. Vol. 51. Encyclopedia Math. Appl. Cambridge Univ. Press, 1994.

[3] F. Borceux and D. Bourn. *Mal'cev, Protomodular, Homological and Semi-Abelian Categories*. Vol. 566. Math. Appl. Springer, 2004.

[4] M. M. Clementino, D. Dikranjan and W. Tholen. 'Torsion theories and radicals in normal categories'. In: *J. Algebra* 305.1 (2006), pp. 98–129.

[5] M. M. Clementino, N. Martins-Ferreira and A. Montoli. 'On the categorical behaviour of preordered groups'. In: *J. Pure Appl. Algebra* 223.10 (2019), pp. 4226–4245.

[6] A. Connes and C. Consani. 'Homological algebra in characteristic one'. In: *High. Struct.* 3.1 (2019), pp. 155–247.

[7] C. Ehresmann. 'Cohomologie à valeurs dans une catégorie dominée'. Extraits du Colloque de Topologie, Bruxelles 1964. In: *Oeuvres complètes et commentées.* III.2. 1980, pp. 531–590.

[8] A. Facchini, Finocchiaro C. and M. Gran. 'Pretorsion theories in general categories'. In: *J. Pure Appl. Algebra* 225.2, 106503 (2021), pp. 1–21.

[9] A. Facchini, C. Finocchiaro and M. Gran. 'A new Galois structure in the category of internal preorders'. In: *Theory Appl. Categ.* 35.11 (2020).

[10] M. Gran. 'An Introduction to Regular Categories'. In: *New Perspectives in Algebra, Topology and Categories.* Ed. by M. M. Clementino, A. Facchini and M. Gran. Springer, 2021.

[11] M. Gran, Z. Janelidze and A. Ursini. 'A good theory of ideals in regular multi-pointed categories'. In: *J. Pure Appl. Algebra* 216.8 (2012), pp. 1905–1919.

[12] M. Gran and A. Michel. 'Torsion theories and coverings of preordered groups'. In: *Algebra Univers.* 82, 22 (2021), pp. 1–30.

[13] M. Grandis. *Homological algebra in strongly non-abelian settings*. World Scientific, 2013.

[14] M. Grandis. 'On the categorical foundations of homological and homotopical algebra'. In: *Cah. Topol. Géom. Différ. Catég.* 33.2 (1992), pp. 135–175.

[15] M. Grandis, G. Janelidze and L. Márki. 'Non-pointed exactness, radicals, closure operators'. In: *J. Aust. Math. Soc.* 94.3 (2013), pp. 348–361.

[16] Z. Janelidze. 'The pointed subobject functor, 3×3 lemmas, and subtractivity of spans'. In: *Theory Appl. Categ.* 33.11 (2010), pp. 221–242.

[17] R. Jansen, M. Qasim and W. Tholen. 'The normal decomposition of a morphism in categories without zeros'. In: *J. Algebra* 678 (2025), pp. 392–425.

[18] M. V. Lawson. *Inverse Semigroups. The Theory of Partial Symmetries*. World Scientific, 1998.

[19] S. Mantovani, M. Messora and E. M. Vitale. 'Homotopy torsion theories'. In: *J. Pure Appl. Algebra* 228.12, 107742 (2024), pp. 1–39.

[20] G. Peschke and T. Van der Linden. *A Homological View of Categorical Algebra*. 2024. arXiv: `2404.15896 [math.CT]`.

[21] Ü. Reimaa. 'An intrinsic approach to kernels in general categories'. Talk presented at the International Category Theory Conference, Brno. 2025.

[22] V. Volpe. 'Teorie di pretorsione e Operatori di chiusura'. (Pretorsion theories and closure operators). Master's thesis. University of Turin, 2024.