A ONE PARAMETER FAMILY OF VOLTERRA-TYPE OPERATORS

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Abstract. For every $\alpha \in (0, +\infty)$ and $p, q \in (1, +\infty)$ let T_α be the operator $L^p[0,1] \to L^q[0,1]$ defined via the equality $(T_\alpha f)(x) := \int_0^{x^\alpha} f(y)\,\mathrm{d}y$. We study the norms of T_α for every p, q. In the case $p = q$ we further study its spectrum, point spectrum, eigenfunctions, and the norms of its iterates. Moreover, for the case $p = q = 2$ we determine the point spectrum and eigenfunctions for $T_\alpha^* T_\alpha$, where T_α^* is the adjoint operator.

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1. Introduction

Let $p, q \in (1, +\infty)$ and let p', q' denote the conjugated exponent of p and q, respectively. Pick any real $\alpha > 0$ and let $T_\alpha \colon L^p[0,1] \to L^q[0,1]$ be defined as

$$(T_\alpha f)(x) := \int_0^{x^\alpha} f(y)\,\mathrm{d}y.$$

This operator can be considered as an interpolation depending on α of the projector on the subspace of constants T_0, and the null operator T_∞, passing through classical Volterra operator $V = T_1$.

In the spirit of earlier introductions by Tonelli [17] and Tikhonov [16] of the class of Volterra operators, an operator T should be called "Volterra type" when $(Tf)(x)$ depends on the values of f only in its "past", i.e. in $[0, x]$. In effect, more general classes have also been considered and termed "Volterra", see for example [5] and the extensive bibliography cited therein, but usually modern presentations of this class of operators still retain in some form that property, plus extra hypotheses about their domain (Hilbert spaces, Banach space, general L^p spaces) and about the regularity of the involved kernel associated with the operator. Just to cite the most relevant for this work, this is the case of Barnes [3], where general properties of the spectrum for Volterra operators on L^p spaces are studied, Eveson [6, 7], where the norm of iterations of Volterra operators with convolutive kernels are studied on $L^2[0,1]$ and in $L^p[0,1]$ cases respectively, or even Adell and Gallardo–Gutierrez [1], where the norm of the Liouville–Riemann fractional integration operators are studied for large values of the parameter, again in L^p spaces.

Under this point of view and despite the name we have adopted to denote them, operators T_α with $\alpha < 1$ represent a deviation to this tradition, since for them $(T_\alpha f)(x)$ depends on the values of f on the strictly larger range $[0, x^\alpha]$. In fact, these operators show more varied and somewhat unexpected behaviours; for example they are not quasi-nilpotent, see Theorem 3.

Every T_α is a compact operator, as one can readily deduce from the Kolmogorov–Riesz Theorem and the fact that

$$|(T_\alpha f)(x) - (T_\alpha f)(y)| \le |x^\alpha - y^\alpha|^{1/p'} \|f\|_p$$

for every $x, y \in [0,1]$ and every f in $L^p[0,1]$. Moreover, the adjoint $T_\alpha^* \colon L^{q'}[0,1] \to L^{p'}[0,1]$ is

$$T_\alpha^* f = \int_{x^{1/\alpha}}^1 f(u)\,\mathrm{d}u,$$

so that we have the equality

$$(T_0 - T_\alpha)^* = T_{1/\alpha}, \tag{1}$$

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where now $T_{1/\alpha}$ is considered as a map $L^{q'}[0,1] \to L^{p'}[0,1]$.
Equation (1) shows that there are relations connecting operators T_α, $T_{1/\alpha}$, T^*_α and $T^*_{1/\alpha}$, so that it is a good idea to consider the full family $\{T_\alpha\}_{\alpha>0}$ as a whole, with the classical Volterra operator $V = T_1$ playing in some sense a pivotal role under the correspondence $\alpha \leftrightarrow 1/\alpha$.

In Section 2 we study the norm of T_α. In Section 3 we determine its spectrum σ, point spectrum σ_0 and eigenfunctions. Section 4 contains a result about the behaviour of the norm of iterates T^n_α of high order. Finally, Section 5 is devoted to the special case $p = q = 2$, where we can explicitly describe the spectrum and the eigenvalues of $T^*_\alpha T_\alpha$, and deduce the exact value of the norm of T_α.

Notations: We frequently use Landau symbols $f(x) = O(g(x))$, $f(x) = o(g(x))$ and $f(x) \asymp g(x)$ as $x \to x_0 \in \mathbb{R} \cup \{\pm\infty\}$ with the meaning that $|f(x)/g(x)|$ stays bounded, $|f(x)/g(x)|$ goes to 0 and both $|f(x)/g(x)|$ and $|g(x)/f(x)|$ stay bounded, respectively. The presence of a subscript in any of such symbols means that the symbol is not uniform in the parameter appearing in the subscript.

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2. Norm

Each operator T_α is positive, so that its norm can be computed using nonnegative test functions. For every such a function f and every fixed x the map $\alpha \to (T_\alpha f)(x)$ is decreasing so that also the map $\alpha \to \|T_\alpha\|_{p,q}$ decreases. The following result proves that the maps $\alpha \to T_\alpha$ and $\alpha \to \|T_\alpha\|_{p,q}$ are Hölder continuous in $[0,+\infty)$ of order $1/p'$, at least.

Theorem 1. *Let $p, q \in (1,+\infty)$ and let p' be the conjugated exponent of p. Let $\alpha, \beta \geq 0$. Then*
$$\big|\|T_\alpha\|_{p,q} - \|T_\beta\|_{p,q}\big| \leq \|T_\alpha - T_\beta\|_{p,q} \leq |\alpha-\beta|^{1/p'}\Gamma(\tfrac{q}{p'}+1)^{1/q}.$$

Proof. Let $f \in L^p[0,1]$. Hölder's inequality gives
$$|(T_\alpha f)(x) - (T_\beta f)(x)| = \Big|\int_{x^\alpha}^{x^\beta} f(y)\,\mathrm{d}y\Big| \leq \|f\|_p |x^\alpha - x^\beta|^{1/p'} \qquad \forall x \in [0,1],$$
so that
$$\|T_\alpha - T_\beta\|_{p,q} \leq \Big[\int_0^1 |x^\alpha - x^\beta|^{q/p'}\,\mathrm{d}x\Big]^{1/q}.$$
By the mean value Theorem, there exists η between α and β and depending on x such that
$$|x^\alpha - x^\beta| = |\alpha-\beta|x^\eta|\log x|,$$
in particular
$$|x^\alpha - x^\beta| \leq |\alpha-\beta||\log x| \qquad \forall x \in [0,1]$$
so that
$$\|T_\alpha - T_\beta\|_{p,q} \leq |\alpha-\beta|^{1/p'}\Big[\int_0^1 |\log x|^{q/p'}\,\mathrm{d}x\Big]^{1/q} = |\alpha-\beta|^{1/p'}\Gamma(\tfrac{q}{p'}+1)^{1/q}.$$
□

The following theorem provides lower/upper bounds for the norm and some hints about its behaviour when α tends to ∞ and to 0.

Theorem 2. *Let $p, q \in (1,+\infty)$ and let p' and q' be the conjugated exponent of p and q, respectively. Then*
$$(\alpha q+1)^{-1/q} \leq \|T_\alpha\|_{p,q} \leq \min\Big\{\big(\alpha\tfrac{q}{p'}+1\big)^{-1/q}, \big[\alpha B\big(\tfrac{p'}{q}+1,\alpha\big)\big]^{1/p'}\Big\}, \tag{2}$$
where $B(a,b) := \int_0^1 x^{a-1}(1-x)^{b-1}\,\mathrm{d}x$, the Euler Beta function. In particular
$$\|T_\alpha\|_{p,q} \asymp \alpha^{-1/q} \qquad \text{as } \alpha \to \infty, \tag{3}$$
$$\|T_\alpha - T_0\|_{p,q} \asymp \alpha^{1/p'} \qquad \text{as } \alpha \to 0, \tag{4}$$
$$\big|\|T_\alpha\|_{p,q} - 1\big| \asymp \alpha \qquad \text{as } \alpha \to 0. \tag{5}$$

The result in (4) refines the case $\beta = 0$ in Theorem 1, since now also the lower bound is proved. Moreover, it is interesting to notice that in the limit $\alpha \to 0$ the operator T_α tends to T_0 as $\alpha^{1/p'}$ by (4), while $\|T_\alpha\|_{p,q}$ tends to $\|T_0\|_{p,q} = 1$ as α by (5), so that the norm converges far better than the operator.

Proof. The lower bound comes quickly by comparing $\|T_\alpha f\|_q$ and $\|f\|_p$ for $f = 1$.
The first upper bound comes from Hölder's inequality: let $f \in L^p[0,1]$, then

$$\begin{aligned}\|T_\alpha f\|_q^q &\leq \int_0^1 \Big(\int_0^{x^\alpha} |f(y)|dy\Big)^q \mathrm{d}x \\ &\leq \int_0^1 \Big[\Big(\int_0^{x^\alpha} |f(y)|^p \,\mathrm{d}y\Big)^{1/p}\Big(\int_0^{x^\alpha} 1^{p'}\,\mathrm{d}y\Big)^{1/p'}\Big]^q \mathrm{d}x \\ &\leq \|f\|_p^q \int_0^1 x^{\alpha q/p'}\,\mathrm{d}x = \|f\|_p^q\big(\alpha\tfrac{q}{p'}+1\big)^{-1}.\end{aligned}$$

Also the second upper bound comes from Hölder's inequality, but used for the operator $T_\alpha^*\colon L^{q'}[0,1] \to L^{p'}[0,1]$, via the equality $\|T_\alpha\|_{p,q} = \|T_\alpha^*\|_{q',p'}$. In fact, let $f \in L^{q'}[0,1]$. Then

$$\begin{aligned}\|T_\alpha^* f\|_{p'}^{p'} &\leq \int_0^1 \Big(\int_{x^{1/\alpha}}^1 |f(y)|\,\mathrm{d}y\Big)^{p'} \mathrm{d}x \leq \int_0^1 \Big[\Big(\int_{x^{1/\alpha}}^1 |f(y)|^{q'}\Big)^{1/q'}\Big(\int_{x^{1/\alpha}}^1 1^q\Big)^{1/q}\Big]^{p'} \mathrm{d}x \\ &\leq \|f\|_{q'}^{p'} \int_0^1 (1-x^{1/\alpha})^{p'/q}\,\mathrm{d}x = \|f\|_{q'}^{p'}\alpha \int_0^1 (1-z)^{p'/q} z^{\alpha-1}\,\mathrm{d}z \\ &= \|f\|_{q'}^{p'}\alpha B\big(\tfrac{p'}{q}+1,\alpha\big).\end{aligned}$$

The comparison of the lower bound and the first upper bound in (2) gives (3) and (5).
To prove (4) we take advantage of the relation (1), so that

$$\|T_\alpha - T_0\|_{p,q} = \|(T_\alpha - T_0)^*\|_{q',p'} = \|T_{1/\alpha}\|_{q',p'}$$

and the claim follows from (3). □

It is immediate to verify that the two upper bounds in Theorem 2 coincide when $q = p'$, for every α. The following proposition shows that this is the unique case where this happens, and describes explicitly when one of the two upper bounds is more convenient than the other.

Proposition 1. *Let $p, q \in (1, +\infty)$ and let p' be the conjugate exponent of p. Then*

$$\begin{aligned}\big[\alpha B\big(\alpha, 1+\tfrac{p'}{q}\big)\big]^{1/p'} &\geq \big(\alpha\tfrac{q}{p'}+1\big)^{-1/q} \qquad \text{if } q \geq p', \\ \big[\alpha B\big(\alpha, 1+\tfrac{p'}{q}\big)\big]^{1/p'} &\leq \big(\alpha\tfrac{q}{p'}+1\big)^{-1/q} \qquad \text{if } q \leq p',\end{aligned}$$

with equality if and only if $q = p'$.

Proof. In terms of $u := p'/q$ the problem is equivalent to studying the sign of the function

$$\begin{aligned}F(\alpha,u) :&= \log\big[\alpha B(\alpha,1+u)\big] + u\log\Big(1+\frac{\alpha}{u}\Big) \\ &= \log\Gamma(\alpha+1) + \log\Gamma(1+u) - \log\Gamma(1+u+\alpha) + u\log\Big(1+\frac{\alpha}{u}\Big),\end{aligned}$$

where the last equality comes from the representation of the Beta function as product of gammas (see [2, Th. 1.1.4]). Since $F(0,u) = 0$ for every $u > 0$, it is sufficient to prove that $\partial_\alpha F(\alpha,u)$ is positive when $u \in (0,1)$ and negative for $u > 1$ (proviso that $\alpha > 0$). Let ψ be the logarithmic derivative of the gamma function, then

$$\partial_\alpha F(\alpha,u) = \psi(1+\alpha) - \psi(1+u+\alpha) + \frac{u}{u+\alpha} = \frac{u}{u+\alpha} - \sum_{n=1}^\infty \frac{u}{(n+\alpha)(n+u+\alpha)},$$

where the series comes from the representation of gamma as Weierstrass product (see [2, Th. 1.2.5]). Since

$$\frac{u}{u+\alpha} = u\sum_{n=1}^\infty \int_{u+n-1}^{u+n} \frac{\mathrm{d}x}{(x+\alpha)^2} = \sum_{n=1}^\infty \frac{u}{(n+u+\alpha-1)(n+u+\alpha)},$$

to conclude it is sufficient to see that for every n

$$\frac{u}{(n+u+\alpha-1)(n+u+\alpha)} \geq \frac{u}{(n+\alpha)(n+u+\alpha)}$$

if and only if $u<1$. □

Howard and Shep [10] proposed general theorems to estimate the norms of positive operators in L^p spaces and used them to compute the exact value of the norm of Volterra operator $V=T_1$ when it is considered as a map $L^p[0,1]\to L^q[0,1]$ for every pair of indexes p, q. Our attempts to estimate the norms of T_α via these results produced values which are larger than what we have stated in Theorem 2 and that we have proved using only basic tools. This is due to the fact that the results in [10] depend on a convenient choice of a test function, for which we have not been able to find a good analogue for the general T_α operator. Also the strategy allowing to compute $\|V\|_{p,q}$ fails for T_α operators, since the equation which should be solved explicitly to detect the best test function becomes a very complicated integro-differential equation in case $\alpha\neq 1$ or when p and q are not 2: for the case $p=q=2$, however, we can compute the norm of T_α as a byproduct of the study of $T_\alpha^* T_\alpha$ in Section 5, see Corollary 1.

3. Spectrum

When $q=p$ one can consider the spectrum of $T_\alpha\colon L^p[0,1]\to L^p[0,1]$. Barnes [3] investigated a general family of Volterra operators and showed that their spectrum is $\{0\}$, so that they have at most 0 as eigenvalue. The operators T_α with $\alpha\geq 1$ belong to this family but the results in [3] do not cover the case $\alpha<1$. In fact, we prove the following result.

Theorem 3. *Fix $p\in(1,+\infty)$. Suppose that $\alpha\geq 1$, then*

$$\sigma(T_\alpha)=\{0\}, \qquad \sigma_0(T_\alpha)=\emptyset.$$

Suppose that $\alpha<1$, then

$$\sigma(T_\alpha)=\sigma_0(T_\alpha)\cup\{0\}, \qquad \sigma_0(T_\alpha)=\{\alpha^n(1-\alpha)\colon n\in\mathbb{N}\}$$

and the eigenspace associated with $\alpha^n(1-\alpha)$ is generated by $x^{\frac{\alpha}{1-\alpha}}P_{n,\alpha}(\log x)$ where $P_{n,\alpha}$ is a suitable polynomial with degree n and depending on α. The span of the family $\{f_n\}_{n\in\mathbb{N}}$ is dense in $L^p[0,1]$.

The proof of this theorem also provides a formula for $P_{n,\alpha}$, see (20).
Note that T_α is not normal, so that the density of the span of the eigenfunctions is not sufficient to prove that they form a Schauder basis for the space; the known conditions that are capable of guaranteeing this property do not appear to be practically verifiable in the present case (see [13, Proposition 1.a.3], and [8, Chapter VI]).

For the proof of this theorem we need a preliminary study of the properties of the iterations of T_α. This is possible since for every $f\in L^p[0,1]$, $T_\alpha f$ can be written as $\int_{[0,1]}\chi_{[0,x^\alpha]}(y)f(y)\,\mathrm{d}y$, so that the iterations of T_α are

$$(T_\alpha^n f)(x)=\int_{[0,1]}K_n(x,y)f(y)\,\mathrm{d}y \qquad \forall n\geq 1,$$

with kernels $K_n(x,y)$ satisfying the recursive formula:

$$K_1(x,y)=\chi_{[0,x^\alpha]}(y), \qquad K_{n+1}(x,y)=\int_{[0,1]}K(x,s)K_n(s,y)\,\mathrm{d}s \quad \forall n\geq 1.$$

The following proposition gives a convenient formula for K_n.

Proposition 2. *Assume $\alpha>0$. For every $n\in\mathbb{N}$ let a_n and b_n be defined as*

$$a_n:=\frac{\alpha-\alpha^n}{1-\alpha} \quad \textit{and} \quad b_1:=1, \quad b_n:=\prod_{k=1}^{n-1}\frac{1-\alpha}{1-\alpha^k} \quad \textit{when } n\geq 2;$$

when $\alpha=1$ these formulas have to be considered as limits, giving $a_n=n-1$ and $b_n=1/(n-1)!$ in that case. Then

$$K_n(x,y)=b_n\chi_{[0,x^{\alpha^n}]}(y)x^{a_n}g_n(x^{-\alpha^n}y),$$

where

$$g_1(z) := 1, \quad \text{and} \quad g_{n+1}(z) := (a_n+1)\int_{z^{1/\alpha^n}}^1 w^{a_n} g_n(w^{-\alpha^n} z)\,\mathrm{d}w \quad \forall n \geq 1.$$

For example,

$$g_2(z) = 1 - z^{1/\alpha}, \qquad g_3(z) = 1 - (\alpha{+}1)z^{1/\alpha} + \alpha z^{1/\alpha+1/\alpha^2},$$
$$g_4(z) = 1 - (\alpha^2{+}\alpha{+}1)z^{1/\alpha} + (\alpha^3{+}\alpha^2{+}\alpha)z^{1/\alpha+1/\alpha^2} - \alpha^3 z^{1/\alpha+1/\alpha^2+1/\alpha^3}.$$

Functions $\{g_n\}_n$ satisfy also the relation

$$g_{n+1}(z) = g_n(z) - \alpha^{n-1} z^{1/\alpha} g_n(z^{1/\alpha}) \tag{6}$$

for every n, which comes from the relation

$$K_{n+1}(x,y) = \frac{1}{a_n+1}[x^\alpha K_n(x^\alpha, y) - \alpha^{n-1} y^{1/\alpha} K_n(x, y^{1/\alpha})]. \tag{7}$$

The equality in (6) can be used to prove the following explicit formulas for g_n and K_n, again valid for every n:

$$g_n(z) = \sum_{k=0}^{n-1} (-1)^k \begin{bmatrix} n-1 \\ k \end{bmatrix}_\alpha \alpha^{\binom{k}{2}} z^{\frac{1-\alpha^{-k}}{\alpha-1}}, \tag{8}$$

$$K_n(x,y) = b_n \chi_{[0,x^{\alpha^n}]}(y) \sum_{k=0}^{n-1} (-1)^k \begin{bmatrix} n-1 \\ k \end{bmatrix}_\alpha \alpha^{\binom{k}{2}} x^{\frac{\alpha^{n-k}-\alpha}{\alpha-1}} y^{\frac{1-\alpha^{-k}}{\alpha-1}}, \tag{9}$$

where $\begin{bmatrix} m \\ k \end{bmatrix}_\alpha := \frac{(1-\alpha^m)(1-\alpha^{m-1})\cdots(1-\alpha^{m-k+1})}{(1-\alpha)(1-\alpha^2)\cdots(1-\alpha^k)}$. This is the so called α-analogue of the binomial coefficient and in spite of its definition it is a polynomial with integral and positive coefficients so that in particular it is positive when $\alpha > 0$ (see [2, Ch. 10]). Formulas (8)-(9) generalize to every α the binomial presentations of identities $g_n(z) = (1-z)^{n-1}$ and $K_n(x,y) = \frac{\chi_{[0,x]}(y)}{(n-1)!}(x-y)^{n-1}$ for $\alpha = 1$. They are useful in case one wants to graph g_n and K_n for some n, but the alternating signs appearing there make them not useful to produce lower/upper bounds, which however is our main interest. For this reason we do not prove relations (6)–(9) here.

Proof of Proposition 2. The claim is evident for $n = 1$. By inductive hypothesis

$$K_{n+1}(x,y) = \int_0^1 K_1(x,s)K_n(s,y)\,\mathrm{d}s = \int_0^1 \chi_{[0,x^\alpha]}(s)\chi_{[0,s^{\alpha^n}]}(y) b_n s^{a_n} g_n(s^{-\alpha^n} y)\,\mathrm{d}s.$$

The product $\chi_{[0,x^\alpha]}(s)\chi_{[0,s^{\alpha^n}]}(y)$ is 1 if and only if $y \leq x^{\alpha^{n+1}}$ and $s \in [y^{1/\alpha^n}, x^\alpha]$, thus

$$K_{n+1}(x,y) = b_n \chi_{[0,x^{\alpha^{n+1}}]}(y) \int_{y^{1/\alpha^n}}^{x^\alpha} s^{a_n} g_n(s^{-\alpha^n} y)\,\mathrm{d}s.$$

Setting $s = x^\alpha w$ this is

$$K_{n+1}(x,y) = b_n \chi_{[0,x^{\alpha^{n+1}}]}(y) x^{\alpha(a_n+1)} \int_{x^{-\alpha} y^{1/\alpha^n}}^1 w^{a_n} g_n(w^{-\alpha^n} x^{-\alpha^{n+1}} y)\,\mathrm{d}w,$$

which is the claim since $a_{n+1} = \alpha(a_n+1)$ and $b_n = b_{n+1}(a_n+1)$. □

Now we can prove Theorem 3.

Proof of Theorem 3. From the recursive formula for g_n it is evident that $g_n(z) \in [0,1]$ for every $z \in [0,1]$ and every n, so that

$$0 \leq K_n(x,y) \leq b_n \cdot \chi_{[0,x^{\alpha^n}]}(y) \cdot x^{a_n}.$$

Thus,

$$\|T_\alpha^n\|_{p,p} \leq \|K_n\|_{L^p([0,1]\times[0,1])} \leq b_n(pa_n + \alpha^n + 1)^{-1/p} \leq b_n \tag{10}$$

and Gelfand's formula for the spectral radius $\rho(T_\alpha)$ produces the bound

$$\rho(T_\alpha) = \lim_{n\to\infty} \|T_\alpha^n\|_{p,p}^{1/n} \leq \lim_{n\to\infty} b_n^{1/n} = \begin{cases} 1-\alpha & \text{if } \alpha \leq 1, \\ 0 & \text{if } \alpha \geq 1. \end{cases} \tag{11}$$

The inclusion $L^p[0,1] \subset L^1[0,1]$ assures that $T_\alpha f$ is absolutely continuous in $[0,1]$ and $\mathcal{C}^\infty$ in $(0,1)$, so that T_α cannot be surjective. However it is injective, by the Lebesgue differentiability Theorem. Thus, the spectrum σ and the point spectrum σ_0 in case $\alpha \geq 1$ are

$$\sigma(T_\alpha) = \{0\}, \qquad \sigma_0(T_\alpha) = \emptyset,$$

respectively, and the problem is completely settled in this case.

Assume now that $\alpha \in (0,1)$. One verifies that $x^{\frac{\alpha}{1-\alpha}} \in L^p[0,1]$ is an eigenfunction for T_α, with eigenvalue $1-\alpha$. With (11) this proves that the spectral radius $\rho(T_\alpha)$ equals $1-\alpha$, and that $\sigma_0(T_\alpha)$ contains $1-\alpha$. The special form of this eigenfunction for the eigenvalue $1-\alpha$ suggests to write the generic eigenfunction $f(x)$ as $x^{\frac{\alpha}{1-\alpha}} h(\log x)$. The regularity of f shows that h is in $\mathcal{C}^\infty(-\infty,0)$ and admits a continuous extension to 0 from the left. In terms of h the equation $T_\alpha f = \lambda f$ becomes

$$\int_{-\infty}^{\alpha y} e^{\frac{w}{1-\alpha}} h(w)\,\mathrm{d}w = \lambda e^{\frac{\alpha y}{1-\alpha}} h(y). \tag{12}$$

A derivation with respect to y of this identity and a rearrangement of the terms produce the equation

$$h'(y) = Ah(\alpha y) - Bh(y) \qquad \text{with } A := \frac{\alpha}{\lambda} \text{ and } B := \frac{\alpha}{1-\alpha}, \tag{13}$$

and the evaluation of the equation at $y=0$ produces the condition

$$\int_{-\infty}^{0} e^{\frac{w}{1-\alpha}} h(w)\,\mathrm{d}w = \lambda h(0). \tag{14}$$

On the contrary, every function h in $\mathcal{C}^\infty(-\infty,0)$ admitting a finite limit to 0^- and satisfying both (13) and (14) also satisfies (12) and hence produces an eigenfunction.
By induction on the order k, one proves that there exist constants $q_{k,j}$ with $j \in \mathbb{Z}$ (depending on parameters A and B) with $q_{k,j} = 0$ when $j < 0$ or $j > k$, such that

$$(-1)^k h^{(k)}(y) = \sum_{j=0}^{k} (-1)^j q_{k,j} h(\alpha^j y). \tag{15}$$

In fact, the formula holds for $k=1$ with $q_{1,0} := B$ and $q_{1,1} := A$. Assume that the formula holds for k. Deriving the formula and plugging (13) into the resulting identity we see that

$$\begin{aligned}
(-1)^{k+1} h^{(k+1)}(y) &= -\sum_{j=0}^{k} (-1)^j q_{k,j} \alpha^j h'(\alpha^j y) \\
&= -\sum_{j=0}^{k} (-1)^j q_{k,j} \alpha^j \big(Ah(\alpha^{j+1} y) - Bh(\alpha^j y)\big) \\
&= -\sum_{j=0}^{k} (-1)^j A q_{k,j} \alpha^j h(\alpha^{j+1} y) + \sum_{j=0}^{k} (-1)^j B q_{k,j} \alpha^j h(\alpha^j y) \\
&= \sum_{j=1}^{k+1} (-1)^j A q_{k,j-1} \alpha^{j-1} h(\alpha^j y) + \sum_{j=0}^{k} (-1)^j B q_{k,j} \alpha^j h(\alpha^j y)
\end{aligned}$$

so that the formula for $k+1$ emerges once we define

$$q_{k+1,j} := A\alpha^{j-1} q_{k,j-1} + B\alpha^j q_{k,j} \qquad \forall j.$$

Let $C := \max(|A|,|B|)$. The recursive definition of constants $q_{k,j}$ shows that

$$|q_{k,j}| \leq C^k \alpha^{\binom{j}{2}} \begin{bmatrix} k \\ j \end{bmatrix}_\alpha \qquad \forall k, j,$$

where $\begin{bmatrix} k \\ j \end{bmatrix}_\alpha$ is the already mentioned Gaussian binomial coefficient (here we use the relation $\begin{bmatrix} k \\ j \end{bmatrix}_\alpha = \begin{bmatrix} k-1 \\ j-1 \end{bmatrix}_\alpha + \alpha^j \begin{bmatrix} k-1 \\ j \end{bmatrix}_\alpha$, see [2, Eq. 10.0.3]). By (15) we see that every derivative $h^{(k)}$ admits a continuation to 0 from the left, with value

$$h^{(k)}(0^-) = (-1)^k h(0^-) \sum_{j=0}^{k} (-1)^j q_{k,j}, \tag{16}$$

and that

$$|h^{(k)}(y)| \le C^k \sum_{j=0}^{k} \alpha^{\binom{j}{2}} \begin{bmatrix} k \\ j \end{bmatrix}_\alpha |h(\alpha^j y)|. \tag{17}$$

Let R be any positive parameter. We are assuming that $\alpha < 1$, therefore $\alpha^j y$ is in $[-R, 0]$ whenever y is in $[-R, 0]$ and $j \ge 0$. As a consequence, by (17) and denoting $\|\cdot\|_{\infty,R}$ the sup norm in $[-R, 0]$, we see that

$$\|h^{(k)}\|_{\infty,R} \le C^k \sum_{j=0}^{k} \alpha^{\binom{j}{2}} \begin{bmatrix} k \\ j \end{bmatrix}_\alpha \|h\|_{\infty,R} = C^k \|h\|_{\infty,R} \prod_{j=0}^{k-1} (1 + \alpha^j),$$

where the equality comes from the identity $\sum_{j=0}^{k} \alpha^{\binom{j}{2}} \begin{bmatrix} k \\ j \end{bmatrix}_\alpha t^j = \prod_{j=0}^{k-1}(1 + \alpha^j t)$ (see [2, Eq. 10.0.9]). The full product $c(\alpha) := \prod_{j=0}^{\infty}(1 + \alpha^j)$ converges, because $\alpha < 1$, therefore we can conclude that

$$\|h^{(k)}\|_{\infty,R} \le c(\alpha) \|h\|_{\infty,R} C^k.$$

This shows that $\|h^{(k)}\|_{\infty,R}$ diverges with the order k as C^k, at most. Since R is arbitrary, this suffices to assure that the power series $\sum_{k=0}^{\infty} \frac{h^{(k)}(0)}{k!} y^k$ converges to h for $y \le 0$ and provides an analytic extension of h as an entire function.
We also need an explicit formula for the coefficients of the power series representing h. They can be recovered from (16) but the following argument is quicker. In fact, writing $h(y) = \sum_{k=0}^{\infty} \beta_k y^k$, the relation in (13) readily shows that

$$\beta_{k+1} = \frac{A\alpha^k - B}{k+1} \beta_k \qquad \forall k \ge 0.$$

By homogeneity we can select $\beta_0 = 1$, because every non zero function is a multiple of what we get under this assumption. This yields

$$h^{(k)}(0) = k! \beta_k = \prod_{j=0}^{k-1} (A\alpha^j - B) = \Big(\frac{-\alpha}{1-\alpha}\Big)^k \prod_{j=0}^{k-1} \Big(1 - \frac{\alpha^j(1-\alpha)}{\lambda}\Big). \tag{18}$$

For it to produce an eigenfunction h must satisfy also the condition (14), therefore λ must be a solution of the equation

$$\lambda = \int_{-\infty}^{0} e^{\frac{w}{1-\alpha}} h(w) \,\mathrm{d}w = \int_{-\infty}^{0} e^{\frac{w}{1-\alpha}} \sum_{k=0}^{\infty} \beta_k w^k \,\mathrm{d}w = \sum_{k=0}^{\infty} (-1)^k k! \beta_k (1-\alpha)^{k+1},$$

where the exchange of the integral and the series is made possible by Fubini's theorem and (18) which shows that $k!|\beta_k| \ll_{\lambda,\alpha} \big(\frac{\alpha}{1-\alpha}\big)^k$ so that the resulting series converges absolutely. Plugging (18) into the previous equation we get that

$$\lambda = (1-\alpha) + (1-\alpha) \sum_{k=1}^{\infty} \alpha^k \prod_{j=0}^{k-1} \Big(1 - \frac{\alpha^j(1-\alpha)}{\lambda}\Big).$$

Setting $(1-\alpha)/\lambda =: U$ it becomes

$$0 = 1 - U - \alpha U \sum_{k=0}^{\infty} \alpha^k \prod_{j=0}^{k} (1 - \alpha^j U). \tag{19}$$

The quantity appearing to the right hand side coincides with $\prod_{j=0}^{\infty}(1 - \alpha^j U)$: in terms of α-analogues symbols, this claim corresponds to the equality $\sum_{k=0}^{\infty} (z;\alpha)_k \alpha^k = (1 - (z;\alpha)_\infty)/z$. We strongly suspect that this formula is well known to every specialist in this area, but we have not been able to locate it precisely in literature. Thus, we provide here a quick proof. Let

$$F(z) := 1 - z - \alpha z \sum_{k=0}^{\infty} \alpha^k \prod_{j=0}^{k} (1 - \alpha^j z).$$

Once again we notice that the inner product can be bounded uniformly in k and $|z| \leq 1$, so that $F(z) \to 1$ as $z \to 0$. In the series defining F we separate the term with $k = 0$ and in each other term with $k \geq 1$ we collect the term $1 - z$ coming from the case $j = 0$. This produces the identity

$$\begin{aligned} F(z) &= 1 - z - \alpha z(1 - z) - \alpha z(1 - z)\Big(\sum_{k=1}^{\infty} \alpha^k \prod_{j=1}^{k} (1 - \alpha^j z)\Big) \\ &= (1 - z)\Big(1 - \alpha z - \alpha^2 z \sum_{k=0}^{\infty} \alpha^k \prod_{j=0}^{k} (1 - \alpha^j (\alpha z))\Big) = (1 - z)F(\alpha z). \end{aligned}$$

Iterating this identity we get that $F(z) = \prod_{j=0}^{L-1} (1 - \alpha^j z) F(\alpha^L z)$, for every L. Setting $L \to \infty$ we get the equality $F(z) = \prod_{j=0}^{\infty} (1 - \alpha^j z)$, since $F(\alpha^L z) \to 1$ as $L \to \infty$.
In this way we see that Equation (19) actually says that

$$0 = \prod_{j=0}^{\infty} (1 - \alpha^j U).$$

The product converges absolutely, hence its unique zeros come from zero factors and this forces $U = \alpha^{-n}$ for $n \in \mathbb{N}$ i.e. $\lambda = \alpha^n(1 - \alpha)$. This proves that the unique (possible) eigenvalues are the numbers $\alpha^n(1 - \alpha)$.
Moreover, suppose that $\lambda = \alpha^n(1 - \alpha)$ for a given $n \in \mathbb{N}$. Formula (18) shows that in this case $h^{(k)}(0) = 0$ whenever $k > n$, so that the previous computations show that the eigenfunction has the shape $x^{\frac{\alpha}{1-\alpha}} h(\log x)$ with h which is actually a polynomial with degree $\leq n$. The same formula also allows an explicit presentation for this polynomial:

$$\begin{aligned} h(y) &= \sum_{k=0}^{n} h^{(k)}(0) \frac{y^k}{k!} = \sum_{k=0}^{n} \Big(\frac{-\alpha}{1 - \alpha}\Big)^k \prod_{j=0}^{k-1} (1 - \alpha^{j-n}) \frac{y^k}{k!} \\ &= \sum_{k=0}^{n} \Big(\prod_{j=0}^{k-1} \frac{1 - \alpha^{j-n}}{1 - \alpha^{-1}} \Big) \frac{y^k}{k!} \end{aligned}$$

showing that its degree is n, and giving the eigenfunction

$$f(x) = x^{\frac{\alpha}{1-\alpha}} \sum_{k=0}^{n} \Big(\prod_{j=0}^{k-1} \frac{1 - \alpha^{j-n}}{1 - \alpha^{-1}} \Big) \frac{(\log x)^k}{k!}. \tag{20}$$

Finally, the eigenfunctions we have found generate the vector space

$$\mathcal{P} := \big\{ x^{\frac{\alpha}{1-\alpha}} P(\log x),\ P \in \mathbb{C}[z] \big\}.$$

Let $f \in L^p[0, 1]$. The change of variable $x := e^w$ gives the identity

$$\int_0^1 \big| f(x) - x^{\frac{\alpha}{1-\alpha}} P(\log x) \big|^p \, \mathrm{d}x = \int_{-\infty}^{0} e^{\frac{1+(p-1)\alpha}{1-\alpha} w} \big| e^{\frac{-\alpha}{1-\alpha} w} f(e^w) - P(w) \big|^p \, \mathrm{d}w.$$

The case $P = 0$ shows that $e^{\frac{-\alpha}{1-\alpha} w} f(e^w)$ is in $L^p((-\infty, 0], e^{\frac{1+(p-1)\alpha}{1-\alpha} w} \, \mathrm{d}w)$. Since polynomials are dense in this space, see [15, p. 40], the previous computation also shows that $\mathcal{P}$ is dense in $L^p[0, 1]$. □

4. Norm of iterates

The study of the behaviour of the norm of the n-th iterated of $V = T_1$ when n diverges is a classical problem which in the space $L^2[0,1]$ has been solved by Lao and Whitley [12] for the order and by Kershaw [11] for the asymptotic: $\|n!V^n\|_{2,2} \to 1/2$. See also [14] for an elementary proof and [4] for a proof with explicit bounds. Later Eveson extended this result to operators of the form $\int_0^x k(x-s)f(s)\,\mathrm{d}s$ under mild conditions for the kernel both in the $L^2[0,1]$ case [6] and for the general $L^p[0,1]$ case [7]. Adell and Gallardo–Gutierrez [1] proved explicit bounds for each s-th Riemann–Liouville fractional integration operators (which coincides with the s-th iteration of V when s is an integer) in the $L^p[0,1]$ case.
We prove a similar result for operators T_α when considered as map from the space $L^p[0,1]$ in itself.

Theorem 4. *Fix $p \in (1,+\infty)$, $\alpha > 0$, and let $n \to \infty$. Then*
$$\log \|T_\alpha^n\|_{p,p} = \begin{cases} n\log(1-\alpha) + o_{p,\alpha}(n) & \text{if } \alpha < 1,\\ -n\log n + O_p(n) & \text{if } \alpha = 1,\\ -\frac{1}{2}n^2\log\alpha + O_{p,\alpha}(n) & \text{if } \alpha > 1. \end{cases}$$
Actually, by the work of the cited authors, in the case $\alpha = 1$ the full expansion is known up to the order $o_p(1)$, so that the conclusions in Theorem 4 are less precise. Nevertheless, they already show in a quantitative way the threshold effect associated with the passage of α through 1: the logarithm of the norm diverges linearly when $\alpha < 1$ and quadratically when $\alpha > 1$.

The case $\alpha < 1$ comes from Gelfand's formula for the spectral radius and the case $\alpha = 1$ is a weak version of the computations in [12], hence only the case $\alpha > 1$ must be proved. For this purpose we use the following lower bound for the functions g_n appearing in Proposition 2.

Proposition 3. *Let $\alpha > 0$ and $n \geq 2$. Then*
$$g_n(z) \geq (1 - z^{1/((n-1)\alpha)})^{n-1} \qquad \forall z \in [0,1].$$
Proof. The inequality for $n = 2$ holds as equality. Assume that the claim is true for n. Then, by the integral relation in Proposition 2 one gets
$$g_{n+1}(z) \geq (a_n+1)\int_{z^{1/\alpha^n}}^1 w^{a_n}(1-(zw^{-\alpha^n})^{1/((n-1)\alpha)})^{n-1}\,\mathrm{d}w.$$
Let γ be a parameter in $[z^{1/\alpha^n},1]$ that we will choose later. Then
$$\begin{aligned} g_{n+1}(z) &\geq (a_n+1)\int_\gamma^1 w^{a_n}(1-(zw^{-\alpha^n})^{1/((n-1)\alpha)})^{n-1}\,\mathrm{d}w \\ &\geq (1-(z\gamma^{-\alpha^n})^{1/((n-1)\alpha)})^{n-1}(a_n+1)\int_\gamma^1 w^{a_n}\,\mathrm{d}w \\ &= (1-(z\gamma^{-\alpha^n})^{1/((n-1)\alpha)})^{n-1}(1-\gamma^{a_n+1}). \end{aligned}$$
We set γ such that $\gamma^{a_n+1} = (z\gamma^{-\alpha^n})^{1/((n-1)\alpha)}$, i.e. $\gamma = z^{1/((n-1)\alpha(a_n+1)+\alpha^n)}$. This value is in the allowed interval $[z^{1/\alpha^n},1]$, hence
$$g_{n+1}(z) \geq (1 - z^{(a_n+1)/((n-1)\alpha(a_n+1)+\alpha^n)})^n.$$
The definition of a_n implies that $(a_n+1)/((n-1)\alpha(a_n+1)+\alpha^n) \geq 1/(n\alpha)$ for every n (because it is equivalent to $a_n + 1 \geq \alpha^{n-1}$ which is true since $a_n + 1 = (\alpha^n-1)/(\alpha-1)$), so that
$$g_{n+1}(z) \geq (1 - z^{1/(n\alpha)})^n.$$
This proves the claim by induction. □

Now we can prove the remaining case $\alpha > 1$ in Theorem 4. By Propositions 2 and 3 we get that
$$K_n(x,y) \geq b_n\chi_{[0,x^{\alpha^n}]}(y)x^{a_n}(1-(x^{-\alpha^n}y)^{1/((n-1)\alpha)})^{n-1}.$$
Let $f(x) = 1$ for every x. Then $\|f\|_p = 1$, and
$$(T_\alpha^n f)(x) = \int_0^1 K_n(x,y)\,\mathrm{d}y \geq b_n x^{a_n}\int_0^{x^{\alpha^n}}(1-(yx^{-\alpha^n})^{1/((n-1)\alpha)})^{n-1}\,\mathrm{d}y.$$

Setting $z := (yx^{-\alpha^n})^{1/((n-1)\alpha)}$, i.e. $y = x^{\alpha^n} z^{(n-1)\alpha}$, this becomes

$$= b_n x^{a_n+\alpha^n}(n-1)\alpha \int_0^1 w^{(n-1)\alpha-1}(1-w)^{n-1}\,\mathrm{d}z$$
$$= b_n x^{a_n+\alpha^n}(n-1)\alpha B((n-1)\alpha, n) = b_n x^{a_n+\alpha^n}(n-1)\alpha\frac{\Gamma((n-1)\alpha)\Gamma(n)}{\Gamma((n-1)\alpha+n)},$$

where we have used the representation of the Beta function in terms of gammas. As a consequence,

$$\|T_\alpha^n\|_{p,p} \geq \|T_\alpha^n f\|_p \geq \frac{(n-1)\alpha\, b_n}{(a_n p + \alpha^n p + 1)^{1/p}}\,\frac{\Gamma((n-1)\alpha)\Gamma(n)}{\Gamma((n-1)\alpha+n)}.$$

Using Stirling asymptotic formula [2, Th. 1.4.1] we deduce that there exists a constant $c > 1$, depending on α and p but independent of n, such that

$$\|T_\alpha^n\|_{p,p} \geq b_n c^{-n} n^{O_{\alpha,p}(1)} \qquad \text{as } n \to \infty.$$

By (10) we already know that $\|T_\alpha^n\|_{p,p} \leq b_n$, hence $\|T_\alpha^n\|_{p,p}{=}b_n \exp(O_{\alpha,p}(n))$. Everything proved up to now holds for any positive α. Suppose that $\alpha > 1$, then the definition of b_n shows that in this case $b_n = \alpha^{-n^2/2}\exp(O_\alpha(n))$, so that

$$\|T_\alpha^n\|_{p,p} = \alpha^{-n^2/2}\exp(O_{\alpha,p}(n)) \qquad \text{as } n \to \infty,$$

as claimed.

5. The case $p = q = 2$

When $T_\alpha\colon L^2[0,1] \to L^2[0,1]$ we can compute exactly spectrum and eigenfunctions for $T_\alpha^* T_\alpha$ (see Theorem 5), in particular we get an exact formula for $\|T_\alpha\|_{2,2}$ and its asymptotic for $\alpha \to 0$ and $\alpha \to \infty$ (see Corollary 1).

Theorem 5. *Let*

$$H_\alpha(z) := \sum_{k=0}^\infty \frac{(-z)^k}{k!\prod_{j=1}^k (j - \frac{1}{1+\alpha})}. \tag{21}$$

All zeros for H_α are real and positive. Let $\{h_n(\alpha)\}_n$ be the sequence of these zeros, ordered by their size. The spectrum for $T_\alpha^* T_\alpha$ coincides with the set

$$\sigma_0(T_\alpha^* T_\alpha) = \Big\{\frac{\alpha}{(1+\alpha)^2}\frac{1}{h_n(\alpha)} : n \in \mathbb{N}\Big\}, \tag{22}$$

each eigenspace is one-dimensional and the n-th eigenspace is generated by the function

$$\sum_{k=0}^\infty \frac{(-h_n(\alpha))^k}{k!\prod_{j=1}^k (j - \frac{1}{1+\alpha})} x^{\frac{(1+\alpha)k}{\alpha}}. \tag{23}$$

This result extends Halmos' computation for the classic Volterra operator $V = T_1$, see [9, Problem 188, p. 100]. In fact, for $\alpha = 1$ the function H_1 as given in (21) becomes

$$H_1(z) = \sum_{k=0}^\infty \frac{(-z)^k}{k!\prod_{j=1}^k (j - \frac{1}{2})} = \sum_{k=0}^\infty \frac{(-4z)^k}{(2k)!} = \cos(2\sqrt{z})$$

so that the spectrum (22) becomes $\{\frac{4}{\pi^2}(1+2n)^{-2}, n \in \mathbb{N}\}$ and the n-th eigenvalue gives the eigenfunction

$$\sum_{k=0}^\infty \frac{(-\frac{\pi^2}{16}(1+2n)^2)^k}{k!\prod_{j=1}^k (j - \frac{1}{2})} x^{2k} = \cos(\tfrac{\pi}{2}(1+2n)x).$$

Proof. The operator $T_\alpha^* T_\alpha$ is compact, selfadjoint and strictly positive since T_α is injective, thus the eigenvalue equation with $\lambda := 1/\omega^2$ says that

$$\omega^2 \int_{x^{1/\alpha}}^1 \int_0^{z^\alpha} f(u)\,\mathrm{d}u\,\mathrm{d}z = f. \tag{24}$$

We know that $T_\alpha f$ is absolutely continuous, thus the equation shows that $f \in C^0([0,1]) \cap C^1((0,1])$. The derivative of this equation produces the equality

$$-\frac{\omega^2}{\alpha} x^{\frac{1-\alpha}{\alpha}} \int_0^x f(u)\,\mathrm{d}u = f'. \tag{25}$$

Since $\frac{1}{x}\int_0^x f(u)\,\mathrm{d}u$ goes to $f(0)$ as $x \to 0$, this equation shows both that actually $f \in C^1([0,1])$, with $f'(0) = 0$, and that $f \in C^2((0,1])$. A further derivation shows that

$$-\frac{(1-\alpha)\omega^2}{\alpha^2} x^{\frac{1-2\alpha}{\alpha}} \int_0^x f(u)\,\mathrm{d}u - \frac{\omega^2}{\alpha} x^{\frac{1-\alpha}{\alpha}} f = f''. \tag{26}$$

When $\alpha < 1$, this equation shows that $f \in C^2([0,1])$, with $f''(0) = 0$ (but for $\alpha = 1$ only the existence of $f''(0)$ can be deduced, and for $\alpha > 1$ also the existence of $f''(0)$ is not evident). Moreover, combining (25) and (26) we get the differential equation

$$f'' - \frac{1-\alpha}{\alpha}\frac{f'}{x} + \frac{\omega^2}{\alpha} x^{\frac{1-\alpha}{\alpha}} f = 0. \tag{27}$$

This is a homogeneous second order differential equation, with regular (i.e., analytic) coefficients in $(0,1)$, and a singularity at 0 coming from the quotient f'/x and from the possible non-analyticity of $x^{\frac{1-\alpha}{\alpha}}$. The first term is actually under control, since we know that the solutions we are looking for have $f'(0) = 0$; the second term is more difficult to deal with, but we can improve it with a suitable change of variable. Suppose $f(x) = g(x^\beta)$, for some $\beta > 0$. This produces the equalities

$$f' = \beta x^{\beta-1} g', \qquad f'' = \beta(\beta-1)x^{\beta-2}g' + \beta^2 x^{2\beta-2} g''$$

which in (27) give the equation

$$\beta^2 x^{2\beta-2} g'' + \beta\Big(\beta - \frac{1}{\alpha}\Big) x^{\beta-2} g' + \frac{\omega^2}{\alpha} x^{\frac{1-\alpha}{\alpha}} g = 0,$$

i.e.,

$$g'' + \Big(1 - \frac{1}{\alpha\beta}\Big) x^{-\beta} g' + \frac{\omega^2}{\alpha\beta^2} x^{\frac{1+\alpha}{\alpha} - 2\beta} g = 0.$$

Thus, setting $\frac{1+\alpha}{\alpha} - 2\beta = \beta$, i.e. $\beta := \frac{1+\alpha}{3\alpha}$, we get the equation

$$g'' - \frac{2-\alpha}{1+\alpha}\frac{g'}{z} + \frac{9\alpha\omega^2 z}{(1+\alpha)^2} g = 0, \tag{28}$$

where derivatives are with respect to z and $f(x) = g(x^{\frac{1+\alpha}{3\alpha}})$ (and $g(z) = f(z^{\frac{3\alpha}{1+\alpha}})$). Note that $\beta = 1$ in case $\alpha = 1/2$ (so that in this case the transformation of x into z is the identity), and that (28) coincides with the equation satisfied by Airy's function when $\alpha = 2$ (after a suitable rescaling of the variable).

When written for the g function, the eigenvalue equation (24) reads

$$\omega^2 \int_{z^{\frac{3}{1+\alpha}}}^1 \int_0^{s^{\frac{1+\alpha}{3}}} v^{\frac{2\alpha-1}{1+\alpha}} g(v)\,\mathrm{d}v\,\mathrm{d}s = \frac{3\alpha}{1+\alpha} g(z).$$

A derivation with respect to z and a division by z show that

$$\omega^2 z^{\frac{1-2\alpha}{1+\alpha}} \int_0^z v^{\frac{2\alpha-1}{1+\alpha}} g(v)\,\mathrm{d}v = -\alpha \frac{g'}{z}.$$

Restoring f in this integral we get

$$\omega^2 \frac{1+\alpha}{3\alpha} z \cdot z^{\frac{-3\alpha}{1+\alpha}} \int_0^{z^{\frac{3\alpha}{1+\alpha}}} f(w)\,\mathrm{d}w = -\alpha \frac{g'}{z}.$$

Since f is continuous at 0, this formula shows that g'/z^2 admits a finite limit as $z \to 0$. In particular, both $g'(0)$ and $g''(0)$ exist and equal zero.

We look for a solution admitting a representation as power series $g(z) = \sum_{k=0}^\infty c_k z^k$. Plugging the power series into (28) and with the assumptions that $c_1 = c_2 = 0$, and that $c_0 = 1$ (by homogeneity), we get

$$c_0 = 1, \quad c_1 = c_2 = 0, \qquad (k+3)\Big(k + \frac{3\alpha}{1+\alpha}\Big)c_{k+3} = -\frac{9\alpha\omega^2}{(1+\alpha)^2}c_k \quad \forall\, k \in \mathbb{N}.$$

Iterating the recursion, we get for the coefficients the explicit formula:

$$c_{3k} = \frac{(-\frac{\alpha\omega^2}{(1+\alpha)^2})^k}{k!\prod_{j=1}^k (j - \frac{1}{1+\alpha})} \quad \forall\, k \in \mathbb{N}, \qquad c_\ell = 0 \quad \text{otherwise.}$$

These coefficients produce the function

$$G_0(z) := \sum_{k=0}^\infty \frac{(-\frac{\alpha\omega^2}{(1+\alpha)^2})^k}{k!\prod_{j=1}^k (j - \frac{1}{1+\alpha})} z^{3k}$$

which converges everywhere and therefore is a true solution of (28). In terms of x, this produces the function

$$f_0(x) := G_0(x^{\frac{1+\alpha}{3\alpha}}) = \sum_{k=0}^\infty \frac{(-\frac{\alpha\omega^2}{(1+\alpha)^2})^k}{k!\prod_{j=1}^k (j - \frac{1}{1+\alpha})} x^{\frac{(1+\alpha)k}{\alpha}}.$$

Now we produce a second and independent solution for (28). Since $G_0(0) = 1$, locally it is not 0. Thus, an independent solution of (28) can be obtained setting $g = G_0 R$ for a suitable R and solving the resulting equation for R. After some computations, the new solution G_1 appears as

$$R(z) := \int_0^z s^{\frac{2-\alpha}{1+\alpha}} G_0(s)^{-2}\,\mathrm{d}s \quad , \quad G_1(z) := G_0(z)R(z) = G_0(z)\int_0^z s^{\frac{2-\alpha}{1+\alpha}} G_0(s)^{-2}\,\mathrm{d}s.$$

The general solution of (28) is a linear combination $aG_0 + bG_1$ with $a, b \in \mathbb{R}$, but only the solutions with $b = 0$ have a chance to produce eigenfunctions of $T_\alpha^* T_\alpha$. In fact, we have proved that g'/z^2 admits a finite limit as $z \to 0$ whenever g is an eigenfunction. The function G_0 satisfies this property, hence the combination $aG_0 + bG_1$ with any $b \neq 0$ has this property if and only if G_1 does the same. We have

$$\frac{G_1'(z)}{z^2} = \frac{G_0'(z)}{z^2}R(z) + G_0(z)\frac{R'(z)}{z^2} = \frac{G_0'(z)}{z^2}R(z) + z^{\frac{-3\alpha}{1+\alpha}}G_0^{-1}(z).$$

Here the first term has a finite limit, but the second diverges for every $\alpha > 0$, so no combination with $b \neq 0$ can be an eigenfunction.

Moreover, Equation (24) shows that every eigenfunction has a zero at $x = 1$, so that for $f_0(x) = G_0(x^{\frac{1+\alpha}{3\alpha}})$ to be an eigenfunction it is necessary to have

$$\sum_{k=0}^\infty \frac{(-\frac{\alpha\omega^2}{(1+\alpha)^2})^k}{k!\prod_{j=1}^k (j - \frac{1}{1+\alpha})} = 0 \tag{29}$$

so that $\frac{\alpha\omega^2}{(1+\alpha)^2}$ is a zero for the function H_α as given in (21). We prove now that this condition is also sufficient. In fact,

$$\omega^2\int_{x^{1/\alpha}}^1\int_0^{z^\alpha} f_0(u)\,\mathrm{d}u\,\mathrm{d}z = \omega^2\int_{x^{1/\alpha}}^1\int_0^{z^\alpha}\Big[\sum_{k=0}^\infty \frac{(-\frac{\alpha\omega^2}{(1+\alpha)^2})^k}{k!\prod_{j=1}^k (j - \frac{1}{1+\alpha})} u^{\frac{(1+\alpha)k}{\alpha}}\Big]\,\mathrm{d}u\,\mathrm{d}z.$$

Everything converges absolutely, thus exchanging the integrals and the series we get

$$= \sum_{k=0}^\infty \frac{(-\frac{\alpha\omega^2}{(1+\alpha)^2})^k\omega^2}{k!\prod_{j=1}^k (j - \frac{1}{1+\alpha})}\int_{x^{1/\alpha}}^1\int_0^{z^\alpha} u^{\frac{(1+\alpha)k}{\alpha}}\,\mathrm{d}u\,\mathrm{d}z.$$

The double integration gives

$$\begin{aligned} &= \sum_{k=0}^\infty \frac{(-\frac{\alpha\omega^2}{(1+\alpha)^2})^k}{k!\prod_{j=1}^k (j - \frac{1}{1+\alpha})}\frac{\frac{\alpha\omega^2}{(1+\alpha)^2}}{(k+1)(k+\frac{\alpha}{1+\alpha})}[1 - x^{\frac{1+\alpha}{\alpha}(k+1)}] \\ &= -\sum_{k=1}^\infty \frac{(-\frac{\alpha\omega^2}{(1+\alpha)^2})^k}{k!\prod_{j=1}^k (j - \frac{1}{1+\alpha})} + \sum_{k=1}^\infty \frac{(-\frac{\alpha\omega^2}{(1+\alpha)^2})^k}{k!\prod_{j=1}^k (j - \frac{1}{1+\alpha})} x^{\frac{1+\alpha}{\alpha}k}, \end{aligned}$$

which is $f_0(x)$ whenever ω satisfies (29), so that f_0 is an eigenfunction. Formula (23) produces f_0 in terms of zeros for H_α, once (29) is taken account.
Finally, we know that eigenvalues exist and must be real and positive, since $T_\alpha^* T_\alpha$ is compact, self-adjoint and injective. However, all computations we have done need only the fact that eigenvalues are not zero. In particular, every zero than H_α has in $\mathbb{C}$ produces an eigenvalue for $T_\alpha^* T_\alpha$. This proves that all complex zeros for H_α are actually real and positive. □

Corollary 1. *Let $h_0(\alpha)$ be the smallest positive zero for H_α, as given in* (21). *Then*

$$\|T_\alpha\|_{2,2} = \frac{1}{1+\alpha}\sqrt{\frac{\alpha}{h_0(\alpha)}}. \tag{30}$$

Moreover,

$$\|T_\alpha\|_{2,2} \sim \frac{1/\sqrt{h_0(\infty)}}{\sqrt{\alpha}} \quad \text{as } \alpha \to \infty, \tag{31}$$

$$\|T_\alpha - T_0\|_{2,2} \sim \sqrt{\alpha/h_0(\infty)} \quad \text{as } \alpha \to 0, \tag{32}$$

where $h_0(\infty) = 1.445796\ldots$ is the smallest positive zero of $H_\infty(z) = \sum_{k=0}^\infty \frac{(-z)^k}{(k!)^2}$, and

$$\|T_\alpha\|_{2,2} = 1 - \frac{3}{4}\alpha + O(\alpha^2) \text{ as } \alpha \to 0. \tag{33}$$

Formulas (31)-(32)-(33) improve the general formulas (3)-(4)-(5).

Proof. The statement (30) is an immediate consequence of (22) in Theorem 5. Claim (31) is deduced from (30) and a localization of the first zero for H_α in (21) which is made possible via an application of Rouché Theorem: we split this argument into four lemmas, where for simplicity we have set $\varepsilon := (1+\alpha)^{-1}$.

Lemma 1. *Let*

$$H(\varepsilon, z) := \sum_{k=0}^\infty \frac{(-z)^k}{k! \prod_{j=1}^k (j-\varepsilon)}.$$

Then

$$|H(\varepsilon, z) - H(0, z)| \le 5\varepsilon e^{|z|}$$

for every $z \in \mathbb{C}$, when $\varepsilon \le 1/8$.

Proof. In fact, for $\varepsilon \le 1/2$ and for a suitable L that we will set later, we have

$$\begin{aligned}|H(\varepsilon, z) - H(0, z)| &\le \sum_{k=1}^L \frac{|z|^k}{k!^2}\Big[\prod_{j=1}^k (1-\varepsilon/j)^{-1} - 1\Big] + \sum_{k=L+1}^\infty \frac{|z|^k}{k!}\Big[\frac{2^k}{(2k-1)!!} + \frac{1}{k!}\Big] \\ &= \sum_{k=1}^L \frac{|z|^k}{k!^2}\Big[\exp\Big(\sum_{j=1}^k\sum_{m=1}^\infty \frac{(\varepsilon/j)^m}{m}\Big) - 1\Big] + \sum_{k=L+1}^\infty \frac{|z|^k}{k!^2}\Big[\frac{4^k k! k!}{(2k)!} + 1\Big].\end{aligned}$$

The term for $m=1$ is estimated using the inequality $\sum_{j\le k} 1/j \le \log(ek)$. The remaining sum $\sum_{m=2}^\infty \sum_{j=1}^k \frac{(\varepsilon/j)^m}{m}$ is estimated by $\frac{\varepsilon^2}{1-\varepsilon}$. The condition $\varepsilon \le 1/2$ is then used to prove that $\frac{\varepsilon^2}{1-\varepsilon} \le \varepsilon\log(ek)$. Moreover, we notice that $\frac{4^k k! k!}{(2k)!} \le 2k-1$ for $k \ge 2$. In this way we obtain the bound

$$\begin{aligned}&\le \sum_{k=1}^L \frac{|z|^k}{k!^2}\Big[\exp\big(2\varepsilon\log(ek)\big) - 1\Big] + 2\sum_{k=L+1}^\infty \frac{k|z|^k}{k!^2} \\ &\le \sum_{k=1}^L \frac{|z|^k}{k!^2}\Big[\exp\big(2\varepsilon\log(ek)\big) - 1\Big] + \frac{2}{L!}e^{|z|}.\end{aligned}$$

We fix the value of L to $\lfloor\exp(1/(2\varepsilon) - 1)\rfloor$; in this way $\exp(2\varepsilon\log(ek)) - 1 \le 4\varepsilon\log(ek)$ inside the first sum because $e^y - 1 \le 2y$ for $y \in [0,1]$. This yields

$$\le 4\varepsilon \sum_{k=1}^L \frac{|z|^k}{k!^2}\log(ek) + \frac{2e^{|z|}}{L!}.$$

When $\varepsilon \leq 1/8$ we have the inequality $L! = \lfloor \exp(1/(2\varepsilon) - 1) \rfloor ! \geq \exp(1/(2\varepsilon) - 1) \geq 2/\varepsilon$, giving

$$\leq 4\varepsilon \sum_{k=1}^{\infty} \frac{|z|^k}{k!^2} \log(ek) + \varepsilon e^{|z|}.$$

The claim now follows, because $\log(ek)/k! \leq 1$ for every $k \geq 1$. □

Lemma 2. *$H(0, z)$ has no zeros on the circle $|z| = 3/2$.*

Proof. In fact,

$$\sum_{k=4}^{\infty} \frac{|z|^k}{k!^2} \leq \frac{1}{4!} \sum_{k=4}^{\infty} \frac{|z|^k}{k!} = \frac{1}{4!} \Big[\exp(|z|) - \sum_{k=0}^{3} \frac{|z|^k}{k!} \Big] \leq 0.02 \quad \text{when } |z| = 3/2.$$

On the other hand, on the circle $|z| = 3/2$, the polynomial $|\sum_{k=0}^{3}(-z)^k/k!^2|$ attains its minimum at $z = 3/2$, with value $1/32 = 0.03125$; hence, for $|z| = 3/2$ one has $|H(0, z)| \geq |\sum_{k=0}^{3}(-z)^k/k!^2| - |\sum_{k=4}^{\infty}(-z)^k/k!^2| \geq 0.03125 - 0.02 > 0$. □

Lemma 3. *$H(0, z)$ has a unique complex zero in $|z| \leq 3/2$.*

Proof. The polynomial $\sum_{k\leq 3} \frac{(-z)^k}{(k!)^2} = 1 - z + \frac{z^2}{4} - \frac{z^3}{36}$ has a unique zero in that disk. Moreover, during the proof of Lemma 2 we have shown that on the circle $|z| = 3/2$

$$\sum_{k=4}^{\infty} \frac{|z|^k}{k!^2} \leq 0.02 < 0.03125 \leq \Big| \sum_{k=0}^{3} \frac{(-z)^k}{k!^2} \Big|.$$

The claim follows by Rouché Theorem. □

By Lemma 1 on the disc $|z| \leq 3/2$ we have that $|H(\varepsilon, z) - H(0, z)| = O(\varepsilon)$ and Lemma 2 shows that the minimum for $|H(0, z)|$ on the circle $|z| = 3/2$ is positive. By Rouché Theorem, these facts prove that functions $H(\varepsilon, \cdot)$ and $H(0, \cdot)$ have the same number of zeros in that disk when ε is small enough, and hence a unique zero, by Lemma (3).

Lemma 4. *$|H'(\varepsilon, z)| \geq 0.02$ for $|z| \leq 3/2$ and $\varepsilon < 1/100$.*

Proof. In fact,

$$\begin{aligned}
|H'(\varepsilon, z)| &= \Big| \sum_{k=1}^{\infty} \frac{k(-z)^{k-1}}{k! \prod_{j=1}^{k}(j - \varepsilon)} \Big| \geq \frac{1}{1 - \varepsilon} - \Big| \sum_{k=2}^{\infty} \frac{k(-z)^{k-1}}{k! \prod_{j=1}^{k}(j - \varepsilon)} \Big| \\
&\geq \frac{1}{1 - \varepsilon} - \sum_{k=2}^{\infty} \frac{k|z|^{k-1}}{k! \prod_{j=1}^{k}(j - \varepsilon)} \geq \frac{1}{1 - \varepsilon} - \sum_{k=2}^{\infty} \frac{k|z|^{k-1}}{k! \prod_{j=1}^{k}(j - 1/100)} \\
&\geq \frac{1}{1 - \varepsilon} - \sum_{k=2}^{\infty} \frac{k(3/2)^{k-1}}{k! \prod_{j=1}^{k}(j - 1/100)} \geq \frac{1}{1 - \varepsilon} - 0.98 \geq 0.02.
\end{aligned}$$

□

Now we can conclude. Assume ε small enough and let $z(\varepsilon)$ be the unique zero of $H(\varepsilon, z)$ in the disk $|z| \leq 3/2$. The mean value Theorem and equalities $H(0, z(0)) = 0 = H(\varepsilon, z(\varepsilon))$ give that

$$(z(0) - z(\varepsilon))H'(\varepsilon, \eta) = H(\varepsilon, z(0)) - H(\varepsilon, z(\varepsilon)) = H(\varepsilon, z(0)) - H(0, z(0))$$

for some η with $|\eta| \leq 3/2$ so that by Lemma 4

$$|z(\varepsilon) - z(0)| \leq 50|H(\varepsilon, z(0)) - H(0, z(0))|.$$

This relation and Lemma 1 prove that $z(\varepsilon) \to z(0)$ as $\varepsilon \to 0$. This conclude the proof of (31).

Claim (32) follows from (31) and the identity $T_0 - T_\alpha = T^*_{1/\alpha}$.

To prove (33) we use directly (29) in order to handle the fact that the term $\prod_{j=1}^{k}(j - \frac{1}{1+\alpha})^{-1}$ contains the factor $(1 - \frac{1}{1+\alpha})^{-1}$ which diverges as $\alpha \to 0$. Equation (29) can be written as

$$f(\alpha, \omega^2) := g(\alpha, \omega^2) - \sum_{k=3}^{\infty} \frac{(\frac{-\alpha}{(1+\alpha)^2})^{k-1} \omega^{2k}}{k! \prod_{j=2}^{k}(j - \frac{1}{1+\alpha})} = 0$$

where

$$g(\alpha,z) := 1+\alpha-z+\frac{\alpha z^2}{2(1+\alpha)(1+2\alpha)}.$$

When $\alpha\to 0$, the function $g(\alpha,z)$ has a unique zero, $z_0(\alpha)$ say, located in $|z|\le 3/2$, and $z_0(\alpha)$ behaves as $1+3\alpha/2+O(\alpha^2)$. Via Rouché Theorem one proves that the same happens to the full function $f(\alpha,z)$. In fact,

$$\big|f(\alpha,z)-g(\alpha,z)\big| \le \sum_{k=3}^{\infty}\frac{(\frac{\alpha}{1+\alpha})^{k-1}|z|^k}{k!\prod_{j=2}^{k}(j-1+j\alpha)} \le \sum_{k=3}^{\infty}\frac{\alpha^{k-1}|z|^k}{k!(k-1)!} \le \alpha^2|z|^3 e^{\alpha|z|}. \tag{34}$$

In particular, the difference is $O(\alpha^2)$ as $\alpha\to 0$ and z is bounded. On the other hand, for $|z|=3/2$

$$\big|g(\alpha,z)\big| \ge |1-z|+O(\alpha)\ge 1/2+O(\alpha)$$

so that (by Rouché Theorem) also $f(\alpha,z)$ has a unique zero in $|z|\le 3/2$, when α is small enough. Moreover, for $|z|\le 3/2$

$$\big|f'(\alpha,z)\big| \ge 1-\sum_{k=2}^{\infty}\frac{(\frac{\alpha}{1+\alpha})^{k-1}k|z|^{k-1}}{k!\prod_{j=2}^{k}(j-1+j\alpha)} \ge 1-\sum_{k=2}^{\infty}\frac{\alpha^{k-1}|z|^{k-1}}{(k-1)!} \ge 1-\alpha|z|e^{\alpha|z|}=1+O(\alpha). \tag{35}$$

Let $z(\alpha)$ denote the zero of $f(\alpha,\cdot)$ in $|z|\le 3/2$. Then $f(\alpha,z(\alpha))=0=g(\alpha,z_0(\alpha))$ and by the mean value Theorem there exists η with $|\eta|\le 3/2$ such that

$$(z(\alpha)-z_0(\alpha))f'(\alpha,\eta)=f(\alpha,z(\alpha))-f(\alpha,z_0(\alpha))=g(\alpha,z_0(\alpha))-f(\alpha,z_0(\alpha))$$

so that by (34) and (35)

$$|z(\alpha)-z_0(\alpha)|\le\frac{O(\alpha^2)}{1+O(\alpha)},$$

i.e.,

$$z(\alpha)=z_0(\alpha)+O(\alpha^2)=1+\frac{3}{2}\alpha+O(\alpha^2).$$

The conclusion follows recalling that the norm is $1/\sqrt{z(\alpha)}$. □

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