



Frequency estimation by frequency jumps



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The frequency of a quantum harmonic oscillator cannot be precisely determined through static measurement strategies on a prepared state. Therefore, dynamical procedures must be employed, involving measurements taken after the system has evolved and encoded the frequency information. This paper explores the precision achievable in a protocol where a known detuning suddenly shifts the oscillator's frequency, which then reverts to its original value after a specific time interval. Our results demonstrate that the squeezing induced by this frequency jump can effectively enhance the encoding of frequency information, significantly improving the quantum signal-to-noise ratio (QSNR) compared to standard free evolution at the same resource (energy and time) cost. The QSNR exhibits minimal dependence on the actual frequency and increases with both the magnitude of the detuning and the overall duration of the protocol. Furthermore, incorporating multiple frequency jumps into the protocol could further enhance precision, particularly for lower frequency values.

Harmonic behavior is ubiquitous in physics, and the quantum harmonic oscillator (QHO) model is relevant in nearly every field of physics. The kinematics and dynamics of the QHO are governed by the frequency parameter, whose determination is essential to characterize the system properly. Indeed, accurate frequency estimation is relevant for several fields, including metrology^{1–6} and quantum sensing^{7–9}, spectroscopy^{10–12}, and precision timekeeping¹³. Additionally, frequency estimation finds application in quantum communication and computation^{14,15}.

As a matter of fact, the dependence of the eigenvalues and eigenstates of the quantum harmonic oscillator on its frequency leads to unfavourable scaling of precision, making static strategies, i.e., measurements on a prepared state, inherently inefficient. Instead, the frequency must be estimated dynamically by performing measurements after the system has evolved and encoded the frequency information. Free evolution may suffice for this purpose, but one may wonder whether more efficient methods exist to encode frequency information in the evolved state, thereby improving the precision of frequency estimation, either in absolute terms or for a fixed amount of resources (energy and time). This is precisely the scope of this paper. In particular, we investigate whether suddenly detuning the oscillator's frequency^{16–19} and then returning it to its original value after a specific time interval can enhance the precision and efficiency of the estimation strategy.

Our results demonstrate that the squeezing induced by a frequency jump can effectively enhance the encoding of frequency information, significantly boosting the quantum signal-to-noise ratio (QSNR) compared to standard free evolution at the same resource cost. The QSNR exhibits minimal dependence on the actual frequency and increases with both the magnitude of the detuning and the overall duration of the protocol.

Furthermore, incorporating multiple frequency jumps into the protocol further improves precision, especially for lower values of frequency.

The manuscript begins with a review of the theoretical description of harmonic systems with time-dependent frequency, focusing particularly on Gaussian states. It then introduces key concepts from local estimation theory relevant to our study. We present results concerning frequency estimation under a single frequency jump, followed by a comparison with the precision bounds obtainable under free evolution without frequency jumps. The analysis is then extended to scenarios involving multiple frequency jumps. Finally, we conclude with a summary of our findings and their implications.

Results

Harmonic oscillator with time-dependent frequency

The harmonic oscillator with n successive frequency jumps can be described by specifying the frequency function $\omega(t)$ as a piece-wise function over n intervals. Let us denote the natural frequency as ω_0 , and the changed frequency as $\omega_1 = \omega_0 + \delta$. The time of the first frequency jump from ω_0 to ω_1 is $t = 0$ and successively the system spend a time interval $t = \tau/n$ at ω_1 , before coming back to ω_0 . Then up to $t = T/n$ the system remains at a frequency ω_0 , before coming back to ω_1 and repeating the frequency jumps cycle. Overall the system spend a time interval $T - \tau = (1 - \alpha)T$ at ω_0 and $\tau = \alpha T$ at ω_1 , where T is the total time evolution and $0 \leq \alpha \leq 1$. The frequency function $\omega(t)$ can be expressed as follows:

$$\omega(t) = \begin{cases} \omega_1 & \text{if } m\tau_n \leq t \leq (m+1)\tau_n \quad 0 \leq m < n \\ \omega_0 & \text{elsewhere,} \end{cases} \quad (1)$$

where $\tau_n = \alpha T/n$.

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This frequency jump model is adopted in a large variety of theoretical frameworks and experimental realizations. In optomechanical systems, the time scale of the frequency switch is determined by the bandwidth of the involved acousto-optical modulator, which is usually much larger than the two involved frequencies, making the switch effectively instantaneous. In other range of frequencies, a smoother description of frequency jumps may instead be more convenient. On the other hand, in situations where there is a linear fast transient between the two frequencies²⁰, a Fourier dependence²¹ or a hyperbolic tangent time dependence²², our model provides a very good approximation. More generally, since squeezing is generated for any rate of frequency change, sudden or smooth²³, the step model may provide at least a qualitative description.

In order to compute the quantum state of the HO at an arbitrary instant $t > 0$, we need to obtain the time evolution operator for this time-dependent Hamiltonian. Following refs. 16–19, the Hamiltonian of the HO with a time-dependent frequency as given in Eq. (1):

$$H = \frac{p^2}{2} + \frac{1}{2}(\omega_0^2 + 2\omega_0\eta(t; \tau, T))q^2 \quad (2)$$

Here, q represents the position operator, p is the momentum operator, ω_0 is the initial frequency, and $\eta(\tau, T)$ is the time-dependent function encoding the frequency variation defined by

$$\eta(t; \tau, T) = \sum_m \eta_0 \left[\Theta\left(t - \frac{m(\tau + T)}{n}\right) - \Theta\left(t - \frac{m(\tau + T) + \tau}{n}\right) \right] \quad (3)$$

where $\Theta(t)$ is the Heaviside step function and

$$\eta_0 = \frac{\omega_1^2 - \omega_0^2}{2\omega_0}, \quad \omega_1 = \sqrt{\omega_0^2 + 2\omega_0\eta_0}. \quad (4)$$

The evolution of the system is piece-wise time-independent, where the two time-independent Hamiltonians are respectively given by

$$H_1 = \frac{p^2}{2} + \frac{1}{2}\omega_1^2 q^2 = \frac{1}{2}\eta_0(a^2 + a^{\dagger 2}) + (\omega_0 + \eta_0)(a^\dagger a + \frac{1}{2}) \quad (5)$$

$$H_0 = \frac{p^2}{2} + \frac{1}{2}\omega_0^2 q^2 = \omega_0(a^\dagger a + \frac{1}{2}), \quad (6)$$

where $[a, a^\dagger] = 1$ are the usual field operators for the QHO. The Hamiltonian H_0 describes free evolution and corresponds to rotation in the phase space, whereas the Hamiltonian H_1 describes (generalized) squeezing. Starting from the ground state of the harmonic oscillator, the evolved state in the case of a single jump in frequency is given by

$$|\psi(t)\rangle = \begin{cases} |\psi_s(t)\rangle & 0 < t < \tau \\ |\psi_\tau(t)\rangle & \tau < t < T \end{cases} \quad (7)$$

where

$$|\psi_s(t)\rangle = N(t) \sum_{n=0}^{\infty} \frac{\sqrt{2n!}}{2^n n!} \Lambda^n(t) |2n\rangle, \quad (8)$$

$$|\psi_\tau(t)\rangle = N(\tau) \sum_{n=0}^{\infty} \frac{\sqrt{2n!}}{2^n n!} \Lambda^n(\tau) e^{-2i\omega_0 n(t-\tau)} |2n\rangle. \quad (9)$$

In other words, $|\psi(t)\rangle$ is a squeezed vacuum state with time-dependent amplitude and phase. In the above formulas we have

$$N(t) = \left| \left[\cosh v(t) - \frac{\lambda(t)}{2v(t)} \sinh v(t) \right] \right|^{-\frac{1}{4}} \quad (10)$$

$$\Lambda(t) = \frac{-4i\eta_0 t \sinh v(t)}{2v \cosh v(t) - \lambda(t) \sinh v(t)}, \quad (11)$$

where

$$\lambda(t) = -2i(\omega_0 + \eta_0 t) \quad (12)$$

$$v(t) = \left(\frac{1}{4}\lambda_3^2 - \eta_0^2 t^2 \right)^{\frac{1}{2}}. \quad (13)$$

In order to solve the dynamics in the case of multiple jumps, it is convenient to describe the dynamics in the phase space²⁴. This is possible because both the Hamiltonians above are quadratic in the field operators with two main consequences: 1. the dynamics maintains the Gaussian character of any initial Gaussian states; 2. for an initial Gaussian state, the dynamics may be entirely described using the symplectic formalism, i.e., by the evolution of the first two moments of the canonical operators q and p , i.e., the vector of mean values $X(t) = (\langle q \rangle, \langle p \rangle)$ and the covariance matrix $\sigma(t)$ with elements $\sigma_{lm} = \frac{1}{2}\langle X_l X_m + X_m X_l \rangle - \langle X_l \rangle \langle X_m \rangle$ where $\langle \dots \rangle = \langle \psi(t) | \dots | \psi(t) \rangle$.

In particular, if we start from the QHO initially in the ground state we have $X(0) = (0, 0)$ and $\sigma(0) = \frac{1}{2}\mathbb{I}_2$, and the state in Eq. (7) may be equivalently described as the Gaussian state having

$$X(t) = (0, 0) \quad (14)$$

$$\sigma(t) = \begin{cases} \frac{1}{2}S(t)^T S(t) & 0 < t < \tau \\ \frac{1}{2}R(t)^T S(\tau)^T S(\tau)R(t) & \tau < t < T \end{cases}, \quad (15)$$

where T denotes transposition. The symplectic matrices corresponding to squeezing and rotation are given by

$$R(t) = \cos \omega_0 t \mathbb{I} + i \sin \omega_0 t \sigma_2 \quad (16)$$

$$S(t) = \cosh 2r \mathbb{I} + \sinh 2r \cos \phi \sigma_3 + \sinh 2r \sin \phi \sigma_1 \quad (17)$$

where the σ 's are the Pauli matrices, and the time-dependent squeezing and phase parameters may be obtained from the relations $\tanh r(t) = |\Lambda(t)|$ and $\tan \phi(t) = \text{Arg } \Lambda(t)$, i.e.,

$$r(t) = \text{ArcTanh} \left(\frac{|\omega_0^2 - \omega_1^2|}{\sqrt{(\omega_0^2 + \omega_1^2)^2 + 4\omega_0^2 \omega_1^2 \cot^2(\omega_1 t)}} \right) \quad (18)$$

$$\phi(t) = \text{ArcTan} \left(\frac{2\omega_0 \omega_1 \cot(\omega_1 t)}{\omega_0^2 + \omega_1^2} \right) \quad (19)$$

In the case of n frequency jumps, at the end of the cycle the vector of mean values still vanishes, whereas the covariance matrix is given by

$$\sigma_n(\alpha, T) = \frac{1}{2} [R^T(T_n) S^T(\tau_n)]^n [S(\tau_n) R(T_n)]^n \quad (20)$$

where

$$\tau_n = \frac{\alpha T}{n} \quad T_n = \frac{(1-\alpha)T}{n} \quad (21)$$

Local quantum estimation theory

By encoding a parameter onto the states of a quantum system, we obtain a family of density matrices ρ_λ , usually referred to as a quantum statistical model, $\lambda \in \Lambda \subset \mathbb{R}$. An estimation strategy consists of an observable to be measured and an estimator to process data. The measurement is described

by a positive operator-valued measure (POVM) $\{\Pi_y\}$, $y \in Y$, such that $\Pi_y > 0$, $\forall y$, and $\sum_{y \in Y} \Pi_y = \mathbb{I}$. The estimator is a function $\hat{\lambda}$ from the data space $Y \times Y \cdots \times Y$ (M times) to the domain Λ , where M denotes the number of repeated measurements. The outcomes of the measurement are distributed according to the Born rule $p(y|\lambda) = \text{Tr}[\rho_\lambda \Pi_y]$. The precision of the estimation strategy is quantified by the variance of the estimator. For unbiased estimators, i.e., $\int_Y dy p(y|\lambda) \hat{\lambda}(y) = \lambda$, the Cramér-Rao theorem establishes a bound on the variance as follows

$$\text{Var} \hat{\lambda} \geq \frac{1}{MF(\lambda)}, \quad (22)$$

where the Fisher information $F(\lambda)$ is defined as

$$F(\lambda) = \int dy p(y|\lambda) [\partial_\lambda \ln p(y|\lambda)]^2. \quad (23)$$

An estimator is said to be *efficient* if it saturates the Cramér-Rao bound. The ultimate bound on the precision of any estimation strategy for λ may be obtained by maximizing the Fisher information over all the possible POVM. The optimization may be actually carried out and the optimal POVM corresponds to the spectral measure of the *symmetric logarithmic derivative* L_λ , which is the self-adjoint operator solving the Lyapunov equation

$$2\partial_\lambda \rho_\lambda = L_\lambda \rho_\lambda + \rho_\lambda L_\lambda. \quad (24)$$

The maximum value of the Fisher information is usually referred to as the Quantum Fisher Information (QFI) $G(\lambda)$, and the corresponding bound as the Quantum Cramér-Rao bound

$$\max_{\{\Pi_y\}} F(\lambda) = G(\lambda) \equiv \text{Tr}[\rho_\lambda L_\lambda^2] \quad (25)$$

$$\text{Var} \hat{\lambda} \geq \frac{1}{MG(\lambda)}. \quad (26)$$

For statistical models made of pure states $\rho_\lambda = |\psi_\lambda\rangle\langle\psi_\lambda|$ the QFI may be written as

$$G(\lambda) = 4[\langle\partial_\lambda \psi_\lambda|\partial_\lambda \psi_\lambda\rangle - |\langle\partial_\lambda \psi_\lambda|\psi_\lambda\rangle|^2]. \quad (27)$$

For Gaussian states having vanishing vector of mean values $X = (0, 0)$ and covariance matrix σ the QFI may be written as

$$G(\lambda) = -\text{Tr}[\Omega^T (\partial_\lambda \sigma) \Omega (\partial_\lambda \sigma)], \quad (28)$$

where $\Omega = i\sigma_2$ is the symplectic matrix.

Finally, in order to fairly compare estimation schemes for small and large actual values of the parameter, we introduce the signal-to-noise ratio (SNR) of an estimation strategy, $R(\lambda) = \lambda^2/\text{Var} \hat{\lambda} \leq \lambda^2 F(\lambda)$. The optimal measurement is characterized by the maximal value of the SNR and, in general, we have $R(\lambda) \leq Q(\lambda)$, where the quantum signal-to-noise ratio (QSNR) is defined by

$$Q(\lambda) = \lambda^2 G(\lambda). \quad (29)$$

Local QET has been successfully applied to find the ultimate bounds to precision for estimation problems in open quantum systems, non-unitary processes, and nonlinear quantities as entanglement^{25–39} and for a closed system, evolving under a unitary transformation^{40–42}. The geometric structure of QET has been exploited to assess the quantum criticality as a resource for quantum estimation^{43–47}.

Frequency estimation by a single frequency jump

In this Section, we analyze the behavior of the quantum signal-to-noise ratio of the quantum Fisher information for the frequency of the oscillator, as obtained by performing measurements on the evolved state after a single frequency jump. Specifically, we optimize the duration of the jump, $\tau = \alpha T$, which represents the fraction of the total evolution time T that induces squeezing, to maximize the QSNR and the QFI, thus enhancing the metrological properties of the probe. In the following, we continue to denote by ω_0 the natural frequency of the oscillator, i.e., the parameter to be estimated, and introduce the symbol $\delta = \omega_1 - \omega_0$ to represent the frequency shift. At first, we set $T = 1$ and defer the discussion about the dependence on

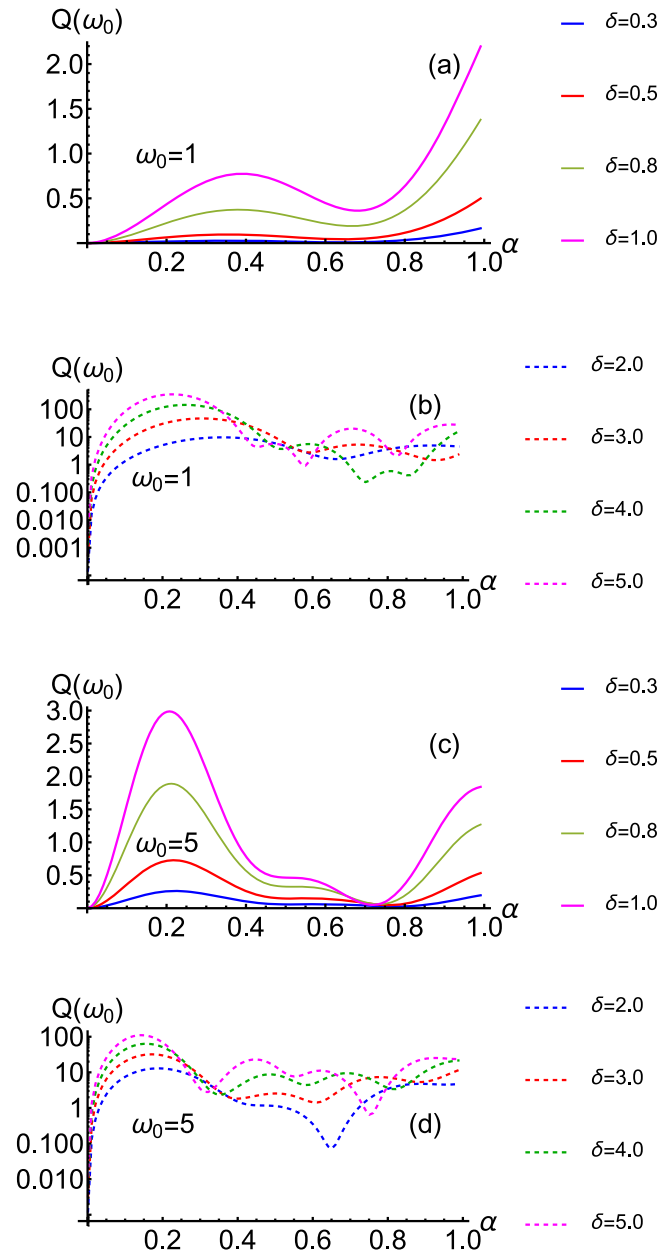
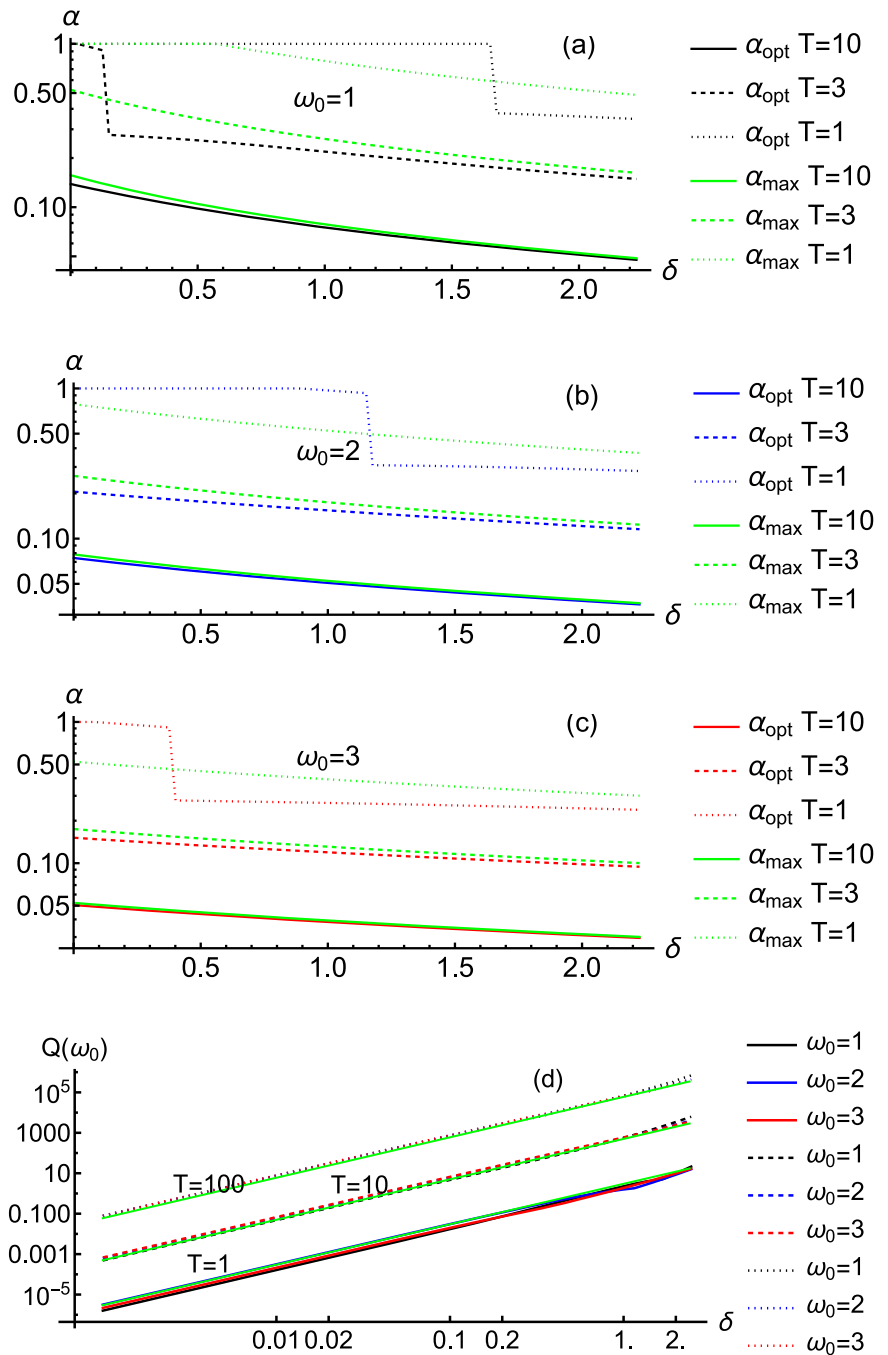


Fig. 1 | Quantum signal-to-noise ratio for a single-frequency jump protocol. The quantum signal-to-noise ratio $Q(\omega_0)$ as a function of the squeezing time fraction α for different values of frequency ω_0 and of the frequency shift δ . **a, b** show results for $\omega_0 = 1.0$ and **c, d** for $\omega_0 = 5.0$. In **(a, c)**, we show linear plots for $\delta = 0.3$ (blue), $\delta = 0.5$ (red), $\delta = 0.8$ (green), $\delta = 1.0$ (magenta), respectively. In **(b, d)**, we show log-plots for $\delta = 2$ (dashed blue), $\delta = 3$ (dashed red), $\delta = 4$ (dashed green), $\delta = 5$ (dashed magenta), respectively.

Fig. 2 | Optimization of quantum signal-to-noise ratio with respect to squeezing time and frequency shift. **a–c** the optimal time fraction α_{opt} maximizing the QSNR as a function of the frequency shift δ for $\omega_0 = 1, 2, 3$, respectively. In each plot, we show results for $T = 1$ (dotted line), $T = 3$ (dashed), and $T = 10$ (solid). The green lines denote the corresponding values of α_{max} , i.e., the time fraction maximizing the squeezing amplitude. **d** The maximized QSNR as a function of the frequency shift for $\omega_0 = 1, 2, 3$ and $T = 1, 10, 100$. The curves for different values of ω_0 nearly overlap for a given value of T . The green lines in **(d)** serve as visual guides proportional to $T^2 \delta^2$.



T to the end of the Section. Results are summarized in the four panels of Fig. 1.

For lower values of the frequency and frequency shift, i.e., $\omega_0 \lesssim 1$ and $\omega_0 \gtrsim \delta$, the optimal procedure is to avoid free evolution. This means choosing $\alpha = 1$ and dedicating the entire evolution time to the squeezing process. This is illustrated in Fig. 1a, where we also observe that the QSNR is not monotonic with respect to α but instead exhibits a local maximum. Additionally, we note that the QSNR increases rapidly with δ . As the frequency shift increases to larger values, $\delta \gtrsim \omega_0$, the local maximum of the QSNR surpasses the value for $\alpha = 1$, and a non-trivial optimal value for the jump duration emerges, as shown in Fig. 1b.

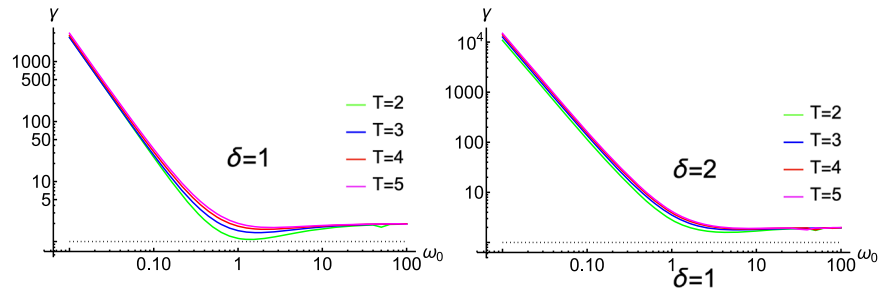
A similar behavior, namely the appearance of a non-trivial optimal value of α , is observed for larger values of the frequency. Results for $\omega_0 = 5$ are displayed in Fig. 1c. This trend is further confirmed for larger δ , as seen in panel (d) of the same figure. The value of the QSNR is not significantly

affected by the specific value of the frequency, indicating that the QFI decreases approximately as $G(\omega_0) \propto \omega_0^{-2}$ with increasing frequency. As δ is further increased, additional peaks appear, although the first maximum, occurring at smaller α , remains the absolute maximum.

Using Eqs. (27) or (28), the analytic expression for $Q(\omega_0)$ can be derived. However, this expression is cumbersome and is not provided here. On the other hand, it is straightforward to show that the amplitude and phase of the squeezing depend on the relevant quantities through the dimensionless combinations $\omega_0 \tau = \alpha \omega_0 T$ and δ/ω_0 , while the QFI and QSNR depend on αT and δ/ω_0 . This implies that the optimal time fraction maximizing the squeezing amplitude, i.e., $\alpha_{max} = \frac{\pi}{2} [T(\omega_0 + \delta)]^{-1}$, corresponding to $r_{max} = \log(1 + \delta/\omega_0)$, is not in general the same as the one maximizing the QSNR, which must be determined numerically.

In panels (a), (b), and (c) of Fig. 2, we show the optimal time fraction α_{opt} as a function of the frequency shift δ for different values of the frequency

Fig. 3 | Comparison of quantum signal-to-noise ratios for jump-based and free evolution protocols. The ratio γ between the QSNRs of jump-based and free evolutions as a function of the actual value of the frequency ω_0 for different values of the overall time duration (from top to bottom, $T = 5$ (magenta), 4 (red), 3 (blue), 2 (green), respectively). The left panel shows results for $\delta = 1.0$ and the right one for $\delta = 2.0$. The asymptotic value for $\omega_0 \gg 1$ is $\gamma \simeq 2$, independent on T . The dotted line denotes the value $\gamma = 1$.



ω_0 ($\omega_0 = 1, 2, 3$, respectively) and the total evolution time T (from top to bottom $T = 1, 3, 10$ in each plot). In the three panels, the green lines denote $\alpha_{\max}$, i.e., the time fraction maximizing the amplitude of squeezing. We see that the two values may be very different for lower T , whereas they become very close to each other for increasing T . The optimal time fraction α_{opt} decreases with δ , consistently with the results shown in Fig. 1.

In (d) of Fig. 2, we show the maximized quantum SNR $Q(\omega_0)$, obtained for $\alpha = \alpha_{\text{opt}}$, as a function of the frequency shift for $\omega_0 = 1, 2, 3$ and $T = 1, 10, 100$. As evident from the plot, the dependence on the frequency is very weak, and the curves for different values of ω_0 nearly overlap for a given value of T . The green lines serve as visual guides, exhibiting a behavior proportional to $T^2 \delta^2$.

Comparison with free evolution

In principle, the frequency can be estimated by performing measurements after free evolution (and no frequency jumps) of an initially prepared state $|\psi_0\rangle$. In this case, the family of states encoding frequency information is given by $|\psi(t)\rangle = e^{-it\omega_0}|\psi_0\rangle$, with $n = a^\dagger a$ and the QFI and QSNR may be easily evaluated as

$$G_f = 4t^2 \Delta n^2 \quad Q_f = 4\omega_0^2 t^2 \Delta n^2 \quad (30)$$

where $\Delta n^2 = \langle \psi_0 | n^2 | \psi_0 \rangle - \langle \psi_0 | n | \psi_0 \rangle^2$. The QFI is independent of the frequency, scales quadratically with the overall evolution time, and depends on the fluctuations of the number operator in the initial state. For the oscillator initially prepared in a coherent state $\Delta n^2 = \bar{n} = \langle a^\dagger a \rangle$.

Estimation strategies should be compared under a fixed amount of resources. In the case of frequency estimation, these resources are the energy of the probe state and the evolution time. For the free evolution, we consider the oscillator initially prepared in a coherent state, allowing us to safely assume that the time required to prepare the initial state is negligible. This enables a fair comparison of the two encoding strategies, both of which has a duration T . Regarding energy, in the jump-based strategy, it corresponds to the mean number of squeezing quanta generated during the jump, which can be evaluated using Eq. (18)

$$\bar{n} = \sinh^2 r(\alpha_{\text{opt}} T) = \left(\frac{\delta}{2\omega_0} \right)^2 \frac{2 + \frac{\delta}{\omega_0}}{1 + \frac{\delta}{\omega_0}} \times \sin^2 \left[\omega_0 \alpha_{\text{opt}} T \left(1 + \frac{\delta}{\omega_0} \right) \right]. \quad (31)$$

For the free evolution, the dynamics is passive, i.e. no energy is added, and the number of quanta is that of the initial coherent state.

In order to compare the two strategies, we introduce the ratio γ between the QSNRs (which is equal to the ratio of the QFIs), using the numerically calculated α_{opt} and evaluate it for the oscillator initially prepared in a coherent state. According to Eq. (30) we have

$$\gamma = \frac{G(\omega_0)}{4T^2 \bar{n}} \quad (32)$$

where one has to insert the expression of $\bar{n}$ in Eq. (31) in order to compare the two strategies using the same amount of resources.

In Fig. 3, we show the ratio γ as a function of the frequency ω_0 for different values of the overall time duration T and frequency shift δ . As it is apparent from the plot, the jump-based strategy outperforms free evolution for a value of ω_0 , largely improving the QSNR for lower frequencies. The asymptotic value for $\omega_0 \gg 1$ is $\gamma \simeq 2$. The ratio is nearly independent on the duration T of the protocol (at least in this range of values) and, for lower values of ω_0 , increases with δ . We conclude that the squeezing induced by this frequency jump can effectively enhance the encoding of frequency information, significantly boosting the QSNR compared to free evolution at the same resource cost.

Frequency estimation by multiple frequency jumps

In section Frequency estimation by a single frequency jump, we demonstrated that the optimal duration of the frequency jump is typically shorter than the total protocol duration T . This finding highlights the positive interplay between frequency jumps and free evolution. A natural question arises: could introducing additional jumps, interspersed with intervals of free evolution, further enhance frequency encoding? In this section, we prove that this is indeed the case. Specifically, we show that multiple jumps can significantly improve precision, particularly for lower frequency values.

To this end, we introduce the ratios

$$\rho_n = \frac{Q_n(\omega_0)}{Q(\omega_0)} \quad (33)$$

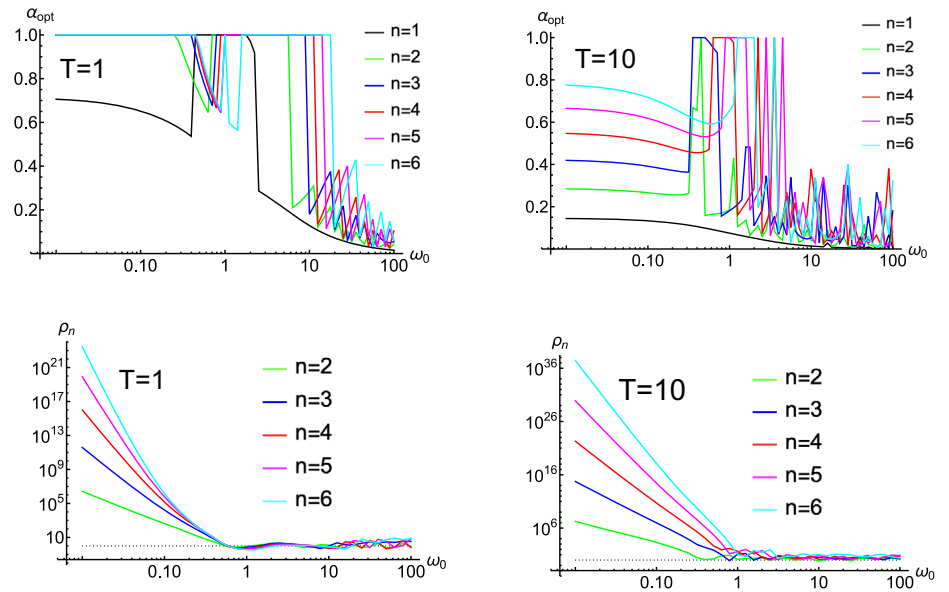
of the QSNR obtained by dividing the total duration T of the evolution into n cycles, each consisting of a frequency jump followed by free evolution, as described in Eq. (21), and the corresponding QSNR obtained from a single jump.

Note that the optimization of the jump duration α yields different values for different values of n . In the upper panels of Fig. 4 we show the optimal values α_{opt} as a function of the frequency for different number n of jumps, $\delta = 1$ and two different values of the total duration T of the evolution. In both cases, the optimal jump duration depends on the number of jumps, although this dependence is not significant. In the lower panels, we present the corresponding ratios ρ_n . As seen in the plots, using multiple frequency jumps consistently improves precision, with the enhancement being particularly pronounced for lower frequency values.

Discussion

In this paper, we have addressed the estimation of frequency of a harmonic oscillator by protocols where a known detuning suddenly shifts the oscillator's frequency, which then returns to its original value after a specific time interval. The squeezing induced by the frequency jump provides a metrologically effective encoding of frequency, which enhances precision and increases the quantum Fisher information of the resulting statistical model. In turn, the quantum signal-to-noise ratio increases compared to standard free evolution at the same resource cost, i.e., using the same amount of time

Fig. 4 | Effect of multiple frequency jumps on optimal squeezing and precision enhancement. The upper panels show the optimal values α_{opt} for $\delta = 1$ as a function of the frequency ω_0 for different number n of jumps: $n = 1$ (black), $n = 2$ (green), $n = 3$ (blue), $n = 4$ (red), $n = 5$ (magenta), $n = 6$ (cyan), and for two different values of the total duration T of the evolution ($T = 1$ on the left and $T = 10$ on the right, respectively). The lower panels show the corresponding ratios ρ_n defined in Eq. (33), illustrating that using multiple frequency jumps consistently improves precision, with the enhancement being particularly pronounced for lower frequencies.



and energy. The QSNR shows minimal dependence on the actual frequency and increases with both the magnitude of the detuning and the duration of the protocol. We have also found that by employing multiple frequency jumps, the estimation precision is further enhanced, in particular for lower values of the frequency.

Squeezing by frequency jumps has been realized experimentally in levitated optomechanical systems^{48,49} and to create squeezed states of atomic motion⁵⁰. In those systems, the protocol presented in this paper may be implemented with current technology. More generally, our results pave the way to more effective encoding of frequency, including nonlinear squeezing and information scrambling.

Methods

Frequency-jump model of the harmonic oscillator

We consider a quantum harmonic oscillator subject to n instantaneous frequency jumps. The frequency alternates between ω_0 and $\omega_1 = \omega_0 + \delta$ in a piecewise manner, with a time-dependent profile $\omega(t)$ defined over n cycles. The time spent at ω_1 is denoted $\tau = \alpha T$, where T is the total evolution time and $0 \leq \alpha \leq 1$. The frequency function is given by Eq. (1).

The system's dynamics is governed by the Hamiltonian in Eq. (2), where the jump structure is encoded via the time-dependent function $\eta(\tau, T)$. For each cycle, the oscillator evolves alternately under two time-independent Hamiltonians H_0 and H_1 , defined in Eqs. (5) and (6), respectively. We solve the time evolution starting from the vacuum state by employing the exact analytical expressions for the squeezed vacuum state given in Eqs. (8) and (9). The resulting state remains Gaussian throughout the dynamics, allowing us to characterize it entirely in terms of the first and second statistical moments. This analysis is carried out within the framework of symplectic formalism, as outlined in Eqs. (14), (15), where the evolution is encoded in the symplectic matrices $S(t)$ and $R(t)$, associated with squeezing and rotation, respectively, and defined explicitly in Eqs. (16) and (17).

Quantum estimation and Fisher information

We apply local quantum estimation theory to compute the quantum Fisher information (QFI) associated with estimating the oscillator's natural frequency ω_0 . Given a parameter-dependent family of quantum states ρ_λ , the QFI $G(\lambda)$ provides the ultimate precision limit for unbiased estimators, as stated by the quantum Cramér-Rao bound (Eqs. (26), (27)).

For pure and Gaussian states, we use both the wavefunction-based expression for QFI (Eq. (27)) and the covariance-matrix-based formula (Eq. (28)). We also evaluate the quantum signal-to-noise ratio (QSNR), $Q(\lambda) =$

$\lambda^2 G(\lambda)$, which quantifies estimation performance relative to the value of the parameter.

Numerical optimization of frequency estimation

We numerically optimize the duration of the frequency jump $\tau = \alpha T$ to maximize the QSNR for various values of δ and ω_0 , keeping the total evolution time T fixed. The optimization is performed for both single and multiple frequency jumps. The performance of these jump-based protocols is then compared to that of free evolution, using a coherent state input as a benchmark.

Data availability

No data were generated in this research beyond those presented in the paper. The codes used in this study are available from the authors upon reasonable request.

Code availability

Codes are available upon request to the authors.

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Author contributions

All the authors contributed equally to the design and performance of the research, as well as writing the manuscript.

Competing interests

The authors declare no competing interests.

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