JEMS

Calum Spicer · Roberto Svaldi

Local and global applications of the Minimal Model Program for co-rank 1 foliations on threefolds

Abstract. We provide several applications of the minimal model program to the local and global study of co-rank 1 foliations on threefolds. Locally, we prove a singular variant of Malgrange's theorem, a classification of terminal foliation singularities and the existence of separatrices for log canonical singularities. Globally, we prove termination of flips, a connectedness theorem on log canonical centres, a non-vanishing theorem and some hyperbolicity properties of foliations.

Keywords. Foliations, minimal model program, singularities, separatrices, hyperbolicity

Introduction

Our primary goal in this paper is to use techniques and ideas from the foliated Minimal Model Program (MMP) to deduce some structural and dynamical results for foliation singularities. Along the way we further develop the MMP and explore some applications of these developments to global properties of foliations.

Local results

To every foliation singularity the MMP associates a numerical invariant, the discrepancy, which measures how the canonical class of the foliation changes under blow ups.

Our local results explore to what extent this numerical invariant characterizes the structural and dynamical behaviour of the foliation singularity. Here we are mostly interested in three classes of foliation singularities which are defined according to the behaviour of the discrepancy: terminal, canonical and log canonical (see Definition 1.4). Terminal singularities can be viewed as the mildest class of singularities of the MMP whereas log canonical are the most severe. Consider the following two quick illustrations of this

Calum Spicer: Department of Mathematics, King's College London, London WC2R 2LS, United Kingdom; calum.spicer@kcl.ac.uk

Roberto Svaldi: EPFL MATH-GE, MA A1 344 (Bâtiment MA), Station 8, 1015 Lausanne, Switzerland; roberto.svaldi@epfl.ch

principle: terminal foliations on smooth surfaces are smooth foliations, and simple singularities (see Definition 1.8) are canonical; we refer the reader to Section 1.9 for a further discussion on the relations and parallels between the classes of singularities of the MMP and other classes of singularities.

The singularities of the MMP are birational generalizations of nice classes of foliation singularities and are natural from the perspective of the geometry of foliations. For example, canonical foliation singularities play a central role in the study of hyperbolicity properties of surfaces by McQuillan [29, 30], and in the classification of foliations with trivial canonical bundle [15, 26].

We remark that simple singularities (which are roughly analogous to the singularities of smooth normal crossings pairs) are not preserved by the operations of the MMP, since the underlying variety may become singular in the course of the MMP. In other words, if one seeks to improve the global geometry of the foliation (by making $K_{\mathcal{F}}$ more positive) one loses some control on the local geometry of the foliation. In light of this, canonical singularities should be viewed as a compromise. They are flexible enough to allow for the operations of the MMP, but mild enough to have many of the desirable properties of simple singularities.

A fundamental result in the study of singular foliations on smooth varieties is a theorem of Malgrange [27] asserting that the classical Frobenius integrability criterion holds even in the presence of foliated singularities, provided that the codimension of the singular locus of the foliation is at least 3. We prove a version of Malgrange's theorem on singular threefolds.

Theorem 0.1 (= Theorem 5.1). *Let $(P \in X)$ be a germ of an isolated* (*analytically*) $\mathbb{Q}$*-factorial threefold singularity with a co-rank* 1 *foliation $\mathcal{F}$. Suppose that $\mathcal{F}$ has an isolated canonical singularity at P. Then $\mathcal{F}$ admits a holomorphic first integral.*

The above statement is close to optimal (cf. [14, Examples 1.1–1.3]).

As a consequence, we obtain the following strong classification result.

Theorem 0.2 (= Theorem 5.20). *Let $(P \in X)$ be a germ of normal threefold with a co-rank* 1 *foliation $\mathcal{F}$ with terminal singularities. Then $\mathcal{F}$ admits a holomorphic first integral.*

Moreover, up to a $\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}$-cover, $\mathcal{F}$ admits a holomorphic first integral $\phi \colon (P \in X) \to (0 \in \mathbb{C})$, where $\phi^{-1}(0)$ is a Du Val surface singularity and $\phi^{-1}(t)$ is smooth for $t \neq 0$. In particular, X is terminal. Furthermore, $(X, \mathcal{F})$ fits into the finite list of families contained in Proposition 5.19.

We remark that in Theorem 0.2 we make no assumption on the singularities of the underlying space other than normality.

We also remark that Theorems 0.1 and 0.2 should be viewed as analogues of the classification of terminal and canonical singularities on threefolds. These classification results have been crucial in understanding the global geometry of threefolds, as well as the moduli space of surfaces, and we expect the above results to play a corresponding

role in the study of the global geometry of foliations of threefolds and moduli of surface foliations.

We next prove a result on the existence of separatrices of log canonical foliation singularities. Loosely speaking, a separatrix may be thought of as a local solution to the differential equation defining the foliation; see Definition 1.11 for a precise definition. It is an interesting and challenging problem to decide when a foliation singularity admits a separatrix. The existence of a (converging) separatrix is an essential element in the study of foliation singularities as it provides a way to "organize" the dynamics around the singularity. While separatrices do not necessarily exist for a general foliation singularity, we prove their existence for log canonical singularities.

Theorem 0.3 (= Theorem 6.1). *Let $\mathcal{F}$ be a germ of a log canonical foliation singularity at $(0 \in \mathbb{C}^3)$. Then $\mathcal{F}$ admits a separatrix.*

Our strategy of proof actually provides a more general version of this result allowing the underlying analytic germ to be singular.

In [8] it is shown that non-dicritical foliation singularities always admit separatrices, confirming a conjecture of R. Thom. Log canonical singularities are in general dicritical and so [8] does not apply to prove existence of separatrices for this class of singularities. Theorem 0.3 is also closely related to a local analogue of a conjecture of Brunella that has been formulated in [12] and explored in [11].

We remark that results analogous to Theorems 0.1 and 0.3 have been shown in [14] and [10], respectively, under differing assumptions on the singularities of the foliation and variety. An advantage of our statements is that they hold for very natural classes of singularities which appear on large classes of foliations.

Classically, the technique of inversion of adjunction has proven crucial for understanding singularities by providing a precise relation between the singularities of a variety and the singularities of a divisor in the variety. We prove a foliated analogue of this result which should prove equally useful in the study of foliation singularities.

Theorem 0.4 (= Theorem 3.12). *Let X be a $\mathbb{Q}$-factorial threefold and let $\mathcal{F}$ be a co-rank 1 foliation. Consider a prime divisor S and an effective $\mathbb{Q}$-divisor Δ on X which does not contain S in its support. Let $v : S^v \to S$ be the normalization and let $\mathcal{G}$ be the restricted foliation to S^v and write $v^*(K_{\mathcal{F}} + \Delta) = K_{\mathcal{G}} + \Theta$. Suppose that*

- *if S is transverse to $\mathcal{F}$, then $(\mathcal{G}, \Theta)$ is log canonical;*
- *if S is $\mathcal{F}$-invariant, then (S^v, Θ) is log canonical.*

Then $(\mathcal{F}, \epsilon(S)S + \Delta)$ is log canonical in a neighbourhood of S.

Here, $\epsilon(S) = 0$ if S is not $\mathcal{F}$-invariant, whereas $\epsilon(S) = 1$ otherwise.

We refer the reader to §3.3 for a discussion of this result and its relationship to the adjunction formula for log canonical pairs (cf. §3.2).

We now take a moment to explain some of the key ideas of the proofs of the above statements. Indeed, our central innovation is the systematic use of F-dlt modifications to study foliation singularities; see Theorem 2.4 for a recollection on the definition and

existence of F-dlt modifications, which was first proved in [13]. An F-dlt modification (which is a foliated analogue of a classical dlt modification) is a special kind of partial resolution which extracts divisors such that the global properties of the foliation restricted to these divisors strongly reflect the local properties of the foliation singularity. In particular, for dicritical singularities, an F-dlt modification will always extract one exceptional geometric valuation transverse to the foliation.

To prove Theorem 0.3 we extract, by way of an F-dlt modification, an exceptional divisor which is transverse to the foliation (in other words, a dicritical component of the singularity). Showing the existence of a separatrix is then reduced to producing a global invariant algebraic divisor for the restricted foliation on this exceptional divisor. An adjunction calculation shows that this restricted foliation has trivial first Chern class, and so the existence of an invariant algebraic divisor is a consequence of the classification of foliations with trivial first Chern class.

To prove Theorem 0.1 we provide a precise bound on the singularities of X (we show that X is klt) by controlling the geometry of the invariant divisors on an F-dlt modification. We then show that the singularities of X are mild enough to allow us to prove the existence of a holomorphic Godbillon–Vey sequence associated to the foliation (§5.2), and we may then conclude roughly along the lines of Malgrange's original proof.

Global results

Until recently, the birational classification of foliated varieties had been understood only for rank 1 foliations on surfaces [4, 30, 32, 33]. In [35] and [13] much of the minimal model program for rank 2 foliations on threefolds was completed, including a cone and contraction theorem, existence of flips and special termination. However, the termination of flips was not proven. In this paper, we prove termination of flips, thereby completing the statement of the MMP for F-dlt pairs. We refer the reader to Definition 1.13 for the definition of F-dlt singularities, but we emphasize here that they are a very large and natural class of foliated singularities; for example, they include pairs $(X, \mathcal{F})$ such that X is smooth and $\mathcal{F}$ has simple singularities.

Theorem 0.5 (= Theorem 2.1). *Let X be a $\mathbb{Q}$-factorial quasi-projective threefold. Let $(\mathcal{F}, \Delta)$ be an F-dlt pair. Then starting at $(\mathcal{F}, \Delta)$ there is no infinite sequence of flips.*

A direct consequence of termination and the work in [13] is the following non-vanishing theorem.

Theorem 0.6 (= Theorem 2.6). *Let $\mathcal{F}$ be a co-rank 1 foliation on a normal projective $\mathbb{Q}$-factorial threefold X. Let Δ be a $\mathbb{Q}$-divisor such that $(\mathcal{F}, \Delta)$ is an F-dlt pair. Let $A, B \geq 0$ be $\mathbb{Q}$-divisors such that $\Delta = A + B$ and A is ample. Assume that $K_{\mathcal{F}} + \Delta$ is pseudo-effective. Then $K_{\mathcal{F}} + \Delta \sim_{\mathbb{Q}} D \geq 0$.*

We then turn our attention to the study of the non-klt centres of foliations. One of our central results in this direction is the proof of a foliated analogue of the connectedness of non-klt centres.

Theorem 0.7 (= Theorem 3.1). *Let X be a projective $\mathbb{Q}$-factorial threefold and let $\mathcal{F}$ be a rank 2 foliation on X. Let $(\mathcal{F}, \Delta)$ be an F-dlt pair on X. Assume that $-(K_{\mathcal{F}} + \Delta)$ is nef and big. Then $\mathrm{Nklt}(\mathcal{F}, \Delta)$ is connected.*

One of the fundamental ideas of the foliated MMP is that the negativity of foliated log pairs $(\mathcal{F}, \Delta)$ with mild singularities is governed by the presence of rational curves (see, for example, [35]). As a final application we prove a foliated version of the main result of [36] which relates the hyperbolicity of a foliation to an analysis of the log canonical singularities of a foliation. Given a foliated pair $(\mathcal{F}, \Delta)$ and a log canonical centre S we will denote by $\bar{S} \subset S$ the locally closed subvariety obtained by removing from S the lc centres of $(\mathcal{F}, \Delta)$ strictly contained in S.

Theorem 0.8 (= Theorem 7.1, foliated Mori hyperbolicity). *Let $(\mathcal{F}, \Delta)$ be a foliated log canonical pair on a normal projective threefold X. Assume that*

- *X is klt;*
- *there is no non-constant morphism $f : \mathbb{A}^1 \to X \setminus \mathrm{Nklt}(\mathcal{F}, \Delta)$ tangent to $\mathcal{F}$; and*
- *for any stratum S of $\mathrm{Nklt}(\mathcal{F}, \Delta)$ there is no non-constant morphism $f : \mathbb{A}^1 \to \bar{S}$ tangent to $\mathcal{F}$.*

Then $K_{\mathcal{F}} + \Delta$ is nef.

Finally, we remark that the central idea in the proof of our connectedness and hyperbolicity results is a refinement of the technique of F-dlt modifications (see Theorem 7.2), and a careful analysis of the properties of F-dlt modifications through adjunction.

1. Preliminaries

Notations and conventions

By *variety*, we will always mean an integral separated scheme over an algebraically closed field k. Unless otherwise stated, it will be understood that $k = \mathbb{C}$. Unless otherwise specified, we adopt the same notations and conventions as in [25].

A *contraction* is a projective morphism $f : X \to Y$ of quasi-projective varieties with $f_* \mathcal{O}_X = \mathcal{O}_Z$. If X is normal, then so is Z and the fibres of f are connected. A proper birational map $f : X \dashrightarrow Y$ of normal quasi-projective varieties is a *birational contraction* if f^{-1} does not contract any divisor.

Given a Weil $\mathbb{R}$-divisor D and a prime divisor E on a normal variety X, we will denote by $\mu_E D$ the coefficient of E in D. If D is a Weil $\mathbb{R}$-divisor on X then for any $c \in \mathbb{R}$ we define

$$D^{*c} := \sum_{\mu_E D * c} \mu_E D \, E,$$

where $*$ is any of $=, \geq, \leq, >, <$. We define the *round down* $\lfloor D \rfloor$ of D to be $\sum \lfloor \mu_E D \rfloor E$, where the sum is taken over all prime divisors E on X. The *fractional part* $\{D\}$ of D is defined as $\{D\} := D - \lfloor D \rfloor$.

The *support* $\mathrm{Supp}(D)$ of an $\mathbb{R}$-divisor D is the union of the prime divisors appearing in D with non-zero coefficient, $\mathrm{Supp}(D) := \bigcup_{\mu_E D \neq 0} E$.

1.1. Recollection on foliations

We refer the reader to [4] for basic notions of foliation theory.

A *foliation* on a normal variety X is a coherent subsheaf $\mathcal{F} \subset T_X$ such that

- $\mathcal{F}$ is saturated, i.e. $T_X / \mathcal{F}$ is torsion free, and
- $\mathcal{F}$ is closed under Lie bracket.

The *rank* of $\mathcal{F}$ is its rank as a sheaf. Its *co-rank* is its co-rank as a subsheaf of T_X.

Let X be a normal variety and let $\mathcal{F}$ be a rank r foliation on X. A *canonical divisor* of $\mathcal{F}$ is a divisor $K_{\mathcal{F}}$ such that

$$\mathcal{O}_X(-K_{\mathcal{F}}) \cong \det(\mathcal{F}).$$

We define the *normal sheaf* of $\mathcal{F}$ as $\mathcal{N}_{\mathcal{F}} := (T_X / \mathcal{F})^{**}$. The *conormal sheaf* $\mathcal{N}_{\mathcal{F}}^*$ of $\mathcal{F}$ is the dual of $\mathcal{N}_{\mathcal{F}}$. If $\mathcal{F}$ is a foliation of co-rank 1 then, by abuse of notation, we denote by $N_{\mathcal{F}}^*$ a divisor associated to $\mathcal{N}_{\mathcal{F}}^*$.

We can associate to $\mathcal{F}$ a morphism

$$\phi \colon \Omega_X^{[r]} \to \mathcal{O}_X(K_{\mathcal{F}})$$

defined by taking the double dual of the r-wedge product of the map $\Omega_X^{[1]} \to \mathcal{F}^*$, induced by the inclusion $\mathcal{F} \subset T_X$. This yields a map

$$\phi' \colon (\Omega_X^{[r]} \otimes \mathcal{O}_X(-K_{\mathcal{F}}))^{**} \to \mathcal{O}_X$$

and we define the *singular locus* of $\mathcal{F}$, denoted by $\mathrm{sing}(\mathcal{F})$, to be the cosupport of the image of ϕ'.

Given a dominant rational map $f \colon Y \dashrightarrow X$ and a foliation $\mathcal{F}$ on X we may pull back $\mathcal{F}$ to a foliation on Y, denoted $f^{-1}\mathcal{F}$.

Remark 1.1. If $q \colon X' \to X$ is a quasi-étale cover and $\mathcal{F}' = q^{-1}\mathcal{F}$ then $K_{\mathcal{F}'} = q^* K_{\mathcal{F}}$, and [15, Proposition 5.13] implies that $\mathcal{F}'$ is non-singular if and only if $\mathcal{F}$ is. In particular, it is not always the case that $\mathrm{sing}(X) \subset \mathrm{sing}(\mathcal{F})$ (cf. Proposition 1.7).

1.2. Invariant subvarieties

Let X be a normal variety and let $\mathcal{F}$ be a rank r foliation on X. Let $S \subset X$ be a subvariety. Then S is said to be $\mathcal{F}$-*invariant*, or *invariant by* $\mathcal{F}$, if for any open subset $U \subset X$ and any section $\partial \in H^0(U, \mathcal{F})$, we have

$$\partial(\mathcal{I}_{S \cap U}) \subset \mathcal{I}_{S \cap U},$$

where $\mathcal{I}_{S \cap U}$ denotes the ideal sheaf of $S \cap U$. If $D \subset X$ is a prime divisor then we define $\epsilon(D) := 1$ if D is not $\mathcal{F}$-invariant and $\epsilon(D) := 0$ if it is $\mathcal{F}$-invariant.

1.3. Foliation singularities

Frequently in birational geometry it is useful to consider pairs (X, Δ) where X is a normal variety and Δ is a $\mathbb{Q}$-Weil divisor such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier. We refer the reader to [25, §2] for the relevant definitions and notations for singularities of pairs. We will use the following non-standard definition.

Definition 1.2. A normal variety X is said to be *potentially log canonical* (resp. *potentially klt*) if there exists an effective $\mathbb{R}$-divisor D on X such that the log pair (X, D) has log canonical singularities (resp. klt singularities).

It is possible to define singularities for pairs also in the foliated world, in analogy with the classical case of pairs.

Definition 1.3. A *foliated pair* $(\mathcal{F}, \Delta)$ is a pair of a foliation and a $\mathbb{Q}$-Weil (resp. $\mathbb{R}$-Weil) divisor such that $K_{\mathcal{F}} + \Delta$ is $\mathbb{Q}$-Cartier (resp. $\mathbb{R}$-Cartier).

Note also that we are typically interested only in the cases when $\Delta \geq 0$, although it simplifies some computations to allow Δ to have negative coefficients.

Given a birational morphism $\pi : \widetilde{X} \to X$ and a foliated pair $(\mathcal{F}, \Delta)$ on X let $\widetilde{\mathcal{F}}$ be the pulled back foliation on $\widetilde{X}$ and $\pi_*^{-1}\Delta$ be the strict transform. We can write

$$K_{\widetilde{\mathcal{F}}} + \pi_*^{-1}\Delta = \pi^*(K_{\mathcal{F}} + \Delta) + \sum a(E_i, \mathcal{F}, \Delta)E_i.$$

Definition 1.4. We say that $(\mathcal{F}, \Delta)$ is *terminal*, *canonical*, *log terminal*, or *log canonical* if $a(E_i, \mathcal{F}, \Delta) > 0, \geq 0, > -\epsilon(E_i)$, or $\geq -\epsilon(E_i)$, respectively, where $\epsilon(D) = 0$ if D is invariant and 1 otherwise and where π varies across all birational morphisms.

If $(\mathcal{F}, \Delta)$ is log terminal and $\lfloor \Delta \rfloor = 0$ we say that $(\mathcal{F}, \Delta)$ is *foliated klt*.

Notice that these notions are well defined, i.e., $\epsilon(E)$ and $a(E, \mathcal{F}, \Delta)$ are independent of π. We say $a(E, \mathcal{F}, \Delta)$ is the *discrepancy* of E (with respect to $(\mathcal{F}, \Delta)$), or the *foliated discrepancy*.

Observe that when $\mathcal{F} = T_X$, no exceptional divisor is invariant, i.e., $\epsilon(E) = 1$ for any prime divisor $E \subset X$, and so this definition recovers the usual definitions of (log) terminal and (log) canonical.

We remark that we will be using the above terminology (terminal, canonical, etc.) to refer to both the singularities of the foliation and the singularities of the underlying variety. If necessary, we will use the term foliation terminal, foliation canonical, etc. to emphasize that we are talking about the singularities of the foliation rather than the variety.

Definition 1.5. Let $(\mathcal{F}, \Delta)$ be a foliated log pair.

(1) We say that $W \subset X$ is a *log canonical centre* (lc centre) of $(\mathcal{F}, \Delta)$ provided $(\mathcal{F}, \Delta)$ is log canonical at the generic point of W and there exists some divisor E of discrepancy $-\epsilon(E)$ on some birational model over X whose centre on X is W.

(2) The *non-klt locus* $\mathrm{Nklt}(\mathcal{F}, \Delta)$ of $(\mathcal{F}, \Delta)$ is the union of the centres of all divisorial valuations E of discrepancy $\leq -\epsilon(E)$.

(3) The *non-lc locus* $\mathrm{Nlc}(\mathcal{F}, \Delta)$ of $(\mathcal{F}, \Delta)$ is the union of the centres of all divisorial valuations E of discrepancy $< -\epsilon(E)$.

Remark 1.6. (1) If $\epsilon(E_i) = 0$ for all exceptional divisors E_i over a centre $W \subset X$, the notions of log canonical and canonical centre coincide for W. In this case, we will refer to canonical centres as log canonical centres.

(2) Any $\mathcal{F}$-invariant divisor D is an lc centre of $(\mathcal{F}, \Delta)$ since D shows up in Δ with coefficient at most $0 = \epsilon(D)$.

Moreover, a direct computation shows that any stratum of a simple singularity is an lc centre.

We have the following nice characterization due to [30, Corollary I.2.2]:

Proposition 1.7. *Let $(0 \in X)$ be a normal surface germ with a terminal foliation $\mathcal{F}$ of rank 1. Then there exists a cyclic cover $\sigma \colon Y \to X$ such that Y is a smooth surface and $\sigma^{-1}\mathcal{F}$ is a smooth foliation. In particular, by Remark 1.1, $0 \notin \mathrm{sing}(\mathcal{F})$.*

We emphasize that the above provides an example of a singular point where even if $0 \in \mathrm{sing}(X)$, then $0 \notin \mathrm{sing}(\mathcal{F})$.

We will also make use of the class of simple foliation singularities [7, Appendix: About simple singularities].

Definition 1.8. We say that $(p \in X)$ with X smooth is a *simple singularity* for $\mathcal{F}$ provided in formal coordinates $x_1, \ldots, x_n$ around p, $N_{\mathcal{F}}^*$ is generated by a 1-form which is in one of the following two forms. Below, $1 \leq r \leq n$.

(1) There are $\lambda_i \in \mathbb{C}^*$ such that

$$\omega = (x_1 \cdots x_r)\left(\sum_{i=1}^{r} \lambda_i \frac{dx_i}{x_i}\right) \tag{1.1}$$

and if $\sum a_i \lambda_i = 0$ for some non-negative integers a_i then $a_i = 0$ for all i.

(2) There is an integer $k \leq r$ such that

$$\omega = (x_1 \cdots x_r)\left(\sum_{i=1}^{k} p_i \frac{dx_i}{x_i} + \psi(x_1^{p_1} \cdots x_k^{p_k})\sum_{i=2}^{r} \lambda_i \frac{dx_i}{x_i}\right) \tag{1.2}$$

where p_i are positive integers without a common factor, $\psi(s)$ is a series which is not a unit, and $\lambda_i \in \mathbb{C}$, and if $\sum a_i \lambda_i = 0$ for some non-negative integers a_i then $a_i = 0$ for all i.

By Cano [7], every foliation on a smooth threefold admits a resolution by blow ups centred in the singular locus of the foliation such that the transformed foliation has only simple singularities.

We recall the definition of non-dicritical foliation singularities [9, §2].

Definition 1.9. Given a foliated pair $(X, \mathcal{F})$ we say that $\mathcal{F}$ has *dicritical singularities* if for some $P \in X$ there exists a germ of a surface $(P \in S)$ such that the restricted foliation $\mathcal{F}|_S$ has infinitely many invariant curves passing through $\operatorname{sing}(\mathcal{F}) \cap S$.

Otherwise, we say that $\mathcal{F}$ has *non-dicritical singularities*.

We remark that the above definition is equivalent on threefolds to the characterization appearing in [13], thanks to the existence of resolution of singularities. Namely, $\mathcal{F}$ has non-dicritical singularities if for any sequence $\pi : (X', \mathcal{F}') \to (X, \mathcal{F})$ of blow ups and any closed $q \in X$, $\pi^{-1}(q)$ is tangent to the foliation.

Remark 1.10. Observe that non-dicriticality implies that if W is $\mathcal{F}$-invariant, then $\pi^{-1}(W)$ is $\mathcal{F}'$-invariant.

Definition 1.11. Given a germ $(0 \in X)$ with a foliation $\mathcal{F}$ such that 0 is a singular point for $\mathcal{F}$ we call a (formal) hypersurface germ $(0 \in S)$ a (*formal*) *separatrix* if it is invariant under $\mathcal{F}$.

Note that away from the singular locus of $\mathcal{F}$ a separatrix is in fact a leaf. Furthermore, a singularity being non-dicritical implies that there are only finitely many separatrices through a singular point. The converse of this statement is false.

Definition 1.12. Given a normal variety X, a co-rank 1 foliation $\mathcal{F}$ and a foliated pair $(\mathcal{F}, \Delta)$ we say that $(\mathcal{F}, \Delta)$ is *foliated log smooth* provided that

(1) (X, Δ) is log smooth;

(2) $\mathcal{F}$ has simple singularities; and

(3) if S is the support of the non-invariant components of Δ then for any $p \in X$ if $\Sigma_1, \ldots, \Sigma_k$ are the separatrices of $\mathcal{F}$ at p (formal or otherwise), then $S \cup \Sigma_1 \cup \cdots \cup \Sigma_k$ is normal crossings at p.

Given a normal variety X, a co-rank 1 foliation $\mathcal{F}$, and a foliated pair $(\mathcal{F}, \Delta)$, a *foliated log resolution* is a high enough model $\pi \colon (Y, \mathcal{G}) \to (X, \mathcal{F})$ such that $(Y, \pi_*^{-1}\Delta + \sum_i E_i)$ is foliated log smooth where the E_i are all the π-exceptional divisors.

Such a resolution on threefolds is known to exist by [7].

We also recall the class of F-dlt singularities introduced in [13, Definition 3.6].

Definition 1.13. Let X be a normal variety and let $\mathcal{F}$ be a co-rank 1 foliation on X. Suppose that $K_{\mathcal{F}} + \Delta$ is $\mathbb{Q}$-Cartier.

We say $(\mathcal{F}, \Delta)$ is *foliated divisorial log terminal* (*F-dlt*) if

- each irreducible component of Δ is transverse to $\mathcal{F}$ and has coefficient at most 1, and

- there exists a foliated log resolution $(Y, \mathcal{G})$ of $(\mathcal{F}, \Delta)$ which only extracts divisors E of discrepancy $> -\epsilon(E)$.

In the case of surfaces, F-dlt singularities have a particularly simple characterization.

Lemma 1.14. *Let X be a normal surface and let $\mathcal{F}$ be a co-rank 1 foliation on X. Suppose that $K_{\mathcal{F}}$ is $\mathbb{Q}$-Cartier and $\mathcal{F}$ is F-dlt. Then for all $P \in X$, either*

- $\mathcal{F}$ *is terminal at P, or*
- *X is smooth at P and $\mathcal{F}$ has a simple singularity at P.*

In particular, if $K_{\mathcal{F}}$ is Cartier then X is smooth.

Proof. The dichotomy is a direct consequence of [13, Lemma 3.8]. The last claim follows from Proposition 1.7 by observing that if $(P \in X)$ is terminal for $\mathcal{F}$ and $K_{\mathcal{F}}$ is Cartier near P then X is smooth. ∎

1.4. Pulling back 1-forms

In §5, we will need the following result.

Proposition 1.15. *Let $(P \in X)$ be an isolated potentially klt singularity and let $\mu \colon \widetilde{X} \to X$ be a resolution of singularities of X. Let E be a prime μ-exceptional divisor. Let $\omega \in \Omega_X^{[1]}$, let*

$$\widetilde{\omega} := d_{\mathrm{refl}} \mu(\omega)$$

and let $\widetilde{\omega}_E$ be the restriction of $\widetilde{\omega}$ to E. Then $\widetilde{\omega}_E = 0$.

Proof. This is a straightforward consequence of the existence of pull back for differential forms on potentially klt varieties [23, Theorem 1.2]. ∎

1.5. Singularities of X vs. singularities of $\mathcal{F}$

The following is [13, Theorem 11.3]. Because we will refer to it frequently we include it here.

Theorem 1.16. *Let $(\mathcal{F}, \Delta)$ be a foliated pair on a quasi-projective threefold X. Assume that either*

- *$(\mathcal{F}, \Delta)$ is F-dlt, or*
- *$(\mathcal{F}, \Delta)$ is canonical.*

Then $\mathcal{F}$ has non-dicritical singularities. Furthermore, if $(\mathcal{F}, \Delta)$ is F-dlt and K_X is $\mathbb{Q}$-Cartier then X is klt.

We also have the following comparison of singularities result, which is a slight modification of [13, Lemma 3.16].

Lemma 1.17. *Let X be a $\mathbb{Q}$-factorial threefold and let $\mathcal{F}$ be a co-rank 1 foliation. Suppose that $(\mathcal{F}, \Delta)$ is F-dlt. Then (X, Δ) is dlt.*

Proof. Let $\pi \colon X' \to X$ be a foliated log resolution of $(\mathcal{F}, \Delta)$ which only extracts divisors of foliation discrepancy $> -\epsilon(E)$. Observe that a foliated log resolution $\pi \colon X' \to X$ of $(\mathcal{F}, \Delta)$ is a log resolution of (X, Δ). By Theorem 1.16, $\mathcal{F}$ has non-dicritical singularities, thus we may apply [35, Lemma 8.14] to conclude that π only extracts divisors of discrepancy > -1 with respect to $K_X + \Delta$, as required. ∎

1.6. Extending separatrices

We recall the following extension of separatrices result.

Lemma 1.18. *Let X be a normal quasi-projective threefold. Let $\mathcal{F}$ be a co-rank 1 foliation on X with non-dicritical singularities. Let $V \subset X$ be a subvariety tangent to $\mathcal{F}$, let $q \in V$ be any point and let S_q be a separatrix at q. Then there exists an analytic open neighbourhood U of V in X and an analytic divisor S on U which contains S_q near q.*

Proof. This is proven in [13, Lemma 3.5] (see also [35, §5.1]). We remark that it is a slight extension of the techniques and ideas utilized in [8, §IV]. ∎

1.7. Special termination

We recall the following theorem [13, Theorem 7.1]:

Theorem 1.19 (Special Termination). *Let X be a $\mathbb{Q}$-factorial quasi-projective threefold. Let $(\mathcal{F}, \Delta)$ be an F-dlt pair. Suppose $(\mathcal{F}_i, \Delta_i)$ is an infinite sequence of $(K_{\mathcal{F}_i} + \Delta_i)$-flips. Then after finitely many flips, the flipping and flipped locus is disjoint from the lc centres of $(\mathcal{F}_i, \Delta_i)$. In particular, $(\mathcal{F}_i, \Delta_i)$ is log terminal in a neighbourhood of each flipping curve.*

1.8. MMP with scaling

A version of the MMP with scaling was proven in [13, §10]; however, for our purposes we will need the MMP with scaling in a slightly different form than presented there. Here we briefly explain the necessary adjustments.

We recall the following lemma proven in [13, Lemma 3.27].

Lemma 1.20. *Let X be a normal projective $\mathbb{Q}$-factorial threefold and let $\mathcal{F}$ be a co-rank 1 foliation on X. Let $\Delta = A + B$ be a $\mathbb{Q}$-divisor such that $(\mathcal{F}, \Delta)$ is an F-dlt pair, $A \geq 0$ is an ample $\mathbb{Q}$-divisor and $B \geq 0$. Let $\varphi \colon X \dashrightarrow X'$ be a sequence of steps of the $(K_{\mathcal{F}} + \Delta)$-MMP and let $\mathcal{F}'$ be the induced foliation on X'. Then there exist $\mathbb{Q}$-divisors $A', C' \geq 0$ on X' such that*

(1) $\varphi_* A \sim_{\mathbb{Q}} A' + C'$;

(2) A' is ample; and

(3) if $\Delta' := A' + C' + \varphi_* B$ then $\Delta' \sim_{\mathbb{Q}} \varphi_* \Delta$ and $(\mathcal{F}', \Delta')$ is F-dlt.

1.8.1. Running the MMP with scaling. Let X be a projective $\mathbb{Q}$-factorial threefold and let $\mathcal{F}$ be a co-rank 1 foliation on X. Let $\Delta = A + B$ be a $\mathbb{Q}$-divisor where $A \geq 0$ is ample and $B \geq 0$ such that $(\mathcal{F}, \Delta)$ is an F-dlt pair. Let H be a divisor on X such that $K_{\mathcal{F}} + \Delta + H$ is nef. In practice we will often take H to be some sufficiently ample divisor on X.

Let $\lambda = \inf \{t > 0 : K_{\mathcal{F}} + \Delta + tH \text{ is nef}\}$. By [13, Lemma 9.2] there exists a $(K_{\mathcal{F}} + \Delta)$-negative extremal ray R such that $(K_{\mathcal{F}} + \Delta + \lambda H) \cdot R = 0$. Let $\phi \colon X \dashrightarrow X'$ be the contraction or flip associated to R.

If ϕ is a fibre type contraction, the MMP terminates and we may assume that ϕ is a divisorial contraction or a flip. Let $\mathcal{F}'$ be the strict transform of $\mathcal{F}$, let $\Delta' = \phi_* \Delta$ and let $H' = \phi_* H$. By Lemma 1.20 we may find $\Theta \sim_{\mathbb{Q}} \Delta'$ such that $(\mathcal{F}', \Theta)$ is F-dlt and $\Theta = A' + B'$ where $A' \geq 0$ is ample and $B' \geq 0$. Thus we can again apply [13, Lemma 9.2] to $K_{\mathcal{F}'} + \Delta' + \lambda H'$, and letting $\lambda' = \inf \{t > 0 : K_{\mathcal{F}'} + \Delta' + tH'$ is nef$\}$ we see that $\lambda' \leq \lambda$ and there exists a $K_{\mathcal{F}'} + \Delta'$-negative extremal ray R' with $(K_{\mathcal{F}'} + \Delta' + \lambda' H') \cdot R' = 0$. We are therefore free to continue this process.

Setting $X_0 := X$, $\mathcal{F}_0 := \mathcal{F}$, $\Delta_0 := \Delta$ and $H_0 := H$ we may produce a sequence $\phi_i \colon X_i \dashrightarrow X_{i+1}$ of divisorial contractions and flips for $K_{\mathcal{F}_i} + \Delta_i$ contracting an extremal ray R_i and rational numbers λ_i such that $K_{\mathcal{F}_i} + \Delta_i + \lambda_i H_i$ is nef and $(K_{\mathcal{F}_i} + \Delta_i + \lambda_i H_i) \cdot R_i = 0$, where $\mathcal{F}_i, \Delta_i, H_i$ are the strict transforms of $\mathcal{F}_0, \Delta_0, H_0$.

Moreover, we have $\lambda_i \geq \lambda_{i+1}$ and $R_i \cdot H_i > 0$ for all i. Assuming the relevant termination of flips we see that this MMP terminates in either a Mori fibre space or a model where $K_{\mathcal{F}_i} + \Delta_i$ is nef. We call this process the MMP of $(\mathcal{F}, \Delta)$ (or $K_{\mathcal{F}} + \Delta$) with scaling of H.

1.9. (Pre) simple vs. (log) canonical

We now briefly discuss some relations and parallels between the classes of singularities defined by the MMP and some of the other classes of singularities described above.

Intuitively, for a given singularity, the smaller its discrepancy the more severe the singularity is. So terminal singularities are, in this sense, the mildest kind of singularities appearing in the MMP. Indeed, terminal singularities on smooth surfaces are in fact smooth foliated points, although this equivalence fails in higher dimensions, as we will see in Theorem 5.20.

We observe that simple singularities are both non-dicritical and canonical, while canonical singularities are in general not simple, as the following example shows.

Example 1.21. The foliation on $\mathbb{C}^2$ defined by the vector field $x \frac{\partial}{\partial x} + (x + y) \frac{\partial}{\partial y}$ has canonical singularities (which may be verified since a single blow up resolves the singularities of the foliation to simple singularities, and this blow up has discrepancy $= 0$). However, since both its eigenvalues are positive integers it is not a simple singularity.

We remark that the above example also shows that canonical singularities are not in general F-dlt singularities. On the other hand, Theorem 1.16 shows that canonical and F-dlt singularities are non-dicritical.

Consider a germ of a vector field ∂ on $\mathbb{C}^2$ and suppose that ∂ is singular at 0, and let $\mathfrak{m}$ be the maximal ideal at 0. We get an induced linear map $\partial \colon \mathfrak{m}/\mathfrak{m}^2 \to \mathfrak{m}/\mathfrak{m}^2$, which is non-nilpotent if and only if the foliation generated by ∂ is log canonical [31, Fact I.ii.4]. To our knowledge there is no similar criterion for characterizing log canonical foliations of rank ≥ 2.

We refer to [7, Definition 3] for the definition of presimple singularities. The difference between simple and presimple is (roughly) the additional non-resonance requirement on the eigenvalues of the foliation. For instance, $x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}$ defines a foliation on $\mathbb{C}^2$ with

a presimple but not simple singularity. A single blow up of this foliation has discrepancy -1 and resolves this foliation to a smooth foliation, which shows that the original foliation has a log canonical but not canonical singularity.

With this example in mind it might be useful to view the relation between simple and presimple singularities as analogous to the relation between canonical and log canonical singularities. However, we do not know if every presimple singularity is log canonical. It also does not seem to be the case that a canonical singularity is a log canonical singularity which satisfies a particular resonance condition, in light of Example 1.21.

We observe that on a smooth threefold a log canonical singularity which is non-dicritical is necessarily canonical. Indeed, by [7] and our assumption of non-dicriticality we may find a resolution of singularities of $\mathcal{F}$, say $\pi\colon X' \to X$, which only extracts $\mathcal{F}'$-invariant divisors for $\mathcal{F}' := \pi^{-1}\mathcal{F}$. If E is any π-exceptional divisor then $a(E, \mathcal{F})$ $\geq -\epsilon(E)$ by log canonicity and so $a(E, \mathcal{F}) \geq 0$ since E is invariant and we may conclude that $\mathcal{F}$ has in fact canonical singularities.

We recall that the class of simple singularities is stable under blow ups contained in strata of the singular locus; however, it is important to realize that a canonical singularity may not remain canonical after a blow up in the singular locus. In fact, it is a subtle problem to decide when the blow up of a canonical singularity remains canonical.

We emphasize that in contrast to (pre)simple singularities the notion of (log) canonical singularities makes sense on singular varieties. Take for instance the foliation on $\mathbb{C}^2$ generated by the vector field $\partial = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}$. This defines a canonical foliation singularity. If we let $X = \mathbb{C}^2/(x, y) \sim (-x, -y)$ then X is singular and ∂ descends to a vector field on X which still defines a canonical singularity.

2. Termination

Our goal in this section is to show the following:

Theorem 2.1 (Termination). *Let X be a $\mathbb{Q}$-factorial quasi-projective threefold and let $(\mathcal{F}, \Delta)$ be an F-dlt pair. Then starting at $(\mathcal{F}, \Delta)$ there is no infinite sequence of flips.*

Together with the existence of flips [13, Theorem 6.4] and divisorial contractions [13, Theorem 6.7], this has the following immediate corollary (whose proof is identical to the proof for the corresponding statement for varieties).

Corollary 2.2. *Let X be a projective $\mathbb{Q}$-factorial threefold and let $(\mathcal{F}, \Delta)$ be an F-dlt pair. Then there is a birational contraction $f : X \dashrightarrow Y$ (which may be factored as a sequence of flips and divisorial contractions) such that if $\mathcal{G}$ is the transformed foliation on Y then either*

(1) *$K_{\mathcal{G}} + f_*\Delta$ is nef, or*

(2) *there is a fibration $g : Y \to Z$ such that $-(K_{\mathcal{G}} + f_*\Delta)$ is g-ample and the fibres of g are tangent to $\mathcal{G}$.*

We call such a contraction an $(\mathcal{F}, \Delta)$-MMP or a $(K_{\mathcal{F}} + \Delta)$-MMP.

We will also frequently need to run the relative MMP over a base variety. The relative MMP can be deduced from the absolute MMP via standard arguments (see for instance [25, §§3.6-7]).

Corollary 2.3. *Let X be a $\mathbb{Q}$-factorial quasi-projective threefold and let $(\mathcal{F}, \Delta)$ be an F-dlt pair. Let $p : X \to S$ be a surjective projective morphism. Then there is a birational contraction $f : X \dashrightarrow Y/S$ (which may be factored as a sequence of flips and divisorial contractions) such that if $\mathcal{G}$ is the transformed foliation and $q : Y \to S$ is the structure map then either*

(1) *$K_{\mathcal{G}} + f_* \Delta$ is q-nef, or*

(2) *there is a fibration $g : Y \to Z/S$ such that $-(K_{\mathcal{G}} + f_* \Delta)$ is g-ample and the fibres of g are tangent to $\mathcal{G}$.*

We call the contraction $f : X \dashrightarrow Y/S$ constructed in the above corollary an $(\mathcal{F}, \Delta)$-MMP or a $(K_{\mathcal{F}} + \Delta)$-MMP *over* S. In case (1) of the above statement, we say that Y (or $(\mathcal{G}, f_* \Delta)$) is a *minimal model* of X (or $(\mathcal{F}, \Delta)$) *over* S; in case (2), instead, we say that Y (or $(\mathcal{G}, f_* \Delta)$) is a *Mori fibre space* for X (or $(\mathcal{F}, \Delta)$) *over* S.

Corollary 2.3 immediately implies the following extension of the existence of an F-dlt modification to the relative setting (cf. [13, Theorem 8.1]).

Theorem 2.4 (Existence of F-dlt modifications). *Let $\mathcal{F}$ be a co-rank 1 foliation on a normal quasi-projective threefold X. Let $(\mathcal{F}, \Delta)$ be a foliated pair. Then there exists a birational morphism $\pi \colon Y \to X$ which only extracts divisors E of foliation discrepancy $-\epsilon(E)$ such that if we write $K_{\mathcal{G}} + \Gamma + F = \pi^*(K_{\mathcal{F}} + \Delta)$, where $\Gamma = \pi_{1*}^{-1}\Delta + \sum_i \epsilon(E_i)E_i$ and where E_i are the π-exceptional divisors, then F is an effective π-exceptional $\mathbb{R}$-divisor and $(\mathcal{G}, \Gamma)$ is F-dlt.*

Furthermore, we may choose $(Y, \mathcal{G})$ so that

(1) *if W is an lc centre of $(\mathcal{G}, \Gamma)$ then W is contained in a codimension 1 lc centre of $(\mathcal{G}, \Gamma)$;*

(2) *Y is $\mathbb{Q}$-factorial; and*

(3) *Y is klt.*

We call such a modification an F-dlt modification.

Proof. The proof is analogous to that of [13, Theorem 8.1]. In particular, it suffices to consider a log resolution $\pi_1 \colon Y_1 \to X$ of $(\mathcal{F}, \Delta)$ and then run the relative $(K_{\mathcal{F}_1} + \Gamma)$-MMP over X, where $\mathcal{F}_1$ is the transform of $\mathcal{F}$ on Y_1 and $\Gamma := \pi_{1*}^{-1}\Delta + \sum \epsilon(E)E$, where the sum is taken over the prime π_1-exceptional divisors. The relative minimal model produced by this MMP is the desired modification $(\mathcal{G}, \Gamma)$. ∎

Remark 2.5. We use the notation of Theorem 2.4.

(1) The equality $K_{\mathcal{G}} + \Gamma + F = \pi^*(K_{\mathcal{F}} + \Delta)$ implies that $\mathrm{Nklt}(\mathcal{F}, \Delta) = \pi(\mathrm{Nklt}(\mathcal{G}, \Gamma + F))$ and $\mathrm{Nlc}(\mathcal{F}, \Delta) = \pi(\mathrm{Nlc}(\mathcal{G}, \Gamma + F))$. Moreover, as $(\mathcal{G}, \Gamma)$ is F-dlt, we have $\mathrm{Nlc}(\mathcal{G}, \Gamma + F)) = \mathrm{Supp}(F)$.

(2) If the foliated log pair $(\mathcal{F}, \Delta)$ is log canonical, then the previous part of the remark implies that $F = 0$ and $K_{\mathcal{G}} + \Gamma = \pi^*(K_{\mathcal{F}} + \Delta)$. Moreover, property (1) in the statement of Theorem 2.4 implies that $\mathrm{Nklt}(\mathcal{G}, \Gamma)$ is the union of all codimension 1 subvarieties contained in it. Hence, $\mathrm{Nklt}(\mathcal{G}, \Gamma) = I \cup \mathrm{Supp}(\lfloor \Gamma \rfloor)$, where I is the union of the $\mathcal{G}$-invariant divisors.

As a consequence of the existence of the MMP we have the following non-vanishing theorem.

Theorem 2.6. *Let $\mathcal{F}$ be a co-rank 1 foliation on a normal projective $\mathbb{Q}$-factorial threefold X. Let Δ be a $\mathbb{Q}$-divisor such that $(\mathcal{F}, \Delta)$ is an F-dlt pair. Let $A, B \geq 0$ be $\mathbb{Q}$-divisors such that $\Delta = A + B$ and A is ample. Assume that $K_{\mathcal{F}} + \Delta$ is pseudo-effective. Then $K_{\mathcal{F}} + \Delta \sim_{\mathbb{Q}} D \geq 0$.*

Proof. We run the $(K_{\mathcal{F}} + \Delta)$-MMP. By Corollary 2.3 and the assumption that $K_{\mathcal{F}} + \Delta$ is pseudo-effective, this MMP terminates with a minimal model $\phi: X \dashrightarrow X'$. Let $\mathcal{F}'$ be the transform of $\mathcal{F}$. By Lemma 1.20 we may find an ample divisor A' and $B' \geq 0$ such that $\phi_* \Delta \sim_{\mathbb{Q}} A' + B'$ and $(\mathcal{F}', A' + B')$ is F-dlt.

Set $\Delta' := A' + B'$. Thus, we may apply [13, Theorem 9.4] to conclude that $K_{\mathcal{F}'} + \Delta'$ is semi-ample and so there exists $0 \leq D \sim_{\mathbb{Q}} K_{\mathcal{F}'} + \Delta'$. For all m sufficiently divisible we have

$$H^0(X, \mathcal{O}(m(K_{\mathcal{F}} + \Delta))) = H^0(X', \mathcal{O}(m(K_{\mathcal{F}'} + \phi_* \Delta)))$$
$$= H^0(X', \mathcal{O}(m(K_{\mathcal{F}'} + \Delta'))),$$

and our result follows. ∎

2.1. Singular Bott partial connections

We recall Bott's partial connections. Let $\mathcal{F}$ be a smooth foliation on a complex manifold X. We can define a partial connection on $N_{\mathcal{F}}$ locally by

$$\nabla : N_{\mathcal{F}} \to \Omega^1_{\mathcal{F}} \otimes N_{\mathcal{F}}, \qquad w \mapsto \sum \omega_i \otimes q([\partial_i, \tilde{w}]),$$

where $\tilde{w}$ is any local lift of w to T_X and ω_i are local generators of $\Omega^1_{\mathcal{F}}$, ∂_i are dual generators of $\mathcal{F}$ and $q : T_X \to N_{\mathcal{F}}$ is the quotient map. One can check that these local connections patch together to give a global connection.

Lemma 2.7. *Let $\mathcal{F}$ be a rank r foliation on a complex analytic variety X. Let $S \subset X$ be a local complete intersection subvariety of dimension r. Assume that S is $\mathcal{F}$-invariant. Let $Z = \mathrm{sing}(X) \cup \mathrm{sing}(\mathcal{F})$. Assume that $Z \cap S$ has codimension at least 2 in S. Then there is a connection*

$$\nabla : N_{S/X} \to \Omega^{[1]}_S \otimes N_{S/X}$$

where $\Omega^{[1]}_S = (\Omega^1_S)^{**}$ is the sheaf of reflexive differentials on S.

Proof. Let $X^\circ = X \setminus Z$ and $S^\circ = S \setminus (Z \cap S)$. Notice that $N_{\mathcal{F}}|_{S^\circ} = N_{S^\circ/X^\circ}$ and that $\Omega^1_{\mathcal{F}}|_{S^\circ} = \Omega^1_{S^\circ}$. Thus, if we restrict Bott's partial connection on X° to S° we get a connection

$$\nabla^\circ : N_{S^\circ/X^\circ} \to \Omega^1_{S^\circ} \otimes N_{S^\circ/X^\circ}.$$

Let $i : S^\circ \to S$ be the inclusion. Since $N_{S/X}$ is locally free we have an isomorphism $i_*(\Omega^1_{S^\circ} \otimes N_{S^\circ/X^\circ}) = \Omega^{[1]}_S \otimes N_{S/X}$ by the push-pull formula [22, Exercise II.5.1 (d)]. Thus we get a map

$$i_*\nabla^\circ : N_{S/X} \to \Omega^{[1]}_S \otimes N_{S/X}$$

and since $i_*\nabla^\circ$ satisfies the Leibniz condition (since it does so away from a set of codimension at least 2), it is the desired connection. $\blacksquare$

2.2. Proof of Theorem 2.1

Lemma 2.8. *Let X be a normal complex analytic threefold and let $(\mathcal{F}, \Delta)$ be a log terminal co-rank 1 foliation on X. Let $C \subset X$ be a compact curve tangent to $\mathcal{F}$. Let S be a germ of an invariant surface containing C. Suppose that K_X, $K_{\mathcal{F}}$ and S are $\mathbb{Q}$-Cartier. Then*

$$S \cdot C = 0.$$

Proof. Let $H \subset X$ be a sufficiently ample divisor meeting C transversely and choose H sufficiently general so that $(\mathcal{F}, \Delta + (1-\epsilon)H)$ is log terminal for all $\epsilon > 0$.

We may then find a Galois cover $\pi : X' \to X$ ramified over H and $\operatorname{sing}(X)$ such that if we write $S' = \pi^{-1}(S)$ and $\mathcal{F}' = \pi^{-1}\mathcal{F}$ then S' and $K_{\mathcal{F}'}$ are both Cartier.

Write $\Delta' = \pi^*\Delta$ and $C' = \pi^{-1}(C)$. We claim that $(\mathcal{F}', \Delta')$ is log terminal. Indeed, let r be the ramification index along H. By foliated Riemann–Hurwitz we have $K_{\mathcal{F}'} + \Delta' = \pi^*(K_{\mathcal{F}} + \Delta + \frac{r-1}{r}H)$. Since $(\mathcal{F}, \Delta + \frac{r-1}{r}H)$ is log terminal, the same is true of $(\mathcal{F}', \Delta')$ (see [25, Proposition 5.20] or [35, proof of Corollary 3.9]).

Since $K_{\mathcal{F}'}$ is Cartier and $(\mathcal{F}', \Delta')$ is log terminal, if we let $Z = \operatorname{sing}(X') \cup \operatorname{sing}(\mathcal{F}')$ we see that $Z \cap S'$ has codimension at least 2 in S' [30, Corollary I.2.2]. By construction S' is Cartier and so we may apply Lemma 2.7 to produce a connection

$$\nabla : N_{S'/X'} \to \Omega^{[1]}_{S'} \otimes N_{S'/X'}.$$

Let $n: B \to C'$ be the normalization of an irreducible component of C'. By [13, Lemma 3.16] we see that $(X', \Delta' + S')$ is plt and so by (usual) adjunction [25, Theorem 5.50], S' is klt. By [20, Theorem 4.3] there exists a non-zero morphism $d_{\mathrm{refl}}n : n^*\Omega^{[1]}_{S'} \to \Omega^1_B$, so we may pull back ∇ to get a connection by composing

$$n^*\mathcal{O}(S') \xrightarrow{n^*\nabla} n^*\Omega^{[1]}_{S'} \otimes n^*\mathcal{O}(S') \xrightarrow{d_{\mathrm{refl}}n \otimes \mathrm{id}} \Omega^1_B \otimes n^*\mathcal{O}(S').$$

In particular, since $n^*\mathcal{O}(S')$ admits a holomorphic connection, it is flat, which implies $0 = S' \cdot n(B) = m(S \cdot C)$. $\blacksquare$

Proof of Theorem 2.1. By Special Termination [13, Theorem 7.1], it suffices to show that any sequence of log terminal flips terminates. Let

$$\phi\colon (X_i, \mathcal{F}_i, \Delta_i) \dashrightarrow (X_{i+1}, \mathcal{F}_{i+1}, \Delta_{i+1})$$

be one such flip and let $C \subset X_i$ be an irreducible component of $\mathrm{exc}(\phi)$. Let $f\colon X_i \to Z$ denote the base of the flip.

Since C is tangent to the foliation, any divisor E dominating C on some model of X_i is invariant, i.e., $\epsilon(E) = 0$. Since $(\mathcal{F}_i, \Delta_i)$ is log terminal we have $a(E, \mathcal{F}_i, \Delta_i) > \epsilon(E) = 0$, i.e., $\mathcal{F}_i$ is terminal at the generic point of C.

By [13, Lemma 3.14] by taking U to be a sufficiently small analytic neighbourhood of $z = f(C)$ we may find a unique $\mathcal{F}_i$-invariant divisor on $X_{i,U} := f^{-1}(U)$ containing C. Call this divisor S.

Since $X_{i,U}$ is klt and projective over U we may find a small $\mathbb{Q}$-factorialization of $X_{i,U}$, denoted $g\colon \overline{X_{i,U}} \to X_{i,U}$. Let $\overline{\mathcal{F}_i}$ be the transformed foliation, write $K_{\overline{\mathcal{F}_i}} + \overline{\Delta_i} = g^*(K_{\mathcal{F}_i} + \Delta_i)$, let $\overline{S}$ be the strict transform of S and let $\overline{C}$ be the strict transform of C. Since g is small, we see that $(\overline{\mathcal{F}_i}, \overline{\Delta_i})$ is still log terminal.

Let $P \in \overline{C}$ be a point and let T be a germ of a $\overline{\mathcal{F}_i}$-invariant divisor at P. We claim that $T = \overline{S}$ (as germs). Indeed, suppose otherwise. Since $\overline{S}$ is $\mathbb{Q}$-Cartier we know that $T \cap \overline{S}$ contains a 1-dimensional component Σ. Since Σ is the intersection of two invariant divisors we see that $\Sigma \subset \mathrm{sing}(\mathcal{F})$ and Σ is tangent to the foliation, in particular $\overline{\mathcal{F}_i}$ is terminal at the generic point of Σ. This, however, contradicts Proposition 1.7, which implies that terminal foliation singularities are non-singular in codimension 2.

By Lemma 2.8 we see that

$$\overline{S} \cdot \overline{C} = 0.$$

On the other hand, by [13, Corollary 3.20] and the observation in the previous paragraph, the collection of $\overline{\mathcal{F}_i}$-invariant divisors meeting $\overline{C}$ is exactly $\overline{S}$ itself and so

$$(K_{\overline{\mathcal{F}_i}} + \overline{\Delta_i}) \cdot \overline{C} = (K_{\overline{X_i}} + \overline{\Delta_i} + \overline{S}) \cdot \overline{C}.$$

Since $K_{\overline{X_i}} + \overline{\Delta_i} = g^*(K_{X_i} + \Delta_i)$, putting these equalities together yields

$$0 > (K_{\overline{\mathcal{F}_i}} + \overline{\Delta_i}) \cdot \overline{C} = (K_{X_i} + \Delta_i) \cdot C.$$

Thus, each $K_{\mathcal{F}_i} + \Delta_i$-flip is in fact a $K_{X_i} + \Delta_i$-flip.

By [13, Lemma 3.16], (X_i, Δ_i) is log terminal and so our result follows by termination for threefold log terminal flips (see for example [25, Theorem 6.17]). ∎

To finish this section we present an example of a foliation flip; another example may be found in [35, Example 9.1].

Example 2.9. Let $b\colon Y \to \mathbb{C}^2$ be the blow up at the origin with exceptional curve C and let $p\colon \widetilde{X} \to Y$ be the total space of the line bundle $\mathcal{O}_Y(C)$. Observe that $\widetilde{X}$ contains a single projective curve, which we will continue to denote by C. Let $\mathcal{G}$ be the foliation on Y given by the transform of the foliation generated by $\frac{\partial}{\partial x_1}$ on $\mathbb{C}^2$ (with coordinates (x_1, x_2)) and let $\widetilde{\mathcal{G}} = p^{-1}\mathcal{F}$. Set $S = p^{-1}(C)$ and let $D_i = p^{-1}(b_*^{-1}\{x_i = 0\})$.

It is straightforward to check that $K_{\widetilde{\mathcal{G}}} \cdot C = 0$ and that $\widetilde{\mathcal{G}}$ is smooth at the generic point of C. Moreover, S and D_2 are $\widetilde{\mathcal{G}}$-invariant whereas D_1 is not.

Consider the map $\sigma \colon \mathbb{C}^2 \to \mathbb{C}^2$ given by $(x_1, x_2) \mapsto (-x_1, x_2)$ and observe that σ lifts to a map $\tau \colon \widetilde{X} \to \widetilde{X}$. Let $X := \widetilde{X}/\langle \tau \rangle$. Then $\pi \colon \widetilde{X} \to X$ is ramified to order 2 along S and D_1. Observe moreover that τ preserves $\widetilde{\mathcal{G}}$ and so it descends to a foliation $\mathcal{F}$ on X. A foliated Riemann–Hurwitz computation shows that $K_{\widetilde{\mathcal{G}}} = \pi^* K_{\mathcal{F}} + D_1$. In particular, if we let $\Sigma = \pi(C)$ we see that $K_{\mathcal{F}} \cdot \Sigma < 0$ and so Σ is a $K_{\mathcal{F}}$-flipping curve.

Notice that Σ meets $\mathrm{sing}(\mathcal{F})$ in a single point which is a $\mathbb{Z}/2$ quotient singularity.

3. Connectedness

3.1. Connectedness of the non-klt locus for foliated pairs

The aim of this section is to prove the following connectedness statement which constitutes one of the pillars in the analysis of the birational structure of foliated singularities. The analogous result in the non-foliated case has a long history and is rather classical; recently, [2, 17] fully settled the Connectedness Principle in full generality for pairs.

Theorem 3.1. *Let $f \colon X \to Y$ be a contraction of normal quasi-projective varieties, with X a $\mathbb{Q}$-factorial threefold. Let $\mathcal{F}$ be a co-rank 1 foliation on X. Let $(\mathcal{F}, \Delta)$ be a foliated log pair with $\Delta = \sum a_i D_i$. Assume that $-(K_{\mathcal{F}} + \Delta)$ is f-nef and f-big and that $(\mathcal{F}, \Delta')$ is F-dlt, where $\Delta' := \sum_{a_i < \epsilon(D_i)} a_i D_i + \sum_{a_j \geq \epsilon(D_j)} \epsilon(D_j) D_j$. Then $\mathrm{Nklt}(\mathcal{F}, \Delta)$ is connected in a neighbourhood of every fibre of f.*

Theorem 3.1 immediately implies the following more general result which makes no assumptions on the singularities of the foliated log pair.

Theorem 3.2. *Let $f \colon X \to Y$ be a contraction of normal quasi-projective varieties. Let $\mathcal{F}$ be a co-rank 1 foliation on X. Let $(\mathcal{F}, \Delta)$ be a foliated log pair. Assume that $-(K_{\mathcal{F}} + \Delta)$ is f-nef and f-big. Then $\mathrm{Nklt}(\mathcal{F}, \Delta)$ is connected in a neighbourhood of every fibre of f.*

Proof. It suffices to consider an F-dlt modification $g \colon X' \to X$, $K_{\mathcal{G}} + \Delta_{X'} = g^*(K_{\mathcal{F}} + \Delta)$ and apply Theorem 3.1 to the pair $(\mathcal{G}, \Delta_{X'})$ and the map $f \circ g \colon X' \to Y$. ∎

We will prove Theorem 3.1 in the course of this section by handling different cases that fit together to provide an argument for it.

Before proving the theorem we indicate a quick application of Theorem 3.1 to the geometry of (weak) Fano foliations (see also [1]).

We will denote by $\mathrm{sing}^*(\mathcal{F})$ the union of all codimension 2 components of $\mathrm{sing}(\mathcal{F})$.

Corollary 3.3. *Let X be a smooth projective threefold and let $\mathcal{F}$ be a co-rank 1 foliation on X. Assume that $-K_{\mathcal{F}}$ is big and nef. Then either*

(1) *$\mathcal{F}$ has an algebraic leaf, or*

(2) *$\mathrm{sing}^*(\mathcal{F})$ is connected.*

Proof. Observe that $\text{Nklt}(\mathcal{F}) = \text{sing}^*(\mathcal{F}) \cup I \cup Z$ where I is the union of all the $\mathcal{F}$-invariant divisors and Z is a finite collection of points.

We take an F-dlt modification $\mu \colon \overline{X} \to X$ of $\mathcal{F}$ and let $\overline{\mathcal{F}}$ be the induced foliation on $\overline{X}$, which exists by Theorem 2.4. Write $K_{\overline{\mathcal{F}}} + \Delta = \mu^* K_{\mathcal{F}}$. Then $\mu(\text{Nklt}(\overline{\mathcal{F}}, \Delta)) = \text{Nklt}(\mathcal{F})$ (see Remark 2.5). If $\mathcal{F}$ has no algebraic leaves then $I = \emptyset$. Applying Theorem 3.1 we conclude that $\text{Nklt}(\overline{\mathcal{F}}, \Delta)$ is connected and our result follows. ∎

Remark 3.4. The statement corresponding to Theorem 3.1 for rank 1 foliations is an essentially trivial consequence of the arguments in [3].

We now turn to the proof of Theorem 3.1.

We will work in the following setting. We denote by $f \colon X \to Y$ a contraction of normal quasi-projective varieties, with X a $\mathbb{Q}$-factorial threefold. Recall that f being a contraction means that f is surjective and projective with $f_* \mathcal{O}_X = \mathcal{O}_Y$.

We assume the existence of a co-rank 1 foliation $\mathcal{F}$ on X and of a foliated log pair $(\mathcal{F}, \Delta)$ with $\Delta = \sum a_i D_i$. We will denote

$$
\begin{aligned}
H &:= -(K_{\mathcal{F}} + \Delta), \\
\Delta' &:= \sum_{a_i < \epsilon(D_i)} a_i D_i + \sum_{a_j \geq \epsilon(D_j)} \epsilon(D_j) D_j, \\
\Delta'' &:= \Delta - \Delta', \\
F &:= \text{Supp}(\Delta^{\geq 1}).
\end{aligned}
$$

We remark that we allow Δ to have $\mathcal{F}$-invariant components in its support, whereas the support of Δ' will contain no $\mathcal{F}$-invariant components.

We start by addressing the birational case.

Lemma 3.5. *With the notation above, assume that f is birational, $\rho(X/Y) = 1$ and $(\mathcal{F}, \Delta')$ is F-dlt. Suppose moreover that every lc centre of $(\mathcal{F}, \Delta')$ is contained in a codimension 1 lc centre of $(\mathcal{F}, \Delta')$. If $-(K_{\mathcal{F}} + \Delta)$ is f-ample then $\text{Nklt}(\mathcal{F}, \Delta)$ is connected in a neighbourhood of any fibre of f.*

Recall that, as observed in Remark 2.5, $\text{Nklt}(\mathcal{F}, \Delta) = \text{Nklt}(\mathcal{F}, \Delta') = \text{Supp}(F + I)$, where I is the sum of the $\mathcal{F}$-invariant divisors.

Proof of Lemma 3.5. To reach a contradiction assume that $\text{Nklt}(\mathcal{F}, \Delta)$ is disconnected in a neighbourhood of some fibre of f. Observe that we may assume that $\Delta'' \geq 0$ is f-nef, otherwise $\text{exc}(f) \subset \text{Supp}(\Delta'')$ and there is nothing to prove. Thus $-(K_{\mathcal{F}} + \Delta') = -(K_{\mathcal{F}} + \Delta) + \Delta''$ is f-ample. By [35, Lemma 8.10] we see that f only contracts curves tangent to $\mathcal{F}$.

Case 1: *The morphism f is a divisorial contraction.* Suppose that f contracts a divisor E. Since $\rho(X/Y) = 1$ we see that E is irreducible. If E is invariant then it is an lc centre and so there is nothing to prove. Thus we may assume that E is not invariant.

If $f(E)$ is a point and if B is a component of $\mathrm{Nklt}(\mathcal{F}, \Delta) \cap E$ then B is ample in E, in particular $\mathrm{Nklt}(\mathcal{F}, \Delta) \cap E$ is connected.
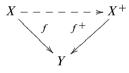
Thus we may assume that $f(E) = C$ is a curve in Y. We may find $t \geq 0$ such that $\Delta' + tE = \Gamma + E$, where $\mathrm{Supp}(\Gamma)$ does not contain E. Since $-E$ is f-ample, $-(K_{\mathcal{F}} + \Gamma + E)$ is f-ample. By the foliated adjunction formula ([13, Lemma 3.18] and [35, Lemma 8.9], or Lemma 3.10 below), we may write $(K_{\mathcal{F}} + \Gamma + E)|_E = K_{\mathcal{G}} + \Gamma_E$ where $\mathcal{G}$ is the restricted foliation, $\Gamma_E \geq 0$ and $\mathrm{Nklt}(\mathcal{F}, \Gamma) \cap E \subset \mathrm{Nklt}(\mathcal{G}, \Gamma_E)$, and so by assumption $\mathrm{Nklt}(\mathcal{G}, \Gamma_E)$ contains at least two components meeting a fibre of f.

Let Σ_0 be an irreducible curve contracted by f. Since Σ_0 is tangent to $\mathcal{G}$, it is a rational curve with $K_{\mathcal{G}} \cdot \Sigma_0 \geq -2$. Moreover, either

(1) Σ_0 meets two distinct components of $\mathrm{Nklt}(\mathcal{G}, \Gamma_E)$, or

(2) the fibre containing Σ_0 is a union of two rational curves meeting at a point and, up to switching the two components of this fibre, we can assume that Σ_0 meets at least one connected component of $\mathrm{Nklt}(\mathcal{G}, \Gamma_E)$.

Hence, $0 > (K_{\mathcal{G}} + \Delta') \cdot \Sigma_0 \geq \deg(K_{\Sigma_0} + p_1 + p_2)$ where, in scenario (1), p_1, p_2 are the intersections of Σ_0 with two distinct connected components of $\mathrm{Nklt}(\mathcal{F}, \Delta')$ along Σ_0, while in scenario (2), p_1 is the intersection of Σ_0 with the other component of the fibre and p_2 is the intersection of Σ_0 with $\mathrm{Nklt}(\mathcal{F}, \Delta')$ (which may be seen by restricting to a germ of an invariant surface containing Σ_0 and applying adjunction). However, $\deg(K_{\Sigma_0} + p_1 + p_2) \geq 0$, which provides a contradiction since $-(K_{\mathcal{G}} + \Gamma_E)$ is $f|_E$-ample.

Case 2: *The morphism f is a flipping contraction.* We denote by

$$X \dashrightarrow X^+$$
$$f \qquad f^+$$
$$Y$$

the flip of f and by Σ a curve in the exceptional locus. Then there exist two divisorial components D_1, D_2 of $\mathrm{Nklt}(\mathcal{F}, \Delta')$ which intersect Σ, and do not contain it. But then on X^+, the strict transforms D_i^+ of the D_i contain the exceptional locus of the map f^+, hence the curves contracted by f^+ must be contained in the intersection of the D_i^+ and so it is a non-klt centre.

Since Σ is tangent to $\mathcal{F}$ we may assume that $(\mathcal{F}, \Delta')$ is terminal along Σ, as otherwise Σ would be an lc centre [13, Lemma 3.14]; on the other hand, the above observation implies that $\mathcal{F}^+$, the birational transform of $\mathcal{F}$ on X^+, is canonical along the exceptional locus of f^+. But this leads to a contradiction, because by the Negativity Lemma the discrepancies of $(\mathcal{F}, \Delta')$ along the f^+-exceptional locus must decrease since $-(K_{\mathcal{F}} + \Delta')$ is f-ample (see, for example, [13, Lemma 2.7]). ∎

Lemma 3.6. *With the notation above, assume that f is birational and $(\mathcal{F}, \Delta')$ is F-dlt. Assume moreover that every lc centre of $(\mathcal{F}, \Delta')$ is contained in a codimension 1 lc centre of $(\mathcal{F}, \Delta')$. If $-(K_{\mathcal{F}} + \Delta)$ is f-ample then $\mathrm{Nklt}(\mathcal{F}, \Delta)$ is connected in a neighbourhood of any fibre of f.*

Proof. Let $y \in Y$ and let X_y denote the fibre of f over Y. Assume that $\text{Nklt}(\mathcal{F}, \Delta)$ is disconnected in a neighbourhood of X_y. As each lc centre of $(\mathcal{F}, \Delta')$ is contained in a codimension 1 lc centre and $\text{Nklt}(\mathcal{F}, \Delta)$ is disconnected in a neighbourhood of X_y, there exist prime divisors E_1, E_2 on E such that

- E_1, E_2 intersect X_y, and

- E_1, E_2 belong to different connected components of $\text{Nklt}(\mathcal{F}, \Delta)$ in a neighbourhood of X_y; in particular, $E_1 \cap E_2 = \emptyset$ in a neighbourhood of X_y.

As H is f-ample, there exists $0 < \epsilon \ll 1$ such that $G := H - \epsilon(E_1 + E_2)$ is f-ample. Then

$$K_{\mathcal{F}} + \Delta' + G \sim_{\mathbb{R}, f} -\epsilon(E_1 + E_2) - \Delta''. \tag{3.1}$$

We can then run the $(K_{\mathcal{F}} + \Delta' + G)$-MMP with scaling of G over Y (see Section 1.8),

$$X = X_0 \xrightarrow{s_1} X_1 \xrightarrow{s_2} \cdots - \xrightarrow{s_i} X_i \xrightarrow{s_{i+1}} X_{i+1} \xrightarrow{s_{i+2}} \cdots \tag{3.2}$$

We quickly explain how to run such an MMP. As we are only interested in what happens over a neighbourhood of $y \in Y$, we can assume that each step of (3.2) is non-trivial in a neighbourhood of $X_{i,y}$. As G is ample, there exists $0 < \eta \ll 1$ such that $G' := G + \eta \lfloor \Delta' \rfloor$ is also ample. Hence, choosing a suitable effective $P \sim_{\mathbb{R}, f} G'$, by Bertini's theorem we see that $K_{\mathcal{F}} + (\Delta' - \eta \lfloor \Delta' \rfloor) + P$ is F-dlt. Hence, the MMP exists for $K_{\mathcal{F}} + (\Delta' - \eta \lfloor \Delta' \rfloor) + P$ and a fortiori for $K_{\mathcal{F}} + \Delta' + G$ as well. Since each step of this MMP is G_i positive, where G_i is the strict transform of G on X_i, each step of this MMP is in fact a step of the $(K_{\mathcal{F}} + \Delta')$-MMP. In particular, we may observe moreover that at each step any lc centre of $(\mathcal{F}_i, \Delta_i')$ is contained in a codimension 1 lc centre of $(\mathcal{F}_i, \Delta_i')$. Indeed, by [13, Lemma 2.7] an lc centre cannot lie in $\text{exc}(s_i^{-1})$ and so if W is an lc centre of $(\mathcal{F}_i, \Delta_i')$ then, for $j \leq i$, each s_j must be an isomorphism at the generic point of W.

Lemma 3.5 shows that the number of connected components of $\text{Nklt}(\mathcal{F}_i, \Delta_i)$ in a neighbourhood of $X_{i,y}$ cannot decrease with i. Assume that at the i-th step $X_{i-1} \dashrightarrow X_i$ of (3.2) the strict transform of one of the E_j, say E_1, gets contracted. Denoting by R_{i-1} the generator of the extremal ray of $\overline{NE}(X_{i-1}/Y)$ contracted at this step and by $E_{1,i-1}$ the strict transform of E_1 on X_{i-1}, we have $R_{i-1} \cdot E_{1,i-1} < 0$. Lemma 3.5 implies that $E_{1,i-1} \cap E_{2,i-1} = \emptyset$ in a neighbourhood of $X_{i-1,y}$, as otherwise the number of connected components would have decreased at some point of the MMP. This observation implies that $E_{2,i-1} \cdot R_{i-1} = 0$. As $-\epsilon(E_{1,i-1} + E_{2,i-1}) \cdot R_{i-1} < \Delta_{i-1}'' \cdot R_{i-1}$, we have $\Delta_{i-1}'' \cdot R_{i-1} > 0$; thus, there exists a component D_1 of Δ_{i-1}'' that intersects $E_{1,i-1}$ and such that $f(D_1) \supset f(E_{1,i-1})$. But then on X_i,

$$K_{\mathcal{F}_i} + \Delta_i' + G' \sim_{\mathbb{R}, f_i} -\epsilon(E_{2,i} + D_{1,i}) - (\Delta_i'' - \epsilon D_{1,i})$$

and we can repeat the argument just illustrated as $E_{2,i} \cap D_{1,i} = \emptyset$ around $X_{i,y}$ and $E_{2,i}, D_{1,i}$ belong to different connected components of $\text{Nklt}(\mathcal{F}_i, \Delta_i)$ in a neighbourhood

of $X_{i,y}$. By Corollary 2.3, the MMP in (3.2) terminates with a minimal model $f': X' \to Y$ of X over Y, since f is birational and hence $K_{\mathcal{F}} + \Delta + G$ is relatively pseudo-effective over Y. Hence the strict transform of $-\epsilon(E_1 + E_2) - \Delta''$ is f'-nef. By the Negativity Lemma [25, Lemma 3.39] it must contain the whole fibre X'_y, which leads to a contradiction. ∎

Proposition 3.7. *With the above notation, assume that f is birational and $(\mathcal{F}, \Delta')$ is F-dlt. If $-(K_{\mathcal{F}} + \Delta)$ is f-big and f-nef then $\mathrm{Nklt}(\mathcal{F}, \Delta)$ is connected in a neighbourhood of any fibre of f.*

Proof. First, observe that we may freely replace $(\mathcal{F}, \Delta')$ by a higher model such that every lc centre of $(\mathcal{F}, \Delta')$ is contained in a codimension 1 lc centre. Indeed, by Theorem 2.4 we may find a modification $\mu: \overline{X} \to X$ such that if we write $K_{\overline{\mathcal{F}}} + \Theta = \mu^*(K_{\mathcal{F}} + \Delta')$ where $\overline{\mathcal{F}} = \mu^{-1}\mathcal{F}$ then $(\overline{\mathcal{F}}, \Theta)$ is F-dlt, every lc centre is contained in a codimension 1 lc centre and $\mathrm{Nklt}(\overline{\mathcal{F}}, \Theta) \subset \mu^{-1}(\mathrm{Nklt}(\mathcal{F}, \Delta'))$. Thus, $\mathrm{Nklt}(\overline{\mathcal{F}}, \Theta)$ is connected in a neighbourhood of a fibre over Y if and only if $\mathrm{Nklt}(\mathcal{F}, \Delta')$ is.

We next reduce the general case to the case of H ample, which then follows from Lemma 3.6. As $H = -(K_{\mathcal{F}} + \Delta)$ is f-big and f-nef there exists an effective $\mathbb{R}$-divisor $B = \sum a_i B_i$ for which $H - \delta B$ is f-ample for any $0 < \delta \ll 1$. We can decompose B as

$$B = B_1 + B_2, \quad \text{where} \quad B_1 := \sum_{B_i \subset \mathrm{Nklt}(\mathcal{F}, \Delta)} a_i B_i, \quad B_2 := B - B_1.$$

We claim that for δ sufficiently small,

$$\mathrm{Nklt}(\mathcal{F}, \Delta') = \mathrm{Nklt}(\mathcal{F}, \Delta' + \delta B_2). \tag{3.3}$$

Indeed, let $r_Z: Z \to X$ be a foliated log resolution of $(\mathcal{F}, \Delta' + \delta B_2)$. We denote by $\mathcal{G}$ the strict transform of $\mathcal{F}$ on Z. Thus,

$$K_{\mathcal{G}} + r_{Z*}^{-1}(\Delta' + \delta B_2) = r_Z^*(K_{\mathcal{F}} + \Delta' + \delta B_2) + \sum_i b_i(\delta) E_i$$

and $\mathrm{Nklt}(\mathcal{F}, \Delta' + \delta B_2) = r_Z(\mathrm{Nklt}(\mathcal{G}, r_{Z*}^{-1}(\Delta' + \delta B_2) - \sum_i b_i(\delta) E_i))$. Each $b_i(\delta)$ depends linearly on δ and $b_i(0) \geq -\epsilon(E_i)$, since $(\mathcal{F}, \Delta')$ is F-dlt. Hence, if $b_i(\delta) \leq -\epsilon(E_i)$ for all $\delta > 0$, then E_i is an lc centre for $(\mathcal{F}, \Delta')$.

For fixed sufficiently small $\delta > 0$ satisfying (3.3), let $\pi: Y \to X$ be an F-dlt modification of $(\mathcal{F}, \Delta' + \delta B_2)$. By Theorem 2.4, writing

$$K_{\mathcal{G}} + \Gamma + C = \pi^*(K_{\mathcal{F}} + \Delta' + \delta B_2),$$

we find that

- $\Gamma = \pi_*^{-1}\Delta' + E$ and $E = \sum_i \epsilon(E_i) E_i$ where we sum over the exceptional divisors of π;
- $(\mathcal{G}, \Gamma)$ is F-dlt;
- $\mathrm{Nklt}(\mathcal{G}, \Gamma) = \lfloor \Gamma \rfloor$;
- $C \geq 0$; and
- the support of C is contained in $\mathrm{Nklt}(\mathcal{G}, \Gamma)$ and is π-exceptional.

Moreover, the $\mathbb{Q}$-factoriality of X implies that there exists an effective π-exceptional divisor $G \geq 0$ such that $-G$ is π-ample. Since $K_{\mathcal{F}} + \Delta + \delta B_2 \sim_{\mathbb{R},X} -(H - \delta B + \delta B_1)$, we get

$$K_{\mathcal{G}} + \Gamma + C + \delta \pi^* B_1 + \pi^*(\Delta - \Delta') + \epsilon G \sim_{\mathbb{R},X} -\pi^*(H - \delta B) + \epsilon G.$$

Let $\Theta := C + \delta \pi^* B_1 + \pi^*(\Delta - \Delta') + \epsilon G$ and observe that

$$\pi^{-1}(\mathrm{Nklt}(\mathcal{F}, \Delta)) = \pi^{-1}(\mathrm{Nklt}(\mathcal{F}, \Delta + \delta B)) = \mathrm{Nklt}(\mathcal{G}, \Gamma + \Theta).$$

For $\epsilon > 0$ sufficiently small we know that $(\Gamma + \Theta)' = \Gamma$ (in the notation at the beginning of the section) and $-(K_{\mathcal{G}} + \Gamma + \Theta) = \pi^*(H - \delta B) - \epsilon G$ is f-ample; this concludes the proof. ∎

Proof of Theorem 3.1. In view of Lemmas 3.5 and 3.6, we only have to handle the case where f is a non-birational contraction. Hence, we assume that $\mathrm{Nklt}(\mathcal{F}, \Delta)$ is disconnected in a neighbourhood of some fibre X_y, $y \in Y$, of f with $\dim X > \dim Y$ and we derive a contradiction.

Step 1. *We first assume that H is f-ample.* Then there exists $0 < \epsilon \ll 1$ such that $G = H - \epsilon F$ is f-ample. We can then run the $(K_{\mathcal{F}} + \Delta')$-MMP with scaling of G over Y (see Section 1.8),

$$X = X_0 \xrightarrow{s_1} X_1 \xrightarrow{s_2} \cdots \xrightarrow{s_i} X_i \xrightarrow{s_{i+1}} X_{i+1} \xrightarrow{s_{i+2}} \cdots \tag{3.4}$$

(with f, f_1, f_i, f_{i+1} mapping to Y)

We denote $\mathcal{F}_i := s_{i*}\mathcal{F}_{i-1}$, $\Delta_i := s_{i*}\Delta_{i-1}$, $\Delta'_i := s_{i*}\Delta'_{i-1}$, $\Delta''_i := s_{i*}\Delta''_{i-1}$, $F_i := s_{i*}F_{i-1}$, $I_i := s_{i*}I_{i-1}$, $G_i := s_{i*}G_{i-1}$, and f_i is the structural map for X_i.

Claim 3.8. *For any i, either $F_i \cdot R_i > 0$ or $\Delta''_i \cdot R_i > 0$.*

Proof of Claim 3.8. By the definition of the MMP with scaling, at each step of (3.4) there exists a positive real number λ_i such that $K_{\mathcal{F}_i} + \Delta'_i + \lambda_i G_i$ is f-nef and moreover

$$(K_{\mathcal{F}_i} + \Delta'_i + \lambda_i G_i) \cdot R_i = 0, \tag{3.5}$$
$$(K_{\mathcal{F}_i} + \Delta'_i) \cdot R_i < 0. \tag{3.6}$$

For any i, we have $\lambda_i > 1$: in fact, assuming $\lambda_i \leq 1$ we reach an immediate contradiction since

$$K_{\mathcal{F}_i} + \Delta'_i + \lambda_i G_i = (1 - \lambda_i)(K_{\mathcal{F}_i} + \Delta'_i) - \lambda_i \epsilon F_i - \lambda_i \Delta''_i$$

would then be non-pseudo-effective over Y, in view of $\dim X_i > \dim Y$. By (3.5),

$$((1 - \lambda_i)(K_{\mathcal{F}_i} + \Delta'_i) \cdot R_i = \lambda_i (\epsilon F_i + \Delta''_i) \cdot R_i,$$

and the condition $\lambda_i > 1$ together with (3.6) implies that $(\epsilon F_i + \Delta''_i) \cdot R_i > 0$, which proves the claim. ∎
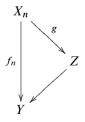
Claim 3.9. *For any i, $\mathrm{Nklt}(\mathscr{F}_i, \Delta_i) = \mathrm{Nklt}(\mathscr{F}_i, \Delta_i') = \mathrm{Supp}(F_i + I_i)$ and the number of connected components of $\mathrm{Nklt}(\mathscr{F}_i, \Delta_i')$ is independent of i.*

Proof of Claim 3.9. For any i, we have $\Delta_i \geq \Delta_i'$, hence $\mathrm{Nklt}(\mathscr{F}_i, \Delta_i) \supset \mathrm{Nklt}(\mathscr{F}_i, \Delta_i')$. On the other hand, as the support of $\Delta_i - \Delta_i'$ is contained in $\Delta_i' + I_i$, it follows that $\mathrm{Nklt}(\mathscr{F}_i, \Delta_i) \subset \mathrm{Nklt}(\mathscr{F}_i, \Delta_i')$. Moreover, since $(\mathscr{F}_i, \Delta_i')$ is F-dlt, we get $\mathrm{Nklt}(\mathscr{F}_i, \Delta_i') = \mathrm{Supp}(F_i + I_i)$.

We now prove the second part of the statement. If $s_{i+1} \colon X_i \to X_{i+1}$ is a divisorial contraction, let E be the prime divisor contracted by s_{i+1}. Since $F_i \cdot R_i > 0$ or $\Delta_i'' \cdot R_i > 0$ it follows that the image of the exceptional locus of s_{i+1} is contained in $\mathrm{Nklt}(\mathscr{F}_{i+1}, \Delta_{i+1}')$. But then Lemma 3.5 implies that the number of connected components of $\mathrm{Nklt}(\mathscr{F}_{i+1}, \Delta_{i+1}')$ in a neighbourhood of $X_{i+1,y}$ must be the same as that of $\mathrm{Nklt}(\mathscr{F}_i, \Delta_i')$ around $X_{i,y}$, since s_{i+1} is $(K_{\mathscr{F}_i} + \Delta_i')$-negative.

If $s_{i+1} \colon X_i \dashrightarrow X_{i+1}$ is a flip, let $z_i^- \colon X_i \to Z_i$ be the associated flipping contraction and $z_{i+1}^+ \colon X_{i+1} \to Z_i$ the other small map involved in the flip. By the first part of the proof, we know that $\mathrm{Nklt}(\mathscr{F}_i, \Delta_i') = F_i + I_i$ and $\mathrm{Nklt}(\mathscr{F}_{i+1}, \Delta_{i+1}') = F_{i+1} + I_{i+1}$. Hence, on Z_i, $z_i^-(\mathrm{Nklt}(\mathscr{F}_i, \Delta_i') = z_i^+(\mathrm{Nklt}(\mathscr{F}_{i+1}, \Delta_{i+1}'))$, as s_{i+1}, z_i^-, z_i^+ are all small maps. Hence it suffices to prove that the number of connected components of $\mathrm{Nklt}(\mathscr{F}_i, \Delta_i')$ (resp. $\mathrm{Nklt}(\mathscr{F}_{i+1}, \Delta_{i+1}')$) around $X_{i,y}$ (resp. $X_{i+1,y}$) is the same as that of $z_i^-(\mathrm{Nklt}(\mathscr{F}_i, \Delta_i'))$ (resp. $z_i^+(\mathrm{Nklt}(\mathscr{F}_{i+1}, \Delta_{i+1}')))$ around $Z_{i,y}$. Lemma 3.5 implies that the number of connected components of $\mathrm{Nklt}(\mathscr{F}_i, \Delta_i')$ in a neighbourhood of $X_{i,y}$ must be the same as that of $z_i^-(\mathrm{Nklt}(\mathscr{F}_i, \Delta_i'))$ around $Z_{i,y}$, since z_i^- is $(K_{\mathscr{F}_i} + \Delta_i')$-negative. On the other, by Claim 3.8 either $F_i \cdot R_i > 0$ or $\Delta_i'' \cdot R_i > 0$, which implies that the exceptional locus of z_i^- is contained in either F_{i+1} or Δ_{i+1}''. Since $F_i = \mathrm{Supp}(\Delta_i'') \subset \mathrm{Nklt}(\mathscr{F}_{i+1}, \Delta_{i+1}')$, we find that $\mathrm{Nklt}(\mathscr{F}_{i+1}, \Delta_{i+1}')$ is connected around every fibre of z_i^-, hence the number of connected components of $\mathrm{Nklt}(\mathscr{F}_{i+1}, \Delta_{i+1}')$ around $X_{i+1,y}$ is the same as that of $z_i^+(\mathrm{Nklt}(\mathscr{F}_{i+1}, \Delta_{i+1}'))$ around $Z_{i,y}$, which concludes the proof. ∎

By Special Termination [13, Theorem 7.1] and Claim 3.8, the run of the MMP in (3.4) must terminate, and since $K_{\mathscr{F}} + \Delta'$ is non-pseudo-effective over Y, the final step will be a Mori fibre space

By Claim 3.9 it suffices to prove that $\mathrm{Nklt}(\mathscr{F}_n, \Delta_n)$ is connected in a neighbourhood of $X_{n,y}$. On X_n, $\mathrm{Nklt}(\mathscr{F}_n, \Delta_n) = \mathrm{Supp}(F_n + I_n)$. As I_n is $\mathscr{F}_n$-invariant, every component of I_n must be vertical over Z. As $F_n \cdot R_n > 0$ or $\Delta_n'' \cdot R_n > 0$, there exists at least one component of F_n which dominates Z and contains only one horizontal component. Let $z \in Z$ be a point and observe that $\dim(g^{-1}(z)) \leq 2$.

If $\dim(g^{-1}(z)) = 2$ for all $z \in Z$, then since $\rho(X/Z) = 1$, it follows that every horizontal component of F_n is g-ample; hence, any two horizontal components of F_n intersect along any fibre of g. If $\dim(g^{-1}(z)) = 1$ for some (equivalently any) z then since $-(K_{\mathcal{F}_n} + \Delta_n)$ is g-ample we see that F_n contains at most one horizontal component. Thus, $\mathrm{Nklt}(\mathcal{F}_n, \Delta_n)$ must be connected in a neighbourhood of $X_{n,y}$. But this gives a contradiction.

Step 2. *We reduce the general case to the case of f-ample H.* Here it suffices to copy the proof of Proposition 3.7 verbatim. ∎

3.2. Adjunction for foliated pairs

The goal of this section is to illustrate adjunction theory for foliated threefolds. Let us highlight the fact that in [13] a $\mathbb{Q}$-factorial threefold X is simply an analytic variety which is (globally) $\mathbb{Q}$-factorial. We will work in this set up throughout §3.2–3.3; the reader should keep this observation in mind when encountering foliated adjunction throughout the paper.

Let us recall the following adjunction for foliations with non-dicritical singularities.

Lemma 3.10 (Adjunction [13, Lemma 3.18]). *Let X be a $\mathbb{Q}$-factorial threefold, and let $\mathcal{F}$ be a co-rank 1 foliation with non-dicritical singularities. Suppose that $(\mathcal{F}, \epsilon(S)S + \Delta)$ is log canonical (resp. log terminal, resp. F-dlt) for a prime divisor S and a $\mathbb{Q}$-divisor $\Delta \geq 0$ on X which does not contain S in its support. Let $v \colon S^v \to S$ be the normalization and let $\mathcal{G}$ be the restricted foliation to S^v. Then there exists $\Theta \geq 0$ on S^v such that*

$$v^*(K_{\mathcal{F}} + \epsilon(S)S + \Delta) = K_{\mathcal{G}} + \Theta. \tag{3.7}$$

Moreover,

- *if $\epsilon(S) = 1$, then $(\mathcal{G}, \Theta)$ is log canonical (resp. log terminal, resp. F-dlt);*
- *if $\epsilon(S) = 0$, $(\mathcal{F}, \Delta)$ is F-dlt, and S and $\mathrm{sing}(\mathcal{F}) \cap S$ are normal, then $(S^v, \Theta' := \lfloor \Theta \rfloor_{\mathrm{red}} + \{\Theta\})$ is lc (resp. lt, resp. dlt).*

We wish to generalize this result to an adjunction formula which holds in full generality.

Lemma 3.11 (General Adjunction). *Let X be a threefold and let $\mathcal{F}$ be a co-rank 1 foliation on X. Suppose that $(\mathcal{F}, \epsilon(S)S + \Delta)$ is a foliated log pair for a prime divisor S and a $\mathbb{Q}$-divisor $\Delta \geq 0$ on X which does not contain S in its support. Let $v \colon S^v \to S$ be the normalization and let $\mathcal{G}$ be the restricted foliation to S^v. Then there exists $\Theta \geq 0$ on S^v such that*

$$v^*(K_{\mathcal{F}} + \epsilon(S)S + \Delta) = K_{\mathcal{G}} + \Theta. \tag{3.8}$$

In the hypotheses of Lemma 3.11, we will refer to Θ as the *different* $\mathrm{Diff}_S \Delta$ of Δ on S.

Proof of Lemma 3.11. Let $\pi\colon Y \to X$ be an F-dlt modification for $(\mathcal{F}, \epsilon(S)S + \Delta)$ and let S' be the strict transform of S on Y. Write

$$K_{\mathcal{F}_Y} + \epsilon(S')S' + \Delta_Y = \pi^*(K_{\mathcal{F}} + \epsilon(S)S + \Delta), \qquad (3.9)$$

Then the pair $(\mathcal{F}_Y, \epsilon(S')S' + \Delta_Y')$ is F-dlt, where $\Delta_Y' := \pi_*^{-1}\Delta + \sum_{\pi\text{-exc}} \epsilon(E)E$. Denoting $\Delta_Y'' := \Delta_Y - \Delta_Y'$, we see immediately that the support of Δ_Y'' does not contain S' and $K_{\mathcal{F}_Y} + \epsilon(S')S' + \Delta_Y' + \Delta_Y'' = \pi^*(K_{\mathcal{F}} + \epsilon(S)S + \Delta)$. As $(\mathcal{F}_Y, \epsilon(S')S' + \Delta_Y')$ is F-dlt, Lemma 3.10 implies that there exists Θ_1 such that

$$(K_{\mathcal{F}_Y} + \epsilon(S')S' + \Delta_Y')|_{S'^\nu} = K_{\mathcal{G}} + \Theta_1.$$

Hence,

$$\begin{aligned}(K_{\mathcal{F}_Y} + \epsilon(S')S' + \Delta_Y)|_{S'^\nu} &= (K_{\mathcal{F}_Y} + \epsilon(S')S' + \Delta_Y' + \Delta_Y'')|_{S'^\nu} \\ &= K_{\mathcal{G}} + \Theta_1 + \Delta_Y''|_{S'^\nu},\end{aligned}$$

so it suffices to take $\Theta := \Theta_1 + \Delta_Y''|_{S'^\nu}$. $\blacksquare$

The two equations (3.7), (3.8) represent the adjunction formula for foliations, where (3.8) is a generalized version of the one proven in [13]. On the other hand, in the more general framework of Lemma 3.11, it is not possible to control the singularities of the restriction of the pair $(\mathcal{F}, \Delta)$ to a codimension 1 log canonical centre.

3.3. Inversion of adjunction

We are now ready to prove inversion of adjunction for foliated pairs.

Theorem 3.12. *Let X be a $\mathbb{Q}$-factorial threefold and let $\mathcal{F}$ be a co-rank 1 foliation. Consider a prime divisor S and an effective $\mathbb{Q}$-divisor Δ on X which does not contain S in its support. Let $\nu\colon S^\nu \to S$ be the normalization and let $\mathcal{G}$ be the restricted foliation on S^ν and Θ be the foliation different for $(\mathcal{F}, \Delta)$ on S^ν. Suppose that*

- *if $\epsilon(S) = 1$ then $(\mathcal{G}, \Theta)$ is lc;*
- *if $\epsilon(S) = 0$ then (S^ν, Θ) is lc.*

Then $(\mathcal{F}, \epsilon(S)S + \Delta)$ is log canonical in a neighbourhood of S.

Proof. Let $\pi\colon Y \to X$ be an F-dlt modification for the pair $(\mathcal{F}, \epsilon(S)S + \Delta)$ and let S' be the strict transform of S on Y. Write

$$K_{\mathcal{F}_Y} + \epsilon(S')S' + \Delta_Y = \pi^*(K_{\mathcal{F}} + \epsilon(S)S + \Delta). \qquad (3.10)$$

Then the pair $(\mathcal{F}_Y, \epsilon(S')S' + \Delta_Y')$ is F-dlt, where $\Delta_Y' := \pi_*^{-1}\Delta + \sum_{\pi\text{-exc}} \epsilon(E)E$ and $K_{\mathcal{F}_Y} + \epsilon(S')S' + \Delta_Y'$ is π-nef [13, proof of Theorem 8.1]. Denote $\Delta_Y'' := \Delta_Y - \Delta_Y'$. Then $-\Delta_Y''$ is π-nef, since by (3.10), $-\Delta_Y'' \sim_{\pi,\mathbb{R}} K_{\mathcal{F}_Y} + \Delta_Y'$. When $\epsilon(S') = 1$, we will denote by $\mathcal{G}'$ the restriction of $\mathcal{F}_Y$ to the normalization $\nu_1\colon S'^\nu \to S'$ of S' and let Ξ' be the different given by adjunction of $(\mathcal{F}_Y, \epsilon(S')S' + \Delta_Y)$.

Step 1. *We prove that* $(\mathcal{F}, \epsilon(S)S + \Delta)$ *is log canonical in a neighbourhood of* S *if and only if* $(\mathcal{F}_Y, \epsilon(S')S' + \Delta_Y)$ *is log canonical in a neighbourhood of* S'. It follows from Definition 1.5 that $(\mathcal{F}, \epsilon(S)S + \Delta)$ is log canonical in a neighbourhood U of S if and only if $S \cap \mathrm{Nlc}(\mathcal{F}, \epsilon(S)S + \Delta) = \emptyset$, since $\mathrm{Nlc}(\mathcal{F}, \epsilon(S)S + \Delta)$ is closed. By Remark 2.5, as $(\mathcal{F}_Y, \epsilon(S')S' + \Delta'_Y)$ is F-dlt, $\mathrm{Nlc}(\mathcal{F}_Y, \epsilon(S')S' + \Delta_Y) = \mathrm{Supp}(\Delta''_Y)$ and $\mathrm{Nlc}(\mathcal{F}, \epsilon(S)S + \Delta) = \pi(\mathrm{Supp}(\Delta''_Y))$. By the Negativity Lemma [38, Lemma 1.3 and Appendix A], since $-\Delta''_Y$ is π-nef, it contains any fibre of π intersecting $\mathrm{Supp}(\Delta''_Y)$; thus, $\mathrm{Nlc}(\mathcal{F}, \epsilon(S)S + \Delta) \cap S = \emptyset$ if and only if $\mathrm{Nlc}(\mathcal{F}_Y, \epsilon(S')S' + \Delta_Y) \cap S' = \emptyset$.

Step 2. *We prove that if* $\epsilon(S') = 1$, *then* $(\mathcal{G}', \Xi')$ *is log canonical. Then we deduce that* $(\mathcal{F}_Y, S' + \Delta_Y)$ *is log canonical in a neighbourhood of* S'. By Lemma 1.17, S' is normal. Hence, $S'^{\nu} = S'$ and $\mathcal{G}' = \mathcal{G}|_{S'}$.

$$K_{\mathcal{G}'} + \Xi' = (K_{\mathcal{F}_Y} + S' + \Delta_Y)|_{S'} = \pi^*(K_{\mathcal{F}} + S + \Delta)|_{S'}.$$

Hence, considering the birational morphism $\psi \colon S' \to S^{\nu}$, we have

$$K_{\mathcal{G}'} + \Xi' = \psi^*(K_{\mathcal{G}} + \Theta).$$

As $(\mathcal{G}, \Theta)$ is log canonical, the same holds for $(\mathcal{G}', \Xi')$. As shown in Step 1, we need to prove that $\mathrm{Supp}(\Delta''_Y) \cap S' = \emptyset$. Seeking a contradiction, let E be a prime component of $\mathrm{Supp}(\Delta''_Y)$ intersecting S', so that $\mu_E \Delta_Y > \epsilon(E)$ and $\mu_E \Delta'_Y = \epsilon(E)$. Let G be any prime component of $E \cap S'$.

Claim 3.13. *G is an lc centre of* $(\mathcal{F}_Y, \Delta'_Y)$.

Proof of Claim 3.13. This fact is an immediate consequence of Lemma 1.17 and its proof if $\epsilon(E) = 1$, while if $\epsilon(E) = 0$ then [13, Lemma 3.16] implies that Y has quotient singularities at the generic point of G at which point the conclusion follows from a local computation on foliated surfaces, upon localizing at the generic point of G. ∎

By [13, Lemma 3.8] and Claim 3.13, $(\mathcal{F}_Y, \Delta'_Y)$ is log smooth at the generic point of G: in particular, Y is smooth at the generic point of G and E meets S' generically transversely along G. Hence, taking $K_{\mathcal{G}'} + \Psi' = (K_{\mathcal{F}_Y} + S' + \Delta'_Y)|_{S'}$, we have $\mu_G \Psi' = \epsilon(G) = \epsilon(E)$. As $\mu_E \Delta''_Y > 0$, it follows that

$$K_{\mathcal{G}'} + \Xi' = (K_{\mathcal{F}_Y} + S' + \Delta'_Y + \Delta''_Y)|_{S'} = K_{\mathcal{G}'} + \Psi' + \Delta''_Y|_{S'}$$

and $\mu_G \Xi' > \epsilon(G)$, which contradicts the fact that $(\mathcal{G}', \Xi')$ is log canonical.

Step 3. *We prove that if* $\epsilon(S') = 0$ *and if* (S^{ν}, Θ) *is log canonical, then* (S'^{ν}, Ξ') *is log canonical; then we prove that* $(\mathcal{F}_Y, \Delta_Y)$ *is log canonical in a neighbourhood of* S'. We have

$$K_{S'^{\nu}} + \Xi' = \nu_1^*(K_{\mathcal{F}_Y} + \Delta_Y) = (\nu_1 \circ \pi)^*(K_{\mathcal{F}} + \Delta).$$

Considering the birational morphism $\psi \colon S'^{\nu} \to S^{\nu}$, we have

$$K_{S'^{\nu}} + \Xi' = \psi^*(K_S + \Theta).$$

As (S^ν, Θ) is log canonical, the same holds for (S'^ν, Ξ'). We need to show $\mathrm{Supp}(\Delta''_Y) \cap S'$ $= \emptyset$. Seeking a contradiction, let E be a prime component of $\mathrm{Supp}(\Delta''_Y)$ intersecting S', so that $\mu_E \Delta_Y > \epsilon(E)$ and $\mu_E \Delta'_Y = \epsilon(E)$. Let G be any prime component of $\nu_1^{-1}(E)$. Set $K_{S'^\nu} + \Psi' = \nu_1^*(K_{\mathcal{F}_Y} + \Delta'_Y)$. Then $\Psi' \le \Xi'$ and [13, Corollary 3.20] implies that $\mu_G \Psi' \ge 1$. As $\mu_E \Delta''_Y > 0$ and $K_{S'^\nu} + \Xi' = \nu_1^*(K_{\mathcal{F}_Y} + S' + \Delta'_Y + \Delta''_Y) = K_{S'^\nu} + \Psi' + \nu_1^* \Delta''_Y$, we have $\mu_G \Xi' > 1$, which contradicts the fact that (S'^ν, Ξ') is log canonical. ∎

This is the adaptation to foliations of the classical statement of inversion of adjunction for log pairs (cf. [25, Theorem 5.50]). Nonetheless, it is not the most general form of inversion of adjunction that one could hope for. In fact, if we look at the statement of Lemma 3.10, we see that the natural divisor to look at when $\epsilon(S) = 0$ would be, in the notation of the lemma, the divisor Θ' rather than the foliated different Θ – recall that $\Theta' := \lfloor \Theta \rfloor_{\mathrm{red}} + \{\Theta\}$. As by definition $\Theta' \le \Theta$, it follows immediately that if (S, Θ) is log canonical, so is (S, Θ'), but it would be even more interesting to have a statement of inversion of adjunction that only assumes (S, Θ') is log canonical.

4. A vanishing result

In this section we prove a relative vanishing theorem for foliations.

We make the following easy observation whose proof is left to the reader.

Lemma 4.1. *Let $f: Y \to X$ be a morphism of varieties. Let $\sigma: X' \to X$ be surjective and étale and let $Y' = Y \times_X X'$. Let $f': Y' \to X'$ and $\tau: Y' \to Y$ be the projections. Let L be a line bundle on Y. Suppose that $R^i f'_* \tau^* L = 0$. Then $R^i f_* L = 0$.*

Lemma 4.2. *Let $f: Y \to X$ be a surjective birational projective morphism of normal varieties of dimension at most 3 and let $(\mathcal{F}, \Delta)$ be an F-dlt foliated pair on Y with $\lfloor \Delta \rfloor = 0$. Suppose that Y is $\mathbb{Q}$-factorial and that every fibre of f is tangent to $\mathcal{F}$.*

Let $P \in X$ be a closed point. Then there exists an étale neighbourhood $\sigma: X' \to X$ of P, a small $\mathbb{Q}$-factorialization $\mu: W \to Y' := Y \times_X X'$ and a reduced divisor $\sum T_i$ on W such that writing $\Delta_W = \mu^* \Delta$ and $\mathcal{F}_W = \mu^{-1} \mathcal{F}$ and $f': W \to X'$ for the induced map we have

(1) $\sum T_i$ *is nef over Y';*

(2) $(K_{\mathcal{F}_W} + \Delta_W) - (K_W + \Delta_W + \sum T_i)$ *is f'-nef; and*

(3) $(W, \Delta_W + (1 - \epsilon) \sum T_i)$ *is klt for all $\epsilon > 0$.*

Proof. First, since Y is $\mathbb{Q}$-factorial we may apply [13, Theorem 11.3] to see that Y is klt. as $(\mathcal{F}, \Delta)$ is F-dlt we also know that $\mathcal{F}$ is non-dicritical by Theorem 1.16.

Let $\{S_1, \ldots, S_N\}$ be the collection of all separatrices of $\mathcal{F}$ meeting $f^{-1}(P)$, formal or otherwise. Fix $n \gg 0$ sufficiently large. By [13, §§4, 5] there is an étale cover $\sigma: X' \to X$ such that we may find divisors R_i on Y' such that $R_i|_{Y'_n} = \overline{S_i}|_{Y'_n}$ where $\overline{S_i} = \tau^* S_i$, Y'_n is the n-th infinitesimal neighbourhood of $\sigma^{-1}(f^{-1}(P))$ and $\tau: Y' \to Y$ is the projec-

tion, and such that $\mathcal{O}_{\widehat{Y}'}(\overline{S}_i) \cong \mathcal{O}_{\widehat{Y}'}(R_i)$ where $\widehat{Y}'$ is the formal completion of Y' along $\sigma^{-1}(f^{-1}(P))$. Let $g \colon Y' \to X'$ be the other projection.

Set $(\mathcal{F}' := \tau^{-1}\mathcal{F}, \Delta' := \tau^*\Delta)$. Since $(\mathcal{F}, \Delta)$ is F-dlt, so is $(\mathcal{F}', \Delta')$, and we may find an F-dlt modification $\pi \colon Z \to Y'$ such that μ is small. Observe that Z is $\mathbb{Q}$-factorial and so $R_i' := \pi_*^{-1}R_i$ and $\overline{S_i}' := \pi_*^{-1}\overline{S}_i$ are $\mathbb{Q}$-Cartier. Set $\mathcal{G} = \pi^{-1}\mathcal{F}$; we may write $K_{\mathcal{G}} + \Gamma = \pi^*(K_{\mathcal{F}'} + \Delta')$. Moreover, $(\mathcal{G}, \Gamma)$ is necessarily terminal at the generic point of a curve $C \subset \mathrm{exc}(\pi)$: otherwise C would be an lc centre of $(\mathcal{G}, \Gamma)$ since it is tangent to $\mathcal{G}$, which by [13, Lemma 3.8] would imply that $(\mathcal{F}', \Delta')$ is log smooth at $\pi(C)$, a contradiction.

Note that $\overline{S_i}'$ are all the separatrices (formal or otherwise) which meet $\pi^{-1}(g^{-1}(p))$ and R_i' still approximate all the $\overline{S_i}'$; in particular, they have the same intersection numbers with all curves contained in $\pi^{-1}(g^{-1}(p))$.

Since Y is klt, the same is true of Z and so we may run the $(K_Z + \delta \sum R_i')$-MMP over Y' for some $\delta > 0$ sufficiently small. Denote this MMP by $\phi \colon Z \dashrightarrow W$ and set $T_i := \phi_* R_i'$ and $\widetilde{S}_i = \phi_* \overline{S_i}'$. Set $\mathcal{F}_W = \phi_*\mathcal{G}$ and $\Delta_W = \phi_*\Gamma$. Observe that each step of this MMP is $K_{\mathcal{G}} + \Gamma$-trivial and $(\mathcal{F}_W, \Delta_W)$ is F-dlt. We claim that W has all the required properties.

Item (1) holds since K_Y is $\mathbb{Q}$-Cartier and so K_Z (and hence K_W) is trivial over Y'. Thus $K_W + \delta \sum T_i$ being nef over Y' implies that $\sum T_i$ is nef over Y'.

To prove (2), let $C \subset \pi^{-1}(P)$. Note that by non-dicriticality of $(\mathcal{F}, \Delta)$ and our assumptions on f we see that C is tangent to $\mathcal{F}_W$. Moreover, if $(\mathcal{F}_W, \Delta_W)$ is canonical at the generic point of C then $(\mathcal{F}_W, \Delta_W)$ is log smooth at a general point of C [13, Lemma 3.8]. So, up to relabelling the S_i we may assume that $C \subset \widetilde{S}_1$ and $\widetilde{S}_1$ gives a strong separatrix at a general point of C if $(\mathcal{F}_W, \Delta_W)$ is canonical at the generic point of C and $(\mathcal{F}_W, \Delta_W)$ has a saddle node at the general point of C; see [4, p. 3] for a recollection on saddle nodes and weak separatrices on surfaces, but which works equally in the current setting.

By [13, Corollary 3.20] we may write

$$(K_{\mathcal{F}_W} + \Delta_W)|_{\widetilde{S}_1} = K_{\widetilde{S}_1} + \Theta, \quad \left(K_W + \Delta_W + \sum \widetilde{S}_i\right)\Big|_{\widetilde{S}_1} = K_{\widetilde{S}_1} + \Theta'$$

where $\Theta \geq \Theta'$ and the coefficient of C in both divisors is the same. It follows that

$$\left((K_{\mathcal{F}_W} + \Delta_W) - \left(K_W + \Delta_W + \sum \widetilde{S}_i\right)\right) \cdot C \geq 0$$

and since $\{T_1, \ldots, T_N\}$ approximate the $\widetilde{S}_1$ we have

$$\left((K_{\mathcal{F}_W} + \Delta_W) - \left(K_W + \Delta_W + \sum T_i\right)\right) \cdot C \geq 0.$$

Since C was arbitrary we get the claimed nefness.

Next, observe that each step of the MMP $\phi \colon Z \dashrightarrow Y$ is $K_{\mathcal{G}} + \Gamma$-trivial so $(\mathcal{F}_W, \Delta_W)$ is still F-dlt and hence $\mathcal{F}_W$ is non-dicritical by Theorem 1.16 and all the log canonical centres of $(\mathcal{F}_W, \Delta_W)$ are contained in $\mathrm{Supp}(\sum T_i)$. Hence we may apply [13, Lemma 3.16] to see that $(W, \Delta_W + (1 - \epsilon)(\sum T_i))$ is klt for all $\epsilon > 0$. This gives item (3). ∎

Theorem 4.3. *In the set up as above, let $(\mathcal{F}, \Delta)$ be an F-dlt pair and let L be a line bundle such that $L - (K_{\mathcal{F}} + \Delta)$ is f-nef and big. Suppose moreover that either*

(1) $L - (K_{\mathcal{F}} + \Delta)$ *is f-ample, or*

(2) $\Delta = A + B$ *where A is f-ample and $B \geq 0$.*

Then $R^i f_* L = 0$ for $i > 0$.

Proof. If $\Delta = A + B$ where A is f-ample then replacing Δ by $\Delta - \delta A$ for $\delta > 0$ small we may assume that $L - (K_{\mathcal{F}} + \Delta)$ is f-ample. Moreover, replacing if necessary Δ by $\Delta - \epsilon \lfloor \Delta \rfloor$ for some $\epsilon > 0$ sufficiently small we may assume that $\lfloor \Delta \rfloor = 0$.

As in Lemma 4.2 we see that Y is klt and so it has rational singularities. By Lemma 4.1 and the fact that Y has rational singularities we see that $R^i f_* L = 0$ provided $R^i f'_* L' = 0$ where $L' = \mu^* \tau^* L$ with μ and τ as in Lemma 4.2 (and its proof).

Next, $L' - (K_{\mathcal{F}_W} + \Delta_W)$ is f'-big and nef and is strictly positive on any curve not contracted by μ. Thus by Lemma 4.2 (2),

$$L' - \left(K_W + \Delta_W + \sum T_i \right)$$
$$= (L' - (K_{\mathcal{F}_W} + \Delta_W)) + \left((K_{\mathcal{F}_W} + \Delta_W) - \left(K_W + \Delta_W + \sum T_i \right) \right)$$

is f'-big and nef and is strictly positive on any curve not contracted by μ.

So for $\epsilon > 0$ sufficiently small since $\sum T_i$ is μ-nef, by Lemma 4.2 (1)

$$L' - \left(K_W + \Delta_W + (1 - \epsilon) \sum T_i \right)$$

is f'-big and nef.

Thus we may apply relative Kawamata–Viehweg vanishing to conclude that $R^i f'_* L' = 0$ for $i > 0$. ∎

5. Malgrange's theorem

In this section we prove a version of Malgrange's theorem on singular threefolds. A weaker version of this statement was proven in [35]. Results in this direction were achieved in [14] and some of our ideas have been inspired by their approach.

Let $(P \in X)$ be a germ of a threefold and let $\mathcal{F}$ be a co-rank 1 foliation on X defined by a holomorphic 1-form ω. We say that $f \in \mathcal{O}_{X,p}$ is a *first integral* for $\mathcal{F}$ if $df \wedge \omega = 0$.

Theorem 5.1. *Let $(P \in X)$ be a germ of an isolated (analytically) $\mathbb{Q}$-factorial threefold singularity with a co-rank 1 foliation $\mathcal{F}$. Suppose that $\mathcal{F}$ has an isolated canonical singularity at P. Then $\mathcal{F}$ admits a holomorphic first integral.*

It would be ideal to drop the $\mathbb{Q}$-factoriality assumption in the theorem; we are able to do this when $\mathcal{F}$ is terminal (see Corollary 5.15).

Theorem 5.1 has the following immediate consequence.

Corollary 5.2. *Let* $(P \in X)$ *be a germ of an isolated threefold singularity with a co-rank* 1 *foliation* $\mathcal{F}$. *Suppose that* X *is* $\mathbb{Q}$-*factorial and* $\mathcal{F}$ *is canonical. Then* $\mathcal{F}$ *has a separatrix at* P.

Proof. If $\mathcal{F}$ is smooth outside of P then this follows directly from Theorem 5.1. Otherwise, let $Z \subset \text{sing}(\mathcal{F})$ be a curve; then Z is tangent to $\mathcal{F}$. Observe that there is a germ of a separatrix for all $Q \in Z \setminus P$. By Theorem 1.16, $\mathcal{F}$ is non-dicritical and so by Lemma 1.18 we may extend S_Q to a neighbourhood of Z, which in turn gives a separatrix at P. ∎

Recall that in general, even for $\mathcal{F}$ non-dicritical, if $P \in X$ is a singular point then there may be no separatrices at P. See [5] for results in this direction on surfaces.

5.1. Controlling the singularities of X and $\mathcal{F}$

The goal of this subsection is to show that under the hypotheses of Theorem 5.1, X has log terminal singularities.

We will need the following version of the classical Camacho–Sad formula for F-dlt foliations. It follows as a special case of the Camacho–Sad formula for foliations on varieties with quotient singularities proven in [16, Proposition 3.12]. We refer to [16, Definition 3.10] for the definition of the Camacho–Sad index.

Lemma 5.3. *Let* X *be a normal surface and* $\mathcal{F}$ *an F-dlt foliation. Let* C *be a compact* $\mathcal{F}$-*invariant curve. Then*

$$C^2 = \sum_{p \in \text{sing}(\mathcal{F}) \cap C} \text{CS}(p, \mathcal{F}, C).$$

Lemma 5.4. *Let* $(P \in X)$ *be a germ of an isolated threefold singularity and let* $\mathcal{F}$ *be a co-rank* 1 *foliation with canonical singularities such that* $\mathcal{F}$ *is smooth away from* P. *Suppose that* K_X *is* $\mathbb{Q}$-*Cartier. Then* X *is log terminal.*

Proof. If $\mathcal{F}$ is terminal then the result follows from Theorem 1.16. So suppose that $\mathcal{F}$ has canonical but not terminal singularities.

Let $\mu : (\overline{X}, \overline{\mathcal{F}}) \to (X, \mathcal{F})$ be an F-dlt modification. Let $E = \sum E_i = \text{exc}(\mu)$. Since $\mathcal{F}$ is canonical we have $\mu^* K_{\mathcal{F}} = K_{\overline{\mathcal{F}}}$. Moreover, we may assume that μ is not the identity. By Theorem 1.16, $\mathcal{F}$ is non-dicritical and so E is $\overline{\mathcal{F}}$-invariant. Let Z be a 1-dimensional component of $\text{sing}(\overline{\mathcal{F}}) \cap E$. By [13, Lemma 3.14] either $\mathcal{F}$ is terminal at the generic point of Z, or $\overline{X}$ is smooth at the generic point of Z, and at a general point of Z, $\overline{\mathcal{F}}$ has simple singularities and there are two separatrices (possibly formal) containing Z. However, Proposition 1.7 applied to a general hyperplane passing through Z and the restricted foliation on this hyperplane implies that $\mathcal{F}$ cannot be terminal at the generic point of Z.

Write $K_{\overline{X}} + \sum E_i = \pi^*(K_X) + \sum a_i E_i$. By [35, Lemma 8.9] we see that $K_{\overline{\mathcal{F}}} - (K_{\overline{X}} + \sum E_i) = -\sum a_i E_i$ is π-nef away from finitely many curves, which implies by the Negativity Lemma [38, Lemma 1.3] that $\sum a_i E_i \geq 0$; and since $\text{Supp}(\sum E_i) = \pi^{-1}(P)$, we have either

(1) $a_i > 0$ for all i, or

(2) $a_i = 0$ for all i.

By [13, Lemma 3.16], $(\overline{X}, (1 - \epsilon) \sum E_i)$ is klt and so if we are in case (1) then we see immediately that X is klt.

So suppose for the sake of contradiction that we are in case (2), i.e., $a_i = 0$ for all i and so $K_{\overline{X}} + E \sim_{\mathbb{Q}} 0$.

We first claim that if $Z \subset \text{sing}(\overline{\mathcal{F}}) \cap E$ is a 1-dimensional component admitting two separatrices contained in E then Z is not a saddle node. Indeed, suppose for the sake of contradiction that there exists $Z \subset E_i$ such that Z is a saddle node and E_i is the weak separatrix of the saddle node; see [4, p. 3] for a recollection on saddle nodes and weak separatrices on surfaces, but which works equally in the current setting. Write

$$K_{\overline{\mathcal{F}}}|_{E_i} = K_{E_i} + \Theta, \quad \left(K_{\overline{X}} + \sum E_j \right)\Big|_{E_i} = K_{E_i} + \Theta'.$$

By Lemma 3.10 we know that $\Theta \geq \Theta'$. Since E_i is the weak separatrix of a saddle node along Z in appropriate (formal) local coordinates around a general point of Z we see that $\overline{\mathcal{F}}$ is generated by a 1-form ω of the form $z(1 + \nu w^k)dw + w^k dz$ where $E_i = \{z = 0\}$, $\nu \in \mathbb{C}$ and $k \geq 2$. The coefficient of Z in Θ is the order of vanishing of $\omega|_{E_i}$ along Z, which is exactly $k \geq 2$. On the other hand, since $(\overline{X}, \sum E_j)$ is log canonical, the coefficient of Z in Θ' is at most 1. However, $K_{\overline{\mathcal{F}}} - (K_{\overline{X}} + \sum E_j)$ cannot then be π-trivial, a contradiction.

A similar argument shows that each 1-dimensional component $Z \subset \text{sing}(\overline{\mathcal{F}}) \cap E$ admits two separatrices, both contained in E. In particular, each 1-dimensional component $Z \subset \text{sing}(\overline{\mathcal{F}}) \cap E$ has two non-zero eigenvalues.

The rest of the argument proceeds essentially as in [30, proof of the first part of Theorem IV.2.2]. We will explain this argument for the reader's convenience. Since $K_{\overline{\mathcal{F}}} \sim_{\mathbb{Q}} 0$ and $K_{\overline{X}} + E \sim_{\mathbb{Q}} 0$ we see that $N^*_{\overline{\mathcal{F}}} + E \sim_{\mathbb{Q}} 0$.

Let $H \subset \overline{X}$ be a general ample divisor, let $\mathcal{G}$ be the restricted foliation on H and let $E \cap H = \bigcup C_i = C$. Set $S = \text{sing}(\mathcal{G}) \cap C$ and notice that $C_i \cap C_j \subset S$ for $i \neq j$.

If H is general enough then $(\overline{\mathcal{F}}, H)$ is F-dlt, and so also is $\mathcal{G}$. Even better, if H is general enough then $N^*_{\overline{\mathcal{F}}}|_H = N^*_{\mathcal{G}}$ and so $N^*_{\mathcal{G}} + \sum C_i \sim_{\mathbb{Q}} 0$. Observe that since $\mathcal{G}$ is F-dlt we see that C is a nodal curve.

For $q \in S$ let ∂_q be a vector field generating $\mathcal{G}$ near q.

Claim 5.5. *The ratio λ_q of the eigenvalues of ∂_q is a root of unity.*

Proof of Claim 5.5. We may check this after taking a cover ramified along a general ample divisor A, and so after taking the index 1 cover associated to $K_{\mathcal{G}}$ on $H \setminus A$ we may assume that $K_{\mathcal{G}}$ is Cartier. By Lemma 1.14 it follows that H is smooth.

For any $p \in C \setminus S$ let U_p be a small open set such that $\mathcal{G}$ is defined by a 1-form $\omega_p = dz_p$ where $\{z_p = 0\} = C \cap U_p$. For any $q \in S$ let U_q be a small open subset such that $\mathcal{G}$ is defined by a 1-form $\omega_q = x_q a_q dy_q + y_q b_q dx_q$ where $\{x_q y_q = 0\} = C \cap U_q$ and where $a_q(q), b_q(q) \neq 0$.

Let $\{(U_{pq}, h_{pq})\}, \{(U_{pq}, g_{pq})\} \in H^1(H, \mathcal{O}_H^*)$ be the cocycles associated to $\mathcal{O}_H(C)$ and $N_\mathcal{G}$ respectively where $U_{pq} = U_p \cap U_q$.

If $p, p' \in C \setminus S$ are such that $U_{pp'} \neq \emptyset$ then $dz_p = h_{pp'} dz_{p'}$ and so $g_{pp'} = h_{pp'}$ when restricted to C. If $p \in C \setminus S$ and $q \in S$ with $U_{pq} \neq \emptyset$ we have $z_p = h_{pq}(x_q y_q)$ and so $dz_p = h_{pq} d(x_q y_q)$ when restricted to C, which in turn gives $dz_p = h_{pq} a_q^{-1} \omega_q$ or $dz_p = h_{pq} b_q^{-1} \omega_q$ depending on whether $p \in \{x_q = 0\}$ or $p \in \{y_q = 0\}$. In particular, after restricting to C we have $g_{pq} = h_{pq} b_q^{-1} = h_{pq} a_q^{-1}$.

Since $N_\mathcal{G}^* + C \sim_\mathbb{Q} 0$ we see that $\{(U_{pq}, h_{pq} g_{qp})\}$ is a torsion cocycle, and hence for some m, $\{(U_{pq}, (h_{pq} g_{qp})^m)\}$ is a trivial cocycle. Set $C_p = U_p \cap C$ and $C_{pq} = U_{pq} \cap C$, and note that $\{(C_{pq}, (h_{pq} g_{qp})^m)\}$ is still a trivial cocycle.

We may therefore find invertible functions f_p on C_p such that $f_p / f_q = (h_{pq} g_{qp})^m$. Without loss of generality we may assume that $f_p = 1$ for $p \in C \setminus S$. From our previous calculations we see that $f_q = a_q^m = b_q^m$ for $q \in S$ (where we consider a_q, b_q as functions restricted to C). In particular, $1 = f_q / f_q = (a_q(q)/b_q(q))^m = \lambda_q^m$ as required. ∎

Since C is contractible we see that $C^2 < 0$, which implies

$$\sum_i C_i^2 < -\sum_{i,j} C_i \cdot C_j = -2 \# S.$$

On the other hand, Lemma 5.3 gives us

$$\left(\sum C_i\right)^2 = \sum_{p \in Z} \mathrm{CS}\left(p, \mathcal{G}, \sum C_i\right) = \sum_{p \in S} 2 + \lambda_p + \frac{1}{\lambda_p},$$

which in turn yields

$$\sum C_i^2 = \sum \lambda_p + \frac{1}{\lambda_p}.$$

However, each λ_p is a root of unity and so the modulus of $\sum_{p \in S}(\lambda_p + \frac{1}{\lambda_p}) = \sum_{p \in S}(\lambda_p + \overline{\lambda_p})$ is bounded by $2 \# S$. This is the sought after contradiction. ∎

5.2. Holomorphic Godbillon–Vey sequences

We say that a 1-form ω is *integrable* provided $\omega \wedge d\omega = 0$.

Definition 5.6. Let M be a complex manifold of dimension ≥ 2 and let ω be an integrable holomorphic 1-form on M. A holomorphic *Godbillon–Vey sequence* for ω is a sequence (ω_k) of holomorphic 1-forms on M such that $\omega_0 = \omega$ and the formal 1-form

$$\Omega = dt + \sum_{j=0}^\infty \frac{t^j}{j!} \omega_j$$

is integrable.

Lemma 5.7. *Let $(P \in X)$ be an analytic germ of an isolated $\mathbb{Q}$-factorial klt singularity with* $\dim X \geq 3$. *Then*

$$H^1(X \setminus P, \mathcal{O}_{X \setminus P}) = 0.$$

Proof. Notice that since X is klt, it is also a rational singularity. Consider the long exact sequence coming from the exponential exact sequence

$$H^1(X \setminus P, \mathbb{Z}) \xrightarrow{a} H^1(X \setminus P, \mathcal{O}_{X \setminus P}) \xrightarrow{b} H^1(X \setminus P, \mathcal{O}^*_{X \setminus P}).$$

By [18, Lemma 6.2] we know that $\operatorname{im}(a) = 0$, in particular b is injective.

On the other hand, we have an injection

$$H^1(X \setminus P, \mathcal{O}^*_{X \setminus P}) = \operatorname{Pic}(X \setminus P) \to \operatorname{Cl}(P \in X)$$

given by $L \mapsto i_* L$ where $i \colon X \setminus P \to X$ is the inclusion; indeed by [34], $i_* L$ is a coherent reflexive sheaf on X. By assumption $\operatorname{Cl}(P \in X)$ is torsion and so the same is true of $H^1(X \setminus P, \mathcal{O}^*_{X \setminus P})$. Since $H^1(X \setminus P, \mathcal{O}_{X \setminus P})$ is a $\mathbb{C}$-vector space, it is a divisible group, which implies that $\operatorname{im}(b) = 0$. Thus $H^1(X \setminus P, \mathcal{O}_{X \setminus P}) = 0$. ∎

The following result is proven in [14, Lemma 2.1.1].

Lemma 5.8. *Let M be a complex manifold of dimension ≥ 3 and let ω be a holomorphic 1-form on M. Assume that the codimension of $\operatorname{sing}(\omega)$ is at least 3 and $H^1(M, \mathcal{O}_M) = 0$. Then ω admits a holomorphic Godbillon–Vey sequence.*

Corollary 5.9. *Let $(P \in X)$ be a germ of an isolated analytically $\mathbb{Q}$-factorial klt three-fold singularity. Let ω be an integrable 1-form on $X \setminus P$ such that $\operatorname{sing}(\omega)$ has codimension at least 3 in $X \setminus P$. Then ω admits a holomorphic Godbillon–Vey sequence.*

Proof. By Lemma 5.7 we have $H^1(X \setminus P, \mathcal{O}_{X \setminus P}) = 0$, in which case we may apply Lemma 5.8 to conclude the proof. ∎

5.3. A few technical lemmas

Lemma 5.10. *Let $(P \in X)$ be an analytic germ of a $\mathbb{Q}$-factorial and klt singularity with $\dim X \geq 3$. Let $\pi \colon (Q \in Y) \to (P \in X)$ be a quasi-étale morphism of germs. Then $(Q \in Y)$ is $\mathbb{Q}$-factorial.*

Proof. Let $\overline{\pi} \colon \overline{Y} \to X$ be the Galois closure of π. Observe that $\overline{\pi}$ is quasi-étale and if $\overline{Y}$ is $\mathbb{Q}$-factorial then so is Y [25, Lemma 5.16]. Thus we may replace Y by $\overline{Y}$ and assume that π is Galois with Galois group G.

Suppose for the sake of contradiction that Y is not $\mathbb{Q}$-factorial. Since π is quasi-étale, Y is klt and therefore it admits a small $\mathbb{Q}$-factorialization $f \colon Y' \to Y$ such that

- G acts on Y';
- f is G-equivariant; and
- f is not the identity.

Indeed, such a Y' can be found by taking a G-equivariant resolution $\mu \colon W \to X$ and running a G-equivariant $(K_W + (1 - \epsilon) \sum E_i)$-MMP over X where $\sum E_i$ is the union of the μ-exceptional divisors and $\epsilon > 0$ is sufficiently small.

Let $X' = Y'/G$ and observe that we have a birational morphism $g: X' \to X$. Moreover, $g: X' \to X$ is small, which contradicts X being $\mathbb{Q}$-factorial. ∎

Lemma 5.11. *Let $\pi: Y \to X$ be a finite morphism of complex varieties and let $\mathcal{F}$ be a co-rank 1 foliation on X. Then $\mathcal{F}$ admits a holomorphic (resp. meromorphic) first integral if and only if $\pi^{-1}\mathcal{F}$ does.*

Proof. We may assume without loss of generality that $\pi: Y \to X$ is Galois, in which case the claim is easy. ∎

We say that $f \in \mathbb{C}[[x_1, \ldots, x_n]]$ is a *power* if there exists $g \in \mathbb{C}[[x_1, \ldots, x_n]]$ and an integer $m \geq 2$ such that $g^m = f$. Observe that if f is a first integral of ω and $g^m = f$ then g is also a first integral of ω. Let $\widehat{\Delta}$ denote the formal completion of $\mathbb{C}$ at the origin.

Lemma 5.12. *Consider $\mathbb{C}^3 \times \mathbb{C}$ with coordinates (z_1, z_2, z_3, t) and let $\Omega = dt + \sum t^i \omega_i$ be a formal 1-form where $\omega_i \in H^0(U, \Omega_U^1)$ is a holomorphic 1-form on $0 \in U \subset \mathbb{C}^3$. Suppose that Ω is integrable. Let $0 \in D \subset U$ be a normal crossings divisor such that ω_i is zero when restricted to D for all i. Let $\widehat{X}$ be the formal completion of $U \times \mathbb{C}$ along $D \times 0$. Then Ω admits a first integral in $H^0(\widehat{X}, \mathcal{O}_{\widehat{X}})$.*

Remark 5.13. A priori, the formal Frobenius theorem only guarantees that Ω admits a first integral in $H^0(\widehat{\mathbb{C}^4}, \mathcal{O}_{\widehat{\mathbb{C}^4}})$, with $\widehat{\mathbb{C}^4}$ the completion of $\mathbb{C}^4$ at the origin.

Proof. Following a change of coordinates and for ease of notation we will assume that $D = \{z_1 z_2 z_3 = 0\}$ (the cases where D has one or two components are simpler).

Since ω_i vanishes when restricted to D, for $j = 1, 2, 3$ we may write

$$\omega_i = f_j^i dz_j + z_j \theta_j^i$$

where f_j^i and θ_j^i are holomorphic. It follows that we may write $\Omega = dt + F_j dz_j + z_j \Theta_j$ where $F_j(z_1, z_2, z_3, t) \in H^0(X_j, \mathcal{O}_{X_j})$, $\Theta_j = \sum H_i^j(z_1, z_2, z_3, t) dz_i \in H^0(X_j, \Omega_{X_j}^1)$ and X_j is the formal completion of $U \times \mathbb{C}$ along $\{t = z_j = 0\}$.

We may then apply [14, Lemma 3.1.1] (or, more precisely, its proof) to find a first integral $G_j \in H^0(X_j, \mathcal{O}_{X_j})$ of Ω. Moreover, if we write $G_j = \sum_{m,n} t^m z_j^n g_{mn}^j$ where g_{mn}^j is a convergent power series in the variables $\{z_1, z_2, z_3\} \setminus \{z_j\}$ then we may choose G_j so that $g_{00}^j = 0$. In particular, this implies that if $\phi \in \text{Aut}(\widehat{\Delta})$, then $\phi \circ G_j$ is still an element of $H^0(X_j, \mathcal{O}_{X_j})$. Indeed, if we write $\phi \circ G_j = \sum_{mn} t^m z_j^n g'_{mn}$ then $g'_{mn} = P(g_{lp}^j)_{l \leq m, p \leq n}$ where P is some polynomial depending on ϕ, in particular, g'_{mn} is convergent provided all the g_{lp}^j are. Without loss of generality we may also assume that G_j is not a power.

By considering G_1, G_2, G_3 as elements in $H^0(\widehat{\mathbb{C}^4}, \mathcal{O}_{\widehat{\mathbb{C}^4}})$, with $\widehat{\mathbb{C}^4}$ the completion of $\mathbb{C}^4$ at the origin, we may apply [28, Théorème de factorisation] to find $\phi_{ij} \in \text{Aut}(\widehat{\Delta})$ such that $G_i = \phi_{ij} \circ G_j$. Thus, replacing G_j by $\phi_{ij} \circ G_j$ if necessary, we may assume that G_1, G_2, G_3 all give the same element in $H^0(\widehat{\mathbb{C}^4}, \mathcal{O}_{\widehat{\mathbb{C}^4}})$, call it G. However, since $G_i \in H^0(X_i, \mathcal{O}_{X_i})$, this implies that G is in fact an element of $H^0(\widehat{X}, \mathcal{O}_{\widehat{X}})$ and we are done. ∎

Lemma 5.14. *Let X be a normal complex variety, let $D \subset X$ be a compact subvariety and let $\widehat{X}$ be the completion of X along D. Let $\mathcal{F}$ be a co-rank 1 formal foliation on $\widehat{X}$ and suppose that D is tangent to $\mathcal{F}$. Suppose that the following hold:*

(1) *for all $p \in D$ there exists an open neighbourhood $p \in U_p \subset X$ and $F_p \in H^0(\widehat{U_p}, \mathcal{O}_{\widehat{U_p}})$*

   with F_p a first integral of $\mathcal{F}$ and where $\widehat{U_p}$ is the formal completion of U_p along D;

(2) *for any $p, q \in D$ we have $\mathrm{sing}(X) \cap U_p \cap U_q = \emptyset$; and*

(3) *for any $p, q \in D$, if $U_p \cap U_q \neq \emptyset$ then $F_p|_{\widehat{U_p \cap U_q}}$ is not a power.*

Then we may produce a representation $\rho : \pi_1(D) \to \mathrm{Aut}(\widehat{\Delta})$ such that if ρ is trivial then $\mathcal{F}$ admits a first integral $F \in H^0(\widehat{X}, \mathcal{O}_{\widehat{X}})$. Moreover, if the F_p can all be taken to be convergent, then F may be taken to be convergent as well.

Proof. Without loss of generality we may assume that $F_p|_D = 0$ for all p.

If $p \neq q$ with $U_p \cap U_q \neq \emptyset$, choose some $z \in U_p \cap U_q \cap D$. By considering F_p, F_q as elements in the completion $\widehat{\mathcal{O}_{X,z}}$ we may apply [28, Théorème de factorisation] to find a $\phi_{p,q} \in \mathrm{Aut}(\widehat{\Delta})$ such that $F_p = \phi_{p,q} \circ F_q$.

We may then produce a representation of $\pi_1(D)$ along the lines of the classical holonomy representation (see for instance [6, Chapter IV]). Let $\gamma \in \pi_1(D)$ be a path $\gamma : [0, 1] \to D$. We may find a collection of points $p_0, \dots, p_{n-1}, p_n = p_0 \in D$ and a partition $0 = t_0 < t_1 < \cdots < t_n = 1$ of $[0, 1]$ such that $\gamma([t_{i-1}, t_i]) \subset U_{p_{i-1}}$. We may then define our representation by setting $\rho(\gamma) = \phi_{p_n, p_{n-1}} \circ \cdots \circ \phi_{p_1, p_0}$.

If $\rho(\gamma) = 1$, then for $1 \leq j \leq n - 1$ we may replace F_{p_j} by $(\phi_{p_j, p_{j-1}} \circ \cdots \circ \phi_{p_1, p_0})^{-1} \circ F_{p_j}$ and so we may assume that $F_{p_j} = F_{p_0}$ for all j. Thus, if the image of ρ is $\{1\}$ then the F_p glue together to give a section $F \in H^0(\widehat{X}, \mathcal{O}_{\widehat{X}})$.

Our claim about convergence follows by observing that if the F_p are all convergent then the $\phi_{p,q}$ may be taken to be convergent as well. ∎

5.4. Proof of Theorem 5.1

First, by Lemma 5.4 we know that X is klt.

Next, by Lemma 5.10 we may replace X by a quasi-étale cyclic cover and so may assume that $N_{\mathcal{F}}^{[*]}$ is Cartier and hence $\mathcal{F}|_{X \backslash P}$ is defined by an integrable 1-form ω which is non-vanishing on $X \setminus P$. By Corollary 5.9, ω admits a holomorphic Godbillon–Vey sequence (ω_k).

Let L_X be the link of X. By [37, Corollary 1.4], $\pi_1(L_X)$ is finite; let $\widetilde{L} \to L_X$ be the universal cover. We may find a Galois étale morphism of complex spaces $Y' \to X \setminus P$ corresponding to this cover and by [21, Proposition 3.13] this cover extends to a Galois quasi-étale cover $\pi : Y \to X$. So by replacing X by Y we may assume that $\pi_1(L_X) = \{1\}$.

Let $\mu : Y \to X$ be a log resolution of X and let $E = \sum E_k$ be the sum of the μ-exceptional divisors. Let $Y^* := Y \setminus \mu^{-1}(P) \cong X \setminus P$. By [20, Theorem 4.3] we see that $\omega_i|_{X \backslash P}$ extends to a holomorphic 1-form $\widetilde{\omega}_i$ on Y.

There exist maps

$$\pi_1(L_X) \cong \pi_1(Y^*) \xrightarrow{a} \pi_1(Y) \xrightarrow{b} \pi_1(E)$$

where a is a surjective and b is an isomorphism, since Y deformation retracts onto E. This implies that $\pi_1(E)$ is trivial.

Define

$$\Omega = dt + \sum_{k=0}^{\infty} \frac{t^k}{k!} \widetilde{\omega}_k$$

and recall that by definition Ω is a non-singular integrable 1-form defined on $\widehat{Y \times \mathbb{C}}$, the completion of $Y \times \mathbb{C}$ along $E \times 0$, and t is a local coordinate on $\mathbb{C}$. Then Ω defines a smooth foliation $\widehat{\mathcal{G}}$ on $\widehat{Y \times \mathbb{C}}$. By construction, $\widehat{\mathcal{G}}|_{\widehat{Y} \times 0} = \mu^{-1}\mathcal{F}|_{\widehat{Y}}$ where $\widehat{Y}$ is the formal completion of Y along E.

Since $\mathcal{F}$ has non-dicritical singularities, E is $\mu^{-1}\mathcal{F}$ invariant, which implies that $E \times 0$ is tangent to $\widehat{\mathcal{G}}$.

By Corollary 1.15, $\widetilde{\omega}_k$ vanishes when restricted to E. Thus we may apply Lemma 5.12 to Ω to find for all $p \in E$ a neighbourhood $p \in U_p \subset Y$ and a first integral of Ω, denoted $F_p \in H^0(\widehat{U_p \times \mathbb{C}}, \mathcal{O}_{\widehat{U_p \times \mathbb{C}}})$ where $\widehat{U_p \times \mathbb{C}}$ is the completion of $U_p \times \mathbb{C}$ along $E \times 0$. Since $\widehat{\mathcal{G}}$ is smooth, without loss of generality we may assume that F_p is not a power on $\widehat{U_p \times \mathbb{C}} \cap \widehat{U_q \times \mathbb{C}}$ for any p, q.

We may therefore apply Lemma 5.14 and since $\pi_1(E) = \{1\}$ we produce a formal first integral $\hat{F} \in H^0(\widehat{Y \times \mathbb{C}}, \mathcal{O}_{\widehat{Y \times C}})$. Restricting $\hat{F}$ to $\widehat{Y} \times 0$ we see that $\widetilde{\omega}_0$ admits a first integral $\hat{f} \in H^0(\widehat{Y}, \mathcal{O}_{\widehat{Y}})$. We now show that we can take this first integral to be convergent.

Write $\hat{f}^*0 = \sum a_i E_i$. By [24, Chapter II] we may find a dominant proper generically finite morphism $W \xrightarrow{\sigma} Y$ such that the central fibre of $\hat{f} \circ \sigma$ is reduced and σ is ramified only over foliation invariant divisors. Write $\widetilde{E} = \sigma^{-1}(E)$, let $\widehat{W}$ be the completion of W along $\widetilde{E}$ and $\tilde{f} = \hat{f} \circ \sigma$.

From the above construction we see that we may write $\tilde{f} = \bar{f}^r$ such that for all $p \in \widetilde{E}$, $\bar{f}$ is not a power in $\mathcal{O}_{\widehat{W}, p}$. Thus we may apply [28, Théorème A] to find a $\phi_p \in \text{Aut}(\widehat{\Delta})$ such that $\phi_p \circ \bar{f}$ is convergent on a neighbourhood U_p of p. We may apply Lemma 5.14 by taking $F_p = \phi_p \circ \bar{f}$ to produce a representation $\rho \colon \pi_1(W) \to \text{Aut}(\widehat{\Delta})$ which vanishes when $\sigma^{-1}\mu^{-1}\mathcal{F}$ admits a convergent first integral.

By taking the Stein factorization of $W \to X$ we produce a birational morphism $W \to X'$ contracting $\widetilde{E}$ to a point and such that $r \colon X' \to X$ is branched only over the separatrices of $\mathcal{F}$. We claim that X' is klt. Indeed, $K_{r^{-1}\mathcal{F}} = r^*K_{\mathcal{F}}$ and so $r^{-1}\mathcal{F}$ has canonical singularities. Let S be a separatrix of $r^{-1}\mathcal{F}$ at $r^{-1}(P)$ (which exists since $r^{-1}\mathcal{F}$ admits a formal first integral). By [13, Lemma 3.16], (X', S) is log canonical, and since S is $\mathbb{Q}$-Cartier it follows that X' is in fact klt.

Thus, passing to a higher quasi-étale cover if necessary we may assume that $\pi_1(W) = 0$. Hence $\sigma^{-1}\mu^{-1}\mathcal{F}$ admits a convergent first integral. By Lemma 5.11 this implies that $\mu^{-1}\mathcal{F}$, and so $\mathcal{F}$, admits a convergent first integral.

5.5. Classification of terminal foliation singularities

We will need the following, which is a direct generalization of [35, Lemma 9.7].

Corollary 5.15. *Let* $(P \in X)$ *be a normal threefold germ and let* $\mathcal{F}$ *be a terminal co-rank* 1 *foliation. Then* $\mathcal{F}$ *admits a holomorphic first integral. In particular,* K_X *is* $\mathbb{Q}$*-Cartier.*

Remark 5.16. A priori we only know that $K_{\mathcal{F}}$ is $\mathbb{Q}$-Cartier.

Proof of Corollary 5.15. After replacing $(P \in X)$ by a finite cover we may assume that $K_{\mathcal{F}}$ is Cartier. Since $\mathcal{F}$ is terminal and $K_{\mathcal{F}}$ is Cartier this implies that $P \in X$ is in fact an isolated singularity. Moreover, perhaps shrinking about P we may assume that $\mathrm{Cl}(P \in X)$ is generated by the classes of divisors $D_1, \ldots, D_N$ on X.

By Theorem 2.4 we may take

$$\mu \colon (Y, \mathcal{G}) \to (X, \mathcal{F}),$$

an F-dlt modification of $\mathcal{F}$. Since $\mathcal{F}$ is terminal we see that μ is small, i.e., $\mu^{-1}(P)$ is a union of curves. Observe that Y is $\mathbb{Q}$-factorial. In particular, $D_i' := \mu_*^{-1} D_i$ is $\mathbb{Q}$-Cartier and so if $P \in U \subset X$ is a smaller germ then $\mu \colon \mu^{-1}(U) \to U$ is also an F-dlt modification of $\mathcal{F}|_U$. Indeed, it suffices to show that $\mu^{-1}(U)$ is globally $\mathbb{Q}$-factorial. If D is any global divisor on U then $\mu_* D \sim \sum a_i D_i$ by assumption and so $D \sim \sum a_i D_i'$ and hence D is $\mathbb{Q}$-Cartier. Thus we may replace X by a smaller germ about P at any point, should we need to do so.

We claim the following:

Claim 5.17. *For all* $Q \in \mu^{-1}(P) \subset Y$, Y *is analytically* $\mathbb{Q}$*-factorial about* Q.

Claim 5.18. Y *is simply connected.*

Proof of Claim 5.18. Let T be a germ of a $\mathcal{G}$-invariant surface containing $\mu^{-1}(P)$. Since $\mathcal{G}$ is terminal and $\mu^{-1}(P)$ is connected, T is irreducible. Let $S = \mu_* T$; by the proper mapping theorem, S is a divisor on X.

Since $\mathcal{F}$, and hence $\mathcal{G}$, is terminal and Gorenstein (i.e., $K_{\mathcal{F}}$ is Cartier), $(X, \mathcal{F})$ and $(Y, \mathcal{G})$ are both smooth in codimension 2 and so $K_{\mathcal{G}}|_T = K_T$ and $K_{\mathcal{F}}|_S = K_S$; hence $\mu^* K_S = K_T$. By [13, Lemma 3.16], T is a log terminal surface and so $(P \in S)$ is a germ of a log terminal singularity. Thus $\mathrm{exc}(T \to S) = \mathrm{exc}(\mu)$ is a tree of rational curves and therefore $\mu^{-1}(P)$ is simply connected. Notice that Y deformation retracts onto $\mu^{-1}(P)$ and so Y is simply connected. $\blacksquare$

Assuming Claim 5.17 we complete the proof of Corollary 5.15 as follows. Observe that $\mu^{-1}\mathcal{F}$ is terminal and so for all $q \in \mu^{-1}(P)$ by Theorem 5.1 there exists a holomorphic first integral F_q defined on a neighbourhood U_q of q so that $F_q|_T = 0$.

Let $s \colon Y' \to Y$ be an index 1 cover associated to T ramified only over T [25, Definition 2.52, Lemma 2.53], and let $\mu' \colon Y' \to X'$ be the Stein factorization of $Y' \to X$. Notice

that $r: X' \to X$ is ramified only along invariant divisors so $K_{r^{-1}\mathcal{F}} = r^* K_{\mathcal{F}}$, in particular $r^{-1}\mathcal{F}$ is still terminal. Replacing X by X' we may freely assume that T is Cartier.

In particular, for any q, up to taking a root, we may assume that $(F_q = 0) = T \cap U_q$, i.e., $(F_q = 0)$ is reduced. Thus for any q and q' with $U_q \cap U_{q'} \neq \emptyset$, we find that F_q is not a power on $U_q \cap U_{q'}$. Moreover, since Y is smooth in codimension 2, we see that $\mu^{-1}(P) \cap \text{sing}(X)$ consists of a finite collection of points, and so by shrinking the U_q if necessary we may also assume that $U_q \cap U_{q'} \cap \text{sing}(X) = \emptyset$. We may then apply Lemma 5.14 to produce a representation $\rho: \pi_1(Y) \to \text{Aut}(\widehat{\Delta})$. Since $\pi_1(Y)$ is trivial we see that ρ is trivial and so we get a global first integral on Y, which descends to X.

To show that K_X is $\mathbb{Q}$-Cartier, let $\phi: (P \in X) \to (0 \in \mathbb{C})$ be a holomorphic first integral for $\mathcal{F}$ where $0 \in \mathbb{C}$ is a (germ of a) curve. Let $F = \phi^{-1}(0)$ and observe that $K_{\mathcal{F}} = K_{X/\mathbb{C}}(-mF)$ where $K_{X/\mathbb{C}} = K_X - \phi^* K_{\mathbb{C}}$ and where $m + 1$ is the multiplicity of the fibre over 0. By assumption, $K_{\mathcal{F}}$ is $\mathbb{Q}$-Cartier, $\phi^* K_{\mathbb{C}}$ is Cartier since $\mathbb{C}$ is a smooth curve and $F = \frac{1}{m+1} \phi^* 0$ is $\mathbb{Q}$-Cartier and so K_X is $\mathbb{Q}$-Cartier as claimed, thus completing the proof.

Proof of Claim 5.17. Let $Q \in \mu^{-1}(P) \subset Y$. We make the following preliminary observation. Let D be any divisor defined in an (analytic) neighbourhood U of Q and suppose that $D \cap \mu^{-1}(P) = Q$. Then, shrinking X to a smaller neighbourhood of P if necessary, we may extend D to a divisor on all of Y. Indeed, for any $Q' \in \mu^{-1}(P) \setminus (\mu^{-1}(P) \cap U)$ we may find an open set $V_{Q'} \subset Y$ such that $V_{Q'} \cap D = \emptyset$. By compactness of $\mu^{-1}(P) \setminus (\mu^{-1}(P) \cap U)$ we may find $Q_1, \ldots, Q_n$ such that $\mu^{-1}(P) \subset U' := U \cup V_{Q_1} \cup \cdots \cup V_{Q_n}$. By construction, D is an analytic divisor defined on all of U', by setting $D \cap V_{Q_i} = \emptyset$. We may then find an open subset W of P in X such that $\mu^{-1}(W) \subset U'$. Replacing X by W we see that our observation follows.

So, let $Q \in \mu^{-1}(P)$; suppose for the sake of contradiction that Y is not analytically $\mathbb{Q}$-factorial about Q and let D be a local divisor defined on a neighbourhood V of Q which is not $\mathbb{Q}$-Cartier. A priori, $D \cap \mu^{-1}(P)$ may be 1-dimensional and so it is not clear if we can extend D to a divisor on all of Y.

Since Y is klt, there exists a small $\mathbb{Q}$-factorialization about Q. Let $f: Z \to (Q \in Y)$ be this $\mathbb{Q}$-factorialization, let D' be the strict transform of D and let $f^{-1}(Q) = \bigcup_i C_i$ be a decomposition into irreducible components.

Observe that for all i we may find an irreducible effective Cartier divisor S_i defined on Z such that $S_i \cdot C_j = \delta_{ij}$ and $S_i \cap f^{-1}(Q)$ is a single point.

By choosing $a_i \in \mathbb{Q}$ appropriately we may assume that $D' + \sum a_i S_i$ is numerically trivial over Y. Since f is small we see that $(D' + \sum a_i S_i) - K_Z$ is nef and big over Y and therefore by the relative basepoint free theorem [25, Theorem 3.24] for $n > 0$ sufficiently divisible we have $n(D' + \sum a_i S_i) \sim_f 0$. In particular, if we let $T_i = f_* S_i$ we see that $D + \sum a_i T_i$ is $\mathbb{Q}$-Cartier near Q.

Since $T_i \cap \mu^{-1}(P)$ is a point, by our observation at the beginning of this proof we may extend T_i to a divisor on all of Y, in particular T_i is $\mathbb{Q}$-Cartier. This in turn implies that D is in fact $\mathbb{Q}$-Cartier, proving our claim. ∎

We can now provide a classification of terminal foliation singularities.

Proposition 5.19. *Let $(P \in X)$ be a normal threefold germ and let $\mathcal{F}$ be a co-rank 1 foliation on $(P \in X)$. Suppose K_X and $K_{\mathcal{F}}$ are Cartier and $\mathcal{F}$ is terminal. Then $\mathcal{F}$ is given by the smoothing of a Du Val surface singularity, i.e., $\mathcal{F}$ admits a first integral $\phi\colon (P \in X) \to (0 \in \mathbb{C})$ where $\phi^{-1}(0)$ is a Du Val surface singularity and $\phi^{-1}(t)$ is smooth for $t \neq 0$. In particular, X is terminal.*

Moreover, one can write down a list of all such smoothings. In an appropriate choice of coordinates we have

$$X = \{\psi(x, y, z) + tg(x, y, z, t) = 0\}$$

and $\mathcal{F}$ is defined by the 1-form dt, i.e., the first integral is just $(x, y, z, t) \mapsto t$ and $\psi(x, y, z)$ is one of the following [25, *Theorem* 4.20]:

(1) $\psi(x, y, z) = x^2 + y^2 + z^{n+1}$ *with* $n \geq 0$;

(2) $\psi(x, y, z) = x^2 + zy^2 + z^{n-1}$ *with* $n \geq 4$;

(3) $\psi(x, y, z) = x^2 + y^3 + z^4$;

(4) $\psi(x, y, z) = x^2 + y^3 + yz^3$;

(5) $\psi(x, y, z) = x^2 + y^3 + z^5$;

(6) $\psi(x, y, z) = x$.

Conversely, if $g(x, y, z, t)$ is such that X has at worst an isolated singularity at P and $\mathcal{F}$ is defined by dt then $\mathcal{F}$ has a terminal singularity at P.

Theorem 5.20. *Let $(P \in X)$ be a threefold germ, let $\mathcal{F}$ be a co-rank 1 foliation on X and suppose that $\mathcal{F}$ is terminal. Then $P \in X$ is a quotient of one of the foliations* (1)–(6) *in the above list by $G = \mathbb{Z}/m \times \mathbb{Z}/n$.*

Proof. By Corollary 5.15 we see that $K_{\mathcal{F}}$ and K_X are both $\mathbb{Q}$-Cartier so we may find a Galois cover $\pi\colon (X', \mathcal{F}') \to (X, \mathcal{F})$ with Galois group $\mathbb{Z}/n \times \mathbb{Z}/m$ such that $K_{\mathcal{F}'}$ and $K_{X'}$ are both Cartier. Indeed, Let $X_1 \to X$ and $X_2 \to X$ be the index 1 covers associated to $K_{\mathcal{F}}$ and K_X, with Galois groups $\mathbb{Z}/m_1$ and $\mathbb{Z}/m_2$ respectively. If X' is the normalization of a component of $X_1 \times_X X_2$ dominating X then $X' \to X$ is Galois and its Galois group is a subgroup of $\mathbb{Z}/m_1 \times \mathbb{Z}/m_2$ as required.

By Proposition 5.19, $(X', \mathcal{F}')$ is one of the foliations (1)–(6) and we can conclude the proof. ∎

Corollary 5.21. *Let $(P \in X)$ be a germ of a normal threefold, let $\mathcal{F}$ be a co-rank 1 foliation on X and suppose that $\mathcal{F}$ is terminal. Then X and $\mathcal{F}$ admit a $\mathbb{Q}$-smoothing, i.e., there exists a family of foliated threefold germs X_t and $\mathcal{F}_t$ such that $(X_0, \mathcal{F}_0) = (X, \mathcal{F})$ and for $t \neq 0$, $(X_t, \mathcal{F}_t)$ is a quotient of a smooth foliation on a smooth variety.*

Proof. This is a direct consequence of the classification in Proposition 5.19. Indeed, in each case we may explicitly construct a smoothing of X and $\mathcal{F}$ by perturbing the defining equations of X and $\mathcal{F}$. ∎

5.6. *Structure of terminal flips*

We finish by providing a rough structural statement for terminal foliated flips.

Theorem 5.22. *Let X be a $\mathbb{Q}$-factorial threefold and let $\mathcal{F}$ be a co-rank 1 foliation on X with terminal singularities. Let $\phi: X \to Z$ be a $K_{\mathcal{F}}$-flipping contraction and let $C = \mathrm{Exc}(\phi)$. Then there exists an analytic open neighbourhood $C \subset U$ and a holomorphic first integral $F: U \to \mathbb{C}$ of $\mathcal{F}$.*

Proof. By Theorem 4.3 we have $R^1 f_* \mathcal{O}_X = 0$, and so C is in fact a tree of rational curves, in particular it is simply connected. For all $p \in C$, by Theorem 5.1, we may find a holomorphic first integral of $\mathcal{F}$ near p. However, since C is simply connected, by arguing as in the proof of Corollary 5.15 we may produce a first integral in a neighbourhood of C. ∎

6. Existence of separatrices for log canonical foliation singularities

The goal of this section is to prove the following.

Theorem 6.1. *Let $(P \in X)$ be an isolated klt singularity. Let $\mathcal{F}$ be a germ of a log canonical co-rank 1 foliation singularity on $(P \in X)$. Then $\mathcal{F}$ admits a separatrix.*

Recall that log canonical foliation singularities which are not canonical are always dicritical, and in general dicritical singularities do not admit separatrices, as the following classical example due to Jouanolou shows.

Example 6.2. The foliation on $(0 \in \mathbb{C}^3)$ defined by

$$(x^m z - y^{m+1})dx + (y^m x - z^{m+1})dy + (z^m y - x^{m+1})dz$$

has no separatrices at the origin for $m \geq 2$. The blow up of this foliation at 0 has discrepancy $-m$, and therefore is not log canonical for $m \geq 2$.

As the next example shows, a log canonical singularity may not admit a separatrix if no assumption is made on the base space.

Example 6.3. Let A be an abelian surface that admits an automorphism τ such that $X := A/\langle \tau \rangle$ is a rational surface and $A \to X$ is étale in codimension 1. We may find a linear foliation on A which admits no algebraic leaves and is τ-invariant and so descends to a foliation $\mathcal{F}$ without algebraic leaves on X.

Let $(P \in Y)$ be the cone over X with vertex P and let $\mathcal{G}$ be the cone over $\mathcal{F}$. It is easy to check that $\mathcal{G}$ is log canonical and admits no separatrices at P. However, $(P \in Y)$ is log canonical and not klt.

We also have the following interesting corollary.

Corollary 6.4. *Let $\mathcal{F}$ be a germ of a foliation on $(0 \in \mathbb{C}^3)$ and let $i\colon (0 \in S) \to (0 \in \mathbb{C}^3)$ be a germ of a surface transverse to $\mathcal{F}$ and such that $i^{-1}\mathcal{F}$ is log canonical, e.g., a radial singularity. Then $\mathcal{F}$ admits a separatrix.*

Proof. This follows by combining Theorems 6.1 and 3.12. ∎

We now proceed with the proof of Theorem 6.1. We will first need the following generalization of Lemma 1.18.

Lemma 6.5. *Let X be a complex threefold with a co-rank 1 foliation $\mathcal{F}$ with non-dicritical singularities. Let $D \subset X$ be a compact subvariety and let $V \subset D$ be a closed proper subvariety of D tangent to $\mathcal{F}$ with the following property:*

($\star$) For all $p \in V$, if S_p is a separatrix of $\mathcal{F}$ at p then $S_p \cap D \subset V$.

Let $q \in V$, let U_q be a neighbourhood of q and let $S_q \subset U_q$ be a separatrix at q. Then there exists an analytic open neighbourhood U of D and an invariant subvariety $S \subset U$ such that $S \cap U_q = S_q$.

Proof. Let $\pi : \overline{X} \to X$ be a resolution of singularities of X and $\mathcal{F}$ and such that $\pi^{-1}(V)$ is an invariant divisor.

Observe that condition ($\star$) still holds for $\pi^{-1}(V)$ and $\pi^{-1}(D)$. Moreover, if $q \in V$ is some point and $\pi^{-1}(S_q)$ admits an extension $\overline{S}$ to a neighbourhood $\overline{U}$ of $\pi^{-1}(D)$ then since π is proper, $\pi(\overline{S}) \subset \pi(\overline{U})$ is an extension of S_q to a neighbourhood of D.

Thus, without loss of generality we may assume that X is smooth, $\mathcal{F}$ has simple singularities and V is a divisor.

Let $q \in V$ and let S_q be any separatrix at q. By Lemma 1.18 we may find a neighbourhood U' of V and an invariant divisor S' which agrees with S_q near q. Let $D' = D \cap U'$. By ($\star$) we see that $S' \cap (D' \setminus V) = \emptyset$. Thus, shrinking U' if necessary, for all $p \in D \setminus V$ there exists a neighbourhood U_p of p such that $U_p \cap S' = \emptyset$.

Taking $U = U' \cup \bigcup_{p \in D \setminus V} U_p$ we see that S' extends to a subvariety of U and we are done. ∎

We recall the following classification result due to [30].

Theorem 6.6. *Let X be a normal projective surface and let $\mathcal{L}$ be a rank 1 foliation on X with canonical foliation singularities. Suppose $c_1(K_{\mathcal{L}}) = 0$. Then there exists a birational morphism $\mu\colon X \to X'$ contracting only rational curves tangent to $\mathcal{L}$, and a cyclic cover $\tau\colon Y \to X'$, étale in codimension 1, such that one of the following holds where $\mathcal{G} = \tau^{-1}\mu_*\mathcal{L}$:*

(1) *μ is an isomorphism, $X = C \times E / G$ where $g(E) = 1$, C is a smooth projective curve, G is a finite group acting on $C \times E$ and $\mathcal{G}$ is the foliation induced by the $\mathcal{G}$-invariant fibration $C \times E \to C$;*

(2) *μ is an isomorphism and $\mathcal{G}$ is a linear foliation on the abelian surface Y;*

(3) *μ is an isomorphism, Y is a $\mathbb{P}^1$-bundle over an elliptic curve and $\mathcal{G}$ is transverse to the bundle structure and leaves at least one section invariant;*

(4) *up to blowing up Y at $P \in \operatorname{sing}(\mathcal{L})$, Y is a compactification of $\mathbb{G}_m \times \mathbb{G}_a$ and $\mathcal{L}$ restricted to this open subset is generated by a $\mathbb{G}_m \times \mathbb{G}_a$-invariant vector field;*

(5) *up to blowing up Y at $P \in \operatorname{sing}(\mathcal{L})$, Y is a compactification of $\mathbb{G}_m \times \mathbb{G}_m$ and $\mathcal{L}$ restricted to this open subset is generated by a $\mathbb{G}_m \times \mathbb{G}_m$-invariant vector field.*

Proof. This follows directly from [30, Theorem IV.3.6] except for the claim in items (1)–(3) that μ is an isomorphism. In each of these cases $\mu_* \mathcal{L}$ is terminal, because for all $P \in X'$ there exists a cyclic cover (namely τ) such that $\tau^{-1} \mu_* \mathcal{L}$ is smooth in a neighbourhood of $\tau^{-1}(P)$ and so we may apply Proposition 1.7.

Since $\mu_* \mathcal{L}$ is terminal and $c_1(K_{\mathcal{L}}) = 0$ this implies that μ is an isomorphism. ∎

Lemma 6.7. *Let S be a surface and let $\mathcal{L}$ be a co-rank 1 foliation on S. Suppose that $c_1(K_{\mathcal{L}}) = 0$ and $\mathcal{L}$ has canonical singularities.*

(1) *For all $p \in \operatorname{sing}(\mathcal{L})$ each separatrix at p is algebraic. In particular, the union of all such separatrices is an algebraic subvariety of S.*

(2) *Either there exists a quasi-étale cover $\tau : A \to S$ where A is an abelian variety, or there exists an algebraic curve $V \subset S$ such that each component of V is $\mathcal{L}$-invariant and if $p \in \operatorname{sing}(\mathcal{L}) \cap V$ then each separatrix at p is contained in V.*

Proof. To prove item (1) observe that in order to check if each separatrix at a singular point is algebraic we may contract curves tangent to the foliation, as well as replace S by a finite cover. Thus, it suffices to check the claim for each of the five types of foliation listed in the statement of Theorem 6.6.

In cases (1)–(3) the foliation is smooth and so there is nothing to prove. Thus it remains to consider cases (4) and (5).

We see that the vector field generating $\mathcal{L}$ on $\mathbb{G}_m \times \mathbb{G}_a$ or $\mathbb{G}_m \times \mathbb{G}_m$, respectively, is smooth. Hence $\operatorname{sing}(\mathcal{L})$ is contained in the boundary of the compactification. Moreover, since $\mathcal{L}$ is invariant under the action of $\mathbb{G}_m \times \mathbb{G}_a$ or $\mathbb{G}_m \times \mathbb{G}_m$, every separatrix of $p \in \operatorname{sing}(\mathcal{L})$ must be contained in the boundary.

To prove item (2) of the lemma, again we may freely contract curves tangent to $\mathcal{L}$ and replace by a finite cover. Thus we may assume that $(S, \mathcal{L})$ is one of the foliations listed in Theorem 6.6. We argue depending on the case.

If we are in case (5) or (4) then $\operatorname{sing}(\mathcal{L})$ is non-empty and so by item (1) proven above we may take V to be the union of all separatrices passing through $\operatorname{sing}(\mathcal{L})$.

If we are in case (1) then $\mathcal{L}$ is algebraically integrable and we may take V to be the closure of a general leaf.

If we are in case (3) let Σ be the invariant section. We claim that $\mathcal{L}$ is smooth along Σ. Indeed, on the one hand $K_{\mathcal{L}} \cdot \Sigma = K_{\Sigma} + \Delta$ where $\Delta \geq 0$ is supported on $\operatorname{sing}(\mathcal{L}) \cap \Sigma$. On the other hand, by assumption $K_{\mathcal{L}} \cdot \Sigma = 0$ and since Σ is an elliptic curve we have $K_{\Sigma} = 0$ and so $\Delta = 0$. This gives $\operatorname{sing}(\mathcal{L}) \cap \Sigma = \emptyset$ and so we may take $V = \Sigma$.

Otherwise S is an abelian variety and there is nothing more to prove. ∎

Lemma 6.8. *Let* $(P \in X)$ *be a germ of a normal threefold and let* $\mathcal{F}$ *be a co-rank* 1 *foliation on* X. *Suppose that* $\mathcal{F}$ *is log canonical but not canonical. Then there exists a birational morphism* $\pi \colon Y \to X$ *and an irreducible* π-*exceptional divisor* E_0 *such that*

(1) E_0 *is a* π-*exceptional divisor transverse to* $\mathcal{G} := \pi^{-1}\mathcal{F}$;

(2) $\pi^{-1}(P) \subset E_0$;

(3) $\mathcal{G}$ *has non-dicritical singularities;*

(4) $K_{\mathcal{G}} + E = \pi^* K_{\mathcal{F}}$ *where* $E = \sum_i \epsilon(E_i) E_i$ *where we sum over all* π-*exceptional divisors;*

(5) $(\mathcal{G}, E)$ *is log canonical and* $(\mathcal{G}, (1 - \epsilon)E)$ *is F-dlt for all* $1 > \epsilon > 0$; *and*

(6) Y *is* $\mathbb{Q}$-*factorial and klt.*

Proof. Let $\mu \colon (\overline{X}, \overline{\mathcal{F}}) \to (X, \mathcal{F})$ be an F-dlt modification of $(X, \mathcal{F})$ and write $K_{\overline{\mathcal{F}}} + \sum \epsilon(E_i') E_i' = \mu^* K_{\mathcal{F}}$ where the E_i' are the μ-exceptional divisors. Observe that $\overline{X}$ is $\mathbb{Q}$-factorial and klt and $\overline{\mathcal{F}}$ has non-dicritical singularities.

Since $\mathcal{F}$ is not canonical, μ extracts some divisor transverse to the foliation. We may therefore assume, after relabelling, that $\epsilon(E_0') = 1$ and $E_0' \cap \mu^{-1}(P) \neq \emptyset$. For $0 < \delta \ll 1$ we know that $(\overline{\mathcal{F}}, \sum \epsilon(E_i') E_i' - \delta E_0' := \Theta)$ is F-dlt and so by Corollary 2.3 we may run a $(K_{\overline{\mathcal{F}}} + \sum \epsilon(E_i') E_i' - \delta E_0')$-MMP over X, say $\phi \colon \overline{X} \dashrightarrow Y$, and let $\pi \colon (Y, \mathcal{G}) \to (X, \mathcal{F})$ be the induced map.

Since the MMP preserves $\mathbb{Q}$-factoriality and klt singularities and the output of the MMP has non-dicritical singularities, we see that items (6) and (3) are satisfied. Item (4) follows by construction and item (5) follows since the MMP preserves F-dlt singularities.

Since

$$K_{\overline{\mathcal{F}}} + \sum \epsilon(E_i') E_i' - \delta E_0' \equiv_\mu -\delta E_0',$$

each ray R contracted by this MMP has positive intersection with the strict transform of E_0', in particular E_0' is not contracted by this MMP. Set $E_0 = \phi_* E_0'$. Since E_0' is transverse to the foliation, so is E_0, proving item (1). Moreover,

$$\left(K_{\mathcal{G}} + \phi_* \Theta\right) - \left(K_{\mathcal{G}} + \phi_* \sum \epsilon(E_i') E_i'\right) = -\delta \phi_* E_0' = -\delta E_0$$

is nef over X. By the Negativity Lemma [38, Lemma 1.3], for all $x \in X$ either $\pi^{-1}(x)$ is disjoint from E_0, or $\pi^{-1}(x)$ is contained in E_0. By our choice of E_0 we have $E_0 \cap \pi^{-1}(P) \neq \emptyset$, which proves item (2). ∎

Lemma 6.9. *Let* $(P \in X)$ *be a germ of a klt singularity with a co-rank* 1 *foliation* $\mathcal{F}$ *with log canonical but not canonical singularities. Let* $\pi : (Y, \mathcal{G}) \to (X, \mathcal{F})$ *be a birational morphism as in Lemma* 6.8 *above. Suppose that* $\dim(\pi^{-1}(P)) = 2$ *and* $\pi^{-1}(P)$ *is the only* π-*exceptional divisor transverse to* $\mathcal{G} := \pi^{-1}\mathcal{F}$. *Then there is a separatrix at* P.

Proof. Let E_0 be a divisor as in Lemma 6.8 containing $\pi^{-1}(P)$. Since E_0 is irreducible this implies that $\pi^{-1}(P) = E_0$.

We will find a closed subset $V \subset E_0$ satisfying the hypotheses of Lemma 6.5 in order to produce a $\mathcal{G}$-invariant divisor in a neighbourhood of E_0 whose push forward will be the desired separatrix.

Let $\{E_i\}$ denote the collection of π-exceptional divisors such that $K_{\mathcal{G}} + E_0 = \pi^* K_{\mathcal{F}}$ and $(\mathcal{G}, E_0)$ is log canonical and E_i is $\mathcal{G}$-invariant for $i \neq 0$. Note that since E_i is invariant, $C_i := \pi(E_i)$ is a curve tangent to $\mathcal{F}$ passing through P.

Since $\mathcal{G}$ is non-dicritical and E_0 is the only π-exceptional divisor which is not $\mathcal{G}$-invariant it follows that $\mathcal{F}$ restricted to $X \setminus P$ is non-dicritical.

By foliation adjunction (Lemma 3.10), we know that

$$0 \sim_{\mathbb{Q}} n^*(K_{\mathcal{G}} + E_0) = K_{\mathcal{G}_0} + \Delta_0$$

where $\Delta_0 \geq 0$ and $n \colon E_0^n \to E_0$ is the normalization.

Next, since X is klt we may write $K_Y + E_0 + B = \pi^* K_X + a E_0$ where $a > 0$ and B is not necessarily effective, but is supported on the $\mathcal{G}$-invariant π-exceptional divisors. Write $n^*(K_Y + E_0 + B) = K_{E_0^n} + \Theta_0$.

We claim that $-E_0|_{E_0}$ is big. Let A be an ample divisor on Y. We may find a divisor $D \geq 0$ on X such that $D + \pi_* A$ is $\mathbb{Q}$-Cartier. We may then write $\pi^*(D + \pi_* A) = t E_0 + A + D'$ where $t > 0$, $D' \geq 0$, $\pi_* D' = D$ and E_0 is not contained in the support of D'. Since $\pi^*(D + \pi_* A)|_{E_0} \sim 0$ we have $-E_0|_{E_0} \sim \frac{1}{t}(A + D')|_{E_0}$, which is big, as required. It follows that $-(K_{E_0} + \Theta_0)$ is big. Observe that Θ_0 is not necessarily effective, but if we write $\Theta_0 = \Theta_0^+ - \Theta_0^-$ where $\Theta_0^+, \Theta_0^- \geq 0$ then Θ_0^- is $\mathcal{G}_0$-invariant since it is supported on $n^{-1}(B \cap E_0)$.

First we handle the case $\Delta_0 \neq 0$. In this case $K_{\mathcal{G}_0}$ is not psef, hence $\mathcal{G}_0$ is algebraically integrable, by [3, Main Theorem]. Take V to be the closure of a general leaf of $\mathcal{G}_0$. Observe that $\mathcal{G}_0$ is non-dicritical since $\pi^{-1}\mathcal{F}$ is, and so V is disjoint from the closure of any other leaf of $\mathcal{G}_0$. Moreover, in this case E_0^n is a $\mathbb{P}^1$-fibration over a curve and V is a general fibre in this fibration. In particular, notice that $K_{E_0^n} \cdot V = -2$.

We claim that $n^{-1}(n(V)) = V$. Indeed, if not then E_0 would not be normal in a neighbourhood of some point of $n(V)$. Let $W \subset \text{sing}(E_0)$ be a 1-dimensional component meeting $n(V)$. Observe that since V is general W is transverse to the foliation. Since $(\mathcal{G}, (1 - \epsilon) E_0)$ is F-dlt for all $1 > \epsilon > 0$ it follows from [35, Lemma 3.11] that $(Y, (1 - \epsilon) E_0)$ is dlt at the generic point of W. It follows by [25, Corollary 5.55] that in a neighbourhood of a general point of W, E_0 consists of two smooth components meeting transversely.

Since V is general, it follows that in an (analytic) neighbourhood of V, $n^{-1}(W)$ consists of two components transverse to $\mathcal{G}_0$. A straightforward calculation shows that the coefficient of each of these components in Δ_0 and in Θ_0 is 1. Notice that $V \cdot \Theta_0^- = 0$ and so $(K_{E_0^n} + \Theta_0) \cdot V \geq 0$. However, V is a movable curve and this contradicts $-(K_{E_0^n} + \Theta_0)$ being big.

Thus for all $q \in n(V)$, if S_q is a separatrix of $\pi^{-1}\mathcal{F}$ at q then $S_q \cap E_0 \subset n(V)$ and so we may apply Lemma 6.5 to produce an extension T of S_q to a neighbourhood U of E_0. Shrinking U if necessary, we may assume that $U = \pi^{-1}(W)$ for some neighbourhood W

of P. Also, since V was chosen to be general we may assume that T is not contained in the union of the π-exceptional divisors. Since $U \to W$ is proper we see that $S = \pi_* T \subset V$ is a divisor and is invariant under $\mathcal{F}$, and hence is the desired separatrix.

Now we handle the case $\Delta_0 = 0$. First observe that $\Delta_0 = 0$ implies that E_0 is normal. By foliation adjunction (Lemma 3.10), $\mathcal{G}_0$ is log canonical, and since $\mathcal{G}_0$ is non-dicritical we see that $\mathcal{G}_0$ is in fact canonical.

First, suppose that there exists a quasi-étale cover $r : Y \to E_0$ such that Y is an abelian variety. We claim that $\mathcal{G}_0$ is then algebraically integrable (in which case we are done by arguing as above). If $\Theta_0^- \neq 0$ then $\mathcal{G}_0$, and hence $r^{-1}\mathcal{G}_0$, admits an invariant algebraic curve and by Theorem 6.6 we see that $\mathcal{G}_0$ is algebraically integrable. So suppose for the sake of contradiction that $\Theta_0^- = 0$. In this case, $-K_{E_0}$ is big and so $-K_Y = -r^* K_{E_0}$ is big, contradicting $K_Y \sim 0$.

Next, suppose that there is no such cover. We may apply Lemma 6.7 to produce $V \subset E_0$ such that each component of V is tangent to $\mathcal{G}_0$ and each separatrix of $\mathcal{G}_0$ meeting V is contained in V. Thus, we may apply Lemma 6.5 to produce an invariant divisor T in a neighbourhood of E_0 and which contains V. We claim that T is not contained in the union of the π-exceptional divisors. Supposing the claim we see that $S = \pi_* T$ is the desired separatrix.

We now prove the claim. First if E_0 is the only exceptional divisor there is nothing to show. So suppose that there is some other π-exceptional divisor E_i. If $Q \in C_i \setminus P$ is a general point then there exists a separatrix of $\mathcal{F}$ at Q, say S_Q (recall $C_i = \pi(E_i)$ is tangent to $\mathcal{F}$). To prove the existence of S_Q first note that in a neighbourhood of Q the variety X has quotient singularities (since klt singularities are quotient singularities outside a subset of codimension ≥ 3) and so (up to replacing X by a cover) we may assume that X is smooth at Q. Since $\mathcal{F}$ is non-dicritical in a neighbourhood of Q we may apply [8, Existence of Separatrix Theorem] to produce S_Q.

Let $S_Q' := \pi_*^{-1} S_Q$. By Lemma 1.18 we may extend S_Q' to an invariant divisor in an (analytic) open neighbourhood of $\sum_{i=1}^m E_i$. Call this extension H; by construction, H is not contained in $\sum_{i=0}^m E_i$. Let $\Sigma = H \cap \sum_{i=1}^m E_i$; it is a closed analytic subset of $\sum E_i$. Let $\Sigma_0 = \Sigma \cap E_0$ and let $x \in \Sigma_0$. We know that $\sum_{i=1}^m E_i \cap E_0 \subset V$ by construction. However, $H \cap E_0$ is a separatrix of $\mathcal{G}_0$ at x intersecting V: in fact, $H \cap E_0 \subset V$. Then, in a neighbourhood of E_0, we have $H \subset T$, in particular T is not contained in the union of the π-exceptional divisors. ∎

We are now ready to prove the main theorem of this section.

Proof of Theorem 6.1. Suppose first that $\mathcal{F}$ has canonical singularities. If X is $\mathbb{Q}$-factorial then we may apply Corollary 5.2 to produce a separatrix. Otherwise, since X is klt, it admits a small $\mathbb{Q}$-factorialization $\mu : X' \to X$. Since $\mathcal{F}$ is non-dicritical $\mu^{-1}(P)$ is tangent to the foliation and is therefore contained in a germ of an invariant surface S. We may then take $\mu_* S$ as the desired separatrix.

So we may assume that $\mathcal{F}$ is not canonical and let $\pi : (Y, \mathcal{G}) \to (X, \mathcal{F})$ be a modification as in Lemma 6.8; let E_0 be a divisor as in the statement of the lemma.

There are two cases: $\pi^{-1}(P)$ can be of dimension 2 or 1. Notice moreover that if there exists some π-exceptional divisor E transverse to $\mathcal{G}$ such that E is centred over a curve in X then by choosing $E = E_0$ in the proof of Lemma 6.8 we see that $\pi^{-1}(P)$ is of dimension 1.

If $\pi^{-1}(P)$ is of dimension 2 we may assume that the only π-exceptional divisor transverse to $\mathcal{G}$ is $\pi^{-1}(P)$. We may then apply Lemma 6.9 to conclude the proof.

Otherwise $C := \pi^{-1}(P) \subset E_0$ is a curve. Let $\mathcal{G}_0$ be the induced foliation on E_0. Suppose first that some component $C_0 \subset C$ is transverse to $\mathcal{G}_0$. Then we may apply Lemma 6.5 with $D = \pi^{-1}(P)$ and V a general point in C_0 to produce an invariant divisor S in a neighbourhood of $\pi^{-1}(P)$. In this case $\pi_* S$ will be the desired separatrix.

Now suppose that each component of C is invariant by $\mathcal{G}_0$. In this case, shrinking X if necessary, we may assume that the union of all convergent separatrices meeting C is an analytic subset of E_0, say $\widetilde{C}$. In this case we may apply Lemma 6.5 with $D = E_0$ and $V = \widetilde{C}$ to produce a separatrix S in a neighbourhood of E_0. Again, $\pi_* S$ is the desired separatrix. ∎

Remark 6.10. In fact, the arguments above prove a slightly stronger claim which may be of interest. In the set up as above, if we let $C \subset \mathrm{sing}(\mathcal{F})$ be a curve of singularities passing through P then C is contained in a separatrix.

7. Foliations and hyperbolicity

The goal of this section is to prove the following foliated version of [36, Theorem 1.1]. Given a foliated pair $(\mathcal{F}, \Delta)$ and an lc centre S we will denote by $\bar{S} \subset S$ the locally closed subvariety obtained by removing from S the lc centres of $(\mathcal{F}, \Delta)$ strictly contained in S.

Theorem 7.1. *Let $(\mathcal{F}, \Delta)$ be a foliated log canonical pair on a normal projective variety X. Assume that*

- *X is potentially klt;*
- *there is no non-constant morphism $f : \mathbb{A}^1 \to X \setminus \mathrm{Nklt}(\mathcal{F}, \Delta)$ tangent to $\mathcal{F}$; and*
- *for any stratum S of $\mathrm{Nklt}(\mathcal{F}, \Delta)$ there is no non-constant morphism $f : \mathbb{A}^1 \to \bar{S}$ which is tangent to $\mathcal{F}$.*

Then $K_{\mathcal{F}} + \Delta$ is nef.

The notions of potentially klt and potentially log canonical have been defined in Definition 1.2.

7.1. A special version of dlt modifications

We prove a refinement of Theorem 2.4, which will be useful in the proof of the main result of this section.

Theorem 7.2 (Existence of special F-dlt modifications). *Let $\mathcal{F}$ be a co-rank 1 foliation on a normal projective variety X of dimension at most 3. Let $(\mathcal{F}, \Delta = \sum a_i D_i)$ be a foliated pair. Set $\Delta' := \sum_{a_i < \epsilon(D_i)} a_i D_i + \sum_{a_j \geq \epsilon(D_j)} \epsilon(D_j) D_j$. Then there exists a birational morphism $\pi \colon Y \to X$ which extracts divisors E of foliation discrepancy $\leq -\epsilon(E)$ such that if we write $K_{\mathcal{G}} + \Gamma = \pi^*(K_{\mathcal{F}} + \Delta)$ then $(\mathcal{G}, \Gamma' := \pi_*^{-1}\Delta' + \sum_{E_i\ \pi\text{-exc.}} \epsilon(E_i) E_i)$ is F-dlt.*

Furthermore, we may choose $(Y, \mathcal{G})$ so that

(1) *if W is a non-klt centre of $(\mathcal{G}, \Gamma)$ then W is contained in a codimension 1 lc centre of $(\mathcal{G}, \Gamma')$;*

(2) *Y is $\mathbb{Q}$-factorial;*

(3) *Y is klt; and*

(4) *$\pi^{-1}\mathrm{Nklt}(\mathcal{F}, \Delta) = \mathrm{Nklt}(\mathcal{G}, \Gamma) = \mathrm{Nklt}(\mathcal{G}, \Gamma')$.*

Proof. For the proof of (1)–(3) we refer to [13, Theorem 8.1]. Let $\pi_Z \colon Z \to X$ be a modification of $(\mathcal{F}, \Delta)$ satisfying these three properties. Let $(\mathcal{H}, \Theta, \Theta')$ be the triple given by the birational transform of $\mathcal{F}$ on Z, and

$$K_{\mathcal{H}} + \Theta = \pi_Z^*(K_{\mathcal{F}} + \Delta), \quad \Theta' := \pi_{Z*}^{-1}\Delta' + \sum_{F_i\ \pi_Z\text{-exc.}} \epsilon(F_i) F_i.$$

With these definitions,

$$K_{\mathcal{H}} + \Theta'^{<1} \sim_{\mathbb{R}, X} -\Theta'', \quad \text{where} \quad \Theta'' := \Theta - \Theta'^{<1}.$$

As $K_{\mathcal{H}} + \Theta'^{<1}$ is big over X, there exists A ample over X and an effective divisor G such that

$$K_{\mathcal{H}} + \Theta'^{<1} \sim_{\mathbb{R}, X} A + G.$$

We can decompose G as

$$G = G_1 + G_2 + G_3,$$

where G_1 is the part of G supported on π_Z-exceptional divisors or $\mathcal{H}$-invariant divisors, G_2 is the part of G supported on those components that are not $\mathcal{H}$-invariant but contain an $\mathcal{H}$-invariant lc centre for $(\mathcal{H}, \Theta)$, and $G_3 := G - G_1 - G_2$. For any $0 < \epsilon \ll 1$ we can write

$$-\Theta'' \sim_{\mathbb{R}, X} K_{\mathcal{H}} + \Theta'^{<1} = (1-\epsilon)(K_{\mathcal{H}} + \Theta'^{<1}) + \epsilon(K_{\mathcal{H}} + \Theta'^{<1})$$

$$\sim_{\mathbb{R}, X} (1-\epsilon)\left(K_{\mathcal{H}} + \Theta'^{<1} + \frac{\epsilon}{1-\epsilon}(A + G)\right)$$

$$\sim_{\mathbb{R}, X} (1-\epsilon)\left(K_{\mathcal{H}} + \Theta'^{<1} + \frac{\epsilon}{1-\epsilon}(A + G_1 + G_2 + G_3)\right),$$

so that

$$K_{\mathcal{H}} + \Theta'^{<1} + \frac{\epsilon}{1-\epsilon}(A + G_2 + G_3) \sim_{\mathbb{R}, X} -\frac{1}{1-\epsilon}\Theta'' - \frac{\epsilon}{1-\epsilon}G_1.$$

Choosing an effective divisor L whose support coincides with the divisorial part of $\mathrm{exc}(\pi_Z)$ such that $A - L$ is ample, then

$$K_{\mathcal{H}} + \Theta'^{<1} + \frac{\epsilon}{1-\epsilon}(G_2 + G_3 + A - L) \sim_{\mathbb{R},X} -\frac{1}{1-\epsilon}\Theta'' - \frac{\epsilon}{1-\epsilon}(G_1 + L).$$

Choose a sufficiently general effective $A' \sim_{\mathbb{R}} A - L$ and define

$$G' := G_2 + G_3 + A', \quad \epsilon' := \frac{\epsilon}{1-\epsilon}, \quad \Xi_{\epsilon'} := \frac{1}{1-\epsilon}\Theta'' + \frac{\epsilon}{1-\epsilon}(G_1 + L).$$

Hence, $K_{\mathcal{H}} + \Theta'^{<1} + \epsilon' G' \sim_{\mathbb{R},X} -\Xi_{\epsilon'}$.

Claim 7.3. *For $0 < \epsilon' \ll 1$, there exists an F-dlt modification $\bar{r}: \bar{Z} \to Z$ of $(\mathcal{H}, \Theta'^{<1} + \epsilon' G')$ such that for any $\bar{r}$-exceptional prime divisor E, $a(E; \mathcal{H}, \Theta') = -\epsilon(E)$.*

Proof of Claim 7.3. Fix $0 < \epsilon' \ll 1$. Let $\tilde{r}: \tilde{Z} \to Z$ be an F-dlt modification in the sense of Theorem 2.4 for $(\mathcal{H}, \Theta'^{<1} + \epsilon' G')$. Define $\tilde{\mathcal{H}}$ to be the birational transform of $\mathcal{H}$ on $\tilde{Z}$. Writing

$$K_{\tilde{\mathcal{H}}} + \tilde{r}_*^{-1}(\Theta'^{<1} + \epsilon' G') + \sum a_i E_i = \tilde{r}^*(K_{\mathcal{H}} + \Theta'^{<1} + \epsilon' G'), \quad a_i \geq \epsilon(E_i),$$
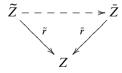
as $\tilde{r}$ is an F-dlt modification, we see that $(\tilde{\mathcal{H}}, \tilde{r}_*^{-1}(\Theta'^{<1} + \epsilon' G') + \sum \epsilon(E_i)E_i)$ is F-dlt. Let E_i be an $\tilde{r}$-exceptional prime divisor such that $a(E_i; \mathcal{H}, \Theta') > -\epsilon(E_i)$; as $\Theta' \geq \Theta'^{<1}$, also $a(E_i; \mathcal{H}, \Theta'^{<1}) > -\epsilon(E_i)$. Since the discrepancy $a(E_i; \mathcal{H}, \Theta'^{<1} + \epsilon' G')$ is a linear function of ϵ', we can choose $0 < \epsilon'' \ll \epsilon'$ such that

$$K_{\tilde{\mathcal{H}}} + \tilde{r}_*^{-1}(\Theta'^{<1} + \epsilon'' G') + \sum b_i E_i = \tilde{r}^*(K_{\mathcal{H}} + \Theta'^{<1} + \epsilon'' G'),$$

and $b_i < \epsilon(E_i)$ whenever $a(E_i; \mathcal{H}, \Theta') > -\epsilon(E_i)$. Hence,

$$K_{\tilde{\mathcal{H}}} + \tilde{r}_*^{-1}(\Theta'^{<1} + \epsilon'' G') + \sum \epsilon(E_i)E_i \sim_{\mathbb{R},Z} P - N,$$

where P, N are effective $\tilde{r}$-exceptional divisors with disjoint supports, and the support of P contains all the E_i with $a(E_i; \mathcal{H}, \Theta') > -\epsilon(E_i)$. The pair $(\tilde{\mathcal{H}}, \tilde{r}_*^{-1}(\Theta'^{<1} + \epsilon'' G') + \sum \epsilon(E_i)E_i)$ is dlt. By Corollary 2.3, we may run the $(K_{\tilde{\mathcal{H}}} + \tilde{r}_*^{-1}(\Theta'^{<1} + \epsilon'' G') + \sum \epsilon(E_i)E_i)$-MMP over Z to obtain a model

$$\tilde{Z} \dashrightarrow \bar{Z}$$
$$\tilde{r} \searrow \quad \swarrow \bar{r}$$
$$Z$$

such that $K_{\bar{\mathcal{H}}} + \bar{r}_*^{-1}(\Theta'^{<1} + \epsilon'' G') + \sum \epsilon(F_i)F_i$ is relatively nef, where $\bar{\mathcal{H}}$ denotes the birational transform of $\mathcal{H}$ on $\bar{Z}$ and the F_i are the strict transforms of the E_i on $\bar{Z}$. The Negativity Lemma implies that

$$K_{\bar{\mathcal{H}}} + \bar{r}_*^{-1}(\Theta'^{<1} + \epsilon'' G') + \sum \epsilon(F_i)F_i \sim_{\mathbb{R},Z} -\bar{N},$$

where $\bar{N}$ is the strict transform of N on $\bar{Z}$. Thus, by construction, $\bar{Z}$ is the model that satisfies the statement of the claim for the chosen value of ϵ''. ∎
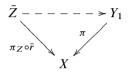
Recall that on Z, $K_{\mathscr{H}} + \Theta'^{<1} + \epsilon'G' \sim_{\mathbb{R},X} -\Xi_{\epsilon'}$. Thus, on $\bar{Z}$ there exists an effective divisor $\bar{F}$ supported on the F_i such that $\bar{F} \geq \sum \epsilon(F_i)F_i$ and

$$\bar{r}^*(K_{\mathscr{H}} + \Theta'^{<1} + \epsilon'G') = K_{\bar{\mathscr{H}}} + \bar{r}_*^{-1}(\Theta'^{<1} + \epsilon'G') + \bar{F} \sim_{\mathbb{R},X} -\bar{r}^*\Xi_{\epsilon'}.$$

Moreover, the support of $\bar{F} + \bar{r}^*\Xi_{\epsilon'}$ is the union of the divisorial part of the exceptional locus of the morphism $\bar{Z} \to X$ together with some $\bar{\mathscr{H}}$-invariant components and

$$K_{\bar{\mathscr{H}}} + \bar{\Theta}' = \bar{r}^*(K_{\mathscr{H}} + \Theta'), \quad \bar{\Theta}' := r_*^{-1}(\Theta') + \sum \epsilon(F_i)F_i.$$

Running the $(K_{\bar{\mathscr{H}}} + \bar{r}_*^{-1}(\Theta'^{<1} + \epsilon'G'))$-MMP over X

terminates with a model $\pi_1 \colon Y_1 \to X$ on which $-(\bar{F} + \bar{r}^*\Xi_{\epsilon'})$ is nef. We denote by $\mathscr{H}_{Y_1}$, Θ'_{Y_1} the strict transforms of $\bar{\mathscr{H}}, \bar{\Theta}'$ on Y_1. To conclude the proof, we take an F-dlt modification $r_Y \colon Y \to Y_1$ of the pair $(\mathscr{H}_{Y_1}, \Theta'_{Y_1})$. The Negativity Lemma [38, Lemma 1.3] and Claim 7.3 imply that Y is the desired model whose existence we claimed in the statement of the theorem. ∎

7.2. Mori hyperbolicity and non-klt locus

We recall the following hyperbolicity result for standard log pairs with dlt support which will be used throughout this section.

Proposition 7.4 ([36, Prop. 5.2]). *Let $(X, \Delta = \sum_i b_i D_i \geq 0)$ be a normal, projective, $\mathbb{Q}$-factorial log pair such that $(X, \Delta' = \sum_{b_i < 1} b_i D_i + \sum_{b_i \geq 1} D_i)$ is dlt. Suppose that $K_X + \Delta$ is nef when restricted to $\mathrm{Supp}(\sum_{b_i \geq 1} b_i D_i) = \mathrm{Nklt}(\Delta') = \mathrm{Nklt}(\Delta)$. Then either*

- *$K_X + \Delta$ is nef, or*
- *$X \setminus \mathrm{Nklt}(\Delta)$ contains an algebraic curve whose normalization is $\mathbb{A}^1$.*

In the case of a general foliated log pair, using dlt modifications we get the following criterion, which will be fundamental in the proof of Theorem 7.1.

Corollary 7.5. *Let X be a normal, projective, $\mathbb{Q}$-factorial threefold. Let $(\mathscr{F}, \Delta = \sum_i b_i D_i \geq 0)$ be a foliated log pair such that $(\mathscr{F}, \Delta' = \sum_{i|b_i < \epsilon(D_i)} b_i D_i + \sum_{i|b_i \geq \epsilon(D_i)} \epsilon(D_i)D_i)$ is F-dlt. Assume that $X \setminus \mathrm{Nklt}(\mathscr{F}, \Delta)$ does not contain algebraic curves tangent to $\mathscr{F}$ whose normalization is $\mathbb{A}^1$. Then $K_{\mathscr{F}} + \Delta$ is nef if and only if it is nef when restricted to $\mathrm{Nklt}(\mathscr{F}, \Delta)$.*

Proof. If $K_{\mathscr{F}} + \Delta$ is nef, then a fortiori it is nef when restricted to any subvariety of X.

We now assume that $K_{\mathcal{F}} + \Delta$ is nef when restricted to Nklt(Δ). As $(\mathcal{F}, \Delta')$ is F-dlt, it follows that

$$\mathrm{Nklt}(\mathcal{F}, \Delta) = \bigcup_{\mu_{D_i} \Delta \geq \epsilon(D_i)} D_i = \mathrm{Nklt}(\mathcal{F}, \Delta'),$$

by definition of Δ'. Now, suppose that $K_{\mathcal{F}} + \Delta$ is not nef. Then there exists a negative extremal ray $R \subset \overline{NE}(X)$. Since $K_{\mathcal{F}} + \Delta$ is nef when restricted along Nklt($\mathcal{F}, \Delta$), it follows that $R \cdot D_i \geq 0$ for any D_i with $\mu_{D_i} \Delta \geq \epsilon(D_i)$. Hence, R is a negative extremal ray also for $K_{\mathcal{F}} + \Delta'$. As $(\mathcal{F}, \Delta')$ is an F-dlt pair, it is non-dicritical by Theorem 1.16; in particular, [13, Lemma 3.30] implies that any curve $C \subset X$ satisfying $[C] \in R$ is tangent to $\mathcal{F}$. Moreover, [13, Theorem 6.7] implies that there exists a contraction $\phi \colon X \to Y$ within the category of projective varieties which only contracts curves in X whose numerical class belongs to R. In particular, as $K_{\mathcal{F}} + \Delta$ is nef along Nklt($\mathcal{F}, \Delta$), it follows that each fibre of ϕ intersects Nklt($\mathcal{F}, \Delta$) in at most finitely many points. As X is $\mathbb{Q}$-factorial, each fibre of ϕ intersecting Nklt($\mathcal{F}, \Delta$) must have dimension at most 1: otherwise, if X_y, for some $y \in Y$, were a 2-dimensional fibre, no component of Δ' could intersect X_y, as this intersection would contain a $(K_{\mathcal{F}} + \Delta)$-negative curve contained in Nklt($\mathcal{F}, \Delta$), hence there would be a rational curve $C \subset X \setminus \mathrm{Nklt}(\mathcal{F}, \Delta)$, thus leading to a contradiction.

Let $\Sigma \subset X$ be an irreducible curve contracted by ϕ. We claim that Σ is a rational curve. Indeed, Σ is tangent to $\mathcal{F}$, thus we may find a germ of an invariant surface S containing Σ. If $\Sigma \not\subset \mathrm{sing}(\mathcal{F})$, then S is simply a leaf containing Σ, while if $\Sigma \subset \mathrm{sing}(\mathcal{F})$, then we may take S to be a strong separatrix at a general point of Σ. As $(\mathcal{F}, \Delta')$ is F-dlt, we can apply Lemma 3.10 to write $\nu^*(K_{\mathcal{F}} + \Delta') = K_{S^\nu} + \Delta'_{S^\nu}$, where $\nu \colon S^\nu \to S$ is the normalization of S, and (S^ν, Δ'_{S^ν}) is lc. If we take T to be the normalization of $\phi(S)$ then the strict transform of Σ on S^ν is a $(K_{S^\nu} + \Delta'_{S^\nu})$-negative curve contracted by the morphism $S^\nu \to T$ and is therefore (by classical adjunction) necessarily a rational curve.

The $\mathbb{Q}$-factoriality of X implies that either

(1) ϕ is a Mori fibre space and all the fibres are 1-dimensional, or

(2) ϕ is birational and the exceptional locus intersects Nklt(Δ).

We claim that in both cases $R^1 \phi_* \mathcal{O}_X = 0$. In fact, in case (1), as all fibres are rational curves, ϕ must be a K_X-negative contraction, while in case 2) the conclusion can be reached by direct application of Theorem 4.3. Thus, Theorem 3.1 implies that Nklt($\mathcal{F}, \Delta$) is connected in a neighbourhood of every fibre of ϕ. In case (1), the generic fibre of μ is a smooth projective rational curve. Theorem 3.1 implies that the generic fibre intersects Nklt(Δ) in at most one point. This concludes the proof in case (1).

In case (2), the positive-dimensional fibres are chains of rational curves and by the vanishing $R^1 \phi_* \mathcal{O}_X = 0$, the generic fibre has to be a tree of smooth rational curves. By Theorem 3.1, Nklt($\mathcal{F}, \Delta$) intersects this chain in at most one point. In particular, there exists a complete rational curve C such that $C \cap (X \setminus \mathrm{Nklt}(\mathcal{F}, \Delta)) = f(\mathbb{A}^1)$, where f is a non-constant morphism, which provides the sought contradiction. ∎

Proof of Theorem 7.1. We divide the proof into two cases.

Case 1: $(\mathcal{F}, \Delta)$ *is F-dlt.* If $K_{\mathcal{F}} + \Delta$ is nef along $\mathrm{Nklt}(\mathcal{F}, \Delta)$ the conclusion follows from Corollary 7.5. Hence, we can assume that there exists a positive-dimensional lc centre W for $(\mathcal{F}, \Delta)$ and $K_{\mathcal{F}} + \Delta$ is not nef along W. By induction on dimension, we can consider W to be minimal (with respect to inclusion) with that property, so that $(K_{\mathcal{F}} + \Delta)|_W$ is nef when restricted to the lc centres of $(\mathcal{F}, \Delta)$ strictly contained in W. Clearly, $\dim W > 0$ and [35, Theorem 4.5] implies that if $\dim W = 1$, then W is tangent to $\mathcal{F}$. As $(\mathcal{F}, \Delta)$ is F-dlt, one of the following conditions hold:

(a) W is a component of Δ of coefficient 1;

(b) W is an invariant divisor;

(c) $W \subset \mathrm{sing}(\mathcal{F})$, $\dim W = 1$, and $\mathcal{F}$ is canonical along W by [13, Lemma 3.12];

(d) $\dim W = 1$ and W is tangent to $\mathcal{F}$, $W \not\subset \mathrm{sing}(\mathcal{F})$, but $W \subset D$, where D is a component of Δ with $\mu_D \Delta = 1$.

Case 1 (a). If W is a component of Δ of coefficient 1, then we can apply the adjunction formula along the normalization $\nu \colon W^{\nu} \to W$:

$$\nu^{*}(K_{\mathcal{F}} + \Delta) = K_{\mathcal{G}} + \Theta,$$

where $\mathcal{G}$ is the restriction of $\mathcal{F}$ to W^{ν} and Θ is the different as defined in Lemma 3.10. The adjunction formula guarantees that $(\mathcal{G}, \Theta)$ is F-dlt (see Lemma 3.10), and that $\nu^{-1}(Z) = \mathrm{Nklt}(\mathcal{G}, \Theta)$, where Z is the union of all lc centres of $(\mathcal{F}, \Delta)$ strictly contained in W. This follows from [13, Lemma 3.8] as (X, Δ) is log smooth in a neighbourhood of any lc centre; in particular, W is normal at the general point of any codimension 2 lc centre contained in it, and thus

$$\mathrm{Nklt}(\mathcal{G}, \Theta) = \nu^{-1}(Z). \tag{7.1}$$

Hence, the conclusion follows from the 2-dimensional case, that is, from Proposition 7.6. In fact, the proposition implies that there is a non-constant map $f \colon \mathbb{A}^1 \to W^{\nu}$ with $f(\mathbb{A}^1) \subset W^{\nu} \setminus \mathrm{Nklt}(\mathcal{G}', \Theta)$, and by (7.1) composing with ν we obtain a map $f' \colon \mathbb{A}^1 \to W \setminus Z$.

Case 1 (b). If W is an invariant divisor, then we can apply the adjunction formula along the normalization $\nu \colon W^{\nu} \to W$:

$$\nu^{*}(K_{\mathcal{F}} + \Delta) = K_{W^{\nu}} + \Theta, \tag{7.2}$$

where Θ is the foliation different. Moreover, $\nu^{-1}(Z) = \mathrm{Nklt}(W^{\nu}, \Theta)$, where Z is the union of all lc centres of $(\mathcal{F}, \Delta)$ strictly contained in W [13, Lemma 3.16]. Hence, by Proposition 7.4, there exists a non-constant map $f \colon \mathbb{A}^1 \to W^{\nu} \setminus \mathrm{Nklt}(W^{\nu}, \Theta)$. As $\nu^{-1}(Z) = \mathrm{Nklt}(W^{\nu})$, $\nu \circ f$ produces the desired curve in $W \setminus Z$.

Case 1 (c). If W is a curve contained in $\mathrm{sing}(\mathcal{F})$ and $\mathcal{F}$ is canonical along W, then by [13, Lemma 3.14] there exist two possibly formal separatrices of $\mathcal{F}$ through W and we can choose one of them, say S, to be the (convergent) strong separatrix [35, Corollary 5.6]. Hence, applying adjunction along S, it follows that

$$\nu^{*}(K_{\mathcal{F}} + \Delta) = K_{S^{\nu}} + W + \Theta,$$

where $\nu \colon S^{\nu} \to S$ is the normalization of S and $W + \Theta$ is the different of $(\mathcal{F}, \Delta)$ along S^{ν}.

So by Lemma 3.10, if P is a non-klt centre of $(S^\nu, W + \Theta)$ then $\nu(P)$ is an lc centre of $(\mathcal{F}, \Delta)$. Let $n\colon V \to W$ be the normalization of W and by (classical) adjunction we may write $n^*(K_{S^\nu} + W + \Theta) = K_V + \Theta_V$ where $\lfloor \Theta_V \rfloor$ is supported on the pre-images of the non-klt centres of $(S^\nu, W + \Theta)$ contained in W. Since $(K_{S^\nu} + W + \Theta) \cdot W < 0$ we have $V \cong \mathbb{P}^1$ and $\lfloor \Theta_V \rfloor$ contains at most one point. Thus we see that the normalization $W \setminus Z'$ is $\mathbb{P}^1$ or $\mathbb{A}^1$ where Z' are the strata of $\mathrm{Nklt}(\mathcal{F}, \Delta)$ contained in W.

Case 1 (d). Let $\mathcal{F}_D$ be the foliation restricted to D and write $(K_\mathcal{F} + \Delta)|_D = K_{\mathcal{F}_D} + \Delta_D$. Again, the result follows directly from Proposition 7.6 and Lemma 3.10 (as in Case 1 (a)), which together imply that the normalization of $W \setminus Z'$ contains a copy of $\mathbb{A}^1$.

Case 2: $(\mathcal{F}, \Delta)$ *is log canonical.* In this case we prove the theorem by reducing to Case 1. By Theorem 7.2, we can take a dlt modification $\pi\colon Y \to X$ with

$$K_{\mathcal{F}_Y} + \Delta_Y = \pi^*(K_\mathcal{F} + \Delta)$$

and $\pi^{-1}(\mathrm{Nklt}(\mathcal{F}, \Delta)) = \mathrm{Nklt}(\mathcal{F}_Y, \Delta_Y)$. If $K_\mathcal{F} + \Delta$ is not nef, then the same must hold for $K_{\mathcal{F}_Y} + \Delta_Y$.

We first discuss the case where $K_\mathcal{F} + \Delta$ is nef along $\mathrm{Nklt}(\mathcal{F}, \Delta)$. Under this assumption, Corollary 7.5 implies the existence of a non-constant map $f\colon \mathbb{A}^1 \to Y \setminus \mathrm{Nklt}(\mathcal{F}_Y, \Delta_Y)$ whose image is tangent to $\mathcal{F}$. This produces the desired contradiction. Hence, we may assume that there is an lc centre $W_Y \subset Y$ of $(\mathcal{F}_Y, \Delta_Y)$ and a non-constant algebraic morphism $f\colon \mathbb{A}^1 \to W_Y$ such that

- $f(\mathbb{A}^1) \subset W_Y \setminus Z_Y$ where Z_Y is the union of all lc centres strictly contained in W_Y, and

- $(K_{\mathcal{F}_Y} + \Delta_Y) \cdot C < 0$, where C is the Zariski closure of $f(\mathbb{A}^1)$.

We define $W := \pi(W_Y)$; this is an lc centre of $(\mathcal{F}, \Delta)$. We wish to show the existence of a non-constant morphism $g\colon \mathbb{A}^1 \to W \setminus Z$, where Z is the union of all lc centres in W.

Let $Z^0 \subset Z$ be the union of all those lc centres Z' in W such that $\pi^{-1}(Z')$ is a union of lc centres. By the above we see that $(\pi \circ f)(\mathbb{A}^1) \subset W \setminus Z^0$.

Notice moreover that if Z' is an lc centre such that $\pi^{-1}(Z')$ has pure codimension 1 then $Z' \subset Z^0$.

We argue by analysing cases depending on the dimension of W. If $\dim W = 0$ there is nothing to show, so suppose for the moment that $\dim W = 1$.

Let T be a codimension 1 lc centre of $(\mathcal{F}_Y, \Delta_Y)$ dominating W and which contains $f(\mathbb{A}^1)$ and denote by $\sigma\colon T \to W$ the projection.

Suppose first that T is transverse to $\mathcal{F}_Y$, write by adjunction $(K_{\mathcal{F}_Y} + \Delta_Y)|_T = K_\mathcal{G} + \Theta$ and let C be as above; notice that $(\mathcal{G}, \Theta)$ is F-dlt. Set Θ_0 to be the part of $\lfloor \Theta \rfloor$ supported on $\sigma^{-1}(Z)$ and set Θ_1 to be those divisors D contained in $\sigma^{-1}(Z)$ with $\epsilon(D) = 1$ and D not contained in the support of $\lfloor \Theta \rfloor$.

Fix $0 < \epsilon, \delta \ll 1$ and run the $(K_\mathcal{G} + \Theta - \epsilon\Theta_0 + \delta\Theta_1)$-MMP over W and denote it by $\phi\colon T \to S$. Let $\mathcal{H}$ be the pushforward of $\mathcal{G}$, let $D = \phi_*C$, let $\Gamma = \phi_*\Theta$ and let $\tau\colon S \to W$ denote the induced map. Then $(\mathcal{H}, \Gamma)$ is log canonical and by the Negativity Lemma [38, Lemma 1.3], we see that $\tau^{-1}(Z) \subset \mathrm{Supp}(\Gamma)$, and so the pre-image of an lc

centre is a union of lc centres. Since $(K_{\mathcal{H}} + \Gamma) \cdot D < 0$ it follows from Proposition 7.6 that there exists a map $\mathbb{A}^1 \to S \setminus \mathrm{Nklt}(\mathcal{H}, \Gamma)$ and we may push forward this map along τ to get a map $\mathbb{A}^1 \to W \setminus Z$.

The case where T is invariant can be proven in a similar manner.

Now suppose that $\dim W = 2$. Let W_Y denote the strict transform of W and let $\sigma: W_Y \to W$ be the induced map.

Suppose first that W is transverse to the foliation and write $(K_{\mathcal{F}_Y} + \Delta_Y)|_{W_Y} = K_{\mathcal{G}} + \Theta$. Let $C \subset W_Y$ be a 1-dimensional lc centre and observe by foliated Riemann–Hurwitz that if $B \subset W_Y$ is a divisor such that $\sigma(B) = C$ then $B \subset W_Y$ is an lc centre of $(\mathcal{G}, \Theta)$. Let $Q \subset W$ be the union of the 0-dimensional lc centres contained in W, set Θ_0 to be the part of $\lfloor \Theta \rfloor$ supported on $\sigma^{-1}(Q)$ and set Θ_1 to be those divisors D contained in $\sigma^{-1}(Z)$ with $\epsilon(D) = 1$ and D not contained in the support of $\lfloor \Theta \rfloor$.

Again, we run the $(K_{\mathcal{G}} + \Theta - \epsilon \Theta_0 + \delta \Theta_1)$-MMP over W for $0 < \epsilon, \delta \ll 1$. Let $\phi: W_Y \to S$ denote this MMP and let $\tau: S \to W$ denote the induced map. Again, notice that the pre-image of an lc centre under τ is a union of lc centres and by applying Proposition 7.6 we may produce a map $\mathbb{A}^1 \to S$ whose push forward along τ gives a map $\mathbb{A}^1 \to W \setminus Z$. Again, the case where W_Y is invariant can be handled in a similar manner.

In all cases, if $K_{\mathcal{F}} + \Delta$ is not nef we have produced a map $\mathbb{A}^1 \to W \setminus Z$. ∎

Proposition 7.6. *Let $(\mathcal{F}, \Delta)$ be a log canonical foliated pair on a normal projective surface X. Assume that*

- *X is potentially log canonical;*
- *there is no non-constant morphism $f: \mathbb{A}^1 \to X \setminus \mathrm{Nklt}(\mathcal{F}, \Delta)$; and*
- *for any stratum S of $\mathrm{Nklt}(\mathcal{F}, \Delta)$ there is no non-constant morphism $f: \mathbb{A}^1 \to \bar{S}$.*

Then $K_{\mathcal{F}} + \Delta$ is nef.

Proof. Assume for the sake of contradiction that $K_{\mathcal{F}} + \Delta$ is not nef and $(\mathcal{F}, \Delta)$ satisfies all the hypotheses in the statement of the proposition. We divide the proof into two distinct cases.

Case 1: $(\mathcal{F}, \Delta)$ *is F-dlt.* By [13, Theorem 3.31], since $K_{\mathcal{F}} + \Delta$ is not nef, there exists a rational curve $C \subset X$ with $(K_{\mathcal{F}} + \Delta) \cdot C < 0$ and C tangent to $\mathcal{F}$. As C is $\mathcal{F}$-invariant, it cannot be contained in $\mathrm{Supp}(\Delta)$. Thus,

$$\nu^*(K_{\mathcal{F}} + \Delta) = K_{C^\nu} + \Delta_{C^\nu},$$

where $\nu: C^\nu \to C$ is the normalization, and $\mathrm{Supp}(\lfloor \Delta_{C^\nu} \rfloor) \supset \nu^{-1}(\mathrm{sing}(\mathcal{F}) \cup \lfloor \Delta \rfloor)$. Finally, observe that $\mathrm{Nklt}(\mathcal{F}, \Delta)$ and all its strata are supported on $\mathrm{sing}(\mathcal{F}) \cup \lfloor \Delta \rfloor$ to conclude that the normalization of $C \setminus Z'$ is $\mathbb{P}^1$ or $\mathbb{A}^1$ where Z' are all the strata of $\mathrm{Nklt}(\mathcal{F}, \Delta)$ meeting C. This is the desired contradiction.

Case 2: $(\mathcal{F}, \Delta)$ *is log canonical.* In this case we reduce the proof to Case 1. Let $\pi: Y \to X$ be an F-dlt modification for the pair $(\mathcal{F}, \Delta)$,

$$K_{\mathcal{F}_Y} + \Gamma = f^*(K_{\mathcal{F}} + \Delta).$$

Hence, also $K_{\mathcal{F}_Y} + \Gamma$ is not nef and by Case 1 there is a rational curve $C \subset Y$ tangent to $\mathcal{F}_Y$ such that $C \cdot (K_{\mathcal{F}_Y} + \Gamma) < 0$; moreover, the normalization morphism $C^\nu \to Y$ induces either a non-constant morphism $f \colon \mathbb{A}^1 \to Y \setminus \mathrm{Nklt}(\mathcal{F}_Y, \Gamma)$, or a non-constant morphism $f \colon \mathbb{A}^1 \to \bar{S}$ for some stratum S of $\mathrm{Nklt}(\mathcal{F}_Y, \Gamma)$. The curve $\pi(C)$ is tangent to $\mathcal{F}$, thus it is $\mathcal{F}$-invariant, since $\mathcal{F}_Y$ has rank 1. If $C \cap (Y \setminus \mathrm{Nklt}(\mathcal{F}_Y, \Gamma)) \neq \emptyset$, it follows from Theorem 7.2 and adjunction that $\pi \circ f \colon \mathbb{A}^1 \to X \setminus \mathrm{Nklt}(\mathcal{F}, \Delta)$ is a well-defined morphism.

Hence we can assume that C is an lc centre of $(\mathcal{F}_Y, \Gamma)$ and that $\bar{C}$ is a copy of $\mathbb{A}^1$ embedded in Y. But then again the adjunction formula and Theorem 7.2 imply that $\pi(\bar{C})$ is also a copy of $\mathbb{A}^1$ embedded in X, thus proving the proposition. ∎

8. Some questions

The proof of Theorem 6.1 and its generalizations and possible applications raise several questions.

Question 8.1. *Let* $(0 \in X)$ *be a germ of a klt singularity and* $\mathcal{F}$ *a log canonical co-rank* 1 *foliation on* X. *Does* $\mathcal{F}$ *admit a separatrix at* 0*?*

Question 8.2. *Let* $\mathcal{F}$ *be a co-rank* 1 *foliation on a klt variety* (X, Δ) *with* $c_1(K_{\mathcal{F}}) = 0$ *and* $-(K_X + \Delta)$ *big.*

(1) *Does* $\mathcal{F}$ *admit an invariant divisor?*

(2) *Is* $\mathrm{sing}(\mathcal{F})$ *non-empty?*

(3) *For* $p \in \mathrm{sing}(\mathcal{F})$, *is every separatrix at* p *algebraic?*

More generally, one may wonder if log canonical singularities of foliations of any dimension admit separatrices. By examples of Gómez-Mont and Luengo [19] it is known that a vector field on $\mathbb{C}^3$ does not always admit a separatrix, but the examples given there are not log canonical.

Question 8.3. *Let* $\mathcal{F}$ *be a foliation of any rank on* $\mathbb{C}^m$. *Let* 0 *be a log canonical singularity of* $\mathcal{F}$. *Does* $\mathcal{F}$ *admit a separatrix at* 0*?*

In the proof of existence of flips given in [13] the existence of separatrices played a central role, and thus the methods given there do not immediately imply the existence of log canonical flips. With Theorem 6.1 in mind we ask the following.

Question 8.4. *Do log canonical foliation flips exist?*

This extension seems to be important to apply the methods of the foliated MMP to several classes of foliations of interest: Fano foliations, for instance, have worse than canonical singularities.

This project was started during a visit of CS and RS to FRIAS; we would like to thank FRIAS and Prof. Stefan Kebekus for their hospitality and the financial support during the course of our visit. Most of this work was completed during several visits of RS to Imperial College, London. RS would like to thank Imperial College for the hospitality and the nice working environment. He would also like to thank University of Cambridge and Churchill College, Cambridge where he was a research fellow when part of this work was completed.

References

[1] Araujo, C., Druel, S.: On Fano foliations. Adv. Math. **238**, 70–118 (2013) Zbl 1282.14085 MR 3033631

[2] Birkar, C.: On connectedness of non-klt loci of singularities of pairs. arXiv:2010.08226 (2020)

[3] Bogomolov, F., McQuillan, M.: Rational curves on foliated varieties. In: Foliation Theory in Algebraic Geometry, Simons Symp., Springer, Cham, 21–51 (2016) Zbl 1337.14041 MR 3644242

[4] Brunella, M.: Birational Geometry of Foliations. Monografías de Matemática, Instituto de Matemática Pura e Aplicada, Rio de Janeiro (2000) Zbl 1073.14022 MR 1948251

[5] Camacho, C.: Quadratic forms and holomorphic foliations on singular surfaces. Math. Ann. **282**, 177–184 (1988) Zbl 0657.32007 MR 963011

[6] Camacho, C., Lins Neto, A.: Geometric Theory of Foliations. Birkhäuser Boston, Boston, MA (1985) Zbl 0568.57002 MR 824240

[7] Cano, F.: Reduction of the singularities of codimension one singular foliations in dimension three. Ann. of Math. (2) **160**, 907–1011 (2004) Zbl 1088.32019 MR 2144971

[8] Cano, F., Cerveau, D.: Desingularization of nondicritical holomorphic foliations and existence of separatrices. Acta Math. **169**, 1–103 (1992) Zbl 0771.32018 MR 1179013

[9] Cano, F., Mattei, J.-F.: Hypersurfaces intégrales des feuilletages holomorphes. Ann. Inst. Fourier (Grenoble) **42**, 49–72 (1992) Zbl 0762.32018 MR 1162557

[10] Cano, F., Molina-Samper, B.: Invariant surfaces for toric type foliations in dimension three. Publ. Mat. **65**, 291–307 (2021) Zbl 07405617 MR 4185834

[11] Cano, F., Ravara-Vago, M.: Local Brunella's alternative II. Partial separatrices. Int. Math. Res. Notices **2015**, 12840–12876 Zbl 1341.32025 MR 3431638

[12] Cano, F., Ravara-Vago, M., Soares, M.: Local Brunella's alternative I. RICH foliations. Int. Math. Res. Notices **2015**, 2525–2575 (2015) Zbl 1317.32064 MR 3344680

[13] Cascini, P., Spicer, C.: MMP for co-rank one foliations on threefolds. Invent. Math. **225**, 603–690 (2021) Zbl 07387419 MR 4285142

[14] Cerveau, D., Lins Neto, A.: Frobenius theorem for foliations on singular varieties. Bull. Brazil. Math. Soc. (N.S.) **39**, 447–469 (2008) Zbl 1194.32018 MR 2473858

[15] Druel, S.: Codimension 1 foliations with numerically trivial canonical class on singular spaces. Duke Math. J. **170**, 95–203 (2021) Zbl 07322635 MR 4194898

[16] Druel, S., Ou, W.: Codimension one foliations with trivial canonical class on singular spaces II. arXiv:1912.07727v2 (2019)

[17] Filipazzi, S., Svaldi, R.: On the connectedness principle and dual complexes for generalized pairs. arXiv:2010.08018v2 (2020)

[18] Flenner, H.: Divisorenklassengruppen quasihomogener Singularitäten. J. Reine Angew. Math. **328**, 128–160 (1981) Zbl 0457.14001 MR 636200

[19] Gómez-Mont, X., Luengo, I.: Germs of holomorphic vector fields in $\mathbf{C}^3$ without a separatrix. Invent. Math. **109**, 211–219 (1992)   Zbl 0774.32019   MR 1172688

[20] Greb, D., Kebekus, S., Kovács, S. J., Peternell, T.: Differential forms on log canonical spaces. Publ. Math. Inst. Hautes Études Sci. **114**, 87–169 (2011)   Zbl 1258.14021   MR 2854859

[21] Greb, D., Kebekus, S., Peternell, T.: Étale fundamental groups of Kawamata log terminal spaces, flat sheaves, and quotients of abelian varieties. Duke Math. J. **165**, 1965–2004 (2016)   Zbl 1360.14094   MR 3522654

[22] Hartshorne, R.: Algebraic Geometry. Grad. Texts in Math. 52, Springer, New York (1977)   Zbl 0367.14001   MR 0463157

[23] Kebekus, S.: Pull-back morphisms for reflexive differential forms. Adv. Math. **245**, 78–112 (2013)   Zbl 1427.14072   MR 3084424

[24] Kempf, G., Knudsen, F. F., Mumford, D., Saint-Donat, B.: Toroidal Embeddings. I. Lecture Notes in Math. 339, Springer, Berlin (1973)   Zbl 0271.14017   MR 0335518

[25] Kollár, J., Mori, S.: Birational Geometry of Algebraic Varieties. Cambridge Tracts in Math. 134, Cambridge Univ. Press, Cambridge (1998)   Zbl 0926.14003   MR 1658959

[26] Loray, F., Pereira, J. V., Touzet, F.: Singular foliations with trivial canonical class. Invent. Math. **213**, 1327–1380 (2018)   Zbl 1426.32014   MR 3842065

[27] Malgrange, B.: Frobenius avec singularités. I. Codimension un. Inst. Hautes Études Sci. Publ. Math. **46**, 163–173 (1976)   Zbl 0355.32013   MR 508169

[28] Mattei, J.-F., Moussu, R.: Holonomie et intégrales premières. Ann. Sci. École Norm. Sup. (4) **13**, 469–523 (1980)   Zbl 0458.32005   MR 608290

[29] McQuillan, M.: Diophantine approximations and foliations. Inst. Hautes Études Sci. Publ. Math. **87**, 121–174 (1998)   Zbl 1006.32020   MR 1659270

[30] McQuillan, M.: Canonical models of foliations. Pure Appl. Math. Quart. **4**, 877–1012 (2008)   Zbl 1166.14010   MR 2435846

[31] McQuillan, M., Panazzolo, D.: Almost étale resolution of foliations. J. Differential Geom. **95**, 279–319 (2013)   Zbl 1295.32041   MR 3128985

[32] Mendes, L. G.: Kodaira dimension of holomorphic singular foliations. Bol. Soc. Brasil. Mat. (N.S.) **31**, 127–143 (2000)   Zbl 0979.32017   MR 1785264

[33] Pereira, J. V., Svaldi, R.: Effective algebraic integration in bounded genus. Algebraic Geom. **6**, 454–485 (2019)   Zbl 1461.37049   MR 3957403

[34] Siu, Y.-t.: Extending coherent analytic sheaves. Ann. of Math. (2) **90**, 108–143 (1969)   Zbl 0181.36301   MR 245837

[35] Spicer, C.: Higher-dimensional foliated Mori theory. Compos. Math. **156**, 1–38 (2020)   Zbl 1428.14025   MR 4036447

[36] Svaldi, R.: Hyperbolicity for log canonical pairs and the cone theorem. Selecta Math. (N.S.) **25**, art. 67, 23 pp. (2019)   Zbl 1442.14057   MR 4030221

[37] Tian, Z., Xu, C.: Finiteness of fundamental groups. Compos. Math. **153**, 257–273 (2017)   Zbl 1453.14014   MR 3604863

[38] Wang, J.: On the Iitaka conjecture $C_{n,m}$ for Kähler fibre spaces. arXiv:1907.06705 (2019)