Well-posedness of a reaction-diffusion model with stochastic dynamical boundary conditions

Mario Maurelli[*], Daniela Morale[†], Stefania Ugolini[‡]

August 15, 2023

Abstract

We study the well-posedness of a nonlinear reaction diffusion partial differential equation system on the half-line coupled with a stochastic dynamical boundary condition, a random system arising in the description of the evolution of the chemical reaction of sulphur dioxide with the surface of calcium carbonate stones. The boundary condition is given by a Jacobi process, solution to a Brownian motion-driven stochastic differential equation with a mean reverting drift and a bounded diffusion coefficient. The main result is the global existence and the pathwise uniqueness of mild solutions. The proof relies on a splitting strategy, which allows to deal with the low regularity of the dynamical boundary condition.

MSC2020: 35R60, 60H15, 35K57, 60H30
Keywords: stochastic dynamical boundary conditions, nonlinear reaction-diffusion PDEs, application to sulphation.

1 Introduction

This paper concerns the study of the well-posedness and regularity of the solution of the following random system:

$$\begin{aligned} \frac{\partial}{\partial t}(\varphi s) &= \nabla \cdot (\varphi \nabla s) - \lambda \varphi s c, \qquad (0,\infty)\times(0,T) \\ \partial_t c &= -\lambda \varphi s c, \end{aligned} \tag{1}$$

with $\lambda \in \mathbb{R}_+$, coupled with initial conditions $s(x,0) = s_0(x)$, $c(x,0) = c_0(x)$ for $x \in \mathbb{R}_+$, $\varphi = \varphi(c)$, where randomness is due to the coupling of (1) with a stochastic dynamical boundary condition for s, given by

$$s(t,0) = \psi_t, \quad t \in [0,T].$$

More precisely, the idea is to consider as boundary condition the stochastic process ψ_t given by the unique bounded solution of the following stochastic differential equation of Itô type

$$d\psi_t = \alpha(\gamma - \psi_t)dt + \sigma\sqrt{\psi_t\,(\eta - \psi_t)}dW_t, \tag{2}$$

where α, γ, σ and η are positive parameters and $W \equiv \{W_t\}_{t\in[0,T]}$ is a Wiener process.

The deterministic system (1), endowed with a constant boundary condition, has been introduced in [2] as an adimensionless model providing a quantitative description of the diffusion and the chemical action of sulphur dioxide (SO2) on the porosity of calcium carbonate stones (CaCO3); it is assumed that polluted air (and so the sulphur dioxide) diffuses through the pores of the stones and interacts with the surface of the pores. In (1) s stands for the porous concentration of SO2, namely the concentration taken with respect to the volume of the

[*]Dipartimento di Matematica, Università di Pisa, Largo Bruno Pontecorvo 5, 56127 Pisa, Italy; email: mario.maurelli@unipi.it
[†]Dipartimento di Matematica, Università degli Studi di Milano, via Cesare Saldini 50, 20133 Milano, Italy, Italy; email: daniela.morale@unimi.it
[‡]Dipartimento di Matematica, Università degli Studi di Milano, via Cesare Saldini 50, 20133 Milano, Italy; email: stefania.ugolini@unimi.it

pores, and c for the local density of CaCO3. The function $\varphi(c)$ is the porosity (volume of pores/total volume), which is supposed to depend only on the density of calcite

$$\varphi(c) = A + Bc.$$

The porosity is related to the concentration ρ_s of SO2 (fraction of total volume occupied by SO2) by the relation $\rho_s = \varphi(c)s$. For a detailed derivation of the system (1), the interested reader may refer to [2], where the authors investigate the qualitative behavior of the solution on the half-line. To the best of our knowledge, all the dynamical systems considered so far describing the degradation of calcium carbonate due to the attack of sulphur dioxide are deterministic ones, with the exception of [7], where a new nonlinear differential system for marble sulphation including the surface rugosity, a function obeying a damage-like equation which describes the microscopic variation of a surface with respect to a flat configuration, is introduced. During the numerical simulations of the deterministic system a random surface rugosity on the boundary based on Weibull'statistics is considered, while here a dynamical model for the boundary condition is proposed.

The mathematical properties of the system (1) on the real line $\mathbb{R}$, again with a constant boundary condition, have been studied in [18, 20]. In [21] the well-posedness of the model is faced in the case $x \in (0, \infty)$, as in the present paper, with a boundary condition ψ for s given by a non negative function with $W^{1,1}$ regularity in time; in the same paper the fast reaction limit and the large time behavior is also discussed. See also [2] for the mathematical study of a generalization of the deterministic model, with Dirichlet boundary conditions for the unknown s expressed in terms of an assigned space-time function. In [2, 18] the boundary condition for s is assumed to be constant, even tough, at least as far as numerical analysis is concerned, the authors state that the same analysis may be performed in the case where $s(0, t)$ is a given pre-selected bounded measurable positive function. They prove existence and uniqueness results and some regularity properties of global solutions, extending existence and uniqueness results of local solutions, at least for smooth initial data as in [11].

Similar evolution problems with stochastic dynamical boundary conditions has been studied with particular attention to the case in which at the boundary the solution acts as a white noise; see, for example, [1,5,8,10,12,13] and the literature therein. Particular attention has been devoted to the case where, at the boundary, the solution acts as a white noise. For example, in [8,12] the framework for the analysis of Poisson and heat equations excited by the white in time and/or space noise on the boundary is considered, and existence and uniqueness of weak solutions in the space of distributions are proven. Closer to our setting are the works in [1,10], where the solution satisfies a stochastic differential equation at the boundary; these works use an approach via semigroup theory to show existence and uniqueness of mild solutions; see also [5]. The difference of the present work with respect to [1, 10] is the presence of the equation for c, which makes the system (1) not strongly parabolic.

In this paper, we introduce the stochastic boundary condition (2) with the aim to study the existence, uniqueness and regularity of the solution of the random system (1)-(2). The motivation of the choice of equation (2) derives from a statistical study of the source term ψ giving the concentration of sulfur dioxide in the air. Indeed, as observed in [3] for the time series of the SO2 concentration (in air) in the Milano area, it is clearly not realistic to consider such concentration as a constant; evidence of a random evolution can be easily recognized. Within a preliminary study, the authors propose the same Pearson process, solution of a mean reverting equation with bounded Jacobi noise as a model for the boundary function ψ_t, taking advantage of an estimation procedure based on the time series of real data, combined with a numerical simulation of the system (1)-(2). This class of diffusion processes (2) is well-studied both in the biological literature, where they are known as Wright-Fisher processes, and in the mathematical one known as Jacoby processes. See, e.g.[19] for a recent approach to these processes via Dirichlet forms. The Pearson processes are interesting both for the modeling and theoretical point of view, since they share the feature that under certain conditions upon the process parameters $\alpha, \gamma, \sigma, \eta$ it can be proved that, when it starts within $[0, \eta], \eta > 0$, it remains positive and bounded in the finite region $[0, \eta]$. Furthermore one can establish that (2) admits an invariant probability distribution, which is a generalized Beta distribution. It is a suitable way to introduce a bounded noise, in spite of the fact that it is driven by a Wiener process W, which is a very important issue in the applications [15, 16].

The main mathematical problem with the stochastic boundary condition (2) is the fact that this condition is no longer in $W^{1,1}$ (P-a.s.), even tough it is more regular than the white noise. More precisely, the diffusion process at the boundary inherits the same regularity of the Wiener process W, that is ψ has almost surely trajectories in C^β, for every $\beta \in (0, 1/2)$, i.e. the class of Hölder continous functions of order less than one half. So the well-posedness of the PDE needs a carefully investigation. Indeed, randomness at the boundary is coupled with a system of nonlinear parabolic equations which is not strongly parabolic and only some energy estimates are available for analysing it, as noticed in [2, 18]. We stress that the strategy of the proof using in [21] cannot be adopted here, essentially because of the non differentiability of the boundary function ψ_t (see Remark 34).

Our approach is based on a splitting strategy, in which the solution (s, c) of the system (1)-(2) is seen as $(u+v, c)$, where from one side u is solution to the heat equation coupled with the stochastic boundary condition (2), not depending upon the less regular function c; on the other side (v, c) is a solution of a non linear system, depending upon u (in a functional way), but endowed with a deterministic constant boundary condition. This strategy allows us to compensate the irregularity of the stochastic boundary path with the regularity of u, coming from the heat equation. In particular, concerning u, the $W^{1,q}$ norm of u can be controlled, in a pathwise sense, by the C^β norm of ψ, for $q < 1/(1-2\beta)$ (see Proposition 13); this is a rather classical result but we give here a self-contained proof based on fractional Sobolev spaces and interpolation theory.

As far as concerns the couple of variables (v, c), the equation satisfied by v is nonlinear and nonlocal, due to the dependence of c on v itself, involving also u and $\partial_x u$. The assumptions of classical existence theorems for nonlinear scalar parabolic equations are not verified because of the lack of L^∞ estimate of the quantity $\partial_x c$, as mentioned in [2, 18, 21], where the authors prove a global existence result for suitable weak solutions, based on the control of $\partial_x c$ in L^2. We partially use their results, but in our case we cannot apply the same strategy, due to the terms containing u and $\partial_x u$. To control these terms, we use the $W^{1,q}$ bound on u in Proposition 13. With this argument, we get the L^2 bound on a linearized system, Proposition 33, and the a priori L^2 estimate on s, Proposition 37, which represent the main novelty of our proof. Our principal result states the well-posedness for the nonlinear system (1) under the hypothesis $\psi \in C^\beta([0, T])$ with $\beta \in (1/4, 1/2)$. The lower bound $\beta > 1/4$ on the Hölder exponent for ψ comes from the L^2 integrability condition required for $\partial_x c$, which in turn forces the solution to the heat equation u to be in $W^{1,2}$.

To sum up, we extend the well-posedness result in [21] by adding a random dynamical boundary condition, given by the solution to an SDE with non Lipschitz diffusion coefficient. In particular, from the analytical point of view, we allow the boundary condition to have lower regularity with respect to the case in [21], and precisely to be only β-Hölder continuous, with $\beta > 1/4$. It would be interesting to investigate the role of an intermediate irregularity between the one typical of Brownian motion and the one of white noise. We plan to work on this direction in a future paper.

The paper is organized as follows. In Section 2 we introduce the PDE-ODE model with a stochastic dynamical boundary condition. We discuss and prove the principal properties of the boundary process, as boundness and Hölder continuity. The splitting strategy and our definition of mild solution are specified. The solution of the heat equation with the stochastic boundary condition above described is analysed in Section 3 with the aim to establish for it continuity results in terms of the C^β norm of the stochastic boundary function. In Section 4 a study of a version of a linear equation for $\tilde{s} = \tilde{u} + \tilde{v}$ with prescribed regularity upon the coefficient is described. In particular the existence of a mild solution $\tilde{v}$, and of $\tilde{s}$ in $L^\infty([0, T], W^{1,p}(\mathbb{R}_+))$ is established. Such results are used in Section 5, where the main well-posedness achievements are illustrated.

Notations. Throught the paper, for sake of simplicity we have introduced in the proofs the following notations for the involved Banach spaces $L^p_x := L^p(\mathbb{R}_+)$ in space and $L^p_t(L^q_x) := L^p([0, T], L^q(\mathbb{R}_+))$ in time and space. Similar notation is considered for the fractional Sobolev spaces. Furthermore, given a function f defined on $[0, T] \times \mathbb{R}_+$, we define the marginals as $f_t := f(t, \cdot)$, for $t \in [0, T]$ and $f_x := f(\cdot, x)$, for $x \in \mathbb{R}_+$.

Acknowledgements. We acknowledge support from Università degli Studi di Milano through the SEED grant 'SciCult: Modellizzazione matematica e analisi SCIentifica per i beni CULTurali: previsione e prevenzione del degrado chimico e meccanico di pietre monumentali in ambienti outdoor'. Part of the work was done while M.M. was at Università degli Studi di Milano. We thank Marco Alessandro Fuhrman for proposing this class of problems, Stefano Bonaccorsi, Cecilia Cavaterra and Roberto Natalini for relevant discussions on the topic.

2 A PDE - ODE model with a stochastic boundary condition

Let us introduce and describe the initial boundary value problem under study. Hence, let us consider the following reaction diffusion system for the couple (s, c)

$$\frac{\partial}{\partial t}(\varphi s) = \nabla \cdot (\varphi \nabla s) - \lambda \varphi s c, \qquad (0, \infty) \times (0, T) \tag{3}$$

$$\partial_t c = -\lambda \varphi s c, \tag{4}$$

where $\lambda \in \mathbb{R}_+$. The initial conditions

$$
\begin{aligned}
s(x,0) &= s_0(x), \qquad x \in (0,\infty) \\
c(x,0) &= c_0(x), \qquad x \in (0,\infty)
\end{aligned}
$$

are such that $s_0, c_0 \in C_0(\mathbb{R}) \cap C^\infty(\mathbb{R})$, where $C_0(\mathbb{R})$ is the class of the continuous function with exponential decay to infinity. We suppose that $0 \leq s_0 \leq M, \quad 0 < m \leq c_0 \leq C_0$.

A stochastic dynamical boundary condition We study the case in which the boundary function for the solution s is random and it is given by a stochastic process

$$
s(0,t) = \psi_t, \qquad \psi_t \in C^\beta, \quad 1/4 < \beta < 1/2. \tag{5}
$$

The specific process we have in mind is a specific Pearson process $\{\psi_t\}_{t\in[0,T]}$, solution to the following mean reverting generalized Jacobi stochastic differential equation

$$
\begin{aligned}
&d\psi_t = \alpha(\gamma - \psi_t)dt + \sigma\sqrt{\psi_t\,(\eta - \psi_t)}dW_t, \\
&\psi_0 \in (0,\eta)
\end{aligned} \tag{6}
$$

with $\alpha, \sigma, \gamma, \eta \in \mathbb{R}_+$, and $\gamma \leq \eta$. Overall we have a PDE problem with a boundary condition which is not only stochastic but with less regularity. Indeed, the following properties hold.

Proposition 1. *Let us consider equation* (6)*, with $\alpha, \sigma, \gamma, \eta \in \mathbb{R}_+$ and $\gamma \leq \eta$. Assume $\psi_0 \in [0,\eta]$. The solution of equation* (6) *exists (globally on $[0,T]$) and is pathwise unique for any $\psi_0 \in [0,\eta]$. Furthermore, let us suppose that the following conditions upon the parameters hold*

$$
\alpha\gamma \geq \frac{\sigma^2\eta}{2}; \qquad \alpha(\eta-\gamma) \geq \frac{\sigma^2\eta}{2}. \tag{7}
$$

Then for any $t \in (0,T]$,

$$
0 \leq \psi_t \leq \eta, \tag{8}
$$

in particular we have $\psi_t \in L^\infty(\mathbb{R}_+)$.

Proof. We give a sketch of the proof. See, for example, [22]. We notice that the drift coefficient $\alpha_1(x) = \alpha(\gamma - x)$ is Lipschitz continuous for any $x \in \mathbb{R}$, while the diffusion coefficient $\sigma_1(x) = \sigma\sqrt{x(\eta - x)}$ is only continuous for any $x \in [0,\eta]$. Furthermore, for any $x \in [0,\eta]$,

$$
\begin{aligned}
|\alpha_1(x)|^2 &= |\alpha\gamma - \alpha x|^2 \leq \alpha^2(|\gamma|^2 + |2\gamma x| + |x|^2) \leq C_1(\alpha,\gamma)\Big(1 + |x|^2\Big) \\
|\sigma_1(x)|^2 &= |\sigma\sqrt{x(\eta - x)}|^2 \leq |\sigma|^2(|\eta x| + |x|^2) \leq C_2(\sigma,\eta)(1 + |x|^2)
\end{aligned}
$$

By classical results due to Skorokhod, we get that the solution exists [28]. The pathwise uniqueness can be proven by applying the Yamada-Watanabe uniqueness criterion in [27], since

$$
|\sigma_1(x) - \sigma_1(y)| \leq \rho(|x-y|), \quad \forall x,y \in \mathbb{R}, \quad i = 1,2,\ldots,
$$

with $\rho(x) = \sqrt{x}$. Indeed,

$$
\begin{aligned}
|\sigma_1(x) - \sigma_1(y)| &\leq \sigma\sqrt{|\eta x - x^2 - \eta y + y^2|} \\
&\leq \sigma\sqrt{|x-y|}\sqrt{|\eta - x - y|} \\
&\leq C_3(\eta)\cdot|x-y|^{1/2}.
\end{aligned}
$$

Hence, the diffusion coefficient $\sigma^2(x)$ is Hölder continuous with index 1/2, for any $x \in [0,\eta]$.

In order to obtain the bound (8) for the solution, we consider the well-known Feller classification of the boundaries [17,22]. In the case under study, given a little generalization of the Jacobi process case, we have that the scale function $scale(x)$ and the speed density $m(x)$ are defined as

$$
scale(x) = \frac{1}{x^p(\eta - x)^q}, \qquad m(x) = \frac{1}{\sigma^2}x^{p-1}(\eta - x)^{q-1},
$$

with

$$p = \frac{2\alpha\gamma}{\sigma^2\eta}, \qquad q = \frac{2\alpha(\eta - \gamma)}{\sigma^2\eta}.$$

The boundary $x = 0$ is an *entrance boundary* if and only if $p \geq 1$ while $x = \eta$ is an *entrance boundary* if and only if $q \geq 1$ and these two condition are satisfied whenever (7) hold (see [22],[4]). □

We also recall the classical Hölder regularity property of solutions to the SDE:

Proposition 2. *The solution of equation* (6) *has trajectories almost surely in* $C^\beta([0,T])$, $\beta \in (0, 1/2)$.

Proof. If ψ is a solution to (6), then, for any $s, t \in [0,T]$, one may write, for $p > 2$

$$|\psi_t - \psi_s|^p \leq \left| \int_s^t \alpha(\gamma - \psi_\tau) d\tau \right|^p + \left| \int_s^t \sigma \sqrt{\psi_\tau (\eta - \psi_\tau)} dW_\tau \right|^p. \tag{9}$$

By Burkholder-Davis-Gundy inequality [26](IV,42), and by the bound (8) one obtained for every $2 < p < \infty$, there exists $C > 0$ such that, for every $s < t$,

$$\mathbb{E}\left[\left| \int_s^t \sigma \sqrt{\psi_\tau (\eta - \psi_\tau)} dW_\tau \right|^p \right] \leq C \mathbb{E}\left[\left(\int_s^t \left| \sigma \sqrt{\psi_\tau (\eta - \psi_\tau)} \right|^2 dr \right)^{p/2} \right] \leq C(\eta) \, |t - s|^{p/2}.$$

Then from (9), we have

$$\mathbb{E}\left[|\psi_t - \psi_s|^p \right] \leq C(\alpha, \gamma) |t - s|^p + C(\eta) \, |t - s|^{p/2} \leq C(\alpha, \gamma, \eta) \, |t - s|^{p/2}$$

By Kolmogorov continuity criterion, with $r_1 = p$ and $1 + r_2 = p/2$, the solution ψ_t is C^β-Hölder continuous for any $\beta \in (0, r_2/r_1) = (0, 1/2 - 1/p)$ and $p > 2$, that is $\beta \in (0, 1/2)$. □

Remark 3. By the previous results, the equation (6) is a good model for a process with the regularity required by (5).

A model for the porosity The function φ in system (3)-(4) refers to the porosity, which is the fraction of void volume with respect to the total one. Considering the literature on the subject, we assume (see, for example, [2])

$$\varphi(c) = A + Bc. \tag{10}$$

with $B = \pm 1$. In the specific case of marble sulphation the value $B = -1$ is more appropriate.

Proposition 4. *If equation* (10) *holds, the system* (3)-(5) *becomes*

$$\begin{aligned} \partial_t s &= \partial_x^2 s + b_c(t,x) \partial_x s + \gamma_c(t,x) s(Bs - 1) & (11) \\ \partial_t c &= -\lambda s(A + Bc)c & (12) \end{aligned}$$

with

$$\begin{aligned} b_c(t,x) &= B \frac{\partial_x c}{(A + Bc)} & (13) \\ \gamma_c(t,x) &= \lambda c & (14) \end{aligned}$$

Proof. Equation (11) is derived in the following simple way

$$\begin{aligned} \partial_t((A + Bc)s) &= \nabla \cdot ((A + Bc) \nabla s) - \lambda(A + Bc)sc \\ A\partial_t s + sB\partial_t c + Bc\partial_t s &= A\partial_x^2 s + B\nabla(c\nabla s) - \lambda(A + Bc)cs \\ (A + Bc)\partial_t s &= A\partial_x^2 s + B\partial_x c \partial_x s + Bc\partial_x^2 s - \lambda c(A + Bc)s + B\lambda c(A + Bc)s^2 \\ (A + Bc)\partial_t s &= (A + Bc)\partial_x^2 s + B\partial_x c \partial_x s + \lambda cs(A + Bc)(Bs - 1) \end{aligned}$$

By dividing by $(A + Bc)$ we have the result. □

We want to prove an existence and uniqueness result for system (11),(12)-(14).

Let us introduce the heat kernel $G(t,x)$ on the full line

$$G(t,x) = t^{-1/2} G_1(t^{-1/2}x) = \frac{1}{\sqrt{4\pi t}} e^{-x^2/4t}, \tag{15}$$

with $G_1(x)$ is the heat kernel $G(t,x)$ evaluated at $t = 1$.

Definition 5 (Bounded positive mild solution)**.** We say that a couple (s,c), with $s \in L^\infty([0,T], W^{1,2}(\mathbb{R}_+)) \cap L^\infty([0,T]\times\mathbb{R}_+)$ and $c \in B_b([0,T]\times\mathbb{R}_+)$, the space of bounded Borel functions, is *a bounded positive mild solution* of the nonlinear PDE (11),(12)-(14) if, for every $x \in \mathbb{R}_+$, the function $c(\cdot,x)$ solves (12)-(14), that is c is explicitly given by

$$c(t,x) = \frac{c_0(x)}{\varphi(c_0(x))e^{\lambda\int_0^t s(x,\tau)d\tau} - Bc_0(x)}, \tag{16}$$

and the function s satisfies

$$0 \le s(t,x) \le \eta \quad \text{for every } (t,x) \in [0,T]\times\mathbb{R}_+,$$

and it is a $L^\infty([0,T], W^{1,2}(\mathbb{R}_+))$-mild solution of (11), that is

$$\begin{aligned} s(t,\cdot) &= -2\int_0^t \partial_x G(t-\tau,\cdot)\psi(\tau)d\tau + G(t,\cdot)*s_0 \\ &+ \int_0^t G(t-\tau,\cdot)*\left(b_c(\tau,\cdot)\partial_x s(\tau,\cdot) + \gamma_c(\tau,\cdot)s(\tau,\cdot)\right)(Bs(\tau,\cdot)-1))ds. \end{aligned} \tag{17}$$

In equation (17) b_c and γ_c are defined as (13) and (14), respectively and $G(t,\cdot)*f$ denotes the convolution-type operator with the heat kernel (15), defined by (50),

The idea to show well-posedness of (11),(12)-(14) is to consider a splitting strategy in order to divide the problem into two sub-problems: in the first one we consider the heat equation with the stochastic, Hölder continuous boundary condition, in the second problem we consider a non linear PDE with zero boundary condition. The initial conditions of the original problems are inherited by the latter system.

A splitting strategy Let us consider the following splitting of the solution s of Equation (11)

$$s = u + v,$$

where the function u is the solution of the following *heat equation with a stochastic boundary condition*

$$\partial_t u = \partial_x^2 u \tag{18}$$
$$u(x,0) = 0, \qquad x \in (0,\infty) \tag{19}$$
$$u(0,t) = \psi_t, \qquad \psi_t \in C^\beta([0,T]), \quad \beta < 1/2 \tag{20}$$

and the function v is the solution of the following *deterministic reaction diffusion problem*

$$\frac{\partial}{\partial t}v = \partial_x^2 v + b(t,x)\partial_x v - \gamma(t,x)v + b(t,x)\partial_x u - \gamma(t,x)u + \gamma(t,x)B(u+v)^2 \tag{21}$$
$$\partial_t c = -\lambda s(A+Bc)c \tag{22}$$

with

$$b(t,x) = B\frac{\partial_x c}{(A+Bc)}, \tag{23}$$
$$\gamma(t,x) = \lambda c. \tag{24}$$

The system (21)-(22) has the following initial and boundary conditions

$$\begin{aligned} c(x,0) &= c_0(x), & x \in (0,\infty); \\ v(x,0) &= s_0(x), & x \in (0,\infty); \\ v(0,t) &= 0, & t \in (0,T]. \end{aligned}$$

Remark 6. Let us note that the system for u (18)-(20) is completely autonomous: it does not depend on the function c of the original system (11)-(14). Differently, the system for v is coupled with that of (18)-(20).

3 Bounds on the solution of the heat equation with stochastic boundary condition

We first study the regularity of the system (18)-(20); though the regularity result for this system is rather classical, we give a self-contained argument. Let us recall that equation (18) admits an explicit solution given in term of the fundamental heat solution. Indeed, by recalling that G is the heat kernel as in 15, we have (see, e.g. [9]):

$$u(t,x) = -2\int_0^t \partial_x G(t-\tau,x)\psi(\tau)d\tau, \tag{25}$$

From (15), the space derivatives of the heat kernel G are given by

$$\partial_x G(t,x) = t^{-1}G_1'(t^{-1/2}x) = -\frac{1}{4\sqrt{\pi}}t^{-3/2}xe^{-x^2/4t}, \tag{26}$$

$$\partial_x^2 G(t,x) = t^{-3/2}G_1''(t^{-1/2}x) = \frac{1}{4\sqrt{\pi}}\left(\frac{1}{2}t^{-1}x^2-1\right)t^{-3/2}e^{-x^2/4t}. \tag{27}$$

Hence, for $p\in[1,\infty)$, there exist some constant $c_p, c_p' \in \mathbb{R}_+$ such that, for any $t\in[0,T]$

$$\begin{aligned}\|G(t,\cdot)\|_{L^p(\mathbb{R}_+)} &= c_p\, t^{-(1-1/p)/2},\\ \|\partial_x G(t,\cdot)\|_{L^p(\mathbb{R}_+)} &= c_p'\, t^{-(2-1/p)/2}.\end{aligned} \tag{28}$$

Proposition 7. *For any ψ bounded Borel function, the solution u given by* (25) *is such that $u\in C^\infty[(0,T)\times(0,\infty)]$ and u and its derivatives admit continuous extensions to $[0,T)\times(0,\infty)$.*

Proof. The statement follows from the fact that the kernel G in (15) is C^∞ on $(0,T)\times(0,\infty)$ and G and its derivatives admit continuous extensions to $[0,T)\times(0,\infty)$. □

The first step is to establish a bound for the norm of u in a fractional Sobolev space, uniformly in time. Hence we recall here the definition of such spaces.

Definition 8 (The fractional Sobolev spaces)**.** We consider the fractional Sobolev space $W^{\alpha,p}$ upon $\mathbb{R}$ for $1\le p\le\infty$ and $0<\alpha<1$ as follows [14, 29]

$$W^{\alpha,p}(\mathbb{R}) = \left\{ f\in L^p(\mathbb{R}) : \frac{|f(x)-f(y)|}{|x-y|^{1/p+\alpha}} \in L^p(\mathbb{R})\right\}, \tag{29}$$

an intermediate Banach space between $L^p(\mathbb{R})$ and $W^{1,p}(\mathbb{R})$, endowed with the natural norm

$$\|f\|^p_{W^{\alpha,p}(\mathbb{R})} = \|f\|^p_{L^p} + [f]^p_{W^{\alpha,p}(\mathbb{R})},$$

where

$$[f]^p_{W^{\alpha,p}(\mathbb{R})} = \iint \frac{|f(x)-f(y)|^p}{|x-y|^{1+\alpha p}}dxdy$$

is the so-called Gagliardo (semi)norm of f. Furthermore, one can define the fractional Sobolev space with exponent $k+\alpha$, with $k\in\mathbb{N}$

$$W^{k+\alpha,p}(\mathbb{R}) = \left\{ f\in W^{k,p}(\mathbb{R}) : \partial^k f\in W^{\alpha,p}(\mathbb{R})\right\}, \tag{30}$$

endowed with the norm

$$\|f\|^p_{W^{k+\alpha,p}(\mathbb{R})} = \|f\|^p_{L^p} + \|\partial^k f\|^p_{L^p} + [\partial^k f]^p_{W^{\alpha,p}(\mathbb{R})}. \tag{31}$$

In the case $\alpha=0$ the previous space is the classical Sobolev space. The previous definitions are extended also to $\mathbb{R}_+$.

Proposition 9. *Let u be the solution of the system* (18)-(20)*, with the boundary condition such that $\psi\in L^\infty([0,T])$. Then, for any $0<\alpha<1$ and $1<p<1/\alpha$, and for a constant C not depending upon ψ, we have P-a.s.*

$$\sup_{t\in[0,T]}\|u(t,\cdot)\|_{W^{\alpha,p}(\mathbb{R}_+)} \le C\|\psi\|_{L^\infty([0,T])}. \tag{32}$$

Proof. By Proposition 7 we already have some continuous properties out from the boundary. Now we study the behavior of the solution near the boundary $x = 0$.

From (26), by considering the self similarity rescaling $\xi = x/\sqrt{t-\tau}$, we analize carefully the behavior of u both close and far from the origin. For some $c > 0$ and a constant C, that may be different from line to line, one may obtain from (25) the following bound for u, for any $(t, x) \in [0, T] \times \mathbb{R}_+$

$$\begin{aligned}
|u(t,x)| &\leq 2\|\psi\|_{L^\infty([0,T])} \int_0^t |\partial_x G(t-\tau, x)|\, d\tau \\
&= C\|\psi\|_{L^\infty([0,T])} \int_{x/\sqrt{t}}^\infty \frac{\xi^2}{x^2} G_1'(\xi) \frac{x^2}{\xi^3} d\xi \\
&\leq C \left(1_{x\leq 1} + e^{-cx} 1_{x>1}\right) \|\psi\|_{L^\infty([0,T])} \int_0^\infty \xi^{-1} G_1'(\xi) d\xi \\
&= C \left(1_{x\leq 1} + e^{-cx} 1_{x>1}\right) \|\psi\|_{L^\infty([0,T])} \int_0^\infty e^{-\xi^2} d\xi \\
&= C \left(1_{x\leq 1} + e^{-cx} 1_{x>1}\right) \|\psi\|_{L^\infty([0,T])}.
\end{aligned} \tag{33}$$

Hence, for any $t \in [0, T]$, we have

$$\|u(t,\cdot)\|_{L^\infty(\mathbb{R}_+)}, \|u(t,\cdot)\|_{L^p(\mathbb{R}_+)} \leq C\, \|\psi\|_{L^\infty([0,T])}, \quad \text{for any } 1 \leq p < \infty. \tag{34}$$

In a similar way, taking the spatial derivative of (25), we get the following bound for the spatial derivative of u, for any $(t, x) \in [0, T] \times \mathbb{R}_+$

$$\begin{aligned}
|\partial_x u(t,x)| &\leq 2\|\psi\|_{L^\infty([0,T])} \int_0^t \left|\partial_x^2 G(t-\tau, x)\right| d\tau \\
&= C\|\psi\|_{L^\infty([0,T])} \int_{x/\sqrt{t}}^\infty \frac{\xi^3}{x^3} G_1''(\xi) \frac{x^2}{\xi^3} d\xi \\
&\leq C\|\psi\|_{L^\infty}([0,T]) \left(\frac{1}{x} 1_{x\leq 1} + e^{-cx} 1_{x>1}\right) \int_0^\infty G_1''(\xi) d\xi \\
&= C \left(\frac{1}{x} 1_{x\leq 1} + e^{-cx} 1_{x>1}\right) \|\psi\|_{L^\infty([0,T])}.
\end{aligned} \tag{35}$$

We can estimate the $W^{\alpha,p}(\mathbb{R}_+)$ seminorm of $u(t,\cdot)$, for any $t \in [0, T]$; indeed, we may write

$$\begin{aligned}
[u(t,\cdot)]^p_{W^{\alpha,p}(\mathbb{R}_+)} &= \iint_{|x-y|<1} \frac{|u(t,x) - u(t,y)|^p}{|x-y|^{1+\alpha p}} dxdy + \iint_{|x-y|>1} \frac{|u(t,x) - u(t,y)|^p}{|x-y|^{1+\alpha p}} dxdy \\
&\leq 2 \iint_{x<y<x+1} \frac{|u(t,x) - u(t,y)|^p}{|x-y|^{1+\alpha p}} dxdy + 2^p \iint_{|x-y|>1} \frac{|u(t,x)|^p}{|x-y|^{1+\alpha p}} dxdy \\
&\leq 2 \iint_{x<y<x+1} \frac{|u(t,x) - u(t,y)|^p}{|x-y|^{1+\alpha p}} dxdy + C\|u\|^p_{L^p(\mathbb{R}_+)},
\end{aligned}$$

where the last line is due to the equation (34) and the fact that the function $x^{-1-\alpha p}$ is integrable at infinity. By using the inequality $e^{-cz} \leq Cz^{-1}$ and by (34) and (35), we have that, for any $t \in [0, T]$,

$$\begin{aligned}
[u(t,\cdot)]^p_{W^{\alpha,p}(\mathbb{R}_+)} &\leq C\|\psi\|^p_{L^\infty([0,T])} + 2 \iint_{x<y<x+1} |x-y|^{-1-\alpha p} \left(\int_x^y |\partial_z u(t,z)| dz\right)^p dxdy \\
&\leq C\|\psi\|^p_{L^\infty([0,T])} + 2 \iint_{x<y<x+1} |x-y|^{-1-\alpha p} \left(\int_x^y C(z^{-1} 1_{z\leq 1} + e^{-cz} 1_{z>1}) \|\psi\|_{L^\infty([0,T])} dz\right)^p dxdy \\
&\leq C\|\psi\|^p_{L^\infty([0,T])} \left(1 + \iint_{x<y<x+1} |x-y|^{-1-\alpha p} \left(|\log(y/x)|^p 1_{x\leq 1} + |x-y|^p e^{-cpx} 1_{x>1}\right) dxdy\right) \\
&\leq C\|\psi\|^p_{L^\infty([0,T])} \left(1 + \int_0^1 dx \int_x^\infty |x|^{-1-\alpha p} |1 - y/x|^{-1-\alpha p} |\log(y/x)|^p dy\right) \\
&\quad + C\|\psi\|^p_{L^\infty([0,T])} \int_1^\infty dx \int_x^{x+1} |x-y|^{-1+(1-\alpha)p} e^{-cpx} dy.
\end{aligned}$$

By the changes of variable $w = y/x$ and $v = y - x$, in the first and second integral, respectively, we obtain,for any $t \in [0, T]$,

$$[u(t,\cdot)]^p_{W^{\alpha,p}(\mathbb{R}_+)} \le C\|\psi\|^p_{L^\infty([0,T])} + C\|\psi\|^p_{L^\infty([0,T])} \int_0^1 dx\, |x|^{-\alpha p} \int_1^\infty |1-w|^{-1-\alpha p} |\log w|^p du$$
$$+ C\|\psi\|^p_{L^\infty([0,T])} \int_1^\infty dx \int_0^1 v^{-1+(1-\alpha)p} e^{-cpx} dv.$$

The second integral in the above expression is clearly finite. Concerning the first integral, note that, for w large, $|1 - uw|^{-1-\alpha p}|\log w|^p$ is integrable, while for w close to 1,

$$|1 - w|^{-1-\alpha p}|\log w|^p \approx |1 - w|^{-\alpha p}.$$

Hence, the first integral is finite if and only if $\alpha p < 1$. Hence, if $0 < \alpha < 1$ and $1 < p < 1/\alpha$, we reach the conclusion. □

Given more regularity on the boundary condition, we get a bound on the norm of u in a $2 + \alpha$ fractional Sobolev space, uniformly in time, with $\alpha \in (0, 1)$.

Proposition 10. *Let u be the solution of the system* (18)-(20)*, with the boundary condition such that ψ is Lipschitz and $\psi(0) = 0$. Then, for any $0 < \alpha < 1$ and $1 < p < 1/\alpha$, and for a constant C not depending upon ψ, we have P-a.s.*

$$\sup_{t\in[0,T]} \|u(t,\cdot)\|_{W^{2+\alpha,p}(\mathbb{R}_+)} \le C\|\psi\|_{W^{1,\infty}([0,T])}. \tag{36}$$

Proof. Note that $\partial_x^2 G = \partial_t G$,due to (15) and (27); furthermore from (19) it follows that $G(0, x) = 0$ for $x > 0$. By an integration by parts in time, we get, for any $(t, x) \in [0, T] \times \mathbb{R}_+$

$$\begin{aligned}
\partial_x u(t,x) &= -2\int_0^t \partial_x^2 G(\tau,x)\psi(t-\tau)d\tau \\
&= -2\int_0^t \partial_\tau G(\tau,x)\psi(t-\tau)d\tau \\
&= -2G(\tau,x)\psi(t-\tau)\,|_{\tau=0}^t -2\int_0^t G(\tau,x)\dot\psi(t-\tau)d\tau \\
&= -2\int_0^t G(\tau,x)\dot\psi(t-\tau)d\tau.
\end{aligned}$$

Differentiating (25) with respect to x, we get

$$\partial_x^2 u(t,x) = -2\int_0^t \partial_x G(\tau,x)\dot\psi(t-\tau)d\tau. \tag{37}$$

Hence, by comparing (15) and (37), one can deduce that for any $(t, x) \in [0, T] \times \mathbb{R}_+$, $\partial_x^2 u(t, x)$ is the solution to the heat equation (18), with boundary condition at $x = 0$ given by ψ. Hence, applying the bound (32) to $\partial_x^2 u$, with $\dot\psi$ replacing ψ, we get

$$\sup_{t\in[0,T]} \|\partial_x^2 u(t,\cdot)\|_{W^{\alpha,p}(\mathbb{R}_+)} \le C\|\dot\psi\|_{L^\infty([0,T])}. \tag{38}$$

By (32) and (38) we finally obtain the bound of the norm (31) with $k = 2$, that is the statement (36).

□

The next step is to give the main result of the section, i.e a continuous result for the solution u of the system (18)-(20) in $W^{1,p}(\mathbb{R}_+)$, uniformly in time, with respect to the boundary condition. In order to obtain the result we consider compatible Banach spaces in Propositions (9) and (10). Let us just recall the definition of interpolation pair of Banach spaces. The interest reader may refer to [6,23]. Here, for the interpolation methods and results, we mainly refer to [25,29].

Definition 11. If (X_0, X_1) is a compatible couple of Banach spaces, we have that

$$X_0 \cap X_1 \subset X_0, X_1 \subset X_0 + X_1,$$

where both $X_0 \cap X_1, X_0 + X_1$ are Banach space with the norms

$$\begin{aligned} \|x\|_{X_0 \cap X_1} &:= \max\left(\|x\|_{X_0}, \|x\|_{X_1}\right), \\ \|x\|_{X_0 + X_1} &:= \inf\left\{\|x_0\|_{X_0} + \|x_1\|_{X_1} \ : \ x = x_0 + x_1, \ x_0 \in X_0, \ x_1 \in X_1\right\}. \end{aligned}$$

An *interpolation space* is a space X that is an intermediate space between X_0 and X_1 in the sense that

$$X_0 \cap X_1 \subset X \subset X_0 + X_1,$$

where the two inclusions maps are continuous. Given two compatible couples (X_0, X_1) and (Y_0, Y_1), an *interpolation pair* is a couple (X, Y) of Banach spaces with the two following properties:

i) The space X is intermediate between X_0 and X_1, and Y is intermediate between Y_0 and Y_1.

ii) If L is any linear operator from $X_0 + X_1$ to $Y_0 + Y_1$, which maps continuously X_0 to Y_0 and X_1 to Y_1, then it also maps continuously X to Y.

Definition 12. Let (X_0, X_1) a compatible couple of Banach spaces , if $0 < \theta < 1$ and $1 \le q \le \infty$ with $(X_0, X_1)_{\theta,q}$ one denotes the Banach space obtained by the real interpolation K method with parameters θ and q. To be more precise, for $t > 0$ and any $x \in X_0 + X_1$, let

$$K(x, t; X_0, X_1) = \inf\left\{\|x_0\|_{X_0} + t\,\|x_1\|_{X_1} \ : \ x = x_0 + x_1, \ x_0 \in X_0, \ x_1 \in X_1\right\}.$$

Let

$$\begin{aligned} \|x\|_{\theta,q} &= \left(\int_0^\infty \left(t^{-\theta} K(x, t; X_0, X_1)\right)^q \frac{dt}{t}\right)^{\frac{1}{q}} = \|K(x, t; X_0, X_1)\|_{L^q_*(\mathbb{R}_+)}, && 0 < \theta < 1, \ 1 \le q < \infty, \\ \|x\|_{\theta,\infty} &= \sup_{t>0} t^{-\theta} K(x, t; X_0, X_1), && 0 \le \theta \le 1, \end{aligned}$$

where $L^q_*(\mathbb{R})$ denotes the L^q space with respect to the measure dt/t. Then $(X_0, X_1)_{\theta,q}$ is the following Banach space

$$(X_0, X_1)_{\theta,q} = \{x \in X_0 + X_1 : \|x\|_{\theta,q} < \infty\}.$$

Proposition 13. *Assume that* $0 < \beta < 1/2$ *and take*

$$1 \le q < \frac{1}{1 - 2\beta}. \tag{39}$$

Let u be the solution of the system (18)-(20)*. Then, for every $\psi \in C^\beta[0, T]$ with $\psi(0) = 0$, and for a constant C not depending upon ψ, we have P-a.s.*

$$\|u\|_{L^\infty([0,T], W^{1,q}(\mathbb{R}_+))} = \sup_{t \in [0,T]} \|u(t, \cdot)\|_{W^{1,q}(\mathbb{R}_+)} \le C\|\psi\|_{C^\beta([0,T])}. \tag{40}$$

Proof. Given the two compatible couples $\left(C([0, T]), C^1([0, T])\right)$ and $\left(W^{\alpha,p}(\mathbb{R}_+), W^{2+\alpha,p}(\mathbb{R}_+)\right)$ we seek for an interpolation couple where the first component is the space to which the initial condition belongs and the second gives the regularity of the solution of the system (18)-(20).

Let us denote by A the solution operator given, where defined, by (25), that is

$$A\psi = u.$$

By (32) and (36), A is a bounded linear operator from $C([0, T]) + C^1([0, T])$ to $W^{\alpha,p}(\mathbb{R}_+) + W^{2+\alpha,p}(\mathbb{R}_+)$, which maps continuously $C([0, T])$ to $W^{\alpha,p}(\mathbb{R}_+)$ and $C^1([0, T])$ to $W^{2+\alpha,p}(\mathbb{R}_+)$. To be precise, A is only defined on subspaces of $C([0, T])$ and $C^1([0, T])$ of functions ψ such that $\psi(0) = 0$. However, we can easily extend A to the full spaces by composing it with the operator $\psi \mapsto \psi - \psi(0)$.

By Theorem 1.1.6 and Proposition 1.1.4 in [25], for any $0 < \beta < 1$ and $\epsilon > 0$ small, A is a bounded operator such that

$$(C([0, T]), C^1([0, T]))_{\beta+\epsilon,\infty} \hookrightarrow (C([0, T]), C^1([0, T]))_{\beta,p} \xrightarrow{A} (W^{\alpha,p}(\mathbb{R}_+), W^{2+\alpha,p}(\mathbb{R}_+))_{\beta,p}. \tag{41}$$

By [29] (Section 4.4.1) and [25] (Examples 1.1.8 and 1.3.7), for any $0 < \beta < 1$ and $\epsilon > 0$ small the interpolation space at the left side of the map (41) is

$$(C([0,T]), C^1([0,T]))_{\beta+\epsilon,\infty} = C^{\beta+\epsilon}([0,T]).. \tag{42}$$

By [29](Sections, 4.3.1. and 4.5.2.) and [25] (Examples 1.3.8 and 1.3.10.), whenever $2\beta + \alpha$ is not an integer, the interpolation space at the right hand side of the map (41) is

$$(W^{\alpha,p}(\mathbb{R}_+), W^{2+\alpha,p}(\mathbb{R}_+))_{\beta,p} = W^{(1-\beta)\alpha+\beta(2+\alpha),p}(\mathbb{R}_+) = W^{2\beta+\alpha,p}(\mathbb{R}_+). \tag{43}$$

Hence, by (41)-(43) we obtain, for $0 < \alpha < 1$, $1 < p < 1/\alpha$, $0 < \beta < 1$ with $2\beta + \alpha \notin \mathbb{N}$,

$$\sup_{t\in[0,T]} \|u(t,\cdot)\|_{W^{2\beta+\alpha,p}(\mathbb{R}_+)} \leq C\|\psi\|_{C^{\beta+\epsilon}([0,T])}. \tag{44}$$

Now, by hypothesis $0 < \beta < 1/2$. By the Sobolev embeddings [25, 29], we may apply Proposition 40. We consider $s = 2\beta + \alpha \in \mathbb{R}_+ \setminus \mathbb{N}$ and $s > 1$, $s' = 1$, and q such that

$$\frac{1}{q} = \frac{1}{p} + 1 - (2\beta + \alpha) > 1 - 2\beta, \tag{45}$$

so that $1 \leq q < 1/(1 - 2\beta)$. Hence, condition (80) is satisfied and from (81), we get

$$W^{2\beta+\alpha,p}(\mathbb{R}_+) \subseteq W^{1,q}(\mathbb{R}_+). \tag{46}$$

In conclusion, from (44) and (46) we can state that for given $0 < \beta < 1/2$, and $1 \leq q < 1/(1 - 2\beta)$, one can always find $0 < \alpha < 1$ and $1 < p < 1/\alpha$ satisfying (45), such that

$$\sup_{t\in[0,T]} \|u(t,\cdot)\|_{W^{1,q}(\mathbb{R}_+)} \leq C\|\psi\|_{C^{\beta+\epsilon}(0,T)}.$$

For the arbitrary of $\epsilon > 0$, the statement is proven. □

4 A priori estimates upon a linear equation

Let f be a function on $[0,T] \times \mathbb{R}_+$ such that $0 \leq f \leq \eta$, with η positive constant. First we study the following linear model for (s, c), where the unknown are $(\tilde{s}, \tilde{c})$

$$\begin{aligned} &\frac{\partial}{\partial t}\tilde{s} = \partial_x^2 \tilde{s} + b_{\tilde{c}}(t,x)\partial_x \tilde{s} + \widetilde{\gamma}_{\tilde{c}}(t,x)\tilde{s}, \\ &\partial_t \tilde{c} = -\lambda f(A + B\tilde{c})\tilde{c}, \end{aligned}$$

with function $b_{\tilde{c}}$ given by (13) and

$$\widetilde{\gamma}_{\tilde{c}} = -\lambda(1 - Bf)\tilde{c}. \tag{47}$$

Under more restrictive hypothesis upon f, that we will consider in Section 5, by the properties of the solution $\tilde{c}$ of the second equation, in [2, 20] it is shown that

$$b_{\tilde{c}} \in L^\infty([0,T], L^2(\mathbb{R}_+)), \quad \widetilde{\gamma}_{\tilde{c}} \in L^\infty([0,T] \times \mathbb{R}_+).$$

Hence, in this section the idea is to study the well posedness of the linear PDE

$$\begin{aligned} &\partial_t \tilde{s} = \partial_x^2 \tilde{s} + b(t,x)\partial_x \tilde{s} + \widetilde{\gamma}(t,x)\tilde{s}, \\ &\tilde{s}(0,x) = s_0 \\ &\tilde{s}(t,0) = \psi(t), \end{aligned} \tag{48}$$

under the previous hypothesis of regularity for the functions $b, \widetilde{\gamma} : [0,T] \times (0,\infty) \to \mathbb{R}$ and the boundary condition $\psi : [0,T] \to \mathbb{R}$ given by a Borel function in C^β. The solution $\tilde{s} : [0,T] \times [0,\infty) \to \mathbb{R}$ is meant as a mild solution [9, 24] according with the following definition.

Definition 14 (Mild solution for $\tilde{s}$)**.** A function $\tilde{s} : [0,T] \times [0,\infty) \to \mathbb{R}$ is a $L^\infty([0,T], W^{1,p}(\mathbb{R}_+))$ *mild solution* of the equation (48) if $\tilde{s} \in L^\infty([0,T], W^{1,p}(\mathbb{R}_+))$ and it is such that, for every $t \in [0,T]$,

$$\tilde{s}(t,\cdot) = -2\int_0^t \partial_x G(t-\tau,\cdot)\psi(\tau)d\tau + G_t * s_0 + \int_0^t G_{t-\tau} * \left(b(\tau,\cdot)\partial_x \tilde{s}(\tau,\cdot) + \tilde{\gamma}(\tau,\cdot)\tilde{s}(\tau,\cdot)\right) d\tau, \tag{49}$$

where G is the heat kernel (15) and $G_t * f(x)$ denotes the following function

$$G_t * f(x) = \int_0^\infty (G(t,x-y) - G(t,x+y))f(y)dy = \int_{\mathbb{R}} G(t,x-y)f^{odd}(y)dy, \tag{50}$$

where $f^{odd}(x) = 1_{x>0}f(x) - 1_{x<0}f(-x)$ is the odd extension of f.

Remark 15. Let us consider some useful facts on $G * f$, as defined in (50). The interest reader may refer to [9] for more details. First, although $G * f$ is not the convolution with f in the strict sense, the Young inequality still holds for $G * f$, i.e. for $1 \le p', q', r' \le \infty$ with

$$\frac{1}{p'} = \frac{1}{r'} + \frac{1}{q'} - 1, \tag{51}$$

for any $f \in L^{q'}(\mathbb{R}_+)$, since by (28) $G \in L^{p'}(\mathbb{R}_+)$, then

$$\|G * f\|_{L^{p'}} \le \|G\|_{L^{r'}} \|f\|_{L^{q'}}. \tag{52}$$

Concerning derivatives, we have that for $f \in L^r(\mathbb{R}_+)$, $1 \le r \le \infty$, $\partial_x(G_t * f) = (\partial_x G_t) * f$. Furthermore, whenever $f \in W^{1,r}(\mathbb{R}_+)$, with $1 \le r \le \infty$ and $f(0) = 0$, we have also

$$\partial_x(G_t * f) = \int_{\mathbb{R}} G(t, \cdot - y)(\partial_x f)^{even}(y)dy := G_t * (\partial_x f)^{even}(y), \tag{53}$$

where $(\partial_x f)^{even}(x) = 1_{x>0}\partial_x f(x) + 1_{x<0}\partial_x f(-x)$ is the even extension of $\partial_x f$.

Now let us go back to the problem (48).

We split the solution $\tilde{s}$ as $\tilde{s} = \tilde{u} + \tilde{v}$ with $\tilde{u}$ again is the solution of the initial boundary problem for the homogeneous heat equation (18)-(20) and $\tilde{v}$ solution of the following non-homogeneous problem

$$\begin{aligned} &\partial_t \tilde{v} = \partial_x^2 \tilde{v} + b\partial_x \tilde{v} + \tilde{\gamma}\tilde{v} + b\partial_x \tilde{u} + \tilde{\gamma}\tilde{u}, && (t,x) \in (0,T] \times \mathbb{R}_+; \\ &\tilde{v}(t,0) = 0, && t \in (0,T]; \\ &\tilde{v}(0,x) = v_0(x) = s_0(x), && x \in \mathbb{R}_+. \end{aligned} \tag{54}$$

Definition 16 (Mild solution for $\tilde{v}$)**.** Let $2 \le p \le \infty$. A function $\tilde{v} : [0,T] \times [0,\infty) \to \mathbb{R}$ is a $L^\infty([0,T], W^{1,p}(\mathbb{R}_+))$ *mild solution* of (48) if it belongs to $L^\infty([0,T], W^{1,p}(\mathbb{R}_+))$ and satisfies, for any $t \in [0,T]$,

$$\begin{aligned} \tilde{v}(t,\cdot) = G_t * v_0 &+ \int_0^t G_{t-\tau} * (b(\tau,\cdot)\partial_x \tilde{v}(\tau,\cdot) + \tilde{\gamma}(\tau,\cdot)\tilde{v}(\tau,\cdot))d\tau \\ &+ \int_0^t G_{t-\tau} * \left(b(\tau,\cdot)\partial_x \tilde{u}(\tau,\cdot) + \tilde{\gamma}(\tau,\cdot)\tilde{u}(\tau,\cdot)\right) d\tau, \end{aligned} \tag{55}$$

where G is the heat kernel (15) and the operator $G_t * f$ is defined by (50).

Our aim is to show that under regularity conditions upon the coefficient functions, the boundary function and initial condition, the equation (54) is well-posed in $L^\infty([0,T], W^{1,p}(\mathbb{R}_+))$ as a mild solution.

Assumption 17. Let us assume the following hypothesis upon the functions in (54) and the boundary condition ψ.

A1. The boundary condition $\psi \in C^\beta([0,T])$, for some $\beta \in (1/4, 1/2)$ and it is such that $\psi(0) = 0$.

A2. The functions b and $\widetilde{\gamma}$ satisfy the following regularity conditions

$$b \in L^\infty([0,T], L^2(\mathbb{R}_+)), \quad \widetilde{\gamma} \in L^\infty([0,T] \times \mathbb{R}_+).$$

A3. The initial condition $v_0 \in W^{1,p}(\mathbb{R}_+)$ for some $2 \le p \le \infty$ and it is such that $v_0(0) = 0$.

First we provide some a priori bounds for $\tilde{v}$.

Proposition 18. *Under Assumption 17, for $p \ge q \ge 2$ such that condition (39) is fulfilled and $r > 0$ such that*

$$\frac{1}{r} = \frac{1}{2} + \frac{1}{p} - \frac{1}{q}, \tag{56}$$

for the function $\tilde{v}$ mild solution to (55) the following bounds hold, for any $t \in [0,T]$

$$\begin{aligned} \|\tilde{v}(t,\cdot)\|_{L^p(\mathbb{R}_+)} \le \|v_0\|_{L^p(\mathbb{R}_+)} + CT^{3/4} \left(\|b\|_{L^\infty([0,T],L^2(\mathbb{R}_+))} + \|\gamma\|_{L^\infty([0,T],L^\infty(\mathbb{R}_+))} \right) \|\tilde{v}\|_{L^\infty([0,T],W^{1,p}(\mathbb{R}_+))} \\ + T^{(1+1/r)/2} \left(\|b\|_{L^\infty([0,T],L^2(\mathbb{R}_+))} + \|\gamma\|_{L^\infty_t(L^\infty_x)} \right) \|\psi\|_{C^\beta([0,T])} \end{aligned} \tag{57}$$

$$\begin{aligned} \|\partial_x \tilde{v}(t,\cdot)\|_{L^p(\mathbb{R}_+)} \le \|\partial_x v_0\|_{L^p(\mathbb{R}_+)} + CT^{1/4} \left(\|b\|_{L^\infty([0,T],L^p(\mathbb{R}_+))} + \|\gamma\|_{L^\infty_t(L^\infty_x)} \right) \|\tilde{v}\|_{L^\infty([0,T],W^{1,p}(\mathbb{R}_+))} \\ + CT^{1/(2r)} \left(\|b\|_{L^\infty([0,T],L^2(\mathbb{R}_+))} + \|\gamma\|_{L^\infty([0,T],L^\infty(\mathbb{R}_+))} \right) \|\psi\|_{C^\beta([0,T])} \end{aligned} \tag{58}$$

Proof. In the following, for any $a, b \ge 2$, we denote by $h(a,b)$ the positive parameter such that $1/h(a,b) = 1/a + 1/b$; hence

$$\frac{1}{b} = \frac{1}{h(a,b)} + \frac{1}{a/(a-1)} - 1,$$

so that (51) is satisfied. In particular, from the previous relation and by hypothesis (56) upon the parameters, one have with $(a = 2, b = p)$

$$\begin{aligned} \frac{1}{p} &= \frac{1}{h(2,p)} + \frac{1}{2} - 1, \\ \frac{1}{p} &= \frac{1}{q} + \frac{1}{r} - \frac{1}{2} = \frac{1}{h(2,q)} + \frac{1}{r} - 1. \end{aligned}$$

Hence, for the set of parameters $(p, h(2,p), 2)$ and $(p, r, h(2,q))$ condition (56) is satisfied and we may apply the Young inequality (52). For simplicity we introduce here the following notations for the involved Banach spaces $L^p_x = L^p(\mathbb{R}_+)$ in space and $L^p_t(L^q_x) = L^p([0,T], L^q(\mathbb{R}_+))$ in time and space.

Note that by Sobolev embedding (e.g. [29] Eq. 4.6.1-(e)])

$$\|\tilde{u}\|_{L^\infty_x} \le C \|\tilde{u}\|_{W^{1,q}_x}, \tag{59}$$

and by Hölder inequality (since $q \le p$)

$$\|\tilde{u}\|_{L^p_x} \le \|\tilde{u}\|^{q/p}_{L^q_x} \|\tilde{u}\|^{1-q/p}_{L^\infty_x},$$

we obtain

$$\|\tilde{u}\|_{L^p_x} \le C \|\tilde{u}\|_{W^{1,q}_x}. \tag{60}$$

In order to estimate the $L^p(\mathbb{R}_+)$ norm of $\tilde{v}$, we apply recursively Young inequality for functions in L^p spaces and their convolution with the kernel G (52) and Hölder inequality (with $p, q \ge 2$). Hence, by considering also the L^p norm of G in (28) and (60), we get from (55)

$$
\begin{aligned}
\|\tilde v(t,\cdot)\|_{L^p_x} &\le \|G_t * v_0\|_{L^p_x} + \int_0^t \big[\|G_{t-\tau} * (b(\tau,\cdot)\partial_x \tilde v(\tau,\cdot))\|_{L^p_x} + \|G_{t-\tau} * (\widetilde\gamma(\tau,\cdot)\tilde v(\tau,\cdot))\|_{L^p_x} \\
&\qquad + \|G_{t-\tau} * (b(\tau,\cdot)\partial_x \tilde u(\tau,\cdot))\|_{L^p_x} + \|G_{t-\tau} * (\widetilde\gamma(\tau,\cdot)\tilde u(\tau,\cdot))\|_{L^p_x}\big]\, d\tau \\
&\le \|G_t\|_{L^1_x}\|v_0\|_{L^p_x} + \int_0^t \Big[\|G_{t-\tau}\|_{L^2_x}\|b(\tau,\cdot)\partial_x\tilde v(\tau,\cdot)\|_{L^{h(2,p)}_x} + \|G_{t-\tau}\|_{L^1_x}\|\widetilde\gamma(\tau,\cdot)\tilde v(\tau,\cdot)\|_{L^p_x} \\
&\qquad + \|G_{t-\tau}\|_{L^r_x}\|b(\tau,\cdot)\partial_x\tilde u(\tau,\cdot)\|_{L^{h(2,q)}_x} + \|G_{t-\tau}\|_{L^1_x}\|\widetilde\gamma(\tau,\cdot)\tilde u(\tau,\cdot)\|_{L^p_x}\Big]\, d\tau \\
&\le c_1\|v_0\|_{L^p_x} + C\int_0^t \Big[(t-\tau)^{-1/4}\|b(\tau,\cdot)\|_{L^2_x}\|\partial_x\tilde v(\tau,\cdot)\|_{L^p_x} + \|\widetilde\gamma(\tau,\cdot)\|_{L^\infty_x}\|\tilde v(\tau,\cdot)\|_{L^p_x} \\
&\qquad + (t-\tau)^{-(1-1/r)/2}\|b(\tau,\cdot)\|_{L^2_x}\|\partial_x\tilde u(\tau,\cdot)\|_{L^q_x} + \|\widetilde\gamma_s\|_{L^\infty_x}\|\tilde u(\tau,\cdot)\|_{L^p_x}\Big]\, d\tau \\
&\le c_1\|v_0\|_{L^p_x} + (CT^{3/4}\|b\|_{L^\infty_t(L^2_x)} + CT\|\widetilde\gamma\|_{L^\infty_t(L^\infty_x)})\|\tilde v(\tau,\cdot)\|_{L^\infty_t(W^{1,p}_x)} \\
&\qquad + (CT^{(1+1/r)/2}\|b\|_{L^\infty_t(L^2_x)} + CT\|\widetilde\gamma\|_{L^\infty_t(L^\infty_x)})\|\tilde u(\tau,\cdot)\|_{L^\infty_t(W^{1,q}_x)}
\end{aligned}
$$

The conclusion (57) follows from the last inequality and Proposition 13, because of Assumption 17-A1.

For obtaining the estimate for the L^p_x norm of $\partial_x\tilde v$, we write the equation for $\partial_x\tilde v$, using the expressions for $\partial_x(G_t * f)$ given in (53) and the fact that $v_0(0) = 0$:

$$
\begin{aligned}
\partial_x\tilde v(t,\cdot) = G_t * (\partial_x v_0)^{even} &+ \int_0^t \partial_x G_{t-\tau} * (b(\tau,\cdot)\partial_x\tilde v(\tau,\cdot) + \widetilde\gamma(\tau,\cdot)\tilde v(\tau,\cdot))\, d\tau \\
&+ \int_0^t \partial_x G_{t-\tau} * (b(\tau,\cdot)\partial_x\tilde u(\tau,\cdot) + \widetilde\gamma(\tau,\cdot)\tilde u(\tau,\cdot))\, d\tau.
\end{aligned}
$$

In a similar way as for inequality (57), we prove the bound (58). Hence, by (51)-(53) ,(56) and (60), we obtain

$$
\begin{aligned}
\|\partial_x\tilde v(t,\cdot)\|_{L^p_x} &\le \|G_t * (\partial_x v_0)^{even}\|_{L^p_x} + \int_0^t \big[\|\partial_x G_{t-\tau} * (b(\tau,\cdot)\partial_x\tilde v(\tau,\cdot))\|_{L^p_x} + \|\partial_x G_{t-\tau} * (\widetilde\gamma(\tau,\cdot)\tilde v(\tau,\cdot))\|_{L^p_x} \\
&\qquad + \|\partial_x G_{t-\tau} * (b(\tau,\cdot)\partial_x\tilde u(\tau,\cdot))\|_{L^p_x} + \|\partial_x G_{t-\tau} * (\widetilde\gamma(\tau,\cdot)\tilde u(\tau,\cdot))\|_{L^p_x}\big]\, d\tau \\
&\le \|G_t\|_{L^1_x}\|\partial_x v_0\|_{L^p_x} + \int_0^t \Big[\|\partial_x G_{t-\tau}\|_{L^2_x}\|b(\tau,\cdot)\partial_x\tilde v(\tau,\cdot)\|_{L^{h(2,p)}_x} + \|\partial_x G_{t-\tau}\|_{L^1_x}\|\widetilde\gamma(\tau,\cdot)\tilde v(\tau,\cdot)\|_{L^p_x} \\
&\qquad + \|\partial_x G_{t-\tau}\|_{L^r_x}\|b(\tau,\cdot)\partial_x\tilde u(\tau,\cdot)\|_{L^{h(2,q)}_x} + \|\partial_x G_{t-\tau}\|_{L^1_x}\|\widetilde\gamma(\tau,\cdot)\tilde u(\tau,\cdot)\|_{L^p_x}\Big]\, d\tau \\
&\le c_1'\|\partial_x v_0\|_{L^p_x} + C\int_0^t \Big[(t-\tau)^{-3/4}\|b(\tau,\cdot)\|_{L^2_x}\|\partial_x\tilde v(\tau,\cdot)\|_{L^p_x} + (t-\tau)^{-1/2}\|\widetilde\gamma(\tau,\cdot)\|_{L^\infty_x}\|\tilde v(\tau,\cdot)\|_{L^p_x} \\
&\qquad + (t-\tau)^{-(2-1/r)/2}\|b(\tau,\cdot)\|_{L^2_x}\|\partial_x\tilde u(\tau,\cdot)\|_{L^q_x} + (t-\tau)^{-1/2}\|\widetilde\gamma_s\|_{L^\infty_x}\|\tilde u(\tau,\cdot)\|_{L^p_x}\Big]\, d\tau \\
&\le c_1'\|\partial_x v_0\|_{L^p_x} + \Big(CT^{1/4}\|b\|_{L^\infty_t(L^2_x)} + CT^{1/2}\|\widetilde\gamma\|_{L^\infty_t(L^\infty_x)}\Big)\|\tilde v(\tau,\cdot)\|_{L^\infty_t(W^{1,p}_x)} \\
&\qquad + \Big(CT^{1/(2r)}\|b\|_{L^\infty_t(L^2_x)} + CT^{1/2}\|\widetilde\gamma\|_{L^\infty_t(L^\infty_x)}\Big)\|\tilde u(\tau,\cdot)\|_{L^\infty_t(W^{1,q}_x)}
\end{aligned}
$$

By Proposition 13 and Assumption 17-A1, we obtain the bound (58).

□

Now we are ready to give a result of existence of a mild solution $\tilde v$ in $L^\infty([0,T], W^{1,p}(\mathbb{R}_+))$, where mild has to be meant as in (55).

Proposition 19. *Under Assumption 17, with $2 \le p < 1/(1-2\beta)$, the PDE* (54) *admits a unique mild solution $\tilde v$ in $L^\infty([0,T], W^{1,p}(\mathbb{R}_+))$. Furthermore, for some constant $C_T > 0$, the solution $\tilde v$ satisfies the following bound*

$$
\begin{aligned}
\sup_{t\in[0,T]} \|\tilde v(t,\cdot)\|_{W^{1,p}(\mathbb{R}_+)} \le\, & C_T(\|b\|_{L^\infty_t(L^2_x)} + \|\widetilde\gamma\|_{L^\infty_{t,x}})^4\|v_0\|_{W^{1,p}(\mathbb{R}_+)} \\
& + C_T\left(\|b\|_{L^\infty([0,T]L^2(\mathbb{R}_+))} + \|\widetilde\gamma\|_{L^\infty([0,T]\times(\mathbb{R}_+))}\right)\|\psi\|_{C^\beta}.
\end{aligned}
\tag{61}
$$

Proof. To show the existence and uniqueness of $\tilde{v}$ satisfying (55), we define a map $F : L^\infty([0,T], W^{1,p}(\mathbb{R}_+)) \to L^\infty([0,T], W^{1,p}(\mathbb{R}_+))$ such that, for any $h \in L^\infty([0,T], W^{1,p}(\mathbb{R}_+))$,

$$F(h) = G_t * v_0 + \int_0^t G_{t-\tau} * (b(\tau,\cdot)\partial_x h(\tau,\cdot) + \widetilde{\gamma}(\tau,\cdot)h(\tau,\cdot))d\tau + \int_0^t G_{t-\tau} * (b(\tau,\cdot)\partial_x \tilde{u}(\tau,\cdot) + \widetilde{\gamma}(\tau,\cdot)\tilde{u}(\tau,\cdot)d\tau.$$

We note that $\tilde{v} \in L^\infty([0,T], W^{1,p}(\mathbb{R}_+))$ satisfies (55) if and only if $\tilde{v}$ is a fixed point for F. By proceeding as for the bounds (57)-(58), we obtain that F is well-defined and, for T sufficiently small, such that

$$CT^{3/4} + CT^{1/4} \le 1/2,$$

F is a contraction on the time interval $[0,T]$. Therefore, by the fixed point theorem, for small $T > 0$, there exists a unique fixed point $\tilde{v}$ of the map F and hence a unique mild solution $\tilde{v}$ in $L^\infty([0,T], W^{1,p}(\mathbb{R}_+))$. The existence and uniqueness for every $T > 0$ follows by a standard iteration argument.

Finally, the bound (61) follows by a Gronwall-type inequality [30](Theorem 3.2.) applied to the bounds (57) on $\|\tilde{v}_t\|_{L^p(\mathbb{R}_+)}$ and (58) on $\|\partial_x \tilde{v}_t\|_{L^p(\mathbb{R}_+)}$, together with Proposition 13. ☐

Corollary 20. *Under Assumption 17, with $2 \le p < 1/(1-2\beta)$, the linear PDE* (48) *admits a unique $L^\infty([0,T], W^{1,p}(\mathbb{R}_+))$ mild solution $\tilde{s}$, i.e. satisfying* (49). *This solution can be decomposed as $\tilde{s} = \tilde{u} + \tilde{v}$, where $\tilde{u}$ is the unique $L^\infty([0,T], W^{1,p}(\mathbb{R}_+))$ mild solution of the system* (18)-(20) *and $\tilde{v}$ is the unique mild solution in $L^\infty([0,T], W^{1,p}(\mathbb{R}_+))$ of* (54).

Proof. If $\tilde{u}$ is the unique $L^\infty([0,T], W^{1,p}(\mathbb{R}_+))$ mild solution of the system (18)-(20), then from Proposition 19 $\tilde{v}$ is a $L^\infty([0,T], W^{1,p}(\mathbb{R}_+))$ mild solution of (55) if and only if $\tilde{s} = \tilde{u} + \tilde{v}$ is a $L^\infty([0,T], W^{1,p}(\mathbb{R}_+))$ mild solution of (49). Hence, well-posedness for $\tilde{s}$ follows from well-posedness of $\tilde{v}$. ☐

Remark 21. If $\psi \in C^1_t$, then well-posedness for mild solutions $\tilde{s}$ to (48) holds also for $p = \infty$, that is it is a $L^\infty([0,T], W^{1,\infty}(\mathbb{R}_+))$, with initial condition v_0 in $W^{1,\infty}(\mathbb{R}_+)$: indeed well-posedness in $L^\infty([0,T], W^{1,\infty}(\mathbb{R}_+))$ holds not only for $\tilde{v}$ solution to (54) but also for $\tilde{u}$ solution to the heat equation (18)-(20), by Proposition 9.

Remark 22. Assume $2 \le p < 1/(1-2\beta)$. The mild solution v to (54) satisfies, for every $s < t$,

$$\tilde{v}_t - \tilde{v}_s = \int_0^s (G_{t-r} - G_{s-r}) * A_r dr + \int_s^t G_{t-r} * A_r dr$$

where $A_r = b_r \partial_x \tilde{v}_r + \widetilde{\gamma}_r \tilde{v}_r + b_r \partial_x \tilde{u}_r + \gamma_r \tilde{u}_r$. Now we take the limit for $t - s \to 0$. For a.e. $r < s$, the term $(G_{t-r} - G_{s-r}) * A_r$ tends to 0 in L^p and, proceeding as in (57), one can show that the L^p norm of $(G_{t-r} - G_{s-r}) * A_r$ is bounded by a time-integrable function, uniformly in s and t, hence $\int_0^s (G_{t-r} - G_{s-r}) * A_r dr$ tends to 0 in L^p. Proceeding again as in (57), one can also show that the $\int_s^t G_{t-r} * A_r dr$ tends to 0 in L^p. Hence $t \mapsto v_t$ is actually continuous with values in L^p. With a similar argument, one gets that $t \mapsto \tilde{u}_t$ is continuous in L^p. Hence also $t \mapsto s_t$ is continuous in L^p, for $2 \le p < 1/(1-2\beta)$.

We conclude the study of the linearized system solution with some stability bounds for both the components of the splitting, which will be useful later.

Proposition 23. *Let b^i, $\widetilde{\gamma}^i$, v_0^i, ψ^i, $i = 1,2$ satisfy Assumption 17 with $p = 2$ and let us denote by $\tilde{u}^i$ and $\tilde{v}^i$, $i = 1,2$, the corresponding $L^\infty([0,T], W^{1,2}(\mathbb{R}_+))$ solutions of* (25) *and* (55)*, respectively. Let us assume in addition that $\widetilde{\gamma}^1$, $\widetilde{\gamma}^2$ are in $C([0,T], L^2(\mathbb{R}_+))$. Then we have*

$$\begin{aligned}
&\sup_{t\in[0,T]} \|\tilde{u}^1 - \tilde{u}^2\|_{W^{1,2}(\mathbb{R}_+)} \le C\|\psi^1 - \psi^2\|_{C^\beta_t},\\
&\sup_{t\in[0,T]} \|\tilde{v}^1 - \tilde{v}^2\|_{W^{1,2}(\mathbb{R}_+)} \le \rho_{1,b,\tilde{\gamma}}\|v_0^1 - v_0^2\|_{W^{1,2}(\mathbb{R}_+)} + \rho_{2,b,\tilde{\gamma}}\|\psi^1 - \psi^2\|_{C^\beta([0,T])}\\
&\qquad\qquad + \rho_{3,b,\tilde{\gamma},v_0,\psi}\left(\|b^1 - b^2\|_{L^\infty([0,T],L^2(\mathbb{R}_+))} + \|\widetilde{\gamma}^1 - \widetilde{\gamma}^2\|_{C([0,T],L^2(\mathbb{R}_+))}\right)
\end{aligned}$$

where $\rho.$ are Borel locally bounded function such that $\rho_{i,b,\tilde{\gamma}} = \rho_i\left(\|b^*\|_{L^\infty([0,T],L^2(\mathbb{R}_+))}, \|\widetilde{\gamma}^*\|_{L^\infty([0,T]\times\mathbb{R}_+)}\right)$, for $i = 1,2$ and $\rho_{3,b,\tilde{\gamma},v_0,\psi} = \rho_3\left(|\tilde{v}_0^i\|_{W^{1,2}(\mathbb{R}_+)}, \|\psi^i\|_{C^\beta([0,T])}, \|b^*\|_{L^\infty([0,T],L^2(\mathbb{R}_+))}, \|\widetilde{\gamma}^*\|_{L^\infty([0,T]\times\mathbb{R}_+)}\right)$. (We have used the notation $\rho.(a^*)$ for $\rho.(a^1, a^2)$.)

Proof. The bound on $\tilde{u}^1 - \tilde{u}^2$ follows immediately from (40) by linearity. Concerning the bound on $\tilde{v}^1 - \tilde{v}^2$, by proceeding as in the a priori bounds (57) and (58), we get

$$\|\tilde{v}_t^1 - \tilde{v}_t^2\|_{W_x^{1,2}} \le \|v_0^1 - v_0^2\|_{W_x^{1,2}} + A + B,$$

where the term A, resp. B, takes into account the contribution of b^i, resp. $\widetilde{\gamma}^i$, $i = 1, 2$. Precisely, for A we have

$$A = C \int_0^t (t-s)^{-3/4} \Big[\|b_s^1\|_{L_x^2} \|\tilde{v}_s^1 - \tilde{v}_s^2\|_{W_x^{1,2}} + \|b_s^1 - b_s^2\|_{L_x^2} \|\tilde{v}_s^2\|_{W_x^{1,2}} \\ + \|b_s^1\|_{L_x^2} \|\tilde{u}_s^1 - \tilde{u}_s^2\|_{W_x^{1,2}} + \|b_s^1 - b_s^2\|_{L_x^2} \|\tilde{u}_s^2\|_{W_x^{1,2}} \Big] ds.$$

For B we have, by (59)

$$\begin{aligned} B &= C \int_0^t (t-s)^{-1/2} \left[\|\widetilde{\gamma}_s^1\|_{L_x^\infty} \|\tilde{v}_s^1 - \tilde{v}_s^2\|_{L_x^2} + \|\widetilde{\gamma}_s^1 - \widetilde{\gamma}_s^2\|_{L_x^2} \|\tilde{v}^2\|_{L_x^\infty} \right. \\ &\qquad \left. \|\widetilde{\gamma}_s^1\|_{L_x^\infty} \|\tilde{u}_s^1 - \tilde{u}_s^2\|_{L_x^2} + \|\widetilde{\gamma}_s^1 - \widetilde{\gamma}_s^2\|_{L_x^2} \|\tilde{u}^2\|_{L_x^\infty} \right] ds \\ &\leq C \int_0^t (t-s)^{-1/2} \Big[\|\widetilde{\gamma}_s^1\|_{L_x^\infty} \|\tilde{v}_s^1 - \tilde{v}_s^2\|_{L_x^2} + \|\widetilde{\gamma}_s^1 - \widetilde{\gamma}_s^2\|_{L_x^2} \|\tilde{v}^2\|_{W_x^{1,2}} \\ &\qquad \|\widetilde{\gamma}_s^1\|_{L_x^\infty} \|\tilde{u}_s^1 - \tilde{u}_s^2\|_{L_x^2} + \|\widetilde{\gamma}_s^1 - \widetilde{\gamma}_s^2\|_{L_x^2} \|\tilde{u}^2\|_{W_x^{1,2}} \Big] ds. \end{aligned}$$

The conclusion derives from a Gronwall-type inequality [30](Theorem 3.2), from the bound (61) on $\tilde{v}^1$ and $\tilde{v}^2$, the bound (40) on $\tilde{u}^1$, $\tilde{u}^2$ and $\tilde{u}^1 - \tilde{u}^2$. □

5 Existence, uniqueness and regularity for the nonlinear system

This section contains the main result of this paper, that is the well-posedness for the nonlinear system (3)-(5). The local well-posedness is proven by a fixed point argument, while the global one by an a priori bound. We follow the proof line in [18], but considering v instead of s, for the energy bound. In the following, we may let the time horizon T vary in a compact interval $[0, T_{fin}]$.

Let us gather together all the assumptions upon the initial and boundary conditions of the dynamical system (3)-(5).

Assumption 24. In the following, η, m, c_0 are non negative constants and $C_0 \geq m > 0$.

B1. The boundary condition ψ satisfies

$$\begin{aligned} &\psi \in C^\beta([0,T]) \text{ for some } 1/4 < \beta < 1/2, \\ &\psi(0) = 0, \\ &0 \leq \psi \leq \eta. \end{aligned}$$

B2. The initial conditions s_0 and c_0 satisfy

$$\begin{aligned} &0 \leq s_0(x) \leq \eta, \quad 0 < m \leq c_0(x) \leq C_0, \quad \forall x \in [0, \infty), \\ &\varphi_m := \min_{c \in [m, C_0]} \varphi(c) > 0, \\ &s_0 \in W_x^{1,2}, \quad C_0 - c_0 \in W_x^{1,2}, \\ &s_0(0) = \psi(0) = 0. \end{aligned}$$

B3. If $B = 1$, then $\eta < 1$.

The linear system. The first step of the analysis is to consider the linear system as in Section 4. In the following we collect all the assumptions on the function f, that help us to decouple the ODE from the solution of the PDE in the system (3)-(4).

Assumption 25. Let us consider the class of Borel function $f : [0,T] \times [0, \infty) \to \mathbb{R}$ which satisfy the following conditions:

$$\begin{aligned} &f \in C\left([0,T], L^2(\mathbb{R}_+)\right) \cap L^2\left([0,T], W^{1,2}(\mathbb{R}_+)\right); \\ &\|f\|^2_{C([0,T],L^2(\mathbb{R}_+))} + \|\partial_x f\|^2_{L^2([0,T]\times\mathbb{R}_+)} \leq K, \quad \text{for some } K > 0; \\ &\|f\|_{L^\infty([0,T]\times\mathbb{R}_+)} \leq \eta; \\ &f(t,0) = \psi(t) \quad \text{for a.e. } t \in [0,T]; \\ &f \geq 0. \end{aligned} \tag{62}$$

Let us consider the following linear PDE-ODE system for (s,g), given a positive Borel function f satisfying Assumption 25

$$\partial_t(\varphi(g)s) = \partial_x(\varphi(g)\partial_x s) - \lambda\varphi(g)sg, \tag{63}$$
$$\partial_t g = -\lambda\varphi(g)fg. \tag{64}$$

In (63)-(64) the function φ is given by (10), the boundary condition ϕ is (5) and the initial conditions are, for any $x \in (0,\infty)$,

$$s(x,0) = s_0(x), \qquad g(x,0) = c_0(x),$$

where s_0, c_0 and ψ satisfy Assumption (24). Equation (63) for s reads as

$$\partial_t s = \partial_x^2 s + b_g\partial_x s + \tilde{\gamma}_g s, \tag{65}$$

where b_g and $\tilde{\gamma}_g$ are defined as in (13) and (47). Note that the solution g of equation (64) admits an explicit form, that is

$$g(x,t) = \frac{c_0(x)}{\varphi(c_0(x))e^{\lambda\int_0^t f(x,\tau)d\tau} - Bc_0(x)}. \tag{66}$$

We seek a solution (s,g) which is a mild solution.

Definition 26 (Mild solution for (s,g))**.** The couple (s,g), with $s \in L^\infty([0,T],W^{1,2}(\mathbb{R}_+)) \cap L^\infty([0,T]\times\mathbb{R}_+)$ and $g \in B_b([0,T]\times\mathbb{R}_+)$, the space of bounded Borel functions, is *a mild solution* of PDE system (63)-(64) if, for every $x \in \mathbb{R}_+$, $g(\cdot,x)$ solves (64), that is g is given by (66) and s is a $L^\infty([0,T],W^{1,2}(\mathbb{R}_+))$ mild solution of (65).

Remark 27. If s is a mild solution of equation (65) and $s \in C^{1,2}([0,T]\times\mathbb{R}_+)$, the space of function C^1 in time and C^2 in space, then s satisfies (63) for every $(t,x) \in (0,T)\times(0,\infty)$; we say in this case that s is a classical solution of equation (63).

As in Section 4, we split the linear solution s of (65) as $s = \tilde{u}+\tilde{v}$, where $\tilde{u}$ solves the heat equation (18)-(20) and $\tilde{v}$ is a $L^\infty([0,T],W^{1,2}(\mathbb{R}_+))$ mild solution of the equation (54) with $b = b_g$ and $\tilde{\gamma} = \tilde{\gamma}_g$.

First of all, we consider some regularity and stability properties for the function g which has an explicit formula given by (66), given the function f satisfying Assumption 62.

Definition 28 (Good Data)**.** We say that s_0, c_0, f, ψ are *good data* if they satisfy Assumption 24 and 25 and if the following regularities are satisfied: there exists an $0<\alpha<1$ such that $f \in C^{1+\alpha/2,2+\alpha}([0,T]\times[0,+\infty))$, $s_0 \in C^{2+\alpha}([0,+\infty))$, $c_0 \in C^{1+\alpha}([0,+\infty))$, $\psi \in C^{1+\alpha/2}([0,T])$ and

$$\dot{\psi}(0) = \partial_x^2 s_0(0) + \frac{\partial_x c_0(0)\partial_x s_0(0)}{\varphi(c_0(0))} + \lambda c_0(0)s_0(0)(f(0,0)-1).$$

We have introduced the definition of good data since the results in [18] and in [21] are obtained under such assumptions of good data and then extended to a more general setting, by density arguments. Furthermore, in the case of good data, we get a higher regularity of the solutions, which is useful in some proofs.

The following proposition proposes time uniform bounds for g and its spatial derivative.

Proposition 29. *Let us assume that f satisfies Assumption* (25)*. Then $0 \le g \le C_0$ and*

$$g \in C([0,T],L^2(\mathbb{R}_+)). \tag{67}$$

Moreover there exists $\kappa > 0$, depending on m, C_0, φ_m, λ, η, T_{fin}, but not on K, such that, for every $T \in [0,T_{fin}]$, for every f satisfying Assumption (25)*, the following bounds hold P-a.s.:*

$$\sup_{t\in[0,T]} \|C_0 - g\|_{L^2(\mathbb{R}_+)} \le \kappa\|C_0 - c_0\|_{L^2(\mathbb{R}_+)} + \kappa\|f\|_{C([0,T],L^2(\mathbb{R}_+))}, \tag{68}$$
$$\sup_{t\in[0,T]} \|\partial_x g\|^2_{L^2(\mathbb{R}_+)} \le \kappa\|\partial_x c_0\|^2_{L^2(\mathbb{R}_+)} + \kappa\|\partial_x f\|^2_{L^2([0,T]\times\mathbb{R}_+)}. \tag{69}$$

Proof. As far as condition (67) concerns, we first note that by (10) and by Assumptions 24 on c_0 and φ, the denominator in (66) is bounded from below, hence g is bounded from above. Since f satisfies (62), we have, for any $0 \le s < t \le T$,

$$\begin{aligned} |g(t,x) - g(s,x)| &\le C \left| e^{\lambda \int_0^t f(r,x)dr} - e^{\lambda \int_0^s f(r,x)dr} \right| \\ &\le C \int_s^t f(r,x)dr \cdot \max\{e^{\lambda \int_0^t f(r,x)dr}, e^{\lambda \int_0^s f(r,x)dr}\} \\ &\le C \int_s^t f(r,x)dr. \end{aligned}$$

The constant C, changing from line to line depends only upon f. We get

$$\|g(t,\cdot) - g(s,\cdot)\|^2_{L^2(\mathbb{R}_+)} \le C \int_0^\infty \left(C \int_s^t f(r,x)dr \right)^2 dx \le (t-s)\|f\|_{L^2_{[0,T]\times(\mathbb{R}_+)}},$$

in particular g is in $C([0,T], L^2(\mathbb{R}_+))$.

If f and c_0 which are good data, in the sense of Definition 28, inequalities (68) and (69) hold true, as shown in [21](Proposition 2.1) and [18](Proposition 3.1), respectively. These bounds can be extended to the general case by a density argument. The proof is complete. ☐

It is possible to establish also some stability results for g.

Proposition 30. *Let us suppose that f_1, f_2 satisfy Assumption* (25) *and that $c_{0,1}, c_{0,2}$ satisfy Assumption 24; let g_1, g_2 be the corresponding solutions to* (64). *Then we have, for some $\tilde{K} > 0$ (depending on m, C_0, φ_m, λ, η, T_{fin}, K), for every $T \in [0, T_{fin}]$, P-a.s.,*

$$\sup_{t\in[0,T]} \|g_1 - g_2\|_{L^\infty(\mathbb{R}_+)} \le \tilde{K}T\|f_1 - f_2\|_{C([0,T],L^2(\mathbb{R}_+))\cap L^2([0,T],W^{1,2}(\mathbb{R}_+))} + \tilde{K}\|c_{0,1} - c_{0,2}\|_{L^\infty(\mathbb{R}_+)}, \tag{70}$$

$$\begin{aligned} \sup_{t\in[0,T]} \|g_1 - g_2\|_{W^{1,2}(\mathbb{R}_+)} &\le \tilde{K}T\|f_1 - f_2\|_{C([0,T],L^2(\mathbb{R}_+))\cap L^2([0,T],W^{1,2}(\mathbb{R}_+))} \\ &\quad + \tilde{K}(1 + \|\partial_x c_{0,1}\|_{L^2(\mathbb{R}_+)})\|c_{0,1} - c_{0,2}\|_{W^{1,2}(\mathbb{R}_+)}. \end{aligned} \tag{71}$$

Proof. It is enough to show the above bounds separately in the case when $c_{0,1} = c_{0,2}$ and in the case when $f_1 = f_2$. In the case when $c_{0,1} = c_{0,2}$, the bound follows from [21](Proposition 2.5) and [18](Proposition 3.4) for f_1, f_2 and $c_{0,1}$ good data and a density argument for general f_1, f_2 and $c_{0,1} = c_{0,2}$. We consider now the case when $f_1 = f_2 =: f$. In the following, $C > 0$ may denote different constants (depending on m, C_0, φ_m, λ, T, η, K). We write, for any solution g to (64),

$$g(t,x) = \rho(t,x,c_0(x)), \quad \rho(t,x,c) = \frac{c}{\varphi(c)h(t,x) - Bc}, \quad h(t,x) = e^{\lambda \int_0^t f(r,x)dr}.$$

In view of bounding $g_1 - g_2$ and $\partial_x g_1 - \partial_x g_2$, we compute (recall $\varphi(c) = A + Bc$, $B = \pm 1$.

$$\begin{aligned} \partial_c \rho(t,x,c) &= \frac{1}{\varphi(c)h(t,x) - Bc} - \frac{(B(h(t,x)-1)c}{(\varphi(c)h(t,x) - Bc)^2}, \\ \partial_x \rho(t,x,c) &= -\frac{c\varphi(c)\partial_x h(t,x)}{(\varphi(c)h(t,x) - Bc)^2}, \\ \partial_c \partial_x \rho(t,x,c) &= \partial_x h(t,x) \left(-\frac{\varphi(c) + Bc}{(\varphi(c)h(t,x) - Bc)^2} + \frac{2c\varphi(c)B(h(t,x)-1)}{(\varphi(c)h(t,x) - Bc)^3} \right), \\ \partial_c^2 \rho(t,x,c) &= -\frac{2B(h(t,x)-1)}{(\varphi(c)h(t,x) - Bc)^2} + \frac{2c(h(t,x)-1)^2}{(\varphi(c)h(t,x) - Bc)^3}. \end{aligned}$$

Since f safisfies (62), we have (with a constant $C > 0$ independent of (t,x))

$$\begin{aligned} \sup_{c\in[0,C_0]} |\partial_c \rho(t,x,c)| &\le C, \\ \sup_{c\in[0,C_0]} |\partial_c \partial_x \rho(t,x,c)| &\le C \int_0^t |\partial_x f(r,x)|dr, \\ \sup_{c\in[0,C_0]} |\partial_c^2 \rho(t,x,c)| &\le C. \end{aligned}$$

Now we can bound $g_1 - g_2$: since $0 \le c_{0,i} \le C_0$, $i = 1, 2$, we have

$$|g_1(t,x) - g_2(t,x)| \le \sup_{c\in[0,C_0]} |\partial_c \rho(t,x,c)||c_{0,1}(x) - c_{0,2}(x)| \le C|c_{0,1}(x) - c_{0,2}(x)|,$$

which implies, for every $2 \le q \le \infty$,

$$\sup_{t\in[0,T]} \|g_1 - g_2\|_{L^q_x} \le C\|c_{0,1} - c_{0,2}\|_{L^q_x}.$$

Concerning $\partial_x g_1 - \partial_x g_2$, we have, for $i = 1, 2$,

$$\partial_x g_i(t,x) = \partial_x \rho(t,x,c_{0,i}(x)) + \partial_c \rho(t,x,c_{0,i}(x))\partial_x c_{0,i}(x)$$

and hence

$$\begin{aligned}
|\partial_x g_1(t,x) - \partial_x g_2(t,x)| &\le |\partial_x \rho(t,x,c_{0,1}(x)) - \partial_x \rho(t,x,c_{0,2}(x))| \\
&\quad + |\partial_c \rho(t,x,c_{0,1}(x)) - \partial_c \rho(t,x,c_{0,2}(x))||\partial_x c_{0,1}(x)| \\
&\quad + |\partial_c \rho(t,x,c_{0,2}(x))||\partial_x c_{0,1}(x) - \partial_x c_{0,2}(x)| \\
&\le \sup_{c\in[0,C_0]} |\partial_c \partial_x \rho(t,x,c)||c_{0,1}(x) - c_{0,2}(x)| \\
&\quad + \sup_{c\in[0,C_0]} |\partial_c^2 \rho(t,x,c)||c_{0,1}(x) - c_{0,2}(x)||\partial_x c_{0,1}(x)| \\
&\quad + \sup_{c\in[0,C_0]} |\partial_c \rho(t,x,c)||\partial_x c_{0,1}(x) - \partial_x c_{0,2}(x)| \\
&\le C\left(\int_0^T |\partial_x f(r,x)|dr + |\partial_x c_{0,1}(x)|\right)|c_{0,1}(x) - c_{0,2}(x)| \\
&\quad + C|\partial_x c_{0,1}(x) - \partial_x c_{0,2}(x)|.
\end{aligned}$$

Therefore we obtain (by Assumption 25 on f)

$$\begin{aligned}
\sup_{t\in[0,T]} \|\partial_x g_1 - \partial_x g_2\|^2_{L^2_x} &\le C(\|\partial_x f\|^2_{L^2_t(L^2_x)} + \|\partial_x c_{0,1}\|^2_{L^2_x})\|c_{0,1} - c_{0,2}\|^2_{L^\infty_x} + C\|\partial_x c_{0,1} - \partial_x c_{0,2}\|^2_{L^2_x} \\
&\le C(1 + \|\partial_x c_{0,1}\|^2_{L^2_x})\|c_{0,1} - c_{0,2}\|^2_{W^{1,2}_x},
\end{aligned}$$

where in the last line we have used the Sobolev embedding of $W^{1,2}_x$ into L^∞_x. The proof is complete. □

Next we show that any function s, mild solution of the linear equation (65), exists in $[0,T]$, it is pathwise unique and it can be approximated by a sequence of solutions of the same system given good data approximating the initial conditions, the boundary conditions and the function f. The latter property is useful to apply the results proven in [18, 21].

Proposition 31. *Let us consider equation* (64)*, where g is the solution of the equation* (64)*, with f satisfying Assumption 25.*

i) There exists a pathwise unique $L^\infty([0,T], W^{1,2}(\mathbb{R}_+))$ mild solution s of the equation (65)*.*

ii) If s_0, c_0, f, ψ are good data as in Definition 28, then the corresponding solution $s \in C^{1,2}([0,T] \times \mathbb{R}_+)$; hence, it is a classical solution to (63)*.*

iii) For any mild solution s of the equation (65)*, there exists a sequence of good data $s_0^n, c_0^n, f^n, \psi^n$, with f^n converging to f in $C([0,T], L^2(\mathbb{R}_+)) \cap L^2([0,T], W^{1,2}(\mathbb{R}_+))$, with corresponding classical solutions s^n of the equation* (65)*, such that $(s^n)_n$ converges to s in $L^\infty([0,T], W^{1,2}(\mathbb{R}_+))$, more precisely*

$$\lim_{n\to\infty} \sup_{t\in[0,T]} \|s^n(t,\cdot) - s(t,\cdot)\|_{W^{1,2}(\mathbb{R}_+)} = 0. \tag{72}$$

In particular, s^n converges to s uniformly on $[0,T] \times (0,\infty)$.

Proof. Condition i) is a consequence of Corollary (20), applied with $p = 2$, and inequality (68). Moreover, if ψ is in C^1, then well-posedness holds also in $L^\infty_t(W^{1,\infty}_x)$, by Remark 21.

The regularity of the solution, stated in condition ii) in the case of good data, i.e. well-posedness of classical solutions in the class $C^{1+\alpha/2,2+\alpha}_{t,x}$ is given in [21](Proposition 2.2). Since the classical solution is also in $L^\infty_t(W^{1,\infty}_x)$, it must coincide with the mild solution.

Let us prove the approximating result iii). Let s be a mild solution of the equation (65). We take a sequence of good data $s_0^n, c_0^n, f^n, \psi^n$ approximating s_0, c_0, f, ψ in the following sense: $(s_0^n)_n$ converges to s_0 in $W_x^{1,2}$, $(C_0 - c^n)_n$ converges to $C_0 - c$ in $W_x^{1,2}$ and hence in L_x^∞ by Sobolev embedding, $(f^n)_n$ converges to f in $C_t(L_x^2) \cap L_t^2(W_x^{1,2})$ and $(f^n)_n$ is uniformly bounded in $L_{t,x}^\infty$, $(\psi^n)_n$ converges to ψ in $C_t^{\beta-\epsilon}$, for $\epsilon > 0$ given and sufficiently small such that $\beta - \epsilon > 1/4$.

Let g^n be the solution to (64) with data c_0^n, f^n. By (71), $(g^n)_n$ converges to g in $L_t^\infty(W_x^{1,2})$ (and in $C_t(L_x^2)$). For any $n \in \mathbb{N}$, we consider the functions $b^n = b_{g_n}$ and $\tilde{\gamma}^n = \tilde{\gamma}_{g_n}$ as in (13) and (47). Therefore, by Assumption 24 and Proposition 29, the family of drifts $(b^n)_n$ converges to $b = b_g$ in $L_t^\infty(L_x^2)$ and the family of coefficients $(\tilde{\gamma}^n)_n$ is uniformly bounded in $L_{t,x}^\infty$ and converges to $\tilde{\gamma} = \tilde{\gamma}_g$ in $C_t(L_x^2)$, where γ is given by (47). Hence, we can apply the stability bound given in Proposition 23: calling s^n the solution to (64) with good data $s_0^n, c_0^n, f^n, \psi^n$, the family $(s^n)_n$ converges to s in the sense of (72). By Sobolev embedding $W_x^{1,2} \to C_x$, the convergence is also uniform on $[0, T] \times (0, \infty)$. The proof is complete. □

Proposition 32. *Let us assume that f satisfies Assumption* (25)*. Then, for any mild solution s of the equation* (65)*, we have P-a.s.*

$$0 \le s(t,x) \le \eta, \qquad \forall (t,x) \in [0,T] \times [0,+\infty), \tag{73}$$

$$\lim_{x \to +\infty} s(t,x) = 0, \qquad \forall t \in (0,T]. \tag{74}$$

Moreover, let consider the splitted representation $s = \tilde{u} + \tilde{v}$. Then, for some constant $C > 0$ (depending only on T_{fin}) we have P-a.s.

$$0 \le \tilde{u}(t,x) \le C\|\psi\|_{C([0,T])} \le C\eta, \qquad \forall (t,x) \in [0,T] \times [0,+\infty), \tag{75}$$

$$|\tilde{v}(t,x)| \le C\eta, \qquad \forall (t,x) \in [0,T] \times [0,+\infty), \tag{76}$$

$$\lim_{x \to +\infty} \tilde{u}(t,x) = \lim_{x \to +\infty} \tilde{v}(t,x) = 0, \qquad \forall\, t \in (0,T]. \tag{77}$$

Proof. If f, s_0, c_0 and ψ are good data, (73) and (74) are given in [21](Proposition 2.3). The general case follows by the approximation result in Proposition 31, namely approximation via solutions with good data. The positivity of u follows from the positivity of ψ, by the representation formula (25) since $\partial_x G \le 0$. The uniform bound (75) and the limit of u in (77) follow from the bound (33). The uniform bound (76) follows from (73) and (75), while the limit of v in (77) follows from (74) and the first of (77).

□

Our main original contribution is given by the following proposition and its corollary 37; the crucial bound on the splitting variable $\tilde{v}$ that, together with the analogous bound for the splitting variable $\tilde{u}$, (Proposition 13), gives the final bound for the solution s of the linear equation. The corollary that follows states the generalization of this bound to the nonlinear equation.

Proposition 33 (Estimate for the linear system)**.** *For every f satisfying Assumption* (25) *with a generic $K > 0$, let s be the solution of the equation* (63)*, coupled with equation* (64)*. Then, there exists a constant $\mu \ge 0$, depending on m, C_0, φ_m, λ, η, T_{fin} but not on K, such that, for every $T \in [0, T_{fin}]$, satisfies the following bound P-a.s.*

$$\sup_{t \in [0,T]} \|s(t,\cdot)\|_{L^2(\mathbb{R}_+)}^2 + \int_0^T \|\partial_x s(t,\cdot)\|_{L^2(\mathbb{R}_+)}^2 dt \le \mu + \frac{1}{2}\int_0^T \|\partial_x f(t,\cdot)\|_{L^2(\mathbb{R}_+)}^2 dt. \tag{78}$$

In particular, for any f satisfying Assumption (25) *with $K \ge 2\mu$, for any $T \in [0, T_{fin}]$, we have*

$$\sup_{t \in [0,T]} \|s(t,\cdot)\|_{L^2(\mathbb{R}_+)}^2 + \int_0^T \|\partial_x s(t,\cdot)\|_{L^2(\mathbb{R}_+)}^2 dt \le K. \tag{79}$$

Proof. By the approximation result in Proposition 31, it is enough to show the two bounds (78) and (79) for classical solutions arising from good data f, s_0, c_0, ψ. Hence we will now consider such classical solutions.

Let us remind that we may consider the splitting $s = \tilde{u} + \tilde{v}$, where $\tilde{u}$ is solution of (18)-(20) and $\tilde{v}$ is solution of (54) We first obtain an equation for the pore concentration for $\varphi(g)\tilde{v}^2$, where φ is given by (10). By the chain rule, equations (54) and (64), we get the following

$$\begin{aligned}
\partial_t(\varphi(g)\tilde{v}^2) &= 2\varphi(g)\tilde{v}\partial_t\tilde{v} + \tilde{v}^2\partial_t g \\
&= 2\tilde{v}\varphi(g)\partial_x^2\tilde{v} + 2\tilde{v}\partial_x(\varphi(g))\partial_x\tilde{v} - 2\lambda\varphi(g)g(1-Bf)\tilde{v}^2 + 2\partial_x g\partial_x\tilde{u}\tilde{v} - 2\lambda\varphi(g)g\tilde{u}(1-Bf)\tilde{v} - \lambda\varphi(g)gf\tilde{v}^2 \\
&= 2\tilde{v}\partial_x(\varphi(g)\partial_x\tilde{v}) - \lambda\varphi(g)g(2+(1-2B)f)\tilde{v}^2 + 2\partial_x g\partial_x\tilde{u}\tilde{v} - 2\lambda\varphi(g)g\tilde{u}(1-Bf)\tilde{v}.
\end{aligned}$$

Let us integrate over $(0,t)\times(0,+\infty)$. By means of the integration by parts in x in the first term at the right hand side, since by Proposition 32 $v(t,0)=0$ and $v(t,+\infty)=0$), we get

$$\begin{aligned}
&\int_0^\infty \varphi(g_t)\tilde v_t^2dx - \int_0^\infty \varphi(g_0)v_0^2dx + 2\int_0^t\int_0^\infty \varphi(g)(\partial_x\tilde v)^2dxdr\\
&= -\int_0^t\int_0^\infty \lambda\varphi(g)g(2+(1-2B)f)\tilde v^2dxdr + \int_0^t\int_0^\infty 2\tilde v\partial_x g\partial_x\tilde u dxdr + \int_0^t\int_0^\infty -2\lambda\varphi(g)g(1-Bf)\tilde u\tilde v dxdr\\
&=: R_1+R_2+R_3.
\end{aligned}$$

By Proposition 32 and Assumption 24, we have $\tilde u\geq 0$, $f\geq 0$ and, for $B=1$, $f\leq\eta<1$ and, for $B=-1$ $2+3f>0$. Hence

$$R_1\leq 0.$$

As the R_2 term concerns, Proposition 32 gives the L^∞ uniform bound on $\tilde v$, inequality (68) gives the L^2 bound on $\partial_x g$, while the L^2 bound on $\partial_x u$ derives by Proposition 13, since $\beta>1/4$; hence, for $\epsilon>0$ to be determined later and the constants C_ϵ and C, depending only on ϵ and only on T_{fin}, respectively, we get what follows

$$\begin{aligned}
R_2 &= \int_0^t\int_0^\infty 2\partial_x g\partial_x\tilde u\tilde v dxdr\\
&\leq C_\epsilon\int_0^T \|v\partial_x\tilde u\|^2_{L^2_x}dr + \epsilon\int_0^T \|\partial_x g\|^2_{L^2_x}dr\\
&\leq C_\epsilon T\|\tilde v\|^2_{L^\infty_{t,x}}\|\partial_x\tilde u\|^2_{L^\infty_t(L^2_x)} + \epsilon\, T\,\|\partial_x g\|^2_{L^\infty_t(L^2_x)}\\
&\leq CC_\epsilon T\eta^2\|\psi\|^2_{C^\beta} + \epsilon\,\kappa T\left(\|\partial_x c_0\|^2_{L^2_x} + \|\partial_x f\|^2_{L^2_t(L^2_x)}\right).
\end{aligned}$$

For the term R_3, we use the L^∞ bound on g given by (68), the L^1 bound (34) on u and the bound $|\psi|\leq\eta$, the uniform bound on v by Proposition 32 and Assumption 24 and get

$$\begin{aligned}
R_3 &= \int_0^t\int_0^\infty -2\lambda\varphi(g)g(1-Bf)\tilde u\tilde v dxdr\\
&\leq 2\lambda T\|g\|_{L^\infty_x}(A+B\|g\|_{L^\infty_x})\|\tilde v\|_{L^\infty_{t,x}}\|\tilde u\|_{L^\infty_t(L^1_x)}\\
&\leq C\lambda T\eta^2 C_0(A+BC_0).
\end{aligned}$$

Hence, by means of the estimates of the terms R_1-R_3 and by considering the lower bound for φ as in condition B2 of the Assumption 24, we get the following space estimate in L^2 for $\tilde v$ and its derivative

$$\begin{aligned}
\int_0^\infty \tilde v_t^2dx + 2\int_0^t\int_0^\infty(\partial_x\tilde v)^2dxdr &\leq \varphi_m^{-1}\int_0^\infty\varphi(g_t)\tilde v_t^2dx + 2\varphi_m^{-1}\int_0^t\int_0^\infty\varphi(g)(\partial_x\tilde v)^2dxdr\\
&\leq \varphi_m^{-1}C_0\|s_0\|^2_{L^2_x} + CC_\epsilon\varphi_m^{-1}T\eta^2\|\psi\|^2_{C^\beta} + \epsilon\varphi_m^{-1}\kappa T\|\partial_x c_0\|^2_{L^2_x}\\
&\quad + C\lambda T\eta^2C_0(A+BC_0) + \epsilon\varphi_m^{-1}\kappa T\|\partial_x f\|^2_{L^2_t(L^2_x)}\\
&=: A_\epsilon + \epsilon\varphi_m^{-1}\kappa T\|\partial_x f\|^2_{L^2_t(L^2_x)}.
\end{aligned}$$

If we combine the latter estimate with the inequality (40) for $\tilde u$, then, for some constant $\bar C$ depending only on T_{fin}, we get the following for $s=\tilde u+\tilde v$,

$$\begin{aligned}
\sup_{t\in[0,T]}\int_0^\infty s_t^2dx + \int_0^T\int_0^\infty(\partial_x s)^2dxdr &\leq 2\sup_{t\in[0,T]}\int_0^\infty\tilde u_t^2dx + 2\int_0^T\int_0^\infty(\partial_x\tilde u)^2dxdr + 2\sup_{t\in[0,T]}\int_0^\infty\tilde v_t^2dx\\
&\quad + 2\int_0^T\int_0^\infty(\partial_x\tilde v)^2dxdr\\
&\leq \bar C\|\psi\|^2_{C^\beta} + 3A_\epsilon + 3\epsilon\varphi_m^{-1}\kappa T\|\partial_x f\|^2_{L^2_t(L^2_x)}.
\end{aligned}$$

Now, let us take $\epsilon>0$, sufficiently small that $3\epsilon\varphi_m^{-1}\kappa T_{fin}\leq 1/2$; we get

$$\sup_{t\in[0,T]}\int_0^\infty s_t^2dx + \int_0^T\int_0^\infty(\partial_x s)^2dxdr \leq \bar C\|\psi\|^2_{C^\beta} + 3A_\epsilon + \frac{1}{2}\|\partial_x f\|^2_{L^2_t(L^2_x)}.$$

So, inequality (78) is proven, with $\mu = \bar{C}\|\psi\|^2_{C^\beta} + 3A_\epsilon$. If we take the parameter K in (62) such that $K \geq 2\mu$, we conclude that

$$\sup_{t\in[0,T]} \int_0^\infty s_t^2 dx + \int_0^T \int_0^\infty (\partial_x s)^2 dxdr \leq \frac{K}{2} + \frac{K}{2} = K.$$

Hence, inequality (79) is proven. □

Remark 34. Proposition 33 is the main point where the irregularity of the boundary condition ψ does not let us use the results already proven in [21]. Precisely, in [21](Proposition 2.4), the estimate on s is obtained by multiplying the equation for $\varphi(g)s$ morally with $s - \psi$ and integrating by parts in time and space; in this procedure the time derivative of ψ appears, something which cannot be controlled in our context. Here instead, in a certain sense we leave the irregularity due to ψ into the u term and consider the equation for v instead, which has smooth (actually zero) boundary condition.

Remark 35. Note that, unlike [21](Proposition 2.4), the bound (78) in Proposition 33 on the solution s to the linear equation (63) depends on the $C([0,T], L^2(\mathbb{R}_+)) \cap L^2([0,T], W^{1,2}(\mathbb{R}_+))$ norm of f. The reason for this dependence is due to the appearance of the term $\partial_x g$, which can only be controlled by the $C([0,T], L^2(\mathbb{R}_+)) \cap L^2([0,T], W^{1,2}(\mathbb{R}_+))$ norm of u. However, we can make this dependence small, that is, with a multiplicative constant less or equal than one half. This is enough to get the a priori estimate in Proposition 37 in the case of the nonlinear equation (11),(12)-(14).

The next result deals with the contraction property of the operator that maps any f in (62) into the solution of the linear equation, for small time T.

Proposition 36. *There exists $\mu > 0$ (as in Proposition 33) such that, for every $K \geq 2\mu$, we have: there exist $T \in [0, T_{fin}]$, $L < 1$ (all depending on m, C_0, φ_m, λ, η, T_{fin} and K) such, that, for every f^1, f^2 satisfying* (62) *with K, calling s^1, s^2 the corresponding solutions, we have*

$$\sup_{t\in[0,T]} \|s_t^1 - s_t^2\|^2_{L^2(\mathbb{R}_+)} + \int_0^T \|\partial_x s_t^1 - \partial_x s_t^2\|^2_{L^2(\mathbb{R}_+)} dt \leq L \left(\sup_{t\in[0,T]} \|f_t^1 - f_t^2\|^2_{L^2(\mathbb{R}_+)} + \int_0^T \|\partial_x f_t^1 - \partial_x f_t^2\|^2_{L^2(\mathbb{R}_+)} dt \right).$$

Proof. If s^1 and s^2 are solutions arising from good data, then one can repeat the arguments in the proof of [21](Proposition 2.6), by replacing the constant K_λ in [21] with the constant $K \geq 2\mu$, where μ is as in Proposition 33. The general case follows by the approximation result in Proposition 31. □

The nonlinear system Let us focus again on the nonlinear PDE (11)-(14). First of all we consider a corollary of Proposition 33.

Proposition 37 (Estimate for the nonlinear system)**.** *If (s, c) is a mild solution to the nonlinear PDE* (11),(12)-(14)*. Then there exists a constant $\mu \geq 0$, depending on m, C_0, φ_m, λ, η, T_{fin} but not on K, such that, for every $T \in [0, T_{fin}]$, if (s, c) is a mild solution to the nonlinear PDE* (11),(12)-(14)*, we have P-a.s.*

$$\sup_{t\in[0,T]} \|s(t,\cdot)\|^2_{L^2(\mathbb{R}_+)} + \int_0^T \|\partial_x s(t,\cdot)\|^2_{L^2(\mathbb{R}_+)} dt \leq 2\mu.$$

Proof. If (s, c) solves the nonlinear PDE (11),(12)-(14), one can consider $f = s$ and the conclusion follows from the bound (78) of Proposition 33. . □

Remark 38. We note that if (s, c) is a mild solution to (11),(12)-(14), then by Remark 22 $s \in C([0,T]; L^2(\mathbb{R}^2))$. Furthermore, we have that (s, c) is a mild solution to (11),(12)-(14) if and only if (s, c) is a mild solution to (63)-(64) with $f = s$.

The global existence of a pathwise unique mild solution for the nonlinear equation is proven in the following theorem.

Theorem 39. *Let us consider system* (11),(12)-(14) *on $[0, T_{fin}] \times \mathbb{R}_+$. For any $T \in [0, T_{fin}]$, there exists a pathwise unique mild solution (s, c), in the sense of Definition 5.*

Proof. Let $\chi_{K,T}$ denote the space of functions f satisfying Assumption 25, with K and T as in Proposition 33 and Proposition 36; it turns out that $\chi_{K,T}$ is a complete metric space endowed with the norm $C_t(L^2_x)\cap L^2_t(W^{1,2}_x)$.

We consider the solution map of the liner system

$$F : \chi_{K,T} \to \chi_{K,T}, \qquad F(f) = s,$$

where, given a function f, s is the first component of the solution (s, g) to the linear system (63)-(64) driven by f. By Proposition 32 and Proposition 33, the operator F is well-defined and it is a contraction on $\chi_{K,T}$ (for small T and large K), due to Proposition 36. Hence, we get existence and uniqueness of a fixed point of F, that is existence and uniqueness for (11),(12)-(14), for small $T > 0$. For general $T > 0$, we can use a strategy which is similar to the one followed in [21]. We take $K = 2\mu$, with μ as in Proposition 33 and Proposition 37 and define

$$T^* = \sup \{t \in [0, T] \mid \exists! \text{ mild solution } (s, c) \text{ to } (11), (12) - (14)\}.$$

First of all, we show that (s, c) can be extended as a mild solution up to T^* included and s is such that it satisfies that $\sup_{t\in[0,T^*]} \|s_t\|_{W^{1,2}_x} < \infty$. In order to do this, we take $f = s$, extended to $t = T^*$ for example taking $f_{T^*} = 0$. With such f, c can be extended up to T^* included by the explicit formula (66) and the bounds (68)-(69) still hold. By Corollary 20, there exists a unique mild solution $\tilde{s}$ on $[0, T^*]$ to (63), in particular $\tilde{s}$ is in $C_t(L^2_x)$ and satisfies $\sup_{t\in[0,T^*]} \|\tilde{s}_t\|_{W^{1,2}_x} < \infty$. By uniqueness of (63), s and $\tilde{s}$ must coincide on $[0, T^*)$. Hence s can be extended to a solution on $[0, T^*]$ by setting $s_{T^*} = \tilde{s}_{T^*}$ and it verifies $\sup_{t\in[0,T^*]} \|s_t\|_{W^{1,2}_x} < \infty$.

Finally, we show that $T^* = T$. By contradiction, if $T^* < T$ then we would like to take s_{T^*} (which is in $W^{1,2}_x$) and c_{T^*} as new initial conditions and apply the local well-posedness result to extend the unique solution for $T > T^*$. Unfortunately here we cannot use exactly this argument, because $s_{T^*} = \psi_{T^*}$ does not need to be zero and so we are outside Assumption 24. However, we can consider a modified fixed point problem as follows. We take the space $\tilde{\chi}^{T^*}_{K,T}$ of functions f satisfying Assumption 25 and such that $f = s$ (the unique solution of (11),(12)-(14) on $[0, T^*] \times [0, \infty)$ and we still consider the map F, restricted on $\tilde{\chi}^{T^*}_{K,T}$, such that $F(f)$ is the solution to the (first equation of) the linear system (63)-(64) driven by f. The proof of Proposition 36 (which is the proof of [21](Proposition 2.6)) does not use the fact that $\psi(0) = 0$ and so one can repeat the proof for f in $\tilde{\chi}^{T^*}_{K,T}$ and get the following fact, for K large enough and $T - T^* > 0$ small enough, for some $L < 1$: for every f^1, f^2 in $\tilde{\chi}^{T^*}_{K,T}$, calling $s^1 = F(f^1)$, $s^2 = F(f^2)$,

$$\sup_{t\in[T,T^*]} \|s^1_t - s^2_t\|^2_{L^2_x} + \int_{T^*}^T \|\partial_x s^1_t - \partial_x s^2_t\|^2_{L^2_x} dt \le L \left(\sup_{t\in[T^*,T]} \|f^1_t - f^2_t\|^2_{L^2_x} + \int_{T^*}^T \|\partial_x f^1_t - \partial_x f^2_t\|^2_{L^2_x} dt \right).$$

Hence, for large K and small $T - T^*$, F is a contraction on $\tilde{\chi}^{T^*}_{K,T}$ and therefore it has a unique fixed point, that is there exists a unique mild solution (s, c) on $[0, T^* + T]$. But this is in contradiction with the definition of T^*. Hence $T^* = T$. The proof is complete.

□

6 Appendix

For convenience of the reader here we recall some classical results of functional analysis related to fractional Sobolev spaces with some embedding results [14,25,29]. Furthermore, we give the basic definition related to the interpolation theory [6,23,25,29].

Let us remark the fact that, as in the classical case with integer exponents, the space $W^{k'+\alpha',p}$ is continuously embedded in $W^{k+\alpha,p}$, with $k + \alpha \le k' + \alpha'$, k, k' integers and $\alpha, \alpha' \in (0, 1)$. The result holds in the limit case $\alpha = 1$ [14].

Proposition 40. *[14] Let $p \in [1, \infty]$, $k, k' \in \mathbb{N}$, and $\alpha, \alpha' \in (0, 1)$, with $k + \alpha \le k' + \alpha'$. Then*

$$\|f\|_{W^{k+\alpha,p}(\mathbb{R})} \le C\|f\|_{W^{k'+\alpha',p}(\mathbb{R})}.$$

In particular,

$$W^{k'+\alpha',p}(\mathbb{R}) \subseteq W^{k+\alpha,p}(\mathbb{R}).$$

The previous hold also in the case $k = k' = 0$. Furthermore,

$$W^{1,p}(\mathbb{R}) \subseteq W^{\alpha,p}(\mathbb{R}).$$

The Sobolev embedding results may be extended by considering different L^p spaces, for various exponents p, provided some relations among the exponents.

Proposition 41. *[25, 29] Let $p, q \in [1, \infty]$, $s, s' \in \mathbb{R}_+$, with $p \leq q$ and $s' \leq s$. If*

$$s - \frac{1}{p} \geq s' - \frac{1}{q}, \tag{80}$$

then

$$W^{s,p}(\mathbb{R}) \subseteq W^{s',q}(\mathbb{R}). \tag{81}$$

All the previous results are applied also to $\mathbb{R}_+$.

References

[1] E. Alòs, and S. Bonaccorsi, Stochastic partial differential equations with Dirichlet white-noise boundary conditions, Ann. I. H. Poincaré (2002), Vol. 38, No. 2, pp. 125–154

[2] D. Aregbda-Driollet, F. Diele, and R. Natalini A mathematical model for the sulphur dioxide aggression to calcium carbonate stones: numerical approximation and asymptotic analysis, Siam J. Appl. Math. (2004), Vol. 64, No. 5, pp. 1936–1667

[3] F. Arceci, L.M. Giordano, M. Maurelli, D.Morale, and S.Ugolini, Randomness in a nonlinear model of sulphation phenomena, in Mathematcal Modeling in Cultural Heritage, MACH2021, (G.Bret, C.Cavaterra, M.Solci, M.Spagnuolo Eds), Springer Nature, (2023)

[4] F. Arceci, D. Morale. S. Ugolini, A numerical study of a reaction-diffusion PDE with stochastic dynamical boundary conditions: nonlinear model of sulphation phenomena (2023), in preparation.

[5] V. Barbu, S. Bonaccorsi, L. Tubaro, Stochastic parabolic equations with nonlinear dynamical boundary conditions, J. Math. Anal. Appl. (2015) 427, 484-–498

[6] Bennett, Colin; Sharpley, Robert (1988), Interpolation of operators, Pure and Applied Mathematics, vol. 129, Academic Press, Inc., Boston (1988)

[7] E. Bonetti, C. Cavaterra, F. Freddi, Maurizio Grasselli, R. Natalini, A nonlinear model for marble sulphation including surface rugosity: theoretical and numerical results, Communications on Pure & Applied Analysis (2019), 18 (2): 977-998

[8] Z. Brzeźniak, B. Goldys, S. Peszat and F. Russo, Second order PDEs with Dirichlet white noise boundary conditions, J. Evol. Equ. 15 (2015), 1–26

[9] Cannon, J. & Browder, F. . The One-Dimensional Heat Equation (Encyclopedia of Mathematics and its Applications). Cambridge: Cambridge University Press. (1984) doi:10.1017/CBO9781139086967

[10] I. Chueshov and B. Schmalfuss, Parabolic stochastic partial differential equations with dynamical boundary conditions, Differential Integral Equations 17 (2004), 751–780

[11] P. D'Ancona, S. Spagnolo, The Cauchy Problemfor Weakly Parabolic System, Math. Ann. 309 (2) (1997), 307–330

[12] G. Da Prato and J, Zabczyk, Evolution equations with whitenoise boundary conditions, Stochastics: An International Journal of Probability and Stochastic Processes 42.3-4 (1993), 167–182

[13] A. Debussche, M. Fuhrman, and G. Tessitore,Optimal control of a stochastic heat equation with boundary-noise and boundary-control, ESAIM: Control, Optimisation and Calculus of Variations, (2007), 13.1, pp. 178–205

[14] E. Di Nezza, G. Palatucci, E.Valdinoci, Hitchhiker's guide to the fractional Sobolev spaces, Bull. Sci. Math. 136 (2012), 521–573

[15] d'Onofrio A. (Ed.), Bounded Noises in Physics, Biology, and Engineering. Birkhäuser (2013)

[16] Domingo, D., A. d'Onofrio, F. Flandoli, Boundedness vs unboundedness of a noise linked to Tsallis q-statistics. The role of the overdamped approximation, J. Mathematical Physics 58(3), (2017)

[17] A. Etheridge, Stochastic analysis and PDEs, HT (2016)

[18] F. R. Guarguaglini, R. Natalini.Global existence of solutions to a nonlinear model of sulphation phenomena in calcium carbonate stones, Nonlinear Analysis: Real World Applications 6 (2005) 477 – 494

[19] Grothaus, M., Dirichlet form analysis of the Jacobi process, Stochastic Processes and their Applications, (2023), 157, 376-412,https://doi.org/10.1016/j.spa.2022.12.002

[20] F. R. Guarguaglini, R. Natalini. Global existence and uniqueness of solutions for multidimensional weakly parabolic systems arising in chemistry and biology. Communications on Pure & Applied Analysis, (2007), 6 (1) : 287-309. doi: 10.3934/cpaa.2007.6.287

[21] F. R. Guarguaglini, R. Natalini Fast Reaction Limit and Large Time Behavior of Solutions to a Nonlinear Model of Sulphation Phenomena. Communications in Partial Differential Equations (2007), 32:2, 163-189.

[22] Karlin S. and Taylor H.M., A Second Course in Stochastic Processes, (1981) Academic Press.

[23] Lions, Jacques-Louis.; Peetre, Jaak, "Sur une classe d'espaces d'interpolation", Inst. Hautes Études Sci. Publ. Math. (1964) 19: 5—68

[24] Liu, W., Röckner, M. . Mild Solutions. In: Stochastic Partial Differential Equations: An Introduction. Universitext. Springer, Cham. (2015)

[25] A. Lunardi, Interpolation Theory, Publications of the Scuola Normale Superiore, Edizioni della Normale Pisa (2018)

[26] L.C.G Rogers and David Williams, Diffusions, Markov Processes, and Martingales. 2 , Cambridge University Presse, (2000)

[27] T. Yamada, S. Watanabe, On the uniqueness of solutions of stochastic differential equations, J. Math. Kyoto Univ., 11-1 (1971) 155-167

[28] A.V. Skorohod, Studies in the theory of random processes, Addison-Wisley, (1965) (Originally published in Kiev).

[29] Triebel, Hans. (1978). Interpolation theory, function spaces, differential operators. Amsterdam ; New York: North-Holland Pub. Co.

[30] J.R.L. Webb, Weakly singular Gronwall inequalities and applications to fractional differential equations, J. Math. Anal. and Appl., (2019), 471 (1–2)