THE ANTICYCLOTOMIC MAIN CONJECTURES FOR ELLIPTIC CURVES

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Contents

1. Introduction

Let $E/\mathbf{Q}$ be a modular elliptic curve of conductor N and let f be the cuspidal eigenform on $\Gamma_0(N)$ associated with E by the modularity theorem. Denote by K_∞ the anticyclotomic $\mathbf{Z}_p$-extension of an imaginary quadratic field K, where p is a prime number. The goal of this article is to obtain a proof of the main conjectures of Iwasawa theory for E over K_∞, both when p is *good ordinary* and when p is *supersingular* for E.

The anticyclotomic setting displays a well-known dichotomy, depending on whether the generic sign of the functional equation of the complex L-function of E/K twisted by finite order characters of the Galois group of K_∞/K is $+1$ or -1. For reasons which will be explained later we call the former case *definite* and the latter case *indefinite*.

Assume first that p is a *good ordinary* prime for E. In the *indefinite* case, a norm-compatible sequence of Heegner points arising from a Shimura curve parametrisation is defined over the finite layers of K_∞/K. Its position in the compact p-adic Selmer group of E/K_∞ is encoded by an element $L_p(f)$ of the anticyclotomic Iwasawa algebra Λ, called the *indefinite anticyclotomic p-adic L-function*. (The notation $L_p(f)$ instead of $L_p(E)$ is adopted throughout, in order to achieve notational uniformity in the modular arguments of this article.) The *indefinite anticyclotomic Iwasawa main conjecture (IAMC)*, formulated by Perrin-Riou [PR87], states that $L_p(f)$ generates the square-root of the characteristic ideal of the Λ-torsion part of the Pontrjagin dual of the p-primary Selmer group of E/K_∞. The proof of this conjecture is one of the main results of this paper. We remark that one divisibility of characteristic ideals – notably the fact that $L_p(f)$ is divisible by the characteristic ideal of the relevant Selmer group – is obtained in Howard's paper [How04], as a direct application of the theory of Euler systems.

We now turn to the *definite* (good ordinary) case, in which the *definite anticyclotomic p-adic L-function* $L_p(f)$ interpolates central critical values of twists of the complex L-function of E/K, described in Section 2 in terms of special points on the Gross curve. The level raising of the modular form f at certain admissible primes yields congruent eigenforms modulo arbitrary powers of p. These eigenforms belong to the indefinite setting and therefore the Heegner construction becomes available on the Shimura curves supporting them. (See Section 3 for the precise definitions.) This basic observation is the opening gambit of the article [BD05] by Bertolini–Darmon, which builds on it by establishing a *first explicit reciprocity law* relating the resulting Heegner cohomology classes to $L_p(f)$. Moreover, with the help of a *second explicit reciprocity law*, this article sets up an inductive procedure (which may be viewed as an analogue of

Kolyvagin's induction) proving that $L_p(f)$ is divisible by the characteristic ideal of the Pontrjagin dual of the p-primary Selmer group of E/K_∞. This shows one divisibility in the *definite Iwasawa anticyclotomic main conjecture (DAMC)*. (The explicit reciprocity laws are reviewed in Section 6.2.) This procedure has been formalised in Howard's paper [How06], leading to the concept of *bipartite Euler system*. The full DAMC proved in this paper is rather based on a refinement of the induction argument in [BD05]. It requires to show the non-vanishing modulo p of values of the definite p-adic L-function attached to an eigenform congruent to f, obtained by raising the level at sufficiently many admissible primes. This maximality property ultimately rests on a fundamental *p-converse* theorem of Skinner–Urban [SU14], as explained in Step 4 of Section 7.1. It should be stressed that both the DAMC and the IAMC are obtained in this article from the same unified approach based on the above-mentioned inductive process. The article [BCK21] by Burungale–Castella–Kim uses directly the techniques of bipartite Euler systems to obtain a proof of the IAMC (that is, Perrin-Riou's Heegner point main conjecture).

Assume now that p is a *supersingular* prime for E. As customary in the supersingular theory, two cases indexed by a sign $\varepsilon = \pm$ need to be distinguished. Depending on the choice of ε, one is led to introduce the concepts of ε-points, ε-Selmer groups and ε-p-adic L-functions $L_p^\varepsilon(f)$. In terms of these objects, this article formulates and proves the analogues of the AMCs outlined above. In the definite setting, the analogous inclusion of [BD05] was obtained by Darmon–Iovita [DI08] when p is split in K and $a_p(E) = 0$, and extended by Burungale–Büyükboduk–Lei [BBL24] without assuming $a_p(E) = 0$ and covering also the case p inert in K.

The following two specific aspects of the supersingular setting are worth noting.

On the one hand, our study of the structure of the ε-Selmer groups rests in a fundamental way on the *control* result stated in Proposition 5.3. The proof of this result is based on Theorem 5.2, which was known for p split in K thanks to the work of Iovita–Pollack [IP06]. For p inert in K, Theorem 5.2 is a consequence of the recent proof of Rubin's conjecture on local points in p-adic towers due to Burungale–Kobayashi–Ota [BKO21]. In a previous version of this article, the control statement of Proposition 5.3 was a running assumption in the inert case.

When p is inert in K, the supersingular setting displays a subcase for $\varepsilon = +$, called *exceptional* in Definition 1.3 of this Introduction. In the exceptional case, the $+$-p-adic L-function acquires an extra-zero of local nature and our approach only allows us to show one divisibility in the AMCs. It would be interesting to further investigate this exceptional zero and the possibility of establishing the full AMC in the exceptional case.

We now formulate our main results more precisely. In order to obtain unified statements, we adopt the convention that $\varepsilon = \emptyset$ in the ordinary case, so that the concept of ε-point, ε-Selmer group and ε-p-adic L-function simply stands for point, Selmer group and p-adic L-function (then in particular $L_p(f) = L_p^\varepsilon(f)$ is this case).

Fix throughout the paper algebraic closures $\bar{\mathbf{Q}}$ of $\mathbf{Q}$ and $\bar{\mathbf{Q}}_p$ of $\mathbf{Q}_p$, as well as embeddings $\bar{\mathbf{Q}} \hookrightarrow \bar{\mathbf{Q}}_p$ and $\bar{\mathbf{Q}} \hookrightarrow \mathbf{C}$.

Our main results are proved under the following assumptions. Let N be as above the conductor of E, assumed to be coprime with the discriminant of K. Factor N as $N = N^+N^-$, where N^+ resp. N^- is divisible only by primes which are split, resp. inert in K.

Hypothesis 1.1.

(1) The rational prime p is $\geqslant 5$ and does not divide N.
(2) The rational prime p does not divide the class number h_K and the discriminant of K.
(3) The representation $\bar{\varrho}_{E,p} : G_{\mathbf{Q}} \to \mathrm{GL}_2(\mathbf{F}_p)$ arising from the p-torsion $E(\bar{\mathbf{Q}})_p$ of E is surjective.
(4) N^- is squarefree.
(5) If E has good ordinary reduction at p, then $a_p(E) \not\equiv \pm 1 \pmod p$ if p is inert in K, and $a_p(E) \not\equiv 1 \pmod p$ if p splits in K.
(6) If q is a prime dividing N^+, then $H^0(G_{\mathbf{Q}_q}, E_p) = 0$ and $\bar{\varrho}_{E,p}$ is ramified at q.
(7) If $q \| N^-$ and $q \equiv \pm 1 \pmod p$, then $\bar{\varrho}_{E,p}$ is ramified at q.

Remark 1.2. The set of conditions in Hypothesis 1.1 are enough to state and prove our main results listed below. Since they are not simultaneously required at all stages, we indicate at the beginning of each section the assumptions under which the results obtained therein hold. In order to simplify our analysis, we do not consider in this paper primes p of multiplicative reduction for E. The assumption $p \nmid h_K$, i.e. K_∞/K is totally ramified at p, is made only to ease notations in the formulas for the compatibility of special points under the trace operators. As customary in Iwasawa theory, p is assumed to be odd. Moreover, in order to quote directly the literature at various points, notably in Section 3, we also require that $p > 3$. The surjectivity of the mod p representation $\bar{\varrho}_{E,p}$, as well as assumption (7), is used for the level raising results of Section 3. The non-anomalous assumption (5) is needed for the control results of Section 5 and for the special value formula of Lemma 4.2. Assumption (6) simplifies the treatment of the Selmer conditions at the primes dividing N^+. The assumption that p is either split or inert in K allows us to quote the existing literature on the structure of local points over the anticyclotomic tower. Finally, assumption (4), i.e. N^- is squarefree, simplifies considerably the moduli description of the Shimura curves and their special points.

Definition 1.3. We say that (E, K, p, ε) is *exceptional* if E has supersingular reduction at p, p is inert in K and $\varepsilon = +$.

For $\varepsilon = \pm, \emptyset$, let $L_p^\varepsilon(f)$ be the anticyclotomic p-adic L-function introduced in Chapter 4 in the definite case and in Section 8.3 in the indefinite case. Moreover, let $\mathrm{Char}_p^\varepsilon(f)$ be the "algebraic anticyclotomic p-adic L-function" defined to be the characteristic ideal of a certain Selmer module in Section 7.3, resp. §8.3 in the definite, resp. indefinite case. Note that in the indefinite case, $L_p^\varepsilon(f)$ describes the position of a Heegner class in a compact Selmer group and $\mathrm{Char}_p^\varepsilon(f)$ refers to the torsion part of an Iwasawa module of rank one.

The next theorem contains our results on the DAMC and IAMC. Although we have strived for maximal notational uniformity, the reader should keep in mind that the nature of the result in the two cases is rather different!

Theorem A (DAMC & IAMC)**.** $(L_p^\varepsilon(f)) \subseteq (\mathrm{Char}_p^\varepsilon(f))$ *with equality in the non-exceptional case.*

The proof of Theorem A is obtained by compiling information from the finite layers of the anticyclotomic tower, via a standard method which will not be recalled in detail in this paper. Specifically, it follows immediately from the ε-Birch and Swinnerton-Dyer (BSD) formulas of Theorem 7.1, resp. of Theorem 8.2 in the definite, resp. indefinite case, by making use of an argument due to Mazur–Rubin [MR04, Section 5.2] and Howard [How04, Section 2.2].

The Birch and Swinnerton-Dyer conjecture leads one to expect BSD formulas for the usual Selmer groups over the finite anticyclotomic layers. These BSD formulas are obtained in Chapter 9 as a consequence of the above-mentioned ε-BSD formulas, via a comparison between the ε-Selmer groups and the standard Selmer groups. We refer the reader to Chapter 7 for the definition of the Selmer group $\mathrm{Sel}(K, A_f(\chi))$ as well as of the Shafarevich–Tate group $Ш(K, A_f(\chi))$, and to Sections 9.2 and 9.3 for an explanation of the constants C (related to certain archimedean periods) and of the regulator $\mathrm{Reg}_\chi(E/K)$, which appear in the statements below.

Theorem B (Definite BSD formulas)**.** *Let χ be a finite order character of conductor p^n of the Galois group of K_∞/K. Then $\mathrm{Sel}(K, A_f(\chi))$ is finite if and only if $L(E/K, \chi, 1) \neq 0$. In this case one has*

$$\mathrm{length}_{\mathscr{O}_\chi}\big(\mathrm{Sel}(K, A_f(\chi))\big) \leqslant \mathrm{ord}_\chi\left(\frac{L(E/K, \chi, 1)}{C}\right)$$

with equality in the non-exceptional case.

Theorem C (Indefinite BSD formulas)**.** *Let χ be a finite order character of conductor p^n of the Galois group of K_∞/K. Then $\mathrm{Sel}(K, A_f(\chi))$ has corank equal to 1 if and only if $L'(E/K, \chi, 1) \neq 0$. In this case one has*

$$\mathrm{length}_{\mathscr{O}_\chi}\big(Ш(K, A_f(\chi))\big) \leqslant \mathrm{ord}_\chi\left(\frac{L'(E/K, \chi, 1)}{C \cdot \mathrm{Reg}_\chi(E/K)}\right)$$

with equality in the non-exceptional case.

Remark 1.4. The case $n = 0$ can be obtained more directly by applying [SU14, FW22] in the setting of Theorem B and the techniques of [Zha14, BBV16] in the setting of Theorem C. In the non-exceptional case it follows as well from the AMCs proved in this paper. The presence of a local zero in the exceptional case prevents us to treat the trivial character on the same ground as the other characters.

Conventions. The following conventions are adopted to lighten the notation (as recalled also in the appropriate parts of the paper).

- Besides denoting with ε one of the signs $+$ or $-$, we also sometimes write $\varepsilon = +1$ when $\varepsilon = +$ and $\varepsilon = -1$ when $\varepsilon = -$. With this convention, the equation $(-1)^n = \varepsilon$ for an integer n implies that n is even if $\varepsilon = +$ and n is odd if $\varepsilon = -$.
- Given a principal ideal (x) of a commutative ring with unity R and an R-module M, we sometimes write M/x to denote $M/(x) = M/xM$.

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2. Special points on Shimura curves and Gross curves

2.1. **Shimura curves and Gross curves.** Fix a positive integer N and a factorisation $N = N^+N^-$ into coprime integers, with N^- squarefree. Let $\mathscr{B}$ be the quaternion algebra over $\mathbf{Q}$ whose discriminant has finite part equal to N^-. The algebra $\mathscr{B}$ (which is unique up to isomorphism) is said to be *indefinite*, (resp., *definite*) if it is split (resp., non-split) at infinity. So $\mathscr{B}$ is indefinite if and only if N^- is divisible by an *even* number of primes.

For every abelian group Z, let $\hat{Z}$ denote $Z \otimes_{\mathbf{Z}} \hat{\mathbf{Z}}$, where $\hat{\mathbf{Z}} = \prod_{\ell \text{ prime}} \mathbf{Z}_\ell$ is the profinite completion of $\mathbf{Z}$. Let $\mathrm{Hom}(\mathbf{C}, \mathscr{B}_\infty)$ be the set of $\mathbf{R}$-algebra morphisms of $\mathbf{C}$ in $\mathscr{B}_\infty = \mathscr{B} \otimes_{\mathbf{Q}} \mathbf{R}$. The group $\mathscr{B}^*$ acts via the diagonal embedding on $\hat{\mathscr{B}}^*$, and via conjugation on $\mathrm{Hom}(\mathbf{C}, \mathscr{B}_\infty)$. Fix a maximal order $\tilde{\mathscr{R}}$ in $\mathscr{B}$, and an Eichler order $\mathscr{R}$ of level N^+ contained in $\tilde{\mathscr{R}}$. Define the set

$$Y_{N^+,N^-}(\mathbf{C}) := \hat{\mathscr{R}}^* \backslash \hat{\mathscr{B}}^* \times \mathrm{Hom}(\mathbf{C}, \mathscr{B}_\infty) \big/ \mathscr{B}^*. \tag{2.1}$$

As notation suggests, $Y_{N^+,N^-}(\mathbf{C})$ is a Riemann surface, arising as the set of complex points of a smooth curve. This curve can be defined over $\mathbf{Q}$, and its description, which we will recall in the next paragraphs, markedly depends on whether $\mathscr{B}$ is definite or indefinite.

In the indefinite case, let $\Gamma_{N^+,N^-} \subset \mathrm{SL}_2(\mathbf{R})$ be the discrete subgroup of $\iota_\infty(\mathscr{R}^*)$ consisting of elements of determinant 1; here $\iota_\infty : \mathscr{B} \cong \mathtt{M}_2(\mathbb{R})$ is a fixed isomorphism. Then the strong approximation theorem shows that

$$Y_{N^+,N^-}(\mathbf{C}) \cong \Gamma_{N^+,N^-} \backslash \mathcal{H},$$

where $\mathcal{H} := \{z \in \mathbf{C} : \Im(z) > 0\}$, and the left action of Γ_{N^+,N^-} on $\mathcal{H}$ is by fractional linear transformations. If $N^- \neq 1$, we set $X_{N^+,N^-}(\mathbf{C}) = Y_{N^+,N^-}(\mathbf{C})$, while if $N^- = 1$ then $Y_{N^+,N^-}(\mathbf{C})$ is the usual modular curve of level $\Gamma_0(N)$, and we let $X_{N^+,N^-}(\mathbf{C})$ denote its standard compactification obtained by adding a finite set of cusps. The Riemann surface $X_{N^+,N^-}(\mathbf{C})$ has a model X_{N^+,N^-} defined over $\mathbf{Q}$, which is called the *Shimura curve* of discriminant N^- and level N^+ (up to isomorphism, it is independent of the choices made).

In the definite case, the double coset space $\hat{\mathscr{R}}^* \backslash \hat{\mathscr{B}}^* / \mathscr{B}^*$ is a finite set, in bijection with the set $\{\mathscr{R}_1, \dots, \mathscr{R}_h\}$ of conjugacy classes of (oriented) Eichler orders of level N^+ in $\mathscr{B}$. For every $j = 1, \dots, h$, set $\Gamma_j := \mathscr{R}_j^* / \mathbf{Z}^*$; each Γ_j is a finite group. Then, again by the strong approximation theorem,

$$Y_{N^+,N^-}(\mathbf{C}) \cong \coprod_{j=1}^{h} \Gamma_j \backslash \mathrm{Hom}(\mathbf{C}, \mathscr{B}_\infty).$$

Attach a conic $\mathscr{C}/\mathbf{Q}$ to $\mathscr{B}$, by the rule

$$\mathscr{C}(A) := \big\{x \in \mathscr{B} \otimes_{\mathbf{Q}} A : x \neq 0, \mathrm{Nr}(x) = \mathrm{Tr}(x) = 0\big\} \big/ A^*,$$

where Nr and Tr denote reduced norm and trace, respectively. There is a natural bijection between $\mathrm{Hom}(\mathbf{C}, \mathscr{B}_\infty)$ and $\mathscr{C}(\mathbf{C})$, from which it follows that $Y_{N^+,N^-}(\mathbf{C})$ is identified with the set of complex points of the disjoint union $X_{N^+,N^-} := \coprod_{j=1}^{h} \mathscr{C}_j$ of the genus zero curves $\mathscr{C}_j := \Gamma_j \backslash \mathscr{C}$ defined over $\mathbf{Q}$. The curve X_{N^+,N^-} is called the *Gross curve* of discriminant N^- and level N^+.

2.2. **Hecke operators.** Since $\mathrm{Pic}(\mathbf{Z}) \cong \hat{\mathbf{Q}}^*/\mathbf{Q}^*\hat{\mathbf{Z}}^*$ is trivial, one has a bijection

$$Y_{N^+,N^-}(\mathbf{C}) \cong \Big(\hat{\mathscr{R}}^* \backslash \hat{\mathscr{B}}^*/\hat{\mathbf{Q}}^* \times \mathrm{Hom}(\mathbf{C}, \mathscr{B}_\infty)\Big)/\mathscr{B}^*.$$

The double coset space $\hat{\mathscr{R}}^* \backslash \hat{\mathscr{B}}^*/\hat{\mathbf{Q}}^*$ is equal to the product over all prime numbers ℓ of the local double coset spaces $\mathcal{T}_\ell = \mathscr{R}_\ell^* \backslash \mathscr{B}_\ell^*/\mathbf{Q}_\ell^*$, where $\mathscr{R}_\ell = \mathscr{R} \otimes_{\mathbf{Z}} \mathbf{Z}_\ell$ and $\mathscr{B}_\ell = \mathscr{B} \otimes_{\mathbf{Q}} \mathbf{Q}_\ell$. If $\ell \nmid N$, then $\mathcal{T}_\ell$ is isomorphic to the set of vertices of the Bruhat-Tits tree $\mathcal{T}_\ell \cong \mathrm{PGL}_2(\mathbf{Z}_\ell) \backslash \mathrm{PGL}_2(\mathbf{Q}_\ell)$ of $\mathrm{PGL}_2(\mathbf{Q}_\ell)$. This decomposition gives rise to an action of Hecke operators T_ℓ, for primes $\ell \nmid N$, and U_ℓ for $\ell \mid N$ by $\mathbf{Q}$-rational correspondences on X_{N^+,N^-}. By covariant functoriality, they induce endomorphisms of the Picard group

$$J_{N^+,N^-} = \mathrm{Pic}(X_{N^+,N^-}/\mathbf{Q})$$

of the curve $X_{N^+,N^-}/\mathbf{Q}$, denoted in the same way. Define $\mathbf{T}_{N^+,N^-}$ to be the $\mathbf{Z}$-subalgebra of the ring $\mathrm{End}_{\mathbf{Q}}(J_{N^+,N^-})$ generated over $\mathbf{Z}$ by the operators T_ℓ and U_ℓ. Note that in the definite case J_{N^+,N^-} is a free $\mathbf{Z}$-module of rank equal to the number of connected components of the Gross curve X_{N^+,N^-}.

2.3. **The Jacquet–Langlands correspondence.** Let $\mathbb{T}_{N^+,N^-}$ be the Hecke algebra acting faithfully on the $\mathbf{C}$-vector space $S_2(\Gamma_0(N))^{N^-\text{-new}}$ of weight-two cusp forms of level N which are new at N^-, generated over $\mathbf{Z}$ by Hecke operators T_ℓ for primes $\ell \nmid N$ and U_ℓ for primes $\ell | N$. The Jacquet–Langlands correspondence states the existence of a canonical isomorphism $\mathbb{T}_{N^+,N^-} \cong \mathbf{T}_{N^+,N^-}$ identifying Hecke operators indexed by the same prime numbers. It follows that $\mathbb{T}_{N^+,N^-}$ acts as a group of $\mathbf{Q}$-rational endomorphisms of J_{N^+,N^-}. See Section 1.6 of [BD96] for details.

2.4. **Special points.** Let $p > 3$ be a prime number such that $p \nmid N$, and $K/\mathbf{Q}$ be an imaginary quadratic field of discriminant D_K coprime with Np. Assume in this subsection that the factorization $N = N^+N^-$ satisfies the following *generalized Heegner hypothesis*: a prime divisor q of N divides N^+ if and only if it is split in K.

The inclusion $\mathrm{Hom}(K, \mathscr{B}) \subset \mathrm{Hom}(\mathbf{C}, \mathscr{B}_\infty)$ arising from extension of scalars induces a map from the set

$$\mathscr{S}_{N^+,N^-}(K) := \hat{\mathscr{R}}^* \backslash \hat{\mathscr{B}}^* \times \mathrm{Hom}(K, \mathscr{B})/\mathscr{B}^*$$

to $Y_{N^+,N^-}(\mathbf{C})$. A *special point* of X_{N^+,N^-} associated with K is any point in the image of this map. When $\mathscr{B}$ is indefinite (resp., definite), so that X_{N^+,N^-} is a Shimura curve (resp., a Gross curve), we say that the points in $\mathscr{S}_{N^+,N^-}(K)$ are *Heegner points* (resp., *Gross points*) associated with K.

Let $P \in \mathscr{S}_{N^+,N^-}(K)$ be represented by $g \times f \in \hat{\mathscr{B}}^* \times \mathrm{Hom}(K, \mathscr{B})$. Then P is a said to be of *conductor* p^n if

$$f(K) \cap g^{-1}\hat{\mathscr{R}}^* g = f(\mathcal{O}_{p^n}),$$

where $\mathcal{O}_{p^n} := \mathbf{Z} + p^n\mathcal{O}_K$ $(n \geqslant 0)$ is the order of K of conductor p^n. Write $\mathscr{S}_{N^+,N^-}(\mathcal{O}_{p^n})$ for the set of special points of conductor p^n in $X_{N^+,N^-}(\mathbf{C})$. The theory of local embeddings guarantees that, under the condition recalled at the beginning of this subsection, the set $\mathscr{S}_{N^+,N^-}(\mathcal{O}_{p^n})$ is not empty for all $n \geqslant 0$ (see [BD96], Section 2.2).

The set of special points $\mathscr{S}_{N^+,N^-}(K)$ is equipped with an algebraic Galois action of the group $\mathrm{Gal}(K^{\mathrm{ab}}/K)$, where K^{ab} is the maximal abelian extension of K. Let $P \in \mathscr{S}_{N^+,N^-}(K)$ be represented by a pair $g \times f \in \hat{\mathscr{B}}^* \times \mathrm{Hom}(K, \mathscr{B})$ and let σ be represented under the inverse of the Artin map by the class of an element $\mathfrak{a} \in \hat{K}^*$. Thadelisationen $\sigma(P)$ is the special point in $\mathscr{S}_{N^+,N^-}(K)$ represented by the pair $g\hat{f}(\mathfrak{a}) \times f$, where $\hat{f}$ is the adelisation of f.

Let $\mathrm{Pic}(\mathcal{O}_{p^n}) = K^* \backslash \hat{K}^*/\hat{\mathcal{O}}_{p^n}$ be the Picard group of $\mathcal{O}_{p^n}$. By class field theory there exists an abelian extension $\tilde{K}_n/K$, the ring class field of conductor p^n, such that the Galois group $\tilde{G}_n = \mathrm{Gal}(\tilde{K}_n/K)$ is isomorphic to $\mathrm{Pic}(\mathcal{O}_{p^n})$ via the inverse of the Artin map. Recall that the Galois group $\mathrm{Gal}(K/\mathbf{Q})$ acts on $\tilde{G}_n$ as inversion.

If X_{N^+,N^-} is a Shimura curve, then the theory of complex multiplication shows that $\mathscr{S}_{N^+,N^-}(\mathcal{O}_{p^n})$ is contained in $X_{N^+,N^-}(\tilde{K}_n)$, for all $n \geqslant 0$, and Shimura's reciprocity law states that the algebraic Galois action on the set of special points $\mathscr{S}_{N^+,N^-}(K)$ described above coincides with the usual geometric action of $\mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})$ on $X_{N^+,N^-}(\bar{\mathbf{Q}})$. In this case, for any extension $H/\mathbf{Q}$ in $\bar{\mathbf{Q}}$, denote as usual $J_{N^+,N^-}(H)$ the subgroup of H-rational divisors of $J_{N^+,N^-}(\bar{\mathbf{Q}})$, i.e. those fixed by $\mathrm{Gal}(\bar{\mathbf{Q}}/H)$.

If X_{N^+,N^-} is a Gross curve, then the algebraic action of $\mathrm{Gal}(K^{\mathrm{ab}}/K)$ on $\mathscr{S}_{N^+,N^-}(K)$ described above does not correspond to any geometric Galois action, since all special points are already defined over K. However, one can check that each element in $\mathscr{S}_{N^+,N^-}(\mathcal{O}_{p^n})$ is fixed by the algebraic action of $\mathrm{Gal}(K^{\mathrm{ab}}/\tilde{K}_n)$, for all $n \geqslant 0$. Extend canonically the action of $\mathrm{Gal}(K^{\mathrm{ab}}/K)$ on $\mathscr{S}_{N^+,N^-}(K)$ defined above to the subgroup of J_{N^+,N^-} generated by the image of $\mathscr{S}_{N^+,N^-}(K)$. Given an abelian extension H of K and an element $D \subseteq J_{N^+,N^-}$ supported on $\mathscr{S}_{N^+,N^-}(K)$, write with an abuse of notation $D \in J_{N^+,N^-}(H)$ to mean that D is fixed by the action of $\mathrm{Gal}(K^{\mathrm{ab}}/H)$.

2.5. **Compatible sequences of special points.** Let K, $N = N^+N^-$ and $p \nmid N$ be fixed as in the previous subsection, and assume that $p \nmid h_K$, the class number of K. Recall from the Introduction the anticyclotomic $\mathbf{Z}_p$-extension K_∞/K, and for any integer $n \geqslant 0$ let K_n be the subfield of K_∞ such that $G_n = \mathrm{Gal}(K_n/K) \cong \mathbf{Z}/p^n\mathbf{Z}$. Since $p \nmid h_K$, we have $\tilde{K}_{n+1} = K_n \cdot \tilde{K}_1$, and $K_n \cap \tilde{K}_1 = K$. In particular $\tilde{G}_{n+1} = \Delta \times G_n$, with $\Delta = \tilde{G}_1$. By convention, set $K_{-1} = K$.

Let $L \geqslant 1$ be a squarefree integer, prime to Np; when $L > 1$, we suppose that L is the product of primes which are inert in K. The set $\mathscr{S}_{N^+,LN^-}(\mathcal{O}_{p^n})$ of special points of conductor p^n in $X_{N^+,LN^-}(\mathbf{C})$ is then non-empty for every $n \geqslant 0$ (note that X_{N^+,LN^-} might be a Gross curve or a Shimura curve, accordingly with the parity of the number of prime divisors of N^-L). As in Section 2.4 of [BD96] fix a *compatible sequence* $\tilde{P}_\infty(L) = (\tilde{P}_n(L))_{n\geqslant 0}$ of special points of p-power conductor, where $\tilde{P}_n(L) \in \mathscr{S}_{N^+,LN^-}(\mathcal{O}_{p^n})$. For every integer $n \geqslant -1$ define

$$P_n(L) = \sum_{\sigma\in\Delta} \sigma\big(\tilde{P}_{n+1}(L)\big) \in J_{N^+,LN^-}(K_n),$$

$$P_K(L) = \sum_{\sigma\in\mathrm{Pic}(\mathcal{O}_K)} \sigma\big(\tilde{P}_0(L)\big) \in J_{N^+,N^-}(K).$$

Let ϵ_K be the quadratic character associated with K, and u_K be one half of the order of the unit group $\mathcal{O}_K^*$. Define

$$u_p = (p - \epsilon_K(p))/u_K. \tag{2.2}$$

Then these points satisfy the following relations:

$$P_{-1}(L) = u_p \cdot P_K(L), \tag{2.3}$$

$$u_K \cdot P_0(L) = \begin{cases} T_pP_K(L) & \text{if } \epsilon_K(p) = -1, \\ T_pP_K(L) - 2P_K(L) & \text{if } \epsilon_K(p) = +1, \end{cases} \tag{2.4}$$

$$\mathrm{Trace}_{K_{n+1}/K_n}(P_{n+1}(L)) = T_pP_n(L) - P_{n-1}(L), \text{ for every } n \geqslant 0, \tag{2.5}$$

where $\mathrm{Trace}_{K_{n+1}/K_n} = \sum_{\sigma\in\mathrm{Gal}(K_{n+1}/K_n)} \sigma$. If $L = 1$ we simply write $P_n = P_n(L)$ and $P_K = P_K(L)$.

3. Admissible primes and raising the level

Fix a positive integer N, a factorisation $N = N^+N^-$ into coprime positive integers, and a rational prime $p > 5$ coprime to N. Assume that N^- is square-free.

3.1. **Eigenforms of level** (N^+, N^-)**.** Recall the Hecke algebra $\mathbb{T}_{N^+,N^-}$ defined in Section 2.3 and let R be a complete local Noetherian ring with finite residue field k_R of characteristic p. A *R-valued (weight two) eigenform of level* (N^+, N^-) is a *surjective* morphism $f : \mathbb{T}_{N^+,N^-} \to R$. Denote by $S_2(N^+, N^-; R)$ the set of such eigenforms. To every eigenform $f \in S_2(N^+, N^-; R)$ is associated a Galois representation

$$\bar{\rho}_f : G_{\mathbf{Q}} \longrightarrow \mathrm{GL}_2(k_R),$$

unramified at every prime $q \nmid Np$, and such that an arithmetic Frobenius $\mathrm{Frob}_q \in G_{\mathbf{Q}}$ at q has characteristic polynomial $\mathrm{char}\,(\bar{\rho}_f(\mathrm{Frob}_q)) = X^2 - \bar{f}(T_q)X + q \in k_R[X]$, where $\bar{f}(T_q)$ denotes the reduction of $f(T_q)$ modulo the maximal ideal of R. The semi-simplification of $\bar{\rho}_f$ is characterised by these properties. Moreover, as proved by Carayol [Car94], if $\bar{\rho}_f$ is *irreducible* (hence absolutely irreducible since p is odd), it can be lifted *uniquely* to a Galois representation

$$\rho_f : G_{\mathbf{Q}} \longrightarrow \mathrm{GL}_2(R),$$

unramified at $q \nmid Np$, and such that $\mathrm{trace}(\mathrm{Frob}_q) = f(T_q)$ and $\det(\mathrm{Frob}_q) = q$ for such a q. Assuming that $\bar{\rho}_f$ is irreducible, write T_f for an R-module giving rise to the representation ρ_f, and, for R a quotient of $\mathbf{Z}_p$, define $A_f = \mathrm{Hom}_{\mathbf{Z}_p}(T_f, \mu_{p^\infty})$, where μ_{p^∞} is the group of p-power roots of unity.

Let $n \in \mathbf{N} \cup \{\infty\}$ and define $R = \mathbf{Z}_p$ if $n = \infty$ and $R = \mathbf{Z}/p^n\mathbf{Z}$ if $n < \infty$. Let $f \in S_2(N^+, N^-; R)$. If $k \in \mathbf{N} \cup \{\infty\}$ and $1 \leqslant k \leqslant n$, let $f_k = f \pmod{p^k}$ denote the reduction of f modulo p^k, with the convention that $f_\infty = f$ if $n = k = \infty$. Let $T_{f,k}$ and $A_{f,k}$ be the modules introduced above for f_k (i.e. $T_{f,k} = T_{f_k}$ and $A_{f,k} = A_{f_k}$). If particular, if $n = \infty$ then $T_{f,\infty} = T_f$ and $A_{f,\infty} = A_f$. Finally, for any $\mathbf{Z}_p$-algebra $\mathscr{O}$, we will write $T_{f,k,\mathscr{O}} = T_{f,k} \otimes_{\mathbf{Z}_p} \mathscr{O}$ and $A_{f,k,\mathscr{O}} = A_{f,k} \otimes_{\mathbf{Z}_p} \mathscr{O}$.

3.2. **Admissible primes.** Let R denote a complete, local Noetherian ring with finite residue field k_R of characteristic p and let $f \in S_2(N^+, N^-; R)$ be an R-valued eigenform of level (N^+, N^-). Fix a quadratic imaginary field $K/\mathbf{Q}$ of discriminant D_K coprime with Np. Following [BD05], we say that a rational prime ℓ is an *admissible prime relative to* (f, K) if the following conditions are satisfied:

*A*1. ℓ does not divide Np;
*A*2. p does not divide $\ell^2 - 1$;
*A*3. $f(T_\ell)^2 = (\ell + 1)^2 \in R$;
*A*4. ℓ is inert in $K/\mathbf{Q}$.

Write $\mathscr{S}(f, K)$ for the set of squarefree products of admissible primes for (f, K).

Let $n \in \mathbf{N} \cup \{\infty\}$, and put $R = \mathbf{Z}_p$ if $n = \infty$ and $R = \mathbf{Z}/p^n\mathbf{Z}$ if $n < \infty$. For $f \in S_2(N^+, N^-; R)$, and $k \in \mathbf{N} \cup \{\infty\}$ with $1 \leqslant k \leqslant n$, call *k-admissible prime* any admissible prime relative to (f_k, K), where recall that f_k is the reduction of f modulo p^k, with the convention that $f_\infty = f$. With an abuse of notation, if no confusion may arise, we write $\mathscr{S}_k$ for $\mathscr{S}(f_k, K)$. We say that $L \in \mathscr{S}_k$ is *definite* if $\epsilon_K(LN^-) = -1$ and *indefinite* if $\epsilon_K(LN^-) = +1$, and write $\mathscr{S}_k^{\mathrm{def}}$ and $\mathscr{S}_k^{\mathrm{ind}}$ for the subsets of $\mathscr{S}_k$ consisting of definite and indefinite integers, respectively; clearly $\mathscr{S}_k = \mathscr{S}_k^{\mathrm{def}} \cup \mathscr{S}_k^{\mathrm{ind}}$.

The following lemma is proved by the same argument appearing in the proof of Lemma 2.6 of [BD05].

Lemma 3.1. *Let ℓ be an admissible prime relative to (f, K), and let $K_\ell/\mathbf{Q}_\ell$ be the completion of K at the unique prime dividing ℓ (so that $K_\ell = \mathbf{Q}_{\ell^2}$ is the quadratic unramified extension of $\mathbf{Q}_\ell$). There is a decomposition of $R[G_{K_\ell}]$-modules $T_f = R(\varepsilon) \oplus R$, where $R(\varepsilon)$ (resp., R) denotes a copy of R on which G_{K_ℓ} acts via the p-adic cyclotomic character ε (resp., acts trivially).*

3.3. **Level raising.** Let n be a positive integer, and let $f \in S_2(N^+, N^-; \mathbf{Z}/p^n\mathbf{Z})$.

Hypothesis 3.2. The data $(\bar{\rho}_f, N^+, N^-, p)$ with $N = N^+N^-$ satisfy the following conditions:

(1) N^- is squarefree, $p \geqslant 5$ and $p \nmid N$;
(2) $\bar{\rho}_f : G_{\mathbf{Q}} \to \mathrm{GL}_2(\mathbf{F}_p)$ is surjective.
(3) If $q \| N^-$ and $q \equiv \pm 1 \bmod p$, then $\bar{\rho}_f$ is ramified at q.

Theorem 3.3. *Assume that $f \in S_2(N^+, N^-; \mathbf{Z}/p^n\mathbf{Z})$ satisfies Assumption 3.2. Let $S \in \mathscr{S}_k$ for some integer $1 \leqslant k \leqslant n$. Then there exists an eigenform*

$$f_S : \mathbb{T}_{N^+, N^-S} \longrightarrow \mathbf{Z}/p^k\mathbf{Z}$$

of level (N^+, N^-S) such that $f_S(T_q) = f_k(T_q)$ for $q \nmid NS$ and $f_S(U_q) = f(U_q)$ for $q \mid N$, where recall that $f_k = f \pmod{p^k}$. Moreover, f_S is unique up to multiplication by a unit in $\left(\mathbf{Z}/p^k\mathbf{Z}\right)^*$.

Proof. Assume that $N^- > 1$ and that N^- has an odd (resp., even) number of prime divisors. In this case Theorem 3.3 is proved in Section 5 (resp., Section 9) of [BD05] under slightly more restrictive assumptions on $(\bar{\rho}_f, N^+, N^-, p)$, subsequently removed in [PW11]. The method of [BD05] builds on work of Ribet (who considered the case $n = k = 1$), and makes essential use of the generalisation of Ihara's Lemma to Shimura curves obtained by Diamond–Taylor [DT94]. We refer to [BD05] for more details and references.

Assume now that $N^- = 1$. If $n = k = 1$, the theorem was proved by Ribet. If $n > 1$, it can be proved by following the arguments of Section 9 of [BD05] (see in particular Proposition 9.2 and Theorem 9.3), using Ihara's Lemma (rather than its generalisation by Diamond–Taylor) in the proof of Proposition 9.2. □

In the situation of Theorem 3.3, we say that f_S is the *level raising of* $f_k = f \pmod{p^n}$ *at* S; it is defined up to units in $\mathbf{Z}/p^k\mathbf{Z}$.

4. p-adic L-functions and special values formulae

Let $E/\mathbf{Q}$ be the elliptic curve fixed in the introduction, and let f be the cuspform associated to E by modularity. In this section we assume the following:

Hypothesis 4.1.

(1) N^- is squarefree, $p \geqslant 5$ and $p \nmid N$.
(2) $p \nmid h_K$.
(3) If $E/\mathbf{Q}_p$ has good ordinary reduction, then $a_p(E) \not\equiv \pm 1 \pmod p$ if p is inert in K and $a_p(E) \not\equiv 1 \pmod p$ if p is split in K.
(4) $\bar{\rho}_f : G_{\mathbf{Q}} \to \mathrm{GL}_2(\mathbf{F}_p)$ is surjective.
(5) If $q \| N^-$ and $q \equiv \pm 1 \bmod p$, then $\bar{\rho}_f$ is ramified at q.

Let $k \in \mathbf{Z}_{\geqslant 1} \cup \{\infty\}$ be an integer or the symbol ∞. If k is an integer let $L \in \mathscr{S}_k^{\mathrm{def}}$ be a *definite* k-admissible integer (i.e. $\epsilon_K(LN^-) = -1$) and denote by $g = f_L \in S_2(N^+, LN^-; \mathbf{Z}/p^k\mathbf{Z})$ the L-level raising of f modulo p^k. If $k = \infty$ assume that f is *definite* (i.e. $\epsilon_K(N^-) = -1$), and set $L = 1$ and $g = f$; under the Jacquet–Langlands isomorphism $\mathbb{T}_{N^+,LN^-} \cong \mathbf{T}_{N^+,LN^-}$ recalled in Section 2, if $k = \infty$ the form g induces a $\mathbf{Z}_p$-valued ring homomorphism $\mathbf{T}_{N^+,LN^-} \to \mathbf{Z}_p$, denoted by the same symbol g. In both cases X_{N^+,LN^-} is a *Gross curve.* Define $R = \mathbf{Z}_p$ if $k = \infty$ and $R = \mathbf{Z}/p^k\mathbf{Z}$ if k is an integer. We may in both cases view g as a morphism

$$g : \mathbf{T}_{N^+,LN^-} \longrightarrow R.$$

Fix a topological generator γ of G_∞. Let $\omega_n = \gamma^{p^n} - 1$, denote $\Phi_{n+1}(T) = \sum_{j=0}^{p-1} T^{j \cdot p^n} \in \mathbf{Z}[T]$ the p^{n+1}-cyclotomic polynomial. Set $\nu_p = 0$ (resp., $\nu_p = 1$) if p is inert (resp., splits) in K. Define

- $\omega_0^+ = \omega_1^+ = (\gamma - 1)^{\nu_p}$.
- $\omega_0^- = \gamma - 1$.
- For each integer $n \geqslant 2$,
$$\omega_n^+ = (\gamma - 1)^{\nu_p} \prod_{1 \leqslant j \leqslant \lfloor \frac{n}{2} \rfloor} \Phi_{2j}(\gamma).$$
- For each integer $n \geqslant 1$.
$$\omega_n^- = (\gamma - 1) \prod_{1 \leqslant j \leqslant \lfloor \frac{n+1}{2} \rfloor} \Phi_{2j-1}(\gamma).$$

4.1. **Modular parametrisations.** Let $\mathfrak{m}_L \subset \mathbf{T}_{N^+,LN^-}$ be the kernel of the reduction

$$\bar{g} : \mathbf{T}_{N^+,LN^-} \longrightarrow \mathbf{F}_p$$

of g modulo p, and let $J_{\mathfrak{m}_L}$ and $\mathbf{T}_{\mathfrak{m}_L}$ denote the completions of J_{N^+,LN^-} and $\mathbf{T}_{N^+,LN^-}$ at $\mathfrak{m}_L$ respectively. Thanks to Ribet's level lowering theorem, Hypothesis 1.1 imply that *Hypothesis CR* in [PW11] holds true, so according to Theorem 6.2 and Proposition 6.5 of [PW11] (relaxing one of the assumptions of [BD05, Lemma 2.2]) it follows that $J_{\mathfrak{m}_L}$ is a free $\mathbf{T}_{\mathfrak{m}_L}$-module of rank one. As a consequence g induces a surjective morphism

$$\psi_g : J_{N^+,LN^-} \otimes_{\mathbf{Z}} \mathbf{Z}_p \longrightarrow R$$

satisfying $\psi_g(t \cdot x) = g(t) \cdot \psi_g(x)$ for every $t \in \mathbf{T}_{N^+,LN^-}$ and every $x \in J_{N^+,LN^-}$, which is uniquely determined by g up to multiplication by a p-adic unit. Let finally $\Lambda_R = \Lambda \otimes_{\mathbf{Z}_p} R = R[\![G_\infty]\!]$ (so that $\Lambda_R = \Lambda$ if $k = \infty$ and $\Lambda_R = \Lambda/p^k\Lambda$ if $k < \infty$) and put $\Lambda_{n,k} = R[G_n]$.

4.2. **The ordinary case.** In the ordinary case, the construction of the p-adic L-function, which we will recall below, has been obtained in [BD96]. Assume in this section that $E/\mathbf{Q}_p$ has *good ordinary* reduction at p, i.e. $p \nmid N$ and $g(T_p) = a_p(E) \pmod{p^k}$ is a p-adic unit. The Hecke polynomial $X^2 - g(T_p)X + p$ has a unique root $\alpha_p(g)$ in R which is congruent to $g(T_p)$ modulo p and hence is a p-adic unit. Recall the compatible sequence $P_\infty(L) = (P_n(L))_{n\geqslant -1}$ of Gross points fixed in Section 2.5. For every $n \geqslant 1$ define

$$\mathcal{L}_{g,n} = \frac{1}{\alpha_p(g)^n} \sum_{\sigma\in G_n} \Big(\psi_g\big(\sigma(P_{n-1}(L))\big) - \alpha_p(g)\cdot\psi_g\big(\sigma(P_n(L))\big)\Big)\cdot\sigma \in \Lambda_{n,k}.$$

Since $\psi_g(T_p x) = a_p(E)\cdot\psi_g(x)$ for every $x \in J_{N^+,LN^-}$, a direct computation based on Equation (2.5) shows that the elements $(\mathcal{L}_{g,n})_{n\geqslant 1}$ are compatible under the natural projection maps $\Lambda_{n+1,k} \twoheadrightarrow \Lambda_{n,k}$. Define the *anticyclotomic square root p-adic L-function*

$$\mathcal{L}_g = \lim_{n\to\infty} \mathcal{L}_{g,n} \in \Lambda_R$$

as the inverse limit of the compatible sequence $(\mathcal{L}_{g,n})_{n\geqslant 1}$ in $\varprojlim \Lambda_{n,k} = \Lambda_R$.

For any $x \in \Lambda$ and any ring homomorphism $\chi : \Lambda \to \mathscr{O}$, define as usual $x(\chi) = \chi(x)$. Denote $\mathbf{1}$ the trivial character. One has (cf. Equation (2.5))

$$\mathcal{L}_g(\mathbf{1}) = \begin{cases} \frac{1}{u_K}\big(1-\alpha_p(g)^2\big)\cdot\psi_g(P_K(L)) & \text{if } \epsilon_K(p) = -1, \\ \frac{-1}{u_K}\big(1-\alpha_p(g)\big)^2\cdot\psi_g(P_K(L)) & \text{if } \epsilon_K(p) = +1. \end{cases} \tag{4.1}$$

Lemma 4.2. *The equality $\mathcal{L}_g(\mathbf{1}) = \psi_g\big(P_K(L)\big)$ holds in R up to multiplication by an element in R^*.*

Proof. This follows from the formulas above and Hypothesis 4.1(3). □

The definition of $\mathcal{L}_g$ depends on the choice of the compatible system of Heegner points $P_\infty(L)$. If $Q_\infty(L)$ is another compatible system, then there exists $\gamma \in G_\infty$ such that $\gamma\big(P_n(L)\big) = Q_n(L)$ for every $n \geqslant 0$ (cf. Section 2 of [BD96]). As a consequence the square root p-adic L-function $\mathcal{L}_g$ is well defined up to multiplication by G_∞. Define the *anticyclotomic p-adic L-function*

$$L_p(g) = \mathcal{L}_g \cdot \mathcal{L}_g^\iota \in \Lambda_R,$$

where ι is Iwasawa's main involution. Note that $L_p(g)$ is independent of the choice of $P_\infty(L)$.

4.3. **The supersingular case.** In the supersingular case, the construction of the p-adic L-function has been obtained in [DI08] when p is split, building on the fundamental work of Pollack [Pol02]. We extend the construction to the inert case. Assume that $E/\mathbf{Q}_p$ has good *supersingular* reduction. As $p > 3$ the Hasse bound gives $a_p(E) = 0$. Let $\Lambda_{n,k} = \Lambda/(\omega_n, p^k) = R[G_n]$, and define $\Lambda^\pm_{n,k} = \Lambda/(\omega^\pm_n, p^k)$. Set

- $\tilde\omega^+_0 = \tilde\omega^+_1 = 1$.
- $\tilde\omega^-_0 = \gamma - 1$.
- For each integer $n \geqslant 2$,
$$\tilde\omega^+_n = \prod_{1\leqslant j\leqslant \lfloor \frac{n}{2}\rfloor} \Phi_{2j}(\gamma).$$
- For each integer $n \geqslant 1$,
$$\tilde\omega^-_n = \prod_{1\leqslant j\leqslant \lfloor \frac{n+1}{2}\rfloor} \Phi_{2j-1}(\gamma).$$

Recall that γ is a topological generator of G_∞, so that $\omega_n = (\gamma-1)\cdot\tilde\omega^+_n\cdot\tilde\omega^-_n$ (and $\tilde\omega^+_n = \omega^+_n$ in the inert case) for each $n \geqslant 0$. For every $n \geqslant 0$ define

$$\mathcal{L}_{g,n} = \sum_{j=0}^{p^n-1} \psi_g\big(\gamma^j(P_n(L))\big)\cdot\gamma^j \in \Lambda_R.$$

Lemma 4.3. *Let $\varepsilon = (-1)^n$. Then $\omega^\varepsilon_n \cdot \mathcal{L}_{g,n} \in \omega_n \cdot \Lambda_R$ for all integers $n \geqslant 0$.*

Proof. The case when p is split is [DI08, Proposition 2.8(1)], and we only need to check the inert case. The proof is by induction. If $n=0$, we have, using (2.4) for the second equality and $a_p(E)=0$ for the last,

$$u_K\cdot\mathcal{L}_{g,0}=\psi_g\big(P_0(L)\big)=a_p(E)\psi_g(P_K(L))=0,$$

so $\mathcal{L}_{g,0}=0$ and the statement is true. If $n=1$, the statement is trivially true because $\omega_1^-=\omega_1$. So we can fix $n\geqslant 1$, assume the statement true for all integers s with $0\leqslant s\leqslant n$ and prove it for $n+1$. We first prove a relation between $\mathcal{L}_{g,m+1}$ and $\mathcal{L}_{g,m-1}$, where $m\geqslant 1$ is an integer. Understanding that congruences are modulo ω_m, we have

$$\begin{aligned}
\mathcal{L}_{g,m+1}&=\sum_{j=0}^{p^m-1}\left(\sum_{i=0}^{p-1}\psi_g(\gamma^{j+ip^m}(P_{m+1}(L)))\cdot\gamma^{ip^m}\right)\cdot\gamma^{j+ip^m}\\
&\equiv\sum_{j=0}^{p^m-1}\psi_g\big(\gamma^j\big(\mathrm{Trace}_{K_{m+1}/K_m}(P_{m+1}(L))\big)\big)\cdot\gamma^j\ \text{(because }\gamma^{ip^m}-1\equiv 0\pmod{\omega_m}\text{)}\\
&\equiv\sum_{j=0}^{p^m-1}\psi_g(T_p\gamma^j(P_m)-\gamma^j(P_{m-1}(L)))\cdot\gamma^j\ \text{(by Equation (2.5))}\\
&\equiv-\sum_{j=0}^{p^m-1}\psi_g\big(\gamma^j(P_{m-1}(L))\big)\cdot\gamma^j\ \text{(because }a_p(E)=0\text{)} \qquad (4.2)\\
&\equiv-\sum_{j=0}^{p^{m-1}-1}\sum_{i=0}^{p-1}\psi_g(\gamma^{j+ip^{m-1}}(P_{m-1}(L)))\cdot\gamma^{j+ip^{m-1}}\\
&\equiv-\sum_{i=0}^{p-1}\left(\sum_{j=0}^{p^{m-1}-1}\psi_g(\gamma^j(P_{m-1}(L)))\cdot\gamma^j\right)\cdot\gamma^{ip^{m-1}}\ \text{(because }\gamma^{p^{m-1}}(P_{m-1}(L))=P_{m-1}(L)\text{)}\\
&\equiv-(\gamma^{(p-1)p^{m-1}}+\cdots+\gamma^{p^{m-1}}+1)\cdot\mathcal{L}_{g,m-1}\\
&\equiv-\Phi_m(\gamma)\cdot\mathcal{L}_{g,m-1}.
\end{aligned}$$

The statement then follows easily. Indeed, if $n=2m+2\geqslant 2$ is even,

$$\begin{aligned}
\omega_{2m+2}^+\cdot\mathcal{L}_{g,2m+2}&=\omega_{2m+2}^+\cdot(-\Phi_{2m+1}(\gamma)\cdot\mathcal{L}_{g,2m}+\omega_{2m+1}\cdot\xi)\ \text{(for some }\xi\in\Lambda_{2m+2,k}\text{)}\\
&=-\Phi_{2m+2}(\gamma)\cdot\Phi_{2m+1}(\gamma)\cdot\omega_{2m}^+\cdot\mathcal{L}_{g,2m}+\omega_{2m+2}^+\cdot\omega_{2m+1}\cdot\xi\\
&\in\underbrace{-\Phi_{2m+2}(\gamma)\cdot\Phi_{2m+1}(\gamma)\cdot\omega_{2m}}_{=\omega_{2m+2}}\cdot\Lambda_{2m+2,k}+\underbrace{\omega_{2m+2}^+\cdot\omega_{2m+1}}_{\text{divisible by }\omega_{2m+2}}\cdot\Lambda_{2m+2,k}
\end{aligned}$$

where the last equality follows by the inductive hypothesis and because $\omega_{2m+2}^+\omega_{2m+1}$ is divisible by ω_{2m+2}, since $\omega_{2m+2}^-=\omega_{2m-1}^-$. If $n=2n+3\geqslant 3$ is odd, we have similarly

$$\begin{aligned}
\omega_{2m+3}^-\cdot\mathcal{L}_{g,2m+3}&=\omega_{2m+3}^-\cdot(-\Phi_{2m+2}(\gamma)\cdot\mathcal{L}_{g,2m+1}+\omega_{2m+2}\cdot\xi)\ \text{(for some }\xi\in\Lambda_R\text{)}\\
&=-\Phi_{2m+3}(\gamma)\cdot\Phi_{p^{2m+2}}(\gamma)\cdot\omega_{2m+1}^-\cdot\mathcal{L}_{g,2m+1}+\omega_{2m+3}^-\cdot\omega_{2m+2}\cdot\xi\\
&\in\underbrace{-\Phi_{2m+3}(\gamma)\cdot\Phi_{2m+2}(\gamma)\cdot\omega_{2m+1}}_{=\omega_{2m+3}}\cdot\Lambda_{2m+3,k}+\underbrace{\omega_{2m+3}^-\cdot\omega_{2m+2}}_{\text{divisible by }\omega_{2m+3}}\cdot\Lambda_{2m+3,k}.
\end{aligned}$$

where the last equality follows by the inductive hypothesis and because $\omega_{2m+3}^-\omega_{2m+2}$ is divisible by ω_{2m+3}, since $\omega_{2m+2}^+=\omega_{2m+3}^+$ □

We will use repeatedly the following elementary result.

Lemma 4.4. *Suppose that R is UFD. Let x,y be non-zero elements of R, and let $z=xy$. Let finally π be an element of R such that π and z do not have common factors. The multiplication by $x:R\to R$ defines an isomorphism $xR/(y,\pi)\cong(R/(z,\pi))[y]$.*

Proof. Multiplication by x induces a map $x : R/(y,\pi) \to R/(z,\pi)$: indeed, suppose that a and b are elements of R which satisfy $a = b + \alpha c$ for some $c \in R$ and $\alpha \in (y,\pi)$; then $xa = xb + x\alpha c$, and $x\alpha \in x(y,\pi) \subseteq (z, x\pi) \subseteq (z,\pi)$, so $x[a] = [xa] = [xb] = x[b]$ in $R/(z,\pi)$. Since $y[xa] = [axy] = [za] = [0]$, the image of the map is contained in $(R/(z,\pi))[y]$. We next show that the map is injective. Suppose $[xa] = [xb]$. Then $xa = xb + c\alpha$ for some $c \in R$ and $\alpha \in (z,\pi)$, so there exist d and e such that $x(a - b - dy) = e\pi$; now π and x do not have common irreducible factors, so since R is a UFD, we have that $x \mid e$. Therefore, we can write $x(a - b - dy - f\pi) = 0$ for some f. Since $x \neq 0$ and R is a domain, we have $a - b - cy - f\pi = 0$, so $[a] = [b]$ in $R/(y,\pi)$. We finally show that the map is surjective. Fix $[c]$ such that $y[c] = [cy] = 0$. Then $cy = d\alpha$ for some $d \in R$ and $\alpha \in (z,\pi)$, so $y(c - ex) = f\pi$ for some e and f, and again, since y and π do not have common factors, we see that $y \mid f$, so we can write $y(c - ex - g\pi) = 0$ for some g, and since $y \neq 0$ and R is a domain, we have $c = ex + g\pi$. So $x[d] = [xd] = [c]$. □

By Lemma 4.4, in the split case multiplication by $\tilde{\omega}_n^{\mp}$ gives an isomorphism between $\Lambda_{n,k}^{\pm}$ and the $\omega_n^{\pm}$-torsion submodule of $\Lambda_{n,k}$ (*cf.* [IP06, Section 4]). In the inert case, again by Lemma 4.4, multiplication by $\omega_n^{+} = \tilde{\omega}_n^{+}$ gives an isomorphism between $\Lambda_{n,k}^{-}$ and the ω_n^{-}-torsion submodule of $\Lambda_{n,k}$, and multiplication by $\omega_n^{-} = (\gamma - 1)\tilde{\omega}_n^{-}$ gives an isomorphism between $\Lambda_{n,k}^{+}$ and the $\omega_n^{+} = \tilde{\omega}_n^{+}$-torsion submodule of $\Lambda_{n,k}$. Lemma 4.3 then implies that if $\varepsilon = (-1)^n$, there exists elements $\mathcal{L}_{n,k}^{\varepsilon} \in \Lambda_{n,k}^{\varepsilon}$ such that

- If p is split in K or p is inert in K and $\varepsilon = -1$ (the non-exceptional case):

$$\mathcal{L}_{g,n} = \begin{cases} (-1)^{n/2}\tilde{\omega}_n^{-}\mathcal{L}_{g,n}^{+}, & \text{if } n \text{ is even,} \\ (-1)^{(n-1)/2}\tilde{\omega}_n^{+}\mathcal{L}_{g,n}^{-}, & \text{if } n \text{ is odd.} \end{cases}$$

- If p is inert in K and $\varepsilon = +1$:

$$\mathcal{L}_{g,n} = (-1)^{n/2}\omega_n^{-}\mathcal{L}_{g,n}^{+}$$

Denote by $\pi_{2m+2}^{+} : \Lambda_{2m+2,k}^{+} \to \Lambda_{2m,k}^{+}$ and $\pi_{2m+3}^{-} : \Lambda_{2m+3,k}^{-} \to \Lambda_{2m+1,k}^{-}$ the natural projections.

Lemma 4.5. *$\pi_{2m+2}^{+}(\mathcal{L}_{g,2m+2}^{+}) = \mathcal{L}_{g,2m}^{+}$ and $\pi_{2m+3}^{-}(\mathcal{L}_{g,2m+3}^{-}) = \mathcal{L}_{g,2m+1}^{-}$ for every $m \geqslant 0$.*

Proof. In the split case, this is Lemma 2.9 of [DI08], so we only need to check the inert case. Equation (4.2) shows that

$$\mathcal{L}_{g,2m+2} = -\Phi_{2m+1}(\gamma) \cdot \mathcal{L}_{g,2m} + \omega_{2m+1} \cdot z \tag{4.3}$$

for some $z \in \Lambda_R$, for each $m \geqslant 0$.

We first prove the statement for $\mathcal{L}_{g,m}^{+}$. From (4.3)

$$(-1)^{m+1} \cdot \omega_{2m+2}^{-} \cdot \mathcal{L}_{g,2m+2}^{+} = (-1)^{m+1}\Phi_{2m+1}(\gamma) \cdot \omega_{2m}^{-} \cdot \mathcal{L}_{g,2m}^{+} + \omega_{2m+1} \cdot z.$$

Both sides of the previous equation are divisible by ω_{2m+2}^{-}. Since Λ_{2m+2}^{+} has no nontrivial ω_{2m+2}^{-}-torsion, dividing by ω_{2m+2}^{-}, we get the result.

We now prove the statement for $\mathcal{L}_{g,m}^{-}$. From (4.3)

$$(-1)^{m+1} \cdot \tilde{\omega}_{2m+3}^{+} \cdot \mathcal{L}_{g,2m+3}^{-} = (-1)^{m+1} \cdot \Phi_{2m+3}(\gamma) \cdot \tilde{\omega}_{2m+1}^{+} \cdot \mathcal{L}_{g,2m+1}^{-} + \omega_{2m+2} \cdot z.$$

Both sides of the previous equation are divisible by $\tilde{\omega}_{2m+3}^{+}$. Since Λ_R^{-} has no nontrivial $\tilde{\omega}_{2m+3}^{+}$-torsion, dividing by $\tilde{\omega}_{2m+3}^{+}$ we get the result. □

Since $\varprojlim \Lambda_{2m,k}^{+} \cong \Lambda_R \cong \varprojlim \Lambda_{2m+1,k}^{-}$ (cf. Section 4 of [IP06]) the previous lemma allows us to define

$$\mathcal{L}_g^{\varepsilon} = \varprojlim_{m \in \mathbf{N}^{\varepsilon}} \mathcal{L}_{g,n}^{\varepsilon} \in \Lambda_R,$$

where $\mathbf{N}^{\varepsilon}$ is the set of natural numbers n satisfying $(-1)^n = \varepsilon$. Every continuous character $\chi : G_\infty \to \bar{\mathbf{Q}}_p^*$ extends uniquely to a morphism $\chi : \Lambda_R \to \mathscr{O}_\chi/p^k\mathscr{O}_\chi$ of $\mathbf{Z}_p$-algebras, where $\mathscr{O}_\chi = \mathbf{Z}_p[\chi(G_\infty)]$. As before, denote by $\mathcal{L}_g^{\pm}(\chi) = \chi(\mathcal{L}_g^{\pm})$ the value of χ at $\mathcal{L}_g^{\pm}$ and by $\mathbf{1}$ the trivial character of G_∞.

Lemma 4.6. *If (f, K, p, ε) is non-exceptional, then the equality $\mathcal{L}_g^{\varepsilon}(\mathbf{1}) = \psi_g(P_K(L))$ holds in R up to multiplication by an element in R^*.*

Proof. This follows from (2.5), after noticing that $P_{-1}(L) = u_p \cdot P_K(L)$ by (2.3) in the non-exceptional case. □

Remark 4.7. In the exceptional case, a result analogous to the equality in Lemma 4.6 is not currently available, to the best knowledge of the authors. Indeed, $\mathcal{L}_{g,0} = u_K^{-1} a_p(E)\psi_g(P_K(L)) = 0$ by Lemma 4.3 (recall $a_p(E) = 0$ under our assumptions). As a consequence, on the one hand $\mathcal{L}_{g,0}$ does not have a direct relation with $\psi_g(P_K(L))$, which is instead directly related to the special value of the L-series of E over K. On the other hand, the equality $\mathcal{L}_{g,0} = 0$, which can be interpreted as an exceptional-zero phenomenon, makes it possible to divide by $\gamma - 1$ to define the anticyclotomic p-adic L-function $\mathcal{L}_g^+$; however, the p-adic L-function thus obtained does not seem to have a clear relation with $\psi_g(P_L(K))$ as well. Therefore, it might be interesting to further investigate an analogue of Lemma 4.6 in the exceptional case, since it seems to require new ideas and a different approach than in the non-exceptional case.

As in the ordinary case, define

$$L_p^\varepsilon(g) = \mathcal{L}_g^\varepsilon \cdot (\mathcal{L}_g^\varepsilon)^\iota \in \Lambda_R,$$

which is independent of the choice of $P_\infty(L)$.

5. Selmer groups

Recall the notation introduced in Section 2.5: for every integer $n \geqslant 0$, K_n/K is the cyclic subextension of K_∞/K of degree p^n, and $G_n = \mathrm{Gal}(K_n/K)$ (as in Section 2.5, we assume that p does not divide the class number of K, cf. Hypothesis 5.1 below). Let $G_\infty = \mathrm{Gal}(K_\infty/K)$, $\Lambda_n = \mathbf{Z}_p[G_n]$ and $\Lambda = \mathbf{Z}_p[\![G_\infty]\!]$. In this section we also fix a finite flat extension $\mathscr{O}/\mathbf{Z}_p$, and define $\Lambda_{\mathscr{O},n} = \mathscr{O}[G_n]$ and $\Lambda_{\mathscr{O}} = \mathscr{O}[\![G_\infty]\!]$. For each prime ideal w of K, denote by K_w the completion of K at w. Fix an algebraic closure $\bar{K}_w$ of K_w, define $G_{K_w} = \mathrm{Gal}(\bar{K}_w/K_w)$, and let I_{K_w} be the inertia subgroup of G_{K_w}.

Let $E/\mathbf{Q}$ be an elliptic curve of conductor N, and let $p \geqslant 5$ be a prime number not dividing N. We also let $N = N^+N^-$ denote the factorization of N as before (a prime divides N^+ if and only if it is split in K, and divides N^- if and only if it is inert in K). Let

$$f \in S_2(\Gamma_0(N))$$

be the newform attached to E by modularity, which we identify, with a slight abuse of notation, with a modular form $f \in S_2(N^+, N^-; \mathbf{Z}_p)$ by the Jacquet–Langlands correspondence (cf. Section 2.3). The representations A_f and T_f associated with f as in Section 3.1 are then the p-divisible group and the p-adic Tate module of E, respectively. In this section we work under the following:

Hypothesis 5.1.

(1) N^- is squarefree, $p \geqslant 5$ and $p \nmid N$.
(2) The rational prime p does not divide the class number and the discriminant of K.
(3) $\bar{\rho}_f : G_{\mathbf{Q}} \to \mathrm{GL}_2(\mathbf{F}_p)$ is irreducible.
(4) If E has good ordinary reduction at p, then $a_p(E) \not\equiv \pm 1 \pmod p$ if p is inert in K, and $a_p(E) \not\equiv 1 \pmod p$ if p is split in K.
(5) If q is a prime dividing N^+, then $H^0(G_{\mathbf{Q}_q}, E_p) = 0$ and $\bar{\rho}_f$ is ramified at q.

5.1. **ε-rational points.** In this subsection we assume that E has supersingular reduction at p. Let $\mathfrak{p}$ be a prime of K dividing p. For every $n \in \mathbf{N} \cup \{\infty\}$ denote by $\Psi_n = K_{n,\mathfrak{p}}$ the completion of K_n at the unique prime dividing $\mathfrak{p}$, by O_n the ring of integers of Ψ_n and by $\mathfrak{m}_n$ its maximal ideal. Set $\Psi = \Psi_0$, $O = O_0$ and $\mathfrak{m} = \mathfrak{m}_0$. Then Ψ_∞ is a totally ramified $\mathbf{Z}_p$-extension of Ψ, whose Galois group can be identified with G_∞ (via $i_p : \bar{\mathbf{Q}} \hookrightarrow \bar{\mathbf{Q}}_p$). Let $\mathbf{E}/O$ be the formal group of E/Ψ, which gives the kernel of the reduction modulo p on E, and let $\log_{\mathbf{E}} : \mathbf{E} \to \mathbf{G}_a$ be the formal group logarithm. The formal group $\mathbf{E}/O$ is a Lubin–Tate group for the uniformiser $-p \in O$ ([Hon70]). Since $E_{p^m} \cong \mathbf{E}_{p^m}$ for every $m \geqslant 1$ as $G_\mathfrak{p}$-modules, the p-adic Tate module T_f of E is isomorphic to the p-adic Tate module of $\mathbf{E}$, hence has a natural structure of $O[\mathrm{Gal}(\bar{\Psi}/\Psi)]$-module.

Denote by Ξ the set of $\bar{\mathbf{Q}}_p^*$-valued finite order characters on G_∞. For every $\chi \in \Xi$ let n_χ be the smallest nonnegative integer such that χ factors through G_{n_χ}, and let

$$\Xi^\pm = \{\chi \in \Xi \mid n_\chi \geqslant 1,\ (-1)^{n_\chi} = \pm 1\}.$$

If p splits in $K/\mathbf{Q}$, set $\Xi_p^{\pm} = \Xi^{\pm}$; if p is inert in $K/\mathbf{Q}$ set $\Xi_p^{+} = \Xi^{+}$ and $\Xi_p^{-} = \Xi^{-} \cup \{\mathbf{1}\}$, where $\mathbf{1} \in \Xi$ is the trivial character on G_∞. Let $\log_\chi : \mathbf{E}(\mathfrak{m}_\infty) \to \bar{\mathbf{Q}}_p$ be the morphism sending y in $\mathbf{E}(\mathfrak{m}_\infty)$ to

$$\log_\chi(y) = p^{-m} \sum_{\sigma \in G_m} \chi(\sigma)^{-1} \log_{\mathbf{E}}(y^\sigma),$$

where $m = m(\chi, y)$ is any positive integer large enough so that $m \geqslant n_\chi$ and y belongs to $\mathbf{E}(\mathfrak{m}_m)$. Following [Rub87] set

$$\mathbf{E}(\mathfrak{m}_\infty)_\pm = \{ y \in \mathbf{E}(\mathfrak{m}_\infty) \mid \log_\chi(y) = 0 \text{ for every } \chi \in \Xi_p^{\mp} \},$$

and for every $1 \leqslant k \leqslant \infty$ define

$$H^1_{\mathrm{fin},\pm}(\Psi_\infty, A_{f,k}) = \mathbf{E}(\mathfrak{m}_\infty)_\pm \otimes_{\mathbf{Z}} (\mathbf{Q}_p/\mathbf{Z}_p)_{p^k},$$

viewed as submodules of $H^1(\Psi_\infty, A_{f,k})$ under the local Kummer map.

Let ε denote one of the signs $+$ or $-$. With an abuse of notation, we sometimes identify the sign $\varepsilon = \pm$ with ± 1, so that the equation $(-1)^n = \varepsilon$ makes sense for each integer n, and states that n is even if $\varepsilon = +$ and n is odd if $\varepsilon = -$. For every integer $n \geqslant 0$ and every prime $\mathfrak{p}$ of K dividing p set

$$\mathbf{E}(\mathfrak{m}_n)_\varepsilon = \mathbf{E}(\mathfrak{m}_n) \cap \mathbf{E}(\mathfrak{m}_\infty)_\varepsilon.$$

If follows from the definitions that $\omega_n^\varepsilon \cdot \mathbf{E}(\mathfrak{m}_n)_\varepsilon = 0$ for every $n \geqslant 0$ and $\mathfrak{p}|p$; in particular, if p is inert in K then $\mathbf{E}(\mathfrak{m})_+ = 0$ and $\mathbf{E}(\mathfrak{m})_- = \mathbf{E}(\mathfrak{m})$, while if p is split in K, we have $\mathbf{E}(\mathfrak{m})_+ = \mathbf{E}(\mathfrak{m})_- = \mathbf{E}(\mathfrak{m})$.

Denote by Λ_O the tensor product of Λ with O. For every Galois extension Ψ'/Ψ the group $\mathbf{E}(\Psi')$ is a module over $O[\mathrm{Gal}(\Psi'/\Psi)]$. The next theorem, which elucidates the structure of the Λ_O-modules $\mathbf{E}(\Psi_n)$ and $\mathbf{E}(\Psi_n)_\varepsilon$, has been obtained by Iovita–Pollack [IP06, Theorem 4.5] in the split case (building on the work of Kobayashi [Kob03]) and by Burungale–Kobayashi–Ota [BKO21, Theorem 5.5] in the inert case. It shows that the ε-local points enjoy trace relations analogous to those satisfied by the families of Heegner points and Gross points intervening in the definition of the ε-p-adic L-functions.

Theorem 5.2.

(1) *$\mathbf{E}(\Psi)$ is a free O-module of rank 1. Choose a generator $\boldsymbol{d}_{\mathfrak{p},0} \in \mathbf{E}(\Psi)$; define $\boldsymbol{d}^+_{\mathfrak{p},0} = \boldsymbol{d}^-_{\mathfrak{p},0} = \boldsymbol{d}_{\mathfrak{p},0}$ if p is split in K and $\boldsymbol{d}^+_{\mathfrak{p},0} = 0$, $\boldsymbol{d}^-_{\mathfrak{p},0} = \boldsymbol{d}_{\mathfrak{p},0}$ if p is inert in K.*

(2) *If $n \geqslant 1$ and $\varepsilon = (-1)^n$, then $\mathbf{E}(\Psi_n)_\varepsilon$ is a free $\Lambda_O/(\omega_n^\varepsilon)$-module of rank 1. We can choose generators $\boldsymbol{d}^\epsilon_{\mathfrak{p},n} \in \mathbf{E}(\Psi_n)_\varepsilon$ satisfying the following trace relations:*
   - $\mathrm{Trace}_{n+2/n+1}(\boldsymbol{d}^\varepsilon_{\mathfrak{p},n+2}) = -\boldsymbol{d}^\varepsilon_{\mathfrak{p},n}$*;*
   - $\mathrm{Trace}_{1/0}(\boldsymbol{d}^-_{\mathfrak{p},1}) = u \cdot \boldsymbol{d}_{\mathfrak{p},0}$*, for some unit* $u \in \mathbf{Z}_p^\times$*.*

(3) *If $n \geqslant 1$ and $\varepsilon = -(-1)^n$, define $\boldsymbol{d}^\varepsilon_{\mathfrak{p},n} = \boldsymbol{d}^\varepsilon_{\mathfrak{p},n-1} \in \mathbf{E}(\Psi_{n-1})_\varepsilon$. Then the Λ_O-module $\mathbf{E}(\Psi_n)$ is generated by $\boldsymbol{d}^\varepsilon_{\mathfrak{p},n}$ and $\boldsymbol{d}^{-\varepsilon}_{\mathfrak{p},n}$.*

Theorem 5.2 furnishes elements $\boldsymbol{d}^\varepsilon_{\mathfrak{p},n}$ defined for all $n \geqslant 0$ and $\varepsilon \in \{\pm\}$ which we consider fixed from now on.

5.2. **Selmer groups.** Let $k \in \mathbf{N} \cup \{\infty\}$. If $k \in \mathbf{N}$ and $L \in \mathscr{S}_k$, let $g : \mathbb{T}_{N^+,N^-L} \to \mathbf{Z}/p^k\mathbf{Z}$ be the level raising of $f_k = f \pmod{p^k}$ at L (cf. Section 3.3). If $k = \infty$, set $L = 1$ and $g = f$. Fix an isomorphism of G_K-modules between $T_{f,k}$ and the p-adic representation T_g associated with g, which also fixes an isomorphism between $A_{f,k}$ and $A_g = \mathrm{Hom}_{\mathbf{Z}_p}(T_g, \mu_{p^\infty})$. We often identify $A_{f,k}$ with $T_f \otimes_{\mathbf{Z}_p} (\mathbf{Q}_p/\mathbf{Z}_p)_{p^k}$, hence A_g with $T_g \otimes_{\mathbf{Z}_p} (\mathbf{Q}_p/\mathbf{Z}_p)_{p^k}$, using the Weil pairing on E (with the convention $p^\infty = 0$). Let $\iota : \Lambda \longrightarrow \Lambda$ be Iwasawa's main involution (acting as inversion on group-like elements), and let

$$\mathbf{T}_g = T_g \otimes_{\mathbf{Z}_p} \Lambda(\epsilon_\infty^{-1}) \quad \text{and} \quad \mathbf{A}_g = \mathrm{Hom}_{\mathrm{cont}}(\mathbf{T}_g^\iota, \mu_{p^\infty}),$$

where $\epsilon_\infty : G_K \to \Lambda^*$ is the tautological representation (obtained by composing the canonical projection $G_K \twoheadrightarrow G_\infty$ with the inclusion $G_\infty \hookrightarrow \Lambda^*$ of group-like elements) and one writes $M^\iota = M \otimes_{\Lambda,\iota} \Lambda$ for every Λ-module M. We also define the scalar extensions

$$\mathbf{T}_{g,\mathscr{O}} = \mathbf{T}_g \otimes_{\mathbf{Z}_p} \mathscr{O} \quad \text{and} \quad \mathbf{A}_{g,\mathscr{O}} = \mathbf{A}_g \otimes_{\mathbf{Z}_p} \mathscr{O}.$$

For every ideal $\mathfrak{P}$ of $\Lambda_{\mathscr{O}}$, set $\mathscr{O}_{\mathfrak{P}} = \Lambda_{\mathscr{O}}/\mathfrak{P}$, and set

$$T_{g,\mathscr{O}}(\mathfrak{P}) = \mathbf{T}_{g,\mathscr{O}}/\mathfrak{P} \cdot \mathbf{T}_{g,\mathscr{O}} \quad \text{and} \quad A_{g,\mathscr{O}}(\mathfrak{P}) = \mathbf{A}_{g,\mathscr{O}}[\mathfrak{P}].$$

Write $T_g(\mathfrak{P})$ for $T_{g,\mathbf{Z}_p}(\mathfrak{P})$ and $A_g(\mathfrak{P})$ for $A_{g,\mathbf{Z}_p}(\mathfrak{P})$. Then $A_{g,\mathscr{O}}(\mathfrak{P})$ is isomorphic as a $\Lambda_{\mathscr{O}}[G_K]$-module to the Kummer dual $\mathrm{Hom}_{\mathscr{O}}(T_{g,\mathscr{O}}(\mathfrak{P}^\iota)^\iota, \mu_{p^\infty} \otimes_{\mathbf{Z}_p} \mathscr{O})$ of $T_{g,\mathscr{O}}(\mathfrak{P}^\iota)^\iota$, where $\mathfrak{P}^\iota = \iota(\mathfrak{P})$. For every finite prime w of K, local Tate duality gives then a perfect $\mathscr{O}$-bilinear pairing

$$\langle -,-\rangle_{\mathfrak{P},w} : H^1(K_w, T_{g,\mathscr{O}}(\mathfrak{P})) \times H^1(K_w, A_{g,\mathscr{O}}(\mathfrak{P}^\iota)) \longrightarrow \mathscr{K}/\mathscr{O}, \tag{5.1}$$

where $\mathscr{K} = \mathrm{Frac}(\mathscr{O})$ is the fraction field of $\mathscr{O}$, such that

$$\langle \lambda \cdot x, y\rangle_{\mathfrak{P},w} = \langle x, \iota(\lambda)\cdot y\rangle_{\mathfrak{P},w}$$

for every $\lambda \in \Lambda_{\mathscr{O}}$, every $x \in H^1(K_w, T_{g,\mathscr{O}}(\mathfrak{P}))$, and every $y \in H^1(K_w, A_{g,\mathscr{O}}(\mathfrak{P}^\iota))$.

5.2.1. *Primes dividing p.* Let $\mathfrak{p}$ be a prime of K dividing p and let $\varepsilon \in \{\emptyset, \pm\}$. For $\varepsilon = \pm$, recall the groups $H^1_{\mathrm{fin},\varepsilon}(K_{\infty,\mathfrak{p}}, A_{f,k})$ defined in Section 5.1, and for $\varepsilon = \emptyset$ define

$$H^1_{\mathrm{fin}}(K_{\infty,\mathfrak{p}}, A_{f,k}) = E(K_{\infty,\mathfrak{p}}) \otimes_{\mathbf{Z}} (\mathbf{Q}_p/\mathbf{Z}_p)_{p^k},$$

viewed as submodules of $H^1(K_{\infty,\mathfrak{p}}, A_{f,k})$ under the Kummer map. Shapiro's Lemma yields a natural isomorphism of Λ-modules $H^1(K_\mathfrak{p}, \mathbf{A}_g) \cong H^1(K_{\infty,\mathfrak{p}}, A_g)$, which we often consider an equality, and we denote by $H^1_{\mathrm{fin},\varepsilon}(K_\mathfrak{p}, \mathbf{A}_g)$ the submodule of $H^1(K_\mathfrak{p}, \mathbf{A}_g)$ corresponding to $H^1_{\mathrm{fin},\varepsilon}(K_{\infty,\mathfrak{p}}, A_{f,k})$ via this isomorphism. We then define

$$H^1_{\mathrm{fin},\varepsilon}(K_\mathfrak{p}, \mathbf{A}_{g,\mathscr{O}}) \subset H^1(K_\mathfrak{p}, \mathbf{A}_{g,\mathscr{O}})$$

as the image of $H^1_{\mathrm{fin},\varepsilon}(K_\mathfrak{p}, \mathbf{A}_g) \otimes_{\mathbf{Z}_p} \mathscr{O}$ via the canonical isomorphism $H^1(K_\mathfrak{p}, \mathbf{A}_g) \otimes_{\mathbf{Z}_p} \mathscr{O} \cong H^1(K_\mathfrak{p}, \mathbf{A}_{g,\mathscr{O}})$. For every ideal $\mathfrak{P}$ of $\Lambda_{\mathscr{O}}$ define

$$H^1_{\mathrm{fin},\varepsilon}(K_\mathfrak{p}, A_{g,\mathscr{O}}(\mathfrak{P})) \subset H^1(K_\mathfrak{p}, A_{g,\mathscr{O}}(\mathfrak{P}))$$

as the inverse image of $H^1_{\mathrm{fin},\varepsilon}(K_\mathfrak{p}, \mathbf{A}_{g,\mathscr{O}})$ under the map $H^1(K_\mathfrak{p}, A_{g,\mathscr{O}}(\mathfrak{P})) \longrightarrow H^1(K_\mathfrak{p}, \mathbf{A}_{g,\mathscr{O}})$ (induced in cohomology by the inclusion $A_{g,\mathscr{O}}(\mathfrak{P}) = \mathbf{A}_{g,\mathscr{O}}[\mathfrak{P}] \hookrightarrow \mathbf{A}_{g,\mathscr{O}}$), and define

$$H^1_{\mathrm{fin},\varepsilon}(K_\mathfrak{p}, T_{g,\mathscr{O}}(\mathfrak{P})) \subset H^1(K_\mathfrak{p}, T_{g,\mathscr{O}}(\mathfrak{P}))$$

as the orthogonal complement of $H^1_{\mathrm{fin},\varepsilon}(K_{\infty,\mathfrak{p}}, A_{g,\mathscr{O}}(\mathfrak{P}^\iota))$ under the local Tate pairing $\langle -,-\rangle_{\mathfrak{P},\mathfrak{p}}$. If $\mathtt{M}_{g,\mathscr{O}}$ denotes either $T_{g,\mathscr{O}}$ or $A_{g,\mathscr{O}}$, let $H^1_{\mathrm{sing},\varepsilon}(K_\mathfrak{p}, \mathtt{M}_{g,\mathscr{O}}(\mathfrak{P}))$ be the quotient of $H^1(K_\mathfrak{p}, \mathtt{M}_{g,\mathscr{O}}(\mathfrak{P}))$ by the finite subgroup $H^1_{\mathrm{fin},\varepsilon}(K_\mathfrak{p}, \mathtt{M}_{g,\mathscr{O}}(\mathfrak{P}))$, so that we have a canonical exact sequence

$$0 \longrightarrow H^1_{\mathrm{fin},\varepsilon}(K_\mathfrak{p}, \mathtt{M}_{g,\mathscr{O}}(\mathfrak{P})) \longrightarrow H^1(K_\mathfrak{p}, \mathtt{M}_{g,\mathscr{O}}(\mathfrak{P})) \longrightarrow H^1_{\mathrm{sing},\varepsilon}(K_\mathfrak{p}, \mathtt{M}_{g,\mathscr{O}}(\mathfrak{P})) \longrightarrow 0.$$

A global class $x \in H^1(K, \mathtt{M}_{g,\mathscr{O}}(\mathfrak{P}))$ is said to be *ε-finite at* $\mathfrak{p}$ if $\mathrm{res}_\mathfrak{p}(x) \in H^1_{\mathrm{fin},\varepsilon}(K_\mathfrak{p}, \mathtt{M}_{g,\mathscr{O}}(\mathfrak{P}))$. For any element $s \in H^1(K, \mathtt{M}_{g,\mathscr{O}}(\mathfrak{P}))$, denote $\partial_\mathfrak{p}(s)$ the projection of the restriction of s at $\mathfrak{p}$ to the singular quotient of $H^1(K_\mathfrak{p}, \mathtt{M}_{g,\mathscr{O}}(\mathfrak{P}))$. We call $\partial_\mathfrak{p}$ the *residue map* at $\mathfrak{p}$, and $\partial_p = \oplus_{\mathfrak{p}|p}\partial_\mathfrak{p}$ the *residue map at p.*

5.2.2. *Primes dividing N^-.* Let $\mathfrak{P}$ be an ideal of $\Lambda_{\mathscr{O}}$ and let ℓ be a rational prime dividing N^-. Then ℓ is inert in $K/\mathbf{Q}$ and $\ell \cdot \mathcal{O}_K$ splits completely in K_∞/K. As a consequence the G_{K_ℓ}-representation $T_{g,\mathscr{O}}(\mathfrak{P})$ is isomorphic to the base change $T_g \otimes_{\mathbf{Z}_p} \mathscr{O}_\mathfrak{P}$ (with G_{K_ℓ} acting trivially on $\mathscr{O}_\mathfrak{P}$). The elliptic curve E/K_ℓ is a Tate curve, i.e. is isomorphic as a rigid analytic variety to the quotient of the multiplicative group $\mathbf{G}_m/K_\ell$ by the lattice $q_\ell^{\mathbf{Z}}$ generated by the Tate period $q_\ell \in \ell \cdot \mathbf{Z}_\ell$ ([Sil94, Chapter 5]). This gives a short exact sequence of G_{K_ℓ}-modules

$$0 \longrightarrow T^{(\ell)}_{f,k} \longrightarrow T_{f,k} \longrightarrow T^{[\ell]}_{f,k} \longrightarrow 0,$$

where $T^{(\ell)}_{f,k} \cong \mathbf{Z}_p/p^k(1)$ and $T^{[\ell]}_{f,k} \cong \mathbf{Z}_p/p^k$, which in turn induces an exact sequence of $\mathscr{O}_\mathfrak{P}[G_{K_\ell}]$-modules

$$0 \longrightarrow T^{(\ell)}_{g,\mathscr{O}}(\mathfrak{P}) \longrightarrow T_{g,\mathscr{O}}(\mathfrak{P}) \longrightarrow T^{[\ell]}_{g,\mathscr{O}}(\mathfrak{P}) \longrightarrow 0,$$

with $T^{(\ell)}_{g,\mathscr{O}}(\mathfrak{P}) \cong \mathscr{O}_\mathfrak{P}(1) \otimes_{\mathbf{Z}_p} \mathbf{Z}_p/p^k$ and $T^{[\ell]}_{g,\mathscr{O}}(\mathfrak{P}) \cong \mathscr{O}_\mathfrak{P} \otimes_{\mathbf{Z}_p} \mathbf{Z}_p/p^k$. Define $A^{(\ell)}_{g,\mathscr{O}}(\mathfrak{P})$ and $A^{[\ell]}_{g,\mathscr{O}}(\mathfrak{P})$ to be the Kummer duals of $T^{[\ell]}_{g,\mathscr{O}}(\mathfrak{P}^\iota)^\iota$ and $T^{(\ell)}_{g,\mathscr{O}}(\mathfrak{P}^\iota)^\iota$ respectively, so that one has an exact sequence

$$0 \longrightarrow A^{(\ell)}_{g,\mathscr{O}}(\mathfrak{P}) \longrightarrow A_{g,\mathscr{O}}(\mathfrak{P}) \longrightarrow A^{[\ell]}_{g,\mathscr{O}}(\mathfrak{P}) \longrightarrow 0$$

of $\mathscr{O}_{\mathfrak{P}}[G_{K_\ell}]$-modules. If $\mathtt{M}_{g,\mathscr{O}}$ is either $T_{g,\mathscr{O}}$ or $A_{g,\mathscr{O}}$, define the *ordinary subspace* of $H^1(K_\ell, \mathtt{M}_{g,\mathscr{O}}(\mathfrak{P}))$ by

$$H^1_{\mathrm{ord}}(K_\ell, \mathtt{M}_{g,\mathscr{O}}(\mathfrak{P})) = \mathrm{Im}\left(H^1(K_\ell, \mathtt{M}^{(\ell)}_{g,\mathscr{O}}(\mathfrak{P})) \longrightarrow H^1(K_\ell, \mathtt{M}_{g,\mathscr{O}}(\mathfrak{P}))\right).$$

As easily proved, $H^1_{\mathrm{ord}}(K_\ell, T_{g,\mathscr{O}}(\mathfrak{P}))$ is the orthogonal complement of $H^1_{\mathrm{ord}}(K_\ell, A_{g,\mathscr{O}}(\mathfrak{P}^\iota))$ under $\langle -, -\rangle_{\mathfrak{P},\ell}$. A global class in $H^1(K, \mathtt{M}_{g,\mathscr{O}}(\mathfrak{P}))$ is said to be *ordinary at* ℓ if its restriction at ℓ belongs to the ordinary subspace $H^1_{\mathrm{ord}}(K_\ell, \mathtt{M}_{g,\mathscr{O}}(\mathfrak{P}))$.

5.2.3. *Primes dividing* L. Let $\mathfrak{P}$ be an ideal of $\Lambda_{\mathscr{O}}$, and let ℓ be a prime divisor of L. As above $\ell \cdot \mathcal{O}_K$ splits completely in K_∞/K and $T_{g,\mathscr{O}}(\mathfrak{P}) = T_g \otimes_{\mathbf{Z}_p} \mathscr{O}_{\mathfrak{P}}$ as G_{K_ℓ}-modules, with G_{K_ℓ} acting trivially on the second factor. Lemma 3.1 then implies that the $\mathscr{O}_{\mathfrak{P}}[G_{K_\ell}]$-module $T_{g,\mathscr{O}}(\mathfrak{P})$ is isomorphic to the direct sum of $T^{(\ell)}_{g,\mathscr{O}}(\mathfrak{P}) = \mathscr{O}_{\mathfrak{P}}(1) \otimes_{\mathbf{Z}_p} \mathbf{Z}_p/p^k$ and $T^{[\ell]}_{g,\mathscr{O}}(\mathfrak{P}) = \mathscr{O}_{\mathfrak{P}} \otimes_{\mathbf{Z}_p} \mathbf{Z}_p/p^k$ (where by definition $\mathscr{O}_{\mathfrak{P}}(1) = \mathbf{Z}_p(1) \otimes_{\mathbf{Z}_p} \mathscr{O}_{\mathfrak{P}}$ as Galois modules). Let as above $A^{(\ell)}_{g,\mathscr{O}}(\mathfrak{P})$ and $A^{[\ell]}_{g,\mathscr{O}}(\mathfrak{P})$ be the Kummer duals of $T^{[\ell]}_{g,\mathscr{O}}(\mathfrak{P}^\iota)^\iota$ and $T^{(\ell)}_{g,\mathscr{O}}(\mathfrak{P}^\iota)^\iota$ respectively. For $\mathtt{M}_{g,\mathscr{O}} \in \{T_{g,\mathscr{O}}, A_{g,\mathscr{O}}\}$, define the *ordinary subspace* of $H^1(K_\ell, \mathtt{M}_{g,\mathscr{O}}(\mathfrak{P}))$ by the equality

$$H^1_{\mathrm{ord}}(K_\ell, \mathtt{M}_{g,\mathscr{O}}(\mathfrak{P})) = H^1(K_\ell, \mathtt{M}^{(\ell)}_{g,\mathscr{O}}(\mathfrak{P})).$$

By Lemma 3.1, $H^1_{\mathrm{ord}}(K_\ell, \mathtt{M}_{g,\mathscr{O}}(\mathfrak{P}))$ is also isomorphic to the *singular quotient*

$$H^1_{\mathrm{sing}}(K_\ell, \mathtt{M}_{g,\mathscr{O}}(\mathfrak{P})) = H^1(K_\ell, \mathtt{M}_{g,\mathscr{O}}(\mathfrak{P}))/H^1_{\mathrm{fin}}(K_\ell, \mathtt{M}_{g,\mathscr{O}}(\mathfrak{P})).$$

Moreover, note that $H^1_{\mathrm{ord}}(K_\ell, T_{g,\mathscr{O}}(\mathfrak{P}^\iota))$ (resp., $H^1_{\mathrm{fin}}(K_\ell, T_{g,\mathscr{O}}(\mathfrak{P}^\iota))$) is the orthogonal complement of $H^1_{\mathrm{ord}}(K_\ell, A_{g,\mathscr{O}}(\mathfrak{P}))$ (resp., $H^1_{\mathrm{fin}}(K_\ell, A_{g,\mathscr{O}}(\mathfrak{P}))$) under $\langle -, -\rangle_{\mathfrak{P},\ell}$. A global class in $H^1(K, \mathtt{M}_{g,\mathscr{O}}(\mathfrak{P}))$ is said to be *ordinary* (resp., *finite*) *at* ℓ if its restriction at ℓ belongs to the ordinary (resp., finite) subspace of $H^1(K_\ell, \mathtt{M}_{g,\mathscr{O}}(\mathfrak{P}))$.

5.2.4. *Primes outside* LN^-p. Let w be a prime of K which does not divide LN^-p, let $\mathfrak{P}$ be an ideal of $\Lambda_{\mathscr{O}}$ and let $\mathtt{M}_{g,\mathscr{O}}$ denote either $T_{g,\mathscr{O}}$ or $A_{g,\mathscr{O}}$. A global class in $H^1(K, \mathtt{M}_{g,\mathscr{O}}(\mathfrak{P}))$ is *finite* (resp., *trivial*) *at* w if its restriction at w belongs to the finite subspace

$$H^1_{\mathrm{fin}}(K_w, \mathtt{M}_{g,\mathscr{O}}(\mathfrak{P})) = H^1(G_{K_w}/I_{K_w}, \mathtt{M}_{g,\mathscr{O}}(\mathfrak{P})^{I_{K_w}})$$

of $H^1(K_w, \mathtt{M}_{g,\mathscr{O}}(\mathfrak{P}))$ (resp., is zero).

5.2.5. *Discrete and compact Selmer groups.* Let S be a positive squarefree integer and let $\mathfrak{P}$ be an ideal of $\Lambda_{\mathscr{O}}$. The *discrete Selmer group*

$$\mathrm{Sel}^S_\varepsilon(K, A_{g,\mathscr{O}}(\mathfrak{P})) \subset H^1(K, A_{g,\mathscr{O}}(\mathfrak{P}))$$

is defined to be the $\mathscr{O}_{\mathfrak{P}}$-module of global cohomology classes in $H^1(K, A_{g,\mathscr{O}}(\mathfrak{P}))$ which are

- ε-finite at primes dividing p;
- ordinary at primes dividing LN^-;
- trivial at primes dividing SN^+;
- finite outside $SLNp$.

The *compact Selmer group*

$$\mathfrak{Sel}^\varepsilon_S(K, T_{g,\mathscr{O}}(\mathfrak{P})) \subset H^1(K, T_{g,\mathscr{O}}(\mathfrak{P}))$$

is the $\mathscr{O}_{\mathfrak{P}}$-module of global cohomology classes in $H^1(K, T_{g,\mathscr{O}}(\mathfrak{P}))$ which are

- ε-finite at primes dividing $p/\text{g.c.d.}(S, p)$;
- ordinary at primes dividing $LN^-/\text{g.c.d.}(S, LN^-)$;
- finite outside $SLNp$.

Write $\mathrm{Sel}_\varepsilon(K, A_{g,\mathscr{O}}(\mathfrak{P}))$ and $\mathfrak{Sel}^\varepsilon(K, T_{g,\mathscr{O}}(\mathfrak{P}))$ as shorthands for $\mathrm{Sel}^1_\varepsilon(K, A_g(\mathfrak{P}))$ and $\mathfrak{Sel}^\varepsilon_1(K, T_{g,\mathscr{O}}(\mathfrak{P}))$ respectively. If $\mathfrak{P}$ is the zero ideal, so that $T_{g,\mathscr{O}}(\mathfrak{P}) = \mathbf{T}_{g,\mathscr{O}}$ and $A_{g,\mathscr{O}}(\mathfrak{P}) = \mathbf{A}_{g,\mathscr{O}}$, set

$$\mathrm{Sel}^S_\varepsilon(K_\infty, A_{g,\mathscr{O}}) = \mathrm{Sel}^S_\varepsilon(K, \mathbf{A}_{g,\mathscr{O}}) \quad \text{and} \quad \mathfrak{Sel}^\varepsilon_S(K_\infty, T_{g,\mathscr{O}}) = \mathfrak{Sel}^\varepsilon_S(K, \mathbf{T}_{g,\mathscr{O}}).$$

Note that if $p \mid S$ then $\mathrm{Sel}^S_\varepsilon(K, \mathbf{A}_{g,\mathscr{O}}) = \mathrm{Sel}^S(K, \mathbf{A}_{g,\mathscr{O}})$ and $\mathfrak{Sel}^\varepsilon_S(K, \mathbf{T}_{g,\mathscr{O}}) = \mathfrak{Sel}_S(K, \mathbf{T}_{g,\mathscr{O}})$.

5.3. **Local properties.** If E has (good) ordinary reduction at p, set $\varepsilon = \emptyset$ and $H^1_{\mathrm{fin},\varepsilon} = H^1_{\mathrm{fin}}$. If E has (good) supersingular reduction at p, let ε denote either $+$ or $-$.

5.3.1. *Primes dividing p.* Fix a prime $\mathfrak{p}$ of K dividing p. We first investigate local properties of points, and then we consider finite and singular subgroups.

Proposition 5.3. *For every nonnegative integer n, the restriction map induces an isomorphism*

$$E(K_{n,\mathfrak{p}})_\varepsilon \otimes_{\mathbf{Z}} \mathbf{Q}_p/\mathbf{Z}_p \cong \big(E(K_{\infty,\mathfrak{p}})_\varepsilon \otimes_{\mathbf{Z}} \mathbf{Q}_p/\mathbf{Z}_p\big)[\omega_n^\varepsilon].$$

Proof. If $E/\mathbf{Q}_p$ has good supersingular reduction, this follows from Theorem 5.2. More precisely, the Pontrjagin dual of the restriction map

$$E(K_{n,\mathfrak{p}})_\varepsilon \otimes_{\mathbf{Z}} \mathbf{Q}_p/\mathbf{Z}_p \longrightarrow \big(E(K_{m,\mathfrak{p}})_\varepsilon \otimes_{\mathbf{Z}} \mathbf{Q}_p/\mathbf{Z}_p\big)[\omega_n^\varepsilon]$$

is a surjective morphism of $\Lambda_\mathfrak{p}$-modules, for all integers $m \geqslant n$. Since Theorem 5.2 implies that its source and target are finite free $\mathbf{Z}_p$-modules of the same rank (indeed both are isomorphic to $\Lambda_\mathfrak{p}/\omega_n^\varepsilon$), it is an isomorphism.

Assume that $E/\mathbf{Q}_p$ has good ordinary reduction and consider the restriction maps

$$r_n : H^1(K_{n,\mathfrak{p}}, A_f) \longrightarrow H^1(K_{\infty,\mathfrak{p}}, A_f)[\omega_n]$$

and

$$r_n^{\mathrm{sing}} : \frac{H^1(K_{n,\mathfrak{p}}, A_f)}{E(K_{n,\mathfrak{p}}) \otimes_{\mathbf{Z}} \mathbf{Q}_p/\mathbf{Z}_p} \longrightarrow \frac{H^1(K_{\infty,\mathfrak{p}}, A_f)}{E(K_{\infty,\mathfrak{p}}) \otimes_{\mathbf{Z}} \mathbf{Q}_p/\mathbf{Z}_p}.$$

Lemma 3.4 of [Gre97] proves that the kernel of r_n^{sing} has cardinality $|E(\mathbf{F}_\mathfrak{p})_p|^2$. (Loc. cit. considers the cyclotomic $\mathbf{Z}_p$-extension F_∞/F of a finite extension $F/\mathbf{Q}_p$, but the argument works for every $\mathbf{Z}_p$-extension F_∞/F such that the inertia subgroup of $\mathrm{Gal}(F_\infty/F)$ has finite index, cf. [Gre97, Proposition 2.4].) The inflation-restriction sequence shows that r_n is surjective and that its kernel is isomorphic to a quotient of $H^0(K_{\infty,\mathfrak{p}}, A_f) = E(K_{\infty,\mathfrak{p}})_{p^\infty}$. Assumption 5.1(4) then implies that r_n is an isomorphism and that r_n^{sing} is injective. The statement follows. □

Fix an ideal $\mathfrak{P}$ of $\Lambda_\mathscr{O}$ generated by a regular sequence.

Proposition 5.4.

(1) $H^1_{\mathrm{fin},\varepsilon}(K_\mathfrak{p}, T_{f,\mathscr{O}}(\mathfrak{P}))$ *and* $H^1_{\mathrm{sing},\varepsilon}(K_\mathfrak{p}, T_{f,\mathscr{O}}(\mathfrak{P}))$ *are free* $\Lambda_\mathscr{O}/\mathfrak{P}$*-modules of rank* $[K_\mathfrak{p} : \mathbf{Q}_p]$.

(2) $H^1_{\mathrm{fin},\varepsilon}(K_\mathfrak{p}, A_{f,\mathscr{O}}(\mathfrak{P}))$ *and* $H^1_{\mathrm{sing},\varepsilon}(K_\mathfrak{p}, A_{f,\mathscr{O}}(\mathfrak{P}))$ *are co-free* $\Lambda_\mathscr{O}/\mathfrak{P}$*-modules of rank* $[K_\mathfrak{p} : \mathbf{Q}_p]$.

Proof. Since $H^1_{\mathrm{fin},\varepsilon}(K_\mathfrak{p}, T_{f,\mathscr{O}}(\mathfrak{P}))$ and $H^1_{\mathrm{sing},\varepsilon}(K_\mathfrak{p}, T_{f,\mathscr{O}}(\mathfrak{P}))$ are isomorphic to the Pontryagin duals of $H^1_{\mathrm{sing},\varepsilon}(K_\mathfrak{p}, A_{f,\mathscr{O}}(\mathfrak{P}^\iota))^\iota$ and $H^1_{\mathrm{fin},\varepsilon}(K_\mathfrak{p}, A_{f,\mathscr{O}})(\mathfrak{P}^\iota))^\iota$ respectively, it is sufficient to prove (2). The proof is divided into three steps.

Step 1. If $\mathscr{D}_\Lambda$ denotes the Pontrjagin dual of Λ, one has isomorphisms of Λ-modules

$$H^1_{\mathrm{fin},\varepsilon}(K_\mathfrak{p}, \mathbf{A}_f) \cong \mathscr{D}_\Lambda^{[K_\mathfrak{p}:\mathbf{Q}_p]} \quad \text{and} \quad H^1_{\mathrm{sing},\varepsilon}(K_\mathfrak{p}, \mathbf{A}_f) \cong \mathscr{D}_\Lambda^{[K_\mathfrak{p}:\mathbf{Q}_p]}. \tag{5.2}$$

If $E/\mathbf{Q}_p$ has good supersingular reduction and p is splits in $K/\mathbf{Q}_p$, (the duals of) Equations (5.2) are proved in Propositions 4.16 of [IP06], which in turn is a slight generalisation of Theorem 6.2 of [Kob03] (see also [Kob03, Proposition 9.2]). If $E/\mathbf{Q}_p$ has good supersingular reduction and p is inert in $K/\mathbf{Q}_p$, this is a consequence of Rubin's conjecture proved in [BKO21]: if, as in Section 5.1, O denotes the valuation ring of $\Psi = K_\mathfrak{p}$, it is proved in [Rub87] that $H^1_{\mathrm{fin},\varepsilon}(K_\mathfrak{p}, \mathbf{A}_f)$ is a co-free Λ_O-module of rank one. Moreover [BKO21] proves that (as conjectured in [Rub87]) $H^1(K_\mathfrak{p}, \mathbf{A}_f)$ is the direct sum of $H^1_{\mathrm{fin},+}(K_\mathfrak{p}, \mathbf{A}_f)$ and $H^1_{\mathrm{fin},-}(K_\mathfrak{p}, \mathbf{A}_f)$. The statement follows.

If $E/\mathbf{Q}_p$ has good ordinary reduction, the representation T_f is ordinary at p, *i.e.* there exists a short exact sequence of $\mathbf{Z}_p[G_{\mathbf{Q}_p}]$-modules

$$0 \longrightarrow T_f^\bullet \longrightarrow T_f \longrightarrow T_f^\circ \longrightarrow 0,$$

arising from the reduction modulo p on $E(\bar{\mathbf{Q}}_p)$. More precisely let $\alpha, \beta \in \mathbf{Z}_p$ be the roots of the Hecke polynomial $X^2 - a_p(E)X + p$. Since $a_p(E)$ is a p-adic unit, one can assume $\alpha \in \mathbf{Z}_p^*$ and $\beta \in p\mathbf{Z}_p$. Then $T_f^\bullet \cong \mathbf{Z}_p(\chi_{\mathrm{cyc}} \cdot \psi^{-1})$ and $T_f^\circ \cong \mathbf{Z}_p(\psi)$, where $\psi : G_{\mathbf{Q}_p} \to \mathbf{Z}_p^*$ is the unramified character which sends an arithmetic Frobenius to α. Set $\mathbf{T}_f^\star = T_f^\star \otimes_{\mathbf{Z}_p} \Lambda(\epsilon_\infty^{-1})$ for $\star \in \{\bullet, \circ\}$, so that there is an exact sequence of $\Lambda[G_K]$-modules $\mathbf{T}_f^\bullet \hookrightarrow \mathbf{T}_f \twoheadrightarrow \mathbf{T}_f^\circ$. According to a result of Greenberg (cf. Proposition 2.4 of [Gre97])

$$H^1_{\mathrm{fin}}(K_\mathfrak{p}, \mathbf{T}_f) = \mathrm{Image}\big(H^1(K_\mathfrak{p}, \mathbf{T}_f^\bullet) \longrightarrow H^1(K_\mathfrak{p}, \mathbf{T}_f)\big) \cong H^1(K_\mathfrak{p}, \mathbf{T}_f^\bullet). \tag{5.3}$$

Let I be the augmentation ideal of Λ. Since $\mathbf{T}_f^\bullet/I = T_f^\bullet$ and $H^0(K_\mathfrak{p}, T_f^\bullet) = 0$, one has $H^1(K_\mathfrak{p}, \mathbf{T}_f^\bullet)[I] = 0$. Moreover $H^1(K_\mathfrak{p}, \mathbf{T}_f^\bullet)/I$ is a free $\mathbf{Z}_p$-module, because it is isomorphic to a submodule of the free $\mathbf{Z}_p$-module $H^1(K_\mathfrak{p}, T_f^\bullet)$. This implies that $H^1(K_\mathfrak{p}, \mathbf{T}_f^\bullet)$ is a free Λ-module of rank $[K_\mathfrak{p} : \mathbf{Q}_p]$, hence so is $H^1_{\mathrm{fin}}(K_\mathfrak{p}, \mathbf{T}_f)$ by Equation (5.3). The short exact sequence $\mathbf{T}_f^\bullet \hookrightarrow \mathbf{T}_f \twoheadrightarrow \mathbf{T}_f^\circ$ and Equation (5.3) induce an exact sequence of Λ-modules

$$0 \longrightarrow H^1_{\mathrm{sing}}(K_\mathfrak{p}, \mathbf{T}_f) \longrightarrow H^1(K_\mathfrak{p}, \mathbf{T}_f^\circ) \longrightarrow H^2(K_\mathfrak{p}, \mathbf{T}_f^\bullet).$$

By Hypothesis 5.1(4), $\psi(\mathrm{Frob}_\mathfrak{p}) \not\equiv 1 \pmod p$, hence $H^2(K_\mathfrak{p}, T_f^\bullet \otimes_{\mathbf{Z}_p} \mathbf{F}_p) = 0$. This implies that $H^2(K_\mathfrak{p}, T_f^\bullet)$ vanishes, hence so does $H^2(K_\mathfrak{p}, \mathbf{T}_f^\bullet)$ by Nakayama's Lemma. It follows that $H^1_{\mathrm{sing}}(K_\mathfrak{p}, \mathbf{T}_f)$ is isomorphic to $H^1(K_\mathfrak{p}, \mathbf{T}_f^\circ)$. Another application of Hypothesis 5.1(4) gives $H^0(K_\mathfrak{p}, T_f^\circ \otimes \mathbf{Q}_p/\mathbf{Z}_p) = 0$, which implies that $H^1(K_\mathfrak{p}, T_f^\circ)$ is a free $\mathbf{Z}_p$-module. As above one deduces that $H^1(K_\mathfrak{p}, \mathbf{T}_f^\circ)$ is a free Λ-module of rank $[K_\mathfrak{p} : \mathbf{Q}_p]$, hence so is $H^1_{\mathrm{sing}}(K_\mathfrak{p}, \mathbf{T}_f)$.

Step 2. The inclusion $A_{f,\mathscr{O}}(\mathfrak{P}) \to \mathbf{A}_{f,\mathscr{O}}$ induces isomorphisms of $\Lambda_\mathscr{O}/\mathfrak{P}$-modules

$$H^1(K_\mathfrak{p}, A_{f,\mathscr{O}}(\mathfrak{P})) \cong H^1(K_\mathfrak{p}, \mathbf{A}_{f,\mathscr{O}})[\mathfrak{P}] \quad \text{and} \quad H^1_{\mathrm{fin},\varepsilon}(K_\mathfrak{p}, A_{f,\mathscr{O}}(\mathfrak{P})) \cong H^1_{\mathrm{fin},\varepsilon}(K_\mathfrak{p}, \mathbf{A}_{f,\mathscr{O}})[\mathfrak{P}].$$

If E has good supersingular reduction the $G_{K_\mathfrak{p}} = \mathrm{Gal}(\bar{K}_\mathfrak{p}/K_\mathfrak{p})$-representation E_p is irreducible (see for example [Ser72, Proposition 12]). If $E/\mathbf{Q}_p$ has good ordinary reduction, the kernel of the reduction modulo p on $E(K_\mathfrak{p})$ is isomorphic to $\mathbf{F}_p(1)$ as a representation of the inertia subgroup of $G_{K_\mathfrak{p}}$. Then Hypothesis 5.1(4) implies that $H^0(K_\mathfrak{p}, E_p)$ vanishes in all cases. Because $\mathbf{A}_f[\mathfrak{m}_\Lambda]$ is isomorphic to E_p this gives $H^0(K_\mathfrak{p}, \mathbf{A}_f) = 0$, and therefore $H^0(K_\mathfrak{p}, \mathbf{A}_{f,\mathscr{O}}) = 0$, which in turn easily implies that $H^1(K_\mathfrak{p}, A_{f,\mathscr{O}}(\mathfrak{P}))$ is isomorphic to the $\mathfrak{P}$-torsion submodule of $H^1(K_\mathfrak{p}, \mathbf{A}_{f,\mathscr{O}})$. The claim follows directly from this and the definitions.

Step 3. Since $H^1(K_\mathfrak{p}, \mathbf{A}_f)$ is a co-free Λ-module of rank $2[K_\mathfrak{p} : \mathbf{Q}_p]$ (cf. Step 1), it follows from Step 2 and the flatness of $\mathscr{O}/\mathbf{Z}_p$ that $H^1(K_\mathfrak{p}, A_{f,\mathscr{O}}(\mathfrak{P}))$ is a co-free $\Lambda_\mathscr{O}/\mathfrak{P}$-module of rank $2[K_\mathfrak{p} : \mathbf{Q}_p]$. To conclude the proof it is then sufficient to show that $H^1_{\mathrm{fin},\varepsilon}(K_\mathfrak{p}, A_{f,\mathscr{O}}(\mathfrak{P}))$ is a co-free $\Lambda_\mathscr{O}/\mathfrak{P}$-module of rank $[K_\mathfrak{p} : \mathbf{Q}_p]$. This is a consequence of the previous two steps. □

Corollary 5.5. *Let $\mathscr{F}$ denote one of the symbols $\emptyset$, «fin,ε» or «sing,ε».*

(1) *The projection $\mathbf{T}_{f,\mathscr{O}} \to T_{f,\mathscr{O}}(\mathfrak{P})$ induces an isomorphism $H^1_\mathscr{F}(K_\mathfrak{p}, \mathbf{T}_{f,\mathscr{O}})/\mathfrak{P} \cong H^1_\mathscr{F}(K_\mathfrak{p}, T_{f,\mathscr{O}}(\mathfrak{P}))$.*
(2) *The inclusion $A_{f,\mathscr{O}}(\mathfrak{P}) \to \mathbf{A}_{f,\mathscr{O}}$ induces an isomorphism $H^1_\mathscr{F}(K_\mathfrak{p}, A_{f,\mathscr{O}}(\mathfrak{P})) \cong H^1_\mathscr{F}(K_\mathfrak{p}, \mathbf{A}_{f,\mathscr{O}})[\mathfrak{P}]$.*

Proof. The second sentence has been proved in *Step 2* of the proof of Proposition 5.4, and the first sentence follows by duality. □

We end this section by proving (following arguments of [Kim07]) that the ε-finite local conditions are suitably self-dual. Set $\Psi = K_\mathfrak{p}$ and $I_n = \omega_n \cdot \Lambda$, so that $T_g(I_n) = T_g \otimes \mathbf{Z}_p[G_n]$ (with g in G_K acting as $g \otimes g^{-1}$), and Shapiro's isomorphism identifies $H^1(\Psi, T_g(I_n))$ with $H^1(\Psi_n, T_g)$. Similarly Shapiro's lemma identifies $H^1(\Psi, A_g(I_n))$ with $H^1(\Psi_n, A_g)$. The Weil pairing then defines a perfect pairing

$$[\cdot,\cdot]_n = [\cdot,\cdot]_{\mathfrak{p},n} : H^1(\Psi, T_g(I_n)) \times H^1(\Psi, T_g(I_n)) \longrightarrow \mathbf{Z}/p^k, \tag{5.4}$$

such that $[\lambda \cdot x, y]_n = [x, \iota(\lambda) \cdot y]_n$ for each λ in Λ and each x and y in $H^1(\Psi, T_g(I_n))$.

Lemma 5.6. *$H^1_{\mathrm{fin},\varepsilon}(\Psi, T_g(I_n))$ is its own orthogonal complement under $[\cdot,\cdot]_n$.*

Proof. Set $L_n^* = H^1_{\mathrm{fin},\varepsilon}(\Psi, T_g(I_n))$. By definition (after identifying A_g with T_g via the Weil pairing, and $H^1(\Psi_n, A_g)$ with the ω_n-torsion submodule of $H^1(\Psi_\infty, A_g)$ via restriction), it is the orthogonal complement of $L_n = \big(\mathbf{E}(\Psi_\infty)_\varepsilon \otimes \mathbf{Q}_p/\mathbf{Z}_p\big)[\omega_n, p^k] = \big(\mathbf{E}(\Psi_\infty)_\varepsilon/p^k\big)[\omega_n]$ under $[\cdot,\cdot]_n$. In addition, L_n and L_n^* have the same cardinality (cf. Proposition 5.4), hence it is sufficient to prove the claim

$$\Big[L_n, \big(\mathbf{E}(\Psi_m)_\varepsilon/p^k\big)[\omega_n]\Big]_n = 0 \tag{5.5}$$

for each integer $m \geqslant n$.

We start by proving Equation (5.5) in the special case $n = m$. Since L_n is co-free over $\Lambda_{n,k}$ (cf. Proposition 5.4), $L_n = L_n[\omega_n^\varepsilon] + L_n[\omega_n/\omega_n^\varepsilon] = L_n[\omega_n^\varepsilon] + \omega_n^\varepsilon \cdot L_n$. As $\iota(\omega_n^\varepsilon)$ kills $\mathbf{E}(\Psi_n)_\varepsilon/p^k$, one has

$[\omega_n^\varepsilon \cdot L_n, \mathbf{E}(\Psi_n)_\varepsilon/p^k]_n = 0$. Moreover, Proposition 5.3 gives $L_n[\omega_n^\varepsilon] = \mathbf{E}(\Psi_n)_\varepsilon/p^k \subset \mathbf{E}(\Psi_n)/p^k$, and the latter is orthogonal to itself under $[\cdot,\cdot]_n$, hence $[L_n[\omega_n^\varepsilon], \mathbf{E}(\Psi_n)_\varepsilon/p^k]_n = 0$. This proves

$$[L_n, \mathbf{E}(\Psi_n)_\varepsilon/p^n]_n = 0. \tag{5.6}$$

Fix now an isomorphism of Λ-modules $H^1(\Psi_\infty, A_g) \simeq \mathscr{D}^a \oplus \mathscr{D}^a$, where $\mathscr{D} = \mathrm{Hom}_{\mathrm{cts}}(\Lambda/p^k, \mathbf{Q}_p/\mathbf{Z}_p)$ is the Pontjagin dual of Λ/p^k and $a = [\Psi : \mathbf{Q}_p]$, which identifies $\mathbf{E}(\Psi_\infty)_\varepsilon/p^k$ with the first copy of $\mathscr{D}^a$ (cf. Proposition 5.4). Set $\mathscr{D}_n = \mathscr{D}[\omega_n] = \mathrm{Hom}_{\mathbf{Z}_p}(\Lambda_n/p^k, \mathbf{Q}_p/\mathbf{Z}_p)$, and for each $m \geqslant n$ let $\mathrm{C}_{m/n} : \mathscr{D}_m \to \mathscr{D}_n$ be the dual of the injective map $\Lambda_n/p^k \to \Lambda_n/p^k$ sending $\lambda + (\omega_n, p^k)$ to $(\omega_m/\omega_n)\cdot\lambda + (\omega_m, p^k)$, for each λ in Λ. Then the surjective map $\mathrm{Cor}_{m/n} = \mathrm{C}^a_{m/n} \oplus \mathrm{C}^a_{m/n}$ corresponds to the corestriction map $H^1(\Psi_m, A_g) \to H^1(\Psi_n, A_g)$ under the fixed isomorphism, once one identifies $H^1(\Psi_s, A_g)$ with the ω_s-torsion submodule of $H^1(\Psi_\infty, A_g)$ (cf. Corollary 5.5). It follows that for every $m \geqslant n$ one has

$$\mathrm{Cor}_{m/n}(L_m) = L_n,$$

which combined with Equation (5.6) proves the Claim (5.5): for each y in $\big(\mathbf{E}(\Psi_m)_\varepsilon/p^k\big)[\omega_n]$ one has

$$[L_n, y]_n = [\mathrm{Cor}_{m/n}(L_m), y]_n = [L_m, y]_m = 0.$$

(Here one identifies $H^1(\Psi_n, A_g)$ with the ω_n-torsion submodule of $H^1(\Psi_m, A_g)$ via the restriction map, and uses the fact that restriction and corestriction are adjoint to each other under the Tate pairing.) □

5.3.2. *Primes not dividing p.* Recall the notation introduced at the beginning of Section 5.2: k is a positive integer or the symbol ∞, $L \in \mathscr{S}_k$ is a square-free product of k-admissible primes, and g is either the level raising of $f_k = f \pmod{p^k}$ at L if k is an integer, or $g = f$ if $k = \infty$. Let $\mathfrak{P}$ be a principal ideal of $\Lambda_{\mathscr{O}}$ generated by a regular sequence.

Lemma 5.7.

(1) *Let ℓ be a prime dividing N^-. Then the morphism*

$$H^1(K_\ell, A^{[\ell]}_{g,\mathscr{O}}(\mathfrak{P})) \longrightarrow H^1(K_\ell, \mathbf{A}^{[\ell]}_{g,\mathscr{O}})$$

induced by the inclusion $A^{[\ell]}_{g,\mathscr{O}}(\mathfrak{P}) = \mathbf{A}^{[\ell]}_{g,\mathscr{O}}[\mathfrak{P}] \hookrightarrow \mathbf{A}^{[\ell]}_{g,\mathscr{O}}$ is injective.

(2) *Let $\ell \in \mathscr{S}_k$ be a k-admissible prime. Then the morphisms*

$$H^1(K_\ell, A^{(\ell)}_{h,\mathscr{O}}(\mathfrak{P})) \longrightarrow H^1(K_\ell, \mathbf{A}^{(\ell)}_{g,\mathscr{O}}) \quad \textit{and} \quad H^1(K_\ell, A^{[\ell]}_{h,\mathscr{O}}(\mathfrak{P})) \longrightarrow H^1(K_\ell, \mathbf{A}^{[\ell]}_{g,\mathscr{O}})$$

induced by the inclusions $A^{(\ell)}_{g,\mathscr{O}}(\mathfrak{P}) \hookrightarrow \mathbf{A}^{(\ell)}_{g,\mathscr{O}}$ and $A^{[\ell]}_{g,\mathscr{O}}(\mathfrak{P}) \hookrightarrow \mathbf{A}^{[\ell]}_{g,\mathscr{O}}$ respectively are injective.

Proof. (1). Let $\mathscr{D}_{\Lambda_\mathscr{O}}$ denote the Pontrjagin dual of $\Lambda_\mathscr{O}$. Then $A^{[\ell]}_{g,\mathscr{O}}(\mathfrak{P})$ is isomorphic to $\mathscr{D}_{\Lambda_\mathscr{O}}[\mathfrak{P}, p^k]$ with trivial Galois action (cf. Section 5.2), hence $H^1(K_\ell, A^{[\ell]}_{g,\mathscr{O}}(\mathfrak{P}))$ is isomorphic to $\mathrm{Hom}_{\mathrm{cont}}(G_{K_\ell}, \mathscr{D}_{\Lambda_\mathscr{O}}[\mathfrak{P}, p^k])$. Similarly, $H^1(K_\ell, \mathbf{A}^{[\ell]}_g) \cong \mathrm{Hom}_{\mathrm{cont}}(G_{K_\ell}, \mathscr{D}_{\Lambda_\mathscr{O}}[p^k])$, and the map $H^1(K_\ell, A^{[\ell]}_{h,\mathscr{O}}(\mathfrak{P})) \to H^1(K_\ell, \mathbf{A}^{[\ell]}_{g,\mathscr{O}})$ corresponds to the morphism induced by the inclusion of $\mathscr{D}_\Lambda[\mathfrak{P}, p^t]$ in $\mathscr{D}_\Lambda[p^k]$.

(2) The injectivity of the map $H^1(K_\ell, A^{[\ell]}_{g,\mathscr{O}}(\mathfrak{P})) \to H^1(K_\ell, \mathbf{A}^{[\ell]}_{g,\mathscr{O}})$ is proved as above. The representation $A^{(\ell)}_{g,\mathscr{O}}(\mathfrak{P})$ is isomorphic to the Kummer dual of $T^{[\ell]}_{g,\mathscr{O}}(\mathfrak{P}^\iota)^\iota \cong \Lambda_\mathscr{O}/(\mathfrak{P}, p^k)$. Since $p \nmid \ell^2 - 1$ because $\ell \in \mathscr{S}_k$, and $\mathscr{O}$ is a flat $\mathbf{Z}_p$-algebra, $H^1(K_\ell, \Lambda_\mathscr{O}/(\mathfrak{P}, p^k)) \cong \Lambda_\mathscr{O}/(\mathfrak{P}, p^k)$ and local Tate duality then gives $H^1(K_\ell, A^{(\ell)}_{g,\mathscr{O}}(\mathfrak{P}, p^k)) \cong \mathscr{D}_{\Lambda_\mathscr{O}}[\mathfrak{P}, p^k]$. Similarly $H^1(K_\ell, \mathbf{A}^{(\ell)}_{g,\mathscr{O}}) \cong \mathscr{D}_{\Lambda_\mathscr{O}}[p^k]$, and the map $H^1(K_\ell, A^{(\ell)}_{h,\mathscr{O}}(\mathfrak{P})) \to H^1(K_\ell, \mathbf{A}^{(\ell)}_{g,\mathscr{O}})$ is identified with the inclusion $\mathscr{D}_{\Lambda_\mathscr{O}}[\mathfrak{P}, p^k] \to \mathscr{D}_{\Lambda_\mathscr{O}}[p^k]$. □

Lemma 5.8. *If w is a prime of K which divides N^+, then $H^j(K_w, A_{h,\mathscr{O}}(\mathfrak{P})) = 0$ for every j.*

Proof. Hypothesis 5.1(5) and local Tate duality imply that $\mathbf{A}_f[\mathfrak{m}_\Lambda] = E_p$ is an acyclic G_w-module. Since $\mathscr{O}/\mathbf{Z}_p$ is flat, this in turn implies that $\mathbf{A}_{f,\mathscr{O}}[J]$ is an acyclic G_w-module for every ideal J of Λ generated by a regular sequence. Because $A_{h,\mathscr{O}}(\mathfrak{P})$ is isomorphic to $\mathbf{A}_{f,\mathscr{O}}[\mathfrak{P}, p^t]$, this concludes the proof. □

5.4. **Control theorems.** Let $\mathfrak{P}$ be an ideal of $\Lambda_{\mathscr{O}}$ generated by a regular sequence.

Proposition 5.9. *Let $S \in \mathscr{S}_k$ be a (possibly empty) squarefree product of k-admissible primes. Then*

$$\mathrm{Sel}_{\varepsilon}^{S}(K, A_{g,\mathscr{O}}(\mathfrak{P})) \cong \mathrm{Sel}_{\varepsilon}^{S}(K, \mathbf{A}_{g,\mathscr{O}})[\mathfrak{P}].$$

Proof. Let $\mathfrak{G} = \mathrm{Gal}(K_{SLNp}/K)$ be the Galois group of the maximal algebraic extension of K which is unramified at every finite place $w \nmid SLNp$ of K. Since $H^0(\mathfrak{G}, E_p)$ vanishes by Hypothesis 5.1.(3), the same is true for the $\mathfrak{G}$-module $E_p \otimes_{\mathbf{Z}_p} \mathscr{O}$, and therefore for every ideal $\mathfrak{P}$ of $\Lambda_{\mathscr{O}}$ generated by a regular sequence the natural map gives an isomorphism $H^1(\mathfrak{G}, \mathbf{A}_{f,\mathscr{O}}[\mathfrak{P}]) \cong H^1(\mathfrak{G}, \mathbf{A}_{f,\mathscr{O}})[\mathfrak{P}]$ (cf. Step 2 in the proof of Proposition 5.4). This implies that the natural map $\mathrm{Sel}_{\varepsilon}^{S}(K, A_{g,\mathscr{O}}(\mathfrak{P})) \to \mathrm{Sel}_{\varepsilon}^{S}(K, \mathbf{A}_{g,\mathscr{O}})[\mathfrak{P}]$ is injective, and its cokernel is isomorphic to a submodule of the kernel of

$$\bigoplus_{w|p} H^1_{\mathrm{sing},\varepsilon}(K_w, A_{g,\mathscr{O}}(\mathfrak{P})) \oplus \bigoplus_{w|\frac{LN^-}{(L,S)}} H^1(K_w, A_{g,\mathscr{O}}^{[w]}(\mathfrak{P})) \oplus \bigoplus_{w|SN^+} H^1(K_w, A_{g,\mathscr{O}}(\mathfrak{P}))$$

$$\longrightarrow \bigoplus_{w|p} H^1_{\mathrm{sing},\varepsilon}(K_w, \mathbf{A}_{g,\mathscr{O}}) \oplus \bigoplus_{w|\frac{LN^-}{(L,S)}} H^1(K_w, \mathbf{A}_{g,\mathscr{O}}^{[w]}) \oplus \bigoplus_{w|SN^+} H^1(K_w, \mathbf{A}_{g,\mathscr{O}}).$$

The proposition then follows from Corollary 5.5(2), Lemma 5.7 and Lemma 5.8. □

Proposition 5.10. *Let S be an integer coprime with LNp. The canonical map*

$$\mathfrak{Sel}_S^{\varepsilon}(K, \mathbf{T}_{g,\mathscr{O}}) \otimes_{\Lambda_{\mathscr{O}}} \Lambda_{\mathscr{O}}/\mathfrak{P} \longrightarrow \mathfrak{Sel}_S^{\varepsilon}(K, T_{g,\mathscr{O}}(\mathfrak{P}))$$

is injective.

Proof. Let $\mathfrak{G}$ be the Galois group of the maximal algebraic extension of K unramified outside $SLNp\infty$. Since $\Lambda_{\mathscr{O}}/p^k$ has no nontrivial $\mathfrak{P}$-torsion the morphism

$$H^1(\mathfrak{G}, \mathbf{T}_{g,\mathscr{O}}) \otimes_{\Lambda_{\mathscr{O}}} \Lambda_{\mathscr{O}}/\mathfrak{P} \longrightarrow H^1(\mathfrak{G}, \mathbf{T}_{g,\mathscr{O}}(\mathfrak{P}))$$

is injective. Moreover the cokernel of the inclusion $\mathfrak{Sel}_S^{\varepsilon}(K, \mathbf{T}_{g,\mathscr{O}}) \to H^1(\mathfrak{G}, \mathbf{T}_{g,\mathscr{O}})$ is isomorphic to a submodule of

$$\bigoplus_{w|p} H^1_{\mathrm{sing},\varepsilon}(K_w, \mathbf{T}_{g,\mathscr{O}}) \oplus \bigoplus_{w|LN^-} H^1(K_w, \mathbf{T}_{g,\mathscr{O}}^{[w]}). \tag{5.7}$$

To prove the first part of the proposition it is then sufficient to show that each summand of the direct sum (5.7) has no non-trivial $\mathfrak{P}$-torsion. For $H^1_{\mathrm{sing},\varepsilon}(K_w, \mathbf{T}_{g,\mathscr{O}})$ this is a consequence of Proposition 5.4. Assume that $w = \ell \cdot \mathcal{O}_K$ for a rational prime ℓ dividing LN^-. In this case $\mathbf{T}_{g,\mathscr{O}}^{[w]}$ is isomorphic to $\Lambda_{\mathscr{O}} \otimes_{\mathbf{Z}_p} \mathbf{Z}_p/p^k$ (with trivial Galois action), hence $H^1(K_w, \mathbf{T}_{g,\mathscr{O}}^{[w]}) \cong H^1(K_\ell, \mathbf{Z}_p/p^k) \otimes_{\mathbf{Z}_p} \Lambda_{\mathscr{O}}$ and $H^1(K_w, \mathbf{T}_{g,\mathscr{O}}^{[w]})[\mathfrak{P}] = 0$. □

In light of the applications to the definite and indefinite main conjectures, we also record the following result due to Mazur–Rubin and Howard. In the rest of this section, let $\mathfrak{P}$ be a height one prime ideal of Λ, and let $\mathscr{O}$ be the integral closure of $\Lambda/\mathfrak{P}$ in its fraction field. Consider the canonical map

$$\mathfrak{Sel}^{\varepsilon}(K, \mathbf{T}_f)/\mathfrak{P} \longrightarrow \mathfrak{Sel}^{\varepsilon}(K, \mathbf{T}_{f,\mathscr{O}}(\mathfrak{P})), \tag{5.8}$$

induced by the composition $\mathrm{sp}_{\mathfrak{P}} : \mathbf{T}_f \to \mathbf{T}_f \otimes_{\mathbf{Z}_p} (\Lambda/\mathfrak{P}) \to \mathbf{T}_{f,\mathscr{O}}(\mathfrak{P})$, and the map

$$\mathrm{Sel}_{\varepsilon}(K, A_{f,\mathscr{O}}(\mathfrak{P})) \longrightarrow \mathrm{Sel}_{\varepsilon}(K, \mathbf{A}_f)[\mathfrak{P}] \tag{5.9}$$

induced by the Kummer dual of $\mathrm{sp}_{\mathfrak{P}^\iota} : \mathbf{T}_f \to \mathbf{T}_{f,\mathscr{O}}(\mathfrak{P}^\iota)$.

Proposition 5.11. *There is a finite set Σ of height one primes in Λ such that for $\mathfrak{P} \notin \Sigma$, the maps (5.8) and (5.9) have finite kernel and cokernels, of order bounded by a constant depending only on $[\mathscr{O} : \Lambda/\mathfrak{P}]$.*

Proof. The proof proceeds as in [MR04, Proposition 5.3.14], replacing the local results in loc. cit. with Corollary 5.5 (for primes dividing p), and following the proof of [How04, Lemma 2.2.7] for the ordinary-type conditions. □

5.5. **Global freeness.** Assume in this section that $k<\infty$. Let $\bar{g}\in S_2(N^+,LN^-;\mathbf{F}_p)$ be the reduction of g modulo p, and let $\mathfrak{m}_\Lambda=(\omega_1,p)\cdot\Lambda$ be the maximal ideal of Λ. Define

$$\mathrm{Sel}^S_\square(K,A_{\bar{g}})\subset H^1(K,\mathbf{A}_g[\mathfrak{m}_\Lambda])$$

to be the Selmer group obtained from $\mathrm{Sel}^S(K,A_{\bar{g}})$ by relaxing the local conditions at the primes of K dividing p (so that $\mathrm{Sel}^S(K,A_{\bar{g}})$ is the set of classes in $\mathrm{Sel}^S_\square(K,A_{\bar{g}})$ which are finite at p.)

Definition 5.12. A *freeing set relative to* g is an integer $S\in\mathscr{S}_k$ such that $\mathrm{Sel}^S_\square(K,A_{\bar{g}})=0$.

A generalization of [BD05, Theorem 3.2] guarantees the existence of infinitely freeing sets relative to g (see also the proof of [BD94, Lemma 2.23], and compare with [KPW17] where a correction including the ramification condition at primes $q\mid N^+$ is made). Let n denote either a nonnegative integer or ∞. Set $\omega_\infty=0$, $I_{\mathscr{O},n}=\omega_n\cdot\Lambda_{\mathscr{O}}$, and $\Lambda_{\mathscr{O},n,k}=\Lambda_{\mathscr{O}}/(I_{\mathscr{O},n},p^k)$, and write

$$\mathfrak{Sel}^{\varepsilon}_S(K_n,T_{g,\mathscr{O}})=\mathfrak{Sel}^{\varepsilon}_S(K,T_{g,\mathscr{O}}(I_{\mathscr{O},n}))\quad\text{and}\quad\mathfrak{Sel}_{Sp}(K_n,T_{g,\mathscr{O}})=\mathfrak{Sel}_{Sp}(K,T_{g,\mathscr{O}}(I_{\mathscr{O},n})),$$

where $\varepsilon=\emptyset$ if E is ordinary at p, and ε denotes one of the symbols $\emptyset$, $+$ of $-$ otherwise.

Proposition 5.13. *Let S be a freeing set relative to g and set $\delta(S)=\#\{$prime divisors of $S\}$.*

(1) *The Selmer group $\mathfrak{Sel}^{\varepsilon}_S(K_n,T_{g,\mathscr{O}})$ is a free $\Lambda_{\mathscr{O},n,k}$-module of rank $\delta(S)$.*
(2) *The Selmer group $\mathfrak{Sel}_{Sp}(K_n,T_{g,\mathscr{O}})$ is a free $\Lambda_{\mathscr{O},n,k}$-module of rank $\delta(S)+2$.*

Proof. If $E/\mathbf{Q}_p$ has ordinary reduction this is a variant of [BD05, Proposition 3.3] (see also Section 3 of [BD94]). We use the computations of the preceding sections to give a proof which works in our more general setting. Without loss of generality, we can (and will) assume $n\neq\infty$ and $\mathscr{O}=\mathbf{Z}_p$ throughout the proof. As usual, we then omit $\mathscr{O}$ from the notation.

Step 1. Set $\mathrm{Sel}^S_\varepsilon(K_n,A_g)=\mathrm{Sel}^S_\varepsilon(K,A_g(I_n))$. We show that

$$\mathrm{Sel}^S_\varepsilon(K_n,A_g)=0. \tag{5.10}$$

Indeed Proposition 5.9 yields

$$\mathrm{Sel}^S_\varepsilon(K,A_{\bar{g}})=\mathrm{Sel}^S_\varepsilon(K,\mathbf{A}_g)[\mathfrak{m}_\Lambda]=\mathrm{Sel}^S_\varepsilon(K_n,A_g)[\mathfrak{m}_\Lambda].$$

As (by assumption) $\mathrm{Sel}^S_\varepsilon(K,A_{\bar{g}})\subset\mathrm{Sel}^S_\square(K,A_{\bar{g}})=0$, Equation (5.10) follows from Nakayama's Lemma.

Step 2. We show that

$$\mathfrak{Sel}_{Sp}(K,T_{\bar{g}})\cong\mathfrak{Sel}_{Sp}(K_n,T_g)[\mathfrak{m}_\Lambda]. \tag{5.11}$$

Denote by K_{SLNp} the maximal algebraic extension of K which is unramified outside $SLNp$ and by $\mathfrak{G}_s$ the Galois group of K_{SLNp}/K_s, for every $0\leqslant s\leqslant\infty$. If one identifies $H^1(\mathfrak{G}_0,T_g(I_n))$ with $H^1(\mathfrak{G}_n,T_g)$ via the Shapiro isomorphism, then

$$\mathfrak{Sel}_{Sp}(K_n,T_g)=\ker\Big(H^1(\mathfrak{G}_n,T_g)\longrightarrow\bigoplus_{\mathfrak{l}\mid LN^-}H^1(K_{n,\mathfrak{l}},T_g^{[\ell]})\Big),$$

where the direct sum is taken over the primes $\mathfrak{l}$ of K_n which divide LN^-. Hypothesis 5.1.(3) guarantees that $H^0(K_s,T_f/p^r)$ vanishes for all $r\geqslant 1$ and $s\geqslant 0$. As a consequence the map

$$H^1(\mathfrak{G}_0,T_{\bar{g}})\longrightarrow H^1(\mathfrak{G}_n,T_g)[\mathfrak{m}_\Lambda]$$

induced by restriction from K to K_n and the inclusion $p^{k-1}:T_{\bar{g}}=T_f/p\hookrightarrow T_f/p^k=T_g$ is an isomorphism. To prove Equation (5.11) it is then sufficient to verify that the map

$$\beta_{\mathfrak{l}}:H^1(K_\ell,T_{\bar{g}}^{[\ell]})\longrightarrow H^1(K_{n,\mathfrak{l}},T_g^{[\ell]})$$

is injective for every prime $\mathfrak{l}$ of K_n lying over $\ell|LN^-$. The G_{K_ℓ}-representation $T_{f,r}^{[\ell]}=T_f^{[\ell]}/p^r$ is isomorphic to $\mathbf{Z}/p^r$ for every $r\geqslant 1$, and one has $K_{n,\mathfrak{l}}=K_\ell$ (since ℓ splits completely in K_n/K). It follows that $H^1(K_\ell,T_{f,r}^{[\ell]})=\mathrm{Hom}_{\mathrm{cont}}(G_{K_\ell},\mathbf{Z}/p^r)$, and (via the fixed isomorphisms $T_{f,k}=T_g$ and $T_{f,1}=T_{\bar{g}}$) the map $\beta_{\mathfrak{l}}$ is (identified with) the injective morphism induced by the inclusion $p^{k-1}:\mathbf{Z}/p\hookrightarrow\mathbf{Z}/p^k$.

Step 3. We prove (2). As the local conditions defining $\mathfrak{Sel}_{Sp}(K_n, T_g)$ are dual to those defining $\mathrm{Sel}^{Sp}(K_n, A_g)$ (via Tate's duality), Theorem 2.19 of [DDT95], Hypothesis 5.1.(3) and Lemma 5.8 yield

$$\frac{\#\mathrm{Sel}^{Sp}(K_n, A_g)}{\#\mathfrak{Sel}_{Sp}(K_n, T_g)} = \#T_g(I_n) \cdot \prod_{w|Sp} \frac{\#H^0(K_w, T_g(I_n))}{\#H^1(K_w, T_g(I_n))} \cdot \prod_{\ell|LN^-/(S,LN^-)} \frac{\#H^0(K_\ell, T_g(I_n))}{\#H^1_{\mathrm{ord}}(K_\ell, T_g(I_n))}.$$

Step 1 implies that $\mathrm{Sel}^{Sp}(K_n, T_g) \subset \mathrm{Sel}^S_\square(K_n, T_g)$ vanishes. For each $w \mid pSLN^-$, define

$$h_w = \frac{\#H^0(K_w, T_g(I_n))}{\#H^1(K_w, T_g(I_n))}$$

and let $h_\infty = \#T_g(I_n)$. For every prime ℓ dividing S, $T_g(I_n) = T_g \otimes \Lambda_{n,k}$, hence Lemma 3.1 and the local Euler characteristic formula give $h_w^{-1} = \#H^2(K_\ell, T_g(I_n)) = \#\Lambda_{n,k}$. For every prime $\ell|L$ Lemma 3.1 similarly gives $h_w = 1$. If $\ell|N^-$, considering the long exact cohomology sequence associated with $\mu_{p^k} \hookrightarrow T_g \twoheadrightarrow \mathbf{Z}/p^k$ one easily proves that $h_\ell = 1$ even in this case. If $\mathfrak{p}$ is a prime dividing p the local Euler characteristic formula shows that $h_\mathfrak{p}^{-1} = \#\Lambda_{n,k}^{2\cdot[K_\mathfrak{p}:\mathbf{Q}_p]} \cdot \#H^2(K_\mathfrak{p}, T_g(I_n))$. On the other hand $H^2(K_\mathfrak{p}, T_g(I_n))$ has the same cardinality as $H^0(K_\mathfrak{p}, A_g(I_n))$, which vanishes since its $\mathfrak{m}_\Lambda$-torsion submodule is equal to $H^0(K_\mathfrak{p}, E_p) = 0$ (thanks to Hypothesis 5.1(4) in the ordinary case). The previous equation then yields

$$\#\mathfrak{Sel}_{Sp}(K_n, T_g) = h_\infty^{-1} \cdot \#\Lambda_{n,k}^{\delta(S)+4} = \#\Lambda_{n,k}^{\delta(S)+2}. \tag{5.12}$$

Since Λ/I_n is a Gorenstein local ring, it is isomorphic to $\mathrm{Hom}_{\mathbf{Z}_p}(\Lambda/I_n, \mathbf{Z}_p)$ as a Λ-module, so that $\Lambda_{n,k}$ is isomorphic to its Pontrjagin dual $\mathscr{D}_{n,k} = \mathrm{Hom}_{\mathbf{Z}_p}(\Lambda_{n,k}, \mathbf{Q}_p/\mathbf{Z}_p)$ as a Λ-module. Taking $n = 0$ and $k = 1$ in Equation (5.12) shows that $\mathfrak{Sel}_{Sp}(K, T_{\bar{g}})$ has dimension $\delta(S) + 2$ over $\mathbf{F}_p$, hence

$$\mathfrak{Sel}_{Sp}(K_n, T_g)[\mathfrak{m}_\Lambda] \cong \mathbf{F}_p^{\delta(S)+2}$$

by Step 2. By Nakayama's Lemma there exists then a surjective morphism

$$\Lambda_{n,k}^{\delta(S)+2} \twoheadrightarrow \mathrm{Hom}_{\mathbf{Z}_p}(\mathfrak{Sel}_{Sp}(K_n, T_g), \mathbf{Q}_p/\mathbf{Z}_p),$$

which is an isomorphism by another application of Equation (5.12). As a consequence $\mathfrak{Sel}_{Sp}(K_n, T_g)$ is isomorphic to $\mathscr{D}_{n,k}^{\delta(S)+2} \cong \Lambda_{n,k}^{\delta(S)+2}$, as was to be shown.

Step 4. There is an exact sequence of $\Lambda_\mathscr{O}$-modules

$$0 \longrightarrow \mathfrak{Sel}^\varepsilon_S(K_n, T_g) \longrightarrow \mathfrak{Sel}_{Sp}(K_n, T_g) \xrightarrow{\partial_p} \bigoplus_{\mathfrak{p}|p} H^1_{\mathrm{sing},\varepsilon}(K_\mathfrak{p}, T_f(I_{n,k})) \longrightarrow 0. \tag{5.13}$$

The only nontrivial fact is the surjectivity of the residue map ∂_p. By construction (cf. Section 5.2) the local conditions defining $\mathfrak{Sel}^\varepsilon_S(K_n, T_g)$ are dual to those defining $\mathrm{Sel}^S_\varepsilon(K_n, A_g)$, hence Poitou–Tate duality implies that the cokernel of ∂_p is isomorphic to a submodule of the Pontrjagin dual of $\mathrm{Sel}^S_\varepsilon(K_n, A_g)$ (see e.g. Theorem 7.3 of [Rub00]), which is trivial according to Step 1.

Step 5. Step 3, Step 4 and Proposition 5.4.(1) prove that $\mathfrak{Sel}^\varepsilon_S(K_n, T_g)$ is a (projective, hence) free $\Lambda_{\mathscr{O},n,k}$-module of rank $\delta(S)$, thus concluding the proof of the proposition. □

We record for future application the following

Corollary 5.14. *Let S be freeing set relative to g. Then, for each integer $1 \leqslant n \leqslant \infty$, the natural projection $\Lambda \to \Lambda_n$ induces isomorphisms of Λ-modules*

$$\mathfrak{Sel}_{Sp}(K_\infty, T_g)/\omega_n \cong \mathfrak{Sel}_{Sp}(K_n, T_g) \quad \textit{and} \quad \mathfrak{Sel}^\varepsilon_S(K_\infty, T_g)/\omega_n \cong \mathfrak{Sel}^\varepsilon_S(K_n, T_g).$$

Proof. This is a direct consequence of Propositions 5.10 and 5.13. □

5.6. **Self-duality.** In this section $k < \infty$. Let $\chi : \Lambda \longrightarrow \mathscr{O}_\chi$ be a morphism of $\mathbf{Z}_p$-algebras into a discrete valuation ring $\mathscr{O}_\chi$ contained in $\bar{\mathbf{Q}}_p$. Set $\Lambda_\chi = \Lambda_{\mathscr{O}_\chi}$, and write again $\chi : \Lambda_\chi \twoheadrightarrow \mathscr{O}_\chi$ for the $\mathscr{O}_\chi$-linear extension of χ. Set $\mathfrak{P}_\chi = \ker(\chi)$, and set

$$T_g(\chi) = T_{g,\mathscr{O}_\chi}(\mathfrak{P}_\chi) \quad \text{and} \quad A_g(\chi) = A_{g,\mathscr{O}_\chi}(\mathfrak{P}_\chi).$$

If $\bar{\chi} = \chi \circ \iota$ denotes the composition of χ with (the $\mathscr{O}_\chi$-linear extension of) Iwasawa's main involution, then $T_g(\chi) = T_g \otimes_{\mathbf{Z}_p} \mathscr{O}_\chi(\bar{\chi})$ and $A_g(\chi) = \mathrm{Hom}_{\mathbf{Z}_p}(T_g, \mu_{p^\infty}) \otimes_{\mathbf{Z}_p} \mathscr{O}_\chi(\bar{\chi})$ as $\mathscr{O}_\chi[G_K]$-modules (where $\mathscr{O}_\chi(\bar{\chi})$ is a copy of $\mathscr{O}_\chi$ on which G_K acts through the composition of $G_K \twoheadrightarrow \mathrm{Gal}(K_\infty/K) \hookrightarrow \Lambda^*$ and $\bar{\chi}$). Via the identifications $T_{f,k} \simeq T_g$ and $A_{f,k} \simeq A_g$ fixed above, the Weil pairing then yields and isomorphism

$$w_\chi : T_g(\chi) \simeq A_g(\chi).$$

of $\mathscr{O}_\chi[G_K]$-modules. For each ε in $\{\emptyset, +, -\}$, one has the following

Proposition 5.15. *The isomorphism induced in G_K-cohomology by w_χ restricts to an isomorphism*

$$w_\chi : \mathfrak{Sel}^\varepsilon(K, T_g(\chi)) \simeq \mathrm{Sel}_\varepsilon(K, A_g(\chi)).$$

More precisely, for each prime v of K dividing NLp, the isomorphism induced in G_{K_v}-cohomology by w_χ identifies the local conditions at v defining $\mathfrak{Sel}_\varepsilon(K, T_g(\chi))$ and $\mathrm{Sel}^\varepsilon(K, A_g(\chi))$.

Proof. For each prime v of K not dividing pNL, denote by $\mathcal{L}_v(\chi)$ (resp., $\mathcal{L}_v^*(\chi)$) the local condition at v satisfied by the classes in the Selmer group $\mathrm{Sel}^\varepsilon(K, A_g(\chi))$ (resp., $\mathfrak{Sel}_\varepsilon(K, T_g(\chi))$). By definition $\mathcal{L}_v^*(\chi)$ is the orthogonal complement of $\mathcal{L}_v(\bar{\chi})$ under the perfect local Tate duality

$$(\cdot,\cdot)_{\chi,v} : H^1(K_v, T_g(\chi)) \times H^1(K_v, A_g(\bar{\chi})) \longrightarrow \mathscr{O}_\chi/p^k$$

arising from the $\mathscr{O}_\chi$-bilinear extension of the evaluation pairing $T_g \times A_g \longrightarrow \mu_{p^k}$. To prove the proposition it is then sufficient to show that $\mathcal{L}_v^*(\bar{\chi})$ is the orthogonal complement of $\mathcal{L}_v^*(\chi)$ under the perfect pairing

$$[\cdot,\cdot]_{\chi,v} : H^1(K_v, T_g(\chi)) \times H^1(K_v, T_g(\bar{\chi})) \longrightarrow \mathscr{O}_\chi/p^k$$

defined as the composition of $(\cdot,\cdot)_{\chi,v}$ and the isomorphism $\mathrm{id} \times w_\chi^{-1}$. This is easily checked when $v \nmid p$, in which case $\mathcal{L}_v(\chi)$ is either an ordinary local condition, or $A_g(\chi)$ is cohomologically trivial. Assume then that $v = \mathfrak{p}$ is a prime of K dividing p, and set $\Psi = K_\mathfrak{p}$.

Since $\chi(\gamma - 1)$ belongs to the maximal ideal of $\mathscr{O}_\chi$, there exists an integer $n \gg 0$ such that $\chi(\omega_n)$ belongs to $p^k\mathscr{O}_\chi$, id est ω_n belongs to $(\mathfrak{P}_\chi, p^k)$. According to Corollary 5.5, one has an isomorphism

$$H^1(\Psi, T_g(\chi)) \simeq H^1(\Psi, \mathbf{T}_g) \otimes_\Lambda \Lambda_\chi/\mathfrak{P}_\chi \simeq H^1(\Psi, T_g(I_n)) \otimes_{\Lambda_n} \Lambda_n/\mathfrak{P}_\chi,$$

which restricts to an isomorphism

$$\mathcal{L}_v^*(\chi) \simeq H^1_{\mathrm{fin},\varepsilon}(\Psi, T_g(I_n)) \otimes_{\Lambda_n} \Lambda_n/\mathfrak{P}_\chi.$$

One has a similar isomorphism for $\bar{\chi}$ in place of χ, and via these isomorphisms $[\cdot,\cdot]_{\chi,v}$ is the reduction modulo $\mathfrak{P}_\chi$ of the $\mathscr{O}_\chi$-linear extension of the pairing

$$\{\cdot,\cdot\}_{\chi,v} : H^1(\Psi, T_g(I_n)) \times H^1(\Psi, T_g(I_n)) \longrightarrow \Lambda_n$$

defined for each x and y in $H^1(\Psi, T_g(I_n))$ by the formula

$$\{x, y\}_{\chi,v} = \sum_{\sigma \in G_n} [x, \sigma \cdot y]_{n,v} \cdot \sigma,$$

where $[\cdot,\cdot]_{n,v}$ is the pairing defined in Equaiton (5.4). Because $L_n^* = H^1_{\mathrm{fin},\varepsilon}(\Psi, T_g(I_n))$ is a Λ_n-submodule of $H^1(\Psi, T_g(I_n))$, Lemma 5.6 yields $\{L_n^*, L_n^*\}_{\chi,v} = [L_n^*, L_n^*]_{n,v} = 0$, hence $[\mathcal{L}_v^*(\chi), \mathcal{L}_v^*(\bar{\chi})]_{\chi,v} = 0$. In other words $\mathcal{L}_v^*(\bar{\chi})$ is contained in the orthogonal complement $\mathcal{L}_v^*(\chi)^\perp$ of $\mathcal{L}_v^*(\chi)$ with respect to the pairing $[\cdot,\cdot]_{\chi,v}$. On the other hand, Proposition 5.4 implies that $\mathcal{L}_v^*(\chi)$, $\mathcal{L}_v^*(\chi)^\perp$ and $\mathcal{L}_v^*(\bar{\chi})$ all have the same cardinality $|\mathscr{O}_\chi/p^k|^{[\Psi:\mathbf{Q}_p]}$, thus $\mathcal{L}_v^*(\bar{\chi}) = \mathcal{L}_v^*(\chi)^\perp$. This concludes the proof of the proposition. □

Remark 5.16. Let τ in $G_\mathbf{Q}$ be complex conjugation, let $\mathrm{Ad}(\tau)$ be conjugation by τ on G_K, and let $[\tau] : T_g(\chi) \to T_g(\bar{\chi})$ be the isomorphism of $\mathscr{O}_\chi$-modules induced by $\tau : T_g \to T_g$. Then the map sending a 1-cocycle $\varphi : G_K \to T_g(\chi)$ to the 1-cocyle $[\tau] \circ \varphi \circ \mathrm{Ad}(\tau) : G_K \to T_g(\bar{\chi})$ induces an isomorphism of $\mathscr{O}_\chi$-modules $\mathfrak{Sel}_\varepsilon(K, T_g(\chi)) \simeq \mathfrak{Sel}_\varepsilon(K, T_g(\bar{\chi}))$.

6. Ramified classes and reciprocity laws

We suppose from now until the end of the paper that Hypothesis 1.1 holds. According to our conventions stated at the end of the Introduction, recall that we often write M/x for M/xM for a module M over a commutative ring with unity R and an element $x \in R$; in particular, we often write $\mathbf{Z}/p^k$ in place of $\mathbf{Z}/p^k\mathbf{Z}$, for an integer $k \geqslant 1$.

6.1. **Global classes.** Let k be a positive integer and $L \in \mathscr{S}_k$ a squarefree product of k-admissible primes. Assume that $L \in \mathscr{S}_k^{\mathrm{ind}}$ is *indefinite* (i.e. $\epsilon_K(LN^-) = +1$), so that J_{N^+,LN^-} is the Picard variety of the Shimura curve X_{N^+,LN^-}. Let $g = f_L \in S_2(N^+, LN^-, \mathbf{Z}/p^k)$ be the L-level raising of f modulo p^k. Let $I_g \subset \mathbf{T}_{N^+,LN^-}$ denote the kernel of g. Proposition 4.4 of [PW11], a slight generalization of [BD05, Theorem 5.15], shows that there is an isomorphism of $\mathbf{Z}_p[G_{\mathbf{Q}}]$-modules

$$\pi_g : \mathrm{Ta}_p(J_{N^+,LN^-})/I_g \cong T_{f,k},$$

which is unique up to multiplication by a p-adic unit by Hypothesis 1.1. For every integer $n \geqslant 0$ define

$$\psi_{g,n} : J_{N^+,LN^-}(K_n) \longrightarrow H^1(K_n, \mathrm{Ta}_p(J_{N^+,LN^-})/I_g) \cong H^1(K_n, T_{f,k}),$$

where the first (resp., second) map is induced by the Kummer map (resp., by π_g). It follows from Proposition 2.7.12 of [Nek12] (see also Section 7 of [BD05] and Theorem 3.10 of [DI08]) that for every $x \in J_{N^+,LN^-}(K_n)$ the class $\psi_{g,n}(x)$ is finite at every prime of K_n dividing p. (To apply Proposition 2.7.12 of [Nek12], note that all results in the Appendix A of loc. cit. on flat cohomology of finite flat group schemes hold for the p-divisible group of the elliptic curve $E/K_{\mathfrak{p}}$ for any prime $\mathfrak{p} \mid p$ of K because $K_{\mathfrak{p}}/\mathbf{Q}_p$, being unramified, has ramification index smaller that $p-1$). Moreover, since J_{N^+,LN^-} has purely toric reduction at every prime divisor of LN^-, Mumford–Tate theory of p-adic uniformisation implies that these classes are ordinary at every such prime. In particular, $\psi_{g,n}$ gives a morphism (cf. Section 5.2.5)

$$\psi_{g,n} : J_{N^+,LN^-}(K_n) \longrightarrow \mathfrak{Sel}(K_n, T_g).$$

Recall the compatible sequence of Heegner points $P_n(L)$, for $n \geqslant 0$, introduced in Section 2.5 and define

$$\tilde{\kappa}_n(g) = \psi_{g,n}(P_n(L)).$$

6.1.1. *Ordinary case.* Suppose that E has ordinary reduction at p. In this case, define

$$\kappa_n(g) = \frac{1}{\alpha_p(g)^n}\Big(\tilde{\kappa}_{n-1}(g) - \alpha_p(g) \cdot \tilde{\kappa}_n(g)\Big). \tag{6.1}$$

By the previous discussion $\kappa_n(g)$ belongs to the compact Selmer group $\mathfrak{Sel}(K_n, T_g) = \mathfrak{Sel}(K, T_g(I_n))$ (with $I_n = \omega_n \cdot \Lambda$). A simple computation using (2.5) shows that the corestriction map takes $\kappa_{n+1}(g)$ to $\kappa_n(g)$ for all $n \geqslant 1$. Since $\mathfrak{Sel}(K_\infty, T_g) = \mathfrak{Sel}(K, \mathbf{T}_g)$ is isomorphic to the inverse limit of the Selmer groups $\mathfrak{Sel}(K_n, T_g)$ under the corestriction maps, we can define

$$\kappa_\infty(g) = \varprojlim_n \kappa_n(g) \in \mathfrak{Sel}(K_\infty, T_g). \tag{6.2}$$

6.1.2. *Supersingular case.* Suppose now that E has supersingular reduction at p. Choose a freeing set $S \in \mathscr{S}_k$ relative to g which is coprime to L. As $a_p(g) = 0$ (in $\mathbf{Z}/p^k$), the norm relations (2.3), (2.4) and (2.5) imply that, if $\varepsilon = (-1)^n$, the class $\tilde{\kappa}_n(g)$ is killed by ω_n^ε (cf. Lemma 4.3):

$$\omega_n^\varepsilon \cdot \tilde{\kappa}_n(g) = 0 \quad \text{if} \quad \varepsilon = (-1)^n. \tag{6.3}$$

If either p splits in K, or p is inert in K and $\varepsilon = -1$, set $\breve{\omega}_n^{-\varepsilon} = \tilde{\omega}_n^{-\varepsilon}$. If p is inert in K and $\varepsilon = +1$ (id est in the exceptional case), set $\breve{\omega}_n^{-\varepsilon} = \omega_n^-$. Since $\mathfrak{Sel}_S(K_n, T_g)$ is free over $\Lambda_{n,k}$ (cf. Proposition 5.13, applied with $\varepsilon = \emptyset$), the previous equation implies that, if $\varepsilon = (-1)^n$, there exists a unique element

$$\kappa_n^\varepsilon(g) \in \mathfrak{Sel}_S(K_n, T_g)/\omega_n^\varepsilon$$

such that

$$(-1)^{\delta(n)} \breve{\omega}_n^{-\varepsilon} \cdot \kappa_n^\varepsilon(g) = \tilde{\kappa}_n(g),$$

where $\delta(n) = n/2$ if n is even (id est $\varepsilon = +1$), and $\delta(n) = (n-1)/2$ if n is odd.

According to Corollary 5.14 (applied again with $\varepsilon = \emptyset$), the projections $\Lambda \twoheadrightarrow \Lambda_n^\varepsilon$ induce isomorphisms

$$\mathfrak{Sel}_S(K_\infty, T_g)/\omega_n^\varepsilon \simeq \mathfrak{Sel}_S(K_n, T_g)/\omega_n^\varepsilon.$$

Via these identifications, the natural projections $\Lambda_{n+2}^{\varepsilon} \twoheadrightarrow \Lambda_n^{\varepsilon}$ then induce surjective maps

$$\pi_{n+2}^{\varepsilon} : \mathfrak{Sel}_S(K_{n+2}, T_g)/\omega_{n+2}^{\varepsilon} \twoheadrightarrow \mathfrak{Sel}_S(K_n, T_g)/\omega_n^{\varepsilon}.$$

As $\mathfrak{Sel}_S(K_\infty, T_g)$ is finite free over Λ, it is equal to the inverse limit of the maps π_{n+2}^{ε}, taken over the set $\mathbf{N}^{\varepsilon}$ of nonnegative integers n satisfying $\varepsilon = (-1)^n$:

$$\mathfrak{Sel}_S(K_\infty, T_g) \cong \varprojlim_{n \in \mathbf{N}^{\varepsilon}} \mathfrak{Sel}_S(K_n, T_g)/\omega_n^{\varepsilon}. \tag{6.4}$$

Lemma 6.1. *For each n in $\mathbf{N}^{\varepsilon}$, one has*

$$\pi_{n+2}^{\varepsilon}(\kappa_{n+2}^{\varepsilon}(g)) = \kappa_n^{\varepsilon}(g).$$

Via the isomorphism defined in Equation (6.4), one then gets a class

$$\kappa_\infty^{\varepsilon}(g) = (\kappa_n^{\varepsilon}(g))_{n \in \mathbf{N}^{\varepsilon}} \in \mathfrak{Sel}_S(K_\infty, T_g).$$

Proof. After identifying $\mathfrak{Sel}_S(K_\infty, T_g)$ with $\Lambda^{\delta(S)}$ (cf. Proposition 5.13), this is proved precisely as the corresponding statement for p-adic theta elements (cf. Lemma 4.3). □

The class $\kappa_\infty^{\varepsilon}(g)$ is finite at every prime divisor q of S. Indeed, as q is a k-admissible prime relative to (f, K), the G_{K_q}-module $T_g(I_n)$ splits as the direct sum of $\Lambda_{k,n} = \Lambda/(p^k, \omega_n) \cdot \Lambda$ (with trivial Galois action) and $\Lambda_{k,n}(1)$, and both $H^1(K_q, \Lambda_{k,n}) = H^1_{\mathrm{fin}}(K_q, T_g(I_n))$ and $H^1(K_q, \Lambda_{k,n}(1)) = H^1_{\mathrm{ord}}(K_q, T_g(I_n))$ are free $\Lambda_{k,n}$-modules of rank one (cf. Section 5.2.3). If $\partial_q : H^1(K_n, T_g) \longrightarrow H^1_{\mathrm{ord}}(K_q, T_g(I_n))$ is the composition of the Shapiro isomorphism $H^1(K_n, T_g) \cong H^1(K, T_g(I_n))$, restriction at q, and projection onto the ordinary subspace, it follows that $\partial_q(\kappa_n^{\varepsilon}(g))$ is the unique class in $H^1_{\mathrm{ord}}(K_q, T_g(I_n))/\omega_n^{\varepsilon}$ mapping to $(-1)^{\delta(n)} \cdot \partial_q(\tilde{\kappa}_n(g))$ under multiplication by $\breve{\omega}_n^{-\varepsilon}$. Finally, by construction the Heegner class $\tilde{\kappa}_n(g)$ is finite at q, id est $\partial_q(\tilde{\kappa}_n(g)) = 0$, hence $\partial_q(\kappa_n^{\varepsilon}(g)) = 0$. Taking the limit for n in $\mathbf{N}^{\varepsilon}$ tending to infinity, this proves that $\kappa_\infty^{\varepsilon}(g)$ is in the kernel of the residue map $\partial_q : \mathfrak{Sel}_S(K_\infty, T_g) \to H^1_{\mathrm{ord}}(K_q, \mathbf{T}_g)$, as claimed.

We now prove that the class $\kappa_\infty^{\varepsilon}$ is ε-finite at p. For each positive integer n, let $K_{n,p}$ be the product of the completions of K_n at the primes dividing p. Since $\tilde{\kappa}_n(g)$ is (by construction) finite at p, and since $\mathbf{E}(K_{n,p})_\varepsilon/p^k$ is equal to the ω_n^{ε}-torsion submodule of the finite local condition $\mathbf{E}(K_{n,p})/p^n$ (cf. Theorem 5.2), Equation (6.3) implies that, if $\varepsilon = (-1)^n$, then the restriction at p of $\tilde{\kappa}_n(g)$ belongs to (the image under the Kummer map of) $\mathbf{E}(K_{n,p})_\varepsilon/p^k$. According to (the proof of) Lemma 5.6 the latter is contained in the ε-finite subspace $H^1_{\mathrm{fin},\varepsilon}(K_{n,p}, T_g) = H^1_{\mathrm{fin},\varepsilon}(K_p, T_g(I_n))$, so that the residue $\partial_p(\tilde{\kappa}_n(g))$ of $\tilde{\kappa}_n(g)$ at p is zero in the singular quotient $H^1_{\mathrm{sing},\varepsilon}(K_{n,p}, T_g) = H^1_{\mathrm{sing},\varepsilon}(K_p, T_g(I_n))$, provided that $\varepsilon = (-1)^n$. On the other hand Proposition 5.4 proves that $H^1_{\mathrm{sing},\varepsilon}(K_{n,p}, T_g)$ is a free $\Lambda_{k,n}$-module, so that multiplication by $\breve{\omega}_n^{-\varepsilon}$ yields an isomorphism between $H^1_{\mathrm{sing},\varepsilon}(K_{n,p}, T_g)/\omega_n^{\varepsilon}$ and $H^1_{\mathrm{sing},\varepsilon}(K_{n,p}, T_g)[\omega_n^{\varepsilon}]$. As by construction $\breve{\omega}_n^{-\varepsilon} \cdot \partial_p(\kappa_n^{\varepsilon}(g))$ equals $(-1)^{\delta(n)} \cdot \partial_p(\tilde{\kappa}_n(g)) = 0$ if $\varepsilon = (-1)^n$, we conclude that $\kappa_n^{\varepsilon}(g)$ belongs to the kernel of the residue map $\partial_p : \mathfrak{Sel}_S(K_n, T_g)/\omega_n^{\varepsilon} \to H^1_{\mathrm{sing},\varepsilon}(K_{n,p}, T_g)/\omega_n^{\varepsilon}$ if $\varepsilon = (-1)^n$. As $H^1_{\mathrm{sing},\varepsilon}(K_{\infty,p}, T_g) = H^1_{\mathrm{sing},\varepsilon}(K_p, \mathbf{T}_g)$ is the inverse limit of the groups $H^1_{\mathrm{sing},\varepsilon}(K_{n,p}, T_g)/\omega_n^{\varepsilon}$ as n tends to infinity in $\mathbf{N}^{\varepsilon}$ (cf. Corollary 5.5), this proves that the class $\kappa_\infty^{\varepsilon}(g)$ belongs to the kernel of the residue map $\partial_p : \mathfrak{Sel}_S(K_\infty, T_g) \to H^1_{\mathrm{sing},\varepsilon}(K_{\infty,p}, T_g)$. We summarise the discussion in the following key

Proposition 6.2. *$\kappa_\infty^{\varepsilon}(g)$ belongs to $\mathfrak{Sel}^{\varepsilon}(K_\infty, T_g)$.*

Remark 6.3. Let $g = f_L$ be the level raising of $f_k = f \pmod{p^k}$ at a definite product L in $\mathscr{S}_k^{\mathrm{def}}$, and let ℓ be a k-admissible prime relative to (f, K) not dividing L, so that $L\ell$ belongs to $\mathscr{S}_k^{\mathrm{ind}}$. Let $g_\ell = f_{L\ell}$ be the level raising at ℓ of g (namely the $L\ell$-level raising of f_k). Then, for $\varepsilon = \pm$ (resp., $\varepsilon = \emptyset$) in the supersingular (resp., ordinary) case, the class $\kappa_\infty^{\varepsilon}(g_\ell) \in \mathfrak{Sel}^{\varepsilon}(K_\infty, T_{g_\ell})$ is also an element of the ε-Selmer group $\mathfrak{Sel}_\ell^{\varepsilon}(K_\infty, T_g)$ of g relaxed at ℓ.

6.2. **Reciprocity laws.** The cohomology classes in Section 6.2 are related the square-root p-adic L-functions by the following explicit reciprocity laws.

Recall that $\varepsilon = \emptyset$ in the ordinary case and $\varepsilon = \pm$ in the supersingular case. Equation (8.1) and Proposition 6.2 define global Selmer classes $\kappa_\infty^{\varepsilon}(g)$ in $\mathfrak{Sel}^{\varepsilon}(K_\infty, T_g)$. Note that each k-admissible prime

ℓ is totally split in K_∞/K, being inert in K. Therefore, $H^1(K_{\infty,\ell},T_g)=H^1(K_\ell,\mathbf{T}_g)$ is isomorphic to $H^1(K_\ell,T_g)\otimes\Lambda$, and Lemma 3.1 allows us to define morphisms

$$v_\ell : H^1(K_\infty,T_g)\longrightarrow H^1_{\mathrm{fin}}(K_\ell,T_g)\otimes\Lambda\cong\Lambda_k \quad\text{and}\quad \partial_\ell : H^1(K_\infty,T_g)\longrightarrow H^1_{\mathrm{ord}}(K_\ell,T_g)\otimes\Lambda\cong\Lambda_k,$$

defined by composing the restriction map at ℓ with the projection onto the finite and the ordinary (or singular) part respectively (cf. Section 5.2.3). Given a global class $x\in H^1(K,T_{f,k})$, we call $v_\ell(x)$ its *finite part* at ℓ, and $\partial_\ell(x)$ its *residue* at ℓ. If $L=\prod_i \ell_i\in\mathscr{S}_k$ is a squarefree product of admissible primes ℓ_i, then we write $\partial_L=\oplus_i\partial_{\ell_i}$ and $v_L=\oplus_i v_{\ell_i}$.

Theorem 6.4 (First Reciprocity Law). *Assume that $L\in\mathscr{S}_k^{\mathrm{def}}$ is* definite*, let $g=f_L$ be the L-level raising of f modulo p^k, and let $\ell\nmid L$ be an admissible prime relative to g and K, so that $L\ell\in\mathscr{S}_k^{\mathrm{ind}}$ is* indefinite*. Let g_ℓ be the ℓ-level raising of g. The following equality*

$$\partial_\ell\left(\kappa^\varepsilon_\infty(g_\ell)\right)=\mathcal{L}^\varepsilon_g$$

holds in Λ/p^k up to units.

Proof. By [BD05, Theorem 4.1], whose proof works both in the ordinary and in the supersingular case, we have $\partial_\ell\left(\kappa_n(g_\ell)\right)=\mathcal{L}_{g,n}$. In the ordinary case, this completes the proof. In the supersingular case, recall that by definition we have for all n such that $\varepsilon=(-1)^n$:

- If p is split in K or p is inert in K and $\varepsilon=-1$ (the non-exceptional case):

$$\kappa_n(g_\ell)=\begin{cases}(-1)^{n/2}\tilde{\omega}_n^{-\varepsilon}\kappa_n^\varepsilon(g_\ell), & \text{if } n \text{ is even;}\\ (-1)^{(n-1)/2}\tilde{\omega}_n^{-\varepsilon}\kappa_n^\varepsilon(g_\ell), & \text{if } n \text{ is odd;}\end{cases}$$

$$\mathcal{L}_{g,n}=\begin{cases}(-1)^{n/2}\tilde{\omega}_n^{-\varepsilon}\mathcal{L}^\varepsilon_{g,n}, & \text{if } n \text{ is even;}\\ (-1)^{(n-1)/2}\tilde{\omega}_n^{-\varepsilon}\mathcal{L}^\varepsilon_{g,n}, & \text{if } n \text{ is odd;}\end{cases}$$

- If p is inert in K and $\varepsilon=+1$ (the exceptional case):

$$\kappa_n(g_\ell)=(-1)^{n/2}\omega_n^-\kappa_n^+(g_\ell);$$

$$\mathcal{L}_{g,n}=(-1)^{n/2}\omega_n^-\mathcal{L}^+_{g,n}.$$

In both cases, since $\Lambda^\varepsilon_{n,k}$ is $\tilde{\omega}_n^{-\varepsilon}$-torsion free (split and non-exceptional case) and is $\omega_n^{-\varepsilon}$-torsion free (exceptional case) it follows from $\partial_\ell\left(\kappa_n(g_\ell)\right)=\mathcal{L}_{g,n}$ that $\partial_\ell\left(\kappa^\varepsilon_n(g_\ell)\right)=\mathcal{L}^\varepsilon_{g,n}$ for all $n\geqslant 0$, and the conclusion follows. □

Theorem 6.5 (Second Reciprocity Law). *Assume that $L\in\mathscr{S}_k^{\mathrm{ind}}$ is* indefinite*, let $g=f_L$ be the L-level raising of f modulo p^k and let $\ell\nmid L$ be an admissible prime relative to g and K, so that $L\ell\in\mathscr{S}_k^{\mathrm{def}}$ is* definite*. Let g_ℓ be the ℓ-level raising of g. Then $\kappa^\varepsilon_\infty(g)$ is finite at ℓ and the equality*

$$v_\ell\left(\kappa^\varepsilon_\infty(g)\right)=\mathcal{L}^\varepsilon_{g_\ell}$$

holds in Λ/p^k up to units.

Proof. The result follows as in the proof of Theorem 6.4 from the relation

$$v_\ell\left(\kappa_\infty(g)\right)=\mathcal{L}_{g_\ell}. \tag{6.5}$$

If $N^-\neq 1$, (6.5) is [BD05, Theorem 4.2], which is proved in Section 9 of loc. cit. using an extension of Ihara's Lemma to indefinite Shimura curves due to Diamond–Taylor [DT94]. The same argument applies when $N^-=1$ using the standard Ihara's Lemma; alternatively, to prove (6.5) when $N^-=1$ one can adapt the arguments in Section 6 of Vatsal's paper [Vat03], where the case $n=1$ is considered. □

7. ε-BSD formulae in the definite case

This section is devoted to the proof of BSD formulae for the ε-Selmer groups. They are a crucial ingredient in the proof of the main results stated in the Introduction. We adopt the abuse of notation introduced in the previous section, thus writing M/x instead of M/xM for any element x of a commutative ring with unity R, and for any R-module M.

Fix a positive integer $k \geqslant 1$ and a (possibly empty) *definite* squarefree product $L \in \mathscr{S}_{2k}^{\mathrm{def}}$ of $2k$-admissible primes relative to (f, K, p) (hence $\epsilon_K(LN^-) = -1$). Denote by $\breve{g} = f_L \in S_2(N^+, LN^-; \mathbf{Z}/p^{2k})$ the L-level raising of the reduction of f modulo p^{2k} (cf. Section 3.3) and by $g \in S_2(N^+, LN^-; \mathbf{Z}/p^k)$ the reduction of $\breve{g}$ modulo p^k.

Let $\chi : \Lambda \to \mathscr{O}_\chi$ be a morphism of $\mathbf{Z}_p$-algebras, where $\mathscr{O}_\chi$ is a discrete valuation ring finite over $\mathbf{Z}_p$. Denote by $\mathfrak{P}_\chi$ the kernel of χ. We assume throughout this section that $\mathscr{O}_\chi$ is the integral closure of $\Lambda/\mathfrak{P}_\chi$ in its fraction field $\mathscr{K}_\chi = \mathrm{Frac}(\mathscr{O}_\chi)$ and, by an abuse of notation, we still denote by

$$\chi : \Lambda_{\mathscr{O}_\chi} \longrightarrow \mathscr{O}_\chi$$

the morphism of $\mathscr{O}_\chi$-algebras induced by χ and by $\mathfrak{P}_\chi \subset \Lambda_{\mathscr{O}_\chi}$ its kernel. Let $\mathrm{ord}_\chi : \mathscr{K}_\chi \twoheadrightarrow \mathbf{Z} \cup \{\infty\}$ be the normalised discrete valuation, let ϖ_χ be a uniformiser of $\mathscr{O}_\chi$ and let $\mathbf{F}_\chi = \mathscr{O}_\chi/\varpi_\chi$ be its residue field. If M is a finite free $\mathscr{O}_\chi/\varpi_\chi^m$-module (for some integer $m \geqslant 1$) and x is a non-zero element of M, denote by $\mathrm{ord}_\chi(x) \in \mathbf{N}$ the largest nonnegative integer $t \geqslant 0$ such that $x \in \varpi_\chi^t \cdot M$. After setting $\mathrm{ord}_\chi(0) = \infty$, this defines an *$\mathscr{O}_\chi$-adic valuation* $\mathrm{ord}_\chi : M \to \{0, 1, \ldots, m-1, \infty\}$. Recall that we already introduced the notation

$$T_g(\chi) = T_{g,\mathscr{O}_\chi}(\mathfrak{P}_\chi) \quad \text{and} \quad A_g(\chi) = A_{g,\mathscr{O}_\chi}(\mathfrak{P}_\chi).$$

Theorem 7.1. *Assume that* $\mathcal{L}_g^\varepsilon(\bar{\chi}) \neq 0$. *Then* $\mathrm{length}_{\mathscr{O}_\chi}\big(\mathrm{Sel}_\varepsilon(K, A_g(\chi))\big) \leqslant 2\mathrm{ord}_\chi\left(\mathcal{L}_g^\varepsilon(\bar{\chi})\right)$, *with equality in the non-exceptional case.*

The rest of this section is devoted to the proof of Theorem 7.1.

7.1. **The Kolyvagin system.** Assume that the value of the p-adic L-function $\mathcal{L}_g^\varepsilon \in \Lambda/p^k$ at χ is non-zero and denote by

$$t_\chi^\varepsilon(g) = \mathrm{ord}_\chi(\mathcal{L}_g^\varepsilon(\chi)) < \infty \tag{7.1}$$

its ϖ_χ-adic valuation. Let $\ell \in \mathscr{S}_{2k}$ be a $2k$-admissible prime not dividing L, so that $\ell \cdot L \in \mathscr{S}_{2k}^{\mathrm{ind}}$ is *indefinite*, and let $S \in \mathscr{S}_{2k}$ be a freeing set relative to $\breve{g}$ which is divisible by $\ell \cdot L$ (cf. Section 5.5). We simplify the notation and write

$$\mathfrak{Sel}_S^\varepsilon(K, \mathbf{T}_{\breve{g}}) \otimes \mathscr{O}_\chi = \mathfrak{Sel}_S^\varepsilon(K, \mathbf{T}_{\breve{g},\mathscr{O}_\chi}) \otimes_{\Lambda_{\mathscr{O}_\chi}} \mathscr{O}_\chi,$$

where the tensor product on the right is taken with respect to the canonical map $\chi : \Lambda_{\mathscr{O}_\chi} \to \mathscr{O}_\chi$ induced by χ. Proposition 5.13 shows that $\mathfrak{Sel}_S^\varepsilon(K, \mathbf{T}_{\breve{g}}) \otimes \mathscr{O}_\chi$ is a free $\mathscr{O}_\chi/p^{2k}$-module of rank $\delta(S)$.

Let $\breve{g}_\ell$ be the level raising of $\breve{g}$ at ℓ. Section 6 attaches to $\breve{g}_\ell$ a global cohomology class $\kappa_\infty^\varepsilon(\breve{g}_\ell)$

$$\kappa_\infty^\varepsilon(\breve{g}_\ell) \in \mathfrak{Sel}_\ell^\varepsilon(K, \mathbf{T}_{\breve{g}}) \subset \mathfrak{Sel}_S^\varepsilon(K, \mathbf{T}_{\breve{g}}) \subset \mathfrak{Sel}_S^\varepsilon(K, \mathbf{T}_{\breve{g},\mathscr{O}_\chi})$$

(cf. Proposition 6.2). To simplify the notation, we write from now on

$$\kappa_\infty^\varepsilon(\ell) = \kappa_\infty^\varepsilon(\breve{g}_\ell)$$

Denote by $\kappa_\chi^\varepsilon(\ell)$ the image of $\kappa_\infty^\varepsilon(\ell)$ in $\mathfrak{Sel}_S^\varepsilon(K, \mathbf{T}_{\breve{g}}) \otimes \mathscr{O}_\chi$ under the natural projection, and by

$$t_\chi^\varepsilon(g, \ell) = \mathrm{ord}_\chi(\kappa_\chi^\varepsilon(\ell)) \tag{7.2}$$

its $\mathscr{O}_\chi$-adic valuation. Note that $t_\chi^\varepsilon(g, \ell)$ is independent of the choice of S and Theorem 6.4 yields

$$t_\chi^\varepsilon(g, \ell) \leqslant \mathrm{ord}_\chi\big(\partial_\ell(\kappa_\chi^\varepsilon(\ell))\big) = \mathrm{ord}_\chi(\mathcal{L}_{\breve{g}}^\varepsilon(\chi)) = t_\chi^\varepsilon(g) < \mathrm{ord}_\chi(p^k), \tag{7.3}$$

where

$$\partial_\ell : \mathfrak{Sel}_S^\varepsilon(K, \mathbf{T}_{\breve{g},\mathscr{O}_\chi}) \longrightarrow H^1_{\mathrm{sing}}(K_\ell, T_{\breve{g},\mathscr{O}_\chi}) \cong \Lambda_{\mathscr{O}_\chi}/p^{2k}$$

is the scalar extension of the residue map at ℓ introduced in Section 6 and the second equality follows from Equation (7.1). In particular there exists $\tilde{\kappa}_\chi^\varepsilon(\ell) \in \mathfrak{Sel}_S^\varepsilon(K, \mathbf{T}_{\breve{g}}) \otimes \mathscr{O}_\chi$ such that

$$\mathrm{ord}_\chi(\tilde{\kappa}_\chi^\varepsilon(\ell)) = 0, \tag{7.4}$$

$$\kappa^{\varepsilon}_{\chi}(\ell) = \varpi_{\chi}^{t^{\varepsilon}_{\chi}(g,\ell)} \cdot \tilde{\kappa}^{\varepsilon}_{\chi}(\ell). \tag{7.5}$$

While $\tilde{\kappa}^{\varepsilon}_{\chi}(\ell)$ is not uniquely determined by the previous equations, its image

$$\hat{\kappa}^{\varepsilon}_{\chi}(\ell) \in \mathfrak{Sel}^{\varepsilon}_{S}(K, \mathbf{T}_g) \otimes \mathscr{O}_{\chi} \overset{\text{def}}{=} \mathfrak{Sel}^{\varepsilon}_{S}(K, \mathbf{T}_{g,\mathscr{O}_{\chi}}) \otimes_{\Lambda_{\mathscr{O}_{\chi}}} \mathscr{O}_{\chi}$$

under the morphism induced by the projection $T_{\breve{g}} \twoheadrightarrow T_g$ is independent of any choice. Let

$$\mathbf{s}_{\chi} : \mathfrak{Sel}^{\varepsilon}_{S}(K, \mathbf{T}_g) \otimes \mathscr{O}_{\chi} \longrightarrow \mathfrak{Sel}^{\varepsilon}_{S}(K, T_g(\chi))$$

be the specialization map. Define

$$\xi^{\varepsilon}_{\chi}(\ell) = \xi^{\varepsilon}_{\chi}(g, \ell) = \mathbf{s}_{\chi}(\hat{\kappa}^{\varepsilon}_{\chi}(\ell)) \in \mathfrak{Sel}^{\varepsilon}_{S}(K, T_g(\chi)).$$

and

$$\bar{\xi}^{\varepsilon}_{\chi}(\ell) = \bar{\xi}^{\varepsilon}_{\ell}(g, \ell) \in H^1(K, T_{\bar{g}}) \otimes_{\mathbf{F}_p} \mathbf{F}_{\chi}$$

as the image of $\hat{\kappa}^{\varepsilon}_{\chi}(\ell)$ under the map induced in cohomology by the reduction map $T_g(\chi) \twoheadrightarrow T_{\bar{g}} \otimes_{\mathbf{F}_p} \mathbf{F}_{\chi}$, where $\bar{g} \in S_2(N^+, LN^-; \mathbf{F}_p)$ is the reduction of g modulo p.

Lemma 7.2 (*cf. Lemma 4.5 of* [BD05])**.**
(1) $0 \neq \xi^{\varepsilon}_{\chi}(\ell) \in \mathfrak{Sel}^{\varepsilon}_{\ell}(K, T_g(\chi))$ *and* $v_{\ell}\big(\xi^{\varepsilon}_{\chi}(\ell)\big) = 0$.
(2) $\mathrm{ord}_{\chi}\big(\partial_{\ell}(\xi^{\varepsilon}_{\chi}(\ell))\big) = t^{\varepsilon}_{\chi}(g) - t^{\varepsilon}_{\chi}(g, \ell)$.
(3) $0 \neq \bar{\xi}^{\varepsilon}_{\chi}(\ell) \in \mathfrak{Sel}^{\varepsilon}_{\ell}(K, T_{\bar{g}}) \otimes_{\mathbf{F}_p} \mathbf{F}_{\chi}$ *and* $\partial_{\ell}(\bar{\xi}^{\varepsilon}_{\chi}(\ell))$ *is non-zero if and only if* $t^{\varepsilon}_{\chi}(g, \ell) = t^{\varepsilon}_{\chi}(g)$.

Proof. (1) Because the kernel of $\mathfrak{Sel}^{\varepsilon}_{S}(K, \mathbf{T}_{\breve{g}}) \to \mathfrak{Sel}^{\varepsilon}_{S}(K, \mathbf{T}_g)$ is killed by p^k and $\mathrm{ord}_{\chi}(\tilde{\kappa}^{\varepsilon}_{\chi}(\ell)) = 0$ by Equation (7.4), the class $\hat{\kappa}^{\varepsilon}_{\chi}(\ell)$ is not zero, hence so is its image $\xi^{\varepsilon}_{\chi}(\ell)$ under the map $\mathbf{s}_{\chi}$, which is injective by Proposition 5.10. Let q be a prime divisor of S/ℓ. To prove the first statement one has to show that the residue

$$\partial_q(\xi^{\varepsilon}_{\chi}(\ell)) \in H^1_{\mathrm{sing}}(K_q, T_{g,\mathscr{O}_{\chi}}) \cong \mathscr{O}_{\chi}/p^k$$

of $\xi^{\varepsilon}_{\chi}(\ell)$ at q is zero. Fix isomorphisms $H^1_{\mathrm{sing}}(K_q, \mathbf{T}_{\breve{g},\mathscr{O}_{\chi}}) \cong \Lambda_{\mathscr{O}_{\chi}}/p^{2k}$ and $H^1_{\mathrm{sing}}(K_q, \mathbf{T}_{g,\mathscr{O}_{\chi}}) \cong \Lambda_{\mathscr{O}_{\chi}}/p^k$ such that the map $H^1_{\mathrm{sing}}(K_q, \mathbf{T}_{\breve{g},\mathscr{O}_{\chi}}) \to H^1_{\mathrm{sing}}(K_q, \mathbf{T}_{g,\mathscr{O}_{\chi}})$ becomes identified with the natural projection $\Lambda_{\mathscr{O}_{\chi}}/p^{2k} \twoheadrightarrow \Lambda_{\mathscr{O}_{\chi}}/p^k$. Since $\partial_q(\kappa^{\varepsilon}_{\infty}(\ell))$ is zero by Proposition 6.2 and $t^{\varepsilon}_{\chi}(g, \ell) < \mathrm{ord}_{\chi}(p^k)$ by Equation (7.3), it follows that $\partial_q(\tilde{\kappa}^{\varepsilon}_{\chi}(\ell)) \in \mathscr{O}_{\chi}/p^{2k}$ has $\mathscr{O}_{\chi}$-adic valuation at least $\mathrm{ord}_{\chi}(p^k)$, hence its projection $\partial_q(\hat{\kappa}^{\varepsilon}_{\chi}(\ell)) \in \mathscr{O}_{\chi}/p^k$ modulo p^k vanishes (here and in the following we wrote ∂_q for the scalar extension $\partial_q \otimes \mathrm{id}$ to simplify the notation as before). This gives

$$\partial_q(\xi^{\varepsilon}_{\chi}(\ell)) = \partial_q \circ \mathbf{s}_{\chi}(\hat{\kappa}^{\varepsilon}_{\chi}(\ell)) = \partial_q(\xi^{\varepsilon}_{\chi}(\ell)) = 0,$$

as was to be shown. The second statement is proved similarly, using that $v_{\ell}(\kappa^{\varepsilon}_{\infty}(\ell)) = 0$ by Proposition 6.2.

(2) Equations (7.3), (7.4) and (7.5) show that $\partial_{\ell}(\tilde{\kappa}^{\varepsilon}_{\chi}(\ell))$ has $\mathscr{O}_{\chi}$-adic valuation $t^{\varepsilon}_{\chi}(g) - t^{\varepsilon}_{\chi}(g, \ell)$. Since $\mathrm{ord}_{\chi}(p^k) > t^{\varepsilon}_{\chi}(g)$ this is also the $\mathscr{O}_{\chi}$-adic valuation of $\partial_{\ell}(\hat{\kappa}^{\varepsilon}_{\chi}(\ell))$, which is equal to that of $\partial_{\ell}(\xi^{\varepsilon}_{\chi}(\ell))$ (*cf.* the proof of (1)).

(3) Note that the class $\bar{\xi}^{\varepsilon}_{\chi}(\ell)$ is equal to the image of $\tilde{\kappa}^{\varepsilon}_{\chi}(\ell)$ under the composition

$$\mathfrak{Sel}^{\varepsilon}_{S}(K, \mathbf{T}_{\breve{g}}) \otimes \mathscr{O}_{\chi} \longrightarrow \mathfrak{Sel}^{\varepsilon}_{S}(K, \mathbf{T}_{\bar{g}}) \otimes \mathscr{O}_{\chi} \xrightarrow{\mathbf{s}_{\chi}} \mathfrak{Sel}^{\varepsilon}_{S}(K, T_{\bar{g}}(\chi)) \otimes_{\mathbf{F}_p} \mathbf{F}_{\chi}.$$

As above this implies that $\bar{\xi}^{\varepsilon}_{\chi}(\ell)$ is not zero, since $\mathbf{s}_{\chi}$ is injective and $\mathrm{ord}_{\chi}(\tilde{\kappa}^{\varepsilon}_{\chi}(\ell)) = 0$. Together with (1) this implies the first statement. Since $\partial_{\ell}\big(\bar{\xi}^{\varepsilon}_{\chi}(\ell)\big) \in H^1_{\mathrm{sing}}(K_{\ell}, T_{\bar{g}}(\chi)) \otimes_{\mathbf{F}_p} \mathbf{F}_{\chi} \cong \mathbf{F}_{\chi}$ is the projection of $\partial_{\ell}(\xi^{\varepsilon}_{\chi}(\ell)) \in \mathscr{O}_{\chi}/p^k$ modulo ϖ_{χ}, the second statement follows from (2). □

7.2. **Proof of Theorem 7.1.** The proof of Theorem 7.1 is divided into several steps. Steps 1, 2 and 3 consist in a generalization to the present context of similar results of [BD05]. The direct generalizations of the techniques in [BD05] only allow one to prove the inequality

$$\mathrm{length}_{\mathscr{O}_{\chi}}\big(\mathrm{Sel}_{\varepsilon}(K, A_g(\chi))\big) \leqslant 2\mathrm{ord}_{\chi}\left(\mathcal{L}^{\varepsilon}_g(\bar{\chi})\right)$$

in Theorem 7.1; this inequality holds in both cases, exceptional and non-exceptional, while the opposite inequality can be shown in the non-exceptional case only with a further inductive argument on the length of $\mathrm{Sel}_\varepsilon(K, A_g(\chi))$, developed in Steps 4, 5, 6 and 7. The key ingredient for the inequality

$$\mathrm{length}_{\mathscr{O}_\chi}\big(\mathrm{Sel}_\varepsilon(K, A_g(\chi))\big) \geqslant 2\mathrm{ord}_\chi\left(\mathcal{L}_g^\varepsilon(\bar{\chi})\right)$$

is Step 4 (the basis of the inductive argument, i.e. the case when $\mathrm{length}_{\mathscr{O}_\chi}\big(\mathrm{Sel}_\varepsilon(K, A_g(\chi))\big) = 0$) which combines Gross formula and Lemma 4.6 with results of Skinner–Urban (ordinary case) and Fouquet-Wan (supersingular case); the inductive argument then follows in Steps 6 and 7 using a a structure theorem for $\mathrm{Sel}_\varepsilon(K, A_g(\chi))$, which we prove in Step 5.

7.2.1. *Step 1.* If $\mathcal{L}_g^\varepsilon(\bar{\chi})$ is a p-adic unit then $\mathrm{Sel}_\varepsilon(K, A_g(\chi))$ is trivial.

Proof. (Cf. [BD05, Proposition 4.7].) Assume *ad absurdum* that there exists a nontrivial class x in the Selmer group $\mathrm{Sel}_\varepsilon(K, A_g(\chi))$. Choose a $2k$-admissible prime ℓ such that $v_\ell(x) \in H^1_{\mathrm{fin}}(K_\ell, A_g(\chi)) \cong \mathscr{O}_\chi/p^k$ is not zero, which exists by Theorem 3.2 of [BD05]. Since $\mathfrak{Sel}^\varepsilon(K, T_g(\bar{\chi}))$ is the dual Selmer group of $\mathrm{Sel}_\varepsilon(K, A_g(\chi))$, Lemma 7.2(1) and the reciprocity law of global class field theory yield

$$0 = \sum_v \left\langle \mathrm{res}_v(x), \mathrm{res}_v(\xi_{\bar{\chi}}^\varepsilon(\ell))\right\rangle_v = \left\langle \mathrm{res}_\ell(x), \mathrm{res}_\ell(\xi_{\bar{\chi}}^\varepsilon(\ell))\right\rangle_\ell,$$

where the sum is taken over all the primes of K and $\langle -, -\rangle_v$ denotes the local Tate pairing at v induced by the duality $T_g(\chi) \times A_g(\chi) \to \mathscr{O}_\chi/p^k(1)$ (cf. [Mil04, Chapter 1]). Since $\partial_\ell(x) = 0$, $v_\ell(x) \neq 0$ and $H^1_{\mathrm{fin}}(K_\ell, T_g(\bar{\chi}))$ is the orthogonal complement of $H^1_{\mathrm{fin}}(K_\ell, A_g(\chi))$ under the perfect pairing $\langle -, -\rangle_\ell$, the previous equation implies that the residue at ℓ of $\xi_{\bar{\chi}}^\varepsilon(\ell)$ has positive $\mathscr{O}_\chi$-adic valuation. According to Lemma 7.2(2) this in turn implies that $\mathcal{L}_g^\varepsilon(\bar{\chi})$ has positive $\mathscr{O}_\chi$-adic valuation, contradicting the assumption. □

7.2.2. *Step 2.* Assume that $\mathrm{Sel}_\varepsilon(K, A_g(\chi))$ is non-trivial. Then there exist two distinct $2k$-admissible primes ℓ_1 and ℓ_2 satisfying the following properties.

I$_1$. $t_{\bar{\chi}}^\varepsilon(g, \ell_1) = t_{\bar{\chi}}^\varepsilon(g, \ell_2) < t_{\bar{\chi}}^\varepsilon(g)$.

I$_2$. If $h \in S_2(N^+, L\ell_1\ell_2N^-; \mathbf{Z}/p^k)$ denotes the $\ell_1\ell_2$-level raising of g, then

$$\mathrm{Sel}_\varepsilon(K, A_h(\chi)) = \mathrm{Sel}_\varepsilon^{\ell_1\ell_2}(K, A_g(\chi)).$$

I$_3$. The ϖ_χ-adic valuation of $\mathcal{L}_h^\varepsilon(\bar{\chi}) \in \mathscr{O}_\chi/p^k$ is equal to $t_{\bar{\chi}}^\varepsilon(g, \ell_i)$ (for $i = 1, 2$):

$$t_{\bar{\chi}}^\varepsilon(h) = \mathrm{ord}_\chi\big(\mathcal{L}_h^\varepsilon(\bar{\chi})\big) = t_{\bar{\chi}}^\varepsilon(g, \ell_i) < \infty.$$

I$_4$. $\mathrm{ord}_\chi\big(v_{\ell_1}\big(\xi_{\bar{\chi}}^\varepsilon(\ell_2)\big)\big) = 0$ and $\mathrm{ord}_\chi\big(v_{\ell_2}\big(\xi_{\bar{\chi}}^\varepsilon(\ell_1)\big)\big) = 0$.

Proof. We first prove that there exist infinitely many $2k$-admissible primes ℓ such that $t_{\bar{\chi}}(g, \ell) < t_{\bar{\chi}}(g)$. Let $\mathfrak{m}_{\Lambda_{\mathscr{O}_\chi}}$ be as above the maximal ideal of $\Lambda_{\mathscr{O}_\chi}$, so that $A_g(\chi)[\mathfrak{m}_{\Lambda_{\mathscr{O}_\chi}}] \cong A_{\bar{g}} \otimes_{\mathbf{F}_p} \mathbf{F}_\chi$ (as $\chi(g) \equiv 1 \pmod{\varpi_\chi}$ for every $g \in G_\infty$). The control theorem of Proposition 5.9 yields

$$\mathrm{Sel}_\varepsilon(K, A_{\bar{g},\mathscr{O}_\chi}) \cong \mathrm{Sel}_\varepsilon(K, A_g(\chi))[\mathfrak{m}_{\Lambda_{\mathscr{O}_\chi}}],$$

hence $\mathrm{Sel}_\varepsilon(K, A_{\bar{g},\mathscr{O}_\chi})$ is nontrivial by Nakayama's Lemma. Fix a non-zero class

$$0 \neq x \in \mathrm{Sel}_\varepsilon(K, A_{\bar{g},\mathscr{O}_\chi}).$$

According to (a slight generalization of) Theorem 3.2 of [BD05] there exist infinitely many $2k$-admissible primes ℓ such that $v_\ell(x) \in H^1_{\mathrm{fin}}(K_\ell, A_{\bar{g},\mathscr{O}_\chi})$ is non zero. We claim that for every such prime ℓ one has

$$t_{\bar{\chi}}^\varepsilon(g, \ell) < t_{\bar{\chi}}^\varepsilon(g). \tag{7.6}$$

Recall the class $\bar{\xi}_{\bar{\chi}}^\varepsilon(\ell) \in H^1(K, T_{\bar{g}}) \otimes_{\mathbf{F}_p} \mathbf{F}_\chi$ constructed in Section 7.1. Lemma 7.2(3) shows that $\bar{\xi}_{\bar{\chi}}^\varepsilon(\ell)$ belongs to $\mathfrak{Sel}_\ell^\varepsilon(K, T_{\bar{g}}) \otimes_{\mathbf{F}_p} \mathbf{F}_\chi$, hence (as in the proof of Step 1) the reciprocity law of global class field theory yields

$$\left\langle \partial_\ell\big(\bar{\xi}_{\bar{\chi}}^\varepsilon(\ell)\big), v_\ell(x)\right\rangle_\ell = 0,$$

where $\langle -, -\rangle_\ell$ is the $\mathbf{F}_\chi$-linear extension of the perfect local Tate pairing

$$H^1_{\mathrm{sing}}(K_\ell, T_{\bar{g}}) \otimes_{\mathbf{F}_p} H^1_{\mathrm{fin}}(K_\ell, A_{\bar{g}}) \longrightarrow \mathbf{F}_p.$$

Since $v_\ell(x) \neq 0$ this gives $\partial_\ell\big(\bar{\xi}^\varepsilon_{\bar{\chi}}(\ell)\big) = 0$, and the claim (7.6) follows from another application of Lemma 7.2(3).

Fix a $2k$-admissible prime ℓ_1 such that $t^\varepsilon_{\bar{\chi}}(g,\ell_1) < t^\varepsilon_{\bar{\chi}}(g)$, and such that $t^\varepsilon_{\bar{\chi}}(g,\ell_1) \leqslant t^\varepsilon_{\bar{\chi}}(g,\ell)$ for every $2k$-admissible prime ℓ. Since $\bar{\xi}^\varepsilon_{\bar{\chi}}(\ell_1)$ is non-zero by Lemma 7.2(3), Theorem 3.2 of [BD05] proves that there exists a $2k$-admissible prime $\ell_2 \neq \ell_1$ such that $v_{\ell_2}\big(\bar{\xi}^\varepsilon_{\bar{\chi}}(\ell_1)\big) \in H^1_{\mathrm{fin}}(K_{\ell_2}, T_{\bar{g}}) \otimes_{\mathbf{F}_p} \mathbf{F}_\chi \cong \mathbf{F}_\chi$ is non-zero. By construction (cf. Section 7.1) the latter condition is equivalent to

$$\mathrm{ord}_\chi\big(v_{\ell_2}(\xi^\varepsilon_{\bar{\chi}}(\ell_1))\big) = 0.$$

The second reciprocity law Theorem 6.5 and the definition of $\xi^\varepsilon_{\bar{\chi}}(\ell_1)$ show that the identities (where we write v_ℓ for $v_\ell \otimes \mathrm{id}$ for $\ell = \ell_1$ and $\ell = \ell_2$ as before)

$$\varpi_\chi^{t^\varepsilon_{\bar{\chi}}(g,\ell_1)} \cdot v_{\ell_2}\big(\xi^\varepsilon_{\bar{\chi}}(\ell_1)\big) = v_{\ell_2}\big(\kappa^\varepsilon_{\bar{\chi}}(\ell_1)\big) \overset{\text{Th. 6.5}}{=} \mathcal{L}^\varepsilon_h(\bar{\chi}) \overset{\text{Th. 6.5}}{=} v_{\ell_1}\big(\kappa^\varepsilon_{\bar{\chi}}(\ell_2)\big) = \varpi_\chi^{t^\varepsilon_{\bar{\chi}}(g,\ell_2)} \cdot v_{\ell_1}\big(\xi^\varepsilon_{\bar{\chi}}(\ell_2)\big) \tag{7.7}$$

hold in $\mathscr{O}_\chi/p^k$ up to multiplication by p-adic units (*cf.* the proof of Lemma 7.2(1) for the first and last identities). Since $t^\varepsilon_{\bar{\chi}}(g,\ell) < \mathrm{ord}_\chi(p^k)$ for $\ell = \ell_1, \ell_2$ by Equation (7.3), and since by construction $t^\varepsilon_{\bar{\chi}}(g,\ell_1) \leqslant t^\varepsilon_{\bar{\chi}}(g,\ell_2)$, the previous two equations and Lemma 7.2(1) show that

$$t^\varepsilon_{\bar{\chi}}(g,\ell_1) = t^\varepsilon_{\bar{\chi}}(g,\ell_2) < t^\varepsilon_{\bar{\chi}}(g) \tag{7.8}$$

and that the identities

$$\big(v_{\ell_1}\big(\xi^\varepsilon_{\bar{\chi}}(\ell_2)\big), v_{\ell_2}\big(\xi^\varepsilon_{\bar{\chi}}(\ell_2)\big)\big) = (1,0), \tag{7.9}$$

$$\big(v_{\ell_1}\big(\xi^\varepsilon_{\bar{\chi}}(\ell_1)\big), v_{\ell_2}\big(\xi^\varepsilon_{\bar{\chi}}(\ell_1)\big)\big) = (0,1) \tag{7.10}$$

hold in $\mathscr{O}_\chi/p^k \oplus \mathscr{O}_\chi/p^k$ up to multiplication by p-adic units. (Here for $\ell = \ell_1$ or $\ell = \ell_2$ one fixes an isomorphism $H^1_{\mathrm{fin}}(K_\ell, T_g(\bar{\chi})) \cong \mathscr{O}_\chi/p^k$.) It follows from the definitions (cf. Section 5.2) that

$$\mathrm{Sel}^{\ell_1\ell_2}_\varepsilon(K, A_g(\chi)) = \mathrm{Sel}^{\ell_1\ell_2}_\varepsilon(K, A_h(\chi)),$$
$$\mathfrak{Sel}^\varepsilon_{\ell_1\ell_2}(K, T_g(\bar{\chi})) = \mathfrak{Sel}^\varepsilon_{\ell_1\ell_2}(K, T_h(\bar{\chi})),$$

and a class $z \in \mathfrak{Sel}^\varepsilon_{\ell_1\ell_2}(K, T_g(\bar{\chi}))$ belongs to $\mathfrak{Sel}^\varepsilon(K, T_h(\bar{\chi}))$ precisely if $v_{\ell_1}(z)$ and $v_{\ell_2}(z)$ are both trivial. Poitou–Tate duality (see Theorem 7.3 of [Rub00] or Chapter 1 of [Mil04]) then yields a short exact sequence of $\mathscr{O}_\chi/p^k$-modules

$$\mathfrak{Sel}^\varepsilon_{\ell_1\ell_2}(K, A_g(\bar{\chi})) \overset{v_{\ell_1}\oplus v_{\ell_2}}{\longrightarrow} \mathscr{O}_\chi/p^k \oplus \mathscr{O}_\chi/p^k \overset{\partial^\vee_{\ell_1}\oplus\partial^\vee_{\ell_2}}{\longrightarrow} \mathrm{Sel}_\varepsilon(K, A_h(\bar{\chi}))^\vee \longrightarrow \mathrm{Sel}^{\ell_1\ell_2}_\varepsilon(K, A_g(\chi))^\vee \longrightarrow 0, \tag{7.11}$$

where $(\cdot)^\vee = \mathrm{Hom}_{\mathbf{Z}_p}(\cdot, \mathbf{Q}_p/\mathbf{Z}_p)$ and for $\ell = \ell_1, \ell_2$ one identifies $H^1_{\mathrm{sing}}(K_\ell, A_g(\chi))$ with the Pontrjagin dual of $H^1_{\mathrm{fin}}(K_\ell, T_g(\bar{\chi})) \cong \mathscr{O}_\chi/p^k$ under the local Tate duality. Equation (7.10) shows that the first map is surjective, hence

$$\mathrm{Sel}_\varepsilon(K, A_h(\chi)) = \mathrm{Sel}^{\ell_1\ell_2}_\varepsilon(K, A_g(\chi)).$$

Together with Equations (7.7)–(7.10) this concludes the proof. □

7.2.3. *Step 3.* $\mathrm{length}_{\mathscr{O}_\chi}\big(\mathrm{Sel}_\varepsilon(K, A_g(\chi))\big) \leqslant 2t^\varepsilon_{\bar{\chi}}(g)$.

Proof. As in [BD05] one proceeds by induction on $t_{\bar{\chi}}(g)$. Step 1 shows that the statement holds if $t_{\bar{\chi}}(g) = 0$. Assume then $t_{\bar{\chi}}(g) > 0$. If $\mathrm{Sel}_\varepsilon(K, A_g(\chi)) = 0$ the statement is trivially verified, hence assume that $\mathrm{Sel}_\varepsilon(K, A_g(\chi))$ is non-trivial. According to Step 2 there exists two distinct $2k$-admissible primes ℓ_1 and ℓ_2 satisfying the properties $\mathbf{I}_1$–$\mathbf{I}_3$. As in loc. cit. denote by $h \in S_2(N^+, L\ell_1\ell_2 L; \mathbf{Z}/p^k)$ the $\ell_1\ell_2$-level raising of g.

Let $\zeta^\varepsilon_{\bar{\chi}}(\ell_1) \in \mathfrak{Sel}^\varepsilon_{\ell_1}(K, T_g(\bar{\chi}))$ be a global class such that $\partial_{\ell_1}\big(\zeta^\varepsilon_{\bar{\chi}}(\ell_1)\big)$ generates the image of the residue map $\partial_{\ell_1} : \mathfrak{Sel}^\varepsilon_{\ell_1}(K, T_g(\bar{\chi})) \to H^1_{\mathrm{sing}}(K_\ell, T_g(\bar{\chi})) \cong \mathscr{O}_\chi/p^k$, viz. ∂_{ℓ_1} induces an isomorphism

$$\partial_{\ell_1} : \mathfrak{Sel}^\varepsilon_{\ell_1}(K, T_g(\bar{\chi}))/\mathfrak{Sel}^\varepsilon(K, T_g(\bar{\chi})) \cong \partial_{\ell_1}\big(\zeta^\varepsilon_{\bar{\chi}}(\ell_1)\big) \cdot \mathscr{O}_\chi/p^k. \tag{7.12}$$

Since $\xi^\varepsilon_{\bar{\chi}}(\ell_1)$ belongs to the Selmer group $\mathfrak{Sel}^\varepsilon_{\ell_1}(K, T_g(\bar{\chi}))$ by Lemma 7.2(1), multiplying $\zeta^\varepsilon_{\bar{\chi}}(\ell_1)$ by a p-adic unit if necessary one can assume that there exists an integer $m_1 \geqslant 0$ such that

$$\xi^\varepsilon_{\bar{\chi}}(\ell_1) - \varpi_\chi^{m_1} \cdot \zeta^\varepsilon_{\bar{\chi}}(\ell_1) \in \mathfrak{Sel}^\varepsilon(K, T_g(\bar{\chi})).$$

Equation (7.12), Lemma 7.2(2) and property $\mathbf{I}_3$ then yield

$$\text{length}_{\mathscr{O}_\chi}\big(\mathfrak{Sel}^{\varepsilon}_{\ell_1}(K,T_g(\bar{\chi}))/\mathfrak{Sel}^{\varepsilon}(K,T_g(\bar{\chi}))\big) = \text{ord}_\chi(p^k) - t^{\varepsilon}_{\bar{\chi}}(g) + t^{\varepsilon}_{\bar{\chi}}(h) + m_1. \tag{7.13}$$

Similarly let $\zeta^{\varepsilon}_{\bar{\chi}}(\ell_2) \in \mathfrak{Sel}^{\varepsilon}_{\ell_1\ell_2}(K,T_g(\bar{\chi}))$ be a class such that the residue map at ℓ_2 induces an isomorphism

$$\partial_{\ell_2} : \mathfrak{Sel}^{\varepsilon}_{\ell_1\ell_2}(K,T_g(\bar{\chi}))/\mathfrak{Sel}^{\varepsilon}_{\ell_1}(K,T_g(\bar{\chi})) \cong \partial_{\ell_2}\big(\zeta^{\varepsilon}_{\bar{\chi}}(\ell_2)\big)\cdot \mathscr{O}_\chi/p^k.$$

Because $\xi^{\varepsilon}_{\bar{\chi}}(\ell_2) \in \mathfrak{Sel}^{\varepsilon}_{\ell_1\ell_2}(K,T_g(\bar{\chi}))$ by Lemma 7.2(1), one can assume that there exists $m_2 \geqslant 0$ such that

$$\xi^{\varepsilon}_{\bar{\chi}}(\ell_2) - \varpi_\chi^{m_2}\cdot \zeta^{\varepsilon}_{\bar{\chi}}(\ell_2) \in \mathfrak{Sel}^{\varepsilon}_{\ell_1}(K,T_g(\bar{\chi})),$$

and apply as above Lemma 7.2(2) and property $\mathbf{I}_3$ to deduce the equality

$$\text{length}_{\mathscr{O}_\chi}\big(\mathfrak{Sel}^{\varepsilon}_{\ell_1\ell_2}(K,T_g(\bar{\chi}))/\mathfrak{Sel}^{\varepsilon}_{\ell_1}(K,T_g(\bar{\chi}))\big) = \text{ord}_\chi(p^k) - t^{\varepsilon}_{\bar{\chi}}(g) + t^{\varepsilon}_{\bar{\chi}}(h) + m_2. \tag{7.14}$$

When combined together Equations (7.13) and (7.14) give the equality

$$\text{length}_{\mathscr{O}_\chi}\big(\mathfrak{Sel}^{\varepsilon}_{\ell_1\ell_2}(K,T_g(\bar{\chi}))/\mathfrak{Sel}^{\varepsilon}(K,T_g(\bar{\chi}))\big) = 2\cdot\text{ord}_\chi(p^k) - 2\cdot t^{\varepsilon}_{\bar{\chi}}(g) + 2\cdot t^{\varepsilon}_{\bar{\chi}}(h) + m_1 + m_2. \tag{7.15}$$

By construction $\text{Sel}^{\cdot}_{\varepsilon}(K,A_g(\chi))$ is the dual Selmer group of $\mathfrak{Sel}^{\varepsilon}_{\cdot}(K,T_g(\bar{\chi}))$, hence Poitou–Tate duality gives a short exact sequence of $\mathscr{O}_\chi/p^k$-modules (cf. Equation (7.11) in the proof of Step 2)

$$0 \longrightarrow \frac{\mathfrak{Sel}^{\varepsilon}_{\ell_1\ell_2}(K,T_g(\bar{\chi}))}{\mathfrak{Sel}^{\varepsilon}(K,T_g(\bar{\chi}))} \overset{\partial_{\ell_1}\oplus\partial_{\ell_2}}{\longrightarrow} \mathscr{O}_\chi/p^k \oplus \mathscr{O}_\chi/p^k \overset{v^\vee_{\ell_1}\oplus v^\vee_{\ell_2}}{\longrightarrow} \left(\frac{\text{Sel}_{\varepsilon}(K,A_g(\chi))}{\text{Sel}^{\ell_1\ell_2}_{\varepsilon}(K,A_g(\chi))}\right)^\vee \longrightarrow 0,$$

where for $\ell = \ell_1,\ell_2$ one identifies $H^1_{\text{sing}}(K_\ell,T_g(\bar{\chi})) \cong H^1_{\text{fin}}(K_\ell,A_g(\chi))^\vee$ with $\mathscr{O}_\chi/p^k$ under a fixed isomorphism. Together with Equation (7.15) and property $\mathbf{I}_2$ this implies

$$\text{length}_{\mathscr{O}_\chi}\big(\text{Sel}_\varepsilon(K,A_g(\chi))\big) - 2\cdot t^{\varepsilon}_{\bar{\chi}}(g) = \text{length}_{\mathscr{O}_\chi}\big(\text{Sel}_\varepsilon(K,A_h(\chi))\big) - 2\cdot t^{\varepsilon}_{\bar{\chi}}(h) - m_1 - m_2. \tag{7.16}$$

Properties $\mathbf{I}_1$ and $\mathbf{I}_3$ give $t^{\varepsilon}_{\bar{\chi}}(h) < t^{\varepsilon}_{\bar{\chi}}(g)$, hence

$$\text{length}_{\mathscr{O}_\chi}\big(\text{Sel}_\varepsilon(K,A_h(\chi))\big) - 2\cdot t^{\varepsilon}_{\bar{\chi}}(h) \leqslant 0 \tag{7.17}$$

by the induction hypothesis. The statement follows from Equations (7.16) and (7.17). □

7.2.4. *Step 4.* Assume that (f,K,p,ε) is not exceptional and that $\text{Sel}_\varepsilon(K,A_g(\chi)) = 0$. Then $t^{\varepsilon}_{\bar{\chi}}(g) = 0$.

Proof. Theorem B of [DT94] implies that there exists a newform $\xi = \sum_{n=1}^\infty a_n(\xi)\cdot q^n$ in $S_2(\Gamma_0(NL))^{\text{new}}$ which is congruent to f modulo p. More precisely, if $\mathbf{Q}(\xi)$ denotes the field generated over $\mathbf{Q}$ by the Fourier coefficients of ξ, then there exists a prime $\bar{\mathfrak{P}}$ of $\bar{\mathbf{Q}}$ dividing p such that $a_l(\xi) \equiv a_l(E) \pmod{\bar{\mathfrak{P}}}$ for every rational prime $l \nmid NLp$. (loc. cit. proves the existence of an eigenform $\xi \in S_2(\Gamma_1(N)\cap\Gamma_0(L))$ of conductor divisible by L which is congruent to f modulo p. It is not difficult to prove that an eigenform with these properties has trivial character and conductor NL.) Let $J^o_\xi/\mathbf{Q}$ be the quotient of $\text{Pic}^0(X_0(NL)/\mathbf{Q})$ associated with ξ by the Eichler–Shimura construction. It is an abelian variety of dimension $[\mathbf{Q}(\xi):\mathbf{Q}]$ equipped with a morphism of $\mathbf{Q}$-algebras $\mathbf{Q}(\xi) \to \text{End}_{\mathbf{Q}}(J^o_\xi)\otimes_{\mathbf{Z}}\mathbf{Q}$. Let $J_\xi/\mathbf{Q}$ be an abelian variety in the isogeny class of J^o_ξ which has real multiplication by the ring of integers $\mathcal{O}$ of $\mathbf{Q}(\xi)$ and set $\mathfrak{P} = \bar{\mathfrak{P}}\cap\mathcal{O}$. Since E_p is an irreducible $\mathbf{F}_p[G_{\mathbf{Q}}]$-module by Hypothesis 1.1(1), the Eichler–Shimura relations and the Brauer–Nesbitt theorem imply that there are isomorphisms of $\mathcal{O}/\mathfrak{P}[G_{\mathbf{Q}}]$-modules

$$J_\xi[\mathfrak{P}] \cong E_p\otimes_{\mathbf{F}_p}\mathcal{O}/\mathfrak{P} \cong A_{\bar{g}}\otimes_{\mathbf{F}_p}\mathcal{O}/\mathfrak{P}.$$

Identify in what follows $J_\xi[\mathfrak{P}]$ and $A_{\bar{g}}\otimes_{\mathbf{F}_p}\mathcal{O}/\mathfrak{P}$ under a fixed isomorphism, and let

$$\text{Sel}_{\mathfrak{P}}(J_\xi/K) \subset H^1(K,A_{\bar{g}})$$

be the $\mathfrak{P}$-Selmer group of J_ξ over K (cf. [GP12]). It follows from the results of [GP12, Sections 3–5] that

$$\text{Sel}_{\mathfrak{P}}(J_\xi/K) = \text{Sel}(K,A_{\bar{g}})\otimes_{\mathbf{F}_p}\mathcal{O}/\mathfrak{P} \tag{7.18}$$

inside $H^1(K,A_{\bar{g}})\otimes_{\mathbf{F}_p}\mathcal{O}/\mathfrak{P}$, where $\text{Sel}(K,A_{\bar{g}})$ is the Selmer group defined by imposing the finite local condition $H^1_{\text{fin}}(K_{\mathfrak{p}},A_{\bar{g}}) \cong E(K_{\mathfrak{p}})\otimes\mathbf{F}_p$ at every prime $\mathfrak{p}$ of K dividing p (viz. $\text{Sel}(K,A_{\bar{g}}) = \text{Sel}_\emptyset(K,A_{\bar{g}})$

with the notation of Section 5.2, independently of whether E has ordinary or supersingular reduction at p). Note that since (f,K,p,ε) is not exceptional, then by definition $E(K_\mathfrak{p})_\varepsilon = E(K_\mathfrak{p})$, hence

$$H^1_{\mathrm{fin},\varepsilon}(K_\mathfrak{p}, A_{\bar g}) = H^1_{\mathrm{fin},\varepsilon}(K_\mathfrak{p}, \mathbf{A}_f)[\mathfrak{m}_\Lambda] = E(K_\mathfrak{p})_\varepsilon \otimes \mathbf{F}_p = E(K_\mathfrak{p}) \otimes \mathbf{F}_p = H^1_{\mathrm{fin}}(K_\mathfrak{p}, A_{\bar g})$$

where the first equality follows from Corollary 5.5(2) and the second from Proposition 5.3. It follows that $\mathrm{Sel}_\varepsilon(K, A_{\bar g}) = \mathrm{Sel}(K, A_{\bar g})$. We have an isomorphism $H^1(K, A_{\bar g}) \otimes_{\mathbf{Z}_p} \mathscr{O}_\chi \cong H^1(K, A_{\bar g, \mathscr{O}_\chi})$ and an injection $\mathrm{Sel}(K, A_{\bar g}) \otimes_{\mathbf{Z}_p} \mathscr{O}_\chi \hookrightarrow H^1(K, A_{\bar g}) \otimes_{\mathbf{Z}_p} \mathscr{O}_\chi$ by the flatness of $\mathscr{O}_\chi/\mathbf{Z}_p$, and therefore $\mathrm{Sel}(K, A_{\bar g}) \otimes_{\mathbf{Z}_p} \mathscr{O}_\chi$ injects into $\mathrm{Sel}(K, A_{\bar g}(\chi))$. Since by assumption $\mathrm{Sel}_\varepsilon(K, A_g(\chi))$ is trivial, it follows from Proposition 5.9 and the irreducibility of $\bar\varrho_{E,p}$ that the same is true for $\mathrm{Sel}(K, A_{\bar g})$ and Equation (7.18) gives

$$\mathrm{Sel}_\mathfrak{P}(J_\xi/K) = 0. \tag{7.19}$$

Let $L(\xi/K, 1)_{\mathrm{alg}}$ denote the algebraic part of the special value of the complex L-function of ξ over K, normalized as in [BBV16, Section 4]. Results of Skinner–Urban and Fouquet–Wan prove the inequality

$$\mathrm{ord}_\mathfrak{P}\big(L(\xi/K,1)_{\mathrm{alg}}\big) \leqslant \mathrm{length}_{\mathcal{O}_\mathfrak{P}}\big(\mathrm{Sel}_{\mathfrak{P}^\infty}(J_\xi/K)\big) + \sum_{q|NL} t_\xi(q), \tag{7.20}$$

where $t_\xi(q)$ is the Tamagawa exponent appearing in [BBV16, Section 4]. See [SU14] in the ordinary case; for the non-ordinary case, the reader is referred to [FW22, Corollaries 1.9, 1.10] and also [BSTW, Theorems 1.5, 1.6], [CCSS, Theorem C].

Since $J_\xi[\mathfrak{P}]$ is an irreducible G_K-module, $\mathrm{Sel}_\mathfrak{P}(J_\xi/K)$ is equal to the $\mathfrak{P}$-torsion submodule of the Selmer group $\mathrm{Sel}_{\mathfrak{P}^\infty}(J_\xi/K)$, so that $\mathrm{Sel}_{\mathfrak{P}^\infty}(J_\xi/K)$ is trivial by Equation (7.19). In addition $t_\xi(q) = 0$ for every prime $q|N^+$ under our assumptions, and the previous equation yields

$$\mathrm{ord}_\mathfrak{P}\big(L(\xi/K,1)_{\mathrm{alg}}\big) \leqslant \sum_{q|LN^-} t_\xi(q).$$

On the other hand, Gross's formula (see Theorem 4.2 of [BBV16] for the formulation in the form required in this paper) gives the identity

$$\mathrm{ord}_\mathfrak{P}\big(L(\xi/K,1)_{\mathrm{alg}}\big) = 2\cdot \mathrm{ord}_\mathfrak{P}\big(\psi_\xi\big(P_K(L)\big)\big) + \sum_{q|LN^-} t_\xi(q).$$

It follows combining the two previous formulas that $\psi_\xi\big(P_K(L)\big)$ has trivial $\mathfrak{P}$-adic valuation:

$$\psi_\xi\big(P_K(L)\big) \in \mathcal{O}_\mathfrak{P}^*. \tag{7.21}$$

Since ξ is congruent to f modulo p, one has $\psi_\xi\big(P_K(L)\big) \equiv \psi_{\bar g}\big(P_K(L)\big) \pmod{\mathfrak{P}}$. In addition, as (f,K,p,ε) is not exceptional, Lemmas 4.2 and 4.6 show that the equalities $\psi_{\bar g}\big(P_K(L)\big) = \mathcal{L}^\varepsilon_{\bar g}(\mathbf{1}) = \mathcal{L}^\varepsilon_g(\mathbf{1}) \pmod{p}$ hold in $\mathbf{F}_p$ up to multiplication by non-zero elements. Equation (7.21) then yields $\mathcal{L}^\varepsilon_g(\mathbf{1}) \in \mathbf{Z}_p^*$. This implies that the p-adic L-function $\mathcal{L}^\varepsilon_g$ is a unit in Λ/p^k, which in turn gives $t^\varepsilon_{\bar\chi}(g) = 0$. □

7.2.5. *Step 5.* There exist an $\mathscr{O}_\chi/p^k$-module $\mathtt{M}$ and an integer $s \in \{0,1\}$ such that

$$\mathrm{Sel}_\varepsilon(K, A_g(\chi)) \cong (\mathscr{O}_\chi/p^k)^s \oplus \mathtt{M} \oplus \mathtt{M} \cong \mathfrak{Sel}^\varepsilon(K, T_g(\chi)).$$

Proof. Thanks to the isomorphism $w_\chi : \mathfrak{Sel}^\varepsilon(K, T_g(\chi)) \simeq \mathrm{Sel}_\varepsilon(K, A_g(\chi))$ of Proposition 5.15, it is enough to prove the statement for $\mathrm{Sel}_\varepsilon(K, T_g(\chi))$. This follows from the results of [How06, Section 2.6] (which in turn grounds on Section 1.4 of [How04]). Precisely, the local conditions defining the Selmer groups $\{\mathfrak{Sel}^\varepsilon(K, T_h(\chi))\}_h$, for h varying through the reductions of g modulo p^j with $1 \leqslant j \leqslant k$, are cartesian in the sense of Definition 2.2.2 of [How06]: this is easily verified for the primes of K not dividing p (where the local condition are unramified, ordinary or the relevant Galois representations are cohomologically trivial), and follows from the local control theorems of Section 5.3.1 for the primes of K dividing p. Moreover, for each prime v of K, denote by $\bar v$ the complex conjugate of v, and by $\mathcal{L}_v$ the local condition at v satisfied by the Selmer classes in $\mathfrak{Sel}^\varepsilon(K, T_g(\chi))$. Then Proposition 5.15 and Remark 5.16 prove that $\mathcal{L}_v$ is the exact orthogonal complement of $\mathcal{L}_{\bar v}$ under the pairing $H^1(K_v, T_g(\chi)) \times H^1(K_{\bar v}, T_g(\chi)) \to \mathscr{O}_\chi/p^k$ induced by the (not G_{K_v}-equivariant) bilinear map $T_g(\chi) \times T_g(\chi) \to \mathscr{O}_\chi/p^k(1)$ arising from the Weil pairing (cf. Equation (4) in Section 2.6 of [How06]). As explained in Section 2.6 of [How06], we can then apply [How06, Proposition 2.2.7] and conclude the proof of Step 5. □

7.2.6. *Step 6.* Assume that (f, K, p, ε) is not exceptional and let ℓ_1 and ℓ_2 be $2k$-admissible primes which satisfy the conditions $\mathbf{I}_1$–$\mathbf{I}_4$ (cf. Step 2). Then (with the notation of loc. cit.)

$$\mathrm{length}_{\mathscr{O}_\chi}\big(\mathrm{Sel}_\varepsilon(K, A_g(\chi))\big) - 2\cdot t^{\varepsilon}_{\bar\chi}(g) = \mathrm{length}_{\mathscr{O}_\chi}\big(\mathrm{Sel}_\varepsilon(K, A_h(\chi))\big) - 2\cdot t^{\varepsilon}_{\bar\chi}(h). \tag{7.22}$$

Proof. We first prove that the dimension of $\mathrm{Sel}_\varepsilon(K, A_{\bar g})$ over $\mathbf{F}_p$ is even:

$$\dim_{\mathbf{F}_p}\big(\mathrm{Sel}_\varepsilon(K, A_{\bar g})\big) \equiv 0 \pmod 2. \tag{7.23}$$

Let $x \in \mathrm{Sel}_\varepsilon(K, A_{\bar g})$ be a nonzero class. Choose an admissible prime ℓ such that

$$v_\ell(x_1) \in H^1_{\mathrm{fin}}(K_{\ell_1}, A_{\bar g}) \cong \mathbf{F}_p$$

is non-zero, which exists by Theorem 3.2 of [BD05]. Let $h \in S_2(N^+, \ell LN^-; \mathbf{F}_p)$ be the ℓ-level raising of $\bar g$. Note that $\mathrm{Sel}_\varepsilon(K, A_{\bar g})$ is identified with $\mathfrak{Sel}^\varepsilon(K, T_{\bar g})$ under the isomorphism $T_{\bar g} \cong A_{\bar g}$ induced by the Weil pairing, viz. $\mathrm{Sel}_\varepsilon(K, A_{\bar g})$ is equal to its own dual Selmer group. (This can either be seen as a special case of Step 5 or, more simply, follows from the discussion in the proof of Step 4 under the current assumptions.) As in the proof of Step 1, Poitou–Tate duality then implies that $\mathrm{Sel}_\varepsilon(K, A_h)$ is equal to $\mathrm{Sel}^\ell_\varepsilon(K, A_{\bar g})$, hence

$$\dim_{\mathbf{F}_p}\big(\mathrm{Sel}_\varepsilon(K, A_h)\big) = \dim_{\mathbf{F}_p}\big(\mathrm{Sel}_\varepsilon(K, A_{\bar g})\big) - 1$$

since $v_\ell(x) \neq 0$. If $\mathrm{Sel}_\varepsilon(K, A_h) \neq 0$ we can apply the same argument after replacing $\bar g$ with h. In this way one constructs a squarefree product $T \in \mathscr{S}_1$ of $\dim_{\mathbf{F}_p}\mathrm{Sel}_\varepsilon(K, A_{\bar g})$ admissible primes such that $\mathrm{Sel}_\varepsilon(K, A_h) = 0$, where $h \in S_2(N^+, TLN^-; \mathbf{F}_p)$ denotes now the T-level raising of $\bar g$. As in the proof of Step 4, the results of Skinner–Urban and Fouquet-Wan then implies that $L(\xi/K, 1) \neq 0$, where ξ is a newform of weight $\Gamma_0(TLN)$ which is congruent to f modulo p. As a consequence

$$-1 = \epsilon_K(TLN) = \epsilon_K(LN^-)\cdot(-1)^{\dim_{\mathbf{F}_p}\left(\mathrm{Sel}_\varepsilon(K, A_{\bar g})\right)},$$

and since by assumption LN^- has an *odd* number of prime divisors, this proves (7.23).

We now show (7.22), which is equivalent to show that the integers m_1 and m_2 in Equation (7.16) are both equal to 0. Preliminarily, note that if $\mathrm{length}_{\mathscr{O}_\chi}\big(\mathrm{Sel}_\varepsilon(K, A_g(\chi))\big) = 2\cdot t^\varepsilon_{\bar\chi}(g)$, then, since $\mathrm{length}_{\mathscr{O}_\chi}\big(\mathrm{Sel}_\varepsilon(K, A_h(\chi))\big) \leqslant 2\cdot t^\varepsilon_{\bar\chi}(h)$ by Step 3, we have $m_1 + m_2 = 0$ (where m_1 and m_2 are defined in Step 3; we also have $\mathrm{length}_{\mathscr{O}_\chi}\big(\mathrm{Sel}_\varepsilon(K, A_h(\chi))\big) = 2\cdot t^\varepsilon_{\bar\chi}(h)$) directly from Equation (7.16). Therefore we assume in the following that $\mathrm{length}_{\mathscr{O}_\chi}\big(\mathrm{Sel}_\varepsilon(K, A_g(\chi))\big) < 2\cdot t^\varepsilon_{\bar\chi}(g)$.

We first show that $m_1 = 0$. By (7.1), $t^\varepsilon_{\bar\chi}(g) < \mathrm{ord}_\chi(p^k)$. Since the $\mathbf{F}_p$-dimension of $\mathrm{Sel}_\varepsilon(K, A_{\bar g})$ is even, combining Step 5 and Nakayama Lemma shows that

$$\mathrm{Sel}_\varepsilon(K, A_g(\chi)) \cong \mathtt{M} \oplus \mathtt{M}.$$

If follows that $\mathrm{length}_{\mathscr{O}_\chi}(\mathtt{M}) < \mathrm{ord}_\chi(p^k)$, and therefore

$$\varpi_\chi^{\mathrm{ord}_\chi(p^k)-1}\cdot \mathrm{Sel}_\varepsilon(K, A_g(\chi)) = 0.$$

We now consider the class $\xi^\varepsilon_{\bar\chi}(\ell_1) - \varpi_\chi^{m_1}\cdot\zeta^\varepsilon_{\bar\chi}(\ell_1)$ in $\mathfrak{Sel}^\varepsilon(K, T_g(\bar\chi))$ appearing in the proof of Step 3. Since $\varpi^{\mathrm{ord}_\chi(p^k)-1}$ kills $\mathrm{Sel}_\varepsilon(K, A_g(\chi))$, and since $\mathrm{Sel}_\varepsilon(K, A_g(\chi))$ and $\mathfrak{Sel}^\varepsilon(K, T_g(\bar\chi))$ are dual to each other, the same is true for $\mathfrak{Sel}^\varepsilon(K, T_g(\bar\chi))$. Therefore we obtain the equality

$$\varpi_\chi^{\mathrm{ord}_\chi(p^k)-1}\cdot\xi^\varepsilon_{\bar\chi}(\ell_1) = \varpi_\chi^{\mathrm{ord}_\chi(p^k)-1+m_1}\cdot\zeta^\varepsilon_{\bar\chi}(\ell_1). \tag{7.24}$$

We now show that the left hand side of this equality is always non-trivial. First, enlarge $\{\ell_1\}$ to a freeing set S as in Section 5.5; then by Proposition 5.13, $\mathfrak{Sel}^\varepsilon_S\big(K, T_g(\chi)\big)$ is free over $\mathscr{O}_\chi/p^k$ of rank $\delta(S)$, the number of prime divisors of S. By Lemma 7.2(3), the class $\bar\xi^\varepsilon_\chi(\ell)$ in $\mathfrak{Sel}^\varepsilon_S(K, T_{\bar g})\otimes_{\mathbf{F}_p}\mathbf{F}_\chi$ is not trivial, therefore ϖ_χ does not divide $\xi^\varepsilon_\chi(\ell)$ and it follows that $\varpi_\chi^{\mathrm{ord}_\chi(p^k)-1}\cdot\xi^\varepsilon_{\bar\chi}(\ell_1) \neq 0$ from the freeness result $\mathfrak{Sel}^\varepsilon_S\big(K, T_g(\chi)\big) \cong (\mathscr{O}_\chi/p^k)^{\delta(S)}$ recalled above. On the other hand, if $m_1 > 0$, then the right hand side of (7.24) is zero, which is a contradiction. Therefore m_1 must be equal to 0.

We now show that $m_2 = 0$ with a similar argument. Consider the class $\xi^\varepsilon_{\bar\chi}(\ell_2) - \varpi_\chi^{m_2}\cdot\zeta^\varepsilon_{\bar\chi}(\ell_2)$ in $\mathfrak{Sel}^\varepsilon_{\ell_1}(K, T_g(\bar\chi))$ appearing in the proof of Step 3. Since $m_1 = 0$, we know that $\partial_{\ell_1}(\xi^\varepsilon_{\bar\chi}(\ell_1))$ generates the image of the residue map $\partial_{\ell_1} : \mathfrak{Sel}^{\varepsilon}_{\ell_1}(K, T_g(\bar{\chi})) \to H^1_{\mathrm{sing}}(K_\ell, T_g(\bar{\chi}))$, and therefore there exists an integer $m_3 \geqslant 0$ such that the class $\partial_{\ell_1}\left(\xi^{\varepsilon}_{\bar{\chi}}(\ell_2) - \varpi^{m_2}_{\chi} \cdot \zeta^{\varepsilon}_{\bar{\chi}}(\ell_2)\right)$ is equal to the class $\varpi^{m_3}_{\chi} \cdot \partial_{\ell_1}(\xi^{\varepsilon}_{\bar{\chi}}(\ell_1))$, i.e.

$$\partial_{\ell_1}\left(\xi^{\varepsilon}_{\bar{\chi}}(\ell_2) - \varpi^{m_2}_{\chi} \cdot \zeta^{\varepsilon}_{\bar{\chi}}(\ell_2) - \varpi^{m_3}_{\chi} \cdot \xi^{\varepsilon}_{\bar{\chi}}(\ell_1)\right) = 0.$$

Therefore by definition the class $\xi^{\varepsilon}_{\bar{\chi}}(\ell_2) - \varpi^{m_2}_{\chi} \cdot \zeta^{\varepsilon}_{\bar{\chi}}(\ell_2) - \varpi^{m_3}_{\chi} \cdot \xi^{\varepsilon}_{\bar{\chi}}(\ell_1)$ belongs to $\mathfrak{Sel}^{\varepsilon}(K, T_g(\bar{\chi}))$. Since this group is annihilated by $\varpi_{\chi}^{\mathrm{ord}_{\chi}(p^k)-1}$ we obtain the equality

$$\varpi_{\chi}^{\mathrm{ord}_{\chi}(p^k)-1} \cdot \xi^{\varepsilon}_{\bar{\chi}}(\ell_2) - \varpi_{\chi}^{\mathrm{ord}_{\chi}(p^k)-1+m_2} \cdot \zeta^{\varepsilon}_{\bar{\chi}}(\ell_2) = \varpi_{\chi}^{\mathrm{ord}_{\chi}(p^k)-1+m_3} \cdot \xi^{\varepsilon}_{\bar{\chi}}(\ell_1).$$

We now suppose *ad absurdum* that $m_2 > 0$. Then the equation above implies

$$\varpi_{\chi}^{\mathrm{ord}_{\chi}(p^k)-1} \cdot \xi^{\varepsilon}_{\bar{\chi}}(\ell_2) = \varpi_{\chi}^{\mathrm{ord}_{\chi}(p^k)-1+m_3} \cdot \xi^{\varepsilon}_{\bar{\chi}}(\ell_1). \tag{7.25}$$

By $\mathbf{I}_4$, $\mathrm{ord}_{\chi}\big(v_{\ell_1}\left(\xi^{\varepsilon}_{\bar{\chi}}(\ell_2)\right)\big) = 0$, and therefore, again using the freeness argument as above, we see that $\varpi_{\chi}^{\mathrm{ord}_{\chi}(p^k)-1} \cdot \xi^{\varepsilon}_{\bar{\chi}}(\ell_2)$ is not trivial. Equation (7.25) then shows that $m_3 = 0$. Therefore, applying v_{ℓ_1} to Equation (7.25), we obtain the equality

$$\varpi_{\chi}^{\mathrm{ord}_{\chi}(p^k)-1} \cdot v_{\ell_1}\left(\xi^{\varepsilon}_{\bar{\chi}}(\ell_2)\right) = \varpi_{\chi}^{\mathrm{ord}_{\chi}(p^k)-1} \cdot v_{\ell_1}\left(\xi^{\varepsilon}_{\bar{\chi}}(\ell_1)\right)$$

By $\mathbf{I}_4$, the left hand side of this equality is not trivial, while by Lemma 7.2(1), the right hand side is trivial, which is a contradiction. Therefore, $m_2 = 0$, concluding the proof of Step 6. □

7.2.7. *Step 7.* Assume that (f, K, p, ε) is not exceptional. Then

$$\mathrm{length}_{\mathscr{O}_\chi}\big(\mathrm{Sel}_{\varepsilon}(K, A_g(\chi))\big) = 2 \cdot t^{\varepsilon}_{\bar{\chi}}(g).$$

Proof. The proof is by induction on $\mathrm{length}_{\mathscr{O}_\chi}\big(\mathrm{Sel}_{\varepsilon}(K, A_g(\chi))\big)$. If $\mathrm{length}_{\mathscr{O}_\chi}\big(\mathrm{Sel}_{\varepsilon}(K, A_g(\chi))\big) = 0$, then the equality follows from Step 4. When $\mathrm{Sel}_{\varepsilon}(K, A_g(\chi))$ is not trivial, choose a pair of $2k$-admissible primes ℓ_1 and ℓ_2 satisfying conditions $\mathbf{I}_1$-$\mathbf{I}_4$, and let h be the $\ell_1\ell_2$-level raising of g. Since $t^{\varepsilon}_{\bar{\chi}}(h) < t^{\varepsilon}_{\bar{\chi}}(g)$, we see from Step 6 that $\mathrm{length}_{\mathscr{O}_\chi}\big(\mathrm{Sel}_{\varepsilon}(K, A_h(\chi))\big)$ is strictly smaller than $\mathrm{length}_{\mathscr{O}_\chi}\big(\mathrm{Sel}_{\varepsilon}(K, A_g(\chi))\big)$, and therefore by the inductive hypothesis $\mathrm{length}_{\mathscr{O}_\chi}\big(\mathrm{Sel}_{\varepsilon}(K, A_h(\chi))\big) = 2t^{\varepsilon}_{\bar{\chi}}(h)$. A further application of the equality in Step 6 implies then the result. □

7.3. Proof of Theorem A in the definite case.

Let $\mathfrak{X}^{\varepsilon}_p(f)$ be the Pontryagin dual of $\mathrm{Sel}_{\varepsilon}(K, \mathbf{A}_f)$, which is a compact torsion Λ-module by Theorem 7.1. Denote $\mathrm{Char}^{\varepsilon}_p(f)$ the characteristic power series of $\mathfrak{X}^{\varepsilon}_p(f)$.

Theorem 7.3 (DAMC)**.** *$(L^{\varepsilon}_p(f)) \subseteq (\mathrm{Char}^{\varepsilon}_p(f))$, with equality in the non-exceptional case.*

Proof. The proof easily follows by combining Theorem 7.1 with Proposition 5.11 and repeating the argument in Mazur–Rubin [MR04, Section 5.3] and Howard [How04, Section 2.2]. □

8. ε-BSD formulas in the indefinite case

Assume that $\mathscr{B}$ is *indefinite* (i.e. $\epsilon_K(N^-) = +1$).

Proposition 8.1. *The compact $\Lambda_{\mathscr{O}}$-module $\mathfrak{Sel}^{\varepsilon}(K, \mathbf{T}_{f,\mathscr{O}})$ is free of finite rank.*

Proof. By Proposition 5.10 (for $\mathfrak{P}$ equal to the augmentation ideal of $\Lambda_{\mathscr{O}}$) and Shapiro's Lemma, the $\Lambda_{\mathscr{O}}$-quotient of G_∞-coinvariants of $\mathfrak{Sel}^{\varepsilon}(K, \mathbf{T}_{f,\mathscr{O}})$ injects into $\mathfrak{Sel}^{\varepsilon}(K, T_{f,\mathscr{O}})$, and therefore is $\mathscr{O}$-free. Thanks to $E_p(K) = 0$, we have $\mathbf{T}^{G_\infty}_{f,\mathscr{O}} = 0$, so the $\Lambda_{\mathscr{O}}$-module $\mathfrak{Sel}^{\varepsilon}(K, \mathbf{T}_{f,\mathscr{O}})$ is torsion free by [PR00, Lemma 1.3.3], hence its $\Lambda_{\mathscr{O}}$-submodule of G_∞-invariants is trivial. The result follows then from a standard argument (*e.g.* [NSW08, Proposition 5.3.19(ii)]). □

8.1. **Λ-adic classes.** We first construct global classes, in a way similar to Section 6.1. Since $\epsilon_K(N^-) = +1$, J_{N^+,N^-} is the Picard variety of the Shimura curve X_{N^+,N^-}. Let $I_f \subset \mathbf{T}_{N^+,N^-}$ denote the kernel of f. Modularity implies that there is an isomorphism of $\mathbf{Z}_p[G_{\mathbf{Q}}]$-modules

$$\pi_f : \mathrm{Ta}_p(J_{N^+,N^-})/I_f \cong T_f,$$

which is unique up to multiplication by a p-adic unit by Hypothesis 1.1(1). For every integer $n \geqslant 0$ define

$$\psi_{f,n} : J_{N^+,N^-}(K_n) \longrightarrow H^1(K_n, \mathrm{Ta}_p(J_{N^+,N^-})/I_f) \cong H^1(K_n, T_f),$$

where the first (resp., second) map is induced by the Kummer map (resp., by π_f). For every point $x \in J_{N^+,N^-}(K_n)$ the class $\psi_{f,n}(x)$ is finite at every prime of K_n dividing p. Moreover, since J_{N^+,N^-} has purely toric reduction at every prime divisor of N^-, Mumford–Tate theory of p-adic uniformisation implies that these classes are ordinary at every such prime. Therefore, we obtain a map

$$\psi_{f,n} : J_{N^+,LN^-}(K_n) \longrightarrow \mathfrak{Sel}(K_n, T_f).$$

Recall the compatible sequence of Heegner points $P_n = P_n(1)$, for $n \geqslant 0$, introduced in Section 2.5 and define

$$\tilde{\kappa}_n = \psi_{f,n}(P_n).$$

8.1.1. *Ordinary case.* Suppose that E has ordinary reduction at p. The classes

$$\kappa_n = \frac{1}{\alpha_p(g)^n}\Big(\tilde{\kappa}_{n-1} - \alpha_p(g) \cdot \tilde{\kappa}_n\Big)$$

belong to $\mathfrak{Sel}(K_n, T_f)$ by the previous discussion, and Equation (2.5) shows that they are norm-compatible. As in Section 6.1, define

$$\kappa_\infty = \varprojlim_n \kappa_n \in \varprojlim_n \mathfrak{Sel}(K_n, T_f) \tag{8.1}$$

where the inverse limit is computed with respect to the canonical norm maps.

8.1.2. *Supersingular case.* Using the freeness result of Proposition 8.1, by the same argument in Section 6.1 one can define classes

$$\tilde{\kappa}_n^\varepsilon \in \mathfrak{Sel}^\varepsilon(K_n, T_f)/\omega_n^\varepsilon$$

such that $\tilde{\omega}_n^{-\varepsilon} \cdot \tilde{\kappa}_n^\varepsilon = \tilde{\kappa}_n$ if p is split in K or p is inert in K and $\varepsilon = -1$ (the non-exceptional case), and $\omega_n^- \cdot \tilde{\kappa}_n^+ = \tilde{\kappa}_n$ if p is inert in K and $\varepsilon = +1$ (the exceptional case). Define $\kappa_n^+ = (-1)^{n/2}\tilde{\kappa}_n^+$ if n is even and $\kappa_n^- = (-1)^{(n-1)/2}\tilde{\kappa}_n^-$ if n is odd. A calculation using Equation (2.5) shows that the classes κ_n^ε are compatible with respect to the canonical projection maps. Define as in Section 6.1

$$\kappa_\infty^\varepsilon = \varprojlim_n \kappa_n^\varepsilon \in \varprojlim_{n \in \mathbf{N}^\varepsilon} \mathfrak{Sel}^\varepsilon(K_n, T_f)/\omega_n^\varepsilon,$$

where $\mathbf{N}^\varepsilon$ is the set of positive integers verifying the condition $(-1)^n = \varepsilon$.

8.2. **Lengths of Selmer groups.** Fix a morphism $\chi : \Lambda \to \mathscr{O}_\chi$ of $\mathbf{Z}_p$-algebras, where as above $\mathscr{O}_\chi$ is the integral closure of $\Lambda/\mathfrak{P}_\chi$, and $\mathfrak{P}_\chi = \ker(\chi)$. Denote

$$\kappa_\chi^\varepsilon \in \mathfrak{Sel}^\varepsilon(K, \mathbf{T}_f) \otimes \mathscr{O}_\chi \stackrel{\text{def}}{=} \mathfrak{Sel}^\varepsilon(K, \mathbf{T}_{f,\mathscr{O}_\chi}) \otimes_{\Lambda_{\mathscr{O}_\chi}} \mathscr{O}_\chi$$

the image of $\kappa_\infty^\varepsilon$ via the canonical map described above, where recall that the tensor product $\otimes_{\Lambda_{\mathscr{O}_\chi}}$ is taken with respect to χ. We assume that $\mathrm{ord}_\chi(\kappa_\chi^\varepsilon)$ is finite. Using that $\mathfrak{Sel}^\varepsilon(K, \mathbf{T}_{f,\mathscr{O}_\chi})$ is $\Lambda_{\mathscr{O}_\chi}$-free by Proposition 8.1, define the integer

$$t_\chi^\varepsilon(f) = \mathrm{ord}_\chi\left(\kappa_\chi^\varepsilon\right) < \infty.$$

For any group p-power torsion group G, let $G_{/\mathrm{div}}$ denote the quotient of G by its maximal p-divisible subgroup.

Theorem 8.2. *Suppose that $t_\chi^\varepsilon(f) < \infty$. Then the $\mathscr{O}_\chi$-corank of $\mathrm{Sel}_\varepsilon(K, A_f(\chi))$ is 1 and we have*

$$\mathrm{length}_{\mathscr{O}_\chi}\left(\mathrm{Sel}_\varepsilon(K, A_f(\chi))_{/\mathrm{div}}\right) \leqslant 2 \cdot \mathrm{length}_{\mathscr{O}_\chi}\left((\mathfrak{Sel}^\varepsilon(K, \mathbf{T}_f) \otimes \mathscr{O}_\chi)/\mathscr{O}_\chi \cdot \kappa_{\bar{\chi}}^\varepsilon\right),$$

and the equality holds if (f, K, p, ε) is not exceptional.

Proof. It follows from the freeness of $\mathfrak{Sel}^\varepsilon(K,\mathbf{T}_{f,\mathscr{O}_\chi})$ that there exists $\tilde\kappa^\varepsilon_\chi$ in $\mathfrak{Sel}^\varepsilon(K,\mathbf{T}_f)\otimes\mathscr{O}_\chi$ such that $\mathrm{ord}_\chi(\tilde\kappa^\varepsilon_\chi)=0$ and $\kappa^\varepsilon_\chi=\varpi_\chi^{t^\varepsilon_\chi(f)}\cdot\tilde\kappa^\varepsilon_\chi$. Define $\xi^\varepsilon_\chi\in\mathfrak{Sel}^\varepsilon(K,T_f(\chi))$ to be the image of $\tilde\kappa^\varepsilon_\chi$ under the (injective) specialization map $\mathbf{s}_\chi:\mathfrak{Sel}^\varepsilon_S(K,\mathbf{T}_f)\otimes\mathscr{O}_\chi\hookrightarrow\mathfrak{Sel}^\varepsilon(K,T_f(\chi))$. We also denote $\kappa^\varepsilon_{\chi,k}$ the image of κ^ε_χ in $\mathfrak{Sel}^\varepsilon(K,T_{f,k}(\chi))$, for all integers $k\geqslant 1$, and $\xi^\varepsilon_{\chi,k}$ the image of ξ^ε_χ in $H^1(K,T_{f,k}(\chi))$. If $k=1$, the element $\xi^\varepsilon_{\chi,1}$ will be denoted $\bar\xi^\varepsilon_\chi$. As before (*cf.* Step 5 in §7.2.5) we have

$$\mathrm{Sel}_\varepsilon(K,A_f(\chi))\cong(\mathscr{K}_\chi/\mathscr{O}_\chi)^s\oplus M_\chi\oplus M_\chi.$$

for some integer s and a finite torsion $\mathscr{O}_\chi$-module M_χ. Choose an integer

$$k>\max\{\mathrm{length}_{\mathscr{O}_\chi}(M_\chi),t^\varepsilon_{\bar\chi}(f)\}.$$

Using [BD05, Theorem 3.2], choose an admissible prime $\ell\in\mathscr{S}_k$ such that $v_\ell(\bar\xi^\varepsilon_{\bar\chi})\neq 0$. Let g be the ℓ-level raising of f. Then since $H^1_{\mathrm{fin}}(K_\ell,T_{f,\mathscr{O}_\chi})$ is a free $\mathscr{O}_\chi$-module of rank 1, and $v_\ell(\bar\xi^\varepsilon_{\bar\chi})\neq 0$, using Proposition 5.5, we have

$$\mathrm{ord}_\chi\left(v_\ell\left(\kappa^\varepsilon_{\bar\chi,k}\right)\right)=\mathrm{ord}_\chi\left(v_\ell\left(\kappa^\varepsilon_{\bar\chi}\right)\right). \tag{8.2}$$

Step 1. Theorem 7.1 for g shows that

$$\mathrm{length}_{\mathscr{O}_\chi}\left(\mathrm{Sel}_\varepsilon(K,A_g(\chi))\right)\leqslant 2\cdot\mathrm{ord}_\chi\left(\mathcal{L}^\varepsilon_g(\bar\chi)\right),$$

with equality in the non-exceptional case, and Theorem 6.5 shows that

$$\mathrm{ord}_\chi\left(v_\ell\left(\kappa^\varepsilon_{\bar\chi,k}\right)\right)=\mathrm{ord}_\chi\left(\mathcal{L}^\varepsilon_g(\bar\chi)\right).$$

Thus, by (8.2) and the injectivity of the map v_ℓ (which follows from $v_\ell(\bar\xi^\varepsilon_{\bar\chi})\neq 0$), we have

$$\mathrm{length}_{\mathscr{O}_\chi}\left(\mathrm{Sel}_\varepsilon(K,A_g(\chi))\right)\leqslant 2\cdot\mathrm{ord}_\chi\left(v_\ell\left(\kappa^\varepsilon_{\bar\chi}\right)\right)=2\cdot\mathrm{length}_{\mathscr{O}_\chi}\left(\mathfrak{Sel}^\varepsilon(K,\mathbf{T}_f)\otimes\mathscr{O}_\chi/\mathscr{O}_\chi\cdot\kappa^\varepsilon_{\bar\chi}\right)$$

with equality in the non-exceptional case.

Step 2. Recall the relaxed Selmer group $\mathrm{Sel}^{(\ell)}_\varepsilon(K,A_{f,k}(\chi))\supseteq\mathrm{Sel}_\varepsilon(K,A_{f,k}(\chi))$, i.e. the set of cohomology classes defined requiring the same conditions as $\mathrm{Sel}_\varepsilon(K,A_{f,k}(\chi))$ at primes different from ℓ, and no condition at ℓ. We claim that

$$\mathrm{Sel}_\varepsilon(K,A_{f,k}(\chi))=\mathrm{Sel}^{(\ell)}_\varepsilon(K,A_{f,k}(\chi)). \tag{8.3}$$

To prove this, let $x\in\mathrm{Sel}^{(\ell)}_\varepsilon(K,A_{f,k}(\chi))$. We have to show that x is in the kernel of the residue map at ℓ. By global class field theory, using the orthogonality of $\mathrm{res}_v(x)$ and $\mathrm{res}_v(\xi^\varepsilon_{\bar\chi,k})$ outside ℓ as in Step 1 of the proof of Theorem 7.1, one then obtains

$$0=\sum_v\left\langle\mathrm{res}_v(x),\mathrm{res}_v\big(\xi^\varepsilon_{\bar\chi,k}\big)\right\rangle_v=\left\langle\mathrm{res}_\ell(x),\mathrm{res}_\ell\big(\xi^\varepsilon_{\bar\chi,k}\big)\right\rangle_\ell=\left\langle\partial_\ell(x),v_\ell\big(\xi^\varepsilon_{\bar\chi,k}\big)\right\rangle_\ell.$$

Since $v_\ell\big(\bar\xi^\varepsilon_{\bar\chi}\big)\neq 0$, and since $\langle-,-\rangle_\ell$ is a perfect pairing, this implies that $\partial_\ell(x)=0$, as was to be shown.

Step 3. We claim that there is an exact sequence:

$$0\longrightarrow\mathrm{Sel}_\varepsilon(K,A_g(\chi))\longrightarrow\mathrm{Sel}_\varepsilon(K,A_{f,k}(\chi))\xrightarrow{v_\ell}H^1_{\mathrm{fin}}(K_\ell,A_{f,k}(\chi))\longrightarrow 0.$$

To show this, first note that

$$\mathrm{Sel}_\varepsilon(K,A_g(\chi))\subseteq\mathrm{Sel}^{(\ell)}_\varepsilon(K,A_g(\chi))=\mathrm{Sel}^{(\ell)}_\varepsilon(K,A_{f,k}(\chi))=\mathrm{Sel}_\varepsilon(K,A_{f,k}(\chi))$$

where the last equality follows from Step 2; this shows the exactness on the left. By definition, the kernel of the map $v_\ell:\mathrm{Sel}_\varepsilon(K,A_{f,k}(\chi))\to H^1_{\mathrm{fin}}(K_\ell,A_{f,k}(\chi))$ is $H^1_{\mathrm{ord}}(K_\ell,A_{f,k}(\chi))$, proving the exactness in the middle. Finally, v_ℓ is surjective because, under the isomorphism $T_{f,k}(\chi)\simeq A_{f,k}(\chi)$, $\xi^\varepsilon_{\bar\chi}$ is a class in $\mathrm{Sel}_\varepsilon(K,A_{f,k}(\chi))$ which satisfies $v_\ell\big(\bar\xi^\varepsilon_{\bar\chi}\big)\neq 0$.

Step 4. From Step 3 we obtain the equality

$$\mathrm{length}_{\mathscr{O}_\chi}\left(\mathrm{Sel}_\varepsilon(K,A_g(\chi))\right)=\mathrm{length}_{\mathscr{O}_\chi}\left(\mathrm{Sel}_\varepsilon(K,A_{f,k}(\chi))\right)-\mathrm{length}_{\mathscr{O}_\chi}\left(\mathscr{O}_\chi/p^k\mathscr{O}_\chi\right)$$

and combinig with Step 1 we get

$$\mathrm{length}_{\mathscr{O}_\chi}\left(\mathrm{Sel}_\varepsilon(K,A_{f,k}(\chi))\right)-\mathrm{length}_{\mathscr{O}_\chi}\left(\mathscr{O}_\chi/p^k\mathscr{O}_\chi\right)\leqslant 2\cdot\mathrm{length}_{\mathscr{O}_\chi}\left(\mathfrak{Sel}^\varepsilon(K,\mathbf{T}_f\otimes\mathscr{O}_\chi)/\mathscr{O}_\chi\cdot\kappa^\varepsilon_{\bar\chi}\right)$$

with equality in the non-exceptional case. Since the left hand side has finite order, bounded independently of k, we see that the $\mathscr{O}_\chi$-corank of $\mathrm{Sel}_\varepsilon(K, A_f(\chi))$ is 1. By the choice of k,

$$\mathrm{length}_{\mathscr{O}_\chi}\left(\mathrm{Sel}_\varepsilon(K, A_{f,k}(\chi))\right) - \mathrm{ord}_\chi\left(\mathscr{O}_\chi/p^k\mathscr{O}_\chi\right) = \mathrm{length}_{\mathscr{O}_\chi}\left(\mathrm{Sel}_\varepsilon(K, A_f(\chi))_{/\mathrm{div}}\right),$$

concluding the proof. □

8.3. **Proof of Theorem A in the indefinite case.** Let $\mathfrak{X}_p^\varepsilon(f)$ be the Pontryagin dual of $\mathrm{Sel}_\varepsilon(K, \mathbf{A}_f)$. Then the compact Λ-module $\mathfrak{X}_p^\varepsilon(f)$ is pseudo-isomorphic to $\Lambda \oplus \mathfrak{M} \oplus \mathfrak{M}$ for a torsion Λ-module $\mathfrak{M}$, supported only on primes of height 1; this follows from Theorem 8.2, the structure results in Step 5 of the proof of Theorem 7.1, and Proposition 5.9. Let $\mathrm{Char}_p^\varepsilon(f)$ be the characteristic ideal of the Λ-module $\mathfrak{M}$.

By Theorem 8.2, $\mathfrak{Sel}^\varepsilon(K, \mathbf{T}_f)$ is a free Λ-module of rank 1 and $\mathfrak{Sel}^\varepsilon(K, \mathbf{T}_f)/\Lambda \cdot \kappa_\infty^\varepsilon$ is a torsion Λ-module. We may then denote $L_p^\varepsilon(f)$ the characteristic power series of $\mathfrak{Sel}^\varepsilon(K, \mathbf{T}_f)/\Lambda \cdot \kappa_\infty^\varepsilon$.

Theorem 8.3 (IAMC)**.** $(L_p^\varepsilon(f)) \subseteq (\mathrm{Char}_p^\varepsilon(f))$, *with equality in the non-exceptional case.*

Proof. As in the definite case, the proof combines Theorem 8.2 with Proposition 5.11 and follows the argument in Mazur–Rubin [MR04, Section 5.2] and Howard [How04, Section 2.2]. □

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9. Proof of Theorems B and C

Fix throughout this section a finite order character $\chi : G_\infty \twoheadrightarrow \mathscr{O}_\chi^\times$ of conductor p^n.

9.1. **Comparison of Selmer groups.** Suppose that p is supersingular. The aim of this section is to compare the discrete Selmer groups $\mathrm{Sel}^\varepsilon(K, A_f(\chi))$ and $\mathrm{Sel}(K, A_f(\chi))$ and the compact Selmer groups $\mathfrak{Sel}^\varepsilon(K, T_f(\chi))$ and $\mathfrak{Sel}(K, T_f(\chi))$. Let $\mathfrak{p} \mid p$ be a prime and fix $k \in \mathbf{N} \cup \{\infty\}$. Set as before $\Psi = K_\mathfrak{p}$, $\Psi_n = K_{n,\mathfrak{p}}$ and $\Psi_\infty = K_{\infty,\mathfrak{p}}$. Let $\mathfrak{P}_\chi = (\mathfrak{p}_\chi)$ be the kernel of the character $\chi : \Lambda_{\mathscr{O}_\chi} \twoheadrightarrow \mathscr{O}_\chi$ obtained from χ, where $\mathfrak{p}_\chi = \gamma - \chi(\gamma)$. We also view χ as a character $\chi : \mathscr{O}_\chi[G_n] \twoheadrightarrow \mathscr{O}_\chi$, whose kernel we still denote by $\mathfrak{P}_\chi = (\mathfrak{p}_\chi)$. To simplify the notation, define

$$\mathbf{E}_{\mathscr{O}_\chi}(\Psi_n)_{\mathrm{div}} = \mathbf{E}(\Psi_n) \otimes_{\mathbf{Z}} (\mathscr{K}_\chi/\mathscr{O}_\chi),$$
$$\mathbf{E}_{\mathscr{O}_\chi}(\Psi_n)_{\pm,\mathrm{div}} = \mathbf{E}(\Psi_n)_\pm \otimes_{\mathbf{Z}} (\mathscr{K}_\chi/\mathscr{O}_\chi).$$

Lemma 9.1. *Let* $\varepsilon = (-1)^n$.

(1) *In the non-exceptional case,* $\frac{(\mathbf{E}_{\mathscr{O}_\chi}(\Psi_n)_{\mathrm{div}})[\mathfrak{P}_\chi]}{(\mathbf{E}_{\mathscr{O}_\chi}(\Psi_n)_{\varepsilon,\mathrm{div}})[\mathfrak{P}_\chi]}$ *is finite and*

$$\mathrm{length}_{\mathscr{O}_\chi}\left(\frac{\left(\mathbf{E}_{\mathscr{O}_\chi}(\Psi_n)_{\mathrm{div}}\right)[\mathfrak{P}_\chi]}{\left(\mathbf{E}_{\mathscr{O}_\chi}(\Psi_n)_{\varepsilon,\mathrm{div}}\right)[\mathfrak{P}_\chi]}\right) = [\Psi : \mathbf{Q}_p] \cdot \mathrm{ord}_\chi(\tilde{\omega}_n^{-\varepsilon}).$$

(2) *In the exceptional case:*
   (a) *if* $n = 0$, *so* $\chi = \mathbf{1}$ *is the trivial character, then*

$$\frac{\left(\mathbf{E}_{\mathbf{Z}_p}(\Psi)_{\mathrm{div}}\right)[(\gamma-1)]}{\left(\mathbf{E}_{\mathbf{Z}_p}(\Psi)_{+,\mathrm{div}}\right)[(\gamma-1)]} = \mathbf{E}(\Psi) \otimes_{\mathbf{Z}_p} \mathbf{Q}_p/\mathbf{Z}_p \cong (\mathbf{Q}_p/\mathbf{Z}_p)^{[\Psi:\mathbf{Q}_p]};$$

   (b) *if* $n \geqslant 2$, *then* $\frac{(\mathbf{E}_{\mathscr{O}_\chi}(\Psi_n)_{\mathrm{div}})[\mathfrak{P}_\chi]}{(\mathbf{E}_{\mathscr{O}_\chi}(\Psi_n)_{+,\mathrm{div}})[\mathfrak{P}_\chi]}$ *is finite and*

$$\mathrm{length}\left(\frac{\left(\mathbf{E}_{\mathscr{O}_\chi}(\Psi_n)_{\mathrm{div}}\right)[\mathfrak{P}_\chi]}{\left(\mathbf{E}_{\mathscr{O}_\chi}(\Psi_n)_{+,\mathrm{div}}\right)[\mathfrak{P}_\chi]}\right) = [\Psi : \mathbf{Q}_p] \cdot \mathrm{ord}_\chi(\omega_n^-).$$

Proof. Suppose first $n = 0$. Then χ is the trivial character, $\mathscr{O}_\chi = \mathbf{Z}_p$ and we suppress the index $\mathscr{O}_\chi$ from the notation, thus writing $\mathbf{E}(\Psi)_{\mathrm{div}}$ for $\mathbf{E}_{\mathbf{Z}_p}(\Psi)_{\mathrm{div}}$ and $\mathbf{E}(\Psi)_{\pm,\mathrm{div}}$ for $\mathbf{E}_{\mathbf{Z}_p}(\Psi)_{\pm,\mathrm{div}}$. If p is split, then $\mathbf{E}(\Psi)_{\mathrm{div}} = \mathbf{E}(\Psi)_{+,\mathrm{div}}$, so the quotient in the statement is trivial; on the other hand, $\tilde{\omega}_0^- = 1$, and the statement is proved. If p is inert, $\mathbf{E}(\Psi)_{\mathrm{div}} = \mathbf{E}(\Psi)_{-,\mathrm{div}}$ and $\mathbf{E}_{\mathscr{O}_\chi}(\Psi)_{+,\mathrm{div}} = 0$, so the quotient in the statement is $\mathbf{E}(\Psi)_{\mathrm{div}}[(\gamma-1)] \cong (\mathbf{Q}_p/\mathbf{Z}_p)^{[\Psi:\mathbf{Q}_p]}$, where the last isomorphism follows from Theorem 5.2.

Suppose $n \geqslant 1$. We first observe that we have an exact sequence:

$$0 \longrightarrow C \longrightarrow \left(\mathbf{E}_{\mathscr{O}_\chi}(\Psi_n)_{\varepsilon,\mathrm{div}}\right)[\mathfrak{P}_\chi] \oplus \left(\mathbf{E}_{\mathscr{O}_\chi}(\Psi_n)_{-\varepsilon,\mathrm{div}}\right)[\mathfrak{P}_\chi] \longrightarrow \left(\mathbf{E}_{\mathscr{O}_\chi}(\Psi_n)_{\mathrm{div}}\right)[\mathfrak{P}_\chi] \longrightarrow 0 \tag{9.1}$$

where $C = 0$ if p is inert in K and $C = \left(\mathbf{E}_{\mathscr{O}_\chi}(\Psi)_{\mathrm{div}}\right)[\mathfrak{P}_\chi]$ if p is split in K; in this exact sequence the second arrow is the map $x \mapsto (x, x)$, and the third arrow is the map $(x, y) \mapsto x - y$. If p is inert in K, it follows from Theorem 5.2 that $\mathbf{E}_{\mathscr{O}_\chi}(\Psi_n)_{\mathrm{div}}$ is the direct sum of $\mathbf{E}_{\mathscr{O}_\chi}(\Psi_n)_{\varepsilon,\mathrm{div}}$ and $\mathbf{E}_{\mathscr{O}_\chi}(\Psi_n)_{-\varepsilon,\mathrm{div}}$, which proves (9.1) (also in the exceptional case). In the split case, it follows again from Theorem 5.2 that

$$\mathbf{E}_{\mathscr{O}_\chi}(\Psi_n)_{\varepsilon,\mathrm{div}} \cap \mathbf{E}_{\mathscr{O}_\chi}(\Psi_n)_{-\varepsilon,\mathrm{div}} = \mathbf{E}_{\mathscr{O}_\chi}(\Psi)_{\mathrm{div}},$$

so we need to show that the exact sequence

$$0 \longrightarrow \mathbf{E}_{\mathscr{O}_\chi}(\Psi)_{\mathrm{div}} \longrightarrow \mathbf{E}_{\mathscr{O}_\chi}(\Psi_n)_{\varepsilon,\mathrm{div}} \oplus \mathbf{E}_{\mathscr{O}_\chi}(\Psi_n)_{-\varepsilon,\mathrm{div}} \longrightarrow \mathbf{E}_{\mathscr{O}_\chi}(\Psi_n)_{\mathrm{div}} \longrightarrow 0$$

remains exact after taking $\mathfrak{P}_\chi$-torsion, so we need to show that the map

$$\left(\mathbf{E}_{\mathscr{O}_\chi}(\Psi_n)_{\varepsilon,\mathrm{div}}\right)[\mathfrak{P}_\chi] \oplus \left(\mathbf{E}_{\mathscr{O}_\chi}(\Psi_n)_{-\varepsilon,\mathrm{div}}\right)[\mathfrak{P}_\chi] \longrightarrow \left(\mathbf{E}_{\mathscr{O}_\chi}(\Psi_n)_{\mathrm{div}}\right)[\mathfrak{P}_\chi]$$

is surjective. The cokernel of this map injects into the quotient $\mathbf{E}_{\mathscr{O}_\chi}(\Psi)_{\mathrm{div}}/\mathfrak{p}_\chi\mathbf{E}_{\mathscr{O}_\chi}(\Psi)_{\mathrm{div}}$, and we need to show that this group is trivial. Since $\mathbf{E}_{\mathscr{O}_\chi}(\Psi)$ is $\mathscr{O}_\chi$-free, it is enough to show that

$$\left(\mathbf{E}_{\mathscr{O}_\chi}(\Psi)/\mathfrak{p}_\chi\mathbf{E}_{\mathscr{O}_\chi}(\Psi)\right) \otimes_{\mathscr{O}_\chi} \mathscr{K}_\chi/\mathscr{O}_\chi = 0. \tag{9.2}$$

Now, $\mathfrak{p}_\chi$ acts on the $\mathscr{O}_\chi$-free module $\mathbf{E}_{\mathscr{O}_\chi}(\Psi)$ as multiplication by $1 - \chi(\gamma)$, and since $\chi(\gamma)$ is a primitive p^n-root of unity, and $n \geqslant 1$, the quotient $\mathbf{E}_{\mathscr{O}_\chi}(\Psi)/\mathfrak{p}_\chi\mathbf{E}_{\mathscr{O}_\chi}(\Psi)$ is finite, and (9.2) follows.

We have therefore a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & 0 & \longrightarrow & \left(\mathbf{E}_{\mathscr{O}_\chi}(\Psi_n)_{\varepsilon,\mathrm{div}}\right)[\mathfrak{P}_\chi] & = & \left(\mathbf{E}_{\mathscr{O}_\chi}(\Psi_n)_{\varepsilon,\mathrm{div}}\right)[\mathfrak{P}_\chi] & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C & \longrightarrow & \left(\mathbf{E}_{\mathscr{O}_\chi}(\Psi_n)_{\varepsilon,\mathrm{div}}\right)[\mathfrak{P}_\chi] \oplus \left(\mathbf{E}_{\mathscr{O}_\chi}(\Psi_n)_{-\varepsilon,\mathrm{div}}\right)[\mathfrak{P}_\chi] & \longrightarrow & \left(\mathbf{E}_{\mathscr{O}_\chi}(\Psi_n)_{\mathrm{div}}\right)[\mathfrak{P}_\chi] & \longrightarrow & 0
\end{array}$$

where C is defined before, and the middle vertical arrow is the map $x \mapsto (x, 0)$. By the snake lemma we obtain an exact sequence

$$0 \longrightarrow C \longrightarrow \left(\mathbf{E}_{\mathscr{O}_\chi}(\Psi_n)_{-\varepsilon,\mathrm{div}}\right)[\mathfrak{P}_\chi] \longrightarrow \frac{\left(\mathbf{E}_{\mathscr{O}_\chi}(\Psi_n)_{\mathrm{div}}\right)[\mathfrak{P}_\chi]}{\left(\mathbf{E}_{\mathscr{O}_\chi}(\Psi_n)_{\varepsilon,\mathrm{div}}\right)[\mathfrak{P}_\chi]} \longrightarrow 0.$$

The Pontryagin dual of the middle term is $\Lambda_{\mathscr{O}_\chi}/(\omega_n^{-\varepsilon}, \mathfrak{p}_\chi)$, because the Pontryagin dual of $\mathbf{E}_{\mathscr{O}_\chi}(\Psi_n)_{-\varepsilon,\mathrm{div}}$ is $\Lambda_{\mathscr{O}_\chi}/(\omega_n^{-\varepsilon})$; since $\Lambda_{\mathscr{O}_\chi}/\mathfrak{P}_\chi \cong \mathscr{O}_\chi$, the length of the middle term is equal to the length of $\mathscr{O}_\chi/\chi(\omega_n^{-\varepsilon})$. Similarly, if p is split, the length of $C = \left(\mathbf{E}_{\mathscr{O}_\chi}(\Psi)_{\mathrm{div}}\right)[\mathfrak{P}_\chi]$ is equal to the length of $\mathscr{O}_\chi/\chi(\gamma - 1)$ and therefore the length of the quotient is equal to the length of $\mathscr{O}_\chi/\chi(\tilde{\omega}_n^{-\varepsilon})$, completing the proof in this case. If p is inert, then C is trivial. If $\varepsilon = -1$ (the non-exceptional case), then $\tilde{\omega}_n^+ = \omega_n^+$, and the length of the last term is equal to the length of $\mathscr{O}_\chi/\chi(\omega_n^+) = \mathscr{O}_\chi/\chi(\tilde{\omega}_n^+)$, while if $\varepsilon = +1$ (the exceptional case) then the length is $\mathscr{O}_\chi/\chi(\omega_n^-)$, completing the proof. □

Proposition 9.2. *In the exceptional case, assume that* $n \neq 0$. *The discrete Selmer groups* $\mathrm{Sel}_\varepsilon(K, A_f(\chi))$ *and* $\mathrm{Sel}(K, A_f(\chi))$ *have the same* $\mathscr{O}_\chi$*-corank. Moreover,*

(1) *If* p *is split in* K *or* p *is inert in* K *and* $\varepsilon = -1$ *(the non-exceptional case),*

$$\mathrm{length}_{\mathscr{O}_\chi}\left(\frac{\mathrm{Sel}(K, A_f(\chi))}{\mathrm{Sel}_\varepsilon(K, A_f(\chi))}\right) = 2 \cdot \mathrm{ord}_\chi(\tilde{\omega}_n^{-\varepsilon}).$$

(2) *If* p *is inert in* K *and* $\varepsilon = +1$ *(the exceptional case),*

$$\mathrm{length}_{\mathscr{O}_\chi}\left(\frac{\mathrm{Sel}(K, A_f(\chi))}{\mathrm{Sel}_+(K, A_f(\chi))}\right) = 2 \cdot \mathrm{ord}_\chi(\omega_n^-).$$

Proof. We have the Poitou–Tate exact sequence

$$0 \longrightarrow \mathrm{Sel}_\varepsilon(K, A_f(\chi)) \longrightarrow \mathrm{Sel}(K, A_f(\chi)) \longrightarrow \prod_{\mathfrak{p}|p} \frac{\left(\mathbf{E}_{\mathscr{O}_\chi}(\Psi)_{\mathrm{div}}\right)[\mathfrak{P}_\chi]}{\left(\mathbf{E}_{\mathscr{O}_\chi}(\Psi_n)_{\varepsilon,\mathrm{div}}\right)[\mathfrak{P}_\chi]} \longrightarrow$$
$$\longrightarrow (\mathfrak{Sel}^\varepsilon(K, T_f(\bar\chi)))^\vee \longrightarrow (\mathfrak{Sel}(K, T_f(\bar\chi)))^\vee \longrightarrow 0.$$

Now $\mathfrak{Sel}^\varepsilon(K, T_f(\bar\chi))$ is $\mathscr{O}_\chi$-free, and therefore $(\mathfrak{Sel}^\varepsilon(K, T_f(\bar\chi)))^\vee$ is p-divisible. If we show that the kernel of the map $(\mathfrak{Sel}^\varepsilon(K, T_f(\bar\chi)))^\vee \to (\mathfrak{Sel}(K, T_f(\bar\chi)))^\vee$ is divisible, then, since the local quotient in the middle of the exact sequence above is finite by Lemma 9.1, we have an exact sequence

$$0 \longrightarrow \mathrm{Sel}_\varepsilon(K, A_f(\chi)) \longrightarrow \mathrm{Sel}(K, A_f(\chi)) \longrightarrow \prod_{\mathfrak{p}|p} \frac{\left(\mathbf{E}_{\mathscr{O}_\chi}(\Psi)_{\mathrm{div}}\right)[\mathfrak{P}_\chi]}{\left(\mathbf{E}_{\mathscr{O}_\chi}(\Psi_n)_{\varepsilon,\mathrm{div}}\right)[\mathfrak{P}_\chi]} \longrightarrow 0$$

and the result follows from Lemma 9.1. So to complete the proof we need to show that the kernel of the (surjective) map

$$(\mathfrak{Sel}^\varepsilon(K, T_f(\bar\chi)))^\vee \longrightarrow\!\!\!\!\!\rightarrow (\mathfrak{Sel}(K, T_f(\bar\chi)))^\vee$$

is divisible. For this, it is enough to show that the cokernel of the (injective) map

$$\mathfrak{Sel}(K, T_f(\bar\chi)) \hookrightarrow \mathfrak{Sel}^\varepsilon(K, T_f(\bar\chi)) \tag{9.3}$$

is torsion free. Let $x \in \mathfrak{Sel}^\varepsilon(K, T_f(\bar\chi))$ and let $M \geqslant 1$ be such that $\varpi_\chi^M \cdot x \in \mathfrak{Sel}(K, T_f(\bar\chi))$; to conclude that the cokernel of the map (9.3) is torsion-free, it is then enough to show that $x \in \mathfrak{Sel}(K, T_f(\bar\chi))$. Since $\varpi_\chi^M \cdot x \in \mathfrak{Sel}(K, T_f(\bar\chi))$, we have (writing $\langle -, -\rangle$ for the local Tate pairing $\langle -, -\rangle_{\mathfrak{P}_\chi,\mathfrak{p}}$ to simplify the notation) $\langle \mathrm{res}_\mathfrak{p}(\varpi_\chi^M \cdot x), y\rangle = 0$ for all $y \in H^1_{\mathrm{fin}}(K_\mathfrak{p}, A_f(\chi))$, and since $\langle \mathrm{res}_\mathfrak{p}(\varpi_\chi^M \cdot x), y\rangle = \langle \mathrm{res}_\mathfrak{p}(x), \varpi_\chi^M \cdot y\rangle$, we also have $\langle \mathrm{res}_\mathfrak{p}(x), \varpi_\chi^M \cdot y\rangle = 0$ for all $y \in H^1_{\mathrm{fin}}(K_\mathfrak{p}, A_f(\chi))$. Recall that $H^1_{\mathrm{fin}}(K_\mathfrak{p}, A_f(\chi))$ is co-free over $\mathscr{O}_\chi$ by Proposition 5.4, hence $H^1_{\mathrm{fin}}(K_\mathfrak{p}, A_f(\chi))$ is ϖ_χ-divisible. So the function $y \mapsto \langle \mathrm{res}_\mathfrak{p}(x), y\rangle$ is zero on $H^1_{\mathrm{fin}}(K_\mathfrak{p}, A_f(\chi))$ and therefore x belongs to $\mathfrak{Sel}(K, T_f(\bar\chi))$, concluding the proof. □

9.2. **Proof of Theorem B.** Recall $L_{p,n}(f) = \mathcal{L}_{f,n} \cdot (\mathcal{L}_{f,n})^\iota \in \mathbf{Z}_p[G_n]$. By Remark 1.4 we may assume $n \neq 0$ in the exceptional case.

Step 1. We first show that

$$\mathrm{length}_{\mathscr{O}_\chi}\big(\mathrm{Sel}(K, A_f(\chi))\big) \leqslant \mathrm{ord}_\chi\big(\chi(L_{p,n}(f))\big)$$

with equality in the non-exceptional case. Take $g = f_k$ for $L = \emptyset$ in Theorem 7.1. By Theorem 7.1, $\mathrm{Sel}_\varepsilon(K, A_{f,k}(\chi))$ is finite, of order bounded independently of k, so $\mathrm{Sel}_\varepsilon(K, A_f(\chi))$ is finite, and by Proposition 9.2 the Selmer group $\mathrm{Sel}(K, A_f(\chi))$ is finite. Let $t^\varepsilon_{\bar\chi}(f) = \mathrm{ord}_{\bar\chi}\big(\bar\chi(\mathcal{L}^\varepsilon_f)\big)$ and choose

$$k > \max\left\{\mathrm{length}_{\mathscr{O}_\chi}\big(\mathrm{Sel}_\varepsilon(K, A_f(\chi))\big), t^\varepsilon_{\bar\chi}(f), \mathrm{ord}_\chi(\tilde\omega_n^{-\varepsilon})\right\}.$$

For such a k, we have $\mathrm{Sel}_\varepsilon(K, A_f(\chi)) \cong \mathrm{Sel}_\varepsilon(K, A_{f,k}(\chi))$, and, by Proposition 9.2,

- $\mathrm{length}_{\mathscr{O}_\chi}(\mathrm{Sel}(K, A_f(\chi))) = \mathrm{length}_{\mathscr{O}_\chi}(\mathrm{Sel}_\varepsilon(K, A_{f,k}(\chi))) + 2 \cdot \mathrm{ord}_\chi(\tilde\omega_n^{-\varepsilon})$ in the non-exceptional case;
- $\mathrm{length}_{\mathscr{O}_\chi}(\mathrm{Sel}(K, A_f(\chi))) = \mathrm{length}_{\mathscr{O}_\chi}(\mathrm{Sel}_\varepsilon(K, A_{f,k}(\chi))) + 2 \cdot \mathrm{ord}_\chi(\omega_n^{-\varepsilon})$ in the exceptional case.

By [BD96, Proposition 2.6], $(\mathcal{L}^\varepsilon_f)^\iota = \pm\gamma_\infty\mathcal{L}^\varepsilon_f$, for some $\gamma_\infty \in G_\infty$, and therefore $t^\varepsilon_\chi(f) = t^\varepsilon_{\bar\chi}(f)$. We thus have $\mathrm{ord}_\chi\big(\chi(L^\varepsilon_p(f))\big) = 2 \cdot t^\varepsilon_{\bar\chi}(f)$. If p is split in K or p is inert in K and $\varepsilon = -1$ (non-exceptional case) we have $\tilde\omega_n^{-\varepsilon} \cdot \mathcal{L}^\varepsilon_{f,n} \equiv \pm\mathcal{L}_{f,n}$ modulo ω_n, and therefore

$$\begin{aligned}\mathrm{length}_{\mathscr{O}_\chi}(\mathrm{Sel}(K, A_f(\chi))) &= \mathrm{length}_{\mathscr{O}_\chi}(\mathrm{Sel}_\varepsilon(K, A_{f,k}(\chi))) + 2 \cdot \mathrm{ord}_\chi(\tilde\omega_n^{-\varepsilon}) \\ &= 2 \cdot \mathrm{ord}_\chi\left(\mathcal{L}^\varepsilon_{f,n}(\bar\chi)\right) + 2 \cdot \mathrm{ord}_\chi(\tilde\omega_n^{-\varepsilon}) \\ &= 2 \cdot \mathrm{ord}_\chi\left(\mathcal{L}_{f,n}(\bar\chi)\right),\end{aligned}$$

where the second equality follows from Theorem 7.1. If p is inert in K and $\varepsilon=+1$ (exceptional case), we have $\omega_n^-\cdot\mathcal{L}_{f,n}^+\equiv\pm\mathcal{L}_{f,n}$ modulo ω_n, and therefore again by Theorem 7.1

$$\begin{aligned}\mathrm{length}_{\mathscr{O}_\chi}\left(\mathrm{Sel}(K,A_f(\chi))\right)&=\mathrm{length}_{\mathscr{O}_\chi}\left(\mathrm{Sel}_\varepsilon(K,A_{f,k}(\chi))\right)+2\cdot\mathrm{ord}_\chi(\omega_n^-)\\&\leqslant 2\cdot\mathrm{ord}_\chi\left(\mathcal{L}_{f,n}^+(\bar\chi)\right)+2\cdot\mathrm{ord}_\chi(\omega_n^-)\\&\leqslant 2\cdot\mathrm{ord}_\chi\left(\mathcal{L}_{f,n}(\bar\chi)\right).\end{aligned}$$

Step 2. We now use explicit formulas for special values to conclude the proof of Theorem B. Recall that Gross's formula gives the equality

$$\frac{L(E/K,\chi,1)}{\Omega}C_\chi=\chi(L_p(f))$$

where $C_\chi=u^2\sqrt{|D_K|}p^n$ with $u=\sharp(\mathcal{O}_{p^n}^\times)/2$, and Ω is Gross's period, defined in [Vat03, Lemma 2.5]. We refer to [Gro87, Proposition 7.7], [Vat03, §2.3] and [CST14, Theorem 1.2] for this result; in particular, in the notation of [CST14], $\Omega=\frac{8\pi^2\langle f,f\rangle_{\Gamma_0(N)}}{\langle\phi,\phi\rangle}$, where $\langle f,f\rangle_{\Gamma_0(N)}$ denotes Petersson inner product, ϕ is the (p-adically normalized) Jacquet–Langland lift of f to the definite quaternion algebra of discriminant N^- which we use to define $L_p(f)$, and $\langle\phi,\phi\rangle$ is the height pairing defined in loc. cit. ; see also [CH18, Theorem 3.11]. From Theorem A and Gross's formula we see that $\mathrm{Sel}(K,A_f(\chi))$ is finite if and only if $L(E/K,\chi,1)\neq 0$ and

$$\mathrm{length}_{\mathscr{O}_\chi}\left(\mathrm{Sel}(K,A_f(\chi))\right)\leqslant\mathrm{ord}_\chi\left(\frac{L(E/K,\chi,1)\cdot C_\chi}{\Omega}\right)$$

with equality in the non-exceptional case, which gives Theorem B for $C=\Omega/C_\chi$.

9.3. **Proof of Theorem C.** As noted in Remark 1.4 we may assume that $n\neq 0$ in the exceptional case. We define the *regulator*

$$\mathrm{Reg}_\chi(E/K)=\frac{h_{\mathrm{NT}}(P_{\bar\chi})}{2\cdot\mathrm{length}_{\mathscr{O}_\chi}\left(\mathfrak{Sel}(K,T_f(\chi))/\mathscr{O}_\chi\cdot\tilde\kappa_{\bar\chi}\right)}$$

and the *Shafarevich–Tate group*

$$\text{Ш}(K,A_f(\chi))=\mathrm{Sel}(K,A_f(\chi))_{/\mathrm{div}}.$$

Step 1. We first show that

$$\mathrm{length}_{\mathscr{O}_\chi}\left(\mathrm{Sel}(K,A_f(\chi))_{/\mathrm{div}}\right)\leqslant\mathrm{length}_{\mathscr{O}_\chi}\left(\mathfrak{Sel}(K,T_f(\chi))/\mathscr{O}_\chi\cdot\tilde\kappa_{\bar\chi})\right)$$

with equality in the non-exceptional case. Combining Theorem 8.2 and Proposition 9.2, if p is split in K or p is inert in K and $\varepsilon=-1$ (non-exceptional case) we have

(9.4)
$$\mathrm{length}_{\mathscr{O}_\chi}\left(\mathrm{Sel}(K,A_f(\chi))_{/\mathrm{div}}\right)=2\cdot\mathrm{length}_{\mathscr{O}_\chi}\left(\left(\mathfrak{Sel}^\varepsilon(K,\mathbf{T}_{f,\mathscr{O}_\chi})\otimes_{\Lambda_{\mathscr{O}_\chi}}\Lambda_{\mathscr{O}_\chi}/\mathfrak{P}_{\bar\chi}\right)/\left(\mathscr{O}_\chi\cdot\kappa_{\bar\chi}^\varepsilon\right)\right)+2\cdot\mathrm{ord}_\chi(\tilde\omega_n^{-\varepsilon}).$$

From (9.6), using Proposition 5.10 and Proposition 9.2(2), we obtain

$$\mathrm{length}_{\mathscr{O}_\chi}\left(\mathrm{Sel}(K,A_f(\chi))_{/\mathrm{div}}\right)=2\cdot\mathrm{length}_{\mathscr{O}_\chi}\left(\mathfrak{Sel}(K,T_f(\chi))/\left(\mathscr{O}_\chi\cdot\kappa_{\bar\chi}^\varepsilon\right)\right)+2\cdot\mathrm{ord}_\chi(\tilde\omega_n^{-\varepsilon}).\tag{9.5}$$

Now $\tilde\kappa_{\bar\chi}=\chi(\tilde\omega_n^{-\varepsilon})\cdot\kappa_{\bar\chi}^\varepsilon$, and the result follows. If p is inert in K and $\varepsilon=+1$ (the exceptional case), combining Theorem 8.2 and Proposition 9.2, we have

(9.6)
$$\mathrm{length}_{\mathscr{O}_\chi}\left(\mathrm{Sel}(K,A_f(\chi))_{/\mathrm{div}}\right)\leqslant 2\cdot\mathrm{length}_{\mathscr{O}_\chi}\left(\left(\mathfrak{Sel}^\varepsilon(K,\mathbf{T}_{f,\mathscr{O}_\chi})\otimes_{\Lambda_{\mathscr{O}_\chi}}\Lambda_{\mathscr{O}_\chi}/\mathfrak{P}_{\bar\chi}\right)/\left(\mathscr{O}_\chi\cdot\kappa_{\bar\chi}^+\right)\right)+2\cdot\mathrm{ord}_\chi(\omega_n^-).$$

From (9.6), using Proposition 5.10 and Proposition 9.2(2), we obtain

$$\mathrm{length}_{\mathscr{O}_\chi}\left(\mathrm{Sel}(K,A_f(\chi))_{/\mathrm{div}}\right)\leqslant 2\cdot\mathrm{length}_{\mathscr{O}_\chi}\left(\mathfrak{Sel}(K,T_f(\chi))/\left(\mathscr{O}_\chi\cdot\kappa_{\bar\chi}^+\right)\right)+2\cdot\mathrm{ord}_\chi(\omega_n^-).\tag{9.7}$$

Now $\tilde\kappa_{\bar\chi}=\chi(\omega_n^{-\varepsilon})\cdot\kappa_{\bar\chi}^\varepsilon$, and the result follows.

Step 2. We now use Gross-Zagier formulas to conclude the proof of Theorem C. By the Gross-Zagier formula we know that

$$L'(E/K,\chi,1)=\frac{8\pi^2\langle f,f\rangle_{\Gamma_0(N)}}{\deg(\phi)C_\chi}\cdot h_{\mathrm{NT}}(P_{\bar\chi})$$

where $\deg(\phi)$ is the degree of a modular parametrization $\phi : X_0(N) \to E$, and, with the same notation introduced in §9.2, $C_\chi = u^2\sqrt{|D_K|}p^n$ with $u = \sharp(\mathcal{O}_{p^n}^\times)/2$ and $\langle f, f\rangle_{\Gamma_0(N)}$ denotes the Petersson inner product. We refer to [GZ86, Theorem 6.3] and [CST14, Theorem 1.1] for the result in this form; see also [YZZ13, Theorem 1.3.1], [Zha01, Theorem 1.2.1], [Zha02, Theorem 8.1]. Combing Step 1 with the definitions of $\mathrm{Reg}_\chi(E/K)$ and $\text{Ш}(K, A_f(\chi))$ introduced above and the Gross–Zagier formula we obtain

$$\begin{aligned}\mathrm{length}_{\mathscr{O}_\chi}(\text{Ш}(K, A_f(\chi))) &= \mathrm{length}_{\mathscr{O}_\chi}\left(\mathrm{Sel}(K, A_f(\chi))_{/\mathrm{div}}\right)\\ &\leqslant 2\cdot \mathrm{length}_{\mathscr{O}_\chi}\left(\mathfrak{Sel}(K, T_f(\chi))/\mathscr{O}_\chi\cdot\tilde{\kappa}_{\bar\chi}\right)\\ &\leqslant \mathrm{ord}_\chi\left(\frac{L'(E/K,\chi,1)\deg(\phi)C_\chi}{8\pi^2\langle f,f\rangle_{\Gamma_0(N)}\cdot \mathrm{Reg}_\chi(E/K)}\right)\end{aligned}$$

with equality in the non-exceptional case, which gives Theorem C for $C = 8\pi^2\langle f, f\rangle_{\Gamma_0(N)}/(\deg(\phi)C_\chi)$.

10. Statements and declarations

- On behalf of all authors, the corresponding author states that there is no conflict of interest.
- Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

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