Asymptotic equivalence of conservative VaR- and ES-based capital charges

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Abstract

We show that the conservative estimate of the Value-at-Risk (VaR) for the sum of d random losses with given identical marginals and finite mean is equivalent to the corresponding conservative estimate of the Expected Shortfall (ES), in the limit as the number of risks becomes arbitrarily large. Examples of interest in quantitative risk management show that the equivalence holds also for relatively small and inhomogeneous risk portfolios. When the individual random losses have infinite first moment, we show that VaR can be arbitrarily large with respect to the corresponding VaR estimate for comonotonic risks if the risk portfolio is large enough.

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1. Motivation and preliminaries

Under the Advanced Measurement Approach (AMA) within the Basel II (becoming Basel III) Accord, banks must set aside a regulatory capital in order to offset their annual risk exposures. In the case of operational risk, the risk capital is typically calculated as the Value-at-Risk (VaR) at a high confidence level for an aggregate loss random variable L having the form

$$L = \sum_{i=1}^{d} L_i,$$

where $L_1, \ldots, L_d$ correspond to the loss random variables for different business lines and/or risk types, over a fixed time period T. The VaR of the aggregate loss L, calculated at a probability level $\alpha \in (0, 1)$, is the α-quantile of its distribution, defined as

$$\mathrm{VaR}_\alpha(L) = F_L^{-1}(\alpha) = \inf\{x \in \mathbb{R} : F_L(x) \geq \alpha\},$$

where $F_L(x) = P(L \leq x)$ is the distribution function of L. Typical values used in the practice of operational risk within Basel II are $\alpha = 0.99, 0.995, 0.999$ and $T = 1$ year. The Solvency II Accord designed by the EU Commission sets $\alpha = 0.995$ and $T = 1$ year.

In the recent years VaR has become the most popular risk measure in financial risk measurement. However, a number of shortcomings have been identified with VaR, including the fact that VaR is not a coherent risk measure (in that it fails the subadditivity criterion) and its inability to capture the impact of low probability events. The recent crisis brought up the question whether VaR is still suitable as a benchmark risk metric and a recent document from the Basel Commitee (see [4]) officially candidates Expected Shortfall as a natural alternative for quantifying not only the low frequency but also the severity of extreme events.

If the loss random variable L satisfies $\mathbb{E}[|L|] < \infty$, the Expected Shortfall (ES) computed at the confidence level $\alpha \in (0, 1)$ is defined as

$$\mathrm{ES}_\alpha(L) = \frac{1}{1-\alpha} \int_\alpha^1 \mathrm{VaR}_q(L)\, dq.$$

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In the literature, there exist several risk measures similar and under some conditions equivalent to ES. For continuous random variables we have for instance that ES is equivalent to the Conditional Tail Expectation (CTE) and Tail Value-at-Risk (TVaR) risk metrics; see [1] and [19].

Unlike VaR, ES accounts for tail risk in a more comprehensive manner considering both the size and likelihood of losses above a certain threshold (e.g. the 0.995-quantile). ES is also a more pessimist risk measure, in the sense that

$$\mathrm{ES}_\alpha(L) \geq \mathrm{VaR}_\alpha(L), \text{ for all } \alpha \in (0, 1).$$

Since ES overcomes the main disadvantages of VaR, the Basel Committee proposes in [4] *the use of ES for the internal models-based approach and also intends to determine risk weights for the standardised approach using an ES methodology*.

The computation of both $\mathrm{VaR}_\alpha(L)$ and $\mathrm{ES}_\alpha(L)$ requires the knowledge of the joint distribution function of the risk portfolio $(L_1, \dots L_d)$. This generally requires a d-variate dataset for the past occurred losses, which is often not available. Typically, only the marginal distribution functions F_i of the L_i can be statistically estimated. Therefore, it is natural to ask for a conservative estimate of $\mathrm{VaR}_\alpha(L)$ and $\mathrm{ES}_\alpha(L)$ when the marginal distributions of the univariate risks L_i are given but no dependence information is known about the joint portfolio $(L_1, \dots, L_d)$. For a fixed $\alpha \in (0, 1)$, and a set of d marginal distributions $F_1, \dots, F_d$ we define the worst-case VaR and the worst-case ES for the aggregate position L as

$$\overline{\mathrm{VaR}}_\alpha(L) = \sup \{\mathrm{VaR}_\alpha(L_1 + \cdots + L_d); L_i \sim F_i, 1 \leq i \leq d\}, \tag{1.1}$$

$$\overline{\mathrm{ES}}_\alpha(L) = \sup \{\mathrm{ES}_\alpha(L_1 + \cdots + L_d); L_i \sim F_i, 1 \leq i \leq d\}. \tag{1.2}$$

Note that the two definitions (1.1) and (1.2) are particular cases of (6.11) in [15]. $\overline{\mathrm{VaR}}_\alpha(L)$ and $\overline{\mathrm{ES}}_\alpha(L)$ represent the largest estimates of $\mathrm{VaR}_\alpha(L)$ and $\mathrm{ES}_\alpha(L)$, respectively, if only the marginal distributions of the random variables $L_1, \dots, L_d$ are known.

Equivalent definitions of $\overline{\mathrm{VaR}}_\alpha(L)$ and $\overline{\mathrm{ES}}_\alpha(L)$ can be given in terms of copulas. A copula C is a d-dimensional distribution function (df) on $[0, 1]^d$ with uniform marginals. Given a copula C and d univariate marginals $F_1, \dots, F_d$, one can always define a df F on $\mathbb{R}^d$ having these marginals by

$$F(x_1, \dots, x_d) = C(F_1(x_1), \dots, F_d(x_d)), \ x_1, \dots, x_d \in \mathbb{R}. \tag{1.3}$$

Sklar's Theorem (see Theorem 5.3 in [15]) states conversely that we can always find a copula C coupling the marginals F_i of a fixed joint distribution F through the above expression (1.3). For continuous marginal dfs, this copula is unique. Sklar's Theorem implies that the copula C of a multivariate distribution F contains all the dependence information of F. As a consequence, one can reformulate the two definitions (1.1) and (1.2) as optimization problems over $\mathfrak{C}_d$, the set of all d-dimensional copulas:

$$\overline{\mathrm{VaR}}_\alpha(L) = \sup \left\{\mathrm{VaR}_\alpha(L_1^C + \cdots + L_d^C) : C \in \mathfrak{C}_d\right\}, \tag{1.4}$$

$$\overline{\mathrm{ES}}_\alpha(L) = \sup \left\{\mathrm{ES}_\alpha(L_1^C + \cdots + L_d^C) : C \in \mathfrak{C}_d\right\}, \tag{1.5}$$

where $(L_1^C, \dots, L_d^C)$ denotes a random vector with given marginals $F_1, \dots, F_d$ and copula C, i.e. with joint distribution function given by $C(F_1, \dots, F_d)$.

Any copula C satisfies the so-called Fréchet bounds

$$\max \left\{\sum_{i=1}^d u_i - d + 1, 0\right\} \leq C(u_1, \dots, u_d) \leq \min\{u_1, \dots, u_d\},$$

for all $u_1, \dots, u_d \in [0, 1]$. The sharp upper Fréchet bound $M(u_1, \dots, u_d) = \min\{u_1, \dots, u_d\}$ is the so-called *comonotonic* copula, which represents perfect positive dependence among the risks. In fact, a risk vector $(L_1, \dots, L_d)$ has copula M if and only if its marginal risks are all almost surely (a.s.) increasing functions of a common random factor. The lower Fréchet bound $W(u_1, \dots, u_d) = [u_1 + \cdots + u_d - d + 1]^+$ is also sharp but it is a well-defined copula only in dimension $d = 2$. In this case, it is called the *countermonotonic* copula and represents perfect negative dependence between two risks. A risk vector (L_1, L_2) has copula W if and only if its marginal risks are a.s. decreasing functions of each other.

Fréchet bounds are important for finding optimal solutions in many optimization problems of interest in quantitative risk management. For instance, it is well known that the upper bound in (1.5) (or, equivalently, in (1.2)) is attained when the risk portfolio $(L_1^C, \dots, L_d^C)$ has comonotonic copula $C = M$, i.e.

$$\overline{\mathrm{ES}}_\alpha(L) = \mathrm{ES}_\alpha(L_1^M + \cdots + L_d^M) = \mathrm{ES}_\alpha(L_1) + \cdots + \mathrm{ES}_\alpha(L_d); \tag{1.6}$$

see for instance [12] and Section 1 in [17].

The computation of $\overline{\mathrm{VaR}}_\alpha(L)$ is more challenging as it is well known that $\mathrm{VaR}_\alpha(L_1^M + \cdots + L_d^M)$ is not the solution of (1.1); see [8]. There are several techniques available in the literature in order to compute $\overline{\mathrm{VaR}}_\alpha(L)$ for a fixed set of marginal distributions. To cite the most relevant: the dual bound technique illustrated in [18] for the analytical computation of the worst VaR of the sum of identically distributed risks; the rearrangement algorithm described in [9] for the numerical computation of the worst VaR in the general case of inhomogeneous portfolios. In this paper we are mostly interested in the asymptotic properties of the conservative VaR- and ES-based capital charges (1.1) and (1.2) when the dimension of the risk portfolio d increases. This case is of particular interest as internal models built by financial institutions to fulfil the Basel and Solvency regulatory frameworks are using a rapidly growing number d of risk factors.

The main result presented in this paper is that when the marginal losses L_i are nonnegative, identically distributed random variables with finite mean, the worst-case estimate of VaR for the aggregate position L is equivalent to the corresponding worst-case ES estimate, in the limit as $d \to \infty$. Examples of interest in quantitative risk management show that this equivalence holds also for relatively small risk portfolios, and for inhomogeneous portfolios. Roughly speaking, if one uses a conservative rule for capital reserving, a VaR-based capital charge will be equivalent to a ES-based reserve for the dimensions d typically used within quantitative risk management. From this worst-case dependence perspective, our main result suggests that the problem of choice between VaR and ES risk metrics might be less relevant than expected.

Under specific assumptions of interdependence among the risks some related results have been discussed in [2] and in [14], where asymptotics for the ratio $\mathrm{ES}_\alpha(L)/\mathrm{VaR}_\alpha(L)$ are given in the limit as $\alpha \to 1$ under some specific assumptions of interdependence among the risks. The comparison between ES and VaR (worst-case) risk measures has a more general domain of application within the context of optimal capital allocations as for instance recently studied in [11; 6; 3; 7].

Our main result also implies asymptotics for the so-called diversification benefit introduced in [5]. We show that for a risk portfolio of d identically distributed, positive random variables with finite mean, the worst-possible diversification benefit approaches a negative constant in the limit as $d \to \infty$. This equivalently means that the worst-case VaR of a sufficiently large portfolio is by a positive factor greater than one larger than the VaR obtained when the risks are comonotonic.

The case of infinite mean is different. When the random losses L_i have infinite first moment, we show that the worst VaR can be arbitrarily large with respect to the corresponding VaR estimate for comonotonic risks if the risk portfolio is large enough. We establish this result in the case of general inhomogeneous distributions.

2. Asymptotics for conservative capital charges: the finite mean case

Throughout this section we assume that the loss random variables $L_1, \dots, L_d$ have a common distribution function F with tail function $\overline{F} = 1 - F$. We also assume that F has a nonnegative support (the L_i's are interpreted as random losses). It is straightforward to see that our results can be extended to the case in which the distribution F assumes also negative values but is bounded from below.

Since $\mathrm{ES}_\alpha(L) \geq \mathrm{VaR}_\alpha(L)$ for any random loss L, for any portfolio $(L_1, \dots, L_d)$ we have that

$$\frac{\overline{\mathrm{ES}}_\alpha(L_1 + \cdots + L_d)}{\overline{\mathrm{VaR}}_\alpha(L_1 + \cdots + L_d)} \geq 1.$$

The main result of this paper is to show that for portfolios of identically distributed risks, the above inequality becomes an equality in the limit as $d \to \infty$. Formally, we prove that if $L_i \sim F, 1 \le i \le d$, we have

$$\lim_{d\to\infty} \frac{\overline{\mathrm{ES}}_\alpha(L_1 + \cdots + L_d)}{\overline{\mathrm{VaR}}_\alpha(L_1 + \cdots + L_d)} = 1. \tag{2.1}$$

This means that the worst-possible ES for the sum of random losses with marginal distributions identical to F is equivalent to the corresponding worst-possible VaR, in the limit as $d \to \infty$.

In order to prove (2.1), it is useful to study the behavior of the maximum tail probability of $L_1 + \cdots + L_d$ evaluated at s, i.e. to determine

$$M(s) = \sup\left\{P(L_1 + \cdots + L_d \ge s); L_i \sim F, 1 \le i \le d\right\}.$$

[8] show that for any nonnegative distribution F and any $s \in \mathbb{R}$

$$M(s) \le D(s) = d \inf_{0\le t<s/d} \min\left\{\frac{\int_t^{s-(d-1)t} \overline{F}(x)\,dx}{s-dt}, 1\right\}. \tag{2.2}$$

Note that under the notation used by [8] we have $m_+(s) = 1 - M(s)$. Under some extra assumptions, [18] prove that the bound in (2.2) is sharp. For a fixed confidence level $\alpha \in (0,1)$, let $\mathfrak{F}_\alpha$ denote the set of nonnegative, unbounded continuous distribution having a positive density f which is decreasing on the interval $(F^{-1}(\alpha), \infty)$. Note that the distributions in $\mathfrak{F}_\alpha$ cover a large domain of applications in quantitative risk management, where $\mathrm{ES}_\alpha(L)$ and $\mathrm{VaR}_\alpha(L)$ are typically calculated for high quantiles α and positive, unbounded loss random variables with a ultimately decreasing density.

Proposition 2.1 ([18]). *Assume that $F \in \mathfrak{F}_\alpha$. For a fixed real threshold s, suppose that it is possible to find $t^* < s/d$ such that*

$$D(s) = \inf_{0\le t<s/d} \frac{d\int_t^{s-(d-1)t} \overline{F}(x)\,dx}{s-dt} = \frac{d\int_{t^*}^{s-(d-1)t^*} \overline{F}(x)\,dx}{s-dt^*},$$

and $t^* \ge F^{-1}(1 - D(s))$. Then we have that

$$M(s) = D(s).$$

[18] show that the extra assumptions of Proposition 2.1 are satisfied by the distributions commonly used in applications of quantitative risk management, for a threshold s large enough. For a distribution $F \in \mathfrak{F}_\alpha$ with finite mean μ, we define the function $h : [\mu, \infty) \to [0, \infty)$, where $h(s)$, $s \ge \mu$ is the unique solution of the equation

$$\mathbb{E}[X|X \ge h(s)] = h(s) + \frac{\int_{h(s)}^\infty \overline{F}(x)\,dx}{\overline{F}(h(s))} = s. \tag{2.3}$$

Note that if $h(s) = F^{-1}(\alpha)$, for any random variable $X \sim F$ we obtain from Lemma 2.16 in [15] that

$$\mathbb{E}[X|X \ge F^{-1}(\alpha)] = \mathrm{ES}_\alpha(X),$$

implying that

$$h^{-1}(F^{-1}(\alpha)) = \mathrm{ES}_\alpha(X). \tag{2.4}$$

The key idea to prove our main result is to study the behavior of the maximum tail probability of $L_1+\cdots+L_d$ evaluated at the threshold ds. The following lemma is needed to prove the main result of our paper.

Lemma 2.2. *Assume that $(L_d, d \in \mathbb{N})$ is an infinite sequence of random variables identically distributed as F, where $F \in \mathfrak{F}_\alpha$ has finite first moment μ. Then, for any $s \ge \mu$ we have*

$$\limsup_{d\to\infty} M(ds) \le \overline{F}(h(s)).$$

Proof. Note that $0 \le h(s) < s$, for any $s \ge \mu$. From (2.2) it follows that

$$M(ds) \le \inf_{0 \le t < s} \frac{\int_t^{d(s-t)+t} \overline{F}(x)\, dx}{s-t} \le \frac{\int_{h(s)}^{d(s-h(s))+h(s)} \overline{F}(x)\, dx}{s-h(s)}.$$

For $d \to \infty$ we have using (2.3):

$$\limsup_{d\to\infty} M(ds) \le \frac{\int_{h(s)}^{\infty} \overline{F}(x)\, dx}{s-h(s)} = \overline{F}(h(s)). \quad \square$$

Theorem 2.3. *Fix $\alpha \in (0,1)$ and assume that $(L_d, d \in \mathbb{N})$ is an infinite sequence of random variables identically distributed as F, where $F \in \mathfrak{F}_\alpha$ has finite first moment μ. Suppose that for a fixed threshold $s \ge \mu$, the extra assumptions of Proposition 2.1 are satisfied for any $d \ge d_0$. Then*

$$\lim_{d\to\infty} \frac{\overline{\mathrm{VaR}}_\alpha(L_1 + \cdots + L_d)}{d} = \mathrm{ES}_\alpha(L_1), \tag{2.5}$$

and consequently

$$\lim_{d\to\infty} \frac{\overline{\mathrm{VaR}}_\alpha(L_1 + \cdots + L_d)}{\overline{\mathrm{ES}}_\alpha(L_1 + \cdots + L_d)} = 1.$$

Proof. For any $d \ge d_0$, let $t_d = \arg\inf_{0\le t<s} f(t,d)$, where

$$f(t,d) = \frac{\int_t^{d(s-t)+t} \overline{F}(x)\, dx}{s-t}.$$

From Proposition 2.1 we have that

$$M(ds) = \inf_{0\le t<s} f(t,d) = f(t_d, d). \tag{2.6}$$

We prove that $\limsup_{d\to\infty} t_d < s$. Assume, on the contrary, that $\limsup_{d\to\infty} t_d = s$. Then it is possible to find a subsequence t_{n_d} for which one of the following four cases occurs:

1. $\lim_{d\to\infty} d(s - t_{n_d}) = 0$. In this case, we have that

$$\lim_{d\to\infty} f(t_{n_d}, d) = \lim_{d\to\infty} \frac{\int_{t_{n_d}}^{d(s-t_{n_d})+t_{n_d}} \overline{F}(x)\, dx}{s - t_{n_d}} = \lim_{d\to\infty} d\,\overline{F}(s) = \infty;$$

2. $\lim_{d\to\infty} d(s - t_{n_d}) = K > 0$. In this case, we have that

$$\lim_{d\to\infty} f(t_{n_d}, d) = \lim_{d\to\infty} \frac{d \int_s^{s+K} \overline{F}(x)\, dx}{K} = \infty;$$

3. $\lim_{d\to\infty} d(s - t_{n_d}) = \infty$. In this case, we have that

$$\lim_{d\to\infty} f(t_{n_d}, d) = \lim_{d\to\infty} \frac{\int_s^{\infty} \overline{F}(x)\, dx}{s - t_{n_d}} = \infty;$$

4. $\lim_{d\to\infty} d(s - t_{n_d})$ does not exist. In this case, there exists a subsequence t'_{n_d} of t_{n_d} such that $t'_{n_d} \to s$ and one of the three cases 1.-3. above is fulfilled for t'_{n_d}. As a consequence we obtain that

$$\limsup_{d\to\infty} f(t_{n_d}, d) = \infty.$$

The above cases 1.-4. are in contrast with (2.6) as we know from Lemma 2.2 that $\limsup_{d\to\infty} M(ds)$ is finite. As a consequence, there exist some positive ϵ, independent of d, such that $\limsup_{d\to\infty} t_d = s-\epsilon$. For $d \geq d_0$, it follows therefore that

$$M(ds) = \inf_{0\leq t\leq s-\epsilon} f(t,d).$$

Using the notation

$$f(t) = \frac{\int_t^\infty \overline{F}(x)\,dx}{s-t},$$

we have that

$$\lim_{d\to\infty} \sup_{0\leq t\leq s-\epsilon} \left| f(t,d) - f(t) \right| = \lim_{d\to\infty} \sup_{0\leq t\leq s-\epsilon} \left| \frac{\int_t^{d(s-t)+t} \overline{F}(x)\,dx}{s-t} - \frac{\int_t^\infty \overline{F}(x)\,dx}{s-t} \right|$$
$$= \lim_{d\to\infty} \sup_{0\leq t\leq s-\epsilon} \left| \frac{\int_{d(s-t)+t}^\infty \overline{F}(x)\,dx}{s-t} \right| \leq \lim_{d\to\infty} \frac{\int_{(d-1)\epsilon+s}^\infty \overline{F}(x)\,dx}{\epsilon} = 0,$$

where the last inequality follows from the finiteness of $\mu = \int_0^\infty \overline{F}(x)\,dx$. As a result, the sequence $f(t,d)$ converges uniformly to $f(t)$ and we finally obtain that

$$\lim_{d\to\infty} M(ds) = \lim_{d\to\infty} \inf_{0\leq t\leq s-\epsilon} f(t,d) = \inf_{0\leq t\leq s-\epsilon} \lim_{d\to\infty} f(t,d) = \inf_{0\leq t\leq s-\epsilon} f(t).$$

Recalling the definition of the function h given in (2.3), it is elementary to check first and second order conditions for $f(t)$ and show that

$$\lim_{d\to\infty} M(ds) = \inf_{0\leq t\leq s-\epsilon} f(t) = \inf_{0\leq t\leq s-\epsilon} \frac{\int_t^\infty \overline{F}(x)\,dx}{s-t} = \frac{\int_{h(s)}^\infty \overline{F}(x)\,dx}{s-h(s)} = \overline{F}(h(s)),$$

from which it directly follows that

$$\lim_{d\to\infty} M\left(d\,h^{-1}(F^{-1}(\alpha)\right) = 1-\alpha. \tag{2.7}$$

From (2.7), using the definition of VaR and (2.4), we obtain equivalently that

$$\lim_{d\to\infty} \frac{\overline{\mathrm{VaR}}_\alpha(L_1+\cdots+L_d)}{d} = h^{-1}(F^{-1}(\alpha)) = \mathrm{ES}_\alpha(L_1).$$

Using (1.6) one obtains $\overline{\mathrm{ES}}_\alpha(L_1+\cdots+L_d) = d\,\mathrm{ES}_\alpha(L_1)$, and hence the theorem. □

Though the approximation

$$\overline{\mathrm{ES}}_\alpha(L_1+\cdots+L_d) \overset{d\to\infty}{\simeq} \overline{\mathrm{VaR}}_\alpha(L_1+\cdots+L_d),$$

is proved to hold in the limit as $d\to\infty$, applications show that the equivalence between conservative ES and VaR estimates holds for relatively small portfolios. In Table 1 we give estimates for $\overline{\mathrm{ES}}_\alpha(L)$ and $\overline{\mathrm{VaR}}_\alpha(L)$, computed at different quantile levels α, for the sum of d Pareto(2) marginals. We recall that a Pareto distribution with tail exponent $\theta > 0$ has tail function $\overline{F}(x) = (1+x)^{-\theta}$, $x > 0$. The VaR estimates are computed via the analytical method described in [18]. The equivalence of capital charges is already visible for $d = 10$. For portfolios of lighter tailed risks, as the LogNormal marginals represented in Table 2, the equivalence of capital charges holds even for small portfolios of $d = 3$ risks. Analogous behavior can be traced for distributions typically used in quantitative risk management.

Additionally, we provide two numerical examples showing that the equivalence between conservative ES- and VaR-based capital charges seems to hold also for portfolios of inhomogeneous marginals, for which an analytical tool to compute $\overline{\mathrm{VaR}}_\alpha(L)$ is not available yet. In Table 3 and Table 4 we give estimates for $\overline{\mathrm{ES}}_\alpha(L)$ and $\overline{\mathrm{VaR}}_\alpha(L)$, computed at different quantile levels α, for the sum of the marginals of the following portfolios:

- *Portfolio A*, in which the marginal distributions $F_i(x) = 1-(1+x)^{-\theta_i}$ are of Pareto type with tail coefficients $\theta_{3k+i} = i+1$, $1\leq i\leq 3$, for $k = 0,\ldots,d/3-1$ (d is a multiple of 3).

d	$\overline{\mathrm{VaR}}_{0.995}\left(\sum_{i=1}^{d} L_i\right)$	$\overline{\mathrm{ES}}_{0.995}\left(\sum_{i=1}^{d} L_i\right)$	ratio	d	$\overline{\mathrm{VaR}}_{0.999}\left(\sum_{i=1}^{d} L_i\right)$	$\overline{\mathrm{ES}}_{0.999}\left(\sum_{i=1}^{d} L_i\right)$	ratio
3	66.2820	81.8528	1.2349	3	151.9194	186.7367	1.2292
10	258.3281	272.8430	1.0562	10	589.9999	622.4555	1.0550
50	1350.0000	1364.2140	1.0105	50	3080.4950	3112.2780	1.0103
100	2714.2490	2728.4270	1.0052	100	6192.8530	6224.5550	1.0051

Table 1: Ratio between the worst ES and the worst VaR for the sum of d Pareto(2) marginals.

d	$\overline{\mathrm{VaR}}_{0.995}\left(\sum_{i=1}^{d} L_i\right)$	$\overline{\mathrm{ES}}_{0.995}\left(\sum_{i=1}^{d} L_i\right)$	ratio	d	$\overline{\mathrm{VaR}}_{0.999}\left(\sum_{i=1}^{d} L_i\right)$	$\overline{\mathrm{ES}}_{0.999}\left(\sum_{i=1}^{d} L_i\right)$	ratio
3	399.0155	420.5341	1.0539	3	640.0679	668.7629	1.0448
10	1398.8790	1401.7800	1.0021	10	2225.8490	2229.2100	1.0015
50	7008.8790	7008.9020	1.0000	50	11146.0300	11146.0500	1.0000
100	14017.8000	14017.8000	1.0000	100	22292.1000	22292.1000	1.0000

Table 2: Ratio between the worst ES and the worst VaR for the sum of d LogNormal(2, 1) marginals.

d	$\overline{\mathrm{VaR}}_{0.995}\left(\sum_{i=1}^{d} L_i\right)$	$\overline{\mathrm{ES}}_{0.995}\left(\sum_{i=1}^{d} L_i\right)$	ratio	d	$\overline{\mathrm{VaR}}_{0.999}\left(\sum_{i=1}^{d} L_i\right)$	$\overline{\mathrm{ES}}_{0.999}\left(\sum_{i=1}^{d} L_i\right)$	ratio
3	29.30	39.07	1.3334	3	59.87	82.74	1.3820
9	106.42	117.21	1.1014	9	222.34	248.23	1.1164
30	379.79	390.70	1.0287	30	801.38	827.43	1.0325
99	1278.42	1289.33	1.0085	99	2704.50	2730.50	1.0096

Table 3: Ratio between the worst ES and the worst VaR for the sum of the marginals in Portfolio A.

d	$\overline{\mathrm{VaR}}_{0.995}\left(\sum_{i=1}^{d} L_i\right)$	$\overline{\mathrm{ES}}_{0.995}\left(\sum_{i=1}^{d} L_i\right)$	ratio	d	$\overline{\mathrm{VaR}}_{0.999}\left(\sum_{i=1}^{d} L_i\right)$	$\overline{\mathrm{ES}}_{0.999}\left(\sum_{i=1}^{d} L_i\right)$	ratio
3	114.70	150.49	1.3120	3	186.49	237.33	1.2726
9	432.97	451.47	1.0427	9	687.09	711.98	1.0362
30	1502.45	1504.91	1.0016	30	2370.39	2373.26	1.0012
99	4966.10	4966.19	1.0000	99	7831.72	7831.75	1.0000

Table 4: Ratio between the worst ES and the worst VaR for the sum of the marginals in Portfolio B.

- *Portfolio B*, in which F_{3k+1} = Pareto(4), F_{3k+2} = LogN(2, 1), F_{3k+3} = Exp(1), for $k = 0, \ldots, d/3 - 1$ (d is a multiple of 3).

For these inhomogeneous portfolios, estimates for $\overline{\mathrm{ES}}_\alpha(L)$ are still available analytically via (1.6), while estimates for $\overline{\mathrm{VaR}}_\alpha(L)$ are computed up to the second decimal digit via the Rearrangement Algorithm as described in [9]. Similar results for the inhomogeneous case can also be found in [17, Table 3]. The examples indicate that our result on the asymptotic equivalence of worst case VaR and ES risk metrics is robust and also seems to be valid in more general inhomogeneous frameworks.

3. Implications for the diversification benefit

Our limit result (2.5) implies asymptotics for the so-called diversification benefit introduced in [5]. The *diversification benefit* for the aggregate loss $L = \sum_{i=1}^{d} L_i$ is defined as

$$D_\alpha(d) = 1 - \frac{\mathrm{VaR}_\alpha\left(\sum_{i=1}^{d} L_i\right)}{\sum_{i=1}^{d} \mathrm{VaR}_\alpha(L_i)}. \tag{3.1}$$

The diversification benefit $D_\alpha(d)$ measures the ratio between the total risk of the portfolio and the sum of risks across the marginal random losses. The diversification benefit depends on the selected level of α, and on the heaviness of the tail of the marginal distributions. A diversification benefit between 0 and 1 indicates that positive diversification effects occur in the portfolio, meaning that the aggregate position L is less risky, the risk being measured via VaR, than the sum of the marginal exposures. This is the typical situation occurring in elliptically distributed risk portfolios, for which it is well known that $D_\alpha(d) \in [0,1]$; see [15, Theorem 6.8]. The case $D_\alpha(d) = 0$ (no diversification) occurs when the risks L_i are *comonotonic*, and $\mathrm{VaR}_\alpha(L) = \sum_{i=1}^d \mathrm{VaR}_\alpha(L_i)$; see [15, Proposition 6.15].

This no-diversification, maximum correlation case is often considered as highly conservative based on the assumption that the random variables L_i are inhomogeneous in nature and cannot be maximally correlated in practice. Contrary to this common sense, the total risk for a financial institution may in general exceed the sum of risks across the individual marginals. Negative diversification benefits are obtained in the case of heavy-tailed and/or skew marginals and/or marginals coupled by a non-elliptical copula; see [16]. This effect occurs even when the losses are independent. For instance, if the random losses $L_1, \dots, L_d$ are independent and each L_i is distributed like a symmetric θ-stable distribution F_i, when $\theta < 1$ we have that

$$\mathrm{VaR}_\alpha(L_1 + \cdots + L_d) = d^{1/\theta}\,\mathrm{VaR}_\alpha(L_1) > d\,\mathrm{VaR}_\alpha(L_1) = \mathrm{VaR}_\alpha(L_1^M + \cdots + L_d^M);$$

see [14]. The non-coherence of VaR as a measure of risk is a consequence of this effect.

A direct interpretation of the diversification benefit can be obtained by rewriting the definition (3.1) as:

$$\mathrm{VaR}_\alpha\left(\sum_{i=1}^d L_i\right) = (1 - D_\alpha(d))\left(\sum_{i=1}^d \mathrm{VaR}_\alpha(L_i)\right). \tag{3.2}$$

From (3.2) it is clear that a negative diversification benefit will imply a superadditive risk within the portfolio. In this sense it is clear what we mean with the term *negative diversification*: having $D_\alpha(d) = 1 - k, k > 1$ means that the aggregate VaR of the portfolio is k times the sum of the marginal VaRs. Analogously to what we have done for the worst-case ES and VaR metrics, for $\alpha \in (0,1)$ and d marginal distributions $F_1, \dots, F_d$ we define the *worst diversification benefit* for L as

$$\overline{D}_\alpha(d) = \inf\left\{1 - \frac{\mathrm{VaR}_\alpha\left(\sum_{i=1}^d L_i\right)}{\sum_{i=1}^d \mathrm{VaR}_\alpha(L_i)}; L_i \sim F_i, 1 \le i \le d\right\} = 1 - \frac{\overline{\mathrm{VaR}}_\alpha\left(\sum_{i=1}^d L_i\right)}{\sum_{i=1}^d \mathrm{VaR}_\alpha(L_i)}.$$

For random losses $L_1, \dots, L_d$ identically distributed as F, the expression for $\overline{D}_\alpha(d)$ simplifies to

$$\overline{D}_\alpha(d) = 1 - \frac{\overline{\mathrm{VaR}}_\alpha\left(\sum_{i=1}^d L_i\right)}{d\,\mathrm{VaR}_\alpha(L_1)},$$

and Theorem 2.3 implies that

$$\lim_{d\to\infty} \overline{D}_\alpha(d) = 1 - \frac{\mathrm{ES}_\alpha(L_1)}{\mathrm{VaR}_\alpha(L_1)}.$$

Since $\mathrm{ES}_\alpha(L_1) \ge \mathrm{VaR}_\alpha(L_1)$ holds always true, under the assumption of Theorem 2.3 we have that the worst diversification benefit goes in the limit to a negative constant $1 - d_\alpha$, where we set

$$d_\alpha = \frac{\mathrm{ES}_\alpha(L_1)}{\mathrm{VaR}_\alpha(L_1)}.$$

Recalling (3.2), this equivalently means that the VaR of a homogeneous risk portfolio could be d_α times larger than the VaR in the comonotonic portfolio. Formally, if $L_i \sim F, 1 \le i \le d$ one obtains:

$$\overline{\mathrm{VaR}}_\alpha(L_1 + \cdots + L_d) \overset{d\to\infty}{\simeq} d_\alpha\,\mathrm{VaR}_\alpha(L_1^M + \cdots + L_d^M). \tag{3.3}$$

Table 5 implies values of the factor d_α in the range $d_\alpha \in [1.4, 11.2]$ for Pareto distributions having tail exponent θ varying between 4 and 1.1. It is also evident from the figures in Table 5 that d_α settles down to a limit when $\alpha \to 1^-$. From (5.2) in [14] we have indeed that if L_1 has a Pareto distribution with tail exponent $\theta > 1$, then

$$\lim_{\alpha\to 1^-} d_\alpha = \lim_{\alpha\to 1^-} \frac{\mathrm{ES}_\alpha(L_1)}{\mathrm{VaR}_\alpha(L_1)} = \frac{\theta}{\theta - 1}.$$

α	$\theta = 1.1$	$\theta = 1.5$	$\theta = 2$	$\theta = 3$	$\theta = 4$
0.99	11.154337	3.097350	2.111111	1.637303	1.487492
0.995	11.081599	3.060242	2.076091	1.603135	1.454080
0.999	11.018773	3.020202	2.032655	1.555556	1.405266

Table 5: Values for the constant d_α for Pareto(θ) distributions.

In Figure 1 we plot the quantity $\overline{D}_\alpha(d)$ as a function of d for Pareto distributions having different tail exponents. For a Pareto distribution a smaller exponent θ corresponds to a heavier tail; for $\theta \leq 1$, the Pareto distribution has infinite first moment. Figure 1 clearly shows that the quantity $\overline{D}_\alpha(d)$ settles down to the limit $1 - d_\alpha$ fairly fast for all the Pareto distributions under study. However, the heavier the tail of the Pareto the slower the convergence. In Figure 2 we confront the behavior of $\overline{D}_\alpha(d)$ for Pareto, Gamma and LogNormal distributions. Again, the heaviness of the tail seems to play a crucial role in determining the speed of convergence of $\overline{D}_\alpha(d)$.

4. Asymptotics for VaR in the infinite mean case

In this section we drop the assumption of having identically distributed random variables and we show that if the random losses L_i have infinite mean, the worst-case VaR capital can be arbitrarily large with respect to $\sum_{i=1}^d \mathrm{VaR}_\alpha(L_i)$, the VaR for comonotonic risks. Formally, we state that

$$\lim_{d\to\infty} \frac{\overline{\mathrm{VaR}}_\alpha(L_1 + \cdots + L_d)}{\sum_{i=1}^d \mathrm{VaR}_\alpha(L_i)} = \infty. \tag{4.1}$$

This result marks the huge difference in risk diversification between the finite- and infinite-mean worlds of distribution functions. Similar benchmark results can be found in [16] and [14].

Theorem 4.1. *Assume that $(L_d, d \in \mathbb{N})$ is an infinite sequence of random variables and fix $\alpha \in (0,1)$. Assume that F_i, the distribution of L_i, satisfies*

$$\lim_{x\to\infty} \inf_{i\geq 1} x\,\overline{F}_i(x) = \infty, \tag{4.2}$$

and, for some constant c,

$$F_i^{-1}(\alpha) \leq c, \;\; i \geq 1. \tag{4.3}$$

Then

$$\lim_{d\to\infty} \frac{\overline{\mathrm{VaR}}_\alpha(L_1 + \cdots + L_d)}{\sum_{i=1}^d \mathrm{VaR}_\alpha(L_i)} = \infty.$$

Proof. For notational simplicity we give the proof only in the case of continuous marginal distributions F_i. The proof for the general case is similar. [13, (3.4)] established that

$$A(x) := \sup\{P(\max\{L_1, \ldots, L_d\} \geq x); L_i \sim F, 1 \leq i \leq d\} = \min\{1, \overline{G}(x)\}, \text{ for all } x \in \mathbb{R}, \tag{4.4}$$

where $\overline{G}(x) := \sum_{i=1}^d \overline{F}_i(x)$. The supremum in (4.4) is attained by a simple recursively defined dependence structure called *maximal dependence*. This implies the inequality

$$M(x) \geq A(x) \geq \min\{1, \overline{G}(x)\}. \tag{4.5}$$

Now denote $y_d := \overline{G}^{-1}(1-\alpha)$, where for $\beta \in (0,1)$ we set $\overline{G}^{-1}(\beta) = \inf\{x \in \mathbb{R} : \overline{G}(x) \leq \beta\}$. From (4.5) and continuity of the F_i's, we have that

$$M(y_d) \geq \min\{1, \overline{G}(y_d)\} = 1 - \alpha.$$

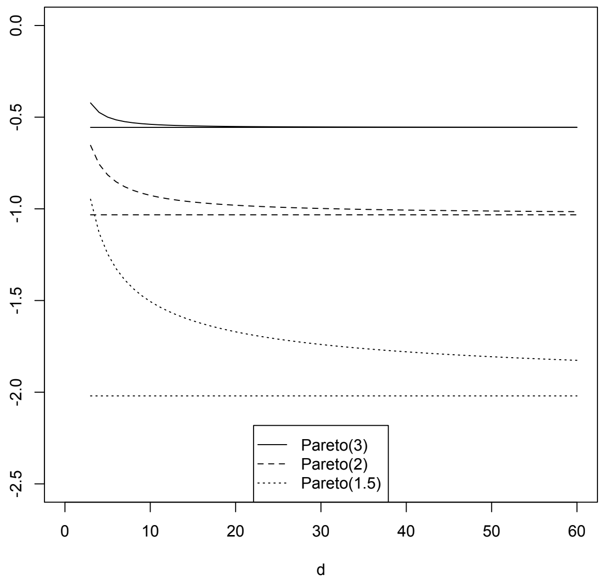

Figure 1: Plot of the function $\overline{D}_\alpha(d)$ versus the dimensionality d for different homogeneous risk portfolios having Pareto marginals and $\alpha = 0.999$. The straight lines represents the corresponding limit constant $(1 - d_{0.999})$.

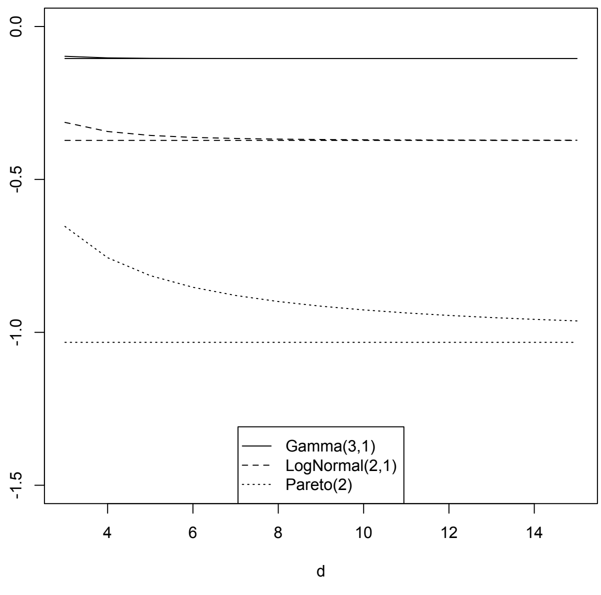

Figure 2: The same as Figure 1 for different homogeneous risk portfolios having Gamma, LogNormal and Pareto marginals.

d	$\sum_{i=1}^d \mathrm{VaR}_{0.995}(L_i)$	$\overline{\mathrm{VaR}}_{0.995}\left(\sum_{i=1}^d L_i\right)$	ratio	d	$\sum_{i=1}^d \mathrm{VaR}_{0.999}(L_i)$	$\overline{\mathrm{VaR}}_{0.999}\left(\sum_{i=1}^d L_i\right)$	ratio
3	377.32	463.73	1.2290	3	2193.06	2458.23	1.1209
9	1131.96	3502.01	3.0938	9	6579.17	20323.17	3.0890
30	3773.20	20788.17	5.5094	30	21930.57	122718.50	5.5958
99	12451.55	99600.49	7.9990	99	72370.90	590636.20	8.1612

Table 6: Ratio between the worst VaR (lower bound) and the sum of marginal VaRs for the sum of Pareto type marginals with tail coefficients $\theta_{3k+1} = 0.9$, $\theta_{3k+2} = 2$, $\theta_{3k+3} = 3$, $k = 0, \dots, d/3 - 1$. The lower bound on the worst VaR has been computed via the Rearrangement Algorithm as described in [9].

As a consequence, we obtain that

$$\overline{\mathrm{VaR}}_\alpha(L_1 + \cdots + L_d) \geq y_d,$$

implying, using (4.3), that

$$\frac{\overline{\mathrm{VaR}}_\alpha(L_1 + \cdots + L_d)}{\sum_{i=1}^d \mathrm{VaR}_\alpha(L_i)} \geq \frac{y_d}{\sum_{i=1}^d \mathrm{VaR}_\alpha(L_i)} \geq \frac{y_d}{d \max_{1\leq i\leq d} \mathrm{VaR}_\alpha(L_i)} \geq \frac{y_d}{d\,c}. \tag{4.6}$$

Since $y_d = \overline{G}^{-1}(1-\alpha)$, we have

$$1 - \alpha = \overline{G}(y_d) = \sum_{i=1}^d \overline{F}_i(y_d) = \frac{\sum_{i=1}^d y_d \overline{F}_i(y_d)}{y_d} \geq \frac{d\ \min_{1\leq i\leq d} y_d \overline{F}_i(y_d)}{y_d}, \tag{4.7}$$

for all d. By (4.2), we have $\min_{1\leq i\leq d} y_d \overline{F}_i(y_d) \to \infty$. It follows from (4.7) that d/y_d converges to zero. Using (4.6) we conclude that

$$\lim_{d\to\infty} \frac{\overline{\mathrm{VaR}}_\alpha(L_1 + \cdots + L_d)}{\sum_{i=1}^d \mathrm{VaR}_\alpha(L_i)} \geq \lim_{d\to\infty} \frac{y_d}{d\,c} = \infty. \quad \square$$

Remark 4.2. *We remark the following points about Theorem 4.1.*

1. *Condition* (4.2) *is a bit stronger than the assumption of an infinite first moment for all the marginal distributions F_i. In the case $\overline{F}_i(x) = (1+x)^{-1}$, $1 \leq i \leq d$, we have infinite first moment but* $\lim_{x\to\infty} x\overline{F}_i(x) = 1$. *It is possible to construct in this case a coupling $(L_1, \dots, L_d)$ such that* (4.1) *holds. It is not clear from the construction whether Theorem 4.1 holds under the assumption of infinite mean only.*

2. *In practice, Theorem 4.1 applies to the distributions F_i with infinite mean typically used in quantitative risk management. For example, the Theorem is valid under the assumption that the tail function F_i is regularly varying with tail index $0 < \theta_i < 1$, for all $i \geq 1$, and $\sup_{i\geq 1} \theta_i < 1$. For the definition of regularly varying distributions we refer for instance to [10].*

3. *Condition* (4.3) *is in particular satisfied in the case the L_i's are identically distributed, or if they assume only a finite set of possible marginal models. In the homogeneous case where $F_i = F, 1 \leq i \leq d$, condition* (4.2) *reduces to* $\lim_{x\to\infty} x\overline{F}(x) = \infty$.

4. *Condition* (4.2) *can be relaxed to*

$$\lim_{x\to\infty} \inf_{i\in J} x\,\overline{F}_i(x) = \infty,$$

where, for some fixed $\epsilon > 0$, $J \subset \mathbb{N}$ satisfies $\#(J \cap \{1, \dots, n\}) \geq \epsilon n, n \geq 1$. As an illustrative example, in Table 6 we give an estimate (from below) of the ratio between the worst VaR and the sum of marginal VaRs for a portfolio of Pareto(0.9,2,3)-distributed marginal random variables.

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