

Macroscopic bound entanglement in thermal graph states

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Abstract. We address the presence of bound entanglement in strongly interacting spin systems at thermal equilibrium. In particular, we consider thermal graph states composed of an arbitrary number of particles. We show that for a certain range of temperatures no entanglement can be extracted by means of local operations and classical communication, even though the system is still entangled. This is found by harnessing the independence of the entanglement in some bipartitions of such states with the system's size. Specific examples for one- and two-dimensional systems are given. Our results thus prove the existence of thermal bound entanglement in an arbitrary large spin system with finite-range local interactions.

Contents

1. Introduction	2
2. Thermal graph states and dephased graph states	3
3. Appearance of multipartite bound entanglement in thermal graph states	5
3.1. The linear cluster with equal couplings	5
3.2. The 2D square cluster with equal couplings	7
3.3. Unequal Hamiltonian couplings	8
4. Conclusion	10
Acknowledgments	10
Appendix. Thermal graph state as a dephased graph state for arbitrary couplings	11
References	11

1. Introduction

The application of tools recently developed in the context of quantum information theory to problems historically native from the area of many-body physics has helped us gain new insight about collective quantum phenomena [1]. In particular, the characterization of the entanglement properties of ground and thermal states of strongly interacting spin Hamiltonians provides a description of these systems from a novel and alternative perspective. Also, in general, it gives us information about how hard it is to simulate their dynamical and statical properties with classical resources [2]. In addition, such characterization is crucial to determine when these systems can in turn assist as resources in some given quantum-information processing task. For all these reasons it is important to classify and characterize standard many-body models in terms of their entanglement properties.

However, the entanglement characterization in multiparticle systems turns out to be formidably hard. On one hand, the calculation of truly multipartite entanglement measures is in general extremely difficult, even for the pure-state case [3, 4]. On the other hand, for realistic many (and especially macroscopically many)-body systems, the interaction with its surrounding environment can additionally never be neglected and mixed states have to be necessarily taken into account. Unfortunately though, the characterization of mixed-state entanglement is to date very poorly developed even for the general bipartite case. One of the most frequent and important types of such interactions—the one on which we will focus here—takes place when the system is embedded in a thermal bath at temperature T and reaches thermal equilibrium with it, a process called thermalization. This process typically causes the system to lose quantum coherence, gain entropy and in most cases also lose entanglement.

An important step forwards in the characterization of mixed-state entanglement was to recognize two different types of entanglement: distillable and bound entanglement [5]. An entangled state ρ is said to be *distillable* if it is possible, by means of local operations and classical communication (LOCC), to obtain from ρ (or, more precisely, several copies thereof) pure-state entanglement. Entangled states for which this task is impossible are said to be *bound-entangled*. Bound-entangled states were for some time believed to be useless for quantum information processing. Nevertheless, they are nowadays known to be useful in some practical situations [6].

Historically, most examples of bound entangled states have been provided without relying on any operational recipe (see e.g. [4]). Yet this kind of entanglement has been recently found to arise in natural processes, in particular thermalization [7]–[11] and other dynamical decoherence processes [11]. For thermal states, the presence of bound entanglement was found in chains of harmonic oscillators in the thermodynamic limit, that is for an arbitrary number of oscillators [7]. On the other hand, in the case of fermionic systems the existence of thermal bound entanglement has been shown only for small systems of up to 12 spins [7]–[9] or for models involving non-local interactions between an arbitrary number of spins [10]. The main contribution of the present paper is thus to show the existence of bound entangled thermal states in strongly correlated systems of arbitrary number N of spin-1/2 particles with local interactions, in particular in the limit $N \rightarrow \infty$.

We show the presence of thermal bound entanglement for the exemplary family of Hamiltonians through which graph states are defined [12]. These Hamiltonians are always frustration-free and typically, depending on the graph, also local (meaning that not all particles interact simultaneously) and of finite-range (meaning that no particle interacts with another one infinitely far away)³. Graph states constitute an extremely important family of states from practical and fundamental points of view. They include cluster states, which are resources for universal measurement-based quantum computation [13], codeword states for quantum error correction [14], and the well-known GHZ states, which are resources for secure quantum communication [15]. Moreover, this family of states can be used in quantum non-locality tests [16]–[19].

The motivation of this work is thus twofold: from a practical point of view, to establish the range of temperatures for which a thermal state of an interesting many-body model proves useful as a resource for some potential quantum information processing task; and, from a fundamental viewpoint, to take this particular case as a concrete example within a broader investigation of the properties of quantum correlations of strongly correlated systems undergoing open-system dynamics.

2. Thermal graph states and dephased graph states

In this section, we define the states under scrutiny and establish the notation. Let us begin by defining a mathematical graph $G \equiv \{\mathcal{V}, \mathcal{E}\}$ as the union of the set \mathcal{V} of vertices $i \in \mathcal{V}$ with the set \mathcal{E} of edges $\{i, j\} \in \mathcal{E}$ connecting each vertex i to some other j , being $1 \leq i, j \leq N$. Next, for each graph G we define Hamiltonian H acting on N spin-1/2 particles as

$$H = -\frac{1}{2} \sum_{i=1}^N B_i X_i \otimes \bigotimes_{j \in \mathcal{N}_i} Z_j, \quad (1)$$

where $B_i > 0$ are arbitrary (strictly positive) coupling strengths in arbitrary units, X_k and Z_k are the usual Pauli operators acting on particle k , and \mathcal{N}_k denotes all neighbouring particles of k —i.e. each particle whose graph representation in G is a vertex j directly connected to k by some edge $\{j, k\} \in \mathcal{E}$. Hamiltonian (1) involves m -body interactions, where m is given by the maximum connectivity of the graph G . Also, since each and all of the N terms in

³ Exceptions are of course the Hamiltonians associated with the maximally connected graphs (that give rise to GHZ states [16]), which are never local; and the ones associated with expander graphs, which can be local but not necessarily of finite range.

summation (1) commute with each other, the eigenstates—here denoted by $|G_{\mu_1 \dots \mu_N}\rangle$, with $\mu_i = 0$ or 1, for $1 \leq i \leq N$ —of each local term are also eigenstates of the whole summation. This implies that Hamiltonian (1) is frustration-free, meaning that the ground state— $|G_{0 \dots 0}\rangle$ —minimizes the energy of each term in H .

The ground state $|G_{0 \dots 0}\rangle$ is the unique (non-degenerate) ground state of Hamiltonian (1) with eigenenergy $-\frac{1}{2} \sum_{i=1}^N B_i$, whereas states $|G_{\mu_1 \dots \mu_N}\rangle$ —related to the former by the local-unitary transformation $|G_{\mu_1 \dots \mu_N}\rangle \equiv \otimes_{i=1}^N Z_i^{\mu_i} |G_{0 \dots 0}\rangle$ —are eigenstates of (1) with $\sum_{i=1}^N \mu_i$ excitations and associated eigenenergies $-\frac{1}{2} \sum_{i=1}^N B_i (-1)^{\mu_i}$ [12]. The eigenstates $|G_{\mu_1 \dots \mu_N}\rangle$ form a complete orthogonal basis of the N -qubit Hilbert space. Since they are local-unitarily related, they all possess exactly the same entanglement properties, extensively studied in [12, 20] and references therein. For historical reasons, the ground state $|G_{0 \dots 0}\rangle$ has been taken however as the defining state for the so-called graph state. Let us recall that there exists also an alternative operational definition for such graph state: it can be physically produced initializing N qubits in the superposition $|+\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$ and subsequently applying control- Z gates $CZ_{ij} = e^{i(\pi/4)(Z_i \otimes Z_j - Z_i - Z_j + \mathbb{1})}$ onto each pair of neighbouring qubits defined by the graph G . Mathematically,

$$|G_{0 \dots 0}\rangle = \bigotimes_{i=1}^N \bigotimes_{j \in \mathcal{N}_i} CZ_{ij} \bigotimes_{k=1}^N |+\rangle_k. \quad (2)$$

We are now in a position to introduce the thermal graph state associated with G as the thermal state of Hamiltonian (1):

$$\rho_T = \frac{e^{-H/T}}{\text{Tr}[e^{-H/T}]}, \quad (3)$$

where T is the temperature of some bath with which our system of interest has reached thermal equilibrium (Boltzmann's constant is set as unit $k_B \equiv 1$ throughout).

Alternatively, ρ_T can be defined as a decohered graph state. In order to show that we need to introduce the completely positive map Λ , acting on any N -qubit density matrix ρ as

$$\Lambda(\rho) \equiv \mathcal{D}_1 \otimes \dots \otimes \mathcal{D}_N(\rho), \quad (4)$$

as the composition of local, independent channels \mathcal{D}_i ,

$$\mathcal{D}_i(\rho) = \left(1 - \frac{p_i}{2}\right) \rho + \frac{p_i}{2} Z_i \rho Z_i, \quad (5)$$

with $0 \leq p_i \leq 1$. Local channel \mathcal{D}_i describes the physical process in which, with probability p_i , an undesired π -phase shift is experienced by qubit i and, with probability $1 - p_i$, the system is left untouched. This process is present in situations where, with probability p_i , there is complete loss of quantum coherence but without any population exchange. In the context of decoherence, map Λ in turn is often referred to as *local* (or *individual*) *dephasing* (or *phase damping*).

It was shown in [21] that thermal state (3)—for the particular case of constant couplings $B_i \equiv B$ —can also be alternatively obtained by individually dephasing graph state (2) with equal probabilities $p_i \equiv p = \frac{2}{1+e^{B/T}}$. As shown in the [appendix](#), this property also holds for general thermal states of Hamiltonian (1) with arbitrary couplings. Indeed,

$$\rho_T \equiv \Lambda(|G_{0 \dots 0}\rangle\langle G_{0 \dots 0}|), \quad (6)$$

with Λ defined according to equations (4) and (5), but with the additional constraint that the local dephasing probabilities satisfy

$$\frac{p_i}{2} \equiv \frac{1}{1 + e^{B_i/T}}. \quad (7)$$

The equivalence mathematically expressed in equations (6) and (7) establishes a very interesting connection between a collective decoherence process (thermalization of systems governed by graph-state Hamiltonians as (1)) and a local one (individual dephasing of systems initialized in graph states as (2)).

3. Appearance of multipartite bound entanglement in thermal graph states

As mentioned before, the characterization of entanglement in multiqubit systems is formidably hard even for pure states. However, a useful tool for the evaluation of the amount of entanglement contained in decohered graph states was developed in [22] for an important family of decoherence processes (see also [21, 23]). Given a certain multipartition of a graph G , the machinery developed in [22] allows us to map the calculation of the entanglement of a decohered (mixed) graph state to the average entanglement of several effective systems, constituted only by the so-called *boundary qubits*—the ones lying on the border of the multipartition and having neighbours on the other side of the border. Solving the former problem involves an optimization over a parameter space exponentially large with N , whereas solving the latter involves only an optimization over the boundary qubits, a task that requires always exponentially less memory space and usually also considerably less computational time, especially when N is large, as in the thermodynamical limit.

For our case of interest, individual dephasing, this formalism works even better as the entanglement contained in an arbitrary multipartition of a locally dephased graph state is equivalent not to the average entanglement of an ensemble of smaller effective boundary systems but just one. Furthermore, such an effective system is simply composed by the locally dephased original system itself but without all non-boundary qubits. The key point behind this idea is that all the control- Z gates that define $|G_{0\dots 0}\rangle$ in equation (2) commute with the dephasing map (4)⁴. Hence, the order in which channels \mathcal{E}_i and the control- Z gates are applied on the product state $\bigotimes_{k=1}^N |+\rangle_k$ to obtain $\Lambda(|G_{0\dots 0}\rangle\langle G_{0\dots 0}|)$ is irrelevant. In particular, state (6) is also obtained if the control- Z gates act after the dephasing channels. Now, all CZ s not crossing any boundary are local unitary operations with respect to the multipartition (see figures 1 and 3 for simple examples). Thus, because every entanglement quantifier is invariant under local unitary operations, as far as what concerns the amount of entanglement in the multipartition one can simply forget about these non-boundary-crossing CZ s.

In what follows we apply this idea to some well-known, paradigmatic examples of graphs to show that there exists a range of temperatures where the associated thermal graph state possesses multipartite bound entanglement. We do it first for the case of constant couplings $B_i \equiv B$ to transmit the essential idea clearly and then move to the arbitrary-coupling case.

3.1. The linear cluster with equal couplings

Let us start with the simplest example: the linear cluster state. Here the defining graph G is the linear graph sketched in figure 1. We denote its thermal state as ρ_{1D_T} . First we consider

⁴ This was indeed the key point behind the results of [21]–[23].

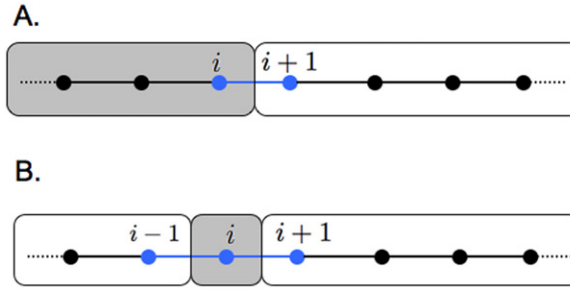


Figure 1. Two possible bipartitions of the linear cluster: the system is split into two subpartitions, represented by the grey and white regions. Note that all control-Z gates corresponding to black edges act locally with respect to the bipartition adopted, and thus do not change its entanglement. (A) The system is divided in two contiguous blocks of spins. The entanglement in this bipartition is equivalent to the entanglement between the two qubits in blue (see the text). (B) Another possible bipartition, this time non-contiguous blocks of spins are part of the same single subpartition (the white one). In this case, all the entanglement is contained in the three-qubit boundary system shown in blue.

a bipartition of the system into two contiguous blocks of spins (figure 1(A)), say from qubit i to the left (grey) and from qubit $i + 1$ to the right (white). We can easily see from the figure that all control-Z gates but one (in blue) act as local unitary operations, and thus have no effect on the entanglement in the bipartition considered. As a result, the entire entanglement between any two contiguous blocks of spins in ρ_{1D_T} is equivalent to that of the simple two-qubit thermal graph state in boundary pair $i-i + 1$. Then, by imposing the entanglement between i and $i + 1$ to vanish we can establish the critical temperature for which the entire thermal cluster is separable with respect to the bipartition under scrutiny. Any entanglement quantifier valid for two-qubit-mixed states would do for this aim, so we choose the simplest one to calculate: the *negativity* [24]. The negativity $\text{Neg}(\rho)$ of a state ρ is the sum of the absolute values of the negative eigenvalues of ρ^Γ , where ρ^Γ is the partial transposition of ρ according to some bipartition. The negativity $\text{Neg}_{i|i+1}(\rho_{1D_T})$ of the thermal state of pair $i|i + 1$ is readily calculated to be $\text{Neg}_{i|i+1}(\rho_{1D_T}) = \frac{1}{4}(2 - 2p_i - 2p_{i+1} + p_i p_{i+1})$. For the case of constant couplings $p_i = p_{i+1} \equiv p = \frac{2}{1+e^{B/T}}$ condition $\text{Neg}_{i|i+1}(\rho_{1D_T}) = 0$ leads to the critical temperature [21]

$$T_{i|i+1}^c(B) = \frac{-B}{\ln(\sqrt{2} - 1)} \approx 1.1B. \quad (8)$$

For $T \geq T_{i|i+1}^c(B)$, since the partition is separable, it is not possible to extract *any type of entanglement* between any two contiguous blocks joined at spins i and $i + 1$ from a thermal one-dimensional (1D) graph state by applying contiguous-block-local operations (arbitrary operations acting on spins 1 to i and on spins $i + 1$ to N). This imposes strong restrictions on the operations one needs to apply to ρ_{1D_T} to distill some entanglement (if any) from some other of its multipartitions. For instance, if individual local operations on each spin are applied, no entanglement can be distilled from this bipartition of ρ_{1D_T} if $T \geq T_{i|i+1}^c(B)$, for these are a particular case of contiguous-block-local operations. Now, since here we have taken all coupling strengths equal, the critical temperature for separability of contiguous blocks is the same for all i . This means that for $T \geq T_{i|i+1}^c(B)$ all contiguous blocks of ρ_{1D_T} are in a separable state.

So no entanglement between any two particles can be extracted by LOCC, as for any two particles a contiguous-block bipartition can be found in which each particle lies on a different side of the partition and is therefore separable from the other. Therefore, *no entanglement at all* can be extracted from ρ_{1D_T} for $T \geq T_{i|i+1}^c(B)$ by LOCC.

We now find another (non-contiguous) family of bipartitions for which the separability temperature, $T_{i|i-1,i+1}^c(B)$, is strictly larger than $T_{i|i+1}^c(B)$. This suffices to prove the non-separability of ρ_{1D_T} —and therefore the presence of multipartite bound entanglement in—for a range of temperatures $T_{i|i+1}^c(B) \leq T \leq T_{i|i-1,i+1}^c(B)$.⁵ Consider for instance the entanglement between the i th qubit in the chain and all the rest (see figure 1(B)). In this case, one can ignore all but two control-Z gates in the calculation of the entanglement. So for these partitions, the entanglement (again quantified here by the negativity) of ρ_{1D_T} is equivalent to that of the central particle versus its two neighbours in a linear thermal cluster state of only three qubits. This negativity vanishes at a critical temperature $T_{i|i-1,i+1}^c(B)$ that turns out to be strictly larger than $T_{i|i+1}^c(B)$ ($T_{i|i-1,i+1}^c(B) \gtrsim 1.6B$, see figure 2). In the range $T_{i|i+1}^c(B) \leq T \leq T_{i|i-1,i+1}^c(B)$, even though ρ_{1D_T} possesses entanglement, we already know that none of it can be distilled through local operations assisted by classical communication. Such entanglement can only be extracted if particles $i-1$ and $i+1$ interact, which is of course not a local operation. Therefore, in this range of temperatures thermal cluster state ρ_{1D_T} possesses multipartite bound entanglement.

It is important to stress that none of the latter results or conclusions depends at all on the size N of the graph. This allows us to guarantee that thermal bound entanglement is also present in macroscopic specimens of these graphs. This independence of the entanglement on the graph's size is precisely the key point behind the method we used to simplify the calculation of the negativity of ρ_{1D_T} . In general, in order to calculate the negativity of an N -qubit mixed state, one would need to diagonalize a $2^N \times 2^N$ matrix, which already for a few tens of qubits cannot even be written down by a current classical computer. In the example studied in this subsection, this method has enabled us to obtain results for arbitrarily large systems calculating only negativities of two-qubit and three-qubit systems.

3.2. The 2D square cluster with equal couplings

Our next example is the 2D thermal cluster state ρ_{2D_T} , for which the associated graph G is a $\sqrt{N} \times \sqrt{N}$ square lattice (see figure 3). The columns of G are labeled by index $1 \leq i \leq \sqrt{N}$ and the rows by index $1 \leq j \leq \sqrt{N}$. At $T = 0$ this state is known to be a universal resource for one-way quantum computation. As a consequence, understanding the entanglement properties of this model under realistic noisy conditions is of course very important from a practical point of view.

To prove the existence of thermal multipartite bound entanglement in this example we follow exactly the same reasoning as in the previous subsection: establish a region of temperatures where the system is separable with respect to any two contiguous blocks—and therefore non-distillable with respect to (contiguous-block) local operations while still being entangled in other partitions. In figure 3(A), we consider a vertical partition of the system from column i to the left (grey) and from column $i+1$ to the right (white). Following the same steps as before, the calculation of entanglement in this bipartition can be reduced to that of a product

⁵ Note that in the first version of arXiv:0902.4343 it was erroneously claimed that $T_{i|i+1}^c(B)$ is the separability temperature for ρ_{1D_T} [25].

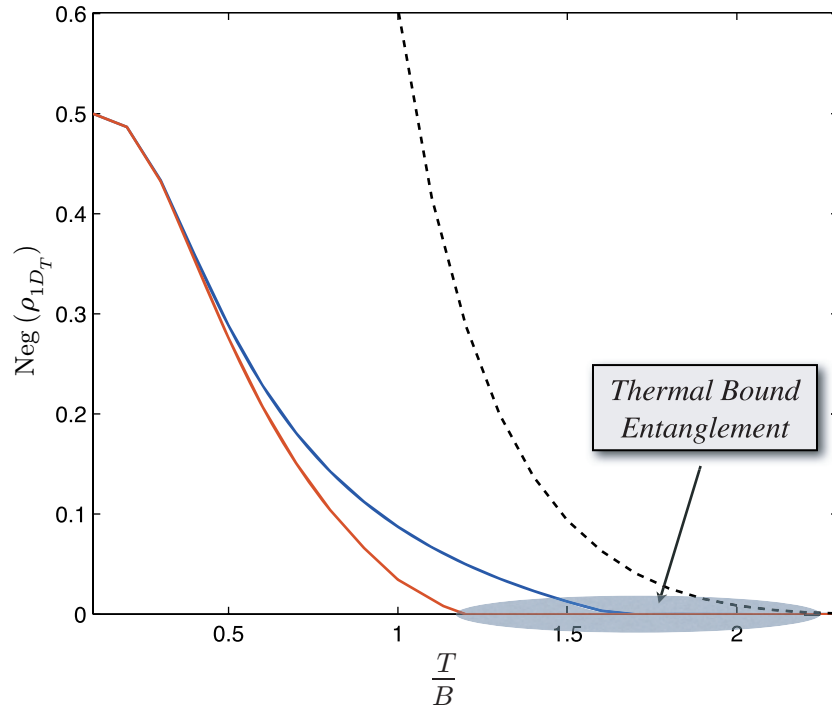


Figure 2. Negativities of the thermal linear cluster state ρ_{1D_T} , as a function of temperature T/B . The considered bipartitions are those of figure 1: any two contiguous blocks, in red and any qubit versus the rest, in blue. None of these curves depends on the size N of the graph. Therefore such thermal bound entanglement is also present in the macroscopic thermodynamical limit. Besides, we also display in dashed line the negativity of the even–odd partition, where particles with even label belong to subsystem A and particles with odd label to subsystem B . In this case, the negativity does depend on the system’s system and the plot is done for $N = 12$. Our numerical investigations suggest that the even–odd partition is the most robust bipartition, i.e. the one with the highest critical temperature of vanishing negativity. The shaded region shows the range of temperatures where the thermal system possesses bound entanglement (see text).

of the \sqrt{N} two-qubit thermal cluster states lying on the boundary. In this way, we see that the critical temperature for separability (of all contiguous blocks) is again given by (8). Once again we consider another family of bipartitions, say, any qubit ij inside the lattice ($1 < i < \sqrt{N}$ and $1 < j < \sqrt{N}$) versus all the rest of the qubits (see figure 3(B)). The entanglement in this bipartition is equivalent to that between the qubit ij and its four neighbours, in a thermal graph state in a star configuration. The temperature for which the negativity vanishes in this case happens to be $T_{ij|i-1,i+1,j+1,j-1}^c(B) \approx 2.5B$, which is again strictly larger than $T_{i|i+1}^c(B)$.

3.3. Unequal Hamiltonian couplings

Consider now arbitrary couplings B_i in Hamiltonian (1) with average value B , that is: $\frac{1}{N} \sum_{i=1}^N B_i = B$. We first restrict ourselves to the familiar 1D graph studied in subsection 3.1,

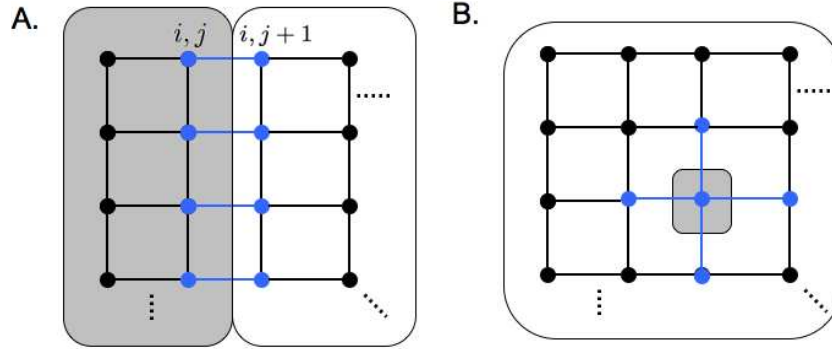


Figure 3. Two different bipartitions of a two-dimensional (2D) cluster. (A) The system is divided vertically into two parts. The only control- Z gates not acting as local unitary operations are the boundary-crossing ones, shown in blue. Thus, the entanglement in any such bipartition of the 2D cluster in a thermal state is equivalent to that of several copies of a two-qubit thermal cluster state (represented in blue). (B) The cluster is partitioned into a central spin and all other ones. The entanglement is this time equivalent to that between the central spin and only its four neighbours in a five-qubit thermal graph state in a star configuration (also in blue).

as this already qualitatively captures all the essential changes that can appear when different couplings are present. The contiguous-block separability condition $\text{Neg}_{i|i+1}(\rho_{1DT}) = \frac{1}{4}(2 - 2p_i - 2p_{i+1} + p_i p_{i+1}) = 0$ now involves $p_i \equiv \frac{2}{1+e^{B_i/T}} \neq p_{i+1} \equiv \frac{2}{1+e^{B_{i+1}/T}}$. Thus, the critical temperature is implicitly expressed by the equation

$$1 = e^{-(B_i/T)} + e^{-(B_{i+1}/T)} + e^{-(B_i+B_{i+1})/T}. \quad (9)$$

Equation (9) above has a unique real solution $T = T_{i|i+1}^c(B_i, B_{i+1})$, but is however non-invertible in general. Clearly, critical temperature $T_{i|i+1}^c(B_i, B_{i+1})$ is symmetric under the exchange of B_i and B_{i+1} . Also, numerical inspections immediately show that it is a monotonously growing function of B_i (B_{i+1}) for fixed B_{i+1} (B_i). Furthermore, under some constraints, as for example $B_i + B_{i+1}$ constant, the critical temperature is maximal when the couplings are equal.

As before, we compare $T_{i|i+1}^c(B_i, B_{i+1})$ with the critical temperature $T_{i|i-1, i+1}^c(B_{i-1}, B_i, B_{i+1})$ corresponding to the bipartition, which distinguishes qubit i versus the rest. For all cases we have studied, $T_{i|i-1, i+1}^c(B_{i-1}, B_i, B_{i+1})$ turns out to be strictly larger than $T_{i|i+1}^c(B_i, B_{i+1}) \forall 1 \leq i \leq N$. If this is true in general, it implies that neither the appearance, nor the range of temperatures, of bound entanglement will be considerably affected by small deviations of B_i from the mean value B .

However, the critical temperatures can be considerably sensitive to the values of the coupling constants at each site i when the deviations are large (say, of the order of B itself). In such case, it might as well happen that $T_{i|i+1}^c(B_i, B_{i+1}) < T_{i|i-1, i+1}^c(B_{i-1}, B_i, B_{i+1}) < T_{k|k+1}^c(B_k, B_{k+1}) < T_{k|k-1, k+1}^c(B_{k-1}, B_k, B_{k+1})$, for some $i \neq k$. Remarkably, also in these cases the system displays a finite range of temperatures for which it is bound entangled. To see this, it suffices to consider the site i_{\max} for which the critical temperature of contiguous-block separability is the largest of all, i.e. $T_{i_{\max}|i_{\max}+1}^c(B_{i_{\max}}, B_{i_{\max}+1}) \geq T_{i|i+1}^c(B_i, B_{i+1}), \forall 1 \leq i \leq N$.

By the same reasoning as before, from $T = T_{i_{\max}|i_{\max}+1}^c(B_{i_{\max}}, B_{i_{\max}+1}) \equiv T_{i_{\max}|i_{\max}+1}^c$ on ρ_{1D_T} is non-distillable, but it is entangled up to $T = T_{i_{\max}|i_{\max}-1, i_{\max}+1}^c(B_{i_{\max}-1}, B_{i_{\max}}, B_{i_{\max}+1}) \equiv T_{i_{\max}|i_{\max}-1, i_{\max}+1}^c$. Therefore (assuming again that $T_{i_{\max}|i_{\max}-1, i_{\max}+1}^c > T_{i_{\max}|i_{\max}+1}^c$ is always true), for $T \in [T_{i_{\max}|i_{\max}+1}^c, T_{i_{\max}|i_{\max}-1, i_{\max}+1}^c)$ the state is bound entangled. The main conclusion to draw from the considerations in this paragraph is that rather than a peculiarity of the (ideal) case of equal coupling strengths, the presence of bound entanglement in a finite range of temperature appears to be a *general phenomenon* for (arbitrarily large) thermal graph states.

Finally, let us stress that similar results hold also for other graphs with different couplings. There again different partitions will give rise to different critical temperatures implying again the presence of bound entanglement along the lines of section 3.2.

4. Conclusion

Considering thermal graph states we have shown the presence of bound entanglement in systems containing a macroscopic number of spins with finite range interactions. This result extends the previous results [7] for bosonic chains to fermionic systems. Our findings suggest that thermal bound entanglement could manifest also in more general systems, since this seems to be a robust feature against, in particular, to modifications in the symmetries of the Hamiltonian.

In this paper, we considered systems with three- (or more)-body interactions. Thus a natural question arises regarding the presence of macroscopic thermal bound entanglement also in spin-1/2 models containing only two-body interactions. This was indeed found in the case of harmonic oscillators systems [7]. Actually, we expect thermal bound entanglement to be a common phenomenon of general many-body systems since it is very unlikely that the negativities of all possible bipartitions of a system vanish at the same temperature. However, proving that for systems in the thermodynamical limit turns to be difficult problem.

It is also worth mentioning that we used the negativity of some bipartitions of the system to detect inseparability of the thermal state. So our method cannot detect bound entanglement in the case that all bipartitions of a system have positive partial transpositions. This was in turn found in the case of small spin systems [8]. We leave the existence of this kind of bound entanglement in macroscopic systems as an open problem.

An interesting question concerns the utility of thermal graph states for information processing. For the region of temperatures such that these states are distillable, one could in principle apply first a distillation protocol (e.g. see [21]) before using the state as a resource for quantum information. However, for the regions of bound entanglement, not even this experimentally demanding strategy would work.

Since in any practical implementation the temperature is always non-null, thermal bound-entangled graph states are ideal probes to explore the limitations of realistic (experimentally feasible) measurement-based quantum computation. This question is certainly of great interest and can be the subject of further analysis.

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Appendix. Thermal graph state as a dephased graph state for arbitrary couplings

Thermal state (3) expressed in the eigenbasis of Hamiltonian (1), $\{|G_{\mu_1 \dots \mu_N}\rangle\}$, reads (disregarding its normalization) $\rho_T \equiv e^{-H/T} = \sum_{\mu_1 \dots \mu_N=0}^1 e^{(1/2T) \sum_{i=1}^N B_i (-1)^{\mu_i}} |G_{\mu_1 \dots \mu_N}\rangle \langle G_{\mu_1 \dots \mu_N}|$. Next, we explicitly evaluate dephased state $\Lambda(|G_{0\dots 0}\rangle \langle G_{0\dots 0}|)$ using equations (4) and (5)

$$\begin{aligned} \Lambda(|G_{0\dots 0}\rangle \langle G_{0\dots 0}|) &\equiv \mathcal{D}_1 \otimes \dots \otimes \mathcal{D}_N (|G_{0\dots 0}\rangle \langle G_{0\dots 0}|) \\ &\equiv \sum_{\mu_1 \dots \mu_N=0}^1 \prod_{i=1}^N (1 - p_i/2)^{|\mu_i-1|_2} (p_i/2)^{|\mu_i|_2} |G_{\mu_1 \dots \mu_N}\rangle \langle G_{\mu_1 \dots \mu_N}|, \end{aligned}$$

where ‘ $|_2$ ’ stands for ‘modulo 2’. The latter is equal to the former expression for ρ_T if and only if each and all of the terms in the summation are equal. That is, if and only if $\prod_{i=1}^N (1 - p_i/2)^{|\mu_i-1|_2} (p_i/2)^{|\mu_i|_2} \equiv e^{(1/2T) \sum_{i=1}^N B_i (-1)^{\mu_i}}$, which when equation (7) holds—and using the fact that $\frac{1}{1+e^{B_i/T}} \equiv \frac{e^{-B_i/2T}}{e^{-B_i/2T} + e^{B_i/2T}}$ and $1 - \frac{1}{1+e^{B_i/T}} \equiv \frac{e^{B_i/2T}}{e^{-B_i/2T} + e^{B_i/2T}}$ —in turn reads $\prod_{i=1}^N \frac{e^{(-1)^{\mu_i} B_i/2T}}{e^{-B_i/2T} + e^{B_i/2T}} \equiv e^{(1/2T) \sum_{i=1}^N B_i (-1)^{\mu_i}}$, which can in turn be immediately checked to be true up to a constant normalization factor.

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