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Vanishing of the Anomaly in Lattice Chiral Gauge Theory

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Abstract. The anomaly cancellation is a basic property of the Standard Model, crucial for its consistence. We consider a lattice chiral gauge theory of massless Wilson fermions interacting with a non-compact massive $U(1)$ field coupled with left- and right-handed fermions in four dimensions. We prove in the infinite volume limit, for weak coupling and inverse lattice step of the order of boson mass, that the anomaly vanishes up to subleading corrections and under the same condition as in the continuum. The proof is based on a combination of exact Renormalization Group, non-perturbative decay bounds of correlations and lattice symmetries.

1. Introduction and Main Results

1.1. Chiral Gauge Theory

The perturbative consistence (renormalizability) of the Standard Model relies on the vanishing of the anomalies, achieved under certain algebraic conditions [1] severely constraining the elementary particles charges and providing a partial explanation of the charge quantization. In order to go beyond a purely perturbative framework in terms of diverging series [2], one needs a lattice formulation with functional integrals with cut-off much higher than the experiments scale; due to triviality [3,4], the cut-off cannot be completely removed, at least in the electroweak sector, and hence, the theory can be seen as an effective one.

One expects a relation between the perturbative renormalizability properties and the size of the cut-off. The electroweak theory is renormalizable [5,6] so that a construction up to exponentially large cut-off could be in principle possible, and such cut-off is much higher than the scales of experiments. However, this requires as a crucial prerequisite that the anomalies cancel, at least to a certain extent. This rises the natural question: does the anomaly cancel

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at a non-perturbative level with finite lattice, under the same condition as in the continuum?

In the continuum, the cancellation is based on compensations at every order [7] based on dimensional regularizations and symmetries, but finite lattice cut-off produce corrections and the question is if they cancel or not. Jacobian arguments are used to support vanishing of higher-order contributions to anomalies but are essentially one loop results, as shown in [8]. Topological arguments explain the anomaly cancellation on a lattice [9] with classical gauge fields, but in the quantum case they work only at lowest order (one loop). The cancellation would be obtained if a non-perturbative regulator for lattice chiral gauge theories could be found, but this is a long-standing unsolved problem and only order by order results are known [10,11].

We consider a lattice chiral gauge theory, given by $2N$ massless fermions in four dimensions, labelled by an index $i = 1, \ldots, 2N$; we also define the indices $i_1 = 1, \ldots, N$ and $i_2 = N+1, \ldots, 2N$. If the gamma matrices are

$$\gamma_0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \quad \gamma_j = \begin{pmatrix} 0 & i\sigma_j \\ -i\sigma_j & 0 \end{pmatrix}, \quad \gamma_5 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \tag{1}$$

and $\sigma^L_\mu = (\sigma_0, i\sigma)$, $\sigma^R_\mu = (\sigma_0, -i\sigma)$,

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2}$$

the formal continuum action is given by the following expression:

$$\begin{aligned} &\int dx F_{\mu,\nu} F_{\mu,\nu} \\ &\quad + \sum_{i_1} \int dx \left[\psi^+_{i_1,L,x} \sigma^L_\mu (\partial_\mu + \lambda Q_{i_1} A_\mu) \psi^-_{i_1,L,x} + \psi^+_{i_1,R,x} \sigma^R_\mu \partial_\mu \psi^-_{i_1,L,x} \right] \\ &\sum_{i_2} \int dx \left[\psi^+_{i_2,R,x} \sigma^R_\mu (\partial_\mu + \lambda Q_{i_2} A_\mu) \psi^-_{i_2,R,x} + \psi^+_{i_2,L,x} \sigma^L_\mu \partial_\mu \psi^-_{i_2,L,x} \right] \end{aligned} \tag{3}$$

with $\mu = (0,1,2,3)$ and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Note that the R fermions of kind i_1 and the L fermions of kind i_2 decouple and are fictitious, non-interacting degrees of freedom, which are convenient to introduce in view of the lattice regularization, see e.g. [12–14]. The total current coupled to A_μ is

$$j^T_\mu = \sum_{i_1} Q_{i_1} \psi^+_{i_1,L,x} \sigma^L_\mu \psi^-_{i_1,L,x} + \sum_{i_2} Q_{i_2} \psi^+_{i_2,R,x} \sigma^R_\mu \psi^-_{i_2,R,x} \tag{4}$$

and the axial and vector part of the current is

$$j^{T,V}_\mu = \frac{1}{2} \sum_i Q_i j_{\mu,i,x} \qquad j^{T,A}_\mu = \frac{1}{2} \sum_i Q_i \widetilde{\varepsilon}_i j^5_{\mu,i,x} \tag{5}$$

with $\widetilde{\varepsilon}_{i_1} = -\widetilde{\varepsilon}_{i_2} = 1$, $j_{\mu,i,x} = \bar{\psi}_{i,x} \gamma_\mu \psi_{i,x}$, $j^5_{\mu,i,x} = \bar{\psi}_{i,x} \gamma_5 \gamma_\mu \psi_{i,x}$ and $\psi_{i,x} = (\psi^-_{i,L,x}, \psi^-_{i,R,x})$, $\bar{\psi}_{i,x} = (\psi^+_{i,L,x}, \psi^+_{i,R,x}) \gamma_0$. Note the chiral nature of the theory, as in the current the fermion with different chiralities has different charges. An example of chiral theory is obtained setting $Q_{i_2} = 0$; in such a case one is describing N fermions with the same chirality interacting with a gauge field. A physically more important example is given by the $U(1)$ sector of the Standard Model with no Higgs and massless fermions; in this case $N = 4$, $i_1 = (\nu_1, e_1, u_1, d_1)$ are the left-handed components and $i_2 = (\nu_2, e_2, u_2, d_2)$ the right-handed of the leptons and quarks. A formal application of Noether theorem with classical fermions and bosons says that the invariance under phase and chiral symmetry, implying the current conservation $\partial_\mu j^T_\mu = 0$. If the fermions are quantum (and the bosons classical), the conservation of current is reflected in Ward identities, and it turns out that *anomalies* generically break the conservation of $j^T_{\mu,i,x}$ unless

$$\sum_{i_1=1}^{N} Q^3_{i_1} - \sum_{i_2=1}^{N} Q^3_{i_2} = 0 \tag{6}$$

In the elecroweak sector, the physical values $Q_{\nu_1} = Q_{e_1} = -1$, $Q_{u_1} = Q_{d_1} = 1/3$, $Q_{\nu_2} = 0$, $Q_{e_2} = -2$, $Q_{u_2} = 4/3$, $Q_{d_2} = -2/3$ verify (6), if Q are the hypercharges and an index for the three colours of quarks is added. Remarkably the hyperchrges (and therefore the charges) are constrained to physical values by purely quantum effects. The question is therefore if in a lattice regularization of (3) and considering A_μ a quantum field, the chiral current is conserved under the same condition (6) at a non-perturbative level.

1.2. The Lattice Chiral Gauge Theory

The lattice chiral gauge theory is defined by its generating function

$$e^{\mathcal{W}(J,J^5,\phi)} = \int P(dA) \int P(d\psi) e^{V(\psi,A,J)+V_c(\psi)+\mathcal{B}(J^5,\psi)+(\psi,\phi)} \tag{7}$$

where $A_{\mu,x} : \Lambda \to \mathbb{R}$, $\Lambda = [0,L]^4 \cap a\mathbb{Z}^4$, $L = Ka$, $K \in \mathbb{N}$ e_μ, $\mu = 0,1,2,3$ an orthonormal basis, $A_{\mu,x} = A_{\mu,x+Le_\mu}$ (periodic boundary conditions) and the bosonic integration is

$$P(dA) = \frac{1}{\mathcal{N}_A} \left[\prod_{x\in\Lambda} \prod_{\mu=0}^{3} dA_{\mu,x} \right] e^{-S_G(A)} \tag{8}$$

with

$$S_G = a^4 \sum_x \left[\frac{1}{4} F_{\mu,\nu,x} F_{\mu,\nu,x} + \frac{M^2}{2} A_{\mu,x} A_{\mu,x} + (1-\xi)(d_\mu A_\mu)^2 \right] \tag{9}$$

is the action of a non-compact lattice $U(1)$ gauge field with a gauge fixing and a mass term, $F_{\mu,\nu} = d_\nu A_\mu - d_\mu A_\nu$ and $d_\nu A_\mu = a^{-1}(A_{\mu,x+e_\nu a} - A_{\mu,x})$, $\mathcal{N}_A$ is the normalization. The *bosonic simple expectation*

$$\mathcal{E}_A(A_{\mu_1,x_1} ... A_{\mu_n,x_n}) = \int P(dA) A_{\mu_1,x_1} ... A_{\mu_n,x_n} \tag{10}$$

is expressed by the Wick rule with covariance

$$g^A_{\mu,\nu}(x,y) = \delta_{\mu,\nu} \frac{1}{L^4} \sum_k \frac{e^{ik(x-y)}}{|\sigma|^2 + M^2} \left(\delta_{\mu,\nu} + \frac{\xi \bar{\sigma}_\mu \sigma_\nu}{(1-\xi)|\sigma|^2 + M^2} \right) \tag{11}$$

with $\sigma_\mu(k) = (e^{ik_\mu a} - 1)a^{-1}$, $k = 2\pi n/L$, $n \in \mathbb{N}^4$ and $k \in [-\pi/a, \pi/a)^4$. The *bosonic truncated expectation*

$$\mathcal{E}^T_A(F; \cdots; F) = \frac{\partial^n}{\partial\lambda^n} \log \int P(dA)\, e^{F(A)}\Big|_{\lambda\equiv 0} \tag{12}$$

is expressed by the Wick rule restricted to the *connected* terms.

We denote by $\psi^\pm_{i,s,x}$ the Grassmann variables, with $i = 1, .., 2N$ the *particle index*; $s = L, R$ the *chiral index*; anti-periodic boundary conditions are imposed and

$$\{\psi^+_{i,s,x}, \psi^+_{i',s',x'}\} = \{\psi^+_{i,s,x}, \psi^-_{i',s',x'}\} = \{\psi^-_{i,s,x}, \psi^-_{i',s',x'}\} = 0 \tag{13}$$

We define $\psi^\pm_{i,s,x} = \frac{1}{L^4}\sum_k e^{\pm ikx}\widehat\psi^\pm_{i,s,k}$, with $\widehat\psi^\pm_{i,s,k}$ another set of Grassmann variable, $k = 2\pi/L(n + 1/2)$, $n \in \mathbb{N}^4$ and $k \in [-\pi/a, \pi/a)^4$. The fermionic Gaussian measure is defined as, $i = 1, .., 2N$, $s = L, R$

$$P(d\psi) = \frac{1}{\mathcal{N}_\psi}\left[\prod_{i,s,x} d\psi^+_{i,s,x} d\psi^-_{i,s,x}\right] e^{-S_F} \tag{14}$$

where $\mathcal{N}_\psi$ a normalization and, if $\psi^\pm_{i,x} = (\psi^\pm_{i,L,x}, \psi^\pm_{i,R,x})$

$$\begin{aligned} S_F = \frac{1}{2a}\sum_{i=1}^{2N} a^4 \sum_x \Bigg[& \sum_\mu (\psi^+_{i,x}\gamma_0\gamma_\mu\psi^-_{i,x+e_\mu a} - \psi^+_{i,s,x+e_\mu a}\gamma_0\gamma_\mu\psi^-_{i,x}) \\ + \quad & r(\psi^+_{i,x}\gamma_0\psi^-_{i,x+e_\mu a} + \psi^+_{i,x+e_\mu a}\gamma_0\psi^-_{i,x} - \psi^+_{i,x}\gamma_0\psi^-_{i,x})\Bigg] \end{aligned} \tag{15}$$

We can write therefore

$$\begin{aligned} S_F = \frac{1}{2a}\sum_{i=1}^{2N} a^4 \sum_x \Bigg[& \sum_\mu \sum_{s=L,R} (\psi^+_{i,s,x}\sigma^s_\mu\psi^-_{i,s,x+e_\mu a} - \psi^+_{i,s,x+e_\mu a}\sigma^s_\mu\psi^-_{i,s,x}) \\ & + r(\psi^+_{i,L,x}\psi^-_{i,R,x+e_\mu a} + \psi^+_{i,L,x+e_\mu a}\psi^-_{i,R,x} - \psi^+_{i,L,x}\psi^-_{i,R,x} \\ & + \psi^+_{i,R,x}\psi^-_{i,L,x+e_\mu a} + \psi^+_{i,R,x+e_\mu a}\psi^-_{i,L,x} - \psi^+_{i,R,x}\psi^-_{i,L,x})\Bigg] \end{aligned} \tag{16}$$

The *fermionic simple expectation*

$$\mathcal{E}_\psi(\psi^{\varepsilon_1}_{i_1,x_1}...\psi^{\varepsilon_n}_{i_n,x_n}) = \int P(d\psi)\psi^{\varepsilon_1}_{i_1,x_1}...\psi^{\varepsilon_n}_{i_n,x_n} \tag{17}$$

is expressed by the anticommutative Wick rule with covariance

$$g^\psi_i(x, y) = \int P(d\psi)\psi_{i,x}\bar\psi_{i,y} = \frac{1}{L^4}\sum_k e^{ik(x-y)}\widehat{g}^\psi_i(k) \tag{18}$$

with

$$\widehat{g}_{i,k} = \left(\sum_\mu i\gamma_0\gamma_\mu a^{-1}\sin(k_\mu a) + a^{-1}\gamma_0 r\sum_\mu (1 - \cos k_\mu a)\right)^{-1} \tag{19}$$

The interaction is

$$\begin{aligned} V(A,\psi,J) &= V_1(A,\psi,J) + V_2(A,\psi,J) \\ V_1(A,\psi,J) &= a^4 \sum_{i,s,x} \left[O^+_{\mu,i,s,x} G^+_{\mu,i,s,x} + O^-_{\mu,i,s,x} G^-_{\mu,i,s}\right] \\ V_2(A,\psi,J) &= \frac{r}{2} a^4 \sum_{i,x} \Big[\psi^+_{i,L,x} H^+_{\mu,i,x} \psi^-_{i,R,x+e_\mu a} \qquad (20) \\ &\quad + \psi^+_{i,L,x+e_\mu a} H^-_{\mu,i,x} \psi^-_{i,R,x} + \psi^+_{i,R,x} H^+_{\mu,i,x} \psi^-_{i,L,x+e_\mu a} \\ &\quad + \psi^+_{i,R,x+e_\mu a} H^-_{\mu,i,x} \psi^-_{i,L,x}\Big] \qquad (21) \end{aligned}$$

with

$$\begin{aligned} G^\pm_{\mu,i,s}(x) &= a^{-1}(: e^{\mp i a Q_i(\lambda b_{i,s} A_{\mu,x} + J_{\mu,x})} : -1) \\ H^\pm_{\mu,i,x} &= a^{-1}(e^{\mp i a Q_i J_{\mu,x}} - 1) \\ O^+_{\mu,i,s,x} &= \frac{1}{2} \psi^+_{i,s,x} \sigma^s_\mu \psi^-_{i,s,x+e_\mu a} \\ O^-_{\mu,i,s,x} &= -\frac{1}{2} \psi^+_{i,s,x+e_\mu a} \sigma^s_\mu \psi^-_{i,s,x} \qquad (22) \end{aligned}$$

with, if $i_1 = 1, N$ and $i_2 = N+1, ..., 2N$

$$b_{i_1,L} = b_{i_2,R} = 1; \qquad b_{i_1,R} = b_{i_2,L} = 0 \qquad (23)$$

and $: e^{\pm i a \lambda Q_i A_\mu(x)} := e^{\pm i \lambda Q_i a A_\mu(x)} e^{\frac{1}{2}(\lambda Q_i)^2 a^2 g^A_{\mu,\mu}(0,0)}$.

The mass counterterm is

$$V_c = \sum_i a^{-1} \nu_i a^4 \sum_x (\psi^+_{i,L,x} \psi^-_{i,R,x} + \psi^+_{i,R,x} \psi^-_{i,L,x}) \qquad (24)$$

Finally, the source term is

$$\mathcal{B} = a^4 \sum_{\mu,x} J^5_{\mu,x} j^5_{\mu,x} \quad j^5_{\mu,x} = \sum_{i,s} \widetilde{\varepsilon}_i \varepsilon_s Q_j Z^5_{i,s} \psi^+_{x,i,s} \sigma^s_\mu \psi^+_{x,i,s}$$

with $\widetilde{\varepsilon}_{i_1} = -\widetilde{\varepsilon}_{i_2} = 1$ and $\varepsilon_L = -\varepsilon_R = 1$. ν_i and $Z^5_{i,s}$ are parameters to be fixed by the renormlization conditions, see below.

Remark. The term proportional to r in S_F (16) is called *Wilson term.* If $r = 0$, the fermionic propagator $\widehat{g}_{i,k}$ has, in the $L \to \infty$ limit, several poles; this has the effect that the low energy behaviour of the lattice theory would not correspond to the continuum target theory (3); the presence of the Wilson term $r \neq 0$ has the effect that only the physical pole $k = 0$ is present but the chiral symmetry is broken [15].

1.3. Physical observables

The fermionic 2-point function is

$$S^\Lambda_{i,s,s'}(x,y) = \frac{\partial^2}{\partial \phi^+_{i,s,x} \partial \phi^-_{i,s',y}} \mathcal{W}_\Lambda(J, J^5, \phi)|_0 \qquad (25)$$

and the Fourier transform is

$$\widehat{S}^{\Lambda}_{i,s,s'}(k) = a^4 \sum_x S^{\Lambda}_{i,s,s'}(x,0)e^{-ikx} \tag{26}$$

The vertex functions are

$$\Gamma^{\Lambda}_{\mu,i',s}(z,x,y) = \frac{\partial^3}{\partial J_{\mu,z}\partial\phi^+_{i',s,x}\partial\phi^-_{i',s,y}}\mathcal{W}(J,J^5,\phi)|_0$$

$$\Gamma^{5,\Lambda}_{\mu,i's}(z,x,y) = \frac{\partial^3}{\partial J^5_{\mu,z}\partial\phi^+_{i',s,x}\partial\phi^-_{i',s,y}}\mathcal{W}(J,J^5,\phi)|_0 \tag{27}$$

The Fourier transform is

$$\widehat{\Gamma}^{\Lambda}_{\mu,i',s}(k,p) = a^4 \sum_z a^4 \sum_y e^{-ipz-iky}\Gamma^{\Lambda}_{\mu,i',s}(z,0,y) \tag{28}$$

and similarly is defined $\widehat{\Gamma}^{5,\Lambda}_{\mu,i's}(k,p)$. The three current vector VVV and axial AVV correlations are:

$$\Pi^{\Lambda}_{\mu,\nu,\rho}(z,y,x) = \frac{\partial^3\mathcal{W}_\Lambda}{\partial J_{\mu,z}\partial J_{\nu,y}\partial J_{\rho,x}}|_0; \quad \Pi^{5,\Lambda}_{\mu,\nu,\rho}(z,y,x) = \frac{\partial^3\mathcal{W}_\Lambda}{\partial J^5_{\mu,z}\partial J_{\nu,y}\partial J_{\rho,x}}|_0 \tag{29}$$

and

$$\widehat{\Pi}^{\Lambda}_{\mu,\nu,\rho}(p_1,p_2) = a^4 \sum_y a^4 \sum_x e^{-ip_1y-ip_2x}\Pi^{\Lambda}_{\mu,\nu,\rho}(0,y,x)$$

$$\Pi^{5,\Lambda}_{\mu,\nu,\rho}(p_1,p_2) = a^4 \sum_y a^4 \sum_x e^{-ip_1y-ip_2x}\Pi^{5,\Lambda}_{\mu,\nu,\rho}(0,y,x) \tag{30}$$

1.4. Ward Identities

The correlations are connected by relations known as *Ward identities*. They can be obtained by performing the change of variables

$$\psi^{\pm}_{i,s,x} \to \psi^{\pm}_{i,s,x}e^{\pm iQ_i\alpha_x} \tag{31}$$

with α_x is a function on $a\mathbb{Z}^4$, with the periodicity of Λ. Let $Q(\psi^+,\psi^-)$ be a monomial in the Grassmann variables and $Q_\alpha(\psi^+,\psi^-)$ be the monomial obtained performing the replacement (31) in $Q(\psi^+,\psi^-)$. It holds that

$$\int \left[\prod_{i,s,x} d\psi^+_{i,s,x}d\psi^-_{i,s,x}\right] Q(\psi^+,\psi^-) = \int \left[\prod_{i,s,x} d\psi^+_{i,s,x}d\psi^-_{i,s,x}\right] Q_\alpha(\psi^+,\psi^-)\,. \tag{32}$$

as both the left-hand side and the right-hand side of (32) are zero unless the same Grassmann field appears once in the monomial; hence, the fields $\psi^+_{i,s,x}$, $\psi^-_{i,s,x}$ come in pairs and the α dependence cancels. By linearity of the Grassmann integration, the property (32) implies the following identity, valid for any function f on the finite Grassmann algebra:

$$\int \left[\prod_{i,s,x} d\psi^+_{i,s,x}d\psi^-_{i,s,x}\right] f(\psi) = \int \left[\prod_{i,s,x} d\psi^+_{i,s,x}d\psi^-_{i,s,x}\right] f_\alpha(\psi)\,, \tag{33}$$

with $f_\alpha(\psi)$ the function obtained from $f(\psi)$, after the transformation (31). We apply now (33) to (7); the phase in the non-local terms can be exactly compensated by modifying J, that is we get

$$W(J, J^5, \phi) = W(J + d_\mu \alpha, J^5, e^{iQ\alpha}\phi) \tag{34}$$

where $J + d_\mu\alpha$ is a shorthand for $J_{\mu,x} + d_\mu \alpha_x$ and $e^{iQ\alpha}\phi$ is a shorthand for $e^{\pm iQ_i\alpha_x}\phi^{\pm}_{i,s,x}$; by differentiating, we get the Ward identities (WI)

$$\sum_\mu \sigma_\mu(p)\widehat{\Pi}^\Lambda_{\mu,\nu_1,..,\nu_n}(p_1,.,p_n) = 0 \quad p = p_1 + ..p_n$$

$$\sum_\mu \sigma_\mu(p)\widehat{\Gamma}^\Lambda_{\mu,i,s}(k,p) = Q_i(\widehat{S}^\Lambda_{i,s,s}(k) - \widehat{S}^\Lambda_{i,s,s}(k+p))$$

$$\sum_\nu \sigma_\nu(p_1)\widehat{\Pi}^{5,\Lambda}_{\mu,\nu,\rho}(p_1,p_2) = \sum_\rho \sigma_\rho(p_2)\widehat{\Pi}^{5,\Lambda}_{\mu,\nu,\rho}(p_1,p_2) = 0 \tag{35}$$

Remark. The above Ward identities represent the conservation of the vector part of the current coupled to the gauge field A_μ; in particular, the first is the lattice counterpart of $\partial_\mu < j^{T,V}_\mu; j^{T,V}_{\nu_1}; \dots j^{T,V}_{\nu_n} >_T = 0$, see (5).

1.5. Main result

Our main result is the following, denoting by $\lim_{L\to\infty} \widehat{S}^\Lambda_{i,s,s'}(k)$ and similarly the other correlations.

Theorem 1.1. *Let us fix $r = 1$ and $Ma \geq 1$. There exists λ_0, C independent on L, a, M such that, for $|\lambda| \leq \lambda_0(Ma)$, it is possible to find ν_i, $Z^5_{i,s}$ continuous functions in λ such that*

1. *The limits $L \to \infty$ of $\widehat{S}^\Lambda_{i,s,s'}(k)$, $\widehat{\Gamma}^\Lambda_{\mu,i',s}(k,p)$, $\widehat{\Gamma}^{5,\Lambda}_{\mu,i',s}(k,p)$ $\Pi^\Lambda_{\mu,\nu,\rho}(p_1,p_2)$, $\Pi^{5,\Lambda}_{\mu,\nu,\rho}(p_1,p_2)$ exist and $\lim_{k\to 0} \widehat{S}^\Lambda_{i,s}(k) = \infty$ and $\lim_{k,p\to 0} \frac{\widehat{\Gamma}^{5,\Lambda}_{\mu,i',s}(k,p)}{\widehat{\Gamma}^\Lambda_{\mu,i',s}(k,p)} = \varepsilon_s I$ where $\varepsilon_L = -\varepsilon_R = 1$.*
2. *The AVV correlation verifies*

$$\sum_\mu \sigma_\mu(p_1+p_2)\widehat{\Pi}^5_{\mu,\rho,\sigma}(p_1,p_2)$$

$$= \sum_{\mu,\nu} \varepsilon_{\mu,\nu,\rho,\sigma}\frac{1}{2\pi^2}\sigma_\mu(p^1)\sigma_\nu(p^2)\left[\sum_{i_1} Q^3_{i_1} - \sum_{i_2} Q^3_{i_2}\right] + r_{\rho,\sigma}(p_1,p_2) \tag{36}$$

with $|r(p_1,p_2)| \leq Ca^\vartheta \bar{p}^{2+\vartheta}$, $\bar{p} = \max(|p_1|,|p_2|)$ and $\vartheta = 1/2$.

Remarks.

(1) The correlations are written in the form of expansions which are convergent in the limit of infinite volume, provided that the lattice cut-off is smaller than the boson mass.

(2) The counterterms ν_i are chosen so that the fermions remain massless in the presence of interactions; the parameters $Z^5_{i,s}$ are fixed so that the charge associated with the vector and axial current are the same, a condition present also at a perturbative level [7].

(3) Under the condition $[\sum_{i_1} Q_{i_1}^3 - \sum_{i_2} Q_{i_2}^3] = 0$ we have $\sum_\mu \sigma_\mu(p)\widehat{\Pi}^\Lambda_{\mu,\nu,\sigma}(p_1,p_2) = 0$ and $\sum_\mu \sigma_\mu(p)\widehat{\Pi}^{5,\Lambda}_{\mu,\rho,\sigma}(p_1,p_2) = O(a^\vartheta \bar{p}^{2+\vartheta})$ expressing the conservation of the chiral current in the sense of correlations and up to subdominant terms for momenta far from the cut-off. The vanishing of the anomaly, obtained up to now only at a purely perturbative level, is proved with a finite lattice cut-off, even if the cut-off breaks important symmetries [15] on which the perturbative cancellation was based, like the Lorentz or the chiral one, and excluding non-perturbative effects. The anomaly cancellation condition is the same as in the continuum case. The lattice regularization plays an essential role; with momentum one a much weaker result holds [16].

(4) Anomalies are strongly connected with transport properties in condensed matter [17–19], and we use indeed techniques recently developed for the proof of universality properties in metals to the anomaly cancellation on a lattice [20–29]. Such methods have their roots in the Gallavotti tree expansion [30], the Battle–Brydges–Federbush formula [31] and the Gawedzki–Kupiainen–Lesniewski formula [32,33] (see e.g. [34] for an introduction).

1.6. Future Perspectives

We have constructed the theory assuming that $1/a \leq (\lambda_0/|\lambda|)M$, that is the cut-off is smaller than the boson mass and we have established (36) for generic values of the coupling. In this regime after the integration of the A_μ the theory have scaling dimension $D = 4 + n - 3n^\psi/2$ if n is the order and n_ψ the number of fields. This requires that the "effective coupling" λ^2/M^2 times the energy cut-off must be not too large so that the expansions are convergent. In order to reach higher cut-off one notes that the boson propagator (11) is composed by two terms: one which behaves as $O(1/k^2)$ for $k^2 >> M^2$ and the other which is $O(1)$ for $k^2 >> M^2$. If the second term does not contribute the scaling dimension improves and it corresponds to a renormalizable theory $D = 4 - 3n^\psi/2 - n^A$, so in principle one can consider cut-offs higher than M and up to an exponentially large values $|\lambda^2 \log a| \leq \varepsilon_0$. In order to have that the second term does not contribute full gauge invariance (broken in our case by the mass and gauge fixing term) is not necessarily required but is sufficient the gauge invariance in the external fields, expressed in the form of Ward identities. It is indeed known that renormalizability is preserved in QED, at the perturbative level, even if a mass is added to photon, see e.g. [35,36]; if one restricts to gauge-invariant observables, the contribution of the not-decaying term of the propagator is vanishing as a consequence of the current conservation. To get exponentially high cut-off in $d = 4$, QED at a non-perturbative level is technically demanding, as it would require a simultaneous decomposition in the bosons and fermions, but the analogous statement can be rigorously proven in $d = 2$ vector models [37].

In the absence of the Wilson term $r = 0$, we get the conservation of the chiral current in the form of a WI given by the first of (35), if $\widehat{\Pi}_{\mu,\nu_1,\ldots,\nu_n}$ is obtained replacing J_μ in $G^{\pm}_{\mu,j,s}$ with $b_{i,s}\widetilde{J}_{\mu,x}$. As a consequence, the averages of invariant observables are ξ independent. This follows from $\partial_\xi \int P(dA)\int [\prod_{i,s,x} d\psi^+_{i,s,x} d\psi^-_{i,s,x}]O = 0$, with $O(A,\psi)$ invariant; indeed,

$$\begin{aligned}\partial_\xi \int P(dA) \int \left[\prod_{i,s,x} d\psi^+_{i,s,x} d\psi^-_{i,s,x}\right] O \\ = \frac{1}{L^4}\sum_p \partial_\xi \widehat{g}^{-1}_{\mu,\nu}(p) \int P(dA)\widehat{A}_{\mu,p}\widehat{A}_{\nu,-p} \int \left[\prod_{i,s,x} d\psi^+_{i,s,x} d\psi^-_{i,s,x}\right] O\end{aligned} \tag{37}$$

from which we get, using that $\widehat{A}_{\mu,p} = \widehat{g}^A_{\mu,\rho}\frac{\partial}{\partial A_{\rho,-p}}$

$$\begin{aligned}\widehat{g}^A_{\mu,\rho'}(p)\partial_\xi(\widehat{g}^A(p))^{-1}_{\mu,\nu}\widehat{g}^A_{\nu,\rho}(p)\frac{\partial^2}{\partial \widehat{J}_{\rho,p}\partial \widehat{J}_{\rho',-p}} \int P(dA) \\ \int \left[\prod_{i,s,x} d\psi^+_{i,s,x} d\psi^-_{i,s,x}\right] O(A+\widetilde{J},\psi)|_0\end{aligned} \tag{38}$$

By noting that $\partial(\widehat{g}^A)^{-1} = -(\widehat{g}^A)^{-1}\partial_\xi\widehat{g}^A(\widehat{g}^A)^{-1}$ and $\partial_\xi \widehat{g}^A$ is proportional to $\bar{\sigma}_\mu\sigma_\nu$, by using $\partial_\alpha \int P(dA)\int [\prod_{i,s,x} d\psi^+_{i,s,x} d\psi^-_{i,s,x}]O(A+d\alpha,\psi)|_0 = 0$; then, $\partial_\xi W$ is vanishing. Therefore, if $r = 0$ in the average of invariant observables one can set $\xi = 0$ and the theory is perturbatively renormalizable. One expects to be able to reach exponentially high cut-off.

The Wilson term $r \neq 0$, physically necessary to avoid fermion doubling [15], breaks the WI and the conservation of chiral current for generic values of the charges, according to (36). Therefore generically the theory is non-renormalizable at scales greater than M and one cannot expect in general to be able to reach exponentially high cut-offs. However, choosing the charges so that $[\sum_{i_1} Q^3_{i_1} - \sum_{i_2} Q^3_{i_2}] = 0$ the contribution of the non-decaying term vanishes up to subdominant terms, making possible in principle to reach exponentially high cut-offs. The anomaly cancellation for $1/a \leq M$ is therefore a prerequisite for reaching higher cut-offs. In the case of the $U(1)$ sector of the Standard Model, one has also to introduce an Higgs boson to generate the fermion mass; one can distinguish a region higher than the boson mass generated by the Higgs, where the second term of the boson propagator does not contribute due to the anomaly cancellation and the WI; and a lower one, when the infinite volume limit can be taken using the infrared freedom of QED and the massive nature of weak forces. Further challenging problems arise considering the anomaly associated to the $SU(2)$ sector.

2. Proof of Theorem 1.1

In the following, we denote by C or by $C_1, C_2..$ generic λ, L, a-independent constants. We integrate the bosonic variables A_μ in (7), obtaining

$$V_F(\psi, J) = \log \int P(dA) e^{V_1(\psi,A,J)} \tag{39}$$

where, by (12)

$$\begin{aligned} V_F(\psi, J) = a^4 \sum_x \sum_{i,\varepsilon=\pm} a^{-1} \left(e^{-ia\varepsilon Q_i J_{\mu,x}} - 1\right) O^\varepsilon_{\mu,s,i} \\ + \sum_{n=2}^\infty a^{4n} \sum_{x_1,\ldots,x_n} \sum_{\substack{\underline{\varepsilon},\underline{i} \\ \underline{\mu},\underline{s}}} \left[\prod_{j=1}^n O^{\varepsilon_j}_{i_j,\mu_j,s_j,x_j} e^{i\varepsilon_j a Q_{i_j} J_{\mu_j,x_j}}\right] \\ \frac{1}{n!} a^{-n} \mathcal{E}^T_A(: e^{i\varepsilon_1 b_{i_1,s_1}\lambda a Q_{i_1} A_{\mu_1,x_1}} :; \ldots ; : e^{i\varepsilon_n b_{i_n,s_n}\lambda a Q_{i_n} A_{\mu_n,x_n}} :) \end{aligned}$$

which can be rewritten as, if $\underline{x} = x_1, .., x_n$, $\underline{i} = i_1, ..., i_n$, $\underline{\mu} = \mu_1, ..., \mu_n$,$\underline{m} = m_1, .., m_n$

$$\begin{aligned} V_F(\psi, J) = a^4 \sum_x \sum_{i,\varepsilon=\pm} a^{-1} \left(e^{-ia\varepsilon Q_i J_{\mu,x}} - 1\right) O^\varepsilon_{\mu,s,i} \\ + \sum_{n=2}^\infty \sum_{m=0}^\infty a^{4n} \sum_{x_1,\ldots,x_n} \sum_{\substack{\underline{\varepsilon},\underline{i},\underline{\mu},\underline{s},\underline{m} \\ \sum_j m_j = m}} \frac{1}{n!} \\ \left[\prod_{j=1}^n O^{\varepsilon_j}_{i_j,\mu_j,s_j,x_j} (J_{\mu_j,x_j})^{m_j}\right] H_{n,m}(\underline{x}, \underline{\varepsilon}, \underline{i}, \underline{\mu}, \underline{s}, \underline{m}) \end{aligned} \tag{40}$$

with

$$\begin{aligned} H_{n,m}(\underline{x}, \underline{\varepsilon}, \underline{i}, \underline{\mu}, \underline{s}, \underline{m}) = \frac{a^{-n}}{n!} \frac{(ia\varepsilon_j Q_{i_j})^{m_j}}{m_j!}] \\ \mathcal{E}^T_A(: e^{i\varepsilon_1 b_{i_1,s_1}\lambda a Q_i A_{\mu_1,x_1}} :; \ldots ; : e^{i\varepsilon_n b_{i_n,s_n}\lambda a Q_{i_n} A_{\mu_n,x_n}}) \end{aligned} \tag{41}$$

and $||H_{n,m}|| = L^{-4} a^{4n} \sup_{\substack{\underline{\varepsilon},\underline{i},\underline{\mu},\underline{s},\underline{m} \\ \sum_j m_j = m}} \sum_{x_1,\ldots,x_n} |H_{n,m}|$.

Lemma 2.1. *The kernels in (40) the following bound, for $n \geq 2$, $m \leq 3$ and uniformly in L*

$$||H_{n,m}|| \leq C^n a^{-(4-3n-m)N} (|\lambda|/(Ma))^{2(n-1)} \tag{42}$$

Proof of Lemma 2.1. We write the truncated expectations in (41) by the Battle–Brydges–Federbush formula, see e.g. Theorem 3.1 in [31] (for completeness a sketch of the proof is in Appendix 1), $n \geq 2$

$$\mathcal{E}_A^T(e^{i\varepsilon_1 b_{i_1,s_1}\lambda Q_{i_1}A_{\mu_1,x_1}};...;e^{i\varepsilon_n b_{i_n,s_n}\lambda a Q_{i_n}A_{\mu_n,x_n}})$$
$$= \sum_{T\in\boldsymbol{T}_n}\prod_{\{j,j'\}\in T}\widetilde{g}^A_{\mu_j\mu_{j'}}(x_j,x_{j'})\int dp_T(\underline{t})e^{-V(X;\underline{t})}, \tag{43}$$

where $X = ((x_1,\varepsilon_1,i_1,\mu_1,s_1,m_1);..;(x_n,\varepsilon_n,i_n,\mu_n,s_n,m_n))$, $\mathbf{T}_n$ is the set of connected tree graphs on $\{1,2,\dots,n\}$, the product $\prod_{\{i,j\}\in T}$ runs over the edges of the tree graph T,

$$\widetilde{g}^A_{\mu_j\mu_{j'}}(x_j,x_{j'}) = \lambda^2 a^2 \varepsilon_j b_{i_j,s_j}Q_{i_j}\varepsilon_{j'}b_{i_{j'},s_{j'}}Q_{i_{j'}}g^A_{\mu_j\mu_{j'}}(x_j,x_{j'}), \tag{44}$$

$V(X;\underline{t})$ is obtained by taking a sequence of convex linear combinations, with parameters $\underline{t}$, of the energies $V(Y)$ of suitable subsets $Y\subseteq X$, defined as

$$V(Y) = \sum_{j,j'\in Y}\lambda^2\varepsilon_j\varepsilon_{j'}b_{i_j,s_j}Q_{i_j}b_{i_{j'},s_{j'}}Q_{i_{j'}}a^2 g^A_{\mu_j,\mu_{j'}}(x_j,x_{j'})$$
$$= \mathcal{E}_A\left(\left[\sum_{j\in Y}\lambda b_{i_j,s_j}Q_{i_j}a\varepsilon_j A_{\mu_j}(x_j)\right]^2\right) \tag{45}$$

and $dp_T(\underline{t})$ is a probability measure, whose explicit form is recalled in the Appendix 1. We use the bounds

$$|g^A_{\mu,\nu}(x,y)|_1 = a^4\sum_x |g^A_{\mu,\nu}(x,y)| \le CM^{-2} \qquad |g^A_{\mu,\mu}(x,y)| \le Ca^{-2} \tag{46}$$

so that

$$||H_{n,m}|| \le C_1^n L^{-4}\sup_{\substack{\underline{\varepsilon},\underline{i},\underline{\mu},\underline{s},\underline{m}\\ \sum_j m_j=m}} a^{4n}\sum_{x_1,\dots,x_n}\frac{a^{-n+m}}{n!}\sum_{T\in\boldsymbol{T}_n}$$
$$\prod_{\{j,j'\}\in T}|\widetilde{g}^A_{\mu_j\mu_{j'}}(x_j,x_{j'})|\int dp_T(\underline{t})e^{-V(X;\underline{t})}| \tag{47}$$

Moreover, $V(Y)$ is stable, that is

$$V(Y) = \mathcal{E}\left(\left[\sum_{j\in Y}\lambda b_{i_j,s_j}Q_{i_j}a\varepsilon_j A_{\mu_j}(x_j)\right]^2\right) \ge 0 \tag{48}$$

hence $V(X;\underline{t}) \ge 0$ and $e^{-V(X;\underline{t})} \le 1$ so that $\int dp_T(\underline{t})e^{-V(X;\underline{t})} < 1$ therefore

$$||H_{n,m}|| \le C_2^n\frac{a^{-n+m}}{n!}\sum_{T\in\boldsymbol{T}_n}\prod_{\{j,j'\}\in T}a^2|g^A_{\mu_j\mu_{j'}}(x_j,x_{j'})|_1$$
$$\le C_3^n\frac{a^{-n+m}}{n!}\sum_{T\in\boldsymbol{T}_n}(aM^{-1})^{2(n-1)} \tag{49}$$

and finally using that $\sum_{T\in\boldsymbol{T}_n}1 \le C_4^n n!$ by Cayley' formula [40] we finally get

$$||H_{n,m}|| \leq C_5^n a^{-n+m}(aM^{-1})^{2(n-1)} \\ = C_5^n a^{-(4-3n-m)N}(|\lambda|/(Ma))^{2(n-1)} \tag{50}$$

□

After the integration of A_μ, the generating function can be written as a Grassmann integral:

$$e^{\mathcal{W}(J,J^5,\phi)} = \int P(d\psi) e^{V^{(N+1)}(\psi,J,J^5,\phi)} \tag{51}$$

with

$$V^{(N+1)}(\psi, J, J^5, \phi) = V_F(\psi, J) + V_2(\psi, J) + V_c(\psi) + \mathcal{B}(J^5, \psi) + (\psi, \phi) \tag{52}$$

The fermionic propagator is massless, that is it has a power law decay at large distances, and this requires a multiscale analysis based on Wilson Renormalization Group.

We introduce parameters $\gamma > 1$ and $N \in \mathbb{N}$ such that[1] $\gamma^N \equiv \pi/(16a)$; moreover, we introduce $\widetilde{f}(t); \mathbb{R}^+ \to \mathbb{R}$ a C^∞ non-decreasing function $= 0$ for $0 \leq t \leq \gamma^{N-1}$ and $= 1$ for $t \geq \gamma^N$; we define also $\chi_N(t) = 1 - \widetilde{f}(t)$ which is therefore non-vanishing for $t \leq \gamma^N$. We introduce the propagator

$$g_i^{(N+1)}(x, y) = \frac{1}{L^4} \sum_k e^{ik(x-y)} \widetilde{f}(|k|_T) \widehat{g}_i^\psi(k) \tag{53}$$

with $|k - k'|_T$ the distance on the 4-dimensional torus $[-\pi/a, \pi/a)^4$. Therefore, for any $K \in \mathbb{N}$ we have

$$|g_i^{(N+1)}(x, y)| \leq \gamma^{3(N+1)} \frac{C_K}{1 + (\gamma^{N+1}|x - y|_{\widetilde{T}})^K} \tag{54}$$

where $|x - y|_{\widetilde{T}}$ is the distance on the $[-L, L)^4$ torus. The above bound is derived by (discrete) integration by parts, see e.g. §3.3 of [34], using that[2] in the support of $\widetilde{f}(|k|_T)$ one has $\sum_\mu (1 - \cos k_\mu a)^2/a^2 \geq C/a^2$ and the volume of the support of $\widetilde{f}$ is $\leq C/a^4$.

We can write therefore, using the addition property of Gaussian Grassmann integrals, see e.g. §2.4 of [34]

$$e^{\mathcal{W}(J,J^5,\phi)} = \int P(d\psi^{(\leq N)}) \int P(d\psi^{(N+1)}) e^{V^{(N+1)}(\psi^{(\leq N)} + \psi^{(N+1)}, J, J^5, \phi)} \\ = \int P(d\psi^{(\leq N)}) e^{V^{(N)}(\psi^{(\leq N)}, J, J^5, \phi)} \tag{55}$$

[1] Any choice for γ^N ensuring that in the support of $1 - \widetilde{f}$ does not include the doubled poles, that is the poles of $\widehat{g}(k)$ with $r = 0$ different from $k = 0$, could be done.

[2] The bound (54)follows from the presence of the Wilson term; if $r = 0$ a power law is found due to the presence of poles in the support of $\widetilde{f}(k)$.

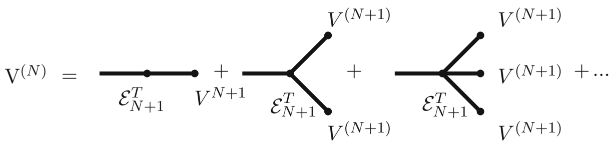

FIGURE 1. Graphical representation of (56); the first term represents $\mathcal{E}^T_{N+1}(V^{(N+1)})$, the second $\frac{1}{2}\mathcal{E}^T_{N+1}(V^{(N+1)}; V^{(N+1)})$ and so on

where

$$V^{(N)}(\psi^{(\leq N)}, J, J^5, \phi) = \sum_{n=1}^{\infty} \frac{1}{n!}\mathcal{E}^T_{N+1}(V^{(N+1)}(\psi^{(\leq N)} + \psi^{(N+1)}, J, J^5, \phi); ...; V^{(N+1)}(\psi^{(\leq N)} + \psi^{(N+1)}, J, J^5, \phi)) \tag{56}$$

and $\mathcal{E}^T_{N+1}$ is the truncated expectation with respect to the integration $P(d\psi^{(N+1)})$ (Fig. 1).

Using the linearity of the truncated expectations, one gets, if $\underline{\gamma} = \underline{\varepsilon}, \underline{s}, \underline{i}, \underline{\mu}, \underline{\beta}$

$$V^{(N)}(\psi^{(\leq N)}, J, J^5, \phi) = a^{4(l_a+l_b+m)} \sum_{\underline{x},\underline{y},\underline{z},\underline{\gamma}} W^{(N)}_{l_a,l_b,m}(\underline{x}, \underline{y}, \underline{z}, \underline{\gamma}) \left[\prod_{j=1}^{l_a} \psi^{\leq N,\varepsilon_j}_{x_j,i_j,s_j}\right] \left[\prod_{j=1}^{l_b} \phi^{\varepsilon_j}_{y_j,i_j,s_j}\right] \left[\prod_{j=1}^{m} J^{\beta_j}_{\mu_j,z_j}\right] \tag{57}$$

with $\varepsilon = \pm$, and $J^{\beta}_{x_j}$ is J_{x_j} or $J^5_{x_j}$ for $\beta = (0, 1)$. Note that the $W^{(N)}$ are a series in the kernels $H_{n,m}$. In the $l_b = 0$ case (the presence of ϕ briefly discussed in the Appendix 2) calling $W^{(N)}_{l_a,0,m} \equiv W^{(N)}_{l_a,m}$, we define $||W^{(N)}_{l,m}|| = L^{-4} \sup_{\underline{\gamma}} a^{4\,l+4\,m} \sum_{\underline{x},\underline{z}} |W^{(N)}_{l,m}(\underline{x}, \underline{z}, \underline{\gamma})|$.

Lemma 2.2. *The kernels in (55) verify, for $|\lambda| \leq \lambda_0(Ma)$, $|\nu_i| \leq C(|\lambda|/(Ma))^2$, λ_0, C, C_1 independent on a, L, N, $l \leq 2, m \leq 3$, $|d|$ is the distance between any coordinate in $\underline{x}, \underline{z}$*

$$||d^s W^{(N)}_{l,m}|| \leq C_1 \gamma^{DN} \tag{58}$$

with $D = 4 - 3l/2 - m - s$.

Proof of Lemma 2.2. We rewrite $V^{(N+1)}$ (52) in a more compact way as

$$V^{(N+1)} = \sum_P \widetilde{\psi}^{(\leq N+1)}(P)\widetilde{J}(P)W^{(N+1)}(P) \tag{59}$$

with P set of field labels and $\widetilde{\psi}(P)=\prod_{f\in P}\psi^{(\leq N)\varepsilon(f)}_{i(f),s(f),x(f)}$, $\widetilde{J}(P)=\prod_{f\in P}J^{\beta(f)}_{x(f)}$. We get therefore, inserting (59) in (56) if $P=Q_1\cup Q_2...\cup Q_n$

$$V^{(N)}(\psi^{\leq N},J,J^5)=\sum_n\frac{1}{n!}\sum_P\sum_{\substack{P_1,\dots P_n\\ Q_1,\dots Q_n}}\widetilde{\psi}^{(\leq N)}(P)$$
$$\mathcal{E}^T_{N+1}(\widetilde{\psi}^{(N+1)}(P_1/Q_1);\dots;\widetilde{\psi}^{(N+1)}(P_n/Q_n))\left[\prod_{i=1}^n W^{(N+1)}(P_i)\widetilde{J}(P_i)\right] \tag{60}$$

We use the Gawedzki–Kupiainen–Lesniewski [32,33] (a sketch of the proof is in Appendix 1; see also (see e.g. §A.3 of [38], §2 of [34] or Appendix D of [39])

$$\mathcal{E}^T_{N+1}(\widetilde{\psi}^{(N+1)}(P_1);...;\widetilde{\psi}^{(N+1)}(P_s))$$
$$=\sum_{T\in\mathcal{T}_n}\prod_{\{i,j\}\in T}g^{(N+1)}(x_i,y_j)\int dP_T(\underline{t})\det G^{N+1,T}(\underline{t}) \tag{61}$$

where $\mathcal{T}_n$ denotes the set of all the 'spanning trees' on $x_{P_1},\dots,x_{P_s}$, that is a set of lines which becomes a tree graph on $\{1,2,\dots,s\}$ if one contracts in a point all the point in $x_P=\cup_{f\in P}x(f)$, the product $\prod_{\{i,j\}\in T}$ runs over the unordered edges of the T, $\underline{t}=\{t_{i,i'}\in[0,1],1\leq i,i'\leq s\}$, $dP_T(\underline{t})$ is a probability measure (whose form is specified in the Appendix 1) with support on a set of $\underline{t}$ such that $t_{i,i'}=u_i\cdot u_{i'}$ for some family of vectors $u_i\in\mathbb{R}^s$ of unit norm and $G^{N+1,T}(\underline{t})$ is a $(n-s+1)\times(n-s+1)$ matrix, whose elements are given by $G^{N+1,T}_{ij,i'j'}=t_{i,i'}g^{(N+1)}(x_{ij},y_{i'j'})$ such that if $=<u_i\otimes A^{(N+1)}_{x(f^-_{ij})},u_{i'}\otimes B^{(N+1)}_{x(f^+_{i'j'})}>$ then the matrix element can be written as a scalar product

$$G^{N+1,T}_{ij,i'j'}=<u_i\otimes A^{(N+1)}_{x(f^-_{ij})},u_{i'}\otimes B^{(N+1)}_{x(f^+_{i'j'})}>$$
$$=(u_i\cdot u_{i'})\frac{1}{L^d}\sum_k\bar{A}^{(N+1)}_{x(f^-_{ij}),k)}B^{(N+1)}_{x(f^+_{i'j'}),k)} \tag{62}$$

with $A^{(N+1)}_{x(f^-_{ij},k)}=e^{ikx(f^-_{ij})}\sqrt{f^N(k)\bar{g}^N(k)}$ and $B^{(N+1)}_{x(f^+_{i'j'}),k)}=e^{ikx(f^+_{i'j'}))}\sqrt{f^N(k)g^N(k)}$.

The determinants are bounded by the *Gram–Hadamard inequality*, see e.g. §2 of [34], stating that, if M is a square matrix with elements M_{ij} of the form $M_{ij}=<A_i,B_j>$, where A_i, B_j are vectors in a Hilbert space with scalar product $<\cdot,\cdot>$, then $|\det M|\leq\prod_i<A_i,A_i>^{\frac{1}{2}}<B_i,B_i>^{\frac{1}{2}}$. Therefore,

$$|\det G^{N+1,T}|\leq C_1^{\sum_i|P_i|}\gamma^{3N\sum_i[|P_i|-(n-1))/2]} \tag{63}$$

We get, setting $|P_i|\equiv n_i$, $|Q_i|\equiv l_i$, $0\leq l_i\leq n_i$,$\sum_i m_i=m$, $l=\sum_i l_i$ and $||x|^s g^{(N+1)}(x)|_1\leq C_2\gamma^{-N-s}$

$$||W^{(N)}_{l,m}||\leq\sum_{n=1}^\infty C_3^n\frac{1}{n!}\sum_{n_1,\dots n_n}\sum_{\substack{l_1,\dots l_n\\ \sum_i l_i=l}}\left[\prod_{i=1}^n\frac{n_i!}{l_i!(n_i-l_i)!}\right]$$

$$\sum_T C_2^s \gamma^{-N((n-1)+s)} C_1^{\sum_i n_i} \gamma^{3N\left[\sum_i (n_i - l_i)/2 - (n-1)\right]}$$
$$\left[\prod_{i=1}^n ||W^{N+1}(P_i)||\right] \tag{64}$$

and $\sum_{T\in\mathcal{T}_n} \leq n! C_4^{\sum_i n_i}$, see e.g. Lemma A3.3 of [38], Lemma 2.4 of [34] or Lemma D.4 of [34], so that

$$||W_{l,m}^{(N)}|| \leq \sum_{n=1}^{\infty} C_6^n \sum_{n_1,\ldots n_n} \sum_{\substack{l_1,\ldots l_n \\ \sum_i l_i = l}} \left[\prod_{i=1}^n (C_5)^{\sum_i n_i} \frac{n_i!}{l_i!(n_i - l_i!)}\right]$$
$$\gamma^{-4N(n-1)} \gamma^{-Ns} \gamma^{3N\sum_i (n_i - l_i)/2} \left[\prod_{i=1}^n \gamma^{N(4-3n_i/2-m_i)}\right]$$
$$\left[\prod_i (|\lambda|/Ma)^{\max(2(n_i/2-1),1-m_i)}\right]$$

Note that $\sum_{l\leq n}[(C_5)^{n-l} C_5^l \frac{n!}{l!(n-l)!}] = (2C_5)^n$ we get

$$||W_{l,m}^{(N)}|| \leq \sum_{n=1}^{\infty} C_8^n \gamma^{N(4-3l/2-m-s)} \prod_i \left[\sum_{n_i} C_7^{n_i} (|\lambda|/Ma)^{\max(2(n_i/2-1),1-m_i)}\right] \tag{65}$$

As $\sum_i m_i = m \leq 3$ the sum over n_i is bounded by

$$\prod_i \left[\sum_{n_i} C_7^{n_i} (|\lambda|/Ma)^{\max(2(n_i/2-1),1-m_i)}\right] \leq C_8^n (|\lambda|/Ma)^{2\max(n-3,1)} \tag{66}$$

so that for λ small enough

$$||W_{l,m}^{(N)}|| \leq \gamma^{N(4-3l-m)} C_9 \left[1 + \sum_{n=4}^{\infty} C_8^n (|\lambda|/Ma)^{2(n-3)}\right]$$
$$\leq C_1 \gamma^{N(4-3l/2-m)} \tag{67}$$

□

In order to integrate $\int P(d\psi^{(\leq N)}) e^{V^{(N)}(\psi^{(\leq N)},J,J^5)}$ (55), we need to take into account the presence of terms with positive or negative scaling dimension $D = 4 - 3l/2 - m$, as can be read from (58).

In order to do that, we extract from $V^{(N)}$ the terms with non-negative dimension. This is done defining an $\mathcal{L}$ (localization) linear operation acting on the kernels of $\widehat{W}_{l,m}^N$ (the Fourier transform of $W_{l,m}^N$ in (55)) in the following way; $\mathcal{L}\widehat{W}_{l,m}^N(\underline{k}) = W_{l,m}^N(\underline{k})$ for $(n,m) \neq (2,0),(2,1)$ and

$$\mathcal{L}\widehat{W}_{2,0}^N(k) = \widehat{W}_{2,0}^N(0) + \frac{\sin k_\mu a}{a} \partial_\mu \widehat{W}_{2,0}^N(0) \qquad \mathcal{L}\widehat{W}_{2,1}^N(k,k+p) = \widehat{W}_{2,1,\mu}^N(0,0) \tag{68}$$

We write therefore

$$e^{\mathcal{W}(J,J^5,0)} = \int P(d\psi^{(\le N)}) e^{\mathcal{L}V^{(N)}(\psi^{(\le N)},J,J^5)+\mathcal{R}V^{(N)}(\psi^{(\le N)},J,J^5)} \tag{69}$$

with $\mathcal{R} = 1 - \mathcal{L}$ (renormalization) and $\mathcal{R}V^{(N)}$ is equal to (87) with $W_{l,m}^{(N)}$ replaced by $\mathcal{R}W_{l,m}^{(N)}$; the $\mathcal{R}$ operation produce an improvement in the bound, see e.g. §4.2 of [34]; for instance, $\mathcal{R}\widehat{W}^N_{2,0}(k)$ admits, by interpolation, a bound similar to the one for $\widehat{W}^N_{2,0}(k)$ times a factor $O(\gamma^{-2N})$ due to the derivatives and an extra $O(\gamma^{2h})$, with h the scale associated with the external fields due to the k^2. Hence, the $\mathcal{R}$ operation produces on such terms an improvement $O(\gamma^{2(h-N)})$. In coordinate space, the action consists in producing a derivative in the external field and a "zero", that is the difference of two coordinates, see, e.g. §3 of [22].

Using symmetry considerations, see Appendix 3, we get

$$\begin{aligned}\mathcal{L}\mathcal{V}^{(N)}(\psi,J,J^5) = a^4 \sum_x \sum_{i,s} \Big[& n_{N,s,i}\gamma^N(\psi^+_{i,L,x}\psi^-_{i,R,x} + \psi^+_{i,R,x}\psi^-_{i,L,x}) \\ & + z_{N,i,s}\sigma^s_\mu \psi^+_{i,s,x}\widetilde{\partial}_\mu \psi^+_{i,s,x} \\ & + \widetilde{Z}^J_{i,s,N} J_{\mu,x}\psi^+_{i,s,x}\sigma^s_\mu\psi^-_{i,s,x} + \varepsilon_s \widetilde{\varepsilon}_i \widetilde{Z}^5_{i,s,N} J^5_{\mu,x}\psi^+_{i,s,x}\sigma^s_\mu \psi^-_{i,s,x}\Big]\end{aligned}$$

with $\varepsilon_L = -\varepsilon_R = 1$, $\widetilde{\varepsilon}_{i_1} = -\widetilde{\varepsilon}_{i_2} = 1$, $n_{N,s,i}\gamma^{-N} = \widehat{W}^N_{2,0}(0)$, $z_{N,s,i} = \partial_\mu \widehat{W}^N_{2,0}(0)$ and $\widetilde{Z}^J_{i,s,N} = \widehat{W}^N_{2,1}(0,0)$, $\widetilde{Z}^5_{i,s,N} = \widehat{W}^N_{2,1}(0,0)$, respectively, with J and J^5.

It is possible to include the marginal quadratic terms in the fermionic Gaussian integration in the following way

$$P(d\psi^{(\le N)}) e^{\sum_{i,s} z_{h,i,s} Z_{h,s,i} a^4 \sum_x \sigma^s_\mu \psi^+_{i,s,x}\widetilde{\partial}_\mu \psi^+_{i,s,x}} \equiv P_{Z_N}(d\psi^{(\le N)}) \tag{70}$$

where $\widetilde{\partial}$ is the discrete derivative and

$$\begin{aligned}\widehat{g}_i^{(\le N)}(k) = \chi_N(k) \Bigg(& \sum_\mu \gamma_0 \widetilde{\gamma}^N_\mu a^{-1} i \sin(k_\mu a) \\ & + a^{-1}\widehat{\gamma}^N_0 \sum_\mu (1 - \cos k_\mu a) \Bigg)^{-1}\end{aligned} \tag{71}$$

$$\begin{aligned}\widetilde{\gamma}^N_0 &= \begin{pmatrix} 0 & Z_{N,L,i}(k) I \\ Z_{N,R,i}(k) I & 0 \end{pmatrix} \\ \widetilde{\gamma}^N_j &= \begin{pmatrix} 0 & iZ_{N,L,i}(k)\sigma_j \\ -iZ_{N,R,i}(k)\sigma_j & 0 \end{pmatrix}\end{aligned} \tag{72}$$

with $Z_{N,s,i}(k) = 1 + \chi_N^{-1}(k) z_{N,s,i}$, and we set $Z_{N,s,i} \equiv 1 + z_{N,s,i}$. We can write therefore

$$e^{\mathcal{W}(J,J^5,0)} = \int P_{Z_N}(d\psi^{(\le N)}) e^{\widetilde{\mathcal{L}}V^{(N)}(\sqrt{Z_N}\psi^{(\le N)},J,J^5)+\mathcal{R}V^{(N)}(\sqrt{Z_N}\psi^{(\le N)},J,J^5)} \tag{73}$$

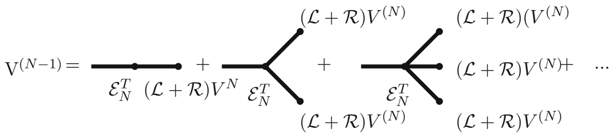

FIGURE 2. Graphical representation of (78); the first term represent $\mathcal{E}_N^T((\mathcal{L}+\mathcal{R})V^{(N+1)})$, the second $\frac{1}{2}\mathcal{E}_N^T((\mathcal{L}+\mathcal{R})V^{(N+1)};(\mathcal{L}+\mathcal{R})V^{(N+1)})$ and so on

where we have rescaled the fields writing

$$\begin{aligned}
&\widetilde{\mathcal{L}}\mathcal{V}^{(N)}(\sqrt{Z_N}\psi, J, J^5)\\
&= a^4\sum_x\sum_{i,s}\Big[\nu_{N,s,i}\gamma^N\sqrt{Z_{N,L,i}Z_{N,R,i}}(\psi^+_{i,L,x}\psi^-_{i,R,x}+\psi^+_{i,R,x}\psi^-_{i,L,x})\\
&\quad +Z^J_{i,s,N}J_{\mu,x}\psi^+_{i,s,x}\sigma^s_\mu\psi^-_{i',s,x}+\varepsilon_s\widetilde{\varepsilon}_i Z^5_{i,s,N}J^5_{\mu,x}\psi^+_{i,s,x}\sigma^s_\mu\psi^-_{i,s,x}\Big]
\end{aligned} \tag{74}$$

with $\nu_{N,s,i}\sqrt{Z_{N,L,i}Z_{N,R,i}} = n_{N,s,i}$ and $\widetilde{Z}^J_{i,s,N}/Z_{i,s,N} = Z^J_{i,s,N}$, $\widetilde{Z}^5_{i,s,N}/Z_{i,s,N} = Z_{i,s,N}$.

We choose $\chi_N(t)\equiv\chi_0(\gamma^{-N}t)$ with $\chi_0(t);\mathbb{R}^+\to\mathbb{R}$ a C^∞ non-increasing function $=1$ for $0\le t\le\gamma^{-1}$ and $=0$ for $t\ge 1$; we write

$$\chi_N(t) = \sum_{h=-\infty}^{N} f_h(t) \qquad f_h(t) = \chi_0(\gamma^{-h}t)-\chi_0(\gamma^{-h+1}t) \tag{75}$$

with $f_h(t)$ with support in $\gamma^{h-1}\le t\le\gamma^{h+1}$. We can write $\chi_N(t)=\chi_{N-1}(t)+f_N(t)$ and

$$\widehat{g}_i^{(\le N)}(k) = \widehat{g}_i^{(\le N-1)}(k)+\widehat{g}_i^{(N)}(k) \tag{76}$$

with $\widehat{g}_i^{(N)}(k)$ given by (71) with $\chi_N(k)$ replaced by $f_N(k)$ and $Z_{i,s,N}(k)$ replaced $Z_{i,s,N}$. We write therefore

$$\begin{aligned}
e^{\mathcal{W}(J,J^5,0)} &= \int P_{Z_N}(d\psi^{(\le N-1)})\int P_{Z_N}(d\psi^{(N)})\\
&\qquad e^{\widetilde{\mathcal{L}}V^{(N)}(\sqrt{Z_N}\psi^{(\le N)},J,J^5)+\mathcal{R}V^{(N)}(\sqrt{Z_N}\psi^{(\le N)},J,J^5)}\\
&= \int P_{Z_N}(d\psi^{(\le N-1)})e^{V^{(N-1)}(\sqrt{Z_N}\psi^{(\le N-1)},J,J^5)}
\end{aligned} \tag{77}$$

where

$$V^{(N-1)} = \sum_n\frac{1}{n!}\mathcal{E}_N^T(\widetilde{\mathcal{L}}V^{(N)}+\mathcal{R}V^{(N)},...;\widetilde{\mathcal{L}}V^{(N)}+\mathcal{R}V^{(N)}) \tag{78}$$

with $\mathcal{V}^N$ given by (60); a graphical representation is shown in Fig. 2. Using more compact notation

$$V^{(N-1)} = \sum_{P,\widetilde{P}}\widetilde{\psi}^{(\le N-1)}(P)\widetilde{J}(P)W^{(N-1)}(P) \tag{79}$$

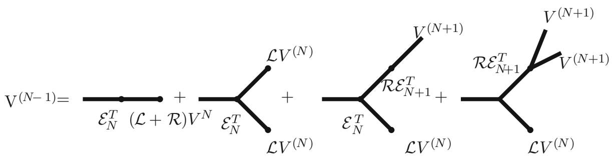

Figure 3. Graphical representation of some term in(80)

By using the linearity of the truncated expectations and expressing $\mathcal{R}V^N$ by (56) we can write, calling $\mathcal{E}^T(V;...;V)=\mathcal{E}^T(V;n)$ (78) as, see Fig. 3

$$V^{(N-1)}=\sum_n \frac{1}{n!}\mathcal{E}_N^T(\widetilde{\mathcal{L}}V^{(N)}+\sum_m \frac{1}{m!}\mathcal{R}\mathcal{E}^T_{N+1}(V^{(N+1)};m);n) \tag{80}$$

From (80), we see that $W^{(N-1)}$ is a function of $W^{(N+1)},\nu_N,Z_NZ_N^J,Z_N^5$.

The procedure can be iterated in a similar way writing

$$P_{Z_N}(d\psi^{(\le N-1)})=P_{Z_N-1}(d\psi^{(\le N-2)})P_{Z_N}(d\psi^{(N-1)}) \tag{81}$$

and $V^{(N-1)}=\mathcal{L}V^{(N-1)}+\mathcal{R}V^{(N-1)}$ with $\mathcal{L}$ acting on the kernels $W^{(N-1)}$ as (68), so that, after modifying the wave function renormalization and rescaling, we get to

$$\int P_{Z_{N-1}}(d\psi^{(\le N-2)})\int P_{Z_{N-1}}(d\psi^{(N-1)})$$
$$e^{\widetilde{\mathcal{L}}V^{(N-1)}(\sqrt{Z_{N-1}}\psi^{(\le N-1)},J,J^5)+\mathcal{R}V^{(N-1)}(\sqrt{Z_{N-1}}\psi^{(\le N-1)},J,J^5)} \tag{82}$$

Therefore, after integrating in the same way $\psi^{(N-1)},\psi^{(N-2)},...,\psi^{(h+1)}$

$$e^{\mathcal{W}(J,J^5,0)}=\int P_{Z_h}(d\psi^{(\le h)})e^{V^{(h)}(\sqrt{Z_h}\psi^{(\le h)},J,J^5)} \tag{83}$$

with $P_{Z_h}(d\psi^{(\le h)})$ with propagator

$$\widehat{g}_i^{(\le h)}(k)=\chi_h(k)\left(\sum_\mu \gamma_0\widetilde{\gamma}^h_\mu a^{-1}i\sin(k_\mu a)\right.$$
$$\left.+a^{-1}\widehat{\gamma}_0^h\sum_\mu(1-\cos k_\mu a)\right)^{-1} \tag{84}$$

$$\widetilde{\gamma}_0^h=\begin{pmatrix}0 & Z_{h,L,i}(k)I\\ Z_{h,R,i}(k)I & 0\end{pmatrix}$$
$$\widetilde{\gamma}_j^h=\begin{pmatrix}0 & iZ_{h,L,i}(k)\sigma_j\\ -iZ_{h,R,i}(k)\sigma_j & 0\end{pmatrix} \tag{85}$$

and

$$\widetilde{\mathcal{L}}\mathcal{V}^{(h)}(\sqrt{Z_h}\psi,J,J^5)$$
$$=a^4\sum_x\sum_{i,s}\Big[\nu_{h,s}\gamma^h\sqrt{Z_{h,L,i}Z_{h,R,i}}(\psi^+_{i,L,x}\psi^-_{i,R,x}+\psi^+_{i,R,x}\psi^-_{i,L,x})$$

$$+Z^J_{i,s,h} J_{\mu,x} \psi^+_{i,s,x} \sigma^s_\mu \psi^-_{i',s,x} + \varepsilon_s \widetilde{\varepsilon}_i Z^5_{i,s,h} J^5_{\mu,x} \psi^+_{i,s,x} \sigma^s_\mu \psi^-_{i,s,x} \Big] \tag{86}$$

and finally, if $\underline{\gamma} = \underline{\alpha}, \underline{s}, \underline{i}, \underline{\mu}, \underline{\beta}$

$$V^{(h-1)}(\sqrt{Z_h}\psi^{(\leq h-1)}, J, J^5)$$
$$= \sum_{l,m} a^{4l+4m} \sum_{\underline{x},\underline{z}} \sum_{\underline{\gamma}} W^{(h-1)}_{l,m}(\underline{x}, \underline{z}, \underline{\gamma}) \left[\prod_{j=1}^{l} \psi^{\leq h-1,\varepsilon_j}_{x_j,i_j,s_j}\right] \left[\prod_{j=1}^{m} J^{\beta_j}_{\mu_j,z_j}\right] \tag{87}$$

and

$$||W^{h-1)}_{l,m}|| = L^{-4} \sup_{\underline{\gamma}} a^{4l+4m} \sum_{\underline{x},\underline{z}} |W^{(h-1)}_{l,m}(\underline{x}, \underline{z})| \tag{88}$$

The $\nu_{k,i}$ is a relevant running coupling constant representing the the renormalization of the mass of the fermion of type i; $\mathcal{Z}_{k,i,s} = (Z_{k,i,s}, Z^J_{k,i,s}, Z^5_{k,i,s})$ are the marginal couplings and represent, respectively, the wave function renormalization of the fermion of type i and chirality s, and the renormalization of the current and of the axial current. By construction, $W^{(h-1)}$ is a function of the kernels $W^{(N+1)}$ in V^{N+1} and of the running coupling constants $\nu_N, \mathcal{Z}_N, ..., \nu_h, \mathcal{Z}_h$; moreover, the running coupling constants verify recursive equations of the form

$$\nu_{h-1,i} = \gamma \nu_{h,i} + \beta^h_{\nu,i}(\nu_N, ..., \nu_h, W^{(N+1)})$$
$$\mathcal{Z}_{h-1,i,s} = \mathcal{Z}_{h,i,s} + \beta^h_{\nu,i,s}(\nu_N, \mathcal{Z}_N ..., \nu_h, \mathcal{Z}_h, W^{(N+1)}) \tag{89}$$

As should be clear from the previous pictures, the W^h and the β^h can be conveniently represented in terms as a sum of labelled trees, called Gallavotti trees, see Fig. 4, defined in the following way (for details, see, e.g. §3 of [34]).

Let us consider the family of all trees which can be constructed by joining a point r, the *root*, with an ordered set of $n \geq 1$ points, the *endpoints* of the *unlabelled tree*, so that r is not a branching point. n will be called the *order* of the unlabelled tree and the branching points will be called the *non-trivial vertices*. The unlabelled trees are partially ordered from the root to the endpoints in the natural way; we shall use the symbol $<$ to denote the partial

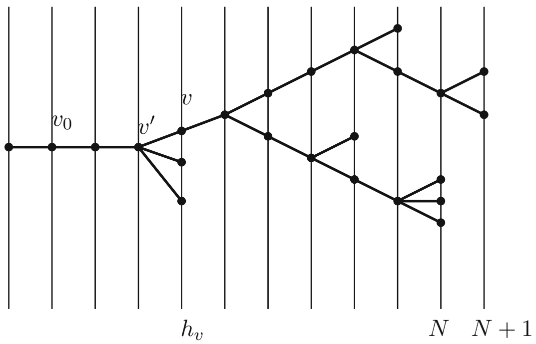

FIGURE 4. A Gallavotti tree

order. The number of unlabelled trees is $\leq 4^n$, see, e.g. §2.1 of [34]. The set of labelled (or Gallavotti) trees $\mathcal{T}_{h,n}$ are defined adding the above labels

(1) We associate a label $h \leq N-1$ with the root and we introduce a family of vertical lines, labelled by an integer taking values in $[h, N+1]$ intersecting all the non-trivial vertices, the endpoints and other points called trivial vertices. The set of the *vertices* v of τ will be the union of the endpoints, the trivial vertices and the non-trivial vertices. The scale label is h_v and, if v_1 and v_2 are two vertices and $v_1 < v_2$, then $h_{v_1} < h_{v_2}$. s_v is the number of subtrees with root v. Moreover, there is only one vertex immediately following the root, which will be denoted v_0 and cannot be an endpoint; its scale is $h+1$.
(2) To the end-points v of scale, $h_v \leq N$ is associated $\widetilde{\mathcal{L}}V^{(h_v)}$; there is the constraint that the vertex v' immediately preceding v, that is $h_{v'} = h_v - 1$ is non-trivial (as $\mathcal{RL} = 0$). The end-points with $h_v \leq N$ can be of type ν or Z.
(3) To the end-points v of scale $h_v = N+1$ is associated one of the terms in $V^{(N+1)}$
(4) Among the end-points, one distinguish between the normal ones, associated with terms not containing J_μ, J^5_μ, whose number is $\bar{n} = n - m$, and the others which are called special.
(5) There is an $\mathcal{R}$ operation associated with each vertex except the end-points and v_0; if the tree contributes to $\mathcal{R}V^h$, it is associated $\mathcal{R}$ while if it contributes to β_h is associated $\mathcal{L}$ and $s_{v_0} \geq 2$.
(6) A subtree with root at scale k is called trivial if contains only the root and an endpoint of scale $k+1$

The effective potential can be written as:

$$V^{(h)}(\sqrt{Z_h}\psi^{(\leq h)}, J, J^5) = \sum_{n=1}^{\infty} \sum_{\tau \in \mathcal{T}_{h,n}} V^{(h)}(\tau, \sqrt{Z_h}\psi^{(\leq h)}, J, J^5)\,, \tag{90}$$

where, if v_0 is the first vertex of τ and $\tau_1, .., \tau_{s_{v_0}}$ are the subtrees of τ with root v_0, $V^{(h)}$ is defined inductively by the relation, $h \leq N-1$

$$V^{(h-1)}(\tau, \sqrt{Z_h}\psi^{(\leq h)}, J, J^5) = \frac{(-1)^{s_{v_0}+1}}{s_{v_0}!} \mathcal{E}^T_h \left[\bar{V}^{(h)}(\tau_1, \sqrt{Z_h}\psi^{(\leq h)}, J, J^5); ..; \bar{V}^{(h)}(\tau_{s_{v_0}}, \sqrt{Z_h}\psi^{(\leq h)}, J, J^5)\right] \tag{91}$$

where $\mathcal{E}^T_h$ is the truncated expectation with propagator $g^{(h)}_i$ and

- if τ_i is non-trivial $\bar{V}^{(h)}(\tau_i, \sqrt{Z_h}\psi^{(\leq h)}, J, J^5) = \mathcal{R}V^{(h)}(\tau_i, \sqrt{Z_h}\psi^{(\leq h)}, J, J^5)$
- if τ is trivial it is equal to one of the terms in $\widetilde{\mathcal{L}}V^{(h)}$ if $h < N$, or to the one of the terms in $V^{(N+1)}$ if $h = N$.

We can write therefore the kernels in (87) as

$$W^{(h)}_{l,m}(\underline{x}, \underline{z}) = \sum_{n=1}^{\infty} \sum_{\tau \in \mathcal{T}_{h,n}} W^{(h)}_{l,m}(\tau, \underline{x}, \underline{z}) \tag{92}$$

It is also convenient to write

$$\mathcal{T}_{h,n} = \mathcal{T}^1_{h,n} \cup \mathcal{T}^2_{h,n} \tag{93}$$

with $\mathcal{T}^1_{h,n}$ is the subset of $\mathcal{T}_{h,n}$ containing all the trees with only end-points associated with $\widetilde{\mathcal{L}}V^k$, while $\mathcal{T}^2_{h,n}$ contains the trees with at least one end-point associated with V^{N+1}. We define

$$W^{i(h)}_{l,m}(\underline{x}, \underline{z}, \underline{\gamma}) = \sum_{n=1}^{\infty} \sum_{\tau\in\mathcal{T}^i_{h,n}} W^{(h)}_{l,m}(\tau; \underline{x}, \underline{z}, \underline{\gamma}) \tag{94}$$

with $i = 1, 2$ and $W^{(h)}_{l,m} = W^{1(h)}_{l,m} + W^{2(h)}_{l,m}$. A similar decomposition can be done for

$$\beta^h_\nu = \sum_{n=1}^{\infty} \sum_{\tau\in\mathcal{T}^2_{h,n}} \beta^h_\nu(\tau) \qquad \beta^h_{\mathcal{Z}} = \sum_{n=1}^{\infty} \sum_{\tau\in\mathcal{T}^2_{h,n}} \beta^h_{\mathcal{Z}}(\tau) \tag{95}$$

In this case by the compact support of the propagator only trees contributing to $\mathcal{T}^2_{h,n}$ are present; the contribution from $\mathcal{T}^1$ are "chain graphs" and the localization corresponds in momentum space to setting $k = 0$, and $\widehat{g}^h(0) = 0$. Finally, we can write

$$\Pi^5_{\mu,\nu,\rho} = \sum_{h=-\infty}^{N} \Pi^{5,1}_{h,\mu,\nu,\rho} + \sum_{h=-\infty}^{N} \Pi^{5,2}_{h,\mu,\nu,\rho} \tag{96}$$

with $\Pi^{5,i}_{h,\mu,\nu,\rho} = \sum_{n=1}^{\infty} \sum_{\tau\in\mathcal{T}^i_{h,n}} W^h_{0,3}(\tau)$. The following lemma holds, see Appendix 2.

Lemma 2.3. *There exists a constant ε such that, for $|\mathcal{Z}_k| \leq e^{\varepsilon(aM)^2}$, $\max(|\nu_N|, \ldots, |\nu_h|, (\lambda/Ma)^2 \leq \varepsilon$ than if $m \leq 3$, d is the distance between any two coordinate*

$$||d^s W^{j(h)}_{l,m}|| \leq C^{l+m} \gamma^{(4-(3/2)l-m-s)h} \gamma^{\vartheta_j(h-N)} \varepsilon^{\max(l/2-1,1)} \tag{97}$$

with $\vartheta_1 = 0$ and $\vartheta_2 = \vartheta$ for a constant $\vartheta = 1/2$; moreover,

$$|\beta^h_\nu| \leq \varepsilon\gamma^{\vartheta(h-N)} \quad |\beta^h_{\mathcal{Z}}| \leq \varepsilon\gamma^{\vartheta(h-N)} \tag{98}$$

The bound is proven showing the convergence of the expansion in ν_k, λ under a smallness condition which is independent from h. Note that if we perform a multiscale integration setting $\mathcal{L} = 0$ then the condition would be that $\lambda \leq \varepsilon_h$ with ε_h going to zero a $h \to -\infty$. The bound is similar to the one in Lemma 1.2, with the same "dimensional" factor $\gamma^{(4-(3/2)l-m)h}$.

A crucial point is that the contributions from trees $\mathcal{T}^2$, that is the terms obtained by the contraction of the irrelevant terms, have a gain $\gamma^{\vartheta(h-N)}$ with respect to the dimensional bounds. This fact, and the bound (98) with $\mathcal{Z}_{h-1} = 1 + \sum_{k=h}^{N} \beta^k_{\mathcal{Z}}$ implies

$$|\mathcal{Z}_{-\infty} - 1| \leq C\varepsilon \qquad |\mathcal{Z}_{-\infty} - \mathcal{Z}_h| \leq C\varepsilon\gamma^{\vartheta(h-N)} \tag{99}$$

that is the wave function and the vertex renormalization is bounded uniformly in h. In addition, we can rewrite (100) as

$$\nu_{h-1,i} = \gamma^{-h}\left(\nu_{N,i} + \sum_{k=h}^{N} \gamma^k \beta^{(h)}_{\nu,i}\right) \tag{100}$$

We consider the system

$$\nu_{h-1,i} = \gamma^{-h}\left(-\sum_{k\le h} \gamma^k \beta^{(h)}_{\nu,i}\right) \tag{101}$$

We can regard the right side of (101) as a function of the whole sequence $\nu_{k,i}$, which we can denote by $\underline{\nu} = \{\nu_k\}_{k\le N}$ so that (101) can be read as a fixed point equation $\underline{\nu} = T(\underline{\nu})$ on the Banach space of sequences ν such that $||\nu|| = \sup_{k\le N} \gamma^{\vartheta(k-N)}|\nu_k| \le C\lambda^2(Ma)^{-2}$. By a standard proof, see, e.g. Appendix A5 of [41], it is possible to prove that there is a choice of ν_i such that the sequence is bounded for any h. With this choice

$$|\nu_h| \le C\gamma^{\vartheta(h-N)}\varepsilon \tag{102}$$

This means that the $\nu_{h,i}$ is bounded so that the condition required in Lemma 2.3 is fulfilled; moreover, it is an easy consequence of the Proof of Lemma 2.3 and of (102) that the limit $L \to \infty$ can be taken; the proof is standard, see Appendix E of [41]. Finally, we can choose $Z^5_{i,s} = 1 + O(\varepsilon)$ so that

$$Z^5_{i,s,-\infty} = Z^J_{i,s,-\infty} \tag{103}$$

We finally to apply the above results and get bounds for the three current function. By (96) and the bound (97) with $l = 0, m = 3, s = 0$, we get

$$\left|\sum_{h=-\infty}^{N} \Pi^5_{h,\mu,\nu,\rho}(x,y,0)\right|_1 \le C \sum_{h=-\infty}^{N} \gamma^h < C_N \tag{104}$$

hence the Fourier transform $\widehat{\Pi}^5_{h,\mu,\nu,\rho}(p_1,p_2)$ is continuous; in addition (97) with $l = 0, m = 3, s = 1 + \vartheta/2$ and $j = 2$

$$\left|\sum_{h=-\infty}^{N}\left(|x|^{1+\vartheta/2} + |y|^{1+\vartheta/2}\right)\Pi^{5,2}_{h,\mu,\nu,\rho}(x,y,0)\right|_1 \le C\sum_{h=-\infty}^{N}\gamma^{-\vartheta/2h}\gamma^{\vartheta(h-N)} < \bar{C}_N \tag{105}$$

hence $\sum_{h=-\infty}^{N}\widehat{\Pi}^{5,2}_{h,\mu,\nu,\rho}$ has continuous derivative.

Note that $\sum_{h=-\infty}^{N}\widehat{\Pi}^{5,1}_{h,\mu,\nu,\rho}$ has a part from trees containing ν_h end-points verifying (102), which by the above argument is again differentiable. We remain then with the contribution from trees with three end points associated with Z^5, Z^J, Z^J. We can write the propagator as

$$g^{(h)}_{i,s,s'}(x,y) = \delta_{s,s'}\frac{1}{L^4}\sum_k \frac{f_h(|k|_T)}{-i\sigma^s_\mu k_\mu}e^{ik(x-y)} + r^h_{i,s,s'}(x,y) \tag{106}$$

where $r^h(x,y)$ is defined by the above equation as the difference; one can verify, again by integration by parts, that for any K

$$|g^{(h)}_{i,s,s'}(x,y)| \leq \frac{1}{Z_{h,i,s}}\gamma^{3(h+1)}\frac{C_K}{1+\left(\gamma^{h+1}|x-y|_{\widetilde{T}}\right)^K}$$
$$|r^{(h)}_{i,s,s'}(x,y)| \leq \gamma^{3(h+1)}\gamma^{h-N}\frac{C_K}{1+\left(\gamma^{h+1}|x-y|_{\widetilde{T}}\right)^K} \tag{107}$$

The above decomposition says that the lattice propagator is equal to the continuum one up to a term with a similar decay with an extra γ^{h-N}. Again the contribution of such terms is differentiable and finally we can replace the $\mathcal{Z}_h$ terms in $\sum_{h=-\infty}^N \widehat{\Pi}^{5,1}_{h,\mu,\nu,\rho}$ with $\mathcal{Z}_{-\infty}$ up again to differentiable terms, by (99). In conclusion, we get, see Fig. 5

$$\widehat{\Pi}_{\mu,\rho,\sigma}(p_1,p_2) = \widehat{\Pi}^a_{\mu,\rho,\sigma}(p_1,p_2) + \widehat{R}_{\mu,\rho,\sigma}(p_1,p_2) \tag{108}$$

with, $p = p_1 + p_2$,

$$\widehat{\Pi}^a_{\mu,\rho,\sigma}(p_1,p_2) = \sum_{\substack{h_1\\h_2,h_3}}\sum_{i,s}\widetilde{\varepsilon}_i\varepsilon_s Q_i^3 \frac{Z^5_{-\infty,i,s}}{Z_{-\infty,i,s}}\frac{Z^J_{-\infty,i,s}}{Z_{-\infty,i,s}}\frac{Z^J_{-\infty,i,s}}{Z_{-\infty,i,s}}$$
$$\int\frac{dk}{(2\pi)^4}\mathrm{Tr}\frac{f_{h_1}(k)}{i\sigma^s_\mu k_\mu}i\sigma^s_\mu\frac{f_{h_2}}{i\sigma^s_\mu(k_\mu+p_\mu)}i\sigma^s_\nu\frac{f_{h_3}}{i\sigma^s_\mu(k_\mu+p^2_\mu)}(i\sigma^s_\rho) \tag{109}$$

(108) says that the Fourier transform of the 3-current correlation can be decomposed in the sum of two terms; the first $\widehat{\Pi}^a_{\mu,\rho,\sigma}(p_1,p_2)$ is continuous and is a sum of triangle graphs equal to the its analogue in the non-interacting continuous case with momentum regularization, with vertex and wave function renormalizations depending on the species and chirality. The second $\widehat{R}_{\mu,\rho,\sigma}(p_1,p_2)$ is a complicate series of terms which is differentiable.

The renormalizations in $\widehat{\Pi}^a_{\mu,\rho,\sigma}(p_1,p_2)$ are, however, the same appearing in the 2-point and vertex correlations so that we can use the Ward identities; we can write, see Appendix 2

$$\widehat{S}_{i,s}(k) = \frac{1}{(i\sigma^s_\mu k_\mu)}\left(\frac{I}{Z_{i,s,-\infty}} + r_1(k)\right) \tag{110}$$

and

$$\widehat{\Gamma}_{\mu,i,s}(k,p) = \frac{1}{(i\sigma^s_\mu k_\mu)}\frac{Z^J_{i,s,-\infty}}{Z^2_{i,s,-\infty}}(i\sigma^s_\mu + r_{2,\mu}(k,p))\frac{1}{(i\sigma^s_\mu(k_\mu+p_\mu))} \tag{111}$$

with $|r_1(k)| \leq C(a|k|)^\vartheta$ and $|r_{2,\mu}(k,p)| \leq C(a|k|)^\vartheta$ with $|p| \leq |k|$.

By inserting (110), (111) in the Ward identities (35), we get exact relations between the wave and vertex renormalizations, that is

$$\frac{Z^J_{-\infty,i,s}}{Z_{-\infty,i,s}} = 1 \tag{112}$$

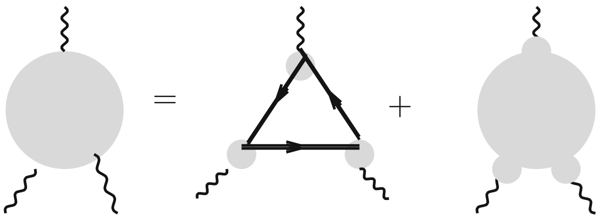

FIGURE 5. Graphical representation of (108)

Note the crucial fact that the contribution from the terms $r_{i,\mu}$, coming from the trees $\mathcal{T}^2$, is subleading. In conclusion, we get

$$\widehat{\Pi}^5_{\mu,\rho,\sigma} = \widehat{I}_{\mu,\rho,\sigma} + \widehat{\mathcal{R}}_{\mu,\rho,\sigma} \tag{113}$$

with $\widehat{\mathcal{R}}$ with Holder continuous derivative and

$$\widehat{I}_{\mu,\rho,\sigma}(p_1,p_2) = \left(\sum_i \widetilde{\varepsilon}_i Q_i^3\right) \int \frac{dk}{(2\pi)^4} \mathrm{Tr} \frac{\chi(k)}{\not k}\gamma_\mu\gamma_5 \frac{\chi(k+p)}{\not k + \not p}\gamma_\nu \frac{\chi(k+p^2)}{\not k + \not p^2}\gamma_\sigma \tag{114}$$

Note that $\widehat{I}_{\mu,\rho,\sigma}(p_1,p_2)$ is the anomaly for non-interacting relativistic continuum fermions with a momentum regularization which violates the vector current conservation, see [29], §3.6 for the explicit computation

$$\begin{aligned}\sum_\mu (p_{1,\mu}+p_{2,\mu})\widehat{I}_{\mu,\nu,\sigma} &= \frac{(\sum_i \widetilde{\varepsilon}_i Q_i^3)}{6\pi^2} p_{1,\alpha}p_{2,\beta}\varepsilon_{\alpha\beta\nu\sigma}\\ \sum_\nu p_{1,\nu}\widehat{I}_{\mu,\nu,\sigma} &= \frac{(\sum_i \widetilde{\varepsilon}_i Q_i^3)}{6\pi^2} p_{1,\alpha}p_{2,\beta}\varepsilon_{\alpha\beta\mu\sigma}\end{aligned} \tag{115}$$

up to $O(a^\vartheta|\bar{p}|^{2+\vartheta})$ corrections. In contrast to $\widehat{I}_{\mu,\rho,\sigma}$, we have that $\widehat{\mathcal{R}}_{\mu,\rho,\sigma}$ has not a simple explicit expression, being expressed in terms of a convergent series depending on all the lattice and interaction details. However, we use the differentiability of $\widehat{\mathcal{R}}_{\mu,\rho,\sigma}(p_1,p_2)$ to expand it at first order obtaining, again up to $O(a^\vartheta|\bar{p}|^{2+\vartheta})$ corrections, using the Ward identity

$$\frac{1}{6\pi^2}\left(\sum_i \widetilde{\varepsilon}_i Q_i^3\right) p_{1,\alpha}p_{2,\beta}\varepsilon_{\alpha\beta\mu\sigma} + \sum_\nu p_{1,\nu}\left(\widehat{\mathcal{R}}_{\mu,\nu,\sigma}(0,0) + \sum_{a=1,2}\sum_\rho p_{a,\rho}\frac{\partial \widehat{\mathcal{R}}_{\mu,\nu,\sigma}}{\partial p_{a,\rho}}(0,0)\right) = 0$$

This implies that

$$\widehat{\mathcal{R}}_{\mu,\nu,\sigma}(0,0) = 0 \tag{116}$$

and

$$\frac{\partial \widehat{\mathcal{R}}_{\mu,\nu,\sigma}}{\partial p_{2,\beta}} = -\frac{1}{6\pi^2}\varepsilon_{\nu\beta\mu\sigma}\left(\sum_i \widetilde{\varepsilon}_i Q_i^3\right)$$

$$\frac{\partial \widehat{\mathcal{R}}_{\mu,\nu,\sigma}}{\partial p_{1,\beta}}(0,0) = \frac{1}{6\pi^2}\varepsilon_{\nu\beta\mu\sigma}\left(\sum_i \widetilde{\varepsilon}_i Q_i^3\right) \tag{117}$$

Finally, using such values we get

$$\begin{aligned}\sum_\mu (p_{1,\mu}+p_{2,\mu})\widehat{\Pi}^5_{\mu,\nu,\sigma}(p_1,p_2) &= \sum_{\alpha,\beta}\frac{(\sum_i \widetilde{\varepsilon}_i Q_i^3)}{6\pi^2}p_{1,\alpha}p_{2,\beta}\varepsilon_{\alpha\beta\nu\sigma}\\ &+\sum_{\mu,\beta}(p_{1,\mu}+p_{2,\mu})\left(\frac{\widehat{\mathcal{R}}_{\mu,\nu,\sigma}}{\partial p_{2,\beta}}(0,0)p_{2,\beta}+\frac{\widehat{\mathcal{R}}_{\mu,\nu,\sigma}}{\partial p_{1,\beta}}(0,0)p_{1,\beta}\right)\end{aligned} \tag{118}$$

and the second term in the r.h.s. is

$$\begin{aligned}&-\frac{1}{6\pi^2}\left(p_{1,\mu}+p_{2,\mu}\right)\sum_{a=1,2}(-1)^a p_{a,\beta}\varepsilon_{\nu\beta\mu\sigma}\left(\sum_i \widetilde{\varepsilon}_i Q_i^3\right)\\ &\quad=\frac{1}{3\pi^2}p_{1,\mu}p_{2,\beta}\varepsilon_{\nu\beta\mu\sigma}\left(\sum_i \widetilde{\varepsilon}_i Q_i^3\right)\end{aligned} \tag{119}$$

which implies Theorem 1.1 □

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3. Appendix 1: Truncated Expectations

3.1. The Brydges–Battle–Federbush Formula

The starting point is the formula

$$\mathcal{E}_A\left(\prod_{i=1}^n e^{i\varepsilon_i\alpha_i A_{\mu_i}(x_i)}\right) = e^{-\frac{1}{2}\sum_{i,j}\varepsilon_i\varepsilon_j\alpha_i\alpha_j g^A_{\mu_i,\mu_j}(x_i,x_j)} \tag{120}$$

Let us define

$$e^{-V} \equiv e^{-\frac{1}{2}\sum_{j,j'\in X}\bar V_{j,j'}} \tag{121}$$

with $X=(1,2,..,n)$ and $\sum_{i,j\in X}\bar V_{i,j}=\sum_{i\le j}V_{i,j}$ $\bar V_{i,i}=V_{i,i}$ and $V_{i,j}=(\bar V_{i,j}+\bar V_{j,i})/2$.

The connected part $e^{-V(X)}|_T$ (corresponding to the truncated expectation) verify

$$e^{-V(X)} = \sum_\pi \prod_{Y\in\pi} e^{-V(Y)}|_T \tag{122}$$

where π are the partitions of X, that is $Y_1, Y_2, ...$ with $Y_1\cup Y_2\cup .. = X$.

If $X_1=\{1\}$, we can define

$$W_X(X_1;t_1) = \sum_\ell t_1(l)V_l \tag{123}$$

where $\ell=(j,j')$ is a pair of elements $j,j'\in X$ and $t_1(l)=t_1$ if l crosses the boundary of X_1 (∂X_1), that is if it connect 1 with $j\neq 1$; $t_1(\ell)=1$ otherwise. More explicitly,

$$\begin{aligned}
W_X(X_1,t_1) &= V_{1,1}+t_1\sum_{k\ge 2}V_{1,k}+\sum_{2\le k\le k'}V_{k,k'}\\
&= t_1\left(V_{1,1}+\sum_{k\ge 2}V_{1,k}+\sum_{2\le k\le k'}V_{k,k'}\right)\\
&\quad +(1-t_1)\left(V_{1,1}+\sum_{2\le k\le k'}V_{k,k'}\right)\\
&= t_1V(X)+(1-t_1)\left(V(X_1)+V(X/X_1)\right)
\end{aligned} \tag{124}$$

We get $W_X(X_1,0)=V(X_1)+V(X/X_1)$, that is if $t_1=0$ X_1 is disconnected from the rest. Therefore, using that $\partial_1 W(X_1,t_1)=\sum_{k\ge 2}V_{1,k}=\sum_{l_1}V_{l_1}$ we can write

$$e^{-V(X)} = \int_0^1 dt_1\partial_1 e^{-W_X(X_1,t_1)} + e^{-W_X(X_1,0)} \tag{125}$$

and

$$e^{-V(X)} = \int_0^1 dt_1\sum_{k\ge 2}V_{1,k}e^{-W_X(X_1,t_1)} + e^{-V(X_1)}e^{-V(X/X_1)} \tag{126}$$

We have therefore expressed $e^{-V(X)}$ as the sum of two terms; in the first there is a bond $(1,k)$ between X_1 and the rest is found, in the second X_1 is decoupled. If $n=2$, the first term is the connected part.

If $n\neq 2$, we further decompose the first term in the r.h.s of (126); we write $X_2=\{1,k\}$ and

$$\int_0^1 dt_1\sum_{k\geq 2}V_{1,k}e^{-W_X(X_1,t_1)}$$
$$=\int_0^1 dt_1\sum_{k\geq 2}V_{1,k}\int_0^1 dt_2\partial_{t_2}e^{-W_X(X_1,X_2;t_1,t_2)}$$
$$+\int_0^1 dt_1\sum_{k\geq 2}V_{1,k}e^{-W_X(X_1,X_2;t_1,0)} \tag{127}$$

where

$$W_X\left(X_1,X_2,t_1,t_2\right)=(1-t_2)\left[W_{X_2}(X_1,t_1)+V(X/X_2)\right]+t_2W_X\left(X_1,t_1\right) \tag{128}$$

and for $X_2=(1,2)$

$$W_X\left(X_1,X_2,t_1,t_2\right)=V_{1,1}+V_{2,2}+t_1t_2\sum_{k\geq 3}V_{1,k}+t_1V_{1,2}$$
$$+t_2\sum_{k\geq 3}V_{2,k}+\sum_{3\leq k\leq k'}V_{k,k'} \tag{129}$$

Suppose that $X=\{1,2,3\}$ and $X_2=\{1,2\}$, then $W_{X_3}(X_1,X_2,t_1,t_2)=V_{1,1}+V_{2,2}+t_1t_2V_{1,3}+t_1V_{1,2}+t_2V_{2,3}+V_{3,3}$ and

$$\int_0^1 dt_1V_{1,2}e^{-W_X(X_1,t_1)}=\int_0^1 dt_1V_{1,2}\int_0^1 dt_2(t_1V_{1,3}+V_{2,3})e^{-W_X(X_1,X_2;t_1,t_2)}$$
$$+\left[\int_0^1 dt_1V_{1,2}e^{-W_{X_2}(X_1;t_1)}\right]e^{-V(X/X_2)} \tag{130}$$

and the first term is connected; similar expressions for $X_2=\{1,3\}$.

Proceeding in this way

$$e^{-V(X)}=\sum_{r=1}^n\sum_{X_r\subset X}\sum_{X_1,..,X_{r-1}}\sum_T\left[\prod_{\ell\in T}V_l\right]$$
$$\left[\sum_{X_1,..,X_{r-1}}\int_0^1 dt_1...\int_0^1 dt_{r-1}\prod_{\ell\in T}\frac{\prod_{k=1}^{r-1}t_k(\ell)}{t_{n(\ell)}}e^{-W_{X_r}(X_1,..,X_{r-1};t_1,..,t_{r-1})}\right]$$
$$e^{-V(X/X_r)} \tag{131}$$

where $X_1\subset X_2\subset ...X_{r-1}$ are sets such that $|X_i|=i$, T is a tree composed by $r-1$ lines $\ell=(j,j')$ such that all the boundaries ∂X_k are intersected at least by a line $\ell=(j,j')$,

$$W_X\left(X_1,\ldots,X_r;t_1,\ldots,t_r\right)=\sum_l t_1(l)t_2(l)\ldots t_r(l)V_l \tag{132}$$

with $t_i(l)=t_i$ if l crosses ∂X_i and $t_i(l)=1$ otherwise, $n(l)$ is the max over k such that l crosses ∂X_k. For instance, in the case (130) the trees are $l_1=$

$(1,2), l_2 = (2,3)$ so that $t_1(l_1) = t_1$, $t_1(l_2) = 1$, $t_2(l_2) = t_2$; and $l_1 = (1,2)$, $l_2 = (1,3)$ so that $t_1(l_1) = t_1$ and $t_1(l_2) = t_1$,$t_2(l_2) = t_2$.

We can reverse the sum over T and X

$$\sum_T \sum_{X_1,\ldots,X_{r-1}} = \sum_{X_1,\ldots,X_{r-1}} \sum_T \tag{133}$$

where in the l.h.s. the sets have to be compatible with T. If $n'(\ell)$ is the minimal k such that ℓ crosses X_k we have $\frac{\prod_{k=1}^{r-1} t_k(\ell)}{t_{n(\ell)}} = t_{n'(\ell)} \ldots t_{n(\ell)-1}$ and, see, e.g. Lemma 2.3 in [34]

$$\sum_{\substack{X_1,\ldots,X_{r-1}\\ fixed T}} \int_0^1 dt_1 \ldots \int_0^1 dt_{r-1} t_{n'(\ell)} \ldots t_{n(\ell)-1} = 1 \tag{134}$$

By calling

$$dp_T(t) = \sum_{\substack{X_1,..,X_{r-1}\\ fixed T}} \frac{\prod_{k=1}^{r-1} t_k(l)}{t_{n(l)}} \tag{135}$$

we get

$$e^{-V(X)}|_T = \sum_T \left[\prod_{\ell\in T} V_l\right] \int_0^1 d\underline{t} dp_T(t) \varepsilon^{e^{\sum_{\ell\in X} t_{n'(\ell)}\ldots t_{n(\ell)-1} V_\ell}} \tag{136}$$

where $\ell \in X$ means $j, j' \in (1,..,n)$.

3.2. The Gawedzki–Kupiainen–Lesniewski Formula

We can write the simple expectations as

$$\mathcal{E}\left(\widetilde{\psi}(P_1)...\widetilde{\psi}(P_r)\right) = \int \prod_{i,j} d\eta_{i,j} e^{-\sum_{j,j'} V_{jj'}} \tag{137}$$

with $V_{jj'} = \sum_{i=1}^{|P_j|} \sum_{i'=1}^{|P'_j|} \eta^+_{x_{ij}} g(x_{ij}, x_{i'j'}) \eta^-_{x_{i'j'}}$ and $\eta^\pm_{i,j}$ is a set of Grassmann variables. Again, we can write $e^{-\sum_{j,j'} V_{jj'}}$ as in (131) obtaining

$$\mathcal{E}^T\left(\widetilde{\psi}(P_1)...\widetilde{\psi}(P_r)\right) = \tag{138}$$

$$\int \prod d\eta^+_{i,j} d\eta^-_{i,j} \sum_T \left[\prod_{l\in T} V_l\right] \int_0^1 d\underline{t} dp_T(\underline{t}) \varepsilon^{-\sum_{\ell\in X} t_{n'(\ell)}\ldots t_{n(\ell)-1} V_\ell} \tag{139}$$

with $V_\ell = \sum_i \sum_{i'} \eta^+_{i,j} g(x_{ij}, x_{i'j'}) \eta^-_{i,j}$, $\ell = (j,j')$. For each tree T, we divide the η in the ones appearing in T, called $\widetilde{\eta}$, and the rest, called $\bar{\eta}$ so that, if $\sum_{\ell\in X'} t_{n'(\ell)}...t_{n(\ell)-1} V_\ell = \widetilde{V}(\underline{t}) + \bar{V}(\underline{t})$ with $\bar{V}(\underline{t})$ obtained setting $\widetilde{\eta} = 0$

$$\mathcal{E}^T\left(\widetilde{\psi}(P_1)...\widetilde{\psi}(P_r)\right) = \sum_T \left[\prod_{l\in T} g_\ell\right] \int_0^1 d\underline{t} dp_T(\underline{t}) \int \prod d\bar{\eta}^+_{i,j} d\bar{\eta}^-_{i,j} \varepsilon^{-\bar{V}(\underline{t})} \tag{140}$$

and $\prod d\bar{\eta}^+_{i,j} d\bar{\eta}^-_{i,j} \varepsilon^{-\bar{V}(\underline{t})} = \det G_T$ with G_T with elements $t_{n'(j,j')}...t_{n(jj')-1}$ $g(x_{ij}, x_{i'j'})$. Fixed T we can relabel the X_k so that $t_j...t_{j'-1} = u_j u_{j'}$ with

$u_1 = v_1$, $u_j = t_{j-1}u_{j-1} + v_j\sqrt{1-t_{j-1}^2}$ with v_j orthonormal, and $u_1 u_2 = t_1$, $u_1 u_3 = t_1 t_2$, $u_2 u_3 = t_2$ and so on.

4. Appendix 2: Proof of Lemma 2.3

The proof is a generalization of the Proof of Lemma 2.2 adapted to the tree structure. We define P_v as the set of field labels of the external fields of v and if $v_1, \ldots, v_{s_v}$ are the s_v vertices immediately following v, we denote by Q_{v_i} the intersection of P_v and P_{v_i}. This definition implies that $P_v = \cup_i Q_{v_i}$. The union of the subsets $P_{v_i} \backslash Q_{v_i}$ are the internal fields of v. The set of all P_v, $v \in \tau$ is called $\mathcal{P}$, and the set of all P_v with $v \in \tau_i$ is called $\mathcal{P}_i$. From (91) we get, if n_{v_0} is the number of coordinate

$$V^{(h)}(\tau) = \sum_{\mathcal{P}} a^{4n_{v_0}} \sum_{x_{v_0}} W^{(h)}_{\tau,\mathcal{P}}(x_{v_0}) \left[\prod_{f\in P_{v_0}} \sqrt{Z_h}\psi^{\varepsilon(f)(\le h)}_{x(f),i(f),s(f)}\right]\left[\prod_f J(x_f)\right] \tag{141}$$

By definition, we have a truncated expectation associated with each v in the tree τ non-associated with an end-point; we can write each of them by the Gawedzki–Kupiainen–Lesniewski formula. The $\mathcal{R}$ operation is applied and by an iterative procedure one can show that the number of zeros associated with propagators of T and the derivative on the fields is bounded by a constant; see e.g. §3 of [22].

The bound is obtained using the Gram bound for the determinant; to each vertex is therefore associated a spanning tree T_v which is used to perform the sum over the coordinate difference, and $T = \cup_v T_v$. The sum over coordinates of the propagators in T and the estimates of the determinants give a factor $\gamma^{-4h_v(s_v-1)}\gamma^{3/2h_v(\sum_i |P_{v_i}|-|P_v|)}$, if S_v is the number of subtrees with root v. The renormalization produces a factor $\prod_v \gamma^{-z_v(h_v-h_{v'})}$ is produced by the $\mathcal{R}$ operation and $z_v = 2$ if $|P_v| = 2$ and there are no J fields, $z_v = 1$ if $|P_v| = 2$ and there is a single J field, $z_v = 0$ otherwise. To the end points not ν, Z is with i ψ fields and j J fields is associated by lemma 2.1 a factor $\gamma^{(4-3i_v/2-j_v)N}(\lambda^{i_v/2}(aM)^{2-i_v})$ with $(4-3i_v/2-j_v) < 0$ and $i_v \ge 4$ and $(aM)^{2-i_v} < (aM)^{-2}$. We get therefore

$$\begin{aligned} a^{4n_{v_0}} \sum_{x_{v_0}} |W_{\tau,\mathbf{P}}(x_{v_0})| \le L^4 \sum_T \prod_{v\, not\, e.p.} \frac{1}{s_v!} C^{\sum_{i=1}^{s_v}|P_{v_i}|-|P_v|}\gamma^{-4h_v(s_v-1)} \\ \gamma^{3/2h_v(\sum_i |P_{v_i}|-|P_v|)}\left[\prod_v \gamma^{-z_v(h_v-h_{v'})}\right] \\ \left[\prod_{v\ e.p.\ not\ \nu,Z} \gamma^{(4-3i_v/2-j_v)N}\right]\left[\prod_{v\ e.p.\ \nu} \gamma^{h_v}\right]\varepsilon^{\bar{n}} \end{aligned} \tag{142}$$

By using that

$$\sum_v (h_v - h)(s_v - 1) = \sum_v (h_v - h_{v'}) \left(\sum_{i,j} m_v^{i,j} - 1 \right)$$

$$\sum_v (h_v - h) \left(\sum_i |P_{v_i}| - |P_v| \right)$$

$$= \sum_v (h_v - h_{v'}) \left(\sum_{i,j} i m_v^{i,j} - |P_v| \right) \tag{143}$$

where $m_v^{i,j}$ is the number of end-points following v with i ψ fields and j J fields, we get

$$a^{4n_{v_0}} \sum_{x_{v_0}} |W_{\tau,\mathbf{P},T}(x_{v_0})| \leq L^4 \gamma^{-h\left[-4+\frac{3|P_{v_0}|}{2} - \sum_{i,j}(3i/2-4) m_{v_0}^{i,j}\right]} \varepsilon^{\bar{n}}$$

$$\prod_{v \text{ not e.p.}} \left\{ \frac{1}{s_v!} C^{\sum_{i=1}^{s_v} |P_{v_i}| - |P_v|} \gamma^{-(-4+\frac{3|P_v|}{2} - \sum_{i,j}(3i/2-4) m_v^{i,j} + z_v)(h_v - h_{v'})} \right\}$$

$$\left[\prod_{v \text{ e.p. not } \nu} \gamma^{(4-3i_v/2-j_v)N} \right] \left[\prod_{v \text{ e.p. } \nu} \gamma^{h_v} \right]$$

We use now that

$$\gamma^{h \sum_{i,j} m_{v_0}^{i,j}} \prod_{v \text{ not e.p.}} \gamma^{\sum_{i,j}(h_v - h_{v'}) m_v^{i,j}} = \prod_{v \text{ e.p.}} \gamma^{h_{v^*}} \tag{144}$$

where v^* is the first non-trivial vertex following v; this implies

$$\gamma^{h \sum_{i,j}(3i/2-4) m_{v_0}^{i,j}} \prod_{v \text{ not e.p.}} \gamma^{\sum_{i,j}(3i/2-4) m_v^{i,j}(h_v - h_{v'})}$$

$$= \prod_{v \text{ e.p not } \nu} \gamma^{h_{v^*}(3i_v/2-4)} \prod_{v \text{ e.p. } \nu} \gamma^{-h_v} \tag{145}$$

so that

$$a^{4n_{v_0}} \sum_{x_{v_0}} |W_{\tau,\mathbf{P},T}(x_{v_0})| \leq L^4 \gamma^{-h\left[-4+\frac{3|P_{v_0}|}{2}\right]} \prod_{v \text{ e.p not } \nu} \gamma^{h_{v^*}(3i_v/2-4)} \varepsilon^{\bar{n}}$$

$$\prod_{v \text{ not e.p.}} \left\{ \frac{1}{s_v!} C^{\sum_{i=1}^{s_v} |P_{v_i}| - |P_v|} \gamma^{-(-4+\frac{3|P_v|}{2} + z_v)(h_v - h_{v'})} \right\}$$

$$\left[\prod_{v \text{ e.p. not } \nu} \gamma^{(4-3i_v/2-j_v)N} \right] \tag{146}$$

Finally, we use the relation

$$\left[\prod_{v \text{ e.p.}} \gamma^{h_{v^*} j_v} \right] \left[\prod_{v \text{ e.p.}} \gamma^{-h_{v^*} j_v} \right]$$

$$= \left[\prod_{v\ e.p.} \gamma^{h_{v^*} j_v}\right] \gamma^{-h \sum_{i,j} j m_{v0}^{i,j}} \prod_{v\ not\ e.p.} \gamma^{-\sum_{i,j}(h_v - h_{v'}) j m_v^{i,j}} \tag{147}$$

and using that $\sum_{i,j} j m_v^{i,j} = n_v^J$, we finally get ($j_v = 0$ if v is a ν-e.p.)

$$a^{4n_{v_0}} \sum_{x_{v_0}} |W_{\tau,\mathbf{P},T}(x_{v_0})| \leq L^4 \gamma^{-h\left[-4+\frac{3|P_{v_0}|}{2}+n_{v_0}^J\right]} \varepsilon_h^{\bar{n}}$$
$$\prod_{v \text{ not e.p.}} \left\{\frac{1}{s_v!} C^{\sum_{i=1}^{s_v}|P_{v_i}|-|P_v|} \gamma^{-\left(-4+\frac{3|P_v|}{2}+z_v+n_v^J\right)(h_v-h_{v'})}\right\}$$
$$\left[\prod_{v\ e.p.\ not\ \nu,Z} \gamma^{(4-3i_v/2-j_v)(N-h_{v^*})}\right]$$

In conclusion,

$$a^{4n_{v_0}} \sum_{x_{v_0}} |W_{\tau,\mathbf{P},T}(\mathbf{x}_{v_0})| \leq L^4 \gamma^{-h d_{v_0}} C^n \varepsilon^{\bar{n}}$$
$$\left[\prod_{\widetilde{v}} \frac{1}{s_{\widetilde{v}}!} \gamma^{-d_{\widetilde{v}}(h_{\widetilde{v}}-h_{\widetilde{v}'})}\right] \left[\prod_{v\ e.p.\ not;\nu,Z} \gamma^{(4-3i_v/2-j_v)(N-h_{v^*})}\right] \tag{148}$$

where $\widetilde{v} \in \widetilde{V}$ are the vertices on the tree such that $\sum_i |P_{v_i}| - |P_v| \neq 0$, $\widetilde{v}'$ is the vertex in $\widetilde{V}$ immediately preceding $\widetilde{v}$ or the root; $d_v = -4 + \frac{3|P_v|}{2} + n_v^J + z_v$. Finally, the number of addenda in $\sum_{T\in\mathbf{T}}$ is bounded by $\prod_v s_v!\, C^{\sum_{i=1}^{s_v}|P_{v_i}|-|P_v|}$, see e.g. §2.1 of [38]. In order to bound the sums over the scale labels and $\mathbf{P}$, we first use the inequality

$$\prod_{\widetilde{v}} \gamma^{-d_{\widetilde{v}}(h_{\widetilde{v}}-h_{\widetilde{v}'})} \leq \left[\prod_{\widetilde{v}} \gamma^{-\frac{1}{2}(h_{\widetilde{v}}-h_{\widetilde{v}'})}\right] \left[\prod_{\widetilde{v}} \gamma^{-\frac{3|P_{\widetilde{v}}|}{4}}\right] \tag{149}$$

where $\widetilde{v}$ are the non-trivial vertices, and $\widetilde{v}'$ is the non-trivial vertex immediately preceding $\widetilde{v}$ or the root. The factors $\gamma^{-\frac{1}{2}(h_{\widetilde{v}}-h_{\widetilde{v}'})}$ in the r.h.s. allow to bound the sums over the scale labels by C^n and $\sum_{\mathbf{P}} \prod_{\widetilde{v}} \gamma^{-\frac{3|P_{\widetilde{v}}|}{4}} \leq C^n$, see §3.7 of of [34].

Let us consider the improvement of the bound. If $\mathcal{T}^*$ is the set of trees with at least an end-point not of ν, Z type then, for $0 < \vartheta < 1$

$$\sum_{\tau\in\mathcal{T}^*} \sum_{\mathbf{P},T} a^{4n_{v_0}} \sum_{x_{v_0}} |W_{\tau,\mathbf{P},T}(x_{v_0})| \leq L^4 \gamma^{(4-(3/2)l-m)h} \gamma^{\vartheta(h-N)} \varepsilon^{\max(l/2-1,1)} \tag{150}$$

To prove (150), let be $\widehat{v}$ the non-trivial vertex following an end-point not of ν, Z type; hence, we can rewrite in (148)

$$\left[\prod_{\widetilde{v}} \gamma^{-d_{\widetilde{v}}(h_{\widetilde{v}}-h_{\widetilde{v}'})}\right] = \left[\prod_{\widetilde{v}} \gamma^{-(d_{\widetilde{v}}-\vartheta)(h_{\widetilde{v}}-h_{\widetilde{v}'})}\right] \gamma^{\vartheta(h-h_{\widehat{v}})} \tag{151}$$

and

$$\gamma^{\vartheta(h-h_{\widehat{v}})}\left[\prod_{v\ e.p.\ not\ \nu,Z}\gamma^{(4-3i_v/2-j_v)(N-h_{v^*})}\right]\le\gamma^{\vartheta(h-N)} \tag{152}$$

as $\prod_{v\ e.p.\ not\ \nu,Z}\gamma^{(4-3i_v/2-j_v)(N-h_{v^*})}\le\gamma^{-\vartheta(N-h_{\widehat{v}})}$ as there is at least an end-point not ν, Z. Noting that $d_{\widetilde{v}}-\vartheta>0$ one can perform the sum as above, and the same bound is obtained with an extra $\gamma^{\vartheta(h-N)}$. □

In the presence of a ϕ term, there is a new relevant coupling proportional to $\psi\phi$, whose local part is vanishing again by the compact support of the propagator. We can compare the bound from the one of a term of the effective potential with $l=2$ with two ν end-points. On each tree there is a vertex with scale $\bar{h}$ which is the root of the subtree to which belong both the end-points associated with $(\psi\phi)$; there is an integral missing giving an extra factor $\gamma^{4\bar{h}}$ and a $\gamma^{-2\bar{h}}$ from the lack of the ν end-points. There is a decay factor proportional to $x-y$ at scale $\gamma^{\bar{h}}$ and, from the trees belonging to $\mathcal{T}^*$, an extra $\gamma^{\vartheta(h-N)}$; see e.g. §3.D of [25]. A similar argument holds for the vertex function. Finally the proof of the $L\to\infty$ limit is an easy corollary of the Proof of Lemma 2.3, see e.g. Appendix D of [25].

5. Appendix 3: Symmetries

By symmetry, there are no quadratic contributions with $i'\neq i$. There is invariance under the transformation $\psi^{\pm}_{k,s}\to\varepsilon_s\psi^{\mp}_{\widetilde{k},s}\sigma_1$, $J_{\mathbf{k}}\,A_{\mathbf{k}}\to J_{\widetilde{\mathbf{k}}}, A_{\widetilde{\mathbf{k}}}$ invariant, if $\widetilde{k}$ is equal to k with k_0,k_1 replaced with $-k_0,-k_1$ and k_2,k_3 invariant. As $j=(2,3)$ $\sigma_1\sigma_j\sigma_1=-\sigma_j$ hence $\sum_k\sin k_j\psi^+_{k,s}\sigma_j\psi^-_{k,s}\to\sum_k\sin k_j\psi^-_{\widetilde{k},s}\sigma_1\sigma_j\sigma_1\psi^+_{\widetilde{k},s}=\sum_k\sin k_j\psi^+_{k,s}\sigma_0\psi^-_k$ and for $j=0,1$ $\sigma_1\sigma_j\sigma_1=\sigma_j$ hence $\sum_k\sin k_j\psi^+_{k,s}\sigma_0\psi^-_k\to\sum_k\sin k_j\psi^-_{\widetilde{k},s}\sigma_1\sigma_j\sigma_1\psi^+_{\widetilde{k},s}=\sum_k\sin k_j\psi^+_{k,s}\sigma_j\psi^-_{k,s}$; and $\sum_k\cos k_j\psi^+_{k,L}\sigma_0\psi^-_{k,R}\to\sum_k-\cos k_i\psi^-_{\widetilde{k},L}\sigma_1\sigma_0\sigma_1\psi^+_{\widetilde{k},R}$. Similarly there is invariance under the transformation $\psi^{\pm}_{k,s}\to\varepsilon_s\psi^{\mp}_{\widetilde{k},s}\sigma_2$, $J_{\mathbf{k}}\,A_{\mathbf{k}}\to J_{\widetilde{\mathbf{k}}}, A_{\widetilde{\mathbf{k}}}$ invariant, if $\widetilde{k}$ is equal to k with k_0,k_2 replaced with $-k_0,-k_2$ and k_1,k_3 invariant. As $\sigma_2\sigma_j\sigma_2=-\sigma_j$ $j=(1,3)$ hence $\sum_k\sin k_j\psi^+_{k,s}\sigma_j\psi^-_{k,s}\to\sum_k\sin k_j\psi^-_{\widetilde{k},s}\sigma_1\sigma_j\sigma_1\psi^+_{\widetilde{k},s}=\sum_k\sin k_j\psi^+_{k,s}\sigma_0\psi^-_k$ and for $j=0,2$ $\sigma_2\sigma_j\sigma_2=\sigma_j$ hence $\sum_k\sin k_j\psi^+_{k,s}\sigma_0\psi^-_k\to\sum_k\sin k_j\psi^-_{\widetilde{k},s}\sigma_1\sigma_j\sigma_1\psi^+_{\widetilde{k},s}=\sum_k\sin k_j\psi^+_{k,s}\sigma_j\psi^-_{k,s}$; and $\sum_k\cos k_j\psi^+_{k,L}\sigma_0\psi^-_{k,R}\to-\sum_k\cos k_i\psi^-_{\widetilde{k},L}\sigma_1\sigma_0\sigma_1\psi^+_{\widetilde{k},R}$.

We can write $\sum_k k_2\psi^+_{\widetilde{k},s}A_2\psi^-_{\widetilde{k},s}=\sum_k k_2[a\sigma_0+b_\mu\sigma_1+c_\mu\sigma_2+d_\mu\sigma_3]$. We apply the first transformation to $\sum_k k_2\psi^+_{k,s}\psi^-_{k,s}(a\sigma_0+b\sigma_1+c\sigma_2+d\sigma_3)\to-\sum_k k_2\psi^+_{\widetilde{k}}\sigma_1(a\sigma_0+b\sigma_1+c\sigma_2+d\sigma_3)\sigma_1\psi^-_{\widetilde{k}}=-\sum_k k_2\psi^+_{k,s}\psi^-_{k,s}(a\sigma_0+b\sigma_1-c\sigma_2-d\sigma_3)\sigma_1\psi^-_k$ hence $a=b=0$. Now we apply the second transformation then $\sum_k k_2\psi^+_{k,s}\psi^-_{k,s}\sigma_2(c\sigma_2+d\sigma_3)\sigma_2\to-\sum_k k_2\psi^+_{\widetilde{k},s}\psi^-_{\widetilde{k},s}(c\sigma_2-d\sigma_3)=\sum_k k_2\psi^+_{\widetilde{k},s}\psi^-_{\widetilde{k},s}(c\sigma_2-d\sigma_3)$ hence $d=0$. Then $\sum_k k_2\psi^+_{\widetilde{k},s}A\psi^-_{\widetilde{k},s}=\sum_k k_2 b\psi^+_{k,s}\sigma_2\psi^-_{k,s}$, and the geeral relation follows from isotropy. Proceeding in a similar way with the terms with different chirality $\sum_k k_2\widetilde{b}\psi^+_{\widetilde{k},L}\sigma_2\psi^-_{\widetilde{k},R} \to -\sum_k k_2\widetilde{b}\psi^+_{\widetilde{k},L}\sigma_2\psi^-_{\widetilde{k},R}$ hence $\widetilde{b}=0$.

Finally by the first tranformation $\sum_k \psi^+_{k,L}\psi^-_{k,R}(a\sigma_0+b\sigma_1+c\sigma_2+d\sigma_3) \to \sum_k \psi^+_{\widetilde{k},L}\sigma_1(a\sigma_0+b\sigma_1+c\sigma_2+d\sigma_3)\sigma_1\psi^-_{\widetilde{k},R} = \sum_k \psi^+_{k,L}(a\sigma_0+b\sigma_1-c\sigma_2-d\sigma_3)\sigma_1\psi^-_{k,R}$ so that $c=d=0$; by the second $\sum_k \psi^+_{k,L}\psi^-_{k,R}(a\sigma_0+b\sigma_2) \to \sum_k \psi^+_{\widetilde{k},L}\sigma_2(a\sigma_0+b\sigma_1)\sigma_2\psi^-_{\widetilde{k},R} = \sum_k \psi^+_L\sigma_2(a\sigma_0-b\sigma_1)\sigma_2\psi^-_{k,R}$; hence, $b=0$.

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