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Failure of the Blok–Esakia Theorem in the monadic setting

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ABSTRACT

The Blok–Esakia Theorem establishes that the lattice of superintuitionistic logics is isomorphic to the lattice of extensions of Grzegorczyk's logic. We prove that the Blok–Esakia isomorphism σ does not extend to the fragments of the corresponding predicate logics of already one fixed variable. In other words, we prove that σ is no longer an isomorphism from the lattice of extensions of the monadic intuitionistic logic to the lattice of extensions of the monadic Grzegorczyk logic.

1. Introduction

It is a classic result of McKinsey and Tarski [32] that the Gödel translation embeds the intuitionistic propositional calculus IPC into Lewis' modal system S4. A systematic study of the relationships between extensions of IPC (superintuitionistic logics) and extensions of S4 was initiated by Dummett and Lemmon [14] and further studied by Maksimova and Rybakov [31], Blok and Dwinger [9], Blok [8], and Esakia [16–18].

Let L be a superintuitionistic logic. Applying the Gödel translation $(-)^t$ to the theorems of L embeds L into the following normal extension of S4:

$$\tau \mathsf{L} = \mathsf{S4} + \{\varphi^t : \mathsf{L} \vdash \varphi\}.$$

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For a normal extension M of $\mathsf{S4}$, let

$$\rho\mathsf{M} = \{\varphi : \mathsf{M} \vdash \varphi^t\}.$$

Then $\rho\mathsf{M}$ is a superintuitionistic logic, and we call M a *modal companion* of a superintuitionistic logic L provided $\mathsf{L} = \rho\mathsf{M}$, in which case L is called the *intuitionistic fragment* of M (see, e.g., [12, Sec. 9.6]).

Esakia [18] proved that all modal companions of a superintuitionistic logic L form the interval $[\tau\mathsf{L}, \sigma\mathsf{L}]$ in the lattice of normal extension of $\mathsf{S4}$, where $\sigma\mathsf{L}$ is obtained from $\tau\mathsf{L}$ by postulating the *Grzegorczyk axiom*

$$\mathsf{grz} = \Box(\Box(p \to \Box p) \to p) \to p.$$

In other words, if $\mathsf{Grz} := \mathsf{S4} + \mathsf{grz}$ is the *Grzegorczyk logic*, then

$$\sigma\mathsf{L} = \mathsf{Grz} + \{\varphi^t : \mathsf{L} \vdash \varphi\}.$$

Let $\Lambda(\mathsf{IPC})$ be the lattice of superintuitionistic logics and $\Lambda(\mathsf{Grz})$ the lattice of normal extensions of Grz. By the celebrated Blok–Esakia Theorem, $\sigma\colon \Lambda(\mathsf{IPC}) \to \Lambda(\mathsf{Grz})$ is an isomorphism (see, e.g., [12, Thm. 9.66]).

The Gödel translation has a natural extension to the predicate setting, and Rasiowa and Sikorski (see, e.g., [37, XI.11.5]) proved that it embeds the intuitionistic predicate calculus IQC into $\mathsf{QS4}$ (the predicate $\mathsf{S4}$). However, the behavior of modal predicate companions of superintuitonistic predicate logics is much less understood. For example, it remains open whether the predicate Grzegorczyk logic QGrz is a modal companion of IQC, let alone the largest modal companion.[1]

Hilbert and Ackermann [28] initiated the study of the monadic fragment of classical predicate logic, where only one variable is allowed in monadic predicates.[2] Wajsberg [38] proved that this fragment is axiomatized by $\mathsf{S5}$, and Halmos [27] conducted an algebraic study of this fragment. Prior [36] introduced the monadic intuitionistic calculus MIPC, which Bull [10] proved to axiomatize the monadic fragment of IQC. Fischer Servi [21] defined $\mathsf{MS4}$ (monadic $\mathsf{S4}$) and proved that the Gödel translation embeds MIPC into $\mathsf{MS4}$.

Monadic logics are better understood than predicate logics. They can be thought of as bimodal logics [21,19,5], and hence can be studied using the standard semantic tools in modal logic (see, e.g., [24]). Because of this, normal extensions of MIPC and $\mathsf{MS4}$ have been studied more extensively than their predicate counterparts. In particular, the monadic Grzegorczyk logic MGrz was introduced in [19], where it was shown that MGrz is a modal companion of MIPC. It is natural to ask whether the Blok–Esakia Theorem extends to the monadic setting. Our main contribution proves that it does **not**. Our main tool is the algebraic semantics for MIPC and $\mathsf{MS4}$ provided by the varieties $\mathbb{MHA}$ of monadic Heyting algebras and $\mathbb{MS4}$ of monadic $\mathsf{S4}$-algebras, which generalize Halmos' monadic boolean algebras [26]. We also heavily use the representation theory for $\mathbb{MHA}$ and $\mathbb{MS4}$, and the corresponding descriptive frames.

The paper is organized as follows. In Section 2 we recall MIPC and its algebraic and descriptive frame semantics, and in Section 3 we do the same for $\mathsf{MS4}$. In Section 4 we generalize ρ, τ, and σ to the monadic setting, and in Section 5 we generalize the functor $\mathcal{O}\colon \mathbb{S4} \to \mathbb{HA}$ associating to each $\mathsf{S4}$-algebra the Heyting algebra of its open elements to the functor $\mathcal{O}\colon \mathbb{MS4} \to \mathbb{MHA}$. We prove that $\mathcal{O}$ is the algebraic counterpart of ρ and give the dual description of $\mathcal{O}$ using the corresponding descriptive frames. While $\mathcal{O}\colon \mathbb{S4} \to \mathbb{HA}$ preserves the class operators H, S, and P of taking homomorphic images, subalgebras, and products, in Section 6 we show that $\mathcal{O}\colon \mathbb{MS4} \to \mathbb{MHA}$ no longer preserves S. It is this key observation that allows us to prove that, although τ and σ remain lattice homomorphisms in the monadic setting, ρ is neither a lattice

[1] In [35] it is claimed that QGrz is a modal companion of IQC, and in [34] that it is not the largest modal companion. However, the proofs in [35,34] rely on the Flagg–Friedman translation [23] that Inoué [29] showed is not faithful. Therefore, these results require further examination (see [25, Rem. 2.11.13] and [6, Rem. 5.16]).

[2] This should not be confused with the monadic fragment, where different variables are allowed in monadic predicates.

homomorphism nor one-to-one. From this we derive that σ is not an isomorphism, thus concluding that the Blok–Esakia Theorem does not extend to the monadic setting.

2. MIPC

Let $\mathcal{L}$ be the propositional language of IPC, and let $\mathcal{L}_{\forall\exists}$ be its extension by two "quantifier modalities" $\forall$ and $\exists$.

Definition 2.1. The *monadic intuitionistic propositional calculus* MIPC is the smallest set of formulas in the language $\mathcal{L}_{\forall\exists}$ containing

(1) all theorems of IPC;
(2) the S4-axioms for $\forall$: $\forall(p \wedge q) \leftrightarrow (\forall p \wedge \forall q)$, $\forall p \to p$, $\forall p \to \forall\forall p$;
(3) the S5-axioms for $\exists$: $\exists(p \vee q) \leftrightarrow (\exists p \vee \exists q)$, $p \to \exists p$, $\exists\exists p \to \exists p$, $(\exists p \wedge \exists q) \to \exists(\exists p \wedge q)$;
(4) the axioms connecting $\forall$ and $\exists$: $\exists\forall p \leftrightarrow \forall p$, $\exists p \leftrightarrow \forall\exists p$;

and closed under the rules of modus ponens, substitution, and necessitation ($\varphi/\forall\varphi$).

The algebraic semantics for MIPC is provided by monadic Heyting algebras, which were first introduced by Monteiro and Varsavsky [33].

Definition 2.2. A *monadic Heyting algebra* is a tuple $\mathfrak{A} = (H, \forall, \exists)$ such that H is a Heyting algebra and $\forall, \exists$ are unary functions on H satisfying the axioms corresponding to the ones in Definition 2.1.

Remark 2.3. For a monadic Heyting algebra $(H, \forall, \exists)$, let $H_0 = \{\forall a : a \in H\}$. Using the axioms of monadic Heyting algebras, it is straightforward to check that

$$H_0 = \{a \in H : a = \forall a\} = \{a \in H : a = \exists a\} = \{\exists a : a \in H\},$$

and that H_0 is a Heyting subalgebra of H. Moreover, $\forall$ is the right adjoint and $\exists$ is the left adjoint of the embedding $H_0 \hookrightarrow H$. Furthermore, each monadic Heyting algebra is represented as a pair (H, H_0), where the Heyting embedding $H_0 \hookrightarrow H$ has both right and left adjoints (see, e.g., [3, Sec. 3]).

Clearly the class of monadic Heyting algebras is equationally definable and hence forms a variety. We denote the corresponding category by $\mathbb{MHA}$.

Since terms in the language of monadic Heyting algebras correspond to formulas in $\mathcal{L}_{\forall\exists}$, we say that a formula φ is *valid* in a monadic Heyting algebra $\mathfrak{A}$ (in symbols $\mathfrak{A} \vDash \varphi$) if the equation $t = 1$ holds in $\mathfrak{A}$, where t is the term corresponding to φ. For a class $\mathbb{K} \subseteq \mathbb{MHA}$ of monadic Heyting algebras, we write $\mathbb{K} \vDash \varphi$ if $\mathfrak{A} \vDash \varphi$ for each $\mathfrak{A} \in \mathbb{K}$. The standard Lindenbaum-Tarski construction then yields:

Theorem 2.4. [21, Thm. 2] *MIPC $\vdash \varphi$ iff $\mathbb{MHA} \vDash \varphi$ for each formula φ of $\mathcal{L}_{\forall\exists}$.*

Definition 2.5. An *extension* L *of* MIPC is a set of formulas in the language $\mathcal{L}_{\forall\exists}$ containing MIPC and closed under modus ponens, substitution, and necessitation.

Each extension L of MIPC gives rise to the variety Alg(L) of monadic Heyting algebras validating all formulas in L. Conversely, each variety $\mathbb{V}$ of monadic Heyting algebras gives rise to the extension Log($\mathbb{V}$) of MIPC consisting of the formulas valid in all members of $\mathbb{V}$. By Λ(MIPC) we denote the complete lattice of extensions of MIPC and by $\Lambda(\mathbb{MHA})$ the complete lattice of subvarieties of $\mathbb{MHA}$. We thus obtain:

Theorem 2.6. [3, Thm. 3] $\Lambda(\mathsf{MIPC})$ *is dually isomorphic to* $\Lambda(\mathbb{MHA})$.

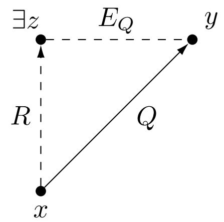
Esakia duality for Heyting algebras [15,20] was generalized to monadic Heyting algebras in [4]. As usual, for a binary relation R on a set X and $S \subseteq X$, we write $R[S]$ for the R-image and $R^{-1}[S]$ for the R-inverse image of S. When $S = \{x\}$, we simply write $R[x]$ and $R^{-1}[x]$. We call S an *R-upset* if $R[S] \subseteq S$ and an *R-downset* if $R^{-1}[S] \subseteq S$. If R is a quasi-order (reflexive and transitive relation), we denote by E_R the equivalence relation given by

$$xE_Ry \iff xRy \ \&\ yRx.$$

A *Stone space* is a topological space X that is compact, Hausdorff, and zero-dimensional. We call a binary relation R on X *continuous* if $R[x]$ is closed for each $x \in X$ and $R^{-1}[U]$ is clopen for each clopen $U \subseteq X$.

Definition 2.7. A *descriptive* MIPC*-frame* is a tuple $\mathfrak{F} = (X, R, Q)$ such that

(1) X is a Stone space,
(2) R is a continuous partial order,
(3) Q is a continuous quasi-order,
(4) U a clopen R-upset $\Longrightarrow Q[U]$ is a clopen R-upset,
(5) $R \subseteq Q$,
(6) $xQy \Longrightarrow \exists z \in X : xRz \ \&\ zE_Qy$.



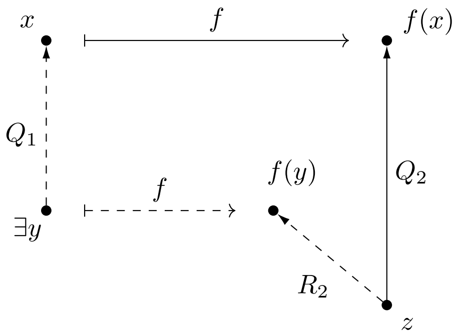
Remark 2.8. The condition in Definition 2.7(6) implies that it is possible to recover the quasi-order Q from R and E_Q in any descriptive MIPC-frame. In fact, descriptive MIPC-frames can be equivalently presented as triples (X, R, E) where X is a Stone space, R is a quasi-order, and E is an equivalence relation satisfying the conditions corresponding to the ones in Definition 2.7 (see [4, Thm. 11(a)]). For our purposes it is more convenient to work with the quasi-order Q, but we will employ this different perspective to work with descriptive frames for MS4.

Definition 2.9. Let $\mathfrak{F}_1 = (X_1, R_1, Q_1)$ and $\mathfrak{F}_2 = (X_2, R_2, Q_2)$ be descriptive MIPC-frames. A map $f: X_1 \to X_2$ is a *morphism of descriptive* MIPC*-frames* if

(1) f is continuous,
(2) $R_2[f(x)] = fR_1[x]$ for each $x \in X_1$,
(3) $Q_2[f(x)] = fQ_1[x]$ for each $x \in X_1$,
(4) $Q_2^{-1}[f(x)] = R_2^{-1}fQ_1^{-1}[x]$ for each $x \in X_1$.

Remark 2.10. Definition 2.9(2) says that f is a p-morphism with respect to R, and Definition 2.9(3) that f is a p-morphism with respect to Q. The left-to-right inclusion of Definition 2.9(4) follows from Definition 2.9(3), the other inclusion can be expressed as follows:

$$zQ_2f(x) \Longrightarrow \exists y \in X_1 : yQ_1x \ \&\ zR_2f(y).$$



We will refer to this condition as being a *weak p-morphism*[3] with respect to Q^{-1}.

Clearly descriptive MIPC-frames and their morphisms form a category, which we denote by $\mathbf{DF}_{\mathsf{MIPC}}$.

Remark 2.11. Since the composition of morphisms in $\mathbf{DF}_{\mathsf{MIPC}}$ is the standard function composition, it follows that isomorphisms in $\mathbf{DF}_{\mathsf{MIPC}}$ are bijections that preserve and reflect the relations R and Q (see [20, Prop. 1.4.15]). As a consequence, they also preserve and reflect the equivalence relation E_Q.

We recall (see, e.g., [1, Def. 3.38]) that two categories $\mathbf{A}$ and $\mathbf{B}$ are *dually equivalent* provided $\mathbf{A}^{\text{op}}$ and $\mathbf{B}$ are equivalent.

Theorem 2.12. [4, Thm. 17] $\mathbb{MHA}$ *is dually equivalent to* $\mathbf{DF}_{\mathsf{MIPC}}$.

Remark 2.13. The functors establishing the above duality generalize the functors establishing Esakia duality [15,20]. If $\mathfrak{F} = (X, R, Q)$ is a descriptive MIPC-frame, we let $\mathfrak{F}^*$ be the Heyting algebra of clopen R-upsets of X and define $\forall, \exists$ on $\mathfrak{F}^*$ by

$$\forall U = X \setminus Q^{-1}[X \setminus U] \quad \text{and} \quad \exists U = Q[U].$$

If $f : \mathfrak{F}_1 \to \mathfrak{F}_2$ is a $\mathbf{DF}_{\mathsf{MIPC}}$-morphism, we define $f^* : \mathfrak{F}_2^* \to \mathfrak{F}_1^*$ by $f^*(U) = f^{-1}[U]$ for each $U \in \mathfrak{F}_2^*$.

If $\mathfrak{A} = (H, \forall, \exists)$ is a monadic Heyting algebra, we let $\mathfrak{A}_*$ be the tuple (X, R, Q) such that (X, R) is the Esakia space[4] of H and Q is defined by

$$xQy \iff x \cap H_0 \subseteq y.$$

If $h : \mathfrak{A}_1 \to \mathfrak{A}_2$ is an $\mathbb{MHA}$-morphism, we define $h_* : (\mathfrak{A}_2)_* \to (\mathfrak{A}_1)_*$ by $h_*(x) = h^{-1}[x]$ for each $x \in (\mathfrak{A}_2)_*$.

3. MS4

Let $\mathcal{L}_{\Box\forall}$ be a propositional modal language with two modalities $\Box$ and $\forall$.

Definition 3.1. The *monadic* S4, denoted MS4, is the smallest set of formulas in $\mathcal{L}_{\Box\forall}$ containing all theorems of the classical propositional calculus CPC, the S4-axioms for $\Box$, the S5-axioms for $\forall$, the left commutativity axiom

[3] Observe that f is a p-morphism with respect to Q^{-1} provided $Q_2^{-1}[f(x)] = fQ_1^{-1}[x]$ for each $x \in X_1$. The above condition is weaker, thus justifying the name.

[4] We recall that X is the set of prime filters of H, R is set-inclusion, and topology is given by the basis $\{\sigma(a) \setminus \sigma(b) : a, b \in H\}$, where $\sigma(a) = \{x \in X : a \in x\}$.

$$\Box\forall p \to \forall\Box p,$$

and closed under the rules of modus ponens, substitution, $\Box$-necessitation, and $\forall$-necessitation.

Notation 3.2. We let $\blacksquare$ denote the compound modality $\Box\forall$.

Remark 3.3. It is straightforward to check that $\blacksquare$ is an S4-modality and that both $\blacksquare p \to \Box p$ and $\blacksquare p \to \forall p$ are provable in MS4. Therefore, $\blacksquare$ is a *master modality* for MS4 (see, e.g., [30, p. 71]).

The algebraic semantics for MS4 is provided by MS4-algebras, called bimodal algebras in [21, p. 145].

Definition 3.4. A *monadic* S4-*algebra*, or an MS4-*algebra* for short, is a tuple $\mathfrak{B} = (B, \Box, \forall)$ such that B is a boolean algebra and $\Box, \forall$ are two unary functions on B satisfying the axioms corresponding to the ones in Definition 3.1.

Remark 3.5.

(1) A tuple $(B, \Box, \forall)$ is an MS4-algebra provided $(B, \Box)$ is an S4-algebra, $(B, \forall)$ is an S5-algebra, and $\Box\forall a \leq \forall\Box a$ for each $a \in B$.
(2) As usual, we define $\exists \colon B \to B$ by $\exists a = \neg\forall\neg a$ for each $a \in B$.
(3) Let $(B, \Box, \forall)$ be an MS4-algebra. We denote by $\blacksquare$ the unary function on B, obtained as the composition $\Box\forall$, corresponding to the master modality $\blacksquare$ of MS4. Then $(B, \blacksquare)$ is an S4-algebra and $\blacksquare a \leq \Box a, \forall a$ for each $a \in B$. As a consequence, the following identities hold for each $a \in B$:

$$\blacksquare\Box a = \Box\blacksquare a = \blacksquare\forall a = \forall\blacksquare a = \blacksquare a.$$

It immediately follows from Definition 3.4 that the class of MS4-algebras forms a variety. We denote the corresponding category by $\mathbb{MS4}$.

Validity of formulas in MS4-algebras is defined as for monadic Heyting algebras, and so are extensions of MS4. We then have the following analogue of Theorems 2.4 and 2.6:

Theorem 3.6.

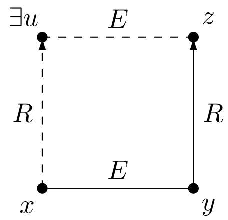
(1) MS4 $\vdash \varphi$ *iff* $\mathbb{MS4} \vDash \varphi$ *for each formula* φ *of* $\mathcal{L}_{\Box\forall}$.
(2) *The complete lattice* Λ(MS4) *of extensions of* MS4 *is dually isomorphic to the complete lattice* $\Lambda(\mathbb{MS4})$ *of subvarieties of* $\mathbb{MS4}$.

As in Theorem 2.6, the above isomorphism is obtained by associating with each extension M of MS4 the variety Alg(M) of MS4-algebras validating all formulas in M. Conversely, each variety $\mathbb{V}$ of MS4-algebras gives rise to the extension Log($\mathbb{V}$) of MS4 consisting of the formulas valid in all members of $\mathbb{V}$.

Jónsson-Tarski duality for BAOs (boolean algebras with operators) takes on the following form for MS4-algebras.

Definition 3.7. A *descriptive* MS4-*frame* is a tuple $\mathfrak{G} = (Y, R, E)$ such that

(1) Y is a Stone space,
(2) R is a continuous quasi-order,
(3) E is a continuous equivalence relation,
(4) $xEy \;\&\; yRz \Longrightarrow \exists u \in Y : xRu \;\&\; uEz$.



Definition 3.8. Let $\mathfrak{G}_1 = (Y_1, R_1, E_1)$ and $\mathfrak{G}_2 = (Y_2, R_2, E_2)$ be descriptive MS4-frames. A map $f : Y_1 \to Y_2$ is a *morphism of descriptive* MS4*-frames* if

(1) f is continuous,
(2) $R_2[f(y)] = fR_1[y]$ for each $y \in Y_1$,
(3) $E_2[f(y)] = fE_1[y]$ for each $y \in Y_1$.

Remark 3.9. Definition 3.8(2) says that f is a p-morphism with respect to R, and Definition 3.8(3) that f is a p-morphism with respect to E.

Clearly descriptive MS4-frames and their morphisms form a category, which we denote by $\mathbf{DF}_{\mathsf{MS4}}$.

Remark 3.10. As in $\mathbf{DF}_{\mathsf{MIPC}}$, the composition of morphisms in $\mathbf{DF}_{\mathsf{MS4}}$ is the standard function composition. Therefore, isomorphisms in $\mathbf{DF}_{\mathsf{MS4}}$ are bijections that preserve and reflect the relations R and E.

We have the following version of Jónsson-Tarski duality for MS4-algebras. For the reader's convenience, we give a sketch of proof (see also [7, Rem. 2.14]).

Theorem 3.11. $\mathbb{MS4}$ *is dually equivalent to* $\mathbf{DF}_{\mathsf{MS4}}$.

Sketch of proof. The functors establishing the dual equivalence are described as follows. The functor $(-)^* : \mathbf{DF}_{\mathsf{MS4}} \to \mathbb{MS4}$ sends a descriptive MS4-frame $\mathfrak{G} = (Y, R, E)$ to $\mathfrak{G}^* = (\mathsf{Clop}(Y), \Box, \forall)$, where $\mathsf{Clop}(Y)$ is the boolean algebra of clopens of Y and $\Box, \forall$ are defined on $\mathsf{Clop}(Y)$ by

$$\Box U = Y \setminus R^{-1}[Y \setminus U] \quad \text{and} \quad \forall U = Y \setminus E[Y \setminus U].$$

Since R is reflexive and transitive, $\Box$ is an S4-operator; since E is an equivalence relation, $\forall$ is an S5-operator; and Definition 3.7(4) yields that $\Box\forall U \le \forall\Box U$. Therefore, $\mathfrak{G}^* \in \mathbb{MS4}$. If f is a $\mathbf{DF}_{\mathsf{MS4}}$-morphism, then f^* is f^{-1}. Clearly, f^{-1} is a boolean homomorphism and Definition 3.8(2,3) yields that f^{-1} is an $\mathbb{MS4}$-morphism. Thus, $(-)^*$ is well defined.

The functor $(-)_* : \mathbb{MS4} \to \mathbf{DF}_{\mathsf{MS4}}$ sends an MS4-algebra $\mathfrak{B} = (B, \Box, \forall)$ to the tuple $\mathfrak{B}_* = (Y, R, E)$ such that (Y, R) is the dual of the S4-algebra $(B, \Box)$[5] and

$$xEy \iff x \cap B_0 = y \cap B_0,$$

where $B_0 = \{\forall a : a \in B\}$. Then Y is a Stone space, R is a continuous quasi-order, and E is a continuous equivalence relation. The commutativity axiom $\Box\forall a \le \forall\Box a$ for every $a \in B$ implies that $ER^{-1}[U] \subseteq R^{-1}E[U]$ for every $U \in \mathsf{Clop}(Y)$. A standard argument using Esakia's Lemma (see [20, Lem. 3.3.12] or [12, p. 350]) then yields that $ER^{-1}[y] \subseteq R^{-1}E[y]$ for every $y \in Y$. It is straightforward to see that this last

[5] That is, Y is the Stone space (the space of ultrafilters) of B and for any $x, y \in Y$ we have xRy iff $\Box^{-1}x \subseteq y$.

condition is equivalent to Definition 3.7(4). Therefore, $\mathfrak{B}_* \in \mathbf{DF}_{\mathsf{MS4}}$. If f is an $\mathbb{MS4}$-morphism, then f_* is f^{-1}. It follows from Jónsson-Tarski duality that f^{-1} is an $\mathbb{MS4}$-morphism, so $(-)_*$ is well defined, and that $(-)^*$ and $(-)_*$ establish the desired dual equivalence between $\mathbb{MS4}$ and $\mathbf{DF}_{\mathsf{MS4}}$. □

Notation 3.12. We use the same notation for the functors establishing duality between $\mathbb{MS4}$ and $\mathbf{DF}_{\mathsf{MS4}}$ and the ones establishing duality between $\mathbb{MHA}$ and $\mathbf{DF}_{\mathsf{MIPC}}$, but it will always be clear from the context which functor we work with. To avoid confusion, we will always denote monadic Heyting algebras by $\mathfrak{A}$ and MS4-algebras by $\mathfrak{B}$. Similarly, we will denote descriptive MIPC-frames by $\mathfrak{F}$ and descriptive MS4-frames by $\mathfrak{G}$.

4. The Gödel translation

We recall (see, e.g., [12, p. 96]) that the Gödel translation $(-)^t$ of IPC into S4 is defined by

$$\begin{aligned}
&\bot^t = \bot \\
&p^t = \Box p \quad \text{for each propositional letter } p \\
&(\varphi \wedge \psi)^t = \varphi^t \wedge \psi^t \\
&(\varphi \vee \psi)^t = \varphi^t \vee \psi^t \\
&(\varphi \to \psi)^t = \Box(\neg\varphi^t \vee \psi^t).
\end{aligned}$$

Fischer Servi [21] (see also [22]) extended the Gödel translation to a translation of MIPC into MS4 as follows:

$$(\forall\varphi)^t = \blacksquare\varphi^t$$
$$(\exists\varphi)^t = \exists\varphi^t.$$

Theorem 4.1. [21] $\mathsf{MIPC} \vdash \varphi$ *iff* $\mathsf{MS4} \vdash \varphi^t$ *for each* φ *in* $\mathcal{L}_{\forall\exists}$.

There are three well-known maps between the lattices of extensions of IPC and S4 (see, e.g., [12, Sec. 9.6]) that have obvious generalizations to the monadic setting.

Definition 4.2.

(1) Define $\rho\colon \Lambda(\mathsf{MS4}) \to \Lambda(\mathsf{MIPC})$ by $\rho\mathsf{M} = \{\varphi : \mathsf{M} \vdash \varphi^t\}$.
(2) Define $\tau\colon \Lambda(\mathsf{MIPC}) \to \Lambda(\mathsf{MS4})$ by $\tau\mathsf{L} = \mathsf{MS4} + \{\varphi^t : \mathsf{L} \vdash \varphi\}$.
(3) Define $\sigma\colon \Lambda(\mathsf{MIPC}) \to \Lambda(\mathsf{MGrz})$ by $\sigma\mathsf{L} = \mathsf{MGrz} + \{\varphi^t : \mathsf{L} \vdash \varphi\}$.

Here $\mathsf{MGrz} \coloneqq \mathsf{MS4} + \mathsf{grz}$ is the *monadic Grzegorczyk logic* introduced by Esakia [19], and $\Lambda(\mathsf{MGrz})$ is the complete lattice of extensions of MGrz.

Proposition 4.3.

(1) τ *is left adjoint to* ρ.
(2) σ *is left adjoint to the restriction of* ρ *to* $\Lambda(\mathsf{MGrz})$.
(3) τ *and* σ *preserve arbitrary joins, and* ρ *preserves arbitrary meets.*

Proof. (1). For all $\mathsf{L} \in \Lambda(\mathsf{MIPC})$ and $\mathsf{M} \in \Lambda(\mathsf{MS4})$,

$$\tau\mathsf{L} \subseteq \mathsf{M} \iff \{\varphi^t : \mathsf{L} \vdash \varphi\} \subseteq \mathsf{M} \iff \mathsf{L} \subseteq \{\varphi : \mathsf{M} \vdash \varphi^t\} \iff \mathsf{L} \subseteq \rho\mathsf{M}.$$

Thus, τ is left adjoint to ρ.

(2). If $\mathsf{M} \in \Lambda(\mathsf{MGrz})$, then (1) yields $\sigma\mathsf{L} \subseteq \mathsf{M}$ iff $\mathsf{L} \subseteq \rho\mathsf{M}$. Thus, σ is left adjoint to the restriction of ρ to $\Lambda(\mathsf{MGrz})$.

(3). This is obvious since left adjoints preserve joins and right adjoints preserve meets (see, e.g., [13, Prop. 7.34]). □

The notions of a modal companion and the intuitionistic fragment have obvious generalizations to the monadic setting:

Definition 4.4. Let $\mathsf{L} \in \Lambda(\mathsf{MIPC})$ and $\mathsf{M} \in \Lambda(\mathsf{MS4})$. If $\mathsf{L} = \rho\mathsf{M}$, then we call M a *modal companion* of L and L the *intuitionistic fragment* of M.

Theorem 4.5.

(1) [21] $\mathsf{MS4}$ *is a modal companion of* MIPC.
(2) [19] MGrz *is a modal companion of* MIPC.

In Section 6 we will show that τ and σ are lattice homomorphisms, thus generalizing the corresponding results of Maksimova and Rybakov [31] (see also [12, Thm. 9.66]). On the other hand, we will prove that ρ is neither a lattice homomorphism nor one-to-one. This yields that σ is not an isomorphism, and hence the Blok–Esakia Theorem fails in the monadic setting. For this we need a semantic characterization of ρ, which will be done in the next section.

5. The functor $\mathcal{O}$

Let $(B, \Box)$ be an $\mathsf{S4}$-algebra. We recall that $a \in B$ is *open* if $\Box a = a$. It is well known (see, e.g., [20, Prop. 2.2.4]) that the set H of open elements of $(B, \Box)$ is a bounded sublattice of B which forms a Heyting algebra, where $a \to_H b = \Box(\neg a \vee b)$. Moreover, this correspondence extends to a functor $\mathcal{O} \colon \mathbb{S4} \to \mathbb{HA}$ (see, e.g., [20, Thm. 2.2.5]). We extend $\mathcal{O}$ to the monadic setting.

Definition 5.1. For an $\mathsf{MS4}$-algebra $\mathfrak{B} = (B, \Box, \forall)$ let $\mathcal{O}(\mathfrak{B}) = (H, \forall_H, \exists_H)$, where H is the Heyting algebra of open elements of $(B, \Box)$, $\forall_H a = \blacksquare a$, and $\exists_H a = \exists a$ for all $a \in H$.

Theorem 5.2. [21, Thm. 4] *If* $\mathfrak{B}$ *is an* $\mathsf{MS4}$-*algebra, then* $\mathcal{O}(\mathfrak{B})$ *is a monadic Heyting algebra.*

For each $\mathsf{MS4}$-morphism $h \colon \mathfrak{B}_1 \to \mathfrak{B}_2$ let $\mathcal{O}(h) \colon \mathcal{O}(\mathfrak{B}_1) \to \mathcal{O}(\mathfrak{B}_2)$ be its restriction. It is straightforward to see that $\mathcal{O}(h)$ is an $\mathbb{MHA}$-morphism and that [20, Thm. 2.2.5] generalizes to the following:

Theorem 5.3. $\mathcal{O} \colon \mathbb{MS4} \to \mathbb{MHA}$ *is a functor.*

Theorem 5.4. [21, Thm. 5] *Let* φ *be a formula in the language* $\mathcal{L}_{\forall\exists}$ *and* $\mathfrak{B}$ *an* $\mathsf{MS4}$-*algebra. Then* $\mathcal{O}(\mathfrak{B}) \vDash \varphi$ *iff* $\mathfrak{B} \vDash \varphi^t$.

In order to see how ρ relates to $\mathcal{O}$, we need to describe the behavior of $\mathcal{O}$ with respect to the class operators H, S, and P of taking homomorphic images, subalgebras, and products.

Let $(B, \Box)$ be an $\mathsf{S4}$-algebra and H the Heyting algebra of its open elements. We recall that a filter G of B is a $\Box$-*filter* if $a \in G$ implies $\Box a \in G$. There is a well-known isomorphism between the congruences of $(B, \Box)$, the $\Box$-filters of $(B, \Box)$, the filters of H, and the congruences of H (see, e.g., [20, Sec. 2.4]). This readily generalizes to the monadic setting.

Definition 5.5. A filter F of a monadic Heyting algebra $\mathfrak{A} = (H, \forall, \exists)$ is *monadic* if $a \in F$ implies $\forall a \in F$; and a $\Box$-filter G of an $\mathsf{MS4}$-algebra $\mathfrak{B} = (B, \Box, \forall)$ is *monadic* if it satisfies the same condition (equivalently, G is a monadic filter provided $a \in G$ implies $\blacksquare a \in G$).

Theorem 5.6. *Let $\mathfrak{B}$ be an $\mathsf{MS4}$-algebra and $\mathfrak{A} = \mathcal{O}(\mathfrak{B})$. There is an isomorphism between the posets of congruences of $\mathfrak{B}$, monadic filters of $\mathfrak{B}$, monadic filters of $\mathfrak{A}$, and congruences of $\mathfrak{A}$.*

Sketch of proof. That the poset of congruences of $\mathfrak{A}$ is isomorphic to the poset of monadic filters of $\mathfrak{A}$ follows from [2, Thm. 2.7]. A similar argument yields that the poset of congruences of $\mathfrak{B}$ is isomorphic to the poset of monadic filters of $\mathfrak{B}$ (see, e.g., [7, Thm. 3.4]). Finally, the isomorphism between the posets of monadic filters of $\mathfrak{B}$ and $\mathfrak{A}$ directly generalizes the well-known isomorphism between the $\Box$-filters of $\mathfrak{B}$ and the filters of $\mathfrak{A}$ obtained by

$$G \mapsto G \cap \mathfrak{A} \quad \text{and} \quad F \mapsto {\uparrow_{\mathfrak{B}}} F,$$

where ${\uparrow_{\mathfrak{B}}} F = \{b \in B : b \geq a \text{ for some } a \in F\}$. $\Box$

As an immediate corollary, we obtain the following correspondence between subdirectly irreducible algebras (for all notions of universal algebra we refer to [11]).

Corollary 5.7. *$\mathfrak{B}$ is a subdirectly irreducible $\mathsf{MS4}$-algebra iff $\mathcal{O}(\mathfrak{B})$ is a subdirectly irreducible monadic Heyting algebra.*

Consequently, both $\mathbb{MS4}$ and $\mathbb{MHA}$ are congruence-distributive and have the congruence extension property (CEP). This will be used in what follows, and so will be the next lemma.

Lemma 5.8. *Let $\mathfrak{B}$ be an $\mathsf{MS4}$-algebra.*

(1) *If G is a monadic filter of $\mathfrak{B}$, then $\mathcal{O}(\mathfrak{B}/G) \cong \mathcal{O}(\mathfrak{B})/(G \cap \mathcal{O}(\mathfrak{B}))$.*
(2) *If F is a monadic filter of $\mathcal{O}(\mathfrak{B})$, then $\mathcal{O}(\mathfrak{B})/F \cong \mathcal{O}(\mathfrak{B}/{\uparrow_{\mathfrak{B}}} F)$.*

Proof. (1). Let $\pi \colon \mathfrak{B} \to \mathfrak{B}/G$ be the quotient map. Then $\mathcal{O}(\pi) \colon \mathcal{O}(\mathfrak{B}) \to \mathcal{O}(\mathfrak{B}/G)$ is an onto $\mathbb{MHA}$-morphism. Moreover, for $b \in \mathcal{O}(\mathfrak{B})$, we have

$$\mathcal{O}(\pi)(b) = 1 \iff \pi(b) = 1 \iff b \in G.$$

Thus, $G \cap \mathcal{O}(\mathfrak{B})$ is the kernel of $\mathcal{O}(\pi)$, and hence the First Isomorphism Theorem [11, Thm. II.6.12] implies that $\mathcal{O}(\mathfrak{B}/G) \cong \mathcal{O}(\mathfrak{B})/(G \cap \mathcal{O}(\mathfrak{B}))$.

(2). Since $F = {\uparrow_{\mathfrak{B}}} F \cap \mathcal{O}(\mathfrak{B})$, (1) implies that

$$\mathcal{O}(\mathfrak{B})/F = \mathcal{O}(\mathfrak{B})/({\uparrow_{\mathfrak{B}}} F \cap \mathcal{O}(\mathfrak{B})) \cong \mathcal{O}(\mathfrak{B}/{\uparrow_{\mathfrak{B}}} F). \quad \Box$$

For a class of algebras $\mathbb{K}$, we denote by $\mathsf{H}(\mathbb{K})$, $\mathsf{S}(\mathbb{K})$, and $\mathsf{P}(\mathbb{K})$, the classes of homomorphic images, subalgebras, and products of algebras from $\mathbb{K}$. Let $\mathsf{Var}(\mathbb{K})$ be the variety generated by $\mathbb{K}$. It is well known (see, e.g., [11, Thm. II.9.5]) that $\mathsf{Var}(\mathbb{K}) = \mathsf{HSP}(\mathbb{K})$. When $\mathbb{K}$ is a class of $\mathsf{MS4}$-algebras, we write $\mathcal{O}(\mathbb{K})$ for $\{\mathcal{O}(\mathfrak{B}) : \mathfrak{B} \in \mathbb{K}\}$.

Proposition 5.9. *For a class $\mathbb{K}$ of $\mathsf{MS4}$-algebras, we have $\mathcal{O}\mathsf{H}(\mathbb{K}) = \mathsf{H}\mathcal{O}(\mathbb{K})$, $\mathcal{O}\mathsf{P}(\mathbb{K}) = \mathsf{P}\mathcal{O}(\mathbb{K})$, and $\mathcal{O}\mathsf{S}(\mathbb{K}) \subseteq \mathsf{S}\mathcal{O}(\mathbb{K})$.*

Proof. First, let $\mathfrak{A} \in \mathcal{O}\mathsf{H}(\mathbb{K})$, so $\mathfrak{A} = \mathcal{O}(\mathfrak{B}')$ for a homomorphic image $\mathfrak{B}'$ of some $\mathfrak{B} \in \mathbb{K}$. Then $\mathfrak{B}' \cong \mathfrak{B}/G$ for some monadic $\Box$-filter G of $\mathfrak{B}$. By Lemma 5.8(1),

$$\mathcal{O}(\mathfrak{B}') \cong \mathcal{O}(\mathfrak{B}/G) \cong \mathcal{O}(\mathfrak{B})/(G \cap \mathcal{O}(\mathfrak{B})),$$

and so $\mathcal{O}(\mathfrak{B}')$ is a homomorphic image of $\mathcal{O}(\mathfrak{B})$. Thus, $\mathfrak{A} \in \mathsf{H}\mathcal{O}(\mathbb{K})$, and so $\mathcal{O}\mathsf{H}(\mathbb{K}) \subseteq \mathsf{H}\mathcal{O}(\mathbb{K})$. For the other inclusion, let $\mathfrak{A} \in \mathsf{H}\mathcal{O}(\mathbb{K})$, so $\mathfrak{A}$ is a homomorphic image of $\mathcal{O}(\mathfrak{B})$ with $\mathfrak{B} \in \mathbb{K}$. Then $\mathfrak{A} \cong \mathcal{O}(\mathfrak{B})/F$ for some monadic filter F of $\mathcal{O}(\mathfrak{B})$. By Lemma 5.8(2),

$$\mathfrak{A} \cong \mathcal{O}(\mathfrak{B})/F \cong \mathcal{O}(\mathfrak{B}/{\uparrow_{\mathfrak{B}}} F),$$

and hence $\mathfrak{A} = \mathcal{O}(\mathfrak{B}')$ for some $\mathfrak{B}' \cong \mathfrak{B}/{\uparrow_{\mathfrak{B}}} F$. Thus, $\mathfrak{A} \in \mathcal{O}\mathsf{H}(\mathbb{K})$, and so $\mathsf{H}\mathcal{O}(\mathbb{K}) \subseteq \mathcal{O}\mathsf{H}(\mathbb{K})$.

Next, let $\{\mathfrak{B}_i : i \in I\}$ be a family of $\mathsf{MS4}$-algebras. By [20, Thm. 2.2.5], $\mathcal{O}(\Pi_{i\in I}\mathfrak{B}_i) = \Pi_{i\in I}\mathcal{O}(\mathfrak{B}_i)$ as Heyting algebras. Moreover, since $\blacksquare$ and $\exists$ are componentwise, the two algebras are equal as monadic Heyting algebras. Thus, $\mathcal{O}\mathsf{P}(\mathbb{K}) = \mathsf{P}\mathcal{O}(\mathbb{K})$.

Finally, let $\mathfrak{A} \in \mathcal{O}\mathsf{S}(\mathbb{K})$, so $\mathfrak{A} = \mathcal{O}(\mathfrak{B}')$ for a subalgebra $\mathfrak{B}'$ of some $\mathfrak{B} \in \mathbb{K}$. Then $\mathcal{O}(\mathfrak{B}')$ is a Heyting subalgebra of $\mathcal{O}(\mathfrak{B})$ (see, e.g., [20, Thm. 2.2.5]). Moreover, $\blacksquare a, \exists a \in \mathfrak{B}'$ for each $a \in \mathcal{O}(\mathfrak{B}')$. Therefore, $\mathcal{O}(\mathfrak{B}')$ is also a monadic subalgebra of $\mathcal{O}(\mathfrak{B})$. Thus, $\mathfrak{A} \in \mathsf{S}\mathcal{O}(\mathbb{K})$, and so $\mathcal{O}\mathsf{S}(\mathbb{K}) \subseteq \mathsf{S}\mathcal{O}(\mathbb{K})$. □

Remark 5.10. In Theorem 6.16(2) we will see that the inclusion $\mathsf{S}\mathcal{O}(\mathbb{K}) \subseteq \mathcal{O}\mathsf{S}(\mathbb{K})$ does not hold in general. This is in contrast with the functor $\mathcal{O}\colon \mathbb{HA} \to \mathbb{S}4$ which is well known to commute with H, S, and P, and hence maps varieties of $\mathsf{S4}$-algebras to varieties of Heyting algebras (see, e.g., [20, Cor. 2.2.6]).

Proposition 5.11.

(1) *If $\mathbb{K}$ is a class of $\mathsf{MS4}$-algebras, then $\mathsf{Var}(\mathcal{O}(\mathbb{K})) = \mathsf{S}\mathcal{O}(\mathsf{Var}(\mathbb{K}))$.*
(2) *If $\mathbb{V}$ is a variety of $\mathsf{MS4}$-algebras, then $\mathsf{S}\mathcal{O}(\mathbb{V})$ is the variety generated by $\mathcal{O}(\mathbb{V})$.*

Proof. (1). By Proposition 5.9, H and $\mathcal{O}$ commute. Since $\mathbb{MHA}$ has the CEP, H and S commute on subclasses of $\mathbb{MHA}$ by [11, p. 62]. Therefore,

$$\mathsf{S}\mathcal{O}(\mathsf{Var}(\mathbb{K})) = \mathsf{S}\mathcal{O}(\mathsf{HSP}(\mathbb{K})) = \mathsf{SH}\mathcal{O}(\mathsf{SP}(\mathbb{K})) = \mathsf{HS}\mathcal{O}(\mathsf{SP}(\mathbb{K})).$$

If $\mathbb{K}'$ is a class of $\mathsf{MS4}$-algebras, then Proposition 5.9 implies that $\mathsf{S}\mathcal{O}(\mathsf{S}(\mathbb{K}')) \subseteq \mathsf{S}\mathcal{O}(\mathbb{K}')$, and hence $\mathsf{S}\mathcal{O}(\mathsf{S}(\mathbb{K}')) = \mathsf{S}\mathcal{O}(\mathbb{K}')$. Thus,

$$\mathsf{HS}\mathcal{O}(\mathsf{SP}(\mathbb{K})) = \mathsf{HS}\mathcal{O}(\mathsf{P}(\mathbb{K})) = \mathsf{HSP}\mathcal{O}(\mathbb{K}) = \mathsf{Var}(\mathcal{O}(\mathbb{K}))$$

because P and $\mathcal{O}$ commute by Proposition 5.9. Consequently, $\mathsf{S}\mathcal{O}(\mathsf{Var}(\mathbb{K})) = \mathsf{Var}(\mathcal{O}(\mathbb{K}))$.

(2). By (1), $\mathsf{S}\mathcal{O}(\mathbb{V}) = \mathsf{S}\mathcal{O}(\mathsf{Var}(\mathbb{V})) = \mathsf{Var}(\mathcal{O}(\mathbb{V}))$. □

Theorem 5.12.

(1) *Let $\mathsf{M} \in \Lambda(\mathsf{MS4})$. Then $\mathsf{Alg}(\rho\mathsf{M}) = \mathsf{S}\mathcal{O}(\mathsf{Alg}(\mathsf{M}))$.*

(2) *Let* $\mathsf{L} \in \Lambda(\mathsf{MIPC})$. *Then* $\mathsf{M} \in \Lambda(\mathsf{MS4})$ *is a modal companion of* L *iff* $\mathsf{Alg}(\mathsf{L}) = \mathsf{S}\mathcal{O}(\mathsf{Alg}(\mathsf{M}))$.
(3) $\mathsf{S}\mathcal{O}$ *commutes with arbitrary joins of varieties of* $\mathsf{MS4}$*-algebras.*

Proof. (1). By algebraic completeness and Theorem 5.4, for each formula φ in $\mathcal{L}_{\forall\exists}$, we have $\mathsf{M} \vdash \varphi^t$ iff $\mathsf{Alg}(\mathsf{M}) \vDash \varphi^t$ iff $\mathcal{O}(\mathsf{Alg}(\mathsf{M})) \vDash \varphi$. Therefore, $\rho\mathsf{M} \vdash \varphi$ iff $\mathcal{O}(\mathsf{Alg}(\mathsf{M})) \vDash \varphi$. Thus, by Proposition 5.11(1),

$$\mathsf{Alg}(\rho\mathsf{M}) = \mathsf{Var}(\mathcal{O}(\mathsf{Alg}(\mathsf{M}))) = \mathsf{S}\mathcal{O}(\mathsf{Alg}(\mathsf{M})).$$

(2). M is a modal companion of L iff $\mathsf{L} = \rho\mathsf{M}$, which is equivalent to $\mathsf{Alg}(\mathsf{L}) = \mathsf{Alg}(\rho\mathsf{M})$. By (1), this is equivalent to $\mathsf{Alg}(\mathsf{L}) = \mathsf{S}\mathcal{O}(\mathsf{Alg}(\mathsf{M}))$.

(3). Let $\{\mathbb{V}_i : i \in I\}$ be a family of varieties of $\mathsf{MS4}$-algebras. Since meets in $\Lambda(\mathsf{MIPC})$ and $\Lambda(\mathsf{MS4})$ are intersections, by (1) and Proposition 4.3(3) we have

$$\begin{aligned}\mathsf{S}\mathcal{O}\left(\bigvee\{\mathbb{V}_i : i \in I\}\right) &= \mathsf{S}\mathcal{O}\left(\bigvee\{\mathsf{Alg}(\mathsf{Log}(\mathbb{V}_i)) : i \in I\}\right) = \mathsf{S}\mathcal{O}\mathsf{Alg}\left(\bigcap\{\mathsf{Log}(\mathbb{V}_i) : i \in I\}\right)\\ &= \mathsf{Alg}\left(\rho\bigcap\{\mathsf{Log}(\mathbb{V}_i) : i \in I\}\right) = \mathsf{Alg}\left(\bigcap\{\rho\mathsf{Log}(\mathbb{V}_i) : i \in I\}\right)\\ &= \bigvee\{\mathsf{Alg}(\rho\mathsf{Log}(\mathbb{V}_i)) : i \in I\} = \bigvee\{\mathsf{S}\mathcal{O}(\mathsf{Alg}(\mathsf{Log}(\mathbb{V}_i))) : i \in I\}\\ &= \bigvee\{\mathsf{S}\mathcal{O}(\mathbb{V}_i) : i \in I\}. \quad \square\end{aligned}$$

We conclude this section by describing a functor from the category of descriptive $\mathsf{MS4}$-frames to the category of descriptive MIPC-frames that is dual to $\mathcal{O}$. For this we generalize the notion of the *skeleton* of an $\mathsf{S4}$-frame (see, e.g., [12, p. 68]) to the monadic setting.

Definition 5.13. For a descriptive $\mathsf{MS4}$-frame $\mathfrak{G} = (Y, R, E)$, define $\rho(\mathfrak{G}) = (X, R', Q')$ as follows. Let $X := Y/E_R$ be the quotient of Y by the equivalence relation E_R on Y induced by R, and let $\pi\colon Y \to X$ be the quotient map. Define R' on X by

$$\pi(x) R' \pi(y) \iff xRy.$$

Also, let Q be the composite $E \circ R$, and define Q' on X by

$$\pi(x) Q' \pi(y) \iff xQy.$$

Theorem 5.14. $\rho(\mathfrak{G})$ *is a descriptive* MIPC*-frame.*

Proof. By [20, Lem. 3.4.13], X is a Stone space and R' is a continuous relation on X. In addition, the quotient map $\pi\colon Y \to X$ is continuous and satisfies $R'[\pi(x)] = \pi[R[x]]$ for each $x \in Y$. By [5, Lem. 3.3], Q' is a well-defined quasi-order satisfying Definition 2.7(5,6). We show that Q' is a continuous relation. Since both R and E are continuous relations on Y, so is Q. For $A \subseteq X$, it is straightforward to see that

$$\pi^{-1}[Q'[A]] = Q[\pi^{-1}[A]] \text{ and } \pi^{-1}[(Q')^{-1}[A]] = Q^{-1}[\pi^{-1}[A]].$$

Let $x \in Y$. Since

$$\pi^{-1}[Q'[\pi(x)]] = Q[\pi^{-1}[\pi(x)]] = Q[E_R[x]] = Q[x]$$

and Q is continuous, it follows from the definition of the quotient topology that $Q'[\pi(x)]$ is closed in X. Let $U \subseteq X$ be clopen. Then

$$\pi^{-1}[(Q')^{-1}[U]] = Q^{-1}[\pi^{-1}[U]],$$

which is clopen in Y because Q is continuous and $\pi^{-1}[U]$ is clopen in Y. Thus, $(Q')^{-1}[U]$ is clopen in X, and hence Q' is continuous.

It is left to verify Definition 2.7(4). Let U be a clopen R'-upset of X. Then $\pi^{-1}[U]$ is a clopen R-upset of Y. Since E is a continuous equivalence relation,

$$\pi^{-1}[Q'[U]] = Q[\pi^{-1}[U]] = ER[\pi^{-1}[U]] = E[\pi^{-1}[U]] = E^{-1}[\pi^{-1}[U]]$$

is clopen. Thus, $Q'[U]$ is clopen, and hence $\rho(\mathfrak{G})$ is a descriptive MIPC-frame. □

As with descriptive S4-frames, if $f\colon \mathfrak{G}_1 \to \mathfrak{G}_2$ is a $\mathbf{DF}_{\mathsf{MS4}}$-morphism, we define $\rho(f)\colon \rho(\mathfrak{G}_1) \to \rho(\mathfrak{G}_2)$ by

$$\rho(f)(\pi_1(x)) = \pi_2(f(x))$$

for each $x \in \mathfrak{G}_1$, where π_1, π_2 are the corresponding quotient maps.

Lemma 5.15. $\rho\colon \mathbf{DF}_{\mathsf{MS4}} \to \mathbf{DF}_{\mathsf{MIPC}}$ *is a well-defined functor.*

Proof. By Theorem 5.14, ρ is well defined on objects. Let $f\colon \mathfrak{G}_1 \to \mathfrak{G}_2$ be a $\mathbf{DF}_{\mathsf{MS4}}$-morphism with $\mathfrak{G}_1 = (Y_1, R_1, E_1)$ and $\mathfrak{G}_2 = (Y_2, R_2, E_2)$. It is well known that $\rho(f)$ is a well-defined continuous p-morphism with respect to R'. For the reader's convenience, we sketch a proof. Since $f[R_1[z]] \subseteq R_2[f(z)]$ for each $z \in Y_1$, we have that $xE_{R_1}y$ implies $f(x)E_{R_2}f(y)$ for each $x, y \in Y_1$. Thus, $\rho(f)$ is a well-defined function. It is continuous because $\pi_2 \circ f = \rho(f) \circ \pi_1$ and $\pi_2 \circ f$ is continuous. For $x \in Y_1$ we have

$$\begin{aligned}\rho(f)[R_1'[\pi_1(x)]] &= \rho(f)[\pi_1[R_1[x]]] = \pi_2[f[R_1[x]]] = \pi_2[R_2[f(x)]]\\ &= R_2'[\pi_2[f(x)]] = R_2'[\rho(f)(\pi_1(x))],\end{aligned}$$

where the first and fourth equalities follow from the definitions of R_1' and R_2', the second and last equalities from the definition of $\rho(f)$, and the third equality holds because f is a p-morphism with respect to R. Thus, $\rho(f)$ is a p-morphism with respect to R'.

Since $Q_1 = E_1 \circ R_1$ and f is a p-morphism with respect to R and E, it is also a p-morphism with respect to Q. Then a similar chain of equalities yields that $\rho(f)$ is a p-morphism with respect to Q'.

We show that $\rho(f)$ is a weak p-morphism with respect to $(Q')^{-1}$. Let $x \in Y_1$. Then

$$Q_2^{-1}[f(x)] = R_2^{-1}E_2[f(x)] = R_2^{-1}f[E_1[x]] = R_2^{-1}[f[R_1^{-1}E_1[x]]] = R_2^{-1}f[Q_1^{-1}[x]], \tag{†}$$

where the first and last equalities follow from the definitions of Q_2 and Q_1, the second equality holds because f is a p-morphism with respect to E, the left to right inclusion in the third equality is a consequence of the reflexivity of R_1, and the right to left inclusion holds because f preserves R_1 and R_2 is transitive. Consequently,

$$\begin{aligned}(Q_2')^{-1}[\rho(f)(\pi_1[x])] &= (Q_2')^{-1}[\pi_2(f(x))] = \pi_2[Q_2^{-1}[f(x)]]\\ &= \pi_2[R_2^{-1}[f[Q_1^{-1}[x]]]] = (R_2')^{-1}[\pi_2[f[Q_1^{-1}[x]]]]\\ &= (R_2')^{-1}[\rho(f)[\pi_1[Q_1^{-1}[x]]]] = (R_2')^{-1}[\rho(f)[(Q_1')^{-1}[\pi_1[x]]]],\end{aligned}$$

where the first and fifth equalities follow from the definition of $\rho(f)$, the second, fourth, and last equalities are consequences of the definitions of Q_2', R_2', and Q_1', and the third equality follows from (†). Thus, $\rho(f)$ is an MIPC-morphism (see Definition 2.9). That ρ preserves compositions and identities is an immediate consequence of its definition. Therefore, $\rho\colon \mathbf{DF}_{\mathsf{MS4}} \to \mathbf{DF}_{\mathsf{MIPC}}$ is a well-defined functor. □

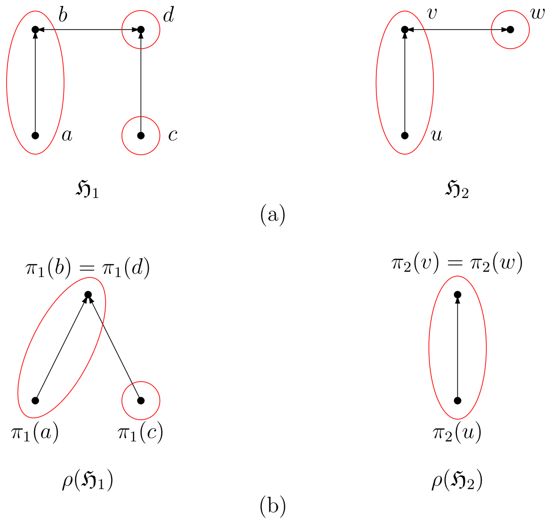


Fig. 1. The counterexample from Example 5.16.

Example 5.16. It is not true in general that if f is a $\mathbf{DF}_{\mathsf{MS4}}$-morphism, then $\rho(f)$ is a p-morphism with respect to $E_{Q'}$. To see this, consider $\mathfrak{H}_1 = (Y_1, R_1, E_1)$ and $\mathfrak{H}_2 = (Y_2, R_2, E_2)$ depicted in Fig. 1(a). The black arrows represent the quasi-orders R_i, the double black arrows the E_{R_i}-equivalence classes, and the red circles the E_i-equivalence classes for $i = 1, 2$.[6]

We show that $\mathfrak{H}_1$ is a descriptive $\mathsf{MS4}$-frame. Clearly R_1 is a quasi-order and E_1 is an equivalence relation. We show that $R_1E_1[x] \subseteq E_1R_1[x]$ for each $x \in Y_1$. If $x = c$ or $x = d$, then $E_1[x] = \{x\}$. Therefore, $R_1E_1[x] = R_1[x] \subseteq E_1R_1[x]$. On the other hand, if $x = a$ or $x = b$, then $E_1[x] = \{a, b\}$ and $R_1[x] = \{a, b, d\}$ or $R_1[x] = \{b, d\}$. Thus, $R_1E_1[x] = \{a, b, d\} \subseteq E_1R_1[x]$. Since $\mathfrak{H}_1$ is finite, its topology is discrete, and hence $\mathfrak{H}_1$ is a descriptive $\mathsf{MS4}$-frame. The proof that $\mathfrak{H}_2$ is a descriptive $\mathsf{MS4}$-frame is similar.

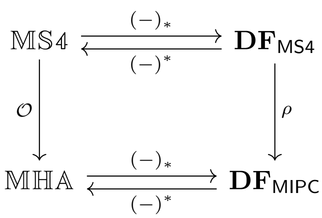
Fig. 1(b) depicts the skeletons $\rho(\mathfrak{H}_1)$ and $\rho(\mathfrak{H}_2)$, where the black arrows represent the partial orders R'_i and the red circles the $E_{Q'_i}$-equivalence classes for $i = 1, 2$. Define $f \colon \mathfrak{H}_1 \to \mathfrak{H}_2$ by

$$f(a) = u, \;\; f(b) = v, \;\; \text{and} \;\; f(c) = f(d) = w.$$

It is straightforward to check that f is a $\mathbf{DF}_{\mathsf{MS4}}$-morphism. However, $\rho(f) \colon \rho(\mathfrak{H}_1) \to \rho(\mathfrak{H}_2)$ is not a p-morphism with respect to $E_{Q'}$ because $\rho(f)(\pi_1(c)) = \pi_2(w)$ and $\pi_2(w) E_{Q'_2} \pi_2(u)$, but $E_{Q'_1}[\pi_1(c)] = \{\pi_1(c)\}$, so there is no $x \in E_{Q'_1}[\pi_1(c)]$ such that $\rho(f)(x) = \pi_2(u)$.

We are ready to prove that $\rho \colon \mathbf{DF}_{\mathsf{MS4}} \to \mathbf{DF}_{\mathsf{MIPC}}$ is dual to $\mathcal{O} \colon \mathbb{MS4} \to \mathbb{MHA}$.

Theorem 5.17. *The following diagram commutes up to natural isomorphism.*



[6] These should be understood as Hasse diagrams of the corresponding frames. For example, aR_1b and bR_1d, so aR_1d, but the arrow from a to d is not drawn.

Proof. It is enough to prove that $(-)^* \circ \rho$ is naturally isomorphic to $\mathcal{O} \circ (-)^*$. Let $\mathfrak{G} = (Y, R, E)$ be a descriptive $\mathsf{MS4}$-frame, $\rho(\mathfrak{G}) = (X, R', Q')$ its skeleton, and $\pi\colon Y \to X$ the quotient map. Clearly, if U is a clopen R-upset of $\rho(\mathfrak{G})$, then $\pi^{-1}[U]$ is a clopen R-upset of $\mathfrak{G}$. In fact, $\pi^{-1}\colon \rho(\mathfrak{G})^* \to \mathcal{O}(\mathfrak{G}^*)$ is an isomorphism of Heyting algebras (see, e.g., [20, Prop. 3.4.15]). It remains to show that π^{-1} commutes with $\forall$ and $\exists$. To simplify notation, let $H = \mathcal{O}(\mathfrak{G}^*)$. For $U \in \rho(\mathfrak{G})^*$, it follows from Definition 5.1 and the proof of Theorem 3.11 that

$$\begin{aligned}\forall_H(\pi^{-1}[U]) &= Y \setminus R^{-1}[Y \setminus [Y \setminus E[Y \setminus \pi^{-1}[U]]]] = Y \setminus R^{-1}E[Y \setminus \pi^{-1}[U]]\\ &= Y \setminus Q^{-1}[Y \setminus \pi^{-1}[U]].\end{aligned}$$

Thus, by Remark 2.13,

$$\begin{aligned}\pi^{-1}[\forall U] &= \pi^{-1}[X \setminus (Q')^{-1}[X \setminus U]] = Y \setminus \pi^{-1}[(Q')^{-1}[X \setminus U]]\\ &= Y \setminus Q^{-1}[\pi^{-1}[X \setminus U]] = Y \setminus Q^{-1}[Y \setminus \pi^{-1}[U]] = \forall_H(\pi^{-1}[U]).\end{aligned}$$

Definition 5.1, Remark 2.13, and the proof of Theorem 3.11 also yield

$$\exists U = Q'[U] \quad \text{and} \quad \exists_H(\pi^{-1}[U]) = E[\pi^{-1}[U]].$$

Therefore,

$$\pi^{-1}[\exists U] = \pi^{-1}[Q'[U]] = Q[\pi^{-1}[U]] = ER[\pi^{-1}[U]] = E[\pi^{-1}[U]] = \exists_H(\pi^{-1}[U]),$$

where the fourth equality follows from the fact that $\pi^{-1}[U]$ is an R-upset. Consequently, π^{-1} is an $\mathbb{MHA}$-isomorphism.

To see the naturality of this isomorphism, we need to show that $\pi_1^{-1} \circ \rho(f)^{-1} = f^{-1} \circ \pi_2^{-1}$ for every $\mathbf{DF}_{\mathsf{MS4}}$-morphism $f\colon \mathfrak{G}_1 \to \mathfrak{G}_2$. For each $U \in \rho(\mathfrak{G}_2)^*$ and $x \in Y_1$, the definition of $\rho(f)$ yields

$$x \in \pi_1^{-1}[\rho(f)^{-1}[U]] \iff \rho(f)(\pi_1(x)) \in U \iff \pi_2(f(x)) \in U \iff x \in f^{-1}[\pi_2^{-1}[U]],$$

and hence $\pi_1^{-1} \circ \rho(f)^{-1}$ and $f^{-1} \circ \pi_2^{-1}$ coincide on U. $\square$

Obtaining an algebraic insight of τ and σ requires investigating the realizability of monadic Heyting algebras as the algebras of open elements of $\mathsf{MS4}$-algebras. Here the situation is more complicated because unlike the classic case of Heyting algebras (see, e.g., [37, Sec. IV.3]), it remains open whether each monadic Heyting algebra can be realized this way. We will discuss this in detail in a forthcoming paper.

6. Failure of Blok–Esakia for MIPC

In this final section we show that τ and σ are lattice homomorphisms, thus generalizing the result of Maksimova and Rybakov [31] (see also [12, Thm. 9.66]) to the monadic setting. On the other hand, we show that ρ is neither a lattice homomorphism nor one-to-one. From this we derive our main result, that σ is not an isomorphism, and hence that the Blok–Esakia Theorem does not extend to the monadic setting.

Lemma 6.1.

(1) *If* $\mathfrak{A} \in \mathbb{MHA}$ *is subdirectly irreducible, then* $\forall a_1 \vee \forall a_2 = 1$ *implies* $a_1 = 1$ *or* $a_2 = 1$ *for any* $a_1, a_2 \in A$.
(2) *If* $\mathfrak{B} \in \mathbb{MS4}$ *is subdirectly irreducible, then* $\blacksquare b_1 \vee \blacksquare b_2 = 1$ *implies* $b_1 = 1$ *or* $b_2 = 1$ *for any* $b_1, b_2 \in B$.

Proof. (1) follows from [2, Thm. 2.11] and (2) is proved similarly. □

Following [31], for two formulas φ and ψ, we write $\varphi \vee' \psi$ for $\varphi \vee \psi'$, where ψ' is obtained by substituting the variables in ψ that occur in φ with fresh variables, so that φ and ψ' have no variables in common.

Lemma 6.2. *Let Γ_1, Γ_2 be sets of formulas in $\mathcal{L}_{\Box\forall}$ and $\mathsf{M}_i = \mathsf{MS4} + \Gamma_i$ for $i = 1, 2$. Then*

$$\mathsf{M}_1 \cap \mathsf{M}_2 = \mathsf{MS4} + \{\blacksquare\gamma_1 \vee' \blacksquare\gamma_2 : \gamma_1 \in \Gamma_1,\ \gamma_2 \in \Gamma_2\}.$$

Proof. It is clear that

$$\{\blacksquare\gamma_1 \vee' \blacksquare\gamma_2 : \gamma_1 \in \Gamma_1,\ \gamma_2 \in \Gamma_2\} \subseteq \mathsf{M}_1 \cap \mathsf{M}_2.$$

We prove the other inclusion by showing that if a subdirectly irreducible MS4-algebra $\mathfrak{B}$ validates $\blacksquare\gamma_1 \vee' \blacksquare\gamma_2$ for every $\gamma_1 \in \Gamma_1$ and $\gamma_2 \in \Gamma_2$, then it validates $\mathsf{M}_1 \cap \mathsf{M}_2$. We argue by contrapositive. Suppose that $\mathfrak{B} \nvDash \mathsf{M}_1 \cap \mathsf{M}_2$. Since $\mathsf{M}_1 \cap \mathsf{M}_2 \subseteq \mathsf{M}_1, \mathsf{M}_2$, we obtain that $\mathfrak{B} \nvDash \mathsf{M}_1, \mathsf{M}_2$. Therefore, there are $\gamma_1 \in \Gamma_1$ and $\gamma_2 \in \Gamma_2$ and two valuations v_1, v_2 on $\mathfrak{B}$ such that $v_1(\gamma_1) \neq 1$ and $v_2(\gamma_2) \neq 1$. Let $p_1, \ldots, p_n$ be the variables occurring in γ_1 and $q_1, \ldots, q_m$ those occurring in γ_2. We let $q_1', \ldots, q_m'$ be the variables that substitute $q_1, \ldots, q_m$ to obtain $\gamma_1 \vee' \gamma_2$. Define a valuation v_2' on $\mathfrak{B}$ by setting $v_2'(q_i') = v_2(q_i)$ for $i = 1, \ldots, m$, and $v_2'(p) = v_2(p)$ for the remaining variables. Then $v_2'(\gamma_2') = v_2(\gamma_2)$. Let v be a new valuation that coincides with v_1 on the variables occurring in γ_1 and with v_2' on the variables occurring in γ_2'. Then $v(\gamma_1) \neq 1$ and $v(\gamma_2') \neq 1$. Since $\mathfrak{B}$ is subdirectly irreducible,

$$v(\blacksquare\gamma_1 \vee' \blacksquare\gamma_2) = \blacksquare v(\gamma_1) \vee \blacksquare v(\gamma_2') \neq 1$$

by Lemma 6.1(2), and hence $\mathfrak{B} \nvDash \blacksquare\gamma_1 \vee' \blacksquare\gamma_2$. Thus,

$$\mathsf{M}_1 \cap \mathsf{M}_2 \subseteq \{\blacksquare\gamma_1 \vee' \blacksquare\gamma_2 : \gamma_1 \in \Gamma_1,\ \gamma_2 \in \Gamma_2\},$$

concluding the proof. □

Theorem 6.3. *τ and σ are lattice homomorphisms.*

Proof. By Proposition 4.3(3), τ and σ preserve arbitrary joins. Thus, it suffices to show that they preserve binary meets. We only prove it for τ because the proof for σ is similar. Let $\mathsf{L}_1, \mathsf{L}_2 \in \Lambda(\mathsf{MIPC})$. Since τ preserves $\subseteq$, we have $\tau(\mathsf{L}_1 \cap \mathsf{L}_2) \subseteq \tau\mathsf{L}_1 \cap \tau\mathsf{L}_2$. For the other inclusion, by Lemma 6.2 it is sufficient to show that $\blacksquare\varphi_1^t \vee' \blacksquare\varphi_2^t \in \tau(\mathsf{L}_1 \cap \mathsf{L}_2)$ for every $\varphi_1 \in \mathsf{L}_1$ and $\varphi_2 \in \mathsf{L}_2$. By the definition of the Gödel translation, $\blacksquare\varphi_1^t \vee' \blacksquare\varphi_2^t = (\forall\varphi_1 \vee' \forall\varphi_2)^t$. Thus, $\blacksquare\varphi_1^t \vee' \blacksquare\varphi_2^t \in \tau(\mathsf{L}_1 \cap \mathsf{L}_2)$ because $\forall\varphi_1 \vee' \forall\varphi_2 \in \mathsf{L}_1 \cap \mathsf{L}_2$. □

Remark 6.4.

(1) Whether τ and σ are complete lattice homomorphisms remains open.
(2) Another open problem is the surjectivity of ρ. Equivalently, the question of whether every extension of MIPC has a modal companion remains open. In turn, this is equivalent to determining whether $\tau\mathsf{L}$ is a modal companion of L for every $\mathsf{L} \in \Lambda(\mathsf{MIPC})$. Indeed, if L has a modal companion, then $\tau\mathsf{L}$ must be the least such. In [6] we will show that Kripke completeness is a sufficient condition for L to have a modal companion.

We now turn our attention to ρ. Since $\mathbb{MHA}$ and $\mathbb{MS4}$ are congruence-distributive varieties, we will freely use Jónsson's Lemma and especially its corollary that if a congruence-distributive variety $\mathbb{V}$ is generated by a finite algebra $\mathfrak{A}$, then subdirectly irreducible algebras in $\mathbb{V}$ are in $\mathsf{HS}(\mathfrak{A})$ (see, e.g., [11, Cor. IV.6.10]). We will also utilize that both $\mathbb{MHA}$ and $\mathbb{MS4}$ have the CEP, and hence that $\mathsf{HS} = \mathsf{SH}$.

The next lemma is a generalization of a similar result for $\mathbb{HA}$ and $\mathbb{S4}$ (see, e.g., [20, Thm. 3.4.16]).

Lemma 6.5. *Let $\mathfrak{A} \in \mathbb{MHA}$ and $\mathfrak{B} \in \mathbb{MS4}$. There are inclusion-reversing bijections between*

(1) *the sets of monadic filters of $\mathfrak{A}$ and closed Q-upsets of $\mathfrak{A}_*$, and*
(2) *the sets of monadic $\Box$-filters of $\mathfrak{B}$ and closed Q-upsets of $\mathfrak{B}_*$.*

Sketch of proof. The proof of (1) can be found in [4, Thm. 18] and (2) is proved similarly (see, e.g., [7, Thm. 3.4]). □

Remark 6.6. Let $\mathfrak{A} \in \mathbb{MHA}$, θ be a congruence on $\mathfrak{A}$, and Z the corresponding closed Q-upset of $\mathfrak{A}_* = (X, R, Q)$. The quotient $\mathfrak{A}/\theta$ is then dual to the descriptive MIPC-frame $(Z, R_{|Z}, Q_{|Z})$ obtained by restricting R and Q to Z. A similar correspondence holds for quotients of $\mathsf{MS4}$-algebras and closed Q-upsets of their duals.

Definition 6.7. A descriptive MIPC-frame (X, R, Q) is *strongly Q-rooted* if there is $x \in X$ such that $Q[x] = X$ and $E_Q[x]$ is clopen.

Strongly Q-rooted descriptive $\mathsf{MS4}$-frames are defined similarly. As a consequence of Lemma 6.5, we have the following dual characterization of subdirectly irreducible algebras in $\mathbb{MHA}$ and $\mathbb{MS4}$, which generalizes a similar characterization of subdirectly irreducible algebras in $\mathbb{HA}$ and $\mathbb{S4}$ (see, e.g., [20, Prop. A.1.2]). The proof of (1) can be found in [4, Thm. 24], and (2) is proved similarly (see, e.g., [7, Thm. 3.5]).

Lemma 6.8.

(1) *$\mathfrak{A} \in \mathbb{MHA}$ is subdirectly irreducible iff $\mathfrak{A}_*$ is strongly Q-rooted.*
(2) *$\mathfrak{B} \in \mathbb{MS4}$ is subdirectly irreducible iff $\mathfrak{B}_*$ is strongly Q-rooted.*

Let $h \colon \mathfrak{B}_1 \to \mathfrak{B}_2$ be a homomorphism of $\mathsf{S4}$-algebras. It is well known that h is one-to-one iff h_* is onto (see, e.g., [20, Lem. 3.3.13]), and that the same holds for Heyting algebra homomorphisms. As an immediate consequence, we obtain:

Lemma 6.9.

(1) *A homomorphism f of monadic Heyting algebras is one-to-one iff f_* is onto.*
(2) *A homomorphism g of $\mathsf{MS4}$-algebras is one-to-one iff g_* is onto.*

The following dual characterization of the composition of the operators H and S is an immediate consequence of Lemma 6.5, Lemma 6.9, and the fact that both $\mathbb{MHA}$ and $\mathbb{MS4}$ have the CEP.

Lemma 6.10. *Let $\mathfrak{F}_1, \mathfrak{F}_2 \in \mathbf{DF}_{\mathsf{MIPC}}$ and $\mathfrak{G}_1, \mathfrak{G}_2 \in \mathbf{DF}_{\mathsf{MS4}}$.*

(1) $\mathfrak{F}_2^* \in \mathsf{HS}(\mathfrak{F}_1^*) \iff \mathfrak{F}_2^* \in \mathsf{SH}(\mathfrak{F}_1^*) \iff$ *there is an onto $\mathbf{DF}_{\mathsf{MIPC}}$-morphism from a closed Q_1-upset of $\mathfrak{F}_1$ to $\mathfrak{F}_2$.*
(2) $\mathfrak{G}_2^* \in \mathsf{HS}(\mathfrak{G}_1^*) \iff \mathfrak{G}_2^* \in \mathsf{SH}(\mathfrak{G}_1^*) \iff$ *there is an onto $\mathbf{DF}_{\mathsf{MS4}}$-morphism from a closed Q_1-upset of $\mathfrak{G}_1$ to $\mathfrak{G}_2$.*

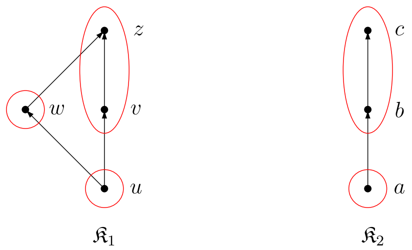


Fig. 2. The frames $\mathfrak{K}_1$ and $\mathfrak{K}_2$.

Definition 6.11. Let $\mathfrak{K}_1 = (Y_1, R_1, E_1)$ and $\mathfrak{K}_2 = (Y_2, R_2, E_2)$ be the descriptive MS4-frames depicted in Fig. 2, where the black arrows represent the partial orders R_i and the red circles the E_i-equivalence classes ($i = 1, 2$).

Remark 6.12. An argument similar to Example 5.16 gives that $\mathfrak{K}_1$ and $\mathfrak{K}_2$ are indeed descriptive MS4-frames.

Let $\mathfrak{B}_1 = \mathfrak{K}_1^*$ and $\mathfrak{B}_2 = \mathfrak{K}_2^*$. Since both $\mathfrak{K}_1$ and $\mathfrak{K}_2$ are finite, their topologies are discrete, and so $\mathfrak{B}_1$ and $\mathfrak{B}_2$ are the powersets of Y_1 and Y_2, respectively. We also define $\mathbb{V}_1 := \mathsf{Var}(\mathfrak{B}_1)$ and $\mathbb{V}_2 := \mathsf{Var}(\mathfrak{B}_2)$. Because each R_i is a partial order, each $\mathfrak{B}_i$ is a finite MGrz-algebra (see, e.g., [20, Cor. 3.5.10]).

Lemma 6.13.

(1) *There is an onto* $\mathbf{DF}_{\mathsf{MIPC}}$*-morphism from* $\rho(\mathfrak{K}_1)$ *to* $\rho(\mathfrak{K}_2)$*.*
(2) *There is no onto* $\mathbf{DF}_{\mathsf{MS4}}$*-morphism from a* Q_1*-upset of* $\mathfrak{K}_1$ *to* $\mathfrak{K}_2$*.*

Proof. (1). Since R_i is a partial order for $i = 1, 2$, we have that $\pi_i \colon \mathfrak{K}_i \to \rho(\mathfrak{K}_i)$ is a bijection that preserves and reflects R_i and Q_i. Thus, we may identify $\rho(\mathfrak{K}_i)$ with $\mathfrak{K}_i$. Define $f \colon \rho(\mathfrak{K}_1) \to \rho(\mathfrak{K}_2)$ by

$$f(u) = a, \quad f(v) = b, \quad \text{and} \quad f(w) = f(z) = c.$$

Clearly f is onto, and it is straightforward to see that f is a p-morphism with respect to R and Q. The following calculations show that f is also a weak p-morphism with respect to Q^{-1}:

$$\begin{aligned}
Q_2^{-1}[f(u)] &= Q_2^{-1}[a] = \{a\} = R_2^{-1}[a] = R_2^{-1}[f(u)] = R_2^{-1} f Q_1^{-1}[u],\\
Q_2^{-1}[f(v)] &= Q_2^{-1}[b] = Y_2 = R_2^{-1}[Y_2] = R_2^{-1} f[Y_1] = R_2^{-1} f Q_1^{-1}[v],\\
Q_2^{-1}[f(w)] &= Q_2^{-1}[c] = Y_2 = R_2^{-1}[\{a, c\}] = R_2^{-1} f[\{u, w\}] = R_2^{-1} f Q_1^{-1}[w],\\
Q_2^{-1}[f(z)] &= Q_2^{-1}[c] = Y_2 = R_2^{-1}[Y_2] = R_2^{-1} f[Y_1] = R_2^{-1} f Q_1^{-1}[z].
\end{aligned}$$

Thus, f is a $\mathbf{DF}_{\mathsf{MIPC}}$-morphism.

(2). Suppose there is a Q_1-upset U of $\mathfrak{K}_1$ and an onto $\mathbf{DF}_{\mathsf{MS4}}$-morphism $g \colon U \to \mathfrak{K}_2$. Since g is onto, there is $x \in U$ such that $g(x) = a$. Because g is a p-morphism with respect to R, we have $g[R_1[x]] = R_2[g(x)] = R_2[a]$. Since $R_2[a]$ has 3 elements, $R_1[x]$ must have at least 3 elements. Therefore, $x = u$, and hence $u \in U$. This implies that $U = Y_1$. Because the cardinality of $R_1[w]$ is 2, the cardinality of $R_2[g(w)] = g[R_1[w]]$ is at most 2. Thus, $g(w) \in \{b, c\}$. That g is a p-morphism with respect to E implies that $g[E_1[w]] = E_2[g[w]] = \{b, c\}$. Consequently, $E_1[w]$ must have at least 2 elements, which is a contradiction, proving that such a g does not exist. $\square$

Lemma 6.14. *Let* $\mathfrak{G}_1, \mathfrak{G}_2$ *be two partially ordered descriptive* MS4*-frames. If* $\rho(\mathfrak{G}_1)$ *and* $\rho(\mathfrak{G}_2)$ *are isomorphic in* $\mathbf{DF}_{\mathsf{MIPC}}$*, then* $\mathfrak{G}_1$ *and* $\mathfrak{G}_2$ *are isomorphic in* $\mathbf{DF}_{\mathsf{MS4}}$*.*

Proof. Since $\mathfrak{G}_i$ is partially ordered for $i = 1, 2$, we have that $\pi_i : \mathfrak{G}_i \to \rho(\mathfrak{G}_i)$ is a bijection that preserves and reflects the relations on $\mathfrak{G}_i$. Because a $\mathbf{DF}_{\mathsf{MIPC}}$-isomorphism $g : \rho(\mathfrak{G}_1) \to \rho(\mathfrak{G}_2)$ is a bijection that preserves and reflects the relations on $\rho(\mathfrak{G}_1)$ (see Remark 2.11), it gives rise to a bijection $f : \mathfrak{G}_1 \to \mathfrak{G}_2$ that preserves and reflects the relations on $\mathfrak{G}_1$. Thus, f is a $\mathbf{DF}_{\mathsf{MS4}}$-isomorphism (see Remark 3.10). □

Lemma 6.15.

(1) $\mathcal{O}(\mathfrak{B}_2)$ *embeds into* $\mathcal{O}(\mathfrak{B}_1)$.
(2) $\mathfrak{B}_2 \notin \mathsf{HS}(\mathfrak{B}_1)$.
(3) $\mathcal{O}(\mathfrak{B}_2) \notin \mathcal{O}(\mathbb{V}_1)$.

Proof. (1). This follows from Lemma 6.9(1) and Lemma 6.13(1).

(2). This follows from Lemma 6.10(2) and Lemma 6.13(2).

(3). Suppose $\mathcal{O}(\mathfrak{B}_2) \in \mathcal{O}(\mathbb{V}_1)$. Then there is $\mathfrak{B} \in \mathbb{V}_1$ such that $\mathcal{O}(\mathfrak{B}_2) = \mathcal{O}(\mathfrak{B})$. Since $Q_2[a] = Y_2$, it follows from Lemma 6.8(2) that $\mathfrak{B}_2$ is subdirectly irreducible, so $\mathcal{O}(\mathfrak{B}_2) = \mathcal{O}(\mathfrak{B})$ is subdirectly irreducible, and hence $\mathfrak{B}$ is subdirectly irreducible by Corollary 5.7. Therefore, since $\mathfrak{B}$ belongs to the variety generated by $\mathfrak{B}_1$, Jónsson's Lemma yields that $\mathfrak{B} \in \mathsf{HS}(\mathfrak{B}_1)$. By Lemma 6.10(2), there are a Q_1-upset U of $\mathfrak{K}_1$ and a $\mathbf{DF}_{\mathsf{MS4}}$-morphism f from U onto the dual $\mathfrak{G} = (Y, R, E)$ of $\mathfrak{B}$. Because R_1 is a partial order, $\mathfrak{B}_1$ is a finite MGrz-algebra. Hence, $\mathfrak{B}$ is a MGrz-algebra since $\mathfrak{B} \in \mathsf{HS}(\mathfrak{B}_1)$. Therefore, R is a partial order. Because $\mathcal{O}(\mathfrak{B}_2) = \mathcal{O}(\mathfrak{B})$, the frames $\rho(\mathfrak{K}_2)$ and $\rho(\mathfrak{G})$ are isomorphic in $\mathbf{DF}_{\mathsf{MIPC}}$. Thus, Lemma 6.14 yields that $\mathfrak{K}_2$ and $\mathfrak{G}$ are isomorphic in $\mathbf{DF}_{\mathsf{MS4}}$, and so there is a $\mathbf{DF}_{\mathsf{MS4}}$-morphism f from U onto $\mathfrak{K}_2$. This contradicts Lemma 6.13(2). □

Proposition 5.11 and Theorem 5.12 show that the operator $\mathsf{S}\mathcal{O}$ plays the same role for varieties of $\mathsf{MS4}$-algebras as $\mathcal{O}$ does for varieties of $\mathsf{S4}$-algebras. The following theorem yields that $\mathsf{S}\mathcal{O}$ is not well behaved, already when restricted to varieties of MGrz-algebras.

Theorem 6.16.

(1) $\mathbb{V}_1$ *is a variety of* MGrz-*algebras such that* $\mathcal{O}(\mathbb{V}_1)$ *is not a variety of monadic Heyting algebras.*
(2) S *and* $\mathcal{O}$ *do not commute.*
(3) $\mathsf{S}\mathcal{O}$ *does not commute with binary intersections.*
(4) $\mathsf{S}\mathcal{O}$ *is not one-to-one.*

Proof. (1). Let $\mathbb{MG}\mathbb{rz}$ be the variety of MGrz-algebras. Since $\mathfrak{B}_1 \in \mathbb{MG}\mathbb{rz}$, we have that $\mathbb{V}_1$ is a subvariety of $\mathbb{MG}\mathbb{rz}$. Because $\mathcal{O}(\mathfrak{B}_1) \in \mathcal{O}(\mathbb{V}_1)$, it follows from Lemma 6.15(1) that $\mathcal{O}(\mathfrak{B}_2) \in \mathsf{S}\mathcal{O}(\mathbb{V}_1)$. By Lemma 6.15(3), $\mathcal{O}(\mathfrak{B}_2) \notin \mathcal{O}(\mathbb{V}_1)$. Consequently, $\mathsf{S}\mathcal{O}(\mathbb{V}_1) \neq \mathcal{O}(\mathbb{V}_1)$, so $\mathcal{O}(\mathbb{V}_1)$ is not closed under S, and hence is not a variety.

(2). As we saw in the proof of (1), $\mathsf{S}\mathcal{O}(\mathbb{V}_1) \neq \mathcal{O}(\mathbb{V}_1) = \mathcal{O}\mathsf{S}(\mathbb{V}_1)$. Thus, S and $\mathcal{O}$ do not commute.

(3). We show that $\mathsf{S}\mathcal{O}(\mathbb{V}_1 \cap \mathbb{V}_2) \neq \mathsf{S}\mathcal{O}(\mathbb{V}_1) \cap \mathsf{S}\mathcal{O}(\mathbb{V}_2)$. By Lemma 6.15(1), $\mathcal{O}(\mathfrak{B}_2) \in \mathsf{S}\mathcal{O}(\mathfrak{B}_1)$, so $\mathcal{O}(\mathfrak{B}_2) \in \mathsf{S}\mathcal{O}(\mathbb{V}_1) \cap \mathsf{S}\mathcal{O}(\mathbb{V}_2)$. It remains to prove that $\mathcal{O}(\mathfrak{B}_2) \notin \mathsf{S}\mathcal{O}(\mathbb{V}_1 \cap \mathbb{V}_2)$. By Jónsson's Lemma, every subdirectly irreducible algebra in $\mathbb{V}_1 \cap \mathbb{V}_2$ is in $\mathsf{HS}(\mathfrak{B}_1) \cap \mathsf{HS}(\mathfrak{B}_2)$. Therefore, $\mathbb{V}_1 \cap \mathbb{V}_2 = \mathsf{Var}(\mathsf{HS}(\mathfrak{B}_1) \cap \mathsf{HS}(\mathfrak{B}_2))$. Thus, by Proposition 5.11(1), $\mathsf{S}\mathcal{O}(\mathbb{V}_1 \cap \mathbb{V}_2) = \mathsf{S}\mathcal{O}(\mathsf{Var}(\mathsf{HS}(\mathfrak{B}_1) \cap \mathsf{HS}(\mathfrak{B}_2))) = \mathsf{Var}(\mathcal{O}(\mathsf{HS}(\mathfrak{B}_1) \cap \mathsf{HS}(\mathfrak{B}_2)))$. Using Jónsson's Lemma again, if $\mathcal{O}(\mathfrak{B}_2) \in \mathsf{Var}(\mathcal{O}(\mathsf{HS}(\mathfrak{B}_1) \cap \mathsf{HS}(\mathfrak{B}_2)))$, then $\mathcal{O}(\mathfrak{B}_2) \in \mathsf{HS}(\mathcal{O}(\mathsf{HS}(\mathfrak{B}_1) \cap \mathsf{HS}(\mathfrak{B}_2)))$. Consequently, there exists $\mathfrak{B} \in \mathsf{HS}(\mathfrak{B}_1) \cap \mathsf{HS}(\mathfrak{B}_2)$ such that $\mathcal{O}(\mathfrak{B}_2) \in \mathsf{HS}(\mathcal{O}(\mathfrak{B}))$. Let $\mathfrak{G} \in \mathbf{DF}_{\mathsf{MS4}}$ be the dual of $\mathfrak{B}$. Since $\mathfrak{B} \in \mathsf{HS}(\mathfrak{B}_2)$, Lemma 6.10(2) yields that the cardinality $|\mathfrak{G}|$ is less than or equal to $|\mathfrak{K}_2| = 3$. Because $\mathcal{O}(\mathfrak{B}_2) \in \mathsf{HS}(\mathcal{O}(\mathfrak{B}))$, from Lemma 6.10(1) it follows that $3 = |\rho(\mathfrak{K}_2)| \leq |\rho(\mathfrak{G})|$. Therefore, $|\mathfrak{G}| = |\mathfrak{K}_2| = 3$, so Lemma 6.10(2) implies that $\mathfrak{G} \cong \mathfrak{K}_2$, and hence $\mathfrak{B} \cong \mathfrak{B}_2$. Thus, from $\mathfrak{B} \in \mathsf{HS}(\mathfrak{B}_1)$ it follows that $\mathfrak{B}_2 \in \mathsf{HS}(\mathfrak{B}_1)$, which contradicts Lemma 6.15(2).

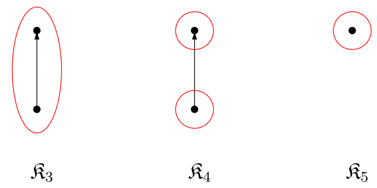


Fig. 3. The frames $\mathfrak{K}_3$, $\mathfrak{K}_4$, and $\mathfrak{K}_5$.

(4). Let $\mathbb{V}$ be the join of $\mathbb{V}_1$ and $\mathbb{V}_2$. We show that $\mathsf{S}\mathcal{O}(\mathbb{V}) = \mathsf{S}\mathcal{O}(\mathbb{V}_1)$ but $\mathbb{V} \neq \mathbb{V}_1$. We have

$$\begin{aligned}\mathsf{S}\mathcal{O}(\mathbb{V}) &= \mathsf{S}\mathcal{O}(\mathsf{Var}(\{\mathfrak{B}_1, \mathfrak{B}_2\})) = \mathsf{Var}(\{\mathcal{O}(\mathfrak{B}_1), \mathcal{O}(\mathfrak{B}_2)\}) \\ &= \mathsf{Var}(\mathcal{O}(\mathfrak{B}_1)) = \mathsf{S}\mathcal{O}(\mathsf{Var}(\mathfrak{B}_1)) = \mathsf{S}\mathcal{O}(\mathbb{V}_1),\end{aligned}$$

where the first and last equalities follow from the definitions of $\mathbb{V}$ and $\mathbb{V}_1$, the second and fourth from Proposition 5.11(1), and the third is a consequence of Lemma 6.15(1). Jónsson's Lemma together with Lemma 6.15(2) yields that $\mathfrak{B}_2 \notin \mathsf{Var}(\mathfrak{B}_1)$. Thus, $\mathbb{V} = \mathsf{Var}(\{\mathfrak{B}_1, \mathfrak{B}_2\}) \neq \mathsf{Var}(\mathfrak{B}_1) = \mathbb{V}_1$. □

Remark 6.17. As we saw in Theorem 6.16(1), $\mathcal{O}(\mathbb{V}_1)$ is not a variety. On the other hand, $\mathcal{O}(\mathbb{V}_2)$ is a variety. This can be seen as follows. For $\mathfrak{G} \in \mathbf{DF}_{\mathsf{MS4}}$, it is straightforward to see that there is an onto $\mathbf{DF}_{\mathsf{MS4}}$-morphism from a closed Q_2-upset of $\mathfrak{K}_2$ to $\mathfrak{G}$ iff $\mathfrak{G} \cong \mathfrak{K}_i$ for $i = 2, \dots, 5$, where $\mathfrak{K}_3, \mathfrak{K}_4, \mathfrak{K}_5$ are shown in Fig. 3. Thus, by Lemma 6.10(2), $\mathfrak{G}^* \in \mathsf{HS}(\mathfrak{B}_2)$ iff $\mathfrak{G} \cong \mathfrak{K}_i$ for $i = 2, \dots, 5$.

Similarly, for $\mathfrak{F} \in \mathbf{DF}_{\mathsf{MIPC}}$, it is straightforward to see that there is an onto $\mathbf{DF}_{\mathsf{MIPC}}$-morphism from a closed Q_2'-upset of $\rho(\mathfrak{K}_2)$ to $\mathfrak{F}$ iff $\mathfrak{F} \cong \rho(\mathfrak{K}_i)$ for $i = 2, \dots, 5$. Thus, by Lemma 6.10(1), $\mathfrak{F}^* \in \mathsf{HS}\mathcal{O}(\mathfrak{B}_2)$ iff $\mathfrak{F} \cong \rho(\mathfrak{K}_i)$ for $i = 2, \dots, 5$.

Jónsson's Lemma yields that the subdirectly irreducible MS4-algebras in $\mathbb{V}_2$ are exactly the ones isomorphic to $\mathfrak{K}_i^*$ for $i = 2, \dots, 5$. Since $\mathbb{V}_2 = \mathsf{Var}(\mathfrak{B}_2)$, by Proposition 5.11 we have

$$\mathsf{Var}(\mathcal{O}(\mathbb{V}_2)) = \mathsf{S}\mathcal{O}(\mathbb{V}_2) = \mathsf{S}\mathcal{O}(\mathsf{Var}(\mathfrak{B}_2)) = \mathsf{Var}(\mathcal{O}(\mathfrak{B}_2)).$$

So, Jónsson's Lemma implies that the subdirectly irreducible monadic Heyting algebras in $\mathsf{Var}(\mathcal{O}(\mathbb{V}_2))$ are exactly the ones isomorphic to $\rho(\mathfrak{K}_i)^*$ for $i = 2, \dots, 5$.

Let $\mathfrak{A} \in \mathsf{Var}(\mathcal{O}(\mathbb{V}_2))$ and $\mathfrak{A}_* = (X, R, Q)$. For each $x \in X$ we have that $Q[x] \cong \rho(\mathfrak{K}_i)$ for $i = 2, \dots, 5$. Therefore, for each $x \in X$, the R-upset $R[x]$ is a chain of at most 3 elements, and $E[x]$ is either a singleton or a 2-element chain whose top element is maximal in X. From this we can derive that $E_Q[U]$ is clopen for each clopen U, and hence that E_Q is a continuous relation on X. Thus, (X, R, E_Q) is a descriptive MS4-frame.

Let $\mathfrak{B}$ be the MS4-algebra dual to (X, R, E_Q). Then $\mathfrak{A} \cong \mathcal{O}(\mathfrak{B})$ by Theorem 5.17. For every $x \in X$ we have that $Q[x]$ is isomorphic to $\rho(\mathfrak{K}_i)$ for some $i = 2, \dots, 5$. Since $\rho(\mathfrak{K}_i) \cong \mathfrak{K}_i$ for every $i = 2, \dots, 5$, it follows that each $Q[x]$ is isomorphic to $\mathfrak{K}_i$ for some $i = 2, \dots, 5$. This means that $\mathfrak{B}$ is a subdirect product of a family of MS4-algebras each isomorphic to $\mathfrak{K}_i^*$ for some $i = 2, \dots, 5$. We observed above that $\mathfrak{K}_i^* \in \mathsf{HS}(\mathfrak{B}_2)$ for every $i = 2, \dots, 5$. Thus, $\mathfrak{B} \in \mathsf{Var}(\mathfrak{B}_2) = \mathbb{V}_2$. Consequently, $\mathfrak{A} \in \mathcal{O}(\mathbb{V}_2)$. This proves that $\mathsf{Var}(\mathcal{O}(\mathbb{V}_2)) \subseteq \mathcal{O}(\mathbb{V}_2)$, and so $\mathcal{O}(\mathbb{V}_2)$ is a variety.

By putting Theorems 2.6, 3.6, 5.12 and 6.16 together, we obtain:

Theorem 6.18.

(1) $\rho\colon \Lambda(\mathsf{MGrz}) \to \Lambda(\mathsf{MIPC})$ *is not a lattice homomorphism.*

(2) $\rho\colon \Lambda(\mathsf{MGrz}) \to \Lambda(\mathsf{MIPC})$ *is not one-to-one.*

Proof. (1). Let $\Lambda(\mathbb{MG}\mathbb{rz})$ be the complete lattice of subvarieties of $\mathbb{MG}\mathbb{rz}$. It follows from Theorem 3.6(2) that $\Lambda(\mathsf{MGrz})$ is dually isomorphic to $\Lambda(\mathbb{MG}\mathbb{rz})$. By Theorem 6.16(3), $\mathsf{S}\mathcal{O}\colon \Lambda(\mathbb{MG}\mathbb{rz}) \to \Lambda(\mathbb{MHA})$ is not a lattice homomorphism. Therefore, neither is ρ by Theorems 2.6 and 5.12(1).

(2). This is proved similarly, but uses Theorem 6.16(4). □

The previous theorem immediately yields:

Corollary 6.19. $\rho\colon \Lambda(\mathsf{MS4}) \to \Lambda(\mathsf{MIPC})$ *is neither a lattice homomorphism nor one-to-one.*

We are ready to prove that the Blok–Esakia Theorem fails in the monadic setting.

Theorem 6.20. $\sigma\colon \Lambda(\mathsf{MIPC}) \to \Lambda(\mathsf{MGrz})$ *is not onto, hence is not an isomorphism.*

Proof. By Proposition 4.3(2), σ is left adjoint to $\rho\colon \Lambda(\mathsf{MGrz}) \to \Lambda(\mathsf{MIPC})$. Thus, $\sigma\rho\sigma = \sigma$. If σ were onto, then $\sigma\rho$ would be the identity on $\Lambda(\mathsf{MGrz})$, and hence ρ would be one-to-one. This contradicts Theorem 6.18(2). □

To summarize, in this paper we showed that the Blok–Esakia isomorphism does not extend to the monadic setting. We conclude by outlining several interesting directions for future research.

- It remains open whether the two lattices $\Lambda(\mathsf{MIPC})$ and $\Lambda(\mathsf{MGrz})$ are isomorphic. Our expectation is that the answer is negative.
- It is also open whether each monadic Heyting algebra can be realized as the algebra of open elements of some monadic S4-algebra (see the end of Section 5).
- A related open problem is whether $\rho\colon \Lambda(\mathsf{MS4}) \to \Lambda(\mathsf{MIPC})$ is surjective. In other words, it remains open whether every extension of MIPC has a modal companion (see Remark 6.4(2)).
- In addition, it is open whether τ and σ are complete lattice homomorphisms (see Remark 6.4(1)).

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Data availability

No data was used for the research described in the article.

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