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Fourier Analysis: A New Computing Approach[a]

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FOURIER EXPANSION is a special case of signal decomposition that decomposes a signal into oscillatory components. In this method, the signal is represented as a linear combination of trigonometric or exponential basis functions. The expansion coefficients (or weights) are then computed by correlating the signal with the corresponding basis functions [1, 2]. The process of computing the coefficients is known as Fourier analysis. In real applications, we are interested to use a few terms of Fourier expansion or it may be impossible to use all the terms to approximate the signal. Therefore, a truncated Fourier expansion is used instead [3]. However, when a truncated Fourier expansion is used to approximate a signal with a jump discontinuity, an overshoot/undershoot at the discontinuity occurs which is known as Gibbs phenomenon. The correct size of the overshoot and the undershoot of truncated Fourier expansion near the point of discontinuity was computed by Gibbs, that is the size of overshoot/undershoot is approximately 9% of the magnitude of the jump [4]. This lecture note shows that the size of overshoot mainly depends on the approach used for computing the Fourier analysis. It shows that in the traditional approach, the Fourier analysis is computed based on the minimization of the mean squares error (MSE) between the signal and its Fourier expansion (*i.e.*, the ℓ_2-norm minimization of the model error). Then it presents a new method to compute the Fourier analysis. In the new approach, the Fourier analysis (expansion coefficients) is obtained by minimizing the mean absolute error (MAE) between the reconstructed signal and the original signal. Since the new approach is defined based on the ℓ_1-norm minimization, we call it ℓ_1 Fourier analysis. Similarly the traditional approach is called ℓ_2 Fourier analysis. Using ℓ_1 Fourier analysis, we observed that the size of overshoot/undershoot for truncated Fourier expansion of signals with jump discontinuities is decreased to 4% of the magnitude of the jump. The effectiveness of the proposed ℓ_1 Fourier analysis, in terms of reduction of Gibbs phenomenon in truncated Fourier-series expansion and filtering the impulsive noise from the signals and images, is showcased using numerical examples.

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Relevance

In signal analysis, one often encounters the so called Fourier analysis. It is one of the most important tools in mathematics, computer science and signal processing where one's need to solve a partial differential equation (PDE) [5], compress music in MP3 players [6], or compress the digital images in JPEG form [7], digital spectral analysis [8] and filter design [9], to just name a few. The method consists of two steps: i) decomposing the signal into oscillatory components by expanding it based on a linear combination of a set of trigonometric or exponential functions with fundamental frequencies; ii) computing the Fourier analysis or finding the expansion coefficients. It was first introduced by Baron Jean Baptiste Joseph Fourier in 1807 to derive the equations of heat propagation using some series of trigonometric function [10]. The original derivation by Fourier was proposed for representing a continuous-time (CT) signal. Nowadays, due to the power of digital computers, the Fourier analysis is mainly presented in the context of discrete-time (DT) signals (sequences) and systems. The most important transform that performs Fourier analysis for discrete signals is the discrete Fourier transform (DFT). The formulations of discrete Fourier analysis (or DFT) and their CT counterparts (continuous Fourier series) are quite similar with some differences. In 1965, Cooley and Tukey jointly developed an implementation of DFT for high speed computers which is known as fast Fourier transform (FFT) [11].

In this article, we concern ourselves mainly with DFT, which is of great practical importance in the analysis of discrete signals and other data. We study the Gibbs phenomenon in truncated Fourier expansions of functions with jump discontinuities and propose a new approach to compute the DFT (Fourier coefficients) that reduces the Gibbs effect. The method is based on the replacement of ℓ_2-norm with ℓ_1-norm. It minimizes the mean absolute error (MAE) between the signal and its Fourier expansion. The replacement of ℓ_2-norm with ℓ_1-norm is a treatment studied for two decades in sparse solutions [12, 13] and compressed sensing [14]. Especially, in many applications in compressed sensing, the measurement matrix is a Fourier matrix [15]. Compressive sensing shows that a compressed signal can be reconstructed from much fewer incoherent measurements. Its aim is to represent the sparse signal without going through the intermediate stage of acquiring all the samples. It is also related to the problem of reconstructing the signal from incomplete frequency information [16]. The ℓ_1-norm has also been attracting more and more attention for the interpolation and approximation of functions and irregular geometric data. In [17], the authors show that the Gibbs phenomenon can be reduced by using ℓ_1 spline fits. In [18], some theoretical results are provided to explain the potential of such methods in avoiding the Gibbs phenomena. While none of the above algorithms use the ℓ_1-norm to identify the expansion coefficients, in this article, the process of computing the Fourier analysis is defined based on the ℓ_1-norm minimization of the error between the reconstructed signal and the original signal. The main problem is that there is no analytic formula for its solution. Therefore, a majorization minimization (MM) approach [19] is used to solve the problem which results in a linear iterative algorithm.

Prerequisites

This lecture note requires basic knowledge of signal and system, engineering mathematics and optimization problem.

Discrete Fourier Transform

The inverse DFT states that a discrete signal can be represented as a linear combination of trigonometric/exponential functions with fundamental frequencies. That is a given signal $x[n]$, $n \in \mathbb{Z}_N = \{0,\ 1, \cdots,\ N-1\}$ can be represented as [20, 9]

$$x[n] = \sum_{k=0}^{N-1} c_k e^{i\frac{2\pi k}{N}n}, \tag{1}$$

where $i = \sqrt{-1}$ and the expansion coefficients c_k are given

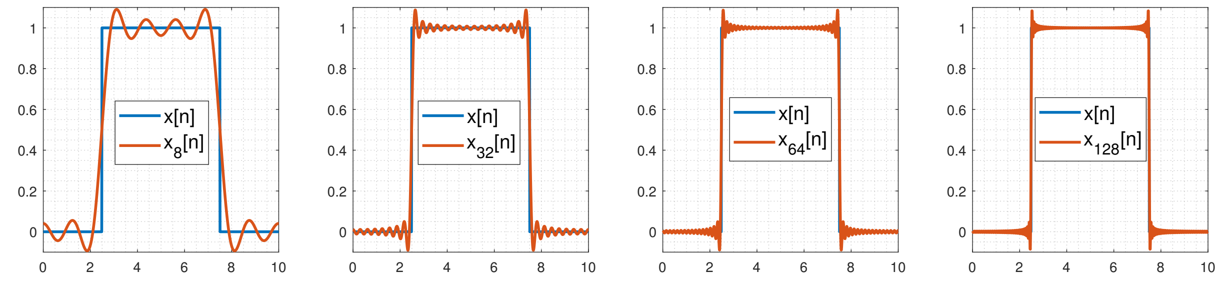


Figure 1: The result of the truncated Fourier expansion when the coefficients are computed using ℓ_2 Fourier analysis ($x_M[n]$ for $M = 8,\ 32,\ 64,\ 128$) in approximating a step function with $N = 1024$.

by

$$c_k = \frac{1}{N}\sum_{n=0}^{N-1} x[n]e^{-i\frac{2\pi k}{N}n}. \tag{2}$$

Some authors include the factor $1/N$ in the definition of $x[n]$ and not in the definition of Fourier analysis [21, 22, 23]. (2) is simply obtained by multiplying both sides of (1) with $e^{i\frac{2\pi j}{N}n}$; taking the sum of the result from 0 to $N-1$; simplifying it while considering the following relation:

$$\frac{1}{N}\sum_{n=0}^{N-1} e^{i\frac{2\pi j}{N}n}e^{i\frac{2\pi k}{N}n} = \begin{cases} 0 & \text{if } j \neq k \\ 1 & \text{if } j = k \end{cases} \tag{3}$$

It is mentionable that when x is a real-valued, its Fourier expansion is usually written in terms of sines and cosines:

$$x[n] = \sum_{k=0}^{N-1} \alpha_k \cos\left(\frac{2\pi k}{N}n\right) + \beta_k \sin\left(\frac{2\pi k}{N}n\right), \tag{4}$$

where the coefficients, α_k and β_k, are computed as

$$\begin{cases} \alpha_k = \dfrac{1}{N}\displaystyle\sum_{n=0}^{N-1} x[n]\cos\left(\frac{2\pi k}{N}n\right) \\ \beta_k = \dfrac{1}{N}\displaystyle\sum_{n=0}^{N-1} x[n]\sin\left(\frac{2\pi k}{N}n\right) \end{cases} \tag{5}$$

Note that c_k, α_k and β_k are related as $c_k = (\alpha_k - i\beta_k)/2$. In real applications, we are interested to use a few terms of Fourier expansion or it may be impossible to use all the terms to approximate the signal. Therefore, a truncated Fourier expansion is used instead. In the following section, the problem of representing the signals with a truncated Fourier expansion and its limitations is discussed.

Problem Statement

Let us consider the problem of approximating $x[n]$ by a truncated Fourier expansion $x_M[n]$ defined by

$$x_M[n] = \sum_{k=0}^{M-1} c_k e^{i\frac{2\pi k}{N}n}, \quad M < N. \tag{6}$$

In truncated Fourier expansion, the number of expansion terms is less than the length of signal. When (6) is used to approximate a signal with a jump discontinuity, an overshoot at the discontinuity occurs. This phenomenon was observed by Michelson when he was using his mechanical machine (called harmonic analyser) to produce graphs of truncated trigonometric series with terms up to 80 sines and cosines. He published his report [24] and the problem was explained by Gibbs [25, 26] in 1899, thus is known as Gibbs phenomenon. However, the history of studying the overshoot and undershoot in the neighborhood of discontinuities of the sums of Fourier series goes back to 1848 where Wilbraham published a paper on this topic for the first time [27]. The Gibbs effect is also seen in other signal decomposition approaches such as wavelet expansion, spline and cubic spline interpolation [28, 17, 18]. As an illustration of Gibbs phenomenon, we consider a step function with length $N = 1024$ ($t = 0 : 0.0098 : 10$) and its Fourier expansion using the truncated model (6). The step function and its truncated Fourier expansion with the first M modes, *i.e.*, $x_M[n]$, for $M = 8,\ 32,\ 64,\ 128$ are shown via blue and red line in Figure 1. The first plot includes only the first 8-th modes in the Fourier expansion, while the last plot includes up to the 128 modes. The more modes we include, the more the curve looks like a step function. However, it introduces strange wiggles (overshoots) near the discontinuity. As the number of modes grows, the wiggles get pushed closer and closer to the discontinuity, in the sense that the amplitude in a given region decreases as the number of modes, M, increases. So in some sense the wiggles go away as M approaches N. However, the overshoot unfortunately never goes away, but it remains roughly the same size (about 9% of the magnitude of the jump). It is provable that this 9% result holds for a discontinuity in any function, not just a step function [4]. So, the question arises as to whether this 9% overshoot is due to the Fourier expansion or due to the approach used for computing the expansion coefficients. If it is due to the later case, is it possible to decrease the Gibbs effect by employing a different computation approach? This article shows that the 9% overshoot of the Fourier expansion reported in the literature is due to the approach used for computing the Fourier analysis and it can be decreased if we employ a proper approach to compute the coefficients.

Solution

As it become clear in the following discussion, the Fourier analysis (expansion coefficients) computed by (2) minimizes the mean square error (MSE) between the signal and its Fourier expansion. MSE corresponds to the ℓ_2-norm. It also gives the maximum likelihood (ML) estimate of the coefficients, under the assumption of a white Gaussian distribution for the modeling error. The problem is that this assumption does not hold in practice. Specially, for the above example, the estimation error confirms a non-Gaussian distribution. We will see that for truncated Fourier expansion of the step function, the Gibbs phenomenon can be reduced if the Fourier coefficients are computed by minimizing the mean absolute error (MAE) between signal and its model.

MSE based Fourier analysis computation

(6) can be written in the following form

$$x_M[n] = \sum_{k=0}^{M-1} c_k \phi_k[n], \tag{7}$$

where

$$\phi_k[n] = e^{i\frac{2\pi k}{N}n}. \tag{8}$$

The expansion error is defined by $e[n] = x[n] - x_M[n]$, *i.e.*,

$$e[n] = x[n] - \sum_{k=0}^{M-1} c_k \phi_k[n]$$

An important property for the expansion model is its ability in signal approximation, that is the error signal should be within an acceptable range. Let us consider that the parameters are found by minimizing the power of the residual error signal, $e[n]$:

$$\hat{c}_k = \underset{c_k}{\text{argmin}} \frac{1}{N} \sum_{n=0}^{N-1} \left(x[n] - \sum_{k=0}^{M-1} c_k \phi_k[n] \right)^2 \tag{9}$$

Then the minimization of (9) with respect to the coefficients c_k, leads to the following solution:

$$\boldsymbol{c}_{opt} = \boldsymbol{\Phi}^{-1} \boldsymbol{x} \tag{10}$$

where $\boldsymbol{\Phi} \in \mathbb{R}^{M \times M}$ and $\boldsymbol{x} \in \mathbb{R}^M$ are, respectively, matrices and vectors with the following entries:

$$\begin{aligned} \boldsymbol{\Phi}_{p,q} &= \frac{1}{N} \sum_{n=0}^{N-1} \phi_p[n] \phi_q^*[n], \quad (p, q = 0, \cdots, M-1) \\ \boldsymbol{x}_q &= \frac{1}{N} \sum_{n=0}^{N-1} x[n] \phi_q^*[n] \end{aligned} \tag{11}$$

Since, the Fourier basis functions $\phi_k[n]$ form an orthonormal basis, the matrix $\boldsymbol{\Phi}$ becomes an identity matrix and the coefficients are found as

$$\boldsymbol{c}_{opt} = \boldsymbol{x}, \tag{12}$$

which is exactly the same as (2) when we set $M = N$. It means that the Fourier analysis obtained by (2) minimizes the MSE between the signal and its expansion (*i.e.*, the ℓ_2-norm of the error). Note that (10) is also the maximum likelihood (ML) estimate of the coefficient vector $\boldsymbol{c}$, under the assumption of a white Gaussian distribution for the modeling error $e[n]$. In the following, we compute the Fourier analysis by minimizing the MAE between the signal and its Fourier expansion (*i.e.*, the ℓ_1-norm minimization of the error).

MAE based Fourier analysis computation

An alternative method is to estimate the expansion coefficients using the following cost function:

$$\hat{c}_k = \underset{c_k}{\text{argmin}} \frac{1}{N} \sum_{n=0}^{N-1} \left| x[n] - \sum_{k=0}^{M-1} c_k \phi_k[n] \right|, \tag{13}$$

which substitutes a mean of absolute errors for the mean of square errors used in the traditional Fourier analysis. Using the proposed Fourier analysis, the truncated Fourier expansion can approximate a function with a jump

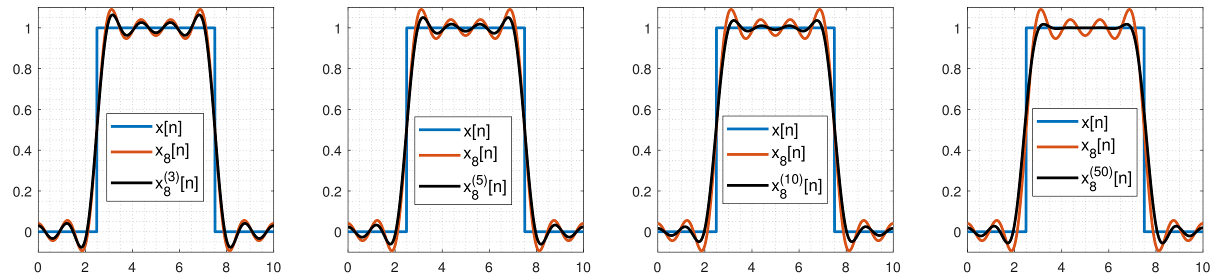


Figure 2: The result of the truncated Fourier expansion of $M = 8$ terms when the coefficients are computed using ℓ_1 Fourier analysis after a certain iteration ($x_8^{(r)}[n]$, $r = 3,\ 5,\ 10,\ 50$) in approximating the step function.

discontinuity with a more robust behaviour against the amplitude changes at discontinuous points. We call it ℓ_1 Fourier analysis or MAE Fourier analysis. Similarly, the traditional Fourier analysis is called ℓ_2 Fourier analysis or MSE Fourier analysis as the coefficients are computed by minimizing the MSE. The optimization problem (13) is convex but there is no analytic formula for its solution (*i.e.*, it is difficult to minimize due to the last term since it is non-differentiable). However, it can be solved numerically in a linear computational complexity. In this article, optimization problem (13) is solved by the MM approach [19]. The key idea of the MM approach is to convert the intractable original problem into a simpler one that can be solved. Specifically, the original cost function is approximated by an iterative tractable surrogate function. Then a solution is found by minimizing the surrogate function with non-increasing cost. The obtained solution converges to a stationary point of the original optimization problem. In order to solve the problem (13), we use a majorizer for the absolute value. That is

$$|x[n]| \leq \frac{1}{2}\frac{x^2[n]}{|x_M^{(r)}[n]|} + \frac{1}{2}\left|x_M^{(r)}[n]\right|, \tag{14}$$

with equality when $x[n] = x_M^{(r)}[n]$ ($x_M^{(r)}[n]$ is the estimated signal after r iterations). In this case, (13) is expressed as

$$\hat{c}_k^{(r+1)} = \underset{c_k}{\text{argmin}} \frac{1}{N}\sum_{n=0}^{N-1}\frac{\left(x[n] - \sum_{k=0}^{M-1} c_k\phi_k[n]\right)^2}{2\left|x[n] - x_M^{(r)}[n]\right|} + \frac{\gamma}{2}, \tag{15}$$

where $\gamma = \left|x[n] - x_M^{(r)}[n]\right|$ and

$$x_M^{(r)}[n] = \sum_{k=0}^{M-1} \hat{c}_k^{(r)}\phi_k[n]$$

The minimization of (15) with respect to the coefficients c_k, leads to the following solution:

$$\boldsymbol{c}_{opt}^{r+1} = \left(\boldsymbol{\Psi}^{(r)}\right)^{-1}\tilde{\boldsymbol{x}}^{(r)} \tag{16}$$

where $\boldsymbol{\Psi}^{(r)} \in \mathbb{R}^{M\times M}$ and $\tilde{\boldsymbol{x}}^{(r)} \in \mathbb{R}^{M}$ are, respectively, matrices and vectors with the following entries:

$$\begin{aligned}\boldsymbol{\Psi}_{p,q}^{(r)} &= \frac{1}{N}\sum_{n=0}^{N-1}\frac{\phi_p[n]\phi_q^*[n]}{\left|x[n] - x_M^{(r)}[n]\right|}, \quad (p,q = 0,\cdots,M-1)\\ \tilde{\boldsymbol{x}}_q^{(r)} &= \frac{1}{N}\sum_{n=0}^{N-1} x[n]\frac{\phi_q^*[n]}{\left|x[n] - x_M^{(r)}[n]\right|}\end{aligned} \tag{17}$$

In this study, we consider the initial condition $x_M^{(0)}[n] = x[n] + \kappa$, where κ is a non-zero constant value. In this case, the traditional approach (*i.e.*, ℓ_2 Fourier analysis) is a special case of ℓ_1 Fourier analysis when κ and the number of iterations, r, are both set to one ($\kappa = r = 1$). The proposed approach (ℓ_1 Fourier analysis) was used to compute the coefficients of the truncated Fourier expansion of the step function in the previous example. Figure 2 illustrates the performance of truncated Fourier series with the first 8 modes. The solid blue and red curve in Figure 2 denote the theoretical step function (*i.e.*, $x[n]$) and its Fourier expansion using the ℓ_2 Fourier analysis (*i.e.*, $x_8[n]$), respectively, which are the same as those in Figure 1. The truncated Fourier expansion with the expansion coefficients obtained by ℓ_1 Fourier analysis after $r = \{3,\ 5,\ 10,\ 50\}$ iterations (*i.e.*, $x_8^{(r)}[n]$) is plotted in Figure 2 with black solid line. The overshoot of the truncated Fourier expansion is reduced when the expansion coefficients are computed using ℓ_1 Fourier analysis. Specially, the overshoot is decreased as the number of iterations is increased. The cost function evolution of MM approach is illustrated in Figure 3. It is seen that the algorithm converges well within a few iterations. In Figure 4(a) and

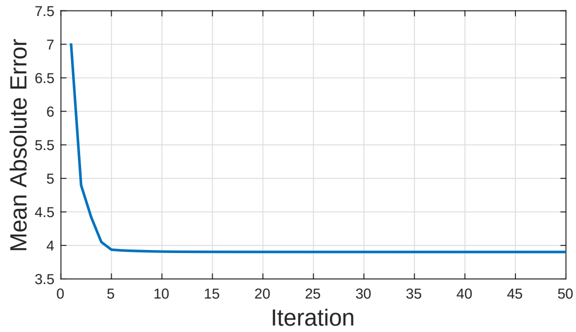
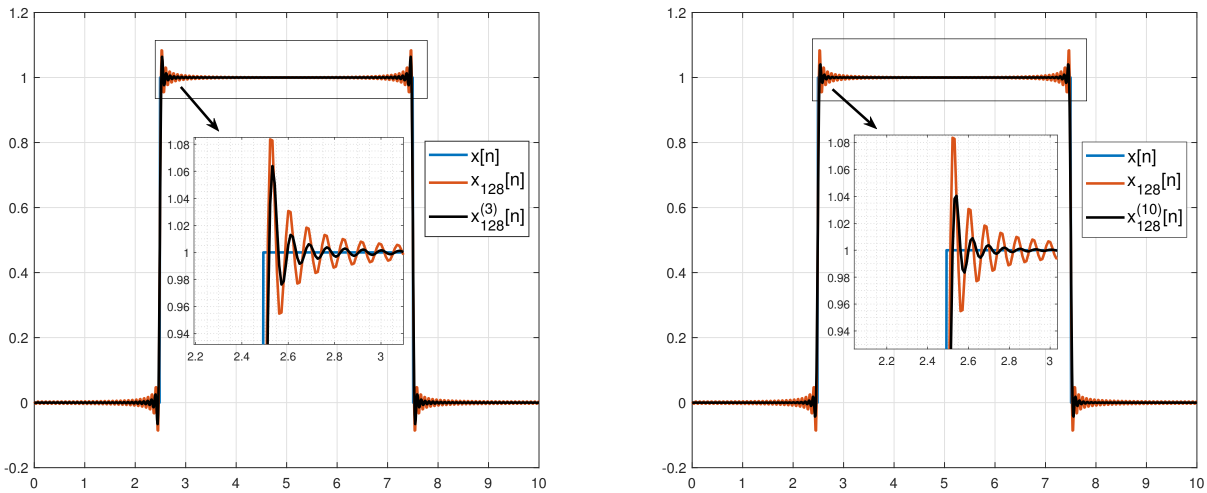


Figure 3: The convergence of MM approach.



(a) r = 3

(b) r = 10

Figure 4: The result of the proposed truncated Fourier expansion of $M = 128$ terms after a certain iteration, $x_8^{(r)}[n]$, in approximating the step function, for $r = 3,\ 10$. The truncated Fourier expansion using ℓ_2 Fourier analysis ($x_M[n]$) is also illustrated via red line for comparison.

4(b), we illustrate the performance of the truncated Fourier series with the first 128 modes. The coefficients are computed by ℓ_1 Fourier analysis after 3 and 10 iterations, respectively. The results show that the overshoot can be decreased to 4% of the magnitude of the jump. Since the matrices $\mathbf{\Psi}^{(r)}$ in (16) are non-orthogonal, the ℓ_1 Fourier analysis involves matrix inversion, which has a complexity of $O(M^3 + MN)$. There are three special spaces in convex optimization: i) the ℓ_1-norm which mostly replaces ℓ_0-norm as ℓ_0-norm is not convex and not well defined, ii) the celebrated ℓ_2-norm that everybody knows and uses, and iii) the ℓ_∞-norm. Other norm spaces mostly produce performances in between these. We also employed the ℓ_∞-norm minimization to compute the Fourier analysis. The truncated Fourier expansion of the step function when the coefficients are computed using ℓ_∞ Fourier analysis is shown via red color in Figure 5. The ℓ_8- and ℓ_4-norm minimization are also employed to compute the Fourier analysis. The truncated Fourier expansion using ℓ_8 and ℓ_4 Fourier analysis are respectively shown via yellow and purple. As expected these two norms produce performance in between ℓ_1- and ℓ_∞-norm minimization. In other words, the size of overshoot/undershoot for truncated Fourier expansion decreases when we decrease the norm of minimization in Fourier analysis computation.

Applications

The new Fourier analysis can find various applications in signal processing. As a proof of concept, we focus on two

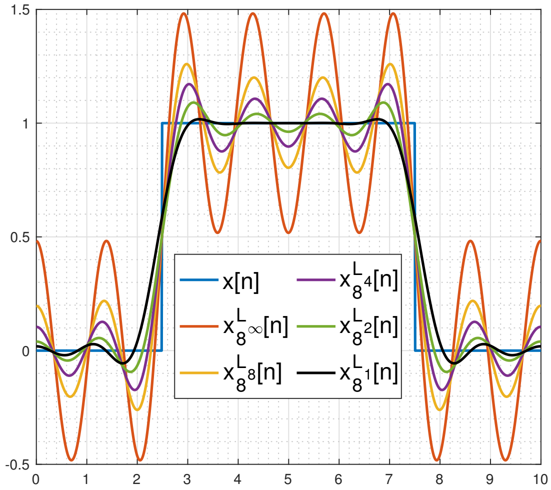


Figure 5: The result of the truncated Fourier expansion of $M = 8$ terms ($x_M^{\ell_p}[n]$) when the coefficients are computed using ℓ_∞, ℓ_8, ℓ_4, ℓ_2 and ℓ_1 Fourier analysis in approximating the step function.

examples.

Reduction the Gibbs phenomena in image filtering

Signal decomposition based method such as Fourier, discrete cosine and wavelet transform is a common approach to image filtering. In this section, we consider the discrete cosine transform (DCT) for low-pass filtering of an image. For a two-dimensional signal $x[n_1, n_2]$, $n_1 = 0,\ 1, \cdots,\ N_1 - 1$, $n_2 = 0,\ 1, \cdots,\ N_2 - 1$, one of the possible two-dimensional inverse discrete cosine transform (iDCT) is defined as

$$x[n_1, n_2] = \sum_{k_1=0}^{N_1-1} \sum_{k_2=0}^{N_2-1} c_{k_1,k_2} \phi_{k_1,k_2}[n_1, n_2], \qquad (18)$$

where

$$\begin{cases} \phi_{k_1,k_2}[n_1, n_2] = \cos \dfrac{k_1 \pi}{N_1} \left(n_1 + \dfrac{1}{2} \right) \cos \dfrac{k_2 \pi}{N_2} \left(n_2 + \dfrac{1}{2} \right) \\ c_{k_1,k_2} = \dfrac{1}{N_1 N_2} \displaystyle\sum_{n_1=0}^{N_1-1} \sum_{n_2=0}^{N_2-1} x[n_1, n_2] \phi_{k_1,k_2}[n_1, n_2] \end{cases} \qquad (19)$$

When we represent an image by iDCT, Gibbs phenomenon is the most common image artifact that arises from truncated iDCT of an image. For instance, consider the original image shown in the left side of Figure 6. We contaminated it with salt-and-pepper noise as shown in the middle of the Figure 6. Salt-and-pepper noise is an impulse noise which is sometimes seen on images. This noise can be caused by sharp and sudden disturbances in the image signal. We employed the truncated iDCT to reconstruct the original image. The result is shown in the right side of Figure 6. The truncated iDCT is not a good model for reconstructing the original image. The noise is not eliminated by the model and the Gibbs effect is clear in the output. The weak performance of the truncated iDCT model is due to the approach used for DCT computation. To show it, we computed the DCT using the ℓ_1 optimization approach. The results of the truncated iDCT using the ℓ_1 DCT after a certain iteration ($r = 2,\ 5,\ 15$) are shown in Figure 7. It is seen that the reconstructed image becomes close to the original image as the number of iterations increases. In other words, the Gibbs effect is reduced when the ℓ_1 optimization is used to compute the coefficients (*i.e.*, DCT). Therefore, the Gibbs effect is mainly due to the DCT computation approach but not the iDCT model. It is decreased in low-pass filtering of images while maintaining the sharp edges features.

Audio filtering

In this section, we compare the truncated Fourier expansion (when the coefficients are computed using ℓ_2 or ℓ_1 Fourier analysis) and zero-phase Butterworth filter for denoising audio signals corrupted by random-valued impulse noise. To this purpose, we consider "guitartune.wav" the standard sample tune that ships with MATLAB. The signal contains 661500 samples which we split it to sub signals each contains 10000 samples. Therefore, we have 66 signals.

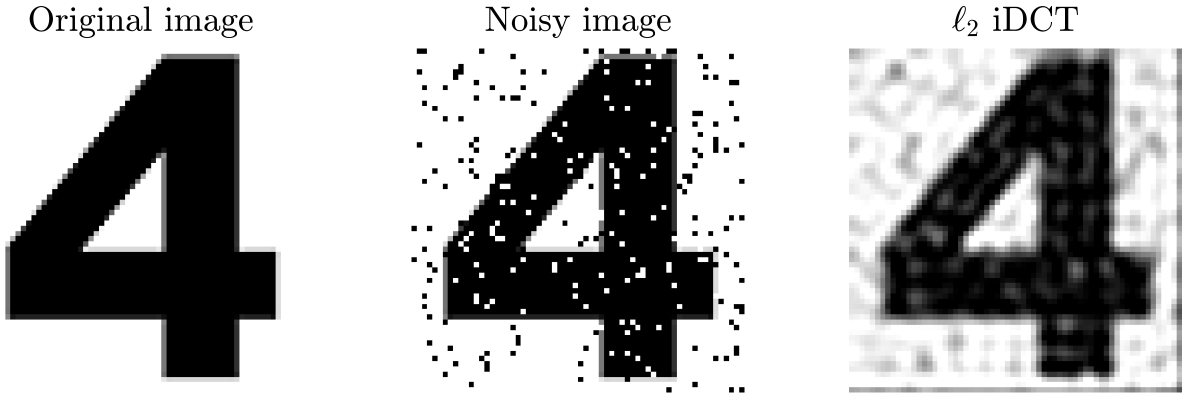


Figure 6: Image filtering using truncated ℓ_2 iDCT. From left to right: original image, noisy image obtained by adding salt-and-pepper noise and denoised image using truncated ℓ_2 iDCT.

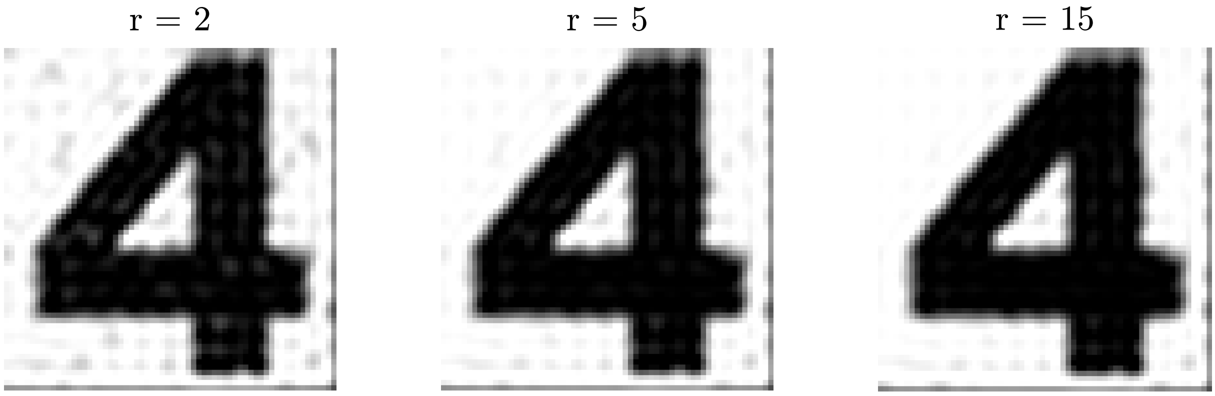
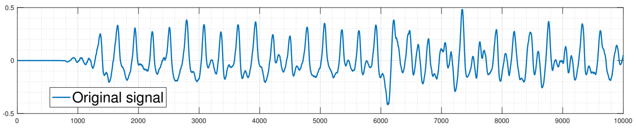


Figure 7: Image denoising using truncated ℓ_1 iDCT after a certain iteration, $r = 2,\ 5,\ 15$.

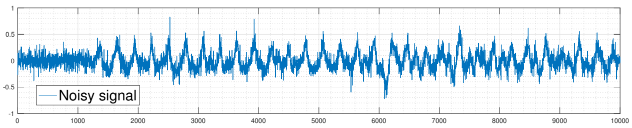
The signals were contaminated with impulse noise. To this purpose, we produced signals varying the power of noise. The signal-to-noise ratio (SNR) was modulated from -5 dB to 10 dB. The truncated Fourier expansion (using ℓ_2 or ℓ_1 Fourier analysis) and the third order zero-phase Butterworth filter were then used to reconstruct the desired signals from the noisy signals. The cutoff frequency was set to 800 Hz. As an example, the first 10000 samples of "guitartune.wav" and its noisy signal with SNR = 5 dB are shown in Figure 8(a) and Figure 8(b), respectively. The results of audio signal filtering with these three methods are illustrated in Figures 8(c)-8(e). For evaluating the performance of the methods, we used the average square relative error (SRE) and the average absolute relative error (ARE) of the estimation accuracy defined by

$$\begin{aligned} \text{SRE} &= \frac{\sum_k (x_k - \hat{x}_k)^2}{\sum_k x_k^2} \\ \text{ARE} &= \frac{\sum_k |x_k - \hat{x}_k|}{\sum_k |x_k|} \end{aligned}, \tag{20}$$

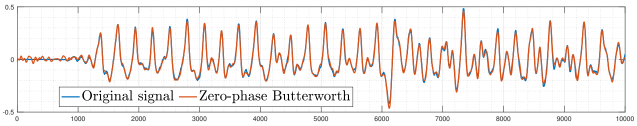
where x and $\hat{x}$ are the original and estimated signal, respectively. The results of the reconstruction procedures using these metrics are reported in Figures 9(a) and 9(b). The SRE and ARE for the truncated Fourier expansion when the coefficients are computed using ℓ_1 Fourier analysis is less than zero-phase Butterworth filter and truncated Fourier expansion when the coefficients are computed using ℓ_2 Fourier analysis which means that the proposed ℓ_1 Fourier expansion outperforms other two methods.



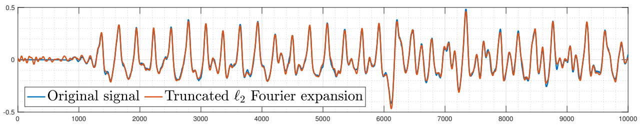
(a) Original signal



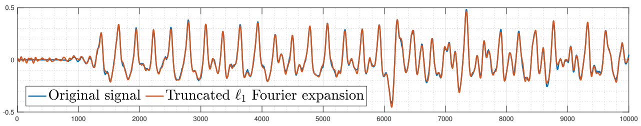
(b) Noisy signal (SNR = 5 dB)



(c) Zero-phase Butterworth filter



(d) Truncated ℓ_2 Fourier expansion



(e) Truncated ℓ_1 Fourier expansion

Figure 8: Audio filtering using zero-phase Butterworth filter (SRE = 0.1092 and ARE = 0.1152), truncated ℓ_2 Fourier expansion (SRE = 0.1118 and ARE = 0.1183) and ℓ_1 Fourier expansion (SRE = 0.0963 and ARE = 0.1000).

What we have learned

In Fourier decomposition, the signal is represented as a linear combination of trigonometric or exponential basis functions and the expansion weights (or coefficients) are computed such that it best fits the signal. In the traditional approach, the Fourier analysis (expansion coefficient) is computed based on the minimization of mean square error between the signal and its expansion (*i.e.*, ℓ_2-norm minimization of the error). In this approach, when a truncated Fourier expansion is used to approximate a signal with a jump discontinuity, an overshoot/undershoot at the discontinuity occurs which is known as Gibbs phenomenon. Using ℓ_2 Fourier analysis, the size of overshoot/undershoot is approximately 9% of the magnitude of the jump. We have learned that the size of overshoot/undershoot is mainly due to the approach used for computing the Fourier analysis. The Fourier analysis can be computed using other ℓ_p-norm minimization. The size of overshoot/undershoot changes if the Fourier analysis is computed based on the ℓ_p-norm minimization of the model error. For $p \geq 1$, other ℓ_p-norm minimization mostly produce performances in between ℓ_1- and ℓ_∞-norm minimization.

Some future directions are summarized as follows:

- Although different ℓ_p-norm optimization approach ($p \geq 1$) were used to compute the Fourier analysis and the ℓ_1 Fourier analysis is the best choice for reducing the size of overshoot/undershoot in the truncated Fourier expansion of the step function, there are some improvements that can be done in the future.
- The extension of the proposed computing approach to ℓ_p Fourier analysis for $0 \leq p < 1$ is interesting. For instance, it is interesting to see what could be the result by using ℓ_0-norm minimization.
- The extension of the proposed method to compute the expansion coefficients of other signal decomposition based methods such as wavelet transform, polynomial and spline interpolation is another possibility.
- Finally, the ℓ_1 Fourier analysis can be used to improve the accuracy of the Fourier expansion at cost of increasing the computational complexity. Improving the computational complexity of the proposed approach can be considered as future work.

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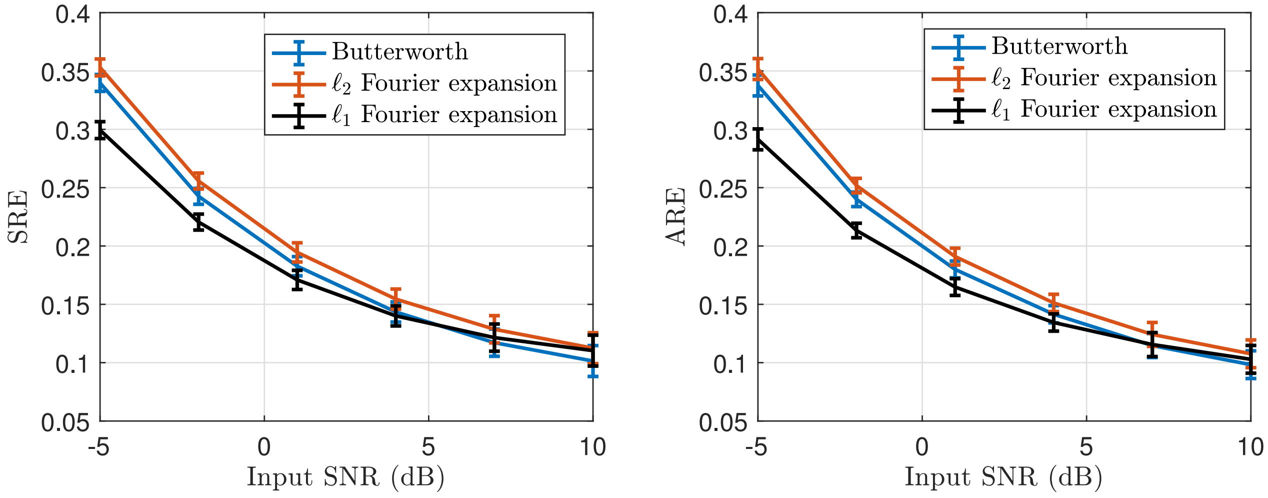

(a) SRE (b) ARE

Figure 9: Audio low-pass filtering using ℓ_2 and ℓ_1 Fourier expansion and zero-phase Butterworth filter.

Fourier and Gibbs phenomena available on YouTube which motivated me to prepare this lecture note.

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